The Pierce Decomposition and Pierce Embedding of Endomorphism Rings of Abelian p-Groups

Brendan Goldsmith
Technological University Dublin, brendan.goldsmith@tudublin.ie

Luigi Salce
Universita di Padova, Paldova, Italy

Funder: No funding was received for this work.

Research Article

Brendan Goldsmith* and Luigi Salce

The Pierce decomposition and Pierce embedding of endomorphism rings of abelian *p*-groups

https://doi.org/10.1515/forum-2022-0208
Received July 18, 2022; revised February 7, 2023

Abstract: The first goal of this paper is to investigate the Pierce decomposition of the endomorphism ring $\mathrm{End}(G) = \widehat{F} \oplus \mathrm{End}_s(G)$ of an abelian p-group G and its application to the recent studies of groups with minimal full inertia and of thick-thin groups. The second goal is to investigate the Pierce embedding

$$\Psi : \mathrm{End}(G)/H(G) \to \prod_n M_{f_n(G)}.$$

We prove that more classes of groups than those described by Pierce have the property that the map Ψ is surjective, and we furnish examples of groups which do not have this property. Several results connecting the Pierce decomposition and the Pierce embedding of $\mathrm{End}(G)$ are obtained that allow one to derive general conditions on a group G which ensure that the Pierce embedding of $\mathrm{End}(G)$ is not surjective.

Keywords: Primary groups, endomorphism rings, Pierce decomposition, Pierce embedding, fully inert subgroups, minimal full inertia, thick-thin groups

MSC 2020: Primary 20K10, 20K27, 20K30; secondary 20K40

Communicated by: Manfred Droste

1 Introduction

All groups considered in this paper are assumed to be additively written abelian p-groups, except if explicitly stated otherwise, where p is a fixed but arbitrary prime number; we denote the ring of p-adic integers by J_p. For unexplained terminology and notions, we refer to the recent monograph [5] by Fuchs.

This paper is heavily based on the pioneering and fundamental paper by Pierce [12], whose review by Hulanicki started with these words: “The great value of the paper lies in the discovery of a new method of investigating p-primary groups and their homomorphisms which enables the author to obtain many important results." Few papers on abelian groups had the same influential impact on the theory of p-groups in the last sixty years as this paper of Pierce.

Two tools play a central role in the “new method of investigating" the endomorphism ring $\mathrm{End}(G)$ of an arbitrary p-group G. The first tool is the so-called *Pierce decomposition* of $\mathrm{End}(G)$, used by many authors for different purposes; see, for instance, the realization theorems by Corner [3]. The second tool, which we call *Pierce embedding*, was used by Pierce to describe properties of direct sums of cyclic groups and torsion-complete groups; it is possible to connect it with the Pierce decomposition. We now describe these two tools in more detail.

The Pierce decomposition of the endomorphism ring of G is the group direct sum $\mathrm{End}(G) = A \oplus E_s(G)$, where $A = \widehat{F}$ is a J_p-algebra which is the completion in the p-adic topology of a free J_p-module F, and $\mathrm{End}_s(G)$

***Corresponding author: Brendan Goldsmith,** Technological University Dublin, City Campus, Lower Grangegorman, Dublin 7, D07T795, Ireland, e-mail: brendan.goldsmith@tudublin.ie
Luigi Salce, Dipartimento di Matematica “Tullio Levi-Civita”, Università di Padova, Via Trieste 63 - 35121 Padova, Italy, e-mail: salce@math.unipd.it

is the two-sided ideal of the small endomorphisms. Recall that an endomorphism $\phi \in \mathrm{End}(G)$ is small if, for every $k \geq 1$, there exists an integer n such that $\phi(p^n G[p^k]) = 0$ (as usual, $p^n G[p^k]$ denotes the intersection $p^n G \cap G[p^k]$).

In order to introduce the Pierce embedding of End(G), recall that in [12] Pierce defines a map

$$\mu : \mathrm{End}(G) \to \prod_{n \in \mathbb{N}} M_{f_n(G)},$$

where G is an unbounded p-group, $f_n(G) = \dim(p^n G[p]/p^{n+1} G[p])$ is the Ulm–Kaplansky invariant of index $n \in \mathbb{N}$, and $M_{f_n(G)}$ is the endomorphism ring of the vector space over the field with p elements of dimension $f_n(G)$. The map μ sends the endomorphism ϕ of G to the sequence $\{\phi_n\}_{n \in \mathbb{N}}$, where

$$\phi_n : p^n G[p]/p^{n+1} G[p] \to p^n G[p]/p^{n+1} G[p]$$

is the map canonically induced by ϕ.

Recall that, if $B = \bigoplus_{n \in \mathbb{N}} B_n$ is a basic subgroup of G, with $B_n \cong \bigoplus_{\alpha_n} \mathbb{Z}(p^n)$ being the n-th homocyclic component (α_n is a cardinal number), then $B_n[p]$ is canonically isomorphic to $p^{n-1} G[p]/p^n G[p]$, so

$$\alpha_n = f_{n-1}(G) = f_{n-1}(B).$$

In the sequel, we will often identify $p^{n-1} G[p]/p^n G[p]$ with $B_n[p]$ and $M_{f_{n-1}(G)}$ with $\mathrm{End}(B_n[p])$.

Pierce proves in [12, Theorem 14.3] that μ is a ring homomorphism, whose kernel (called today the *Pierce radical* of End(G)) is the two-sided ideal

$$H(G) = \{\phi \mid \phi(p^n G[p]) \leq p^{n+1} G[p] \text{ for all } n \in \mathbb{N}\},$$

and that $\bigoplus_{n \in \mathbb{N}} M_{f_n(G)} \subseteq \mathrm{Im}(\mu)$. Therefore, there is a ring embedding

$$\Psi : \frac{\mathrm{End}(G)}{H(G)} \to \prod_n M_{f_n(G)}$$

that we call the *Pierce embedding* of End(G), with abuse of terminology since it is not an embedding of End(G), but of a factor ring of it (see also [5, Proposition 2.12, p. 626]).

Recently, the Pierce decomposition was applied in the paper [7], which investigates groups with minimal full inertia, that is, groups whose fully inert subgroups are commensurable with fully invariant subgroups (for these notions we refer to [7]). One notion which does not appear in [5] and which is central in the next theorem proved in [7], is the notion of a *semi-standard* p-group. This terminology, which seems to have originated in the work of Corner [3], is as follows: a p-group G is said to be semi-standard if for each $n < \omega$ the Ulm–Kaplansky invariant $f_n(G)$ is finite; equivalently, each homocyclic component of exponent $n + 1$ of a basic subgroup of G is a direct sum of finitely many cyclic groups of order p^{n+1}.

Theorem 1.1 (Goldsmith–Salce). *Let G be a separable semi-standard p-group such that in the Pierce decomposition* $\mathrm{End}(G) = \widehat{F} \oplus \mathrm{End}_s(G)$*, $\widehat{F}$ is the completion in the p-adic topology of a J_p-subalgebra F which is a free J_p-module of at most countable rank. If H is a countably infinite subgroup of G, then the higher socles $H^F[p^k]$ ($k \geq 1$) of the subgroup $H^F = \sum_{\alpha \in F} \alpha(H)$ are fully inert in G, but not commensurable with any fully invariant subgroup of G.*

Thus the groups satisfying the hypotheses of Theorem 1.1 do not have minimal full inertia. Surprisingly enough, a similar hypothesis on the Pierce decomposition furnished in [9] the key tool to obtain groups which are at the same time thick and thin, consequently called *thick-thin* (for these notions we refer to [9]). In fact, the following result was proved.

Theorem 1.2 (Keef–Salce). *Suppose G is an unbounded separable group and* $\mathrm{End}(G) = \widehat{F} \oplus \mathrm{End}_s(G)$ *is the Pierce decomposition of its endomorphism ring. If F is a free J_p-module of rank strictly less than the continuum $c = 2^{\aleph_0}$, then G is thick-thin.*

We now present an outline of the contents of the paper indicating the principal new results obtained in each section.

Section 2 is devoted to weakening the hypotheses in Theorem 1.1. Our principal result there is Theorem 2.1, in which we remove the restriction that F be closed under multiplication and replace the rank restriction of countability by the much weaker restriction of being smaller than the continuum. The significance of this for the study of groups with minimal full inertia is noted in Remark 2.5. We also provide an example of a complete J_p-algebra $\widehat{F}$ that is the completion in the p-adic topology of a free J_p-module F which fails to be an algebra with the induced multiplication.

In Section 3, we move to consideration of the Pierce embedding; recall that Pierce states in [12, Remark, p. 288] that the map Ψ is epic (hence an isomorphism) provided that G is either a direct sum of cyclic groups or is torsion-complete. In [5, Proposition 2.12 (ii), p. 627] it is stated that Ψ is epic if and only if G is torsion-complete, a claim which is clearly in conflict with Pierce's remark. In fact, our principal result in this section, Theorem 3.6, shows that for a countably infinite family of groups having the property that for each group the Pierce embedding is surjective, then their direct sum will also have a surjective Pierce embedding provided their Ulm–Kaplansky invariants satisfy a suitable independence condition. As a consequence, a large class of groups including direct sums of cyclic groups and certain direct sums of torsion-complete groups will have a surjective Pierce embedding. We also establish in Corollary 3.9 the hitherto unobserved fact that a direct summand of a group with surjective Pierce embedding also has this surjective property.

In Section 4, we investigate a question arising naturally from our results in Section 3: does an arbitrary direct sum of torsion-complete groups have a surjective Pierce embedding? We provide examples showing that this question has a negative answer. Our main results in this section, Theorem 4.4 and Theorem 4.5, show that the Pierce embedding of a direct sum may fail to be surjective if there are infinitely many common non-zero Ulm–Kaplansky invariants.

In Section 5, we investigate the relationship between the Pierce decomposition and the Pierce embedding. In Theorem 5.2, we establish that if the rank of the free J_p-module appearing in the Pierce decomposition is "not too large", then the Pierce embedding will fail to be surjective. Our final result, Theorem 5.4, shows that certain separable groups which are "close to being torsion-complete", and are in fact thick-thin groups, may also fail to have a surjective Pierce embedding.

2 Applications of the Pierce decomposition

Theorem 1.1 can be improved as follows.

Theorem 2.1. *Let G be a separable semi-standard p-group such that in the Pierce decomposition*

$$\mathrm{End}(G) = A \oplus \mathrm{End}_s(G),$$

A is the completion in the p-adic topology of a free J_p-module F of rank strictly smaller than the continuum. If H is a countably infinite subgroup of G, then the higher socles $H^F[p^k]$ $(k \geq 1)$ of the subgroup $H^F = \sum_{\alpha\in F} \alpha(H)$ are fully inert in G, but not commensurable with any fully invariant subgroup of G.

The proof of Theorem 1.1 made use of the following lemma, whose hypothesis of countability will be modified in the next Lemma 2.3.

Lemma 2.2 (Goldsmith–Salce). *Let G be an unbounded separable semi-standard p-group with Pierce decomposition of its endomorphism ring* $\mathrm{End}(G) = A \oplus \mathrm{End}_s(G)$. *If A is the completion in the p-adic topology of a free J_p-module F of countable rank, then G is uncountable.*

The next lemma provides the new version of Lemma 2.2; although its proof is similar to that of Lemma 2.2, we include it for the sake of completeness.

Lemma 2.3. *Let G be an unbounded separable semi-standard p-group with Pierce decomposition of its endomorphism ring* $\mathrm{End}(G) = A \oplus \mathrm{End}_s(G)$. *If A is the completion in the p-adic topology of a free J_p-module F of rank ρ strictly smaller than the continuum, c, then G is uncountable.*

Proof. Assume, by way of contradiction, that G is countable. Then, by [5, Theorem 5.3, p. 96], G is a direct sum of cyclic groups, say $G = \bigoplus_{i \in \mathbb{N}} C_i$.

Let $\chi : \mathrm{End}(G) \to \mathrm{Hom}(G[p], G)$ be the restriction map sending $\phi \in \mathrm{End}(G)$ into $\phi \restriction G[p]$. We claim that $\chi(\mathrm{End}(G))$ has cardinality at least c. In fact, every element of $\mathrm{Hom}(G[p], G)$ can be thought of as an infinite vector of the form $\delta = (\delta_1, \delta_2, \dots)$ where the δ_i correspond to homomorphisms from $C_i[p]$ into G. If we choose the δ_i to be either the zero map or the identity map, we can clearly produce c homomorphisms in $\mathrm{Hom}(G[p], G)$.

Furthermore, since the zero map and the identity map from $C_i[p] \to G$ both extend trivially to maps from $C_i \to G$, the vector δ clearly extends to a map $\gamma : G \to G$ which satisfies $\chi(\gamma) = \delta$. This proves that $|\chi(\mathrm{End}(G))| \geq c$.

However, the image of A under the map χ has cardinality at most $\rho = \max\{\mathrm{rk}_{J_p} F, \aleph_0\}$, since $\chi(pA) = 0$ and $A/pA \cong F/pF \cong (J_p/pJ_p)^{(\rho)}$ has cardinality at most ρ. Additionally, as G is semi-standard, every small endomorphism vanishes on a cofinite subgroup $p^n G[p]$ of $G[p]$ for some n. Consequently, the image $\chi(\mathrm{End}_s(G))$ is countable. Thus $\chi(A \oplus \mathrm{End}_s(G))$ has cardinality at most ρ, and so is strictly smaller than c, which is a contradiction. We conclude that G must be uncountable. □

In Lemma 2.3, we could also replace the hypothesis that G is semi-standard by the hypothesis that a basic subgroup of G has cardinality strictly smaller than c, deducing that $|\chi(\mathrm{End}_s(G))| < c$. However, the hypothesis that G is semi-standard is unavoidable in the proof of Theorem 2.1.

Using the notation of Theorem 1.1, the hypothesis that F is closed under multiplication is replaced by the following lemma.

Lemma 2.4. *H^F is F-invariant.*

Proof. Let $\phi \in F$. Since $H^F = \sum_{\alpha \in F} \alpha(H)$, it is enough to prove that $(\phi \cdot \alpha)(H) \leq H^F$ for each $\alpha \in F$; note that $\phi \cdot \alpha \in \widehat{F}$. Let $x \in H$ be such that $p^k x = 0$. As F is dense in $\widehat{F}$, we have that $\phi \cdot \alpha = \psi + p^k \beta$ for some $\psi \in F$ and $\beta \in \widehat{F}$. Then we have

$$(\phi \cdot \alpha)(x) = \psi(x) + p^k \beta(x).$$

But $p^k \beta(x) = \beta(p^k x) = 0$, and therefore $(\phi \cdot \alpha)(x) = \psi(x) \in H^F$, as desired. □

The proof of Theorem 2.1 is now identical to the proof of Theorem 1.1, just using Lemma 2.3 instead than Lemma 2.2, eliminating the words "because F is closed under multiplication" and justifying the F-invariance of H^F by Lemma 2.4.

Remark 2.5. The condition in Theorem 2.1 that the J_p-module F in the Pierce decomposition

$$\mathrm{End}(G) = \widehat{F} \oplus \mathrm{End}_s(G)$$

has rank smaller than c, in order that G does not have minimal full inertia, is a sufficient condition, but not necessary. In fact, the group $G = B \oplus \overline{B}$, where $B = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$ and $\overline{B}$ is its torsion-completion, does not have minimal full inertia, by [7, Proposition 3.7], and $\mathrm{End}(G)$ contains $\mathrm{End}(\overline{B}) = \widehat{F} \oplus \mathrm{End}_s(\overline{B})$, where F is a free J_p-module of rank c, the continuum.

The following question arises naturally: assuming that the completion $\widehat{F}$ in the p-adic topology of a free J_p-module F is a J_p-algebra, is F necessarily an algebra with the induced multiplication? The next example shows that the answer is in the negative.

Example 2.6. Let $R = J_p[X]$ be the ring of polynomials over J_p and let $\widehat{R}$ be its completion in the p-adic topology. Then $\widehat{R}$ is clearly a J_p-algebra. Consider $R = \bigoplus_{n \in \mathbb{N}} J_p X^n$ as a free J_p-module of countable rank and let $\mathbb{Q}_p$ be the field of the p-adic numbers, which is a divisible torsion-free group. As a J_p-module, $\mathbb{Q}_p$ is generated by the elements $\{p^{-n}\}_{n \in \mathbb{N}}$. Let $\Phi : R \to \mathbb{Q}_p$ be the surjective J_p-morphism defined as follows: $\Phi(X) = 0$ and $\Phi(X^n) = p^{-n}$ for $n > 1$. Let $F = \mathrm{Ker}\,\Phi$. Since $R/F \cong \mathbb{Q}_p$ is a divisible torsion-free group, F is a pure and dense subgroup of R, whence $\widehat{F} = \widehat{R}$. Furthermore, F, as a submodule of a free J_p-module, is also a free J_p-module. To conclude, note that F is not closed under multiplication, since $X \in F$ but $X^2 \notin F$, because $\Phi(X^2) \neq 0$.

3 When the Pierce embedding is surjective

In order to extend Pierce's claim that the Pierce embedding

$$\Psi : \mathrm{End}(G)/H(G) \to \prod_n M_{f_n(G)}$$

is epic if one considers G either a direct sum of cyclic groups or a torsion-complete group, it is convenient to introduce a new notation, analogous to the Pierce radical. If G and A are two groups, we set

$$H(G, A) = \{\phi \in \mathrm{Hom}(G, A) \mid \phi(p^n G[p]) \leq p^{n+1} A[p] \text{ for all } n \in \mathbb{N}\}.$$

Notice that, if $G = A$, then $H(G, A)$ coincides with the Pierce radical $H(G)$ of $\mathrm{End}(G)$.

Every homomorphism $\phi : G \to A$ induces maps

$$\phi_n : \frac{p^n G[p]}{p^{n+1} G[p]} \to \frac{p^n A[p]}{p^{n+1} A[p]} \quad \text{for all } n \in \mathbb{N}.$$

Let $B = \bigoplus_{n\in\mathbb{N}} B_n$ and $C = \bigoplus_{n\in\mathbb{N}} C_n$ be basic subgroups of G and A, respectively, where B_n and C_n are the homocyclic components. Consider the group homomorphism

$$\omega : \mathrm{Hom}(G, A) \to \prod_n \mathrm{Hom}(B_n[p], C_n[p])$$

sending $\phi \in \mathrm{Hom}(G, A)$ to the sequence $(\phi_n)_{n\in N}$; we identify $B_n[p]$ with $p^n G[p]/p^{n+1} G[p]$, and $C_n[p]$ with $p^n A[p]/p^{n+1} A[p]$, in view of the natural isomorphisms

$$p^n G[p]/p^{n+1} G[p] \cong B_n[p] \quad \text{and} \quad p^n A[p]/p^{n+1} A[p] \cong C_n[p].$$

Clearly, ω has kernel $H(G, A)$, and therefore we have a group embedding

$$\Omega : \frac{\mathrm{Hom}(G, A)}{H(G, A)} \to \prod_n \mathrm{Hom}(B_n[p], C_n[p]).$$

Given two groups G and A, we say that they are *UK-independent* (short for Ulm–Kaplansky independent) if, for every positive integer n, $f_n(G) > 0$ implies $f_n(A) = 0$, and consequently $f_n(A) > 0$ implies $f_n(G) = 0$. The following lemma connects the UK-independence of two groups G and A with the vanishing of the two quotient groups $\mathrm{Hom}(G, A)/H(G, A)$ and $\mathrm{Hom}(A, G)/H(A, G)$. A key point is that the two implications

$$f_n(G) > 0 \implies f_n(A) = 0$$

and

$$f_n(A) > 0 \implies f_n(G) = 0$$

are equivalent.

Lemma 3.1. *Let G and A be two groups. The following assertions are equivalent:*
(i) $\mathrm{Hom}(G, A) = H(G, A)$.
(ii) *G and A are UK-independent.*
(iii) $\mathrm{Hom}(A, G) = H(A, G)$.

Proof. “(i) $\Longrightarrow$ (ii)”: Let $B = \bigoplus_{n\in\mathbb{N}} B_n$ and $C = \bigoplus_{n\in\mathbb{N}} C_n$ be basic subgroups of G and A, respectively. Assume, by way of contradiction, that $f_n(G) > 0$ and $f_n(A) > 0$ for a fixed $n \in \mathbb{N}$. Then there is a non-zero homomorphism $\phi_n : B_n[p] \to C_n[p]$, which obviously extends to a non-zero homomorphism from B_n to C_n. Sending the complement in G of B_n to zero, we get a homomorphism $\phi : G \to A$ which induces the sequence $(\phi_m)_{m\in\mathbb{N}}$ with $\phi_m = 0$ for $m \neq n$. Clearly, this ϕ does not belong to $H(G, A)$. Thus we get that $H(G, A) \subsetneq \mathrm{Hom}(G, A)$, which is a contradiction.

“(ii) $\Longrightarrow$ (iii)”: Let G and A be UK-independent and assume, by way of contradiction, that there exists a map $\phi : A \to G$ such that $\phi(p^n A[p])$ is not contained in $p^{n+1} G[p[$. This implies that $p^{n+1} A[p] \neq p^n A[p]$ so that

$f_n(A) > 0$, and so by hypothesis $f_n(G) = 0$. Thus $p^n G[p] = p^{n+1}G[p]$, giving us the desired contradiction since

$$\phi(p^n A[p]) \leq p^n G[p] = p^{n+1} G[p].$$

"(iii) $\Longrightarrow$ (ii)" is similar to "(i) $\Longrightarrow$ (ii)", and "(ii) $\Longrightarrow$ (i)" is similar to "(ii) $\Longrightarrow$ (iii)". □

Remark 3.2. Observe that the relation of UK-independence between two groups G and A is symmetric, and this is confirmed in Lemma 3.1 by the equivalence with the two equalities $\mathrm{Hom}(G, A) = H(G, A)$ and $\mathrm{Hom}(A, G) = H(A, G)$.

This contrasts with what happens if we replace $H(G, A)$ with the subgroup of the small homomorphisms $\mathrm{Hom}_s(G, A)$. In fact, the equality $\mathrm{Hom}(G, A) = \mathrm{Hom}_s(G, A)$ is the object of the investigation in [9], and it gives rise to *small pairs* of abelian p-groups and to the classes of fully-thick and fully-thin groups.

We introduce a new property for direct sums of groups concerning their Ulm–Kaplansky invariants.

Definition 3.3. Let $G = \bigoplus_{i\in I} G_i$ be a direct sum of groups. We say that G has the property $(\mathcal{P})$ if G_i and G_j are UK-independent for $i \neq j$.

Note that, if G has the property $(\mathcal{P})$, then the index set I is countable, since, if $f_n(G) > 0$, for every $i \in I$ there is a unique $n \in \mathbb{N}$ such that $f_n(G_i) > 0$. So we will assume that $I = \{1, 2, \ldots\}$. We prove now a lemma that is the essence of this approach.

Lemma 3.4. *Let $G = \bigoplus_{i=1}^{\infty} G_i$ be a direct sum of groups with the property $(\mathcal{P})$. If $\psi \in \mathrm{End}(G)$ is an arbitrary endomorphism and $\Delta = (\psi_{ij})$ is a matrix representation of ψ, then the matrix $\Delta - \mathrm{diag}\{\psi_{11}, \psi_{22}, \ldots, \psi_{nn}, \ldots\}$ belongs to the Pierce radical $H(G)$.*

Proof. Let

$$\Phi = \Delta - \mathrm{diag}\{\psi_{11}, \psi_{22}, \ldots, \psi_{nn}, \ldots\}$$

and consider an arbitrary element $\mathbf{x} \in G[p]$ so that $\mathbf{x} = (g_1, g_2, \ldots, g_n, 0, \ldots)$ for some n (depending on $\mathbf{x}$), where $g_i \in G_i[p]$; the height of $\mathbf{x}$ is $t = \min\{ht(g_i)\}$. Then $\Phi.\mathbf{x}$ is a vector whose j-th component is of the form $\sum_{i\neq j} \alpha_{ji}(g_i)$ with each $\alpha_{ji} \in \mathrm{Hom}(G_i, G_j)$. Since we are assuming that G has the property $(\mathcal{P})$, it follows from Lemma 3.1 that $\mathrm{Hom}(G_i, G_j) = H(G_i, G_j)$, so that the height satisfies $ht(\alpha_{ji}(g_i)) > ht(g_i) \geq t$. Thus each component of $\Phi.\mathbf{x}$ has height greater than t, so that the height of $\Phi.\mathbf{x}$ must exceed the height of $\mathbf{x}$. Thus Φ is height-increasing on the socle $G[p]$ of G and $\Phi \in H(G)$, as required. □

Applying Lemma 3.4 to the Pierce embedding of the endomorphism ring of $G = \bigoplus_{i=1}^{\infty} G_i$ we have the next proposition.

Proposition 3.5. *If $G = \bigoplus_{i=1}^{\infty} G_i$ has the property $(\mathcal{P})$, then $\mathrm{End}(G)/H(G)$ is isomorphic (as a ring) to the ring direct product $\prod_{i=1}^{\infty}(\mathrm{End}(G_i)/H(G_i))$.*

Proof. Let $\Delta + H(G)$ be an arbitrary element of $\mathrm{End}(G)/H(G)$. Then, by the previous Lemma 3.4, we have that $\Delta + H(G)$ can be expressed uniquely in the form $\Delta_1 + H(G)$ where $\Delta_1 = \mathrm{diag}\{\psi_{11}, \psi_{22}, \ldots, \psi_{nn}, \ldots\}$. Now define

$$\lambda : \mathrm{End}(G)/H(G) \to \prod_{i=1}^{\infty}(\mathrm{End}(G_i)/H(G_i))$$

by

$$\lambda(\Delta_1 + H(G)) = (\psi_{11} + H(G_1), \psi_{22} + H(G_2), \ldots, \psi_{nn} + H(G_n), \ldots).$$

It is easy to check that λ is a well-defined surjective ring homomorphism. Furthermore, the kernel of λ consists of those $\Delta_1 + H(G)$ where $\psi_{jj} \in H(G_j)$ for all $j \geq 1$, and hence $\Delta_1 \in H(G)$ follows easily. □

As a main consequence, we can prove the following theorem.

Theorem 3.6. *Assume that $G = \bigoplus_{i=1}^{\infty} G_i$ has the property $(\mathcal{P})$. If each G_i has the property that the Pierce embedding of $\mathrm{End}(G_i)$ is surjective, so too is the Pierce embedding of $\mathrm{End}(G)$.*

Proof. By Proposition 3.5,

$$\mathrm{End}(G)/H(G) \cong \prod_{i=1}^{\infty} \mathrm{End}(G_i)/H(G_i).$$

From the hypotheses we have that, for each i, $\mathrm{End}(G_i)/H(G_i)$ is canonically isomorphic (as a ring) via a Pierce embedding to $\prod_n M_{f_n(G_i)}$. Since the groups G_i are pairwise UK-independent, for every $n \in \mathbb{N}$ there exists exactly one index i_n such that $M_{f_n(G)} = M_{f_n(G_{i_n})}$ and $M_{f_n(G_i)} = 0$ for $i \neq i_n$, whence we get

$$\mathrm{End}(G)/H(G) \cong \prod_{i=1}^{\infty} \prod_n M_{f_n(G_i)} = \prod_n \prod_{i=1}^{\infty} M_{f_n(G_i)} = \prod_n M_{f_n(G_{i_n})} = \prod_n M_{f_n(G)}.$$

This concludes the proof. □

The particular case of Theorem 3.6 when the groups G_i are torsion-complete is considered in the next result.

Corollary 3.7. *Assume that $G = \bigoplus_{i=1}^{\infty} G_i$ has the property* ($\mathcal{P}$)*, where each G_i is torsion-complete. Then the Pierce embedding of* $\mathrm{End}(G_i)$ *is surjective.*

Proof. The proof is an immediate consequence of Theorem 3.6, since the Pierce embedding for endomorphism rings of torsion-complete groups is epic. We provide here an alternative, more direct proof.

The hypothesis of UK-independence on the direct summands ensures that, for every $n \in \mathbb{N}$ for which $f_n(G) > 0$, there exists exactly one index $i(n) \in I$ such that $f_n(G) = f_n(G_{i(n)})$. Therefore, if $B = \bigoplus_n B_n$ is a basic subgroup of G and $B^{(i)} = \bigoplus_n B_n^{(i)}$ is a basic subgroup of G_i, for every $n \in \mathbb{N}$ there exists exactly one index $i(n) \in I$ such that $B_n = B_n^{(i(n))}$. In order to prove that the Pierce embedding Ψ is epic, we must show that, given a sequence $(\phi_n)_{n\in\mathbb{N}}$ of endomorphisms of $p^{n-1}G[p]/p^n G[p] \cong B_n[p]$, there exists an endomorphism ϕ of G that induces the maps ϕ_n for all n. But each ϕ_n extends to an endomorphism of $B_n = B_n^{(i(n))}$; collect all the B_n's whose corresponding indices $i(n) \in I$ are the same. Then the corresponding endomorphisms ϕ_n give rise to an endomorphism of the basic subgroup of $G_{i(n)}$, and since $G_{i(n)}$ is torsion-complete, this endomorphism extends uniquely to an endomorphism of $G_{i(n)}$. All these endomorphisms produce the desired endomorphism of G, since clearly $G = \bigoplus_n G_{i(n)}$. □

The hypothesis of Corollary 3.7 also covers direct sums of cyclic groups, since they are direct sums of homocyclic bounded groups which are torsion-complete and UK-independent. Notice that the hypothesis of UK-independence may be assumed considering the Ulm–Kaplansky invariants of index $n \geq n_0$ for a fixed $n_0 \in \mathbb{N}$. We illustrate this with the following simple result.

Proposition 3.8. *Let $G = A \oplus C$, where A, C are separable p-groups. If $J = \{n \geq 0 \mid f_n(A) \neq 0 \neq f_n(C)\}$ is finite, then $G = E \oplus X$, where E is a bounded group, $X = A' \oplus C'$ with A', C' being UK-independent direct summands of A and C, respectively, and the Pierce embedding of* $\mathrm{End}(G)$ *is surjective if and only if the Pierce embedding of* $\mathrm{End}(X)$ *is surjective.*

Proof. Let $B = \bigoplus_{n\in\mathbb{N}} B_n$ and $D = \bigoplus_{n\in\mathbb{N}} D_n$ be basic subgroups of A and C, respectively. Set E to be the bounded summand of G generated by the homocyclic components B_n, D_n with $n \in J$. Note that if X is a fixed complement of E, then E, X are UK-independent. Thus, if $\mathrm{End}(X)$ has a surjective Pierce embedding, it follows from Theorem 3.6 that the Pierce embedding of $\mathrm{End}(G)$ is also surjective.

Conversely, suppose for a contradiction that the Pierce embedding of $\mathrm{End}(G)$ is surjective but that of $\mathrm{End}(X)$ is not. Let Y be basic in X with homocyclic components Y_k, so that there are endomorphisms $\phi_k \in \mathrm{End}(Y_k[p])$ but no $\phi \in \mathrm{End}(X)$ exists such that $\phi \restriction Y_k[p] = \phi_k$. Every endomorphism of G can be represented as a matrix $\left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ with $\alpha \in \mathrm{End}(E)$, $\beta \in \mathrm{End}(X)$, $\gamma : X \to E$ and $\delta : E \to X$. Now consider the endomorphism ψ of G given by $\psi \restriction E = 0$ and $\psi \restriction Y_k[p] = \phi_k$. Thus $\psi(e, 0) = (\alpha(e), \delta(e)) = (0, 0)$ for all $e \in E$, so that δ must be the zero map. However, for all $y_k \in Y_k[p]$ we would then have that

$$(0, \phi(y_k)) = \psi((0, y_k)) = (\gamma(y_k), \beta(y_k)),$$

so that $\beta(y_k) = \phi_k(y_k) = \phi_k(y_k)$ for all $y_k \in Y_k$. Hence β is an endomorphism of X with $\beta \restriction Y_k[p] = \phi_k$, which is a contradiction. The result follows. □

Note that the argument used in the proof of necessity in Proposition 3.8 made no use of the fact that the subgroup E was bounded, and thus we have an immediate corollary to the proof.

Corollary 3.9. *If X is a direct summand of the group G and the Pierce embedding of* $\mathrm{End}(G)$ *is surjective, then so too is the Pierce embedding of* $\mathrm{End}(X)$.

Corollary 3.7 contrasts with a result proved by Kemoklidze in [10], which states that, given a separable p-group G with basic subgroup B, every endomorphism of B extends to an endomorphism of G if and only if either $G = B$ or $G = \overline{B}$. A similar result, with automorphisms replacing endomorphisms, was proved by Leptin in 1960.

As an application of Corollary 3.7, we obtain in the next example a negative answer to the following question.

Question. Does the hypothesis that the Pierce embedding $\Psi : \mathrm{End}(G)/H(G) \to \prod_n M_{f_n(G)}$ is surjective imply that G has minimal full inertia?

Recall that the answer to this question is positive if G is either a direct sum of cyclic groups or is torsion-complete.

Example 3.10. Let $G = B_1 \oplus \overline{B}_2$, where B_1 is an unbounded direct sum of cyclic groups and $\overline{B}_2$ is unbounded torsion-complete and semi-standard. Furthermore, assume that B_1 and $\overline{B}_2$ are UK-independent. Then G does not have minimal full inertia, by [7, Proposition 3.7], but the Pierce embedding of $\mathrm{End}(G)$ is epic, by Theorem 3.6.

4 When the Pierce embedding is not surjective

The results in the preceding section naturally pose the question whether there exist more classes of groups with the surjectivity property of the Pierce embedding of their endomorphism rings, in particular, if this property holds for arbitrary direct sums of torsion-complete groups. The next example provides a negative answer to the latter question by employing two rather different approaches which will be exploited later.

Example 4.1. (i) Let $B = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)$ and $G = \bigoplus_{i\in\mathbb{N}} \overline{B}_i$, where $B_i = B$ for all $i \in \mathbb{N}$. Then the Pierce embedding of $\mathrm{End}(G)$ is not surjective.

(ii) Let $B = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n) = C$. Then, if $G = \overline{C} \oplus B$, the Pierce embedding of $\mathrm{End}(G)$ is not surjective.

To prove the claim in (i), let $\langle b_{i,n}\rangle$ be the copy of $\mathbb{Z}(p^n)$ in B_i for every $i \in \mathbb{N}$. A basic subgroup of G is $C = \bigoplus_{i\in\mathbb{N}} B_i$. If $C = \bigoplus_{n\in\mathbb{N}} C_n$ is a decomposition of C as direct sum of its homocyclic summands, then $C_n = \bigoplus_{i\in\mathbb{N}} \langle b_{i,n}\rangle$; setting $p^{n-1}b_{i,n} = x_{i,n}$, we clearly have $x_{i,n} \in p^{n-1}C_n[p]$, and the elements $x_{i,n}$ ($i \in \mathbb{N}$) form a basis of $C_n[p]$.

For every $n \in \mathbb{N}$, define the endomorphism $\phi_n : C_n[p] \to C_n[p]$ as follows:

$$\phi_n(x_{i,n}) = x_{i+n,n} \quad \text{for all } i \in \mathbb{N}.$$

Note that the map ϕ_n is injective. We claim that the element $(\phi_n)_{n\in\mathbb{N}} \in \prod_n M_{f_n(G)}$ does not belong to the image of the Pierce embedding of $\mathrm{End}(G)$. Assume, by way of contradiction, that the endomorphism ϕ of G acts as ϕ_n on $C_n[p]$ for each n. By Lemma 4.3 below, the map ϕ is injective: The subgroup $\phi(\overline{B}_1)$ of G is torsion-complete, being isomorphic to $\overline{B}_1$. By a result due to Enochs in [4] (see also [5, Lemma 4.2, p. 318]), there exist an $m \in \mathbb{N}$ and indices $i_1, i_2, \dots, i_k$ such that $(p^m\phi(\overline{B}_1))[p]$ is contained in $\overline{B}_{i_1} \oplus \overline{B}_{i_2} \oplus \cdots \oplus \overline{B}_{i_k}$. It follows that

$$\phi(p^m B_1)[p] \subseteq \overline{B}_{i_1}[p] \oplus \overline{B}_{i_2}[p] \oplus \cdots \oplus \overline{B}_{i_k}[p].$$

As ϕ is injective, we have $\phi(p^m B_1)[p] = \phi(p^m B_1[p])$. Thus we get the desired contradiction since, for n sufficiently large, $x_{1,n} \in p^m B_1[p]$ and $\phi(x_{1,n}) = \phi_n(x_{1,n}) = x_{1+n,n}$ does not belong to $\overline{B}_{i_1}[p] \oplus \overline{B}_{i_2}[p] \oplus \cdots \oplus \overline{B}_{i_k}[p]$.

For (ii), observe that each homocyclic component G_n of a basis of G is of the form $\langle c_n\rangle \oplus \langle b_n\rangle$, where each of c_n, b_n is of order p^n. Define endomorphisms ϕ_n of $G_n[p]$ by setting $\phi(p^{n-1}c_n) = p^{n-1}b_n$, while $\phi(p^{n-1}b_n) = 0$. We claim that there does not exist an endomorphism ϕ of G such that $\phi \restriction G_n[p] = \phi_n$; this is clearly enough to prove the desired result. Suppose, for a contradiction, that such a ϕ exists. Then ϕ has a matrix representation $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where δ is a homomorphism from $\overline{C} \to B$; it is a well-known result of Megibben that such a homomorphism is necessarily small. However, $\Delta(p^{n-1}c_n) = p^{n-1}b_n \neq 0$, so that $\delta(p^{n-1}\overline{C}[p]) \neq 0$ for all n. As δ is small, this is impossible, and the result follows.

Elaborating on the idea in Example 4.1 (i), we obtain a more general result in the next Theorem 4.4. First, we need two preliminary lemmas.

Lemma 4.2. *Let $C = \bigoplus_{n\in\mathbb{N}} C_n$ be an unbounded direct sum of cyclic groups, where C_n is its n-th homocyclic component, and let $\overline{C}$ be its torsion-completion. Let $\chi : C \to C$ be an endomorphism and $\overline{\chi}$ its unique extension to $\overline{C}$. If $\overline{\chi}(C_n[p]) = 0$ for all n, then $\overline{\chi} = 0$*

Proof. It is enough to prove that $\overline{\chi} \restriction \overline{C}[p] = 0$. Let $x = \sum_n c_n \in \overline{C}[p]$, with $c_n \in C_n[p]$ for all n. Then

$$\overline{\chi}(x) = \overline{\chi}\Big(\sum_n c_n\Big) = \sum_n \overline{\chi}(c_n) = \sum_n \chi(c_n) = 0,$$

where the second equality follows by the continuity of $\overline{\chi}$ with respect to the p-adic topologies. So the claim follows. □

Lemma 4.3. *Let $C = \bigoplus_{n\in\mathbb{N}} C_n$ be an unbounded direct sum of cyclic groups, where C_n is its n-th homocyclic component, and let $\overline{C}$ be its torsion-completion. For each $n \in \mathbb{N}$, let $\phi_n : C_n[p] \to C_n[p]$ be an injective endomorphism.*

(i) *If $\overline{\phi} : \overline{C} \to \overline{C}$ is an endomorphism such that $\overline{\phi} \restriction C_n[p] = \phi_n$ for all n, then $\overline{\phi}$ is injective.*
(ii) *If C is a basic subgroup of a group G and $\psi : G \to G$ is an endomorphism such that $\psi \restriction C_n[p] = \phi_n$ for all n, then ψ is an injective map.*

Proof. (i) It is enough to prove that $\overline{\phi} \restriction \overline{C}[p]$ is injective. Let $0 \neq x = \sum_n c_n \in \overline{C}[p]$, where $c_n \in C_n[p]$ for all n, and let c_k be the first c_n not equal to 0. Assume, by way of contradiction, that $\overline{\phi}(x) = 0$. Then

$$0 \neq \overline{\phi}(c_k) = -\overline{\phi}\Big(\sum_{n>k} c_n\Big) \in C_k[p] \cap \overline{\sum_{n>k} C_n},$$

which is manifestly absurd since

$$C_k \cap \overline{\sum_{n>k} C_n} = 0.$$

(ii) Let $\phi = \bigoplus_n \phi_n$. The maps $\overline{\phi}, \overline{\psi} : \overline{G} \to \overline{G}$ coincide on $C_n[p]$ for all n. Therefore, setting $\chi = \phi - \psi$, we get that $\overline{\chi}(C_n[p]) = 0$ for all $n \in \mathbb{N}$. Thus $\overline{\chi} = \overline{\phi} - \overline{\psi} = 0$ by Lemma 4.2. But $\overline{\phi}$ is injective, by point (i), so $\overline{\psi}$ as well as ψ are injective maps. □

We are now able to prove that the Pierce embedding of the endomorphism ring of a direct sum of torsion-complete groups with enough positive Ulm–Kaplansky invariants of the same index is not surjective.

Theorem 4.4. *Let $G = \bigoplus_{i\in\mathbb{N}} \overline{B_i}$ be a direct sum of countably many torsion-complete groups, with the B_i being unbounded direct sums of cyclic p-groups. Assume that there is an infinite subset J of $\mathbb{N}$ such that, for every $n \in J$, $f_{n-1}(B_i) > 0$ for $i \in I_{n-1}$, an infinite subset of $\mathbb{N}$ depending on n. If there exists an index i belonging to infinitely many subsets I_{n-1}, then the Pierce embedding of* End(G) *is not surjective.*

Proof. For each $n \in J$, let $\langle b_{i,n}\rangle$ be a cyclic summand of B_i of order p^n, with i ranging over I_{n-1}. A basic subgroup of G is $C = \bigoplus_{i\in\mathbb{N}} B_i$, so that, for each $n \in J$, C has a homocyclic component C_n which has $\bigoplus_{i\in I_{n-1}} \langle b_{i,n}\rangle$ as a summand: $C_n = \bigoplus_{i\in I_{n-1}} \langle b_{i,n}\rangle \oplus Y_n$ for some Y_n.

For each $n \in J$, define the endomorphism $\phi_n : C_n[p] \to C_n[p]$ by setting ϕ_n to be the identity map on Y_n and, denoting $p^{n-1}b_{i,n}$ by $x_{i,n}$,

$$\phi_n(x_{i,n}) = x_{j_i,n} \quad \text{for all } i \in I_{n-1},$$

where $i + n \leq j_i \in I_{n-1}$ and $j_i > j_{i'}$ for all $i' < i$ in I_{n-1}; such an index j_i exists by the hypothesis that I_{n-1} is infinite. If $n \notin J$, define the endomorphism $\phi_n : C_n[p] \to C_n[p]$ to be the identity map. Note that the maps ϕ_n are injective for all $n \in \mathbb{N}$.

We claim that the element $(\phi_n)_{n\in\mathbb{N}} \in \prod_n M_{f_n(G)}$ does not belong to the image of the Pierce embedding of End(G). Assume, by way of contradiction, that the endomorphism ψ of G acts as ϕ_n on $C_n[p]$ for each n, that is, $\psi \restriction C_n[p] = \phi_n$. By Lemma 4.3 (ii), ψ is an injective map.

By hypothesis, there exists an index i belonging to infinitely many I_{n-1}. The subgroup $\psi(\overline{B_i})$ of G is torsion-complete, being isomorphic to $\overline{B_i}$ by the injectivity of ψ. By the result due to Enochs in [4] quoted above, there exist an $m \in \mathbb{N}$ and indices $i_1, i_2, \dots, i_k$ such that $(p^m\psi(\overline{B}_i))[p]$ is contained in $\overline{B}_{i_1} \oplus \overline{B}_{i_2} \oplus \cdots \oplus \overline{B}_{i_k}$. It follows that

$$\psi(p^m B_i)[p] \subseteq \overline{B}_{i_1}[p] \oplus \overline{B}_{i_2}[p] \oplus \cdots \oplus \overline{B}_{i_k}[p].$$

But $\psi(p^m B_i[p]) = \psi(p^m B_i)[p]$. Therefore, we get the desired contradiction since, for n sufficiently large, $0 \neq x_{i,n} \in p^m B_i[p]$ and $\psi(x_{i,n}) = \psi_n(x_{i,n}) = x_{j_i,n}$ does not belong to $B_{i_1}[p] \oplus B_{i_2}[p] \oplus \cdots \oplus B_{i_k}[p]$. □

Example 4.1 (i) is, of course, a special case of Theorem 4.4. The condition that there exists an index i belonging to infinitely many subsets I_{n-1} ensures that one can find one of the subgroups $\overline{B_i}$ with appropriate Ulm–Kaplansky invariants. An alternative approach would be to start with a given fixed $\overline{B_1}$ and using this as group to define J, and then construct the remaining groups B_i using the properties in the hypothesis on J.

In a similar way we also have the following result, which clearly is based on Example 4.1 (ii).

Theorem 4.5. *Let A, C be separable p-groups such the invariants $f_n(A), f_n(C)$ are both non-zero for $n \in I$, an infinite set, and* $\mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C)$. *If $G = A \oplus C$, then the Pierce embedding of* $\mathrm{End}(G)$ *is not surjective.*

Proof. Let B_n ($n \in I$) be a homocyclic component of a basic subgroup of G so that B_n has a summand of the form $\langle a_n \rangle \oplus \langle c_n \rangle$ where $p^n a_n = 0 = p^n c_n$, with $a_n \in A$, $c_n \in C$. For each $n \in I$, define an endomorphism ϕ_n of $B_n[p]$ by sending $p^{n-1}a_n \mapsto p^{n-1}c_n$ and extending this trivially to the whole of $B_n[p]$. Now, if the Pierce embedding of $\mathrm{End}(G)$ were surjective, there would exist an endomorphism ϕ of G with $\phi \restriction B_n[p] = \phi_n$. However, since $G = A \oplus C$, the action of ϕ on $p^{n-1}a_n$ is given by the action of γ on $p^{n-1}a_n$, where γ is a fixed homomorphism in $\mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C)$. Thus, we have that

$$0 \neq p^{n-1}c_n = \phi(p^{n-1}a_n) = \gamma(p^{n-1}a_n) \quad \text{for all } n \in I.$$

Since, by assumption, I is infinite, this contradicts the fact that $\gamma \in \mathrm{Hom}_s(A, C)$, and therefore establishes the claim. □

If we strengthen the hypotheses in Theorem 4.5, we obtain the following satisfactory equivalence:

Corollary 4.6. *Let A, C be separable p-groups such that* $\mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C)$ *and the Pierce embeddings of both* $\mathrm{End}(A)$, $\mathrm{End}(C)$ *are surjective. If $G = A \oplus C$, then the Pierce embedding of* $\mathrm{End}(G)$ *is surjective if and only if the set $J = \{n \mid f_n(A) \neq 0 \neq f_n(C)\}$ is finite.*

Proof. If J is not finite, then it follows immediately from Theorem 4.5 that the Pierce embedding of $\mathrm{End}(G)$ is not surjective.

Conversely, suppose that J is finite. Then, as in the proof of Proposition 3.8, we may write $G = E \oplus X$, where E is bounded and X has the form $X = A' \oplus C'$ with A', C' being summands of A, C, respectively. Furthermore, the Pierce embedding of $\mathrm{End}(G)$ is surjective precisely if that of $\mathrm{End}(X)$ is surjective. It follows from Corollary 3.9 that the Pierce embeddings of $\mathrm{End}(A')$, $\mathrm{End}(C')$ are both surjective and, moreover, A', C' are UK-independent, so that an application of Theorem 3.6 yields the desired result that the Pierce embedding of $\mathrm{End}(X)$ is surjective. □

There are many concrete examples arising from Theorem 4.5: if $(\mathcal{A}, \mathcal{C})$ is a small pair in the sense of [9], we can take any $A \in \mathcal{A}$ and $C \in \mathcal{C}$ satisfying the hypothesis on the Ulm–Kaplansky invariants required by the theorem. There are 2^{2^c} (c denotes the continuum) examples of such pairs, the most important being the one formed by thick and fully thin groups, and that formed by fully thick and thin groups (we refer to [9] for these notions). The choice of $A = \overline{B}$ and $C = B$, a standard basic group, provides a simple example.

5 Relationships between the Pierce decomposition and the Pierce embedding

Now, we look for general conditions ensuring that the map Ψ is not surjective. These conditions can be deduced from some relationships between the Pierce decomposition and the Pierce embedding of the endomorphism ring.

First, we prove the following lemma, making use of the notation above and of the fact that Pierce in [12] proved that $H(G) = J(E(G), E_s(G))$, where

$$J(E(G), E_s(G)) = \{\phi \in \operatorname{End}(G) \mid \phi\theta \in J(\operatorname{End}_s(G)) \text{ for all } \theta \in \operatorname{End}_s(G)\}$$

where, as usual, $J(\cdot)$ denotes the Jacobson radical.

Lemma 5.1. *Let G be an unbounded group and let $\operatorname{End}(G) = \widehat{F}_G \oplus \operatorname{End}_s(G)$ be the Pierce decomposition of its endomorphism ring.*

(i) $\Psi((\widehat{F}_G + H(G)/H(G))$ has cardinality at most $p \cdot \operatorname{rk}_{J_p} F_G$.

(ii) $\Psi((\operatorname{End}_s(G) + H(G))/H(G))$ is contained in the direct sum $\bigoplus_{n\in\mathbb{N}} M_{f_n(G)}$ and is isomorphic to $\bigoplus_{n\in\mathbb{N}} M_{f_n(G)}$.

Proof. (i) Since $p\operatorname{End}(G) \subseteq H(G)$, $(\widehat{F}_G + H(G))/H(G)$ is an epimorphic image of

$$(\widehat{F}_G + p\operatorname{End}(G)/p\operatorname{End}(G)) \cong \widehat{F}_G/p\widehat{F}_G \cong F_G/pF_G,$$

and F_G/pF_G has cardinality $p \cdot \operatorname{rk}_{J_p} F_G$ (which coincides with $\operatorname{rk}_{J_p} F_G$ if this rank is infinite).

(ii) If $\phi \in \operatorname{End}_s(G)$, there exists a k such that $\phi(p^n G[p]) = 0$ for all $n \geq k$. Hence, $\mu(\phi) \in \bigoplus_{n\in\mathbb{N}} M_{f_n(G)}$ because $\phi_n = 0$ for all $n \geq k$. Therefore, $\mu(\operatorname{End}_s(G)) \leq \bigoplus_{n\in\mathbb{N}} M_{f_n(G)}$, and consequently

$$\Psi((\operatorname{End}_s(G) + H(G))/H(G)) \leq \bigoplus_{n\in\mathbb{N}} M_{f_n(G)}.$$

The final statement was proved in [12, Theorem 13.6]; we include here the proof for the sake of completeness. Recalling that $H(G) = J(E(G), E_s(G))$, by [12, Lemma 14.2 (e)], one has that $H(G) \cap \operatorname{End}_s(G) = J(\operatorname{End}_s(G))$. However, Ψ is monic, so that

$$\begin{aligned}\Psi((\operatorname{End}_s(G) + H(G))/H(G)) &\cong (\operatorname{End}_s(G) + H(G))/H(G)\\ &\cong \operatorname{End}_s(G)/(H(G) \cap \operatorname{End}_s(G))\\ &= \operatorname{End}_s(G)/J(\operatorname{End}_s(G))\\ &\cong \bigoplus_{n\in\mathbb{N}} M_{f_n(G)},\end{aligned}$$

where the final isomorphism comes from [12, Theorem 13.6]. □

Theorem 5.2. *Let G be an unbounded group and let $\operatorname{End}(G) = \widehat{F}_G \oplus \operatorname{End}_s(G)$ be the Pierce decomposition of its endomorphism ring.*

(i) *If $\operatorname{rk}_{J_p} F_G$ is countable and G is semi-standard, then $|\operatorname{Im}(\Psi)|$ is countable.*

(ii) *If $\operatorname{rk}_{J_p} F_G < c$ and $f_n(G) < c$ for all $n \in \mathbb{N}$, then $|\operatorname{Im}(\Psi)| < c$.*

In both cases, Ψ is not epic.

Proof. In both cases, apply Lemma 5.1, noting that if G is semi-standard, then $M_{f_n(G)}$ is finite for all n since B_n is such. Since the cardinality of $\prod_n M_{f_n(G)}$ is c, in both cases Ψ cannot be epic. □

Note that, if in Theorem 5.2 we assume in addition that $\operatorname{rk}_{J_p} F_G$ is finite, then $\operatorname{Im}(\Psi)$ is a finite extension of a subgroup of $\bigoplus_{n\in\mathbb{N}} M_{f_n(G)}$, and hence, as an additive group, is a semi-standard direct sum of cyclic groups.

Our next example shows that even when a semi-standard separable p-group has the completion of a "large" free J_p-module in its Pierce decomposition, the Pierce embedding of its endomorphism ring may fail to be surjective.

Example 5.3. There exists a semi-standard separable p-group G such that the Pierce decomposition of G has the form $\operatorname{End}(G) = A \oplus \operatorname{End}_s(G)$, where A is the completion in the p-adic topology of a free J_p-module of rank c, the continuum, but the Pierce embedding of $\operatorname{End}(G)$ is not surjective. Furthermore, the group G does not have minimal full inertia.

To exhibit such a group, we utilize an example constructed by Braun and Strüngmann in their work on Hopfian and co-Hopfian groups [2]. The group in question has the following properties: a basic subgroup of G is of the form $B = \bigoplus_{n\in\mathbb{N}}(\mathbb{Z}(p^n))^{(n)}$ and $\operatorname{End}(G) = J_p[[\phi]] \oplus \operatorname{End}_s(G)$, where ϕ is transcendental over J_p and satisfies $\phi(G[p]) = 0$. Note that $J_p[[\phi]]$ is the completion in the p-adic topology of a free J_p-module F of rank c, the continuum:

$$F/pF \cong \widehat{F}/p\widehat{F} = \prod_{\aleph_0} J_p/p\prod_{\aleph_0} J_p \cong \prod_{\aleph_0} \mathbb{Z}(p).$$

Suppose, for a contradiction, that the Pierce embedding of End(G) is surjective. Each homocyclic component of B is of the form $\mathbb{Z}(p^n)^{(n)}$, so we may choose for each $n > 1$ a non-zero endomorphism ψ_n of $B_n[p]$ which is not an automorphism of $B_n[p]$. If the Pierce embedding of End(G) is surjective, then there exists an endomorphism α of G such that $\alpha \restriction B_n[p] = \psi_n$ for all n. Now α has the form

$$\alpha = (\pi_0 + \pi_1\phi + \cdots + \pi_k\phi^k + \dots) + \theta,$$

where $\pi_i \in J_p$ and θ is a small endomorphism of G. Thus there is an integer, $N > 1$ say, with $\theta(p^N G[p]) = 0$. Since $B_{N+1}[p] \leq p^N G[p]$, we have that

$$0 \neq \psi_{N+1}(B_{N+1}[p]) = \pi_0(B_{N+1}[p]),$$

so that $\pi_0 \neq 0$ and $p \nmid \pi_0$, whence π_0 is a unit in J_p, forcing α to be an automorphism of G. Since

$$\alpha \restriction B_{N+1}[p] = \psi_{N+1},$$

this contradicts the choice of ψ_{N+1}, and the Pierce embedding of End(G) is not surjective.

Finally, since the group G is separable and uncountable, every fully invariant subgroup H of G has the form $H = \{x \in G \mid U_G(x) \geq \mathbf{u}\}$ for some U-sequence $\mathbf{u}$ and is then necessarily uncountable since it contains a subgroup of the form $p^n G[p]$ for some integer n. The subgroup $B[p]$ is countable, and thus cannot be commensurable with any fully invariant subgroup of G. However, $B[p]$ is fully inert, since for every $\alpha \in \mathrm{End}(G)$, arguing as above, we have that $\alpha(B[p]) = (\pi_0 + \theta)(B[p])$. Thus $(B[p] + \alpha(B[p]))/B[p]$ is finite since B is semi-standard and θ is small. This concludes our example.

Suppose that G is a separable p-group and fix a basic subgroup B of G so that one may consider G as a pure subgroup of $\overline{B}$ containing B. For convenience, we will write $D = G/B$ so that $D \cong \bigoplus_{\mu_G} \mathbb{Z}(p^\infty)$ is a divisible group of rank μ_G, and $D_1 = \overline{B}/G \cong \bigoplus_{\lambda_G} \mathbb{Z}(p^\infty)$ is a divisible group of rank λ_G.

It is well known (see [5, Theorem 3.5, p. 313]) that, if $\mu_G \leq \aleph_0$, then G is Σ-cyclic. In the next theorem, we will see that, if G is thin, the finiteness of λ_G implies the finiteness of the J_p-rank of F_G, where $\mathrm{End}(G) = \widehat{F}_G \oplus \mathrm{End}_s(G)$ is the Pierce decomposition of End(G). Recall that G is thin if every homomorphism from a torsion-complete group T to G is small.

Theorem 5.4. *Let G be a separable thin group such that λ_G is finite, and let* $\mathrm{End}(G) = \widehat{F}_G \oplus \mathrm{End}_s(G)$ *be the Pierce decomposition of* End(G). *Then the J_p-rank of F_G is finite, and consequently G is also thick and Ψ is not epic.*

Proof. From the pure exact sequence $0 \to G \to \overline{B} \to D_1 \to 0$, we get the following two sequences:

$$0 \to \mathrm{Hom}(\overline{B}, G) \xrightarrow{\alpha} \mathrm{End}(G) \to \mathrm{PExt}(D_1, G), \quad \text{where } \alpha \text{ is the obvious restriction map,} \tag{5.1}$$
$$0 \to \mathrm{End}(D_1) \to \mathrm{PExt}(D_1, G) \to \mathrm{PExt}(D_1, \bar{B}) = 0.$$

Thus

$$\mathrm{PExt}(D_1, G) \cong \mathrm{End}(D_1) \cong \prod_n \widehat{\bigoplus_n J_p},$$

where $n = \mathrm{rk}(D_1)$.

So we have that $\mathrm{End}(G)/\alpha(\mathrm{Hom}(\overline{B}, G))$ is isomorphic to a free J_p-module of finite rank at most n^2. As G is thin, $\mathrm{Hom}(\overline{B}, G) = \mathrm{Hom}_s(\overline{B}, G)$, and as α is the restriction map, it maps $\mathrm{Hom}(\overline{B}, G)$ into $\mathrm{End}_s(G)$. In fact, the image is precisely $\mathrm{End}_s(G)$ since, due to G being pure in $\overline{B}$, small endomorphisms of G extend to small homomorphisms in $\mathrm{Hom}(\overline{B}, G)$; see, for example [12, Theorem 4.4]. Since $\mathrm{End}(G)/\mathrm{End}_s(G)$ is isomorphic to $\widehat{F_G}$, we conclude that F_G is a free J_P-module of finite rank. The last claim follows by Theorem 1.2. □

An immediate consequence of Theorem 5.4 and of Theorem 1.1 is that a semi-standard thin group with λ_G finite does not have minimal full inertia. Unfortunately, it is not clear how to proceed in the case where λ_G is countable, since in this case

$$\mathrm{PExt}(D_1, G) \cong \mathrm{End}(D_1) \cong \widehat{\bigoplus_c J_p}.$$

Concrete examples of separable thin groups G satisfying the hypotheses of Theorem 5.4 are easily constructed. In fact, the proof of that theorem actually shows that for *any* separable group G with λ_G finite, the endomorphism ring satisfies $\mathrm{End}(G) = F \oplus \alpha(\mathrm{Hom}(\overline{B}, G))$, where F is a free J_p-module of finite rank and α is the restriction map in the exact sequence (5.1). The proof also shows that, to obtain the desired Pierce decomposition, it suffices to show that every homomorphism from $\overline{B}$ to G is small. In the situation where $\lambda_G = 1$, so that G is a maximal pure subgroup of $\overline{B}$ containing B, this is easily achieved using an old argument of Beaumont and Pierce [1, Lemma 5.2]; further details may be found in the paper [6] by the first author.

Another example of a thin group G with $\lambda_G = 1$ is provided by Hill and Megibben in [8], where a proper quasi-complete group is constructed with $\lambda_G = 1$. Recall that a quasi-complete group is a reduced p-group G such that the closure in the p-adic topology of every pure subgroup is still pure; it is well known that torsion-complete groups are quasi-complete, and a quasi-complete group is proper if it is not torsion-complete. Megibben proved in [11] that a proper quasi-complete group is thin.

References

[1] R. A. Beaumont and R. S. Pierce, Some invariants of p-groups, *Michigan Math. J.* **11** (1964), 137–149.

[2] G. Braun and L. Strüngmann, The independence of the notions of Hopfian and co-Hopfian Abelian p-groups, *Proc. Amer. Math. Soc.* **143** (2015), no. 8, 3331–3341.

[3] A. L. S. Corner, On endomorphism rings of primary abelian groups, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 277–296.

[4] E. Enochs, Isomorphic refinements of decompositions of a primary group into closed groups, *Bull. Soc. Math. France* **91** (1963), 63–75.

[5] L. Fuchs, *Abelian Groups*, Springer Monogr. Math., Springer, Cham, 2015.

[6] B. Goldsmith, Essentially-rigid families of abelian p-groups, *J. Lond. Math. Soc. (2)* **18** (1978), no. 1, 70–74.

[7] B. Goldsmith and L. Salce, Abelian p-groups with minimal full inertia, *Period. Math. Hungar.* **85** (2022), no. 1, 1–13.

[8] P. D. Hill and C. K. Megibben, Quasi-closed primary groups, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 271–274.

[9] P. Keef and L. Salce, Small pairs of abelian p-groups, *J. Pure Appl. Algebra* **227** (2023), no. 4, Paper No. 107266.

[10] T. Kemoklidze, On the full transitivity and fully invariant subgroups of cotorsion hulls of separable p-groups, *J. Math. Si.* **153** (2008), 506–517.

[11] C. Megibben, Large subgroups and small homomorphisms, *Michigan Math. J.* **13** (1966), 153–160.

[12] R. S. Pierce, Homomorphisms of primary abelian groups, in: *Topics in Abelian Groups*, Scott, Foresman & Co., Chicago (1963), 215–310.