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Daniel Slilaty

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GRAPHS WITHOUT A $2C_3$-MINOR AND BICIRCULAR MATROIDS WITHOUT A $U_{3,6}$-MINOR

DANIEL SLILATY

Abstract. In this note we characterize all graphs without a $2C_3$-minor. A consequence of this result is a characterization of the bicircular matroids with no $U_{3,6}$-minor.

1. Introduction

We assume the reader has a basic familiarity with matroid theory as in [5]; however, it isn't completely necessary to read this note. Given a fixed graph H, results characterizing the structure of graphs G without an H-minor have a well-established history going back as far as 1937 with Wagner's seminal result [7] for $H = K_5$. Recently Ding and Liu [3] surveyed the known results for 3-connected graphs H and an older survey by Diestel [2] lists results for some other small graphs. In all of the results listed in [2, 3], the graph H is simple. The graph $2C_3$ is obtained from the cycle of length 3 by doubling each edge. The graph $2C_3$ is of interest in matroid theory in that a bicircular matroid $B(G)$ is isomorphic to $U_{3,6}$ if and only if $G \cong 2C_3$ up to removal of isolated vertices (see [8, Lemma 2.12] or [1, Theorem 4.11]).

The main result of this note is Theorem 1.1 which describes the very limited structure that a graph with no $2C_3$-minor can have. We remark that Theorem 1.1 is enough to characterize all graphs without a $2C_3$-minor because: G has a $2C_3$-minor if and only if some block of G has a $2C_3$-minor and if G has a vertex v of degree 2, then G has a $2C_3$-minor if and only if the graph obtained from G by smoothing out v has a $2C_3$-minor. We also prove Theorem 1.2.

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Let G be an outerplanar simple graph. Thus G consists of a Hamilton cycle H along with a set of chords C. Let C' be a disjoint copy of C. Embed $G \cup C'$ with chords C inside H and chords C' outside H. A *doubled outerplanar embedding* is any graph K contained between H and $G \cup C'$ with embedding inherited from $G \cup C'$.

Theorem 1.1. *If G is a connected and nonseparable graph with minimum degree 3, then G has no $2C_3$-minor if and only if*

(1) *$G \cong K_4$ or*

(2) *G is the topological dual graph of some doubled outerplanar embedding.*

Theorem 1.2. *If G is 3-connected and loopless, then $G \cong K_4$ or G contains a $2C_3$-minor.*

2. Proofs

Given a graph G, a *k-separation* is an expression $G = G_1 \cup G_2$ in which each G_i has at least k edges and $G_1 \cap G_2$ is a set of k vertices. A connected graph is *separable* when it has a 1-separation. A graph G is *nonseparable* when it is connected and has no 1-separation. A *link* is an edge in a graph that is not a loop. Note that every edge in a nonseparable graph is a link. A connected graph G is *k-connected* when it has at least $k+1$ vertices and it has no set of $t < k$ vertices whose removal leaves a disconnected subgraph.

Proof of Theorem 1.1. Assume that $G \cong K_4$ or $G = H^*$ where H is a doubled outerplanar embedding. It is important to note that the graph of a doubled outerplanar embedding is still an outerplanar graph. If $G \cong K_4$, then G has no $2C_3$-minor. If $G = H^*$, then H has no $K_{2,3}$-minor. (It is well known that a graph G is outerplanar if and only if it has no $K_{2,3}$- or K_4-minor.) Since any embedding of $2C_3$ in the plane has topological dual graph isomorphic to $K_{2,3}$, we get that G has no $2C_3$-minor.

Conversely, suppose that G has no $2C_3$-minor. The reader can check that $K_{3,3}$ and K_5 both contain $2C_3$-minors and hence G is planar. Let H be the topological dual graph of some embedding of G in the plane. Note that H has no faces of length two because G has minimum degree 3. Furthermore, since G is nonseparable, so must be H. Now $|V(H)| > 2$ because G has minimum degree 3. Since $|V(H)| \geq 3$, H is 2-connected. Let H_v be the graph obtained from H by adjoining an apex vertex to all other vertices of H. Thus H_v is 3-connected. If H_v is planar, then H is outerplanar and has an embedding in the plane without faces of length 2. Thus H is a doubled outerplanar embedding, a desired result. If H_v is non-planar, then by a theorem of D.W. Hall ([4] or see [5, 12.2.11]) either $H_v \cong K_5$ along with maybe some doubled edges or H_v contains a $K_{3,3}$-subdivision. In the former case, $H \cong K_4$ along with maybe some doubled edges. If an edge of K_4 is doubled, however, the resulting graph has a $2C_3$-minor, a contradiction. Thus $G \cong K_4$, a desired outcome. In the latter case H contains a $K_{2,3}$-subdivision and so G contains a $2C_3$-minor, a contradiction. □

Proof of Theorem 1.2. Let $\hat{G}$ be the simplification of G; that is, for each class of parallel links, delete all but one of them. Thus $\hat{G}$ is 3-connected and simple. By Tutte's Wheel Theorem ([6] or see

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[5, Theorem 8.8.4]) there is a sequence of 3-connected simple graphs $G_1, \ldots, G_t$ such that $G_1 = \hat{G}$, $G_{i+1} = G_i/e$ or $G_i\backslash e$, and $G_t \cong W_n$ for $n \geq 3$ where W_n is the n-spoked wheel. If $n \geq 4$, then G_t has a $2C_3$-minor and therefore so does G. So suppose that $G_t \cong W_3 \cong K_4$. If $G = G_t$, then we are done. So suppose that G_t is a proper minor of G. Since there is no 3-connected simple graph H for which H/e or $H\backslash e$ is K_4, we must have that $\hat{G} = G_1 = G_t \cong K_4$. Since G_t is a proper minor of G, G contains K_4 along with one doubled edge. This contains a $2C_3$-minor, as required. □

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Daniel Slilaty

Department of Mathematics and Statistics, Wright State University Dayton, OH 45435, USA

Email: `daniel.slilaty@wright.edu`