Robust coalitional model predictive control with predicted topology transitions

Eva Masero, José M. Maestre, Antonio Ferramosca, Mario Francisco, and Eduardo F. Camacho

***Abstract*—This paper presents a novel clustering model predictive control technique where transitions to the best cooperation topology are planned over the prediction horizon. A new variable, the so-called transition horizon, is added to the optimization problem to calculate the optimal instant to introduce the next topology. Accordingly, agents can predict topology transitions to adapt their trajectories while optimizing their goals. Moreover, conditions to guarantee recursive feasibility and robust stability of the system are provided. Finally, the proposed control method is tested via a simulated eight-coupled tanks plant.**

***Index Terms*—Model predictive control; Control by clustering; Distributed control; Coalitional control; Networked control.**

I. Introduction

Model predictive control (MPC) is an optimization-control method that uses a model to predict the system behavior through a given time horizon. At each time step, the control sequence that minimizes a cost function based on the system evolution is computed subject to a set of constraints. Only the first control signal of the obtained sequence is applied, and the rest is discarded. In the next instant, the time window of the problem is displaced one step ahead following a receding horizon strategy, and the same procedure is repeated. Compared with other methods, MPC presents major advantages, such as the ability to deal explicitly with delays and dead times. It also handles the multi-variable, non-minimum phase, and unstable systems [1]. For these reasons, this method is widely applied in the process industry [2].

Nevertheless, the application of MPC in large-scale processes is sometimes impossible due to the incapacity to find a centralized model of the overall system and the high computational requirements, to name two typical problems in this context. In these cases, a distributed approach may be required. Indeed, the improvements in computational and communication technologies made distributed MPC (DMPC) a popular framework with applications in large-scale systems such as road traffic networks [3], power grids [4], gas networks [5], irrigation canals [6], water distribution systems [7], supply chains [8], and energy management in buildings [9], [10], among others.

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The main idea of DMPC is to divide the overall system into subsystems governed by local controllers or agents with individual control goals that may cooperate to improve performance; see [11], [12] for further details. Recently, it has been proposed to adapt the cooperation burden in real-time depending on the coupling between control tasks by promoting the formation of coalitions or clusters of cooperating controllers. This approach is useful for autonomous systems that perform cooperative tasks and deal with saving energy, such as UAVs and robots. For example, [13] proposed an adaptive cluster formation scheme based on active coupling constraints, and [14] presented hierarchical controllers that are dynamically adapted to the operational conditions. Furthermore, the controller topology can be selected by a supervisory controller (top-down approach) [15], [16], or be obtained disabling communication links with a poor contribution to the local performance (bottom-up architecture) [17], [18].

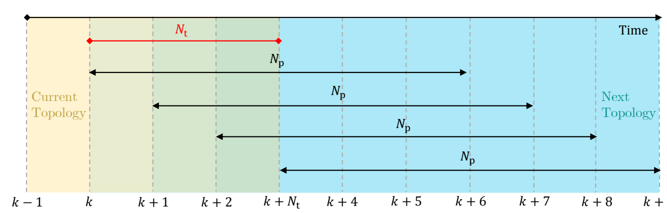


Fig. 1. Example scheme of the prediction horizon N_p and the transition horizon N_t.

This work presents a novel hierarchical coalitional MPC method, where the information is exchanged between two control layers. In the lower layer, local controllers optimize their control and state sequences, and the upper layer evaluates the information received from agents and chooses the best cooperation topology to attain a global objective, including performance and cooperation costs. Once the topology is selected, it is sent to the lower layer, where local controllers must follow it. The instant when they adopt the new topology affects the results. Hitherto, the new topology was chosen and abruptly implemented by local controllers at the same instant, as presented in [17], [15]. In this work, the moment to switch topology becomes a variable of the optimization problem, the so-called transition horizon. Unlike the prediction horizon, which is displaced toward the future at each time instant, the transition horizon is not moved ahead but actualized at each time instant as depicted in Fig. 1.

In other words, the optimization problem solved by the upper control layer considers the cost over the prediction horizon in a receding horizon fashion with the moment to switch topology being also optimized. Hence, our approach provides more degrees of freedom for optimization, and the resulting performance will always be no worse regarding the MPC receding horizon objective. It also results in the controller's capacity to gradually prepare for a switch of topology within the prediction horizon (see Fig. 1) and, hence, adapt its control sequence accordingly.

Regarding robustness, several techniques can be found in the literature to deal with constrained systems' stability in the presence of bounded additive disturbances. Some of them are the input-to-state (ISS) property [19], widely used to analyze non-linear systems; the min-max strategy, where the optimization problem is solved for the worst-case disturbances [20]; and the tube approach [21], [22], which is suitable for linear systems, and based on computing a region around the nominal trajectory that contains the system state under any uncertainties. Our approach relies on the inherent robustness of the predictive controllers due to the feedback nature of MPC [23], [24], [25].

To illustrate the benefits of our strategy, an eight-coupled tanks plant, which is an extension of the quadruple-tank process proposed by [26], has been designed. The quadruple-tank process has popularly been used as a benchmark, e.g, to study the effects of multi-variable plants with dead times [27].

Finally, a much earlier and abridged version of the proposed method was accepted for presentation at a conference [28]. There are significant differences between the current article and the conference paper: i) the conference version does not include proofs for its stability claims; ii) robustness is now considered and guarantees are given and proved in this regard; iii) we present here new improved methods for the selection of topology; iv) several refinements in the main algorithm have been introduced in the current version; v) the plant configuration has been modified to couple subsystems by the states; and vi) the experiments performed are new for the article version and include the previously mentioned novelties.

Index of contents: Section II introduces the problem formulation. Section III defines the control objective and the coalitional MPC algorithm. Controllers are designed in Section IV. Section V provides recursive feasibility and stability guarantees. Section VI details the eight-coupled tanks used as a benchmark, the control techniques assessed, and results via simulation. Section VII provides conclusions and discussion.

Notation: $\mathbb{N}_{0+}$ and $\mathbb{N}_+$ are the sets of non-negative and positive integers. $\mathbb{R}^n$ refers to an n-dimension set of real numbers. For sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, the Cartesian product is $\mathcal{X} \times \mathcal{Y} \triangleq \{(x, y) : x \in \mathcal{X},\ y \in \mathcal{Y}\}$. If $\{\mathcal{X}_i\}_{i\in\mathcal{N}}$ is a finite family of sets indexed by $\mathcal{N}$, then the Cartesian product $\bigtimes_{i\in\mathcal{N}} \mathcal{X}_i$ is $\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N = \{(x_1, x_2, \ldots, x_N) : x_1 \in \mathcal{X}_i, \ldots, x_N \in \mathcal{X}_N\}$. The set subtraction operator is $\backslash$. The image of set $\mathcal{X} \subseteq \mathbb{R}^n$ under a linear mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is $A\mathcal{X} \triangleq \{Ax : x \in \mathcal{X}\}$. For sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, the Minkowski sum is $\mathcal{X} \oplus \mathcal{Y} \triangleq \{x + y : x \in \mathcal{X},\ y \in \mathcal{Y}\}$; and for $\mathcal{Y} \subseteq \mathcal{X}$, the Pontryagin difference is $\mathcal{X} \ominus \mathcal{Y} \triangleq \{z \in \mathbb{R}^n : \mathcal{Y} \oplus \{z\} \subseteq \mathcal{X}\}$. Matrix I is the identity matrix. The l_z-norm defined in $\mathbb{R}^n$ is represented as $||\cdot||_z$ with $z \in \mathbb{N}_+$.

II. Problem Setting

In this section, the problem formulation is provided. The system dynamics are firstly introduced, and, subsequently, it is explained how information is exchanged between agents.

A. System dynamics and constraints

The system is divided into $\mathcal{N} = \{1, 2, \ldots, N\}$ coupled subsystems whose discrete-time dynamics are modeled as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_{ii}u_i(k) + w_i(k), \\ w_i(k) &= \sum_{j\in\mathcal{M}_i} \big(A_{ij}x_j(k) + B_{ij}u_j(k)\big) + w_i^{\mathrm{e}}(k), \end{aligned} \tag{1}$$

where $k \in \mathbb{N}_+$ is the discrete-time index, $x_i \in \mathbb{R}^{q_i}$ and $u_i \in \mathbb{R}^{r_i}$ are, respectively, the state and input vectors of each subsystem $i \in \mathcal{N}$, which are assumed to be constrained in convex sets containing the origin $\mathcal{X}_i \triangleq \{x_i \in \mathbb{R}^{q_i} : A_i^x x_i \leqslant b_i^x\}$ and $\mathcal{U}_i \triangleq \{u_i \in \mathbb{R}^{r_i} : A_i^u u_i \leqslant b_i^u\}$. Moreover, $A_{ii} \in \mathbb{R}^{q_i\times q_i}$, $B_{ii} \in \mathbb{R}^{q_i\times r_i}$ are matrices of the proper dimension for each subsystem. The variable $w_i \in \mathbb{R}^{q_i}$ comprises neighbor coupling through state vector $x_j \in \mathcal{X}_j$ and input vector $u_j \in \mathcal{U}_j$ with $j \in \mathcal{M}_i \triangleq \{j \in \mathcal{N}\backslash\{i\} : A_{ij} \neq 0 \vee B_{ij} \neq 0\}$, and the external disturbance w_i^{e}, which is assumed to be bounded by the convex set $\mathcal{W}_i^{\mathrm{e}}$. Therefore, w_i is bounded by $\mathcal{W}_i \triangleq \bigoplus_{j\in\mathcal{M}_i} \big(A_{ij}\mathcal{X}_j \oplus B_{ij}\mathcal{U}_j\big) \oplus \mathcal{W}_i^{\mathrm{e}}$, where $\mathcal{W}_i \subset \mathbb{R}^{q_i}$.

Aggregating all the subsystems states and inputs as $x_{\mathcal{N}} = (x_i)_{i\in\mathcal{N}}$ and $u_{\mathcal{N}} = (u_i)_{i\in\mathcal{N}}$, the global system evolution is

$$x_{\mathcal{N}}(k+1) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}(k) + w_{\mathcal{N}}(k), \tag{2}$$

where $A_{\mathcal{N}} = [A_{ij}]_{i,j\in\mathcal{N}}$, and $B_{\mathcal{N}} = [B_{ij}]_{i,j\in\mathcal{N}}$. Note that only external uncertainties affect the system, i.e. $w_{\mathcal{N}} = (w_i^{\mathrm{e}})_{i\in\mathcal{N}}$ because disturbances by coupling are implicitly included in the overall system dynamics. Thus, global additive disturbances are constrained in set $\mathcal{W}_{\mathcal{N}} = \bigtimes_{i\in\mathcal{N}} \mathcal{W}_i^{\mathrm{e}}$. Additionally, global states and inputs are constrained in sets $\mathcal{X}_{\mathcal{N}} = \bigtimes_{i\in\mathcal{N}} \mathcal{X}_i$ and $\mathcal{U}_{\mathcal{N}} = \bigtimes_{i\in\mathcal{N}} \mathcal{U}_i$, respectively.

B. Information exchange and network controllability

Each subsystem $i \in \mathcal{N}$ is managed by a local controller or agent with partial information of the overall system. Thus, agents must communicate and cooperate to attain their control goals. The cooperation network is described by the undirected graph $(\mathcal{N}, \mathcal{L})$, where the set of subsystems is $\mathcal{N}$, and the set of links is $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$. Each link $l_{ij} = \{i, j\} = \{j, i\} = l_{ji} \in \mathcal{L}$, which connects agents i and j, provides a bidirectional information flow that can be enabled or disabled by the control scheme. Each enabled link involves a fixed cooperation cost $\mathrm{c}_{\mathrm{link}} \in \mathbb{R}_{>0}$. As a result of the trade-off between performance and cooperation costs, the set of active links at instant k defines the network topology $\Lambda \subseteq \mathcal{L}$. Considering $L = |\mathcal{L}|$ the number of links, there are 2^L different topologies, which are grouped in a set $\mathcal{T} \triangleq \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2^L}\}$. For convenience, Λ_1 denotes decentralized topology, which corresponds to all cooperation links being disabled, and Λ_{2^L} corresponds to the centralized topology, i.e. when full network cooperation is established.

The disjoint set of cooperation clusters of subsystems, referred to as *coalitions*, resulting from a given topology

$\Lambda \in \mathcal{T}$ is denoted by $\mathcal{N}/\Lambda \triangleq \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{|\mathcal{N}/\Lambda|}\}$. Coalitions $\mathcal{C}$ are non-empty disjoint sets, and satisfy $\bigcup_{\mathcal{C}\in\mathcal{N}/\Lambda} \mathcal{C} = \mathcal{N}$.

Each coalition $\mathcal{C} \in \mathcal{N}/\Lambda$ is described by the discrete-time dynamics

$$\begin{aligned} x_{\mathcal{C}}(k+1) &= A_{\mathcal{C}} x_{\mathcal{C}}(k) + B_{\mathcal{C}} u_{\mathcal{C}}(k) + w_{\mathcal{C}}(k), \\ w_{\mathcal{C}}(k) &= \sum_{j\in\mathcal{M}_{\mathcal{C}}} \big(A_{\mathcal{C}j} x_j(k) + B_{\mathcal{C}j} u_j(k)\big) + w^{\mathrm{e}}_{\mathcal{C}}(k), \end{aligned} \tag{3}$$

where $x_{\mathcal{C}} = (x_i)_{i\in\mathcal{C}}$ and $u_{\mathcal{C}} = (u_i)_{i\in\mathcal{C}}$ are, respectively, the aggregate state and input vectors of the subsystems in $\mathcal{C}$, which are constrained in the sets $\mathcal{X}_{\mathcal{C}} = \times_{i\in\mathcal{C}} \mathcal{X}_i$ and $\mathcal{U}_{\mathcal{C}} = \times_{i\in\mathcal{C}} \mathcal{U}_i$. The corresponding state matrix is $A_{\mathcal{C}} = [A_{ij}]_{i,j\in\mathcal{C}}$, and the input matrix is $B_{\mathcal{C}} = [B_{ij}]_{i,j\in\mathcal{C}}$. The vector $w_{\mathcal{C}} = (w_i)_{i\in\mathcal{C}}$, which is contained in the set $\mathcal{W}_{\mathcal{C}} = \times_{i\in\mathcal{C}} \mathcal{W}_i$, represents the disturbances caused by the coupling with neighbors plus the external disturbances $w^{\mathrm{e}}_{\mathcal{C}}$. The set of neighbors that affects coalition $\mathcal{C}$ is $\mathcal{M}_{\mathcal{C}} \triangleq \{j \in \mathcal{N}\backslash\mathcal{C} : A_{\mathcal{C}j} \neq 0 \vee B_{\mathcal{C}j} \neq 0\}$.

From a centralized viewpoint, coalitions dynamics can be aggregated as (2) defining matrices $A_{\mathcal{N}} = \mathrm{diag}(A_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda}$, $B_{\mathcal{N}} = \mathrm{diag}(B_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda}$ and the overall state and input vectors as $x_{\mathcal{N}} = (x_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda}$ and $u_{\mathcal{N}} = (u_{\mathcal{C}})_{\mathcal{C}\in\mathcal{N}/\Lambda}$, respectively.

Assumption 1. *Each coalition $(A_{\mathcal{C}}, B_{\mathcal{C}})$, for $\mathcal{C} \in \mathcal{N}/\Lambda$, is controllable for any topology $\Lambda \in \mathcal{T}$.*

Remark 1. *The coalitions' controllability matrices need to be full rank for any topology as a necessary and sufficient network controllability condition [29]. Although it is a conservative assumption, the objective of this work is to evaluate the performance when the controller can optimize the time instant of topology changes.*

III. Control Algorithm

Coalitional control balances control performance and cooperation costs adapting the topology in real-time, enabling and disabling links. The price per active cooperation link $\mathrm{c_{link}}$ is a tuning parameter in the design of the coalitional controller and plays a significant role in deciding the new topology. Its value needs to be selected properly to represent the cooperation effort. The effect of this parameter was studied in [30].

A. Control goal

At time instant k, the goal of each coalition $\mathcal{C} \in \mathcal{N}/\Lambda$ is the minimization of a function $J^{\Lambda}_{\mathcal{C}}(\cdot)$, which sums a stage cost $l_{\mathcal{C}}(\cdot)$ and a cooperation cost $g^{\Lambda}_{\mathcal{C}}(\cdot)$ over a prediction horizon N_{p}, and also includes a terminal cost function $f^{\Lambda}_{\mathcal{C}}(\cdot)$, i.e.,

$$\begin{aligned} &J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k), u_{\mathcal{C}}(0:N_{\mathrm{p}}-1|k), \Lambda_{\mathcal{C}}(0:N_{\mathrm{p}}|k)\big) \\ &= f^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(N_{\mathrm{p}}|k), \Lambda_{\mathcal{C}}(N_{\mathrm{p}}|k)\big) + \sum_{t=0}^{N_{\mathrm{p}}-1} l_{\mathcal{C}}\big(x_{\mathcal{C}}(t|k), u_{\mathcal{C}}(t|k)\big) + \sum_{t=0}^{N_{\mathrm{p}}} g^{\Lambda}_{\mathcal{C}}\big(\Lambda_{\mathcal{C}}(t|k)\big), \end{aligned} \tag{4}$$

with

$$\begin{aligned} f^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(N_{\mathrm{p}}|k), \Lambda_{\mathcal{C}}(N_{\mathrm{p}}|k)\big) &= x_{\mathcal{C}}(N_{\mathrm{p}}|k)^{\top} P_{\mathcal{C}}\big(\Lambda_{\mathcal{C}}(N_{\mathrm{p}}|k)\big) x_{\mathcal{C}}(N_{\mathrm{p}}|k), \\ l_{\mathcal{C}}\big(x_{\mathcal{C}}(t|k), u_{\mathcal{C}}(t|k)\big) &= x_{\mathcal{C}}(t|k)^{\top} Q_{\mathcal{C}} x_{\mathcal{C}}(t|k) + u_{\mathcal{C}}(t|k)^{\top} R_{\mathcal{C}} u_{\mathcal{C}}(t|k), \\ g^{\Lambda}_{\mathcal{C}}\big(\Lambda_{\mathcal{C}}(t|k)\big) &= \mathrm{c_{link}} |\Lambda_{\mathcal{C}}(t|k)|, \end{aligned}$$

where $u_{\mathcal{C}}(0 : N_{\mathrm{p}}-1|k) \triangleq [u_{\mathcal{C}}(0|k), \ldots, u_{\mathcal{C}}(N_{\mathrm{p}}-1|k)]$ and $\Lambda_{\mathcal{C}}(0 : N_{\mathrm{p}}|k) \triangleq [\Lambda_{\mathcal{C}}(0|k), \Lambda_{\mathcal{C}}(1|k), \ldots, \Lambda_{\mathcal{C}}(N_{\mathrm{p}}|k)]$ are, respectively, the sequences of control actions and topologies in the prediction horizon[1]; $(t|k)$ indicates the prediction step t computed at instant k; $Q_{\mathcal{C}} \geqslant 0$ and $R_{\mathcal{C}}, P_{\mathcal{C}} > 0$ are weighting matrices; the cost per active link $\mathrm{c_{link}} \in \mathbb{R}_{>0}$ is a tuning parameter; $|\Lambda_{\mathcal{C}}(t|k)|$ represents the number of links in coalition $\mathcal{C}$ at prediction step t; and superscript Λ denotes a topology dependency. Note that the size of coalitions can range from an agent $\mathcal{C} = \{i\}$ to even the global system $\mathcal{C} = \mathcal{N}$. Thus, the overall cost can be calculated aggregating (4) as $J_{\mathcal{N}}(\cdot) = \sum_{\mathcal{C}\in\mathcal{N}/\Lambda} J^{\Lambda}_{\mathcal{C}}(\cdot)$.

The optimization of (4) has to be performed subject to the coalition dynamics and disturbances (3), constraints $\mathcal{X}_{\mathcal{C}}$ and $\mathcal{U}_{\mathcal{C}}$, and topology $\Lambda \subseteq \mathcal{L}$. Furthermore, the final state of coalition $\mathcal{C}$ must be constrained in a terminal region, which is a polyhedron defined as $\mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}} \triangleq \{x_{\mathcal{C}} \in \mathbb{R}^{q_{\mathcal{C}}} : A^{\mathcal{X}_{\mathrm{f}}}_{\mathcal{C}} x_{\mathcal{C}} \leqslant b^{\mathcal{X}_{\mathrm{f}}}_{\mathcal{C}}\}$, and whose calculation will be detailed later. Finally, note that since $J^{\Lambda}_{\mathcal{C}}(\cdot)$ is convex by definition for any possible topology sequence, we conclude the subsection with some properties of interest of $J^{\Lambda}_{\mathcal{C}}(\cdot)$, which will become useful later.

Definition 1 (N_{p}-step stabilizable set [23]). *Let $\mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$ be a terminal region for the nominal system $x_{\mathcal{C}}(k+1) = A_{\mathcal{C}} x_{\mathcal{C}}(k) + B_{\mathcal{C}} u_{\mathcal{C}}(k)$ of coalition $\mathcal{C} \in \mathcal{N}/\Lambda$, subject to $\mathcal{X}_{\mathcal{C}}$ and $\mathcal{U}_{\mathcal{C}}$. The N_{p}-step stabilizable set $\mathcal{S}^{\Lambda}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{C}}, \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}})$ is the set of states in $\mathcal{X}_{\mathcal{C}}$ that evolves to $\mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$ in a trajectory of N_{p} steps by feasible control sequences.*

Definition 2 (Robust positively invariant (RPI) set). *Given topology Λ, the set $\mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$ is an RPI set for $x_{\mathcal{C}}(k+1) = A_{\mathcal{C}} x_{\mathcal{C}}(k) + B_{\mathcal{C}} u_{\mathcal{C}}(k) + w_{\mathcal{C}}(k)$ if and only if there exists a linear control law $u_{\mathcal{C}} = K^{\Lambda}_{\mathcal{C}} x_{\mathcal{C}}$ such that its evolution satisfies: $x_{\mathcal{C}}(k) \in \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}} \rightarrow x_{\mathcal{C}}(k+1) = (A_{\mathcal{C}} + B_{\mathcal{C}} K^{\Lambda}_{\mathcal{C}}) x_{\mathcal{C}}(k) + w_{\mathcal{C}}(k) \in \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$, $\forall w_{\mathcal{C}}(k) \in \mathcal{W}^{\Lambda}_{\mathcal{C}}$ and $\forall k \in \mathbb{N}_{+}$.*

Following [23], it can be proved that the perturbed coalition $\mathcal{C}$ evolves to an RPI set w.r.t. disturbances $w_{\mathcal{C}}$. This fact is summarized in the following lemma.

Lemma 1 (Robust Stability [23]).

1) *Let the nominal model of coalition be $x_{\mathcal{C}}(k+1) = F\big(x_{\mathcal{C}}(k)\big)$, such that $F(\cdot)$ is continuous, and the origin is a fixed point. Let $J^{\Lambda}_{\mathcal{C}}(x_{\mathcal{C}})$ be a Lyapunov function of the closed-loop system such that*

$$\begin{aligned} a^{\Lambda}_{\mathcal{C}} ||x_{\mathcal{C}}(k)||^{\sigma} \leqslant J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k)\big) &\leqslant b^{\Lambda}_{\mathcal{C}} ||x_{\mathcal{C}}(k)||^{\sigma}, \\ J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k+1)\big) - J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k)\big) &\leqslant -c^{\Lambda}_{\mathcal{C}} ||x_{\mathcal{C}}||^{\sigma}, \end{aligned} \tag{5}$$

where $a^{\Lambda}_{\mathcal{C}}, b^{\Lambda}_{\mathcal{C}}, c^{\Lambda}_{\mathcal{C}}$ are positive constants, and $\sigma \geqslant 1$. Then, there exists a positive constant $\rho^{\Lambda}_{\mathcal{C}} = 1 - (c^{\Lambda}_{\mathcal{C}}/b^{\Lambda}_{\mathcal{C}}) < 1$ that satisfies $J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k+1)\big) \leqslant \rho^{\Lambda}_{\mathcal{C}} J^{\Lambda}_{\mathcal{C}}\big(x_{\mathcal{C}}(k)\big)$.

2) *Let $J^{\Lambda}_{\mathcal{C}}(\cdot)$ be also Lipschitz in a neighborhood of the origin $\Omega^{r}_{\mathcal{C}} \triangleq \{x_{\mathcal{C}} \in \mathcal{X}_{\mathcal{C}} : J^{\Lambda}_{\mathcal{C}}(\cdot) \leqslant r^{\Lambda}_{\mathcal{C}}\}$ and $\lambda^{\Lambda}_{\mathcal{C}}$ be the Lipschitz constant of $J^{\Lambda}_{\mathcal{C}}(\cdot)$ in $\Omega^{r}_{\mathcal{C}}$, i.e, $J^{\Lambda}_{\mathcal{C}}(x_{\mathcal{C}}(k) + w_{\mathcal{C}}) \leqslant J^{\Lambda}_{\mathcal{C}}(x_{\mathcal{C}}(k)) + \lambda^{\Lambda}_{\mathcal{C}} ||w_{\mathcal{C}}||$. Then, there exists a constant $\mu^{\Lambda}_{\mathcal{C}} > 0$ that satisfies*

$$\mu^{\Lambda}_{\mathcal{C}} \leqslant \big((1 - \rho^{\Lambda}_{\mathcal{C}})/\lambda^{\Lambda}_{\mathcal{C}}\big)\, r^{\Lambda}_{\mathcal{C}}, \tag{6}$$

such that if $w_{\mathcal{C}}(k) \in \mathcal{B}^{\mu}_{\mathcal{C}} \triangleq \{w_{\mathcal{C}} \in \mathbb{R}^{q_{\mathcal{C}}} : ||w_{\mathcal{C}}(k)|| < \mu^{\Lambda}_{\mathcal{C}}\}$ for all k, then the perturbed coalition $x_{\mathcal{C}}(k+1) = F\big(x_{\mathcal{C}}(k)\big) + w_{\mathcal{C}}$ is asymptotically ultimately bounded $\forall x_{\mathcal{C}}(0) \in \Omega^{r}_{\mathcal{C}}$.

[1] The topology is also relevant for the terminal cost of the controller. For this reason, its sequence is considered until the end of the prediction horizon.

As shown in Section V, our goal is to assure that the closed-loop system stays asymptotically ultimately bounded when $t \to \infty$ despite the disturbances by the topology switchings.

B. Coalitional MPC algorithm

The coalitional method can be prohibitive for large-scale systems since it solves a convex optimization problem for every possible topology. Nevertheless, it can be employed a subset of topologies $\mathcal{T}_{\mathrm{new}} \subset \mathcal{T}$ to relieve this issue.

Let $k \in \mathbb{N}_+$ denote the time instant, $T_{\mathrm{up}} \in \mathbb{N}_+$ indicate how often the upper control layer is executed, $\Lambda_{\mathrm{cur}} \in \mathcal{T}_{\mathrm{new}}(k)$ be the current topology, and $\mathcal{T}_{\mathrm{new}}(k) \subseteq \mathcal{T}$ denote the set of potential successor topologies at instant k according to a given criterion, e.g., considering only the topologies whose links differ in one with regard to the current topology.

At each time instant k, the proposed hierarchical coalitional MPC algorithm, which is divided into upper and lower control layers, is implemented as detailed in Algorithm 1. Every T_{up} time instants, the upper control layer is executed, where a centralized controller optimizes *Problem* 1 obtaining the optimal control sequence, the new topology, and the transition horizon to attain the global objective.

Algorithm 1 (Upper control layer)

Inputs: $k,\ T_{\mathrm{up}},\ x_{\mathcal{N}}(k),\ \Lambda_{\mathrm{cur}}$
Outputs: $u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k),\ \Lambda_{\mathrm{new}},\ N_{\mathrm{t}}$
1: The upper control layer measures $x_{\mathcal{C}}, \forall \mathcal{C} \in \mathcal{N}/\Lambda$.
2: The upper layer solves the following mixed-integer optimization problem:
<u>*Problem* 1:</u>

$$\min_{u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k),\, \Lambda_{\mathrm{new}},\, N_{\mathrm{t}}} J_{\mathcal{N}}\big(x_{\mathcal{N}}(k), u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k), \Lambda(0:N_{\mathrm{p}}|k)\big),$$

subject to

$$\begin{aligned} x_{\mathcal{C}}(t+1|k) =&\ A_{\mathcal{C}} x_{\mathcal{C}}(t|k) + B_{\mathcal{C}} u_{\mathcal{C}}(t|k),\ \ t = 0, \ldots, N_{\mathrm{p}}-1, \\ x_{\mathcal{C}}(k) =&\ \tilde{x}_{\mathcal{C}}(k),\ \ \forall \mathcal{C} \in \mathcal{N}/\Lambda, \\ x_{\mathcal{C}}(t|k) \in&\ \mathcal{X}_{\mathcal{C}} \ominus \mathcal{W}_{\mathcal{C}}\big(\Lambda_{\mathcal{C}}(t|k)\big),\ \ t = 1, \ldots, N_{\mathrm{p}}-1, \\ u_{\mathcal{C}}(t|k) \in&\ \mathcal{U}_{\mathcal{C}},\ \ t = 0, \ldots, N_{\mathrm{p}}-1, \\ x_{\mathcal{C}}(N_{\mathrm{p}}|k) \in&\ \mathcal{X}_{\mathcal{C}\mathrm{f}}\big(\Lambda_{\mathcal{C}}(t|k)\big), \end{aligned}$$

for all $\mathcal{C} \in \mathcal{N}/\Lambda$, and

$$\Lambda(t|k) = \begin{cases} \Lambda_{\mathrm{cur}} & \text{if } t < N_{\mathrm{t}} \\ \Lambda_{\mathrm{new}} & \text{if } t \geqslant N_{\mathrm{t}} \end{cases}, \quad t = 0, \ldots, N_{\mathrm{p}},$$
$$\Lambda_{\mathrm{new}} \in \mathcal{T}_{\mathrm{new}}(k),$$

where $\tilde{x}_{\mathcal{C}}(k)$ is the coalition state measurement; $\mathcal{W}_{\mathcal{C}}$ refers to the disturbances received by coalition $\mathcal{C}$; $\mathcal{X}_{\mathcal{C}\mathrm{f}}$ is the robust invariant terminal region computed offline and imposed as a terminal state constraint (see Section IV for details of its calculation); Λ_{cur} and Λ_{new} are the current and the next topology, respectively.
3: The resulting control sequence, the new topology, and the transition horizon are sent to the lower layer.

In the lower control layer, each coalition solves *Problem* 2 to compute its control action in accordance with the last information (i.e., the new topology and transition horizon) received by the upper layer:

Algorithm 1 (Lower control layer)

Inputs: $k,\ x_{\mathcal{C}}(k),\ \Lambda_{\mathrm{cur}},\ \Lambda_{\mathrm{new}},\ N_{\mathrm{t}}$
Outputs: $k,\ N_{\mathrm{t}},\ x_{\mathcal{C}}(k+1)$
1: The state $x_{\mathcal{C}}$ is measured for each $\mathcal{C} \in \mathcal{N}/\Lambda$.
2: Each coalition $\mathcal{C}$ calculates its control actions aware of the topology will switch after N_{t} time instants, as commanded by the upper layer. To this end, it is minimized (4) as:
<u>*Problem* 2:</u>

$$\min_{u_{\mathcal{C}}(0:N_{\mathrm{p}}-1|k)} J_{\mathcal{C}}^{\Lambda}\big(x_{\mathcal{C}}(k), u_{\mathcal{C}}(0:N_{\mathrm{p}}-1|k), \Lambda_{\mathcal{C}}(0:N_{\mathrm{p}}|k)\big),$$

subject to

$$\begin{aligned} x_{\mathcal{C}}(t+1|k) =&\ A_{\mathcal{C}} x_{\mathcal{C}}(t|k) + B_{\mathcal{C}} u_{\mathcal{C}}(t|k),\ \ t = 0, \ldots, N_{\mathrm{p}}-1, \\ x_{\mathcal{C}}(k) =&\ \tilde{x}_{\mathcal{C}}(k), \\ x_{\mathcal{C}}(t|k) \in&\ \mathcal{X}_{\mathcal{C}} \ominus \mathcal{W}_{\mathcal{C}}\big(\Lambda_{\mathcal{C}}(t|k)\big),\ \ t = 1, \ldots, N_{\mathrm{p}}-1, \\ u_{\mathcal{C}}(t|k) \in&\ \mathcal{U}_{\mathcal{C}},\ \ t = 0, \ldots, N_{\mathrm{p}}-1, \\ x_{\mathcal{C}}(N_{\mathrm{p}}|k) \in&\ \mathcal{X}_{\mathcal{C}\mathrm{f}}\big(\Lambda_{\mathcal{C}}(t|k)\big), \end{aligned}$$

and $\quad \Lambda_{\mathcal{C}}(t|k) = \begin{cases} \Lambda_{\mathcal{C}}^{\mathrm{cur}} & \text{if } t < N_{\mathrm{t}} \\ \Lambda_{\mathcal{C}}^{\mathrm{new}} & \text{if } t \geqslant N_{\mathrm{t}} \end{cases}, \quad t = 0, \ldots, N_{\mathrm{p}},$
where $\Lambda_{\mathcal{C}}^{\mathrm{cur}}$ and $\Lambda_{\mathcal{C}}^{\mathrm{new}}$ denote coalition $\mathcal{C}$ in the current and next topology, respectively.
3: Each coalition applies the first element of its optimal control sequence $u_{\mathcal{C}}(0 : N_{\mathrm{p}}-1|k)$ to obtain $x_{\mathcal{C}}(k+1)\ \forall \mathcal{C} \in \mathcal{N}/\Lambda$.
4: Set $k = k+1$, and update the transition horizon as $N_{\mathrm{t}} = N_{\mathrm{t}} - 1$ for the next time instant.

Remark 2. *Since disturbances are non-manipulable inputs, the feasibility of the state can be endangered. For that reason, a tightened constraint set $\mathcal{X}_{\mathcal{C}} \ominus \mathcal{W}_{\mathcal{C}}$ is considered to avoid violating the state constraints. Conversely, the input $u_{\mathcal{C}}$ is an optimization variable and, thus, the controller holds it in $\mathcal{U}_{\mathcal{C}}$.*

Remark 3. *The transition horizon N_{t} does not appear explicitly in the cost but in the topology constraints of the optimization problems. The problem solved by the upper layer has an extra degree of freedom on choosing the moment to switch topology. Hence, the resulting performance will always be optimal for the MPC receding horizon objective.*

Remark 4. *This basic algorithm may be varied in the way that the transition horizon is determined, e.g., N_{t} can be a fixed value, computed in the upper control layer, or obtained in the lower control layer. Note that, in the latter case, feasibility guarantees may be endangered, as detailed later. These alternatives will be assessed in the simulation section.*

IV. Controller Design Procedure

This section provides requirements to ensure that the closed-loop system is feasible and stable with the proposed coalitional MPC scheme. Details regarding the controller design procedure, based on linear matrix inequalities (LMI), are given.

A. Feedback design

Global sparse matrices K_{Λ} and P_{Λ} adapted to topology Λ are designed to solve an LMI problem following [17]. These matrices are sufficient to guarantee stability and to find the invariant terminal region, as shown later.

Lemma 2 (Coalitional feedback controller design [17])**.** *Let the global discrete system matrices be $A_{\mathcal{N}} = [A_{ij}]_{i,j\in\mathcal{N}}$ and $B_{\mathcal{N}} = [B_{ij}]_{i,j\in\mathcal{N}}$, with the stage cost matrices $Q_{\mathcal{N}} = \mathrm{diag}(Q_i)_{i\in\mathcal{N}}$ and $R_{\mathcal{N}} = \mathrm{diag}(R_i)_{i\in\mathcal{N}}$. If there are matrices $H_\Lambda = H_\Lambda^\top = [H_{ij}]_{i,j\in\mathcal{N}}$, where $H_{ij} \in \mathbb{R}^{q_i\times q_j}$, and $Y_\Lambda = [Y_{ij}]_{i,j\in\mathcal{N}}$, where $Y_{ij} \in \mathbb{R}^{r_i\times q_j}$ with $H_{ij} = 0$ and $Y_{ij} = 0$ if the link $l_{ij} \in \Lambda$ is disabled, i.e., if $i \in \mathcal{C}$ and $j \notin \mathcal{C}$, such that the following constraint is satisfied*

$$\begin{bmatrix} H_\Lambda & H_\Lambda A_{\mathcal{N}}^\top + Y_\Lambda^\top B_{\mathcal{N}}^\top & H_\Lambda Q_{\mathcal{N}}^{1/2} & Y_\Lambda^\top R_{\mathcal{N}}^{1/2} \\ A_{\mathcal{N}} H_\Lambda + B_{\mathcal{N}} Y_\Lambda & H_\Lambda & 0 & 0 \\ Q_{\mathcal{N}}^{1/2} H_\Lambda & 0 & I & 0 \\ R_{\mathcal{N}}^{1/2} Y_\Lambda & 0 & 0 & I \end{bmatrix} \geqslant 0, \tag{7}$$

then there exist feedback control law $K_\Lambda = [K_{\mathcal{C}}^\Lambda]_{\mathcal{C}\in\mathcal{N}/\Lambda} = Y_\Lambda H_\Lambda^{-1}$ that stabilizes the closed-loop system, and a Lyapunov function $f_{\mathcal{N}}\big(x_{\mathcal{N}}(k)\big) = x_{\mathcal{N}}(k)^\top P_\Lambda x_{\mathcal{N}}(k)$, with $P_\Lambda = [P_{\mathcal{C}}^\Lambda]_{\mathcal{C}\in\mathcal{N}/\Lambda} = H_\Lambda^{-1}$ that satisfies

$$f_{\mathcal{N}}\big(x_{\mathcal{N}}(k)\big) \geqslant \sum_{t=0}^{\infty} l_{\mathcal{N}}\big(x_{\mathcal{N}}(k+t), u_{\mathcal{N}}(k+t)\big). \tag{8}$$

As proved in [17], if (7) remains true, then the nominal system model $x_{\mathcal{N}}(k+1) = A_{\mathcal{N}} x_{\mathcal{N}}(k) + B_{\mathcal{N}} u_{\mathcal{N}}(k)$ is exponentially stable with the linear control law $u_{\mathcal{N}}(k) = K_\Lambda x_{\mathcal{N}}(k)$, with P_Λ satisfying

$$x_{\mathcal{N}}(k)^\top P_\Lambda x_{\mathcal{N}}(k) - x_{\mathcal{N}}(k+1)^\top P_\Lambda x_{\mathcal{N}}(k+1) \geqslant l_{\mathcal{N}}\big(x_{\mathcal{N}}(k)\big), \tag{9}$$

i.e., $x_{\mathcal{N}}(k)^\top P_\Lambda x_{\mathcal{N}}(k)$ is a a Lyapunov function. Consequently, (8) holds, and the exponential stability of the nominal system is guaranteed.

Assumption 2. *There exist matrices H_Λ and Y_Λ such that (7) holds for the decentralized topology, which presents the highest disturbances due to couplings, and also provides a feasible solution for any other topology.*

In order to design the controller, we maximize the trace of H_Λ subject to (7) to indirectly minimize $P_\Lambda = H_\Lambda^{-1}$ and, therefore, the control cost-to-go.

Remark 5. *Feedbacks K_Λ are defined for all cases ranging from decentralized to centralized topologies. Nevertheless, this method can be implemented with just a subset of topologies $\mathcal{T}_{\mathrm{new}} \subset \mathcal{T}$ to avoid combinatorial explosion issues in large-scale systems.*

B. Invariant set design

From the coalitions' viewpoint, the effect of their neighboring agents can be considered as unknown bounded disturbances when calculating invariant set $\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda$ to decouple the terminal regions and solve a more manageable problem.

Assumption 3. *For each $\mathcal{C}$ in $\mathcal{N}/\Lambda$, there exists an RPI set $\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda$ (whose size depends on the disturbance set $\mathcal{W}_{\mathcal{C}}^\Lambda$ that relies on the coupling with its neighbors in topology Λ) that satisfies $(A_{\mathcal{C}} + B_{\mathcal{C}} K_{\mathcal{C}}^\Lambda)\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda \oplus \mathcal{W}_{\mathcal{C}}^\Lambda \subseteq \mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda$, $\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda \subseteq \mathcal{X}_{\mathcal{C}}$, and $K_{\mathcal{C}}^\Lambda \mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda \subseteq \mathcal{U}_{\mathcal{C}}$ under the linear control law $u_{\mathcal{C}} = K_{\mathcal{C}}^\Lambda x_{\mathcal{C}}$. Hence, the robust positively invariant set for the overall system is computed as*

$$\mathcal{X}_{\mathcal{N}\mathrm{f}} = \bigtimes_{\mathcal{C}\in\mathcal{N}/\Lambda} \mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda, \tag{10}$$

which satisfies $(A_{\mathcal{N}} + B_{\mathcal{N}} K_\Lambda)\mathcal{X}_{\mathcal{N}\mathrm{f}} \oplus \mathcal{W}_{\mathcal{N}} \subseteq \mathcal{X}_{\mathcal{N}\mathrm{f}}$, $\mathcal{X}_{\mathcal{N}\mathrm{f}} \subseteq \mathcal{X}_{\mathcal{N}}$, and $K_\Lambda \mathcal{X}_{\mathcal{N}\mathrm{f}} \subseteq \mathcal{U}_{\mathcal{N}}$ under the linear control law $u_{\mathcal{N}} = K_\Lambda x_{\mathcal{N}}$.

Theorem 1. *The RPI set $\mathcal{X}_{\mathcal{N}\mathrm{f}} = \bigtimes_{\mathcal{C}\in\mathcal{N}/\Lambda_1}(\mathcal{X}_{\mathcal{C}\mathrm{f}}^{\Lambda_1})$ corresponding to the decentralized topology Λ_1 can be used for any other topology.*

Proof. For topology Λ_1, each coalition $\mathcal{C} = \{i\}$, $\forall i \in \mathcal{N}$ presents the highest disturbances set due to the highest coupling with neighbors, i.e., $\mathcal{W}_{\mathcal{C}}^\Lambda \subseteq \mathcal{W}_{\mathcal{C}}^{\Lambda_1}$, $\forall \Lambda \in \mathcal{T}$. Since greater $\mathcal{W}_{\mathcal{C}}^\Lambda$ involves greater $\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda$, it is satisfied that $\mathcal{X}_{\mathcal{C}\mathrm{f}}^\Lambda \subseteq \mathcal{X}_{\mathcal{C}\mathrm{f}}^{\Lambda_1}$. ☐

There are numerous procedures to find RPI sets given unknown bounded disturbances [31]. The Multi-Parametric Toolbox (MPT) of MATLAB® [32] can be employed to compute the maximal RPI set for each coalition of topology $\Lambda \in \mathcal{T}$ taking into account its constraints and disturbances.

For a conservative approach, the RPI set for decentralized topology can be generally used for any topology because it is computed for the worst-case disturbances. Note that, since an RPI set can be computed for each agent, the advantage of this approach is to avoid combinatorial explosion issues in large-scale systems.

V. Controller Feasibility and Stability

Hitherto, it has been proved the stability of the nominal system for a fixed topology and its corresponding feedback in the unconstrained case. In this section, recursive feasibility of the system is ensured when the coalitional MPC is implemented. Due to the persistent bounded disturbances, asymptotic stability cannot be assured. Consequently, we desire the system to converge to a bounded set around the origin [25].

A. Recursive feasibility

Assumption 4. *At time instant $k = 0$, Problem 1 has a feasible control sequence for the centralized topology, i.e., $\Lambda_{\mathrm{cur}} = \Lambda_{2^L}$.*

Theorem 2. *Let Assumption 2, which implies the existence of the most conservative K_Λ and P_Λ (feasible solution for any topology), and Assumption 4 hold. Given topology $\Lambda \in \mathcal{T}$ and matrices K_Λ, P_Λ computed by maximizing the trace of H_Λ subject to the LMI problem (7), then recursive feasibility of Algorithm 1 is always ensured.*

Proof. First, the proof deals with the recursive feasibility for a fixed topology $\Lambda \in \mathcal{T}$ from the global and coalitions viewpoints. Secondly, it is proved that the system is feasible despite topology switching.

Part 1. Consider the state $x_{\mathcal{N}}(k) \in \mathcal{X}_{\mathcal{N}}$ $\forall k$, the predicted nominal state is $x_{\mathcal{N}}(1|k) = A_{\mathcal{N}} x_{\mathcal{N}}(k) + B_{\mathcal{N}} u_{\mathcal{N}}(k)$. We can prove that $x_{\mathcal{N}}(1|k)$ fulfills constraints despite disturbances. Since $x_{\mathcal{N}}(k+1) = x_{\mathcal{N}}(1|k) + w_{\mathcal{N}}(k)$, and due to the problem constraints: $x_{\mathcal{N}}(1|k) \in \mathcal{X}_{\mathcal{N}} \ominus \mathcal{W}_{\mathcal{N}}$ and $w_{\mathcal{N}}(k) \in \mathcal{W}_{\mathcal{N}}$, then

$x_{\mathcal{N}}(k+1) \in \mathcal{X}_{\mathcal{N}}$. Thus, the recursive feasibility of the state is guaranteed.

Under Assumption 4, at least a topology Λ has an overall feasible control solution at instant $k=0$:

$$u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k) = [u_{\mathcal{N}}(0|k),\dots,u_{\mathcal{N}}(N_{\mathrm{p}}-1|k)], \quad (11)$$

which fulfills constraints $x_{\mathcal{N}}(k) \in \mathcal{X}_{\mathcal{N}}$, $u_{\mathcal{N}}(k) \in \mathcal{U}_{\mathcal{N}}$, and $x_{\mathcal{N}}(N_{\mathrm{p}}|k) \in \mathcal{X}_{\mathcal{N}\mathrm{f}}$ for all disturbance sequences. At next instant $k+1$, a shifted feasible sequence $u^{\mathrm{s}}_{\mathcal{N}}$ can be formed as the tail of (11) as

$$\begin{aligned} u^{\mathrm{s}}_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k+1) = & \ [u^{\mathrm{s}}_{\mathcal{N}}(0|k+1),\dots,u^{\mathrm{s}}_{\mathcal{N}}(N_{\mathrm{p}}-1|k+1)] = \\ = & \ [u_{\mathcal{N}}(1|k),\dots,u_{\mathcal{N}}(N_{\mathrm{p}}-1|k), K_{\Lambda}x_{\mathcal{N}}(N_{\mathrm{p}}|k)]. \end{aligned}$$

Since the overall feasible control solution $u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k)$ fulfills the terminal state constraint, the term $x_{\mathcal{N}}(N_{\mathrm{p}}|k) \in \mathcal{X}_{\mathcal{N}\mathrm{f}}$, and, at $k+1$ it holds

$$(A_{\mathcal{N}} + B_{\mathcal{N}}K_{\Lambda})x_{\mathcal{N}}(N_{\mathrm{p}}|k) + w_{\mathcal{N}} \in \mathcal{X}_{\mathcal{N}\mathrm{f}}, \quad \forall w_{\mathcal{N}} \in \mathcal{W}_{\mathcal{N}}.$$

It follows from applying this procedure recursively that the feasibility of *Problem* 1 is guaranteed for topology Λ and $\forall k$. Note that K_{Λ} and $\mathcal{X}_{\mathcal{N}\mathrm{f}}$ can be respectively dis-aggregated in $K^{\Lambda}_{\mathcal{C}}$ and $\mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$ for each $\mathcal{C} \in \mathcal{N}/\Lambda$. Thus, there are also feasible sequences for any $\mathcal{C} \in \mathcal{N}/\Lambda$ at instant $k = T_{\mathrm{up}}$, i.e.,

$$u_{\mathcal{C}}(0:N_{\mathrm{p}}-1|T_{\mathrm{up}}) = [u_{\mathcal{C}}(0|T_{\mathrm{up}}), u_{\mathcal{C}}(1|T_{\mathrm{up}}),\dots,u_{\mathcal{C}}(N_{\mathrm{p}}-1|T_{\mathrm{up}})],$$

subject to $x_{\mathcal{C}}(T_{\mathrm{up}}) \in \mathcal{X}_{\mathcal{C}}$, $u_{\mathcal{C}}(T_{\mathrm{up}}) \in \mathcal{U}_{\mathcal{C}}$, and $x_{\mathcal{C}}(N_{\mathrm{p}}|T_{\mathrm{up}}) \in \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$ for all disturbance sequences. At the following instant, a shifted feasible sequence $u^{\mathrm{s}}_{\mathcal{C}}$ can be composed with the feedback gain $K^{\Lambda}_{\mathcal{C}}$ as

$$u^{\mathrm{s}}_{\mathcal{C}}(0:N_{\mathrm{p}}-1|T_{\mathrm{up}}+1) = [u_{\mathcal{C}}(1|T_{\mathrm{up}}),\dots,u_{\mathcal{C}}(N_{\mathrm{p}}-1|T_{\mathrm{up}}), K^{\Lambda}_{\mathcal{C}}x_{\mathcal{C}}(N_{\mathrm{p}}|T_{\mathrm{up}})].$$

Given that $x_{\mathcal{C}}(N_{\mathrm{p}}|T_{\mathrm{up}}) \in \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}$, at the next instant, it holds

$$(A_{\mathcal{C}} + B_{\mathcal{C}}K^{\Lambda}_{\mathcal{C}})x_{\mathcal{C}}(N_{\mathrm{p}}|T_{\mathrm{up}}) + w_{\mathcal{C}} \in \mathcal{X}^{\Lambda}_{\mathcal{C}\mathrm{f}}, \quad \forall w_{\mathcal{C}} \in \mathcal{W}^{\Lambda}_{\mathcal{C}}.$$

Therefore, there are feasible solutions available for *Problem* 2 at time instant $T_{\mathrm{up}}+1$. Recall that if the recursive feasibility from the centralized viewpoint is assured, it can also be guaranteed from the coalitional viewpoint because K_{Λ} is aggregated from the $K^{\Lambda}_{\mathcal{C}}$ for any $\mathcal{C} \in \mathcal{N}/\Lambda$ and $\mathcal{X}_{\mathcal{N}\mathrm{f}}$ is computed by (10). Applying this procedure recursively, it can be proved that coalitional feasibility holds for all k.

Part 2. Switching between different topologies requires that the overall feasibility is not endangered. First, note that if a change of topology occurs at instant k_{c}, a feasible global input sequence must exist according to the constraints of *Problem* 1. Otherwise, the same topology Λ_{cur} could be used because it is a feasible control sequence, as proved in *Part* 1. Secondly, if global feasibility is not endangered when switching to topology Λ_{new}, neither is the coalitional one endangered. In particular, global feasibility implicitly considers the dynamics and constraints of all coalitions $\mathcal{C} \in \mathcal{N}/\Lambda_{\mathrm{new}}$. Therefore, feasibility is also guaranteed despite topology changes. □

B. Robust stability

Here it is analyzed the asymptotic ultimate boundedness of the constrained linear system in the presence of bounded additive uncertainties for our proposed coalitional MPC scheme. In particular, we consider the worst-case coupling between subsystems. By using this conservative approach, we can guarantee ultimate boundedness for any topology.

Assumption 5. *The most disturbed case is bounded by the constant* $\bar{\mu}_{\mathcal{N}} \leqslant (1-\bar{\rho}_{\mathcal{N}})/\bar{\lambda}_{\mathcal{N}}\,\bar{r}_{\mathcal{N}}$*, such that* $w_i(k) \in \mathcal{B}^{\mu}_i \triangleq \{w_i \in \mathbb{R}^{q_i} : ||w_i(k)|| < \bar{\mu}_{\mathcal{N}}\}\,\forall k$*, for any* $i \in \mathcal{N}$ *and* $\Lambda \in \mathcal{T}$*. Here,* $\bar{\rho}_{\mathcal{N}}, \bar{\lambda}_{\mathcal{N}}, \bar{r}_{\mathcal{N}}$ *are calculated according the Lemma 1 based on a common upper bound of the cost function* $\bar{J}_{\mathcal{N}}(\cdot)$ *for all system topologies, which is defined as*

$$\bar{J}_{\mathcal{N}}(\cdot) = l_{\mathcal{N}}(\cdot) + \bar{g}_{\mathcal{N}}(\Lambda_{2^L}) + \bar{f}_{\mathcal{N}}(\cdot), \quad (12)$$

where $l_{\mathcal{N}}(\cdot)$ *is the stage cost,* $\bar{g}_{\mathcal{N}}(\Lambda_{2^L})$ *is the cooperation cost of the centralized topology (all cooperation links enabled), and* $\bar{f}_{\mathcal{N}}(\cdot)$ *is the terminal cost, whose weighting matrix* $\bar{P} > P_{\Lambda}\ \forall\Lambda$ *is computed such that (7) is satisfied for any* K_{Λ}*.*[2]

Note that, although Assumption 5 is conservative, it is needed to ensure robust stability and feasibility.

Theorem 3 (Asymptotic ultimate boundedness)**.** *Let* $x_{\mathcal{N}}(k+1) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}(k)$ *be the global model of the nominal system, subject to* $x_{\mathcal{N}}(k) \in \mathcal{X}_{\mathcal{N}}$*,* $u_{\mathcal{N}}(k) \in \mathcal{U}_{\mathcal{N}}$*, and let the linear control law* $u_{\mathcal{N}}(k) = K_{\Lambda}x_{\mathcal{N}}(k)$ *stabilize the nominal system exponentially in* $\mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}})$*. Considering that its cost function* $J_{\mathcal{N}}(\cdot)$ *is Lipschitz in* $\mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}})$*, and Assumption 5 holds, then system (2) with the linear control law* $u_{\mathcal{N}}(k) = K_{\Lambda}x_{\mathcal{N}}(k)$ *is asymptotically ultimately bounded* $\forall x_{\mathcal{N}}(0) \in \bar{\Omega}^r_{\mathcal{N}}$*, with* $\bar{\Omega}^r_{\mathcal{N}} \triangleq \{x_{\mathcal{N}} \in \mathcal{X}_{\mathcal{N}} : \bar{J}_{\mathcal{N}}(\cdot) \leqslant \bar{r}_{\mathcal{N}}\} \subseteq \mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}}) \subseteq \mathcal{X}_{\mathcal{N}}$*.*

Proof. Firstly, given a fixed topology $\Lambda \in \mathcal{T}$, it is proved exponential stability and asymptotic ultimate boundedness for the nominal and the uncertain system, respectively. Secondly, we prove that the system is asymptotically ultimately bounded despite the switching of topologies.

Part 1. With respect to Lemma 1, the overall nominal system is continuous, and we have to prove that $\bar{J}_{\mathcal{N}}(\cdot)$ is Lyapunov function for the nominal system, i.e., a non-increasing MPC cost. This upper-bound cost function decreases in time if

$$\begin{gathered} \bar{J}_{\mathcal{N}}\big(x_{\mathcal{N}}(k+1), u^{\mathrm{s}}_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k+1), \Lambda(0:N_{\mathrm{p}}|k+1)\big) \leqslant \\ \bar{J}_{\mathcal{N}}\big(x_{\mathcal{N}}(k), u_{\mathcal{N}}(0:N_{\mathrm{p}}-1|k), \Lambda(0:N_{\mathrm{p}}|k)\big) \end{gathered}$$

holds, which can be rewritten as

$$\begin{gathered} \sum_{t=0}^{N_{\mathrm{p}}-1} l_{\mathcal{N}}\big(x_{\mathcal{N}}(t|k+1), u_{\mathcal{N}}(t|k+1)\big) + \sum_{t=0}^{N_{\mathrm{p}}} \bar{g}_{\mathcal{N}}\big(|\Lambda_{2^L}(t|k+1)|\big) + \bar{f}_{\mathcal{N}}\big(x_{\mathcal{N}}(N_{\mathrm{p}}|k+1)\big) \leqslant \\ \sum_{t=0}^{N_{\mathrm{p}}-1} l_{\mathcal{N}}\big(x_{\mathcal{N}}(t|k), u_{\mathcal{N}}(t|k)\big) + \sum_{t=0}^{N_{\mathrm{p}}} \bar{g}_{\mathcal{N}}\big(|\Lambda_{2^L}(t|k)|\big) + \bar{f}_{\mathcal{N}}\big(x_{\mathcal{N}}(N_{\mathrm{p}}|k)\big). \end{gathered}$$

Since in the nominal case $x_{\mathcal{N}}(t+1|k) = x_{\mathcal{N}}(t|k+1)$, deleting terms in common and rearranging terms, we have

$$\begin{gathered} l_{\mathcal{N}}\big(x_{\mathcal{N}}(N_{\mathrm{p}}|k), u_{\mathcal{N}}(N_{\mathrm{p}}|k)\big) + \bar{f}_{\mathcal{N}}\big(x_{\mathcal{N}}(N_{\mathrm{p}}|k+1)\big) - \bar{f}_{\mathcal{N}}\big(x_{\mathcal{N}}(N_{\mathrm{p}}|k)\big) \\ \leqslant l_{\mathcal{N}}\big(x_{\mathcal{N}}(0|k), u_{\mathcal{N}}(0|k)\big), \end{gathered}$$

[2]It is straightforward to prove that such weighting matrix exists. A very simple and conservative choice would be to take $\bar{P} = \sum_{\Lambda\in\mathcal{T}} P_{\Lambda}$.

where the left-hand side is lower than or equal to zero according to (9). Likewise, the right-hand side is positive definite by construction. Since $\bar{J}_{\mathcal{N}}(\cdot)$ is positive definite by (12), it is proved that $\bar{J}_{\mathcal{N}}(\cdot)$ is a Lyapunov function and fulfill the condition of Lemma 1. Moreover, since $\bar{J}_{\mathcal{N}}(\cdot)$ is a quadratic function with $Q_{\mathcal{N}} \geqslant 0$ and $R_{\mathcal{N}}, \bar{P} > 0$, and the system is linear, we claim that *Problem* 1 ensures exponential stability in $\mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}})$ for the nominal model system (see Theorem 2.24 (c) in [33]).

Nevertheless, when additive disturbances are considered, exponential stability may be endangered. Regarding Lemma 1, asymptotic ultimate boundedness can be assured for system (2), which has bounded additive uncertainties, if $\bar{J}_{\mathcal{N}}(\cdot)$ is also Lipschitz in $\mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}})$. Since the system is linear and continuous, there exists a Lipschitz constant $\bar{\lambda}_{\mathcal{N}}$ of $\bar{J}_{\mathcal{N}}(\cdot)$ in $\mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}})$ such that $\bar{J}_{\mathcal{N}}(x_{\mathcal{N}}(k) + w_{\mathcal{N}}) \leqslant \bar{J}_{\mathcal{N}}(x_{\mathcal{N}}(k)) + \bar{\lambda}_{\mathcal{N}}||w_{\mathcal{N}}||$. Since previous conditions and Assumption 5 hold, the overall perturbed system (2) is asymptotically ultimately bounded for any $x_{\mathcal{N}}(0) \in \bar{\Omega}^{r}_{\mathcal{N}} \subseteq \mathcal{S}_{N_{\mathrm{p}}}(\mathcal{X}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}\mathrm{f}}) \subseteq \mathcal{X}_{\mathcal{N}}$. Likewise, coalitions states are also ultimately bounded while satisfying constraints of *Problem* 2.

Part 2. When a change from Λ_{cur} to Λ_{new} is performed, the disturbances that affect the subsystems change. Since the asymptotic ultimate boundedness of the overall system and coalitions are guaranteed for the most disturbed case and a common Lyapunov function, we conclude that the closed-loop system remains asymptotically ultimately bounded despite the topology switchings. □

VI. SIMULATION

This section presents the eight-coupled tanks plant and describes the techniques used to assess the proposed algorithm.

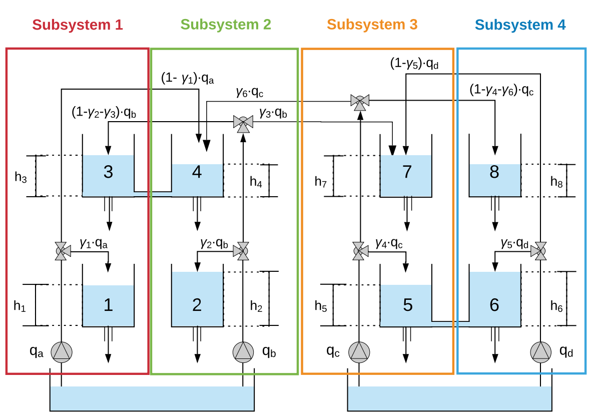


Fig. 2. Schematic diagram of the eight-coupled tanks plant.

A. Plant description

The plant is formed by eight interconnected tanks, as shown in Fig. 2. The four upper tanks (#3, #4, #7, and #8) discharge flows to the lower ones (#1, #2, #5, and #6), and these, in turn into sinking tanks. Moreover, the third and the fourth tanks are connected through a pipe, and also the fifth and the sixth tanks. The plant is controlled by four pumps that keep the water circulation between tanks. Additionally, six three-way valves, whose task is to divide into two ways the flow on arrival, are provided.

The overall system is divided into $N = 4$ subsystems: the first one is composed of tanks #1 and #3; the second one by tanks #2 and #4; the third one by tanks #5 and #7; and the last one by tanks #6 and #8. The control objective is that the lower tanks reach target levels taking into account control and cooperation costs subject to operational constraints. Consequently, it is considered a multi-variable control problem with four inputs (q_{a}, q_{b}, q_{c}, q_{d}) and four outputs (h_1, h_2, h_5, h_6).

B. Plant model

The model used in the simulation is non-linear and can be obtained applying mass balance and Bernoulli's law. It is given by the following equations:

$$
\begin{aligned}
S_1 \tfrac{dh_1}{dt} &= a_3\sqrt{2gh_3} - a_1\sqrt{2gh_1} + \gamma_1 \tfrac{q_{\mathrm{a}}}{3600},\\
S_2 \tfrac{dh_2}{dt} &= a_4\sqrt{2gh_4} - a_2\sqrt{2gh_2} + \gamma_2 \tfrac{q_{\mathrm{b}}}{3600},\\
S_3 \tfrac{dh_3}{dt} &= -a_3\sqrt{2gh_3} - a_{34}\sqrt{2g(h_3 - h_4)} + (1-\gamma_2-\gamma_3)\tfrac{q_{\mathrm{b}}}{3600},\\
S_4 \tfrac{dh_4}{dt} &= -a_4\sqrt{2gh_4} + a_{34}\sqrt{2g(h_3 - h_4)} + (1-\gamma_1)\tfrac{q_{\mathrm{a}}}{3600} + \gamma_6 \tfrac{q_{\mathrm{c}}}{3600},\\
S_5 \tfrac{dh_5}{dt} &= a_7\sqrt{2gh_7} - a_5\sqrt{2gh_5} + a_{56}\sqrt{2g(h_6 - h_5)} + \gamma_4 \tfrac{q_{\mathrm{c}}}{3600},\\
S_6 \tfrac{dh_6}{dt} &= a_8\sqrt{2gh_8} - a_6\sqrt{2gh_6} - a_{56}\sqrt{2g(h_6 - h_5)} + \gamma_5 \tfrac{q_{\mathrm{d}}}{3600},\\
S_7 \tfrac{dh_7}{dt} &= -a_7\sqrt{2gh_7} + (1-\gamma_5)\tfrac{q_{\mathrm{d}}}{3600} + \gamma_3 \tfrac{q_{\mathrm{b}}}{3600},\\
S_8 \tfrac{dh_8}{dt} &= -a_8\sqrt{2gh_8} + (1-\gamma_4-\gamma_6)\tfrac{q_{\mathrm{c}}}{3600},
\end{aligned}
\tag{13}
$$

where h_n is the water level of tank $n \in \{1, 2, \ldots, 8\}$; $S_n = 13.89 \cdot 10^{-3}\,\mathrm{m}^2$ is its corresponding cross-section, which is assumed to be equal for all tanks; $a_n = 50.265 \cdot 10^{-6}\,\mathrm{m}^2$ stands for the cross-section of the outlet pipes; and $a_{34}, a_{56} = 50.265 \cdot 10^{-7}\,\mathrm{m}^2$ refers to the cross-section of pipes which are connecting tanks 3-4 and 5-6. The parameter $\gamma_i \in [0, 1]$, with $i \in \{1, 2, \ldots, 6\}$, refers to the opening of the six three-way valves ($\gamma_1, \gamma_4 = 0.3$; $\gamma_2, \gamma_5 = 0.4$ and $\gamma_3, \gamma_6 = 0.1$), $g = 9.81\,\mathrm{m/s}^2$ corresponds to the gravity, and q_m refers to the caudal flow provided by pump $m \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$.

In order to obtain the control model, let us define an operating point of each tank, measured in meters, by $h_1^0 = 2.7831$; $h_2^0 = 4.8066$; $h_3^0 = 1.3388$; $h_4^0 = 1.7346$; $h_5^0 = 2.5740$; $h_6^0 = 2.6479$; $h_7^0 = 1.0317$; $h_8^0 = 1.2613$, and the operating point of each pump, measured in cubic meters per hour, by $q_1^0 = 1.2975$; $q_2^0 = 1.6663$; $q_3^0 = 1.4253$; $q_4^0 = 1.0113$. Hence, the discrete linear state-space model can be expressed as

$$
\bar{x}_{\mathcal{N}}(k+1) = A_{\mathcal{N}}\bar{x}_{\mathcal{N}}(k) + B_{\mathcal{N}}\bar{u}_{\mathcal{N}}(k) + \hat{d}_{\mathcal{N}}(k), \tag{14}
$$

where $\bar{x}_{\mathcal{N}} = [h_1(k) - h_1^0, \ldots, h_8(k) - h_8^0]^\top$ becomes the state vector, and $\bar{u}_{\mathcal{N}} = [q_{\mathrm{a}}(k) - q_{\mathrm{a}}^0, \ldots, q_{\mathrm{d}}(k) - q_{\mathrm{d}}^0]^\top$ is the input vector. Note that the system is linearized around the operating point. We have striven to minimize the linearization error using offset free concept proposed in [34], where it is estimated a disturbance term $\hat{d}_{\mathcal{N}}(k) = x_{\mathcal{N}}(k) - \bar{x}_{\mathcal{N}}(k)$ at each time instant k. Note that the system-modeling error $\hat{d}_{\mathcal{N}}$ can also be considered as an external disturbance $w_{\mathcal{N}} = \hat{d}_{\mathcal{N}}$.

Likewise, the state evolution of each subsystem $i \in \mathcal{N} = \{1, 2, 3, 4\}$ is given by

$$\begin{aligned} \bar{x}_i(k+1) &= A_{ii}\bar{x}_i(k) + B_{ii}\bar{u}_i(k) + \bar{w}_i(k) + \hat{d}_i(k), \\ \bar{w}_i &= \textstyle\sum_{j \in \mathcal{M}_i} \big(A_{ij}\bar{x}_j(k) + B_{ij}\bar{u}_j(k)\big), \end{aligned} \tag{15}$$

where $\bar{w}_i$ refers to the coupling through states and inputs with its neighbors $j \in \mathcal{M}_i$, which is constrained in the convex set $\mathcal{W}_i \triangleq \bigoplus_{j \in \mathcal{M}_i} \big(A_{ij}\mathcal{X}_j \oplus B_{ij}\mathcal{U}_j\big)$, and the subsystem-modeling error is defined as $\hat{d}_i(k) = x_i(k) - \bar{x}_i(k)$, $\forall k$.

The subsystem matrices are given by:

$$A_{11} = \begin{bmatrix} 0.9763 & 0.0336 \\ 0 & 0.9659 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0.0062 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.0296 \\ 0.002 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.0008 \\ 0.0491 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0.9819 & 0.0297 \\ 0 & 0.9700 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & -0.0062 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.0010 \\ 0.0689 \end{bmatrix}, B_{22} = \begin{bmatrix} 0.0396 \\ 0 \end{bmatrix}, B_{23} = \begin{bmatrix} 0.0001 \\ 0.0098 \end{bmatrix},$$

$$A_{33} = \begin{bmatrix} 0.9752 & 0.0382 \\ 0 & 0.9613 \end{bmatrix}, A_{34} = \begin{bmatrix} 0.0144 & 0 \\ 0 & 0 \end{bmatrix}, B_{32} = \begin{bmatrix} 0.0002 \\ 0.0098 \end{bmatrix}, B_{33} = \begin{bmatrix} 0.0296 \\ 0 \end{bmatrix}, B_{34} = \begin{bmatrix} 0.0014 \\ 0.0588 \end{bmatrix},$$

$$A_{44} = \begin{bmatrix} 0.9756 & 0.0346 \\ 0 & 0.9642 \end{bmatrix}, A_{43} = \begin{bmatrix} -0.0144 & 0 \\ 0 & 0 \end{bmatrix}, B_{43} = \begin{bmatrix} 0.0008 \\ 0.0589 \end{bmatrix}, B_{44} = \begin{bmatrix} 0.0395 \\ 0 \end{bmatrix},$$

Additionally, states and inputs are constrained by

$$\begin{aligned} 0.02 \text{ m} < h_n \leqslant 5 \text{ m}, \quad \forall n \in \{1, 2, \ldots, 8\}, \\ 0 \text{ m}^3/\text{h} < q_m \leqslant 5 \text{ m}^3/\text{h}, \quad \forall m \in \{\text{a}, \text{b}, \text{c}, \text{d}\}. \end{aligned}$$

C. Control methods

Each agent i can cooperate through a network formed by the edges $\mathcal{L} = \big\{\{1, 2\}, \{2, 3\}, \{3, 4\}\big\}$, which can be enabled or disabled. Since the number of links is $L = 3$, there are eight cooperation topologies $\mathcal{T} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_8\}$, which are determined by the set of active links as displayed in Fig. 3. For example, all disabled links $\mathcal{L} = \{0, 0, 0\}$ define the decentralized topology Λ_1, and all active links $\mathcal{L} = \{1, 1, 1\}$ refer to centralized topology Λ_8.

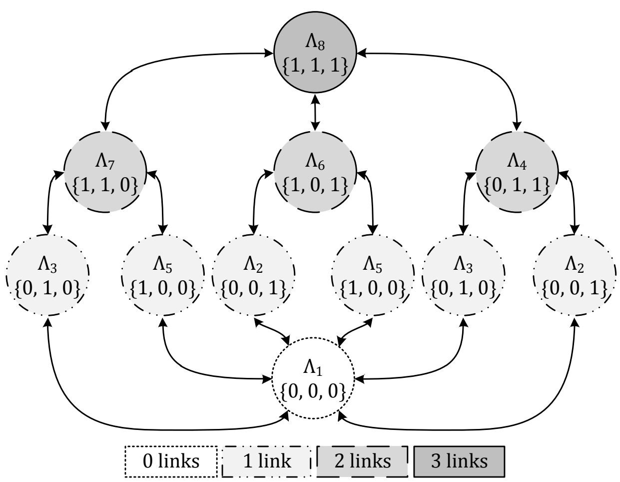


Fig. 3. The partially ordered set containing the eight cooperation topologies.

In this work, the set of potential successor topologies $\mathcal{T}_{\text{new}} \subseteq \mathcal{T}$ is formed by those topologies that only differ in zero or one link with respect to the current topology Λ_{cur}, i.e., those $\Lambda \in \mathcal{T}$ whose Hamming distance is 0 or 1, as follows

$$\begin{aligned} \mathcal{T}_{\text{new}} \triangleq \quad & \{\Lambda \subset \Lambda_{\text{cur}} : |\Lambda| = |\Lambda_{\text{cur}}| - 1\} \\ & \cup \{\Lambda_{\text{cur}} \subset \Lambda : |\Lambda| = |\Lambda_{\text{cur}}| + 1\} \cup \{\Lambda_{\text{cur}}\}. \end{aligned}$$

The following five MPC control methods are assessed:

1) **CEN** consists of a centralized MPC scheme characterized by full system information.
2) **ABRUPT** presents a coalitional MPC algorithm where the topology Λ_{cur} is instantly switched to $\Lambda_{\text{new}} \in \mathcal{T}_{\text{new}}$, i.e., $N_{\text{t}} = 0$, which corresponds to schemes in [15], [35]. In this method, the topology Λ^*_{new} is chosen every $T_{\text{up}} = 20$ samples after evaluating by exhaustive search $|\mathcal{T}_{\text{new}}|$ quadratic problems.
3) $\mathbf{PRED_{10}}$ implements a coalitional MPC scheme where the transition to Λ_{new} is made throughout a fixed transition horizon $N_{\text{t}} = 10$. Every $T_{\text{up}} = 20$ time steps from $k = 10$, the upper control layer assesses $|\mathcal{T}_{\text{new}}|$ problems and selects the optimal topology Λ^*_{new} to make transitions at the same time that ABRUPT.
4) $\mathbf{PRED_{N_t up}}$ is a coalitional MPC algorithm that every $T_{\text{up}} = 20$ samples solves $|\mathcal{T}_{\text{new}}| \cdot |N_{\text{t}}|$ convex problems and chooses that with the lowest cost to find the pair $\{\Lambda^*_{\text{new}}, N^*_{\text{t}}\}$. The variable $N_{\text{t}} \in \mathbb{N}_+$ takes values between 0-10, and the considered number of topologies is $|\mathcal{T}_{\text{new}}|$.
5) $\mathbf{PRED_{N_t low}}$ presents a coalitional MPC scheme where the transition horizon N_{t} is defined as an optimization variable of the lower control layer. The variable N_{t} can take values between 0 and 10. Note that the new topology Λ^*_{new} is still chosen by the upper layer every $T_{\text{up}} = 20$ time steps.

Table I displays a comparative summary of these methods.

Remark 6. *Feasibility guarantees may be endangered in $PRED_{N_t low}$ since the lower layer can delay the instant in which the new topology, chosen by the upper layer, is implemented. However, we did not have any feasibility problem in the implementation of the $PRED_{N_t low}$ method.*

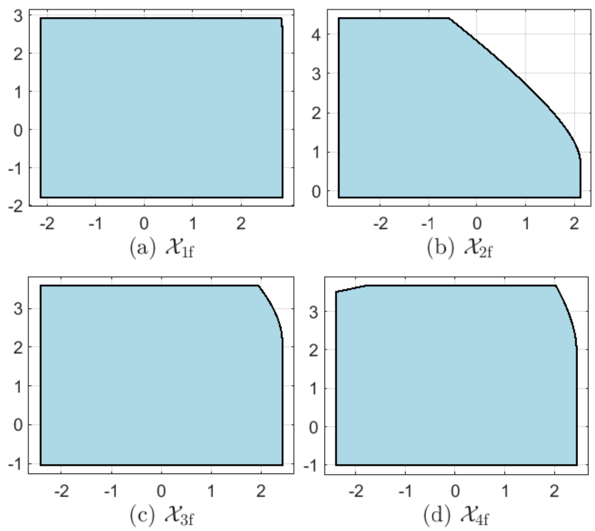
The weighting Q_i and R_i used for the controller design are

$$Q_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_i = 0.1, \quad \forall i \in \mathcal{N} = \{1, 2, 3, 4\}.$$

The global feedback gain $K_\Lambda = [K_i]_{i \in \mathcal{N}}$ and global weighting matrix $P_\Lambda = [P_i]_{i \in \mathcal{N}}$, which defines the terminal cost, are obtained according to Lemma 2. As explained before, to compute the RPI set $\mathcal{X}_{\mathcal{N}\text{f}} = \times_{i \in \mathcal{N}} \mathcal{X}_{i\text{f}}$, standard methods [36] have been implemented with the Multi-Parametric Toolbox (MPT) of MATLAB® [32]. To reduce complexity, the RPI set $\mathcal{X}_{\mathcal{N}\text{f}}$ employed is the one of the decentralized topology Λ_1 because it is the most conservative (see Theorem 1). Fig. 4 shows its corresponding invariant sets $\mathcal{X}_{i\text{f}}$, $\forall i \in \{1, 2, 3, 4\}$.

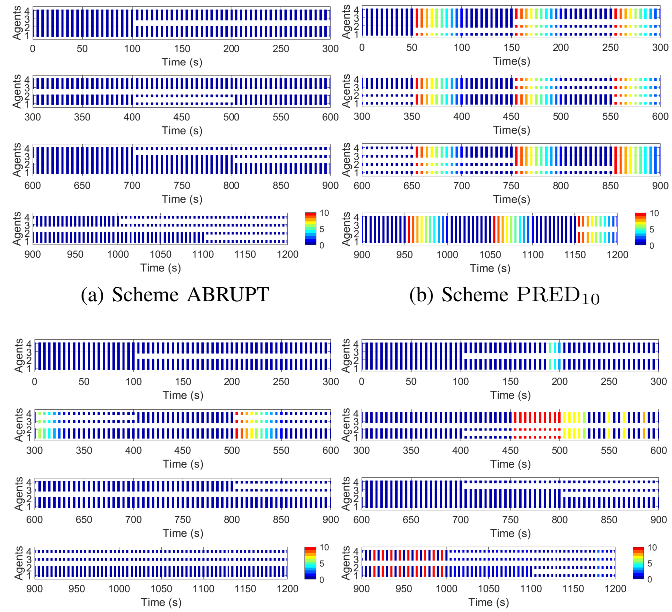
D. Results

The sampling time used in the simulation is $T_{\text{m}} = 5$ s, with a simulation length of $N_{\text{sim}} = 240$ time steps. The initial topology is the centralized one Λ_8. Recall that the topology will gradually change according Fig. 3. The MPC methods consider a prediction horizon $N_{\text{p}} = 20$ samples, and a cost per active cooperation link $\text{c}_{\text{link}} = 2 \cdot 10^{-3}$. N.B.: This value is chosen large enough to represent cooperation costs, considering that a small value and a huge value will always lead to a centralized and a decentralized network, respectively. Further details about the effect of the tuning



(a) $\mathcal{X}_{1f}$ (b) $\mathcal{X}_{2f}$ (c) $\mathcal{X}_{3f}$ (d) $\mathcal{X}_{4f}$

Fig. 4. Invariant sets of the four subsystems in topology Λ_1.



(a) Scheme ABRUPT (b) Scheme PRED_{10} (c) Scheme $\mathrm{PRED}_{\mathrm{N_t up}}$ (d) Scheme $\mathrm{PRED}_{\mathrm{N_t low}}$

Fig. 5. Formation of coalitions and transition horizons N_t used in each coalitional scheme. The transition horizon is depicted with a color linked to a value between $0 - 10$ as identified in the colorbar.

parameter c_{link} can be found in [30]. The computational time τ_{total} has been measured by implementing all optimization problems in one PC, using MATLAB® on Windows with a PC Intel® Core™ i7-8700 CPU at 3.20 GHz and 16 GB RAM. The computation time per average number of coalitions is computed as

$$\tau_{\mathcal{C}} = \frac{\tau_{\text{total}}}{\bar{\mathcal{C}}}, \text{ with } \bar{\mathcal{C}} = \frac{\sum_{k=1}^{N_{\text{sim}}} \sum_{\mathcal{C} \in \mathcal{N}/\Lambda} \mathcal{C}(k)}{N_{\text{sim}}},$$

to compare the schemes presented. The control goal of all schemes is to regulate the state to a setpoint different from the origin. Moreover, small reference changes are considered, and RPIs corresponding to Λ_1 are computed for each setpoint to comply with the assumptions required for the methods.

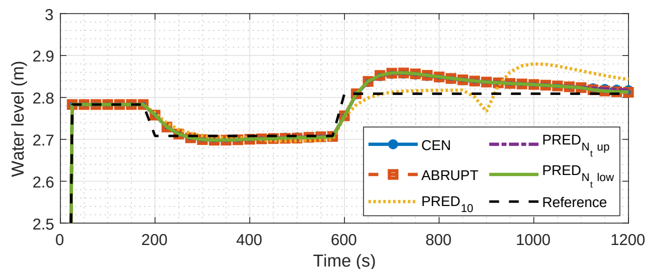
The formation of clusters and the value of N_t are illustrated in Fig. 5. For example, the scheme ABRUPT (Fig. 5 (a)) shows that the four agents have cooperated for the first 100 s (centralized topology Λ_8). Then, the topology changes to Λ_6 at time instant $t = 105$ s, i.e., agents #1 and #2 form a coalition, and the same holds for agents #3 and #4 for 300 s. At $t = 405$ s, agents #1 and #2 start to work independently in topology Λ_2. From $t = 505$ to $t = 600$ s, topology changes to Λ_6, where there are two coalitions $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{3, 4\}$. Afterwards, centralized topology Λ_8 is adopted from $t = 605$ s to $t = 700$ s, and so on. In scheme $\mathrm{PRED}_{\mathrm{N_t up}}$ (Fig. 5 (c)), topology Λ_5 is selected at $t = 305$ s to be fully implemented in $N_t = 6$, i.e., $t = 30$ s time instant head because the sampling time is $T_m = 5$ s. Moreover, this scheme selects topology Λ_5 at $t = 500$ s, which will be completely adopted in $N_t = 10$ i.e., $t = 50$ s time instant head.

Figures 6-7 show several outputs (h_1, h_2) and inputs (q_a, q_b) of the plant for the five control schemes implemented. The significantly different behavior of PRED_{10} can be explained as follows. Let us focus on the system behavior between $t = 850$-950 s. At $t = 850$ s, scheme PRED_{10} chooses the centralized topology $\Lambda_{\text{new}} = \Lambda_8$. In this case, the switch of topology is not entirely implemented until $N_t = 10$ time instant later, i.e., after 50 s (recall that the sampling time is $T_m = 5$ s). In other words, there is a transition from $\Lambda_{\text{cur}} = \Lambda_4$, where agent #1 works independently, to $\Lambda_{\text{new}} = \Lambda_8$, with agent #1 cooperating with the rest. This situation generates a conflict with the values of q_a and q_b to maintain h_1 and h_2 close to their references. As soon as agent #1 starts to fully cooperate with the rest (at $t = 900$ s), this behavior is corrected. In particular, q_a is increased to raise water levels h_1 and h_4, and q_b is simultaneously decreased not to raise h_2 and h_3, which would increase h_1 in consequence. Finally, the centralized controller finds the appropriate control sequences to reach both references conveniently.

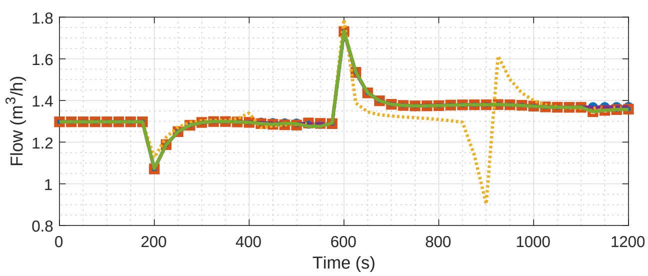
Numerical results are shown in Table I. The method CEN provides the best performance cost at the expense of keeping all links enabled. Reducing cooperation costs and the size of coalitions is crucial in some applications. Thus, the online formation of coalitions permits to reduce cooperation costs while satisfying the system requirements. Note that the upper control layer of the coalitional methods establishes full communication every T_{up} time instants to know the current state of the overall system. Hence, the cooperative costs of these centralized communications are also considered. With regard to the total cost of the centralized MPC, ABRUPT provides an improvement of 21.25 %, PRED_{10} of 6 %, $\mathrm{PRED}_{\mathrm{N_t up}}$ of 24 %, and $\mathrm{PRED}_{\mathrm{N_t low}}$ of 21.30 %. As shown, after optimizing dynamically the moment of switch topology, the lowest cost is achieved. Hence, our results show that it may be interesting to anticipate the switching of topologies from a global cost viewpoint. In particular, methods $\mathrm{PRED}_{\mathrm{N_t up}}$ and $\mathrm{PRED}_{\mathrm{N_t low}}$ outperform the control techniques with fixed N_t when cooperation costs are considered.

TABLE I
NUMERICAL RESULTS FROM APPLYING THE FIVE CONTROL METHODS

Control scheme	$\mathbf{N_t^*}$	$\mathbf{\Lambda^*_{new}}$	# Problems to compute the topology	τ_C	Perform. cost	Coop. cost	Total cost
CEN	–	Fixed	–	6.1172 s	1.0595	1.4400	2.4995
ABRUPT	0	Variable	$\lvert\mathcal{T}_{\rm new}\rvert$ problems to choose $\Lambda^*_{\rm new}$	2.3610 s	1.0623	0.9060	1.9683
$\mathbf{PRED_{10}}$	10	Variable	$\lvert\mathcal{T}_{\rm new}\rvert$ problems to select $\Lambda^*_{\rm new}$	1.4121 s	1.4121	0.9375	2.3496
$\mathbf{PRED_{N_t up}}$	Variable	Variable	$\lvert\mathcal{T}_{\rm new}\rvert \cdot \lvert N_{\rm t}\rvert$ problems to choose $\{\Lambda^*_{\rm new}, N_{\rm t}^*\}$	5.3266 s	1.0661	0.8336	1.8997
$\mathbf{PRED_{N_t low}}$	Variable	Variable	$\lvert\mathcal{T}_{\rm new}\rvert$ to set $\Lambda^*_{\rm new}$, and $\Lambda^*_{\rm new} \cdot \lvert N_{\rm t}\rvert$ to select $N_{\rm t}^*$	20.052 s	1.0632	0.9040	1.9672

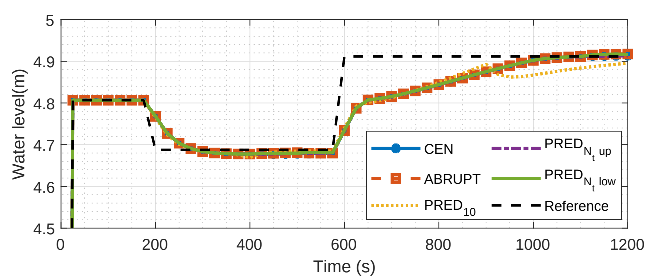


(a) Water level h_1

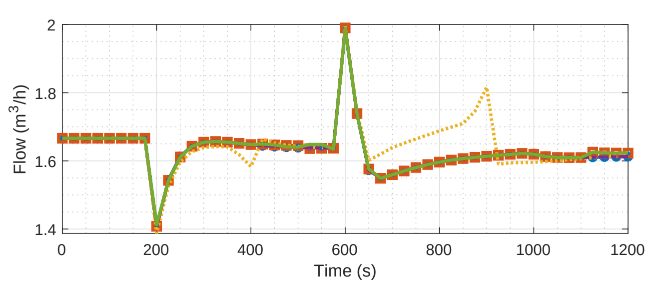


(b) Flow rate $q_{\rm a}$

Fig. 6. Trajectories obtained from applying the MPC control methods.



(a) Water level h_2



(b) Flow rate $q_{\rm b}$

Fig. 7. Trajectories obtained from applying the MPC control methods.

VII. CONCLUSIONS AND DISCUSSION

A novel coalitional MPC algorithm with a hierarchical architecture where local controllers can predict topology transitions over the prediction horizon was presented. A new variable, the so-called transition horizon, is added to the optimization problem to obtain the optimal time instant to switch the topology. The proposed coalitional schemes promote cooperation between agents to minimize costs and provide a smoother system evolution. Conditions to assure the recursive feasibility and asymptotic ultimate boundedness of the closed-loop system are also considered in the controller design. Additionally, the algorithm employs a subset of network topologies to mitigate scalability issues in large-scale systems.

Numerical results in an eight-coupled tanks plant show that there are improvements in total costs after $N_{\rm t}$ is introduced as a variable in the optimization problem. Our comparison shows that, in the methods $\mathrm{PRED_{N_t up}}$, and $\mathrm{PRED_{N_t low}}$ where the topology-switching time is optimized, the accumulated total costs are lower than the control schemes with fixed switching time and, in turn, are lower than the centralized MPC cost when cooperation costs are explicitly considered. Therefore, our results suggests that additional flexibility in introducing the new topology plays an outstanding role in decreasing the total cost. However, guaranteeing that our proposed methodology always results in a lower cost-to-go of the infinite-horizon problem is challenging due to the disturbances. Although disturbances might occasionally benefit the closed-loop cost of an abrupt change of topology, it is preferable to let the controller have this additional degree of freedom because it makes the optimization cost no worse. Additionally, it also results in the controllers' capacity to gradually prepare for a topology switch within the prediction horizon and compute their control actions accordingly. Coalitional MPC methods with fixed switching time are likely to offer worse performance in situations where its lack of flexibility becomes an issue, e.g., when there is a forecasted event several time steps ahead within the prediction horizon (for instance, a change of reference or a disturbance that can be anticipated), but the decision regarding the change of topology must be performed at that specific time step. Consequently, topology changes may be introduced too early or too late with the consequent decrease of optimality. Hence, adjusting the switch of topology within the prediction horizon allows the controller to benefit more from future knowledge, which is a key point of the proactive nature of MPC.

Future research will deal with a fully distributed implementation of the proposed approach. Likewise, plug-and-play features for the proposed scheme will also be explored.

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