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CR embeddings of CR manifolds

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Abstract
We improve results of Baouendi, Rothschild and Treves and of Hill and Nacinovich by finding a much weaker sufficient condition for a CR manifold of type (n, k) to admit a local CR embedding into a CR manifold of type $(n+\ell, k-\ell)$. While their results require the existence of a finite dimensional solvable transverse Lie algebra of vector fields, we require only a finite dimensional extension.

Keywords CR manifold · CR embedding

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1 Introduction and notation

Consider a CR manifold (M, D, J) of type (n, k). This means that M is a manifold of dimension $2n + k$ with a rank $2n$ distribution D and a field of endomorphisms $J : D \to D$ such that $J^2 = -\mathrm{id}$. We assume that M is integrable, meaning that the $-\mathrm{i}$ eigendistribution $D^{0,1}$ of J is involutive, that is,

$$[\mathfrak{X}^{0,1}, \mathfrak{X}^{0,1}] \subseteq \mathfrak{X}^{0,1},$$

where

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$$\mathfrak{X}^{0,1} = \{Z \in \Gamma(D)_{\mathbb{C}} : Z = X + iJX, X \in \Gamma(D)\}.$$

As usual, Γ indicates a space of sections and a subscript $\mathbb{C}$ indicates a complexification. We write $\mathfrak{X}$ for the space of vector fields on M.

We say that $F : M \to \tilde{M}$ is a CR embedding of a CR-manifold (M, D, J) of type (n, k) into another CR-manifold $(\tilde{M}, \tilde{D}, \tilde{J})$ of type $(n + \ell, k - \ell)$ if F is a smooth embedding and $F_* : D^{0,1} \to \tilde{D}^{0,1}$ satisfies

$$F_* J = \tilde{J} F_*.$$

The case where $\ell = k$ is of particular interest, as this corresponds to an embedding into a complex manifold.

Finding embeddings of the type envisaged above is one of the fundamental problems in CR geometry. We are going to consider the local problem only; that is, we fix a point p and look for an embedding of a neighbourhood of p. This means that we can replace M by a small open neighbourhood of p at any stage.

We mention a few contributions that are particularly relevant. It is well known that analytic CR-manifolds can always be locally embedded in complex space. Baouendi, Rothschild and Treves [1] consider the case where there is an abelian Lie algebra $\mathfrak{g}$ of real vector fields that is transverse, in the sense that

$$\mathfrak{X} = \Gamma(D) \oplus \mathfrak{g},$$

and normalising, in the sense that

$$[\mathfrak{g}, \mathfrak{X}^{0,1}] \subseteq \mathfrak{X}^{0,1}, \tag{1}$$

and construct an embedding into a complex space. Baouendi and Rothschild [2] extended this result to deal with the case where the Lie algebra $\mathfrak{g}$ is no longer required to be abelian. Jacobowitz [4] considers the case where $\ell = 1$ and finds a condition for the existence of an embedding into $\mathbb{C}^{n+1}$. Finally, Hill and Nacinovich [3] treat the case where there is a solvable transverse normalising Lie algebra of complex vector fields of dimension ℓ, and construct an embedding into a manifold of type $(n + \ell, k - \ell)$; they use solvability to extend by induction on dimension. We are going to treat the case of a finite-dimensional Lie algebra extension of $\mathfrak{X}^{0,1}$ in $\mathfrak{X}_{\mathbb{C}}$ by nonvanishing complex vector fields X_1, ..., X_s, and show that M embeds into a CR manifold $\tilde{M}$ of type $(n + \ell, k - \ell)$ if $\dim(\mathfrak{X}^{(0,1)} + \langle X_1, \dots, X_s \rangle) = n + \ell$.

2 Main results

We state our theorem more precisely.

Theorem 1 *Let (M, D, J) be a CR-manifold of type (n, k). Suppose that $X_1, \dots, X_s$ are nonvanishing complex vector fields that normalise $\mathfrak{X}^{0,1}$ (as in (1)) and satisfy*

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma \mod \mathfrak{X}^{0,1},$$

where the $c^\gamma_{\alpha\beta}$ are constants. If

$$\dim(\mathfrak{X}^{0,1} + \operatorname{span}\{X_1, \dots, X_s\}) = n + \ell,$$

then there is a (local) CR-embedding of M into a CR-manifold $\tilde{M}$ of type $(n+\ell, k-\ell)$.

Proof Fix $p \in M$. Without loss of generality we assume that each $X_\alpha(p)$ is not purely imaginary. The $c^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra $\mathfrak{g}$ defined by

$$\mathfrak{g} := (\mathrm{span}\{X_1, \dots, X_s\} + \mathfrak{X}^{0,1})/\mathfrak{X}^{0,1}.$$

By renumbering the vector fields and passing to a submanifold of M containing p if necessary, we may suppose that

$$\mathfrak{X}^{0,1} + \mathrm{span}\{X_1, \dots, X_s\} = \mathfrak{X}^{0,1} \oplus \mathrm{span}\{X_1, \dots, X_\ell\},$$

where $\ell \le s$. We shall construct complex vector fields

$$Y_\alpha = \lambda^\gamma_\alpha(t) X_\gamma + \mathrm{i}\partial_\alpha,$$

where $\alpha = 1, \dots, s$ on a neighbourhood of $(p, 0)$ in $M \times \mathbb{R}^s$ such that

$$[Y_\alpha, Y_\beta] = 0 \mod \mathfrak{X}^{0,1}; \tag{2}$$

here $t^1, \dots, t^s$ are coordinates in $\mathbb{R}^s$ and ∂_α means $\partial/\partial t^\alpha$. Then we shall show that the functions λ^γ_α can be chosen in such a way that

$$\lambda^\gamma_\alpha(t^1, \dots, t^s) = \lambda^\gamma_\alpha(t^1, \dots, t^\ell) \tag{3}$$

when $\alpha \le \ell$. It will then follow quickly that the vector fields Y_α with $\alpha \le \ell$ define a CR structure on $M \times V$, where V is a neighbourhood of 0 in $\mathbb{R}^\ell$, and there is a CR-embedding of M in $M \times V$.

To show that (2) holds, we observe that the Y_α preserve the (lifted) $\mathfrak{X}^{0,1}$, and choose the functions $\lambda^\gamma_\alpha(t)$ such that $\lambda^\gamma_\alpha(0) = \delta^\gamma_\alpha$ and the Y_α commute modulo $\mathfrak{X}^{0,1}$. Equivalently,

$$\partial_\alpha \lambda^\gamma_\beta - \partial_\beta \lambda^\gamma_\alpha = \mathrm{i}\lambda^\mu_\alpha \lambda^\nu_\beta c^\gamma_{\mu\nu}. \tag{4}$$

Consider this system of PDE. Let $\{\xi_\gamma\}$ be a basis of an abstract copy of the Lie algebra $\mathfrak{g}$ and $(t^1, \dots, t^s)$ be coordinates of $\mathfrak{g}$ with respect to this basis. Then $\Lambda := \lambda^\gamma_\alpha \xi_\gamma dt^\alpha$ is a Lie algebra valued 1-form, and the system (4) may be rewritten as

$$d\Lambda = \frac{\mathrm{i}}{2}[\Lambda, \Lambda], \tag{5}$$

where d is the exterior derivative with respect to the t variables. This nonlinear autonomous system of PDE is similar to the structure equation of the Maurer–Cartan form, and this similarity allows us to solve the system (5). Let Ω be the left-invariant Maurer–Cartan form on the simply connected Lie group G with Lie algebra $\mathfrak{g}$. Then Ω satisfies the Maurer–Cartan equation

$$d\Omega = -\frac{1}{2}[\Omega, \Omega].$$

Let t be real-analytic local coordinates on a neighbourhood of the identity e in G such that 0 corresponds to e and define $\Omega := \omega^\gamma_\alpha(t)\xi_\gamma dt^\alpha$. Then $\Omega(0) = \xi_\alpha dt^\alpha$. Let

$$\lambda^\gamma_\alpha(t) = \omega^\gamma_\alpha(-\mathrm{i}t).$$

This is well defined since the ω^{γ}_{α} are real-analytic, and the $\lambda^{\gamma}_{\alpha}$ defined in this way satisfy the equations (4) and $\omega^{\gamma}_{\alpha}(0) = \delta^{\gamma}_{\alpha}$.

To arrange that (3) holds, we suppose that t^1, ..., t^s are exponential coordinates of the second kind in some neighbourhood of e in G, that is,

$$g = \exp(t^s \xi_s) \dots \exp(t^1 \xi_1).$$

We observe that the dt^{α} component of the Maurer–Cartan form depends on t^1, ..., $t^{\alpha-1}$ only. Indeed, the (left-invariant) Maurer–Cartan form is

$$dL_{g^{-1}} dg$$

and the dt^{α} component is

$$\begin{aligned} dL_{g^{-1}} \frac{\partial g}{\partial t^{\alpha}} &= dL_{\exp(-t_1 \xi_1)} \cdots dL_{\exp(-t^s \xi_s)} \\ &\qquad \frac{\partial}{\partial t_{\alpha}} L_{\exp(t^s \xi_s)} \cdots L_{\exp(t^{\alpha+1} \xi_{\alpha+1})} R_{\exp(t^1 \xi_1)} \cdots R_{\exp(t^{\alpha-1} \xi_{\alpha-1})} \exp(t^{\alpha} \xi_{\alpha}) \\ &= dL_{\exp(-t^1 \xi_1)} \cdots dL_{\exp(-t^{\alpha-1} \xi_{\alpha-1})} dR_{\exp(t^1 \xi_1)} \cdots dR_{\exp(t^{\alpha-1} \xi_{\alpha-1})} \xi_{\alpha} \\ &= \mathrm{Ad}_{\exp(-t^1 \xi_1)} \cdots \mathrm{Ad}_{\exp(-t^{\alpha-1} \xi_{\alpha-1})} \xi_{\alpha}. \end{aligned}$$

Here L and R denote left and right translations. Therefore the functions $\lambda^{\gamma}_{\alpha}$ do indeed depend only on the variables t^{μ} with $\mu < \alpha$.

It follows that $\mathfrak{X}^{0,1} \oplus \langle Y_1, \dots, Y_{\ell} \rangle$ is well defined on $M \times V$, where V is a suitable neighbourhood of 0 in $\mathbb{R}^{\ell}$. It remains to show that $\mathfrak{Y}^{0,1}$, the span of (the lift of) $\mathfrak{X}^{0,1}$ and the vector fields $Y_1, \dots, Y_{\ell}$ defines a CR structure on $\tilde{M} = M \times V$, that is,

$$\mathfrak{Y}^{0,1} \cap \overline{\mathfrak{Y}^{0,1}} = \{0\}.$$

Suppose that $V + a^j Y_j \in \mathfrak{Y}^{0,1}$ and $W + b^k \overline{Y}_k \in \overline{\mathfrak{Y}^{0,1}}$. If

$$V + a^j Y_j = W + b^k \overline{Y}_k,$$

then

$$W - V = a^j Y_j - b^k \overline{Y}_k = 0,$$

that is $V = W = 0$, and also

$$a^j (X_j + \mathrm{i}\partial_j) - b^j (\overline{X}_j - \mathrm{i}\partial_j) = 0.$$

Therefore $a_j = -b_j$ and hence

$$a_j (X_j + \overline{X}_j) = 0.$$

Since $X_j(p)$ is not purely imaginary and the $X_j(p)$ are linearly independent, $X_j + \overline{X}_j \neq 0$ in a neighbourhood of p, and so (again passing to a submanifold if necessary) $a_j = 0$ is the only solution. □

It may be worth remarking that if the CR manifold (M, D, J) admits a CR embedding into a complex space, then in fact the conditions of Baouendi, Rothschild and Treves [1] are satisfied, and *a fortiori* those of Hill and Nacinovich [3], and ours too. It is less clear what happens when there is a CR embedding into another CR manifold that is not a complex space.

It may also be helpful to note that in the special case where the vector fields X_1, ..., X_s are real, then they generate (local) flows that preserve the CR structure; if they also generate a Lie algebra, then this generates a (local) group of transformations that preserves the structure. Further, even in the more special case where there is a transverse normalising Lie algebra of real vector fields, then our result extends that of [1]; in this case, the use of exponential coordinates of the second kind is not necessary.

Here is a corollary of the proof of our theorem.

Corollary 1 *Let (G, D, J) be a left-invariant CR structure on a Lie group G. Then G can be locally embedded into complex space.*

Proof Let $\{X_1, \dots, X_s\}$ be a basis of right-invariant vector fields such that X_1, ..., X_ℓ are transverse to D at $e \in G$.

As before we can find (complex) functions $\lambda_{\alpha\beta}(t)$ such that the vector fields Y_α, given by

$$Y_\alpha := \sum \lambda_{\alpha\beta}(t) X_\beta + \mathrm{i}\frac{\partial}{\partial t_\alpha},$$

commute, and let $t^1, \dots, t^s$ be exponential coordinates of the second kind on G. Then $\mathfrak{X}^{0,1} + \langle Y_1, \dots, Y_\ell \rangle$ determines an integrable complex structure on (a neighbourhood of $e \times 0$ in) $G \times \mathbb{R}^\ell$. □

Of course, this was already known, as everything is analytic in this case, but arguably this proof is simpler.

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