

ALMOST SURE POINTWISE CONVERGENCE OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION ON \mathbb{T}^2

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ABSTRACT. We revisit a result from “Pointwise convergence of the Schrödinger flow, E. Compaan, R. Lucà, G. Staffilani, *International Mathematics Research Notices*, 2021 (1), 596-647” regarding the pointwise convergence of solutions to the periodic cubic nonlinear Schrödinger equation in dimension $d = 2$.

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1. INTRODUCTION

We consider the question of pointwise almost everywhere (a.e.) convergence of solutions to the cubic nonlinear Schrödinger equation (NLS) on \mathbb{T}^2 , namely

$$(1) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^2 u, \\ u(x, 0) &= f(x), \end{cases}$$

If $f \in H^s$, for what s do we have that $u(x, t) \rightarrow f(x)$ as $t \rightarrow 0$ for (Lebesgue) almost every x ?

In the linear Euclidean setting, namely when the linear Schrödinger equation is posed on \mathbb{R}^d , this question was first posed by Carleson [8]. He proved Lebesgue (a.e.) convergence $e^{it\Delta} f(x)$ to $f(x)$ for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. Dahlberg–Kenig [11] showed that this one dimensional result is sharp, proving the necessity of the regularity condition $s \geq \frac{1}{4}$ in any dimension. The (considerably more difficult) higher dimensional problem has been studied by many authors [10, 28, 34, 1, 26, 31, 32, 29, 20, 4, 22, 23, 24, 12, 16]. Recently, Bourgain [5] proved that $s \geq \frac{d}{2(d+1)}$ is necessary (see also [24, 21] for some refinements of this result). This has been proved to be sharp, up to the endpoint, by Du–Guth–Li [15] on \mathbb{R}^2 and by Du–Zhang [14] in higher dimensions (the endpoint case is still open in dimensions $d \geq 2$).

In the periodic case much less is known. When $d = 1$ Mouya–Vega [27] proved the sufficiency of $s > \frac{1}{3}$ and necessity of $s \geq \frac{1}{4}$. Their proof, based on Strichartz estimates, has been extended to dimension $d = 2$ in [35] and to higher dimension

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in [9]. In fact, together with recent improvements in periodic Strichartz estimates [6], one can show that $s > \frac{d}{d+2}$ is a sufficient condition for almost everywhere convergence to initial data. On the other hand there are several counterexamples showing that we have the same necessary conditions than on \mathbb{R}^d [27, 9, 17], namely the necessity of $s \geq \frac{d}{2(d+1)}$; in particular, one can “adapt” the counterexamples from \mathbb{R}^d to the periodic setting. At the moment, in the periodic case almost sure convergence when $s \in \left[\frac{d}{2(d+1)}, \frac{d}{d+2} \right]$ remains an open question.

In the first part of this paper we show how to extend the a.e. converge statement

$$(2) \quad \lim_{t \rightarrow 0} e^{it\Delta} f(x) = f(x), \quad \text{for a.e. } x \in \mathbb{T}^2 \text{ and for all } f \in H^s(\mathbb{T}^2), s > 1/2$$

to the case of the cubic equation.

Theorem 1.1. *If $f \in H^s(\mathbb{T}^2)$ with $s > 1/2$ and u is the corresponding solution to (1), then*

$$(3) \quad \lim_{t \rightarrow 0} u(x, t) = f(x) \quad \text{for a.e. } x \in \mathbb{T}^2.$$

Remark 1.2. *By the proof will be clear that any improvement of the amount of Sobolev regularity which is sufficient for the convergence of the linear Schrödinger flow on \mathbb{T}^2 would imply an analogous improvement in the statement of Theorem 1.1 as well.*

In the second part of this paper we consider probabilistic improvements to the convergence problem. More precisely, we will show that a randomization of the Fourier coefficients of the initial data guarantees a better pointwise behavior of the associated linear (and also nonlinear) evolution. To explain why we may expect this, it is worth mentioning that counterexamples to the linear pointwise convergence problem in the periodic setting has been constructed in [17] using as building block for the initial datum the (squared) Dirichlet kernels

$$(4) \quad \prod_{\ell=1, \dots, d} \sum_{k_\ell \in \mathbb{Z}, |k_\ell| \leq N} e^{ik_\ell \cdot x_\ell}, \quad x := (x_1, \dots, x_d).$$

where $N \gg 1$ is a large frequency parameter. It is worth recalling that the pointwise convergence problem is essentially¹ equivalent to establish an $L^2(\mathbb{T}^2)$ estimate for the maximal Schrödinger operator

$$(5) \quad \left\| \sup_{t \in [0, 1]} |e^{it\Delta} f| \right\|_{L^2(\mathbb{T}^2)} \lesssim \|f\|_{H^s(\mathbb{T}^2)}$$

It has been observed in [27, 17] that (5) behaves particularly bad with data of the form (4). It is in fact seen to be false for $s < \frac{n}{2(n+1)}$, taking $N \rightarrow \infty$. The moral is that if the bad counterexamples are characterized by having a very rigid

¹It is indeed easy to check that the maximal estimate (5) with $s > 1/2$ implies (2) (the argument is the same used in the proof of Proposition 2.1). The opposite implication requires the Stein maximal principle. Strictly speaking there is an equivalence with a weak L^2 estimate. On the other hand, the weak L^2 estimate can be easily promoted to a strong one with an ε -regularity loss. Thus, since we are not interested in an endpoint results, we see that (2) and (5) are indeed equivalent.

structure. The Fourier coefficients in (4) are indeed all equal to 1. This suggest to consider as “good” initial data the following randomized Fourier series

$$(6) \quad f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2} + \alpha}} e^{in \cdot x}, \quad \alpha > 0,$$

where g_n^ω are independent (complex) standard Gaussian variables. The Japanese brackets are defined as usual as $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

It is easy to see that if we fix $t \in \mathbb{R}$, then $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s < \alpha} H^s(\mathbb{T}^d)$ \mathbb{P} -almost surely (a.s.), where \mathbb{P} is the probability measure induced by the sequence $\{g_n^\omega\}_{n \in \mathbb{Z}^d}$. Thus we are working at the $H^{\alpha-}$ level. In fact more is true, namely that $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s < \alpha} C^s(\mathbb{T}^d)$, \mathbb{P} -a.s.; in particular $e^{it\Delta} f^\omega$ is \mathbb{P} -a. s. a continuous function of the x variable, On the other hand, the randomization does not improve the regularity, in the sense that $\|f^\omega\|_{H^\alpha(\mathbb{T}^d)} = \infty$ also holds \mathbb{P} -a. s.; see for example Remark 1.2 in [7] and the introduction of [25].

We have the following improved pointwise (a.e.) convergence result for randomized initial data.

Proposition 1.3. *Let $\alpha > 0$ and let f^ω of the form (6). We have \mathbb{P} -a. s. the following. For all $t \in \mathbb{R}$ the free solution $e^{it\Delta} f^\omega$ belongs to $\bigcap_{s < \alpha} C^s(\mathbb{T}^d)$ and*

$$e^{it\Delta} f^\omega(x) \rightarrow f^\omega(x) \quad \text{as } t \rightarrow 0$$

for every $x \in \mathbb{T}^d$ and uniformly.

Finally, we want to prove a similar statement for the cubic NLS (1). In fact, it will be more convenient working with the Wick ordered version of the equation (WNLS)

$$(7) \quad \begin{cases} i\partial_t u + \Delta u &= \mathcal{N}(u), \\ u(x, 0) &= f(x), \end{cases}$$

where

$$(8) \quad \mathcal{N}(u) := \pm u (|u|^2 - 2\mu), \quad \mu := \int_{\mathbb{T}^2} |u(x, t)|^2 dx = \int_{\mathbb{T}^2} |f(x)|^2 dx,$$

(recal that μ is a conserved quantity). Since solutions to WNLS are related to that of the cubic NLS by multiplication with a factor $e^{i2\mu t}$, the study of pointwise convergence turns out to be equivalent to that of NLS.

Theorem 1.4. *Let $\alpha > 0$ and let f^ω of the form (6). Let u be the solution to WNLS (7) with initial data f^ω . We have \mathbb{P} -almost surely :*

$$(9) \quad \lim_{t \rightarrow 0} u(x, t) = f^\omega(x) \quad \text{for a.e. } x \in \mathbb{T}^d.$$

Thus the same is true for solutions to the cubic NLS.

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1.2. Notations and terminology. For a fixed $p \in \mathbb{R}$ we often use the notation $p+ := p + \varepsilon$, $p- := p - \varepsilon$, where ε is any sufficiently small strictly positive real number. When in the same inequality we have two such quantities we use the following notation to compare them. We write $p + \dots + := p + \varepsilon \cdot$ (number of $+$), $p - \dots - := p - \varepsilon \cdot$ (number of $-$). We will use $C > 0$ to denote several constants depending only on fixed parameters, like for instance the dimension d . The value of C may clearly differ from line to line. Let $A, B > 0$. We may write $A \lesssim B$ if $A \leq CB$ when $C > 0$ is such a constant. We write $A \gtrsim B$ if $B \lesssim A$ and $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$. We write $A \ll B$ if $A \leq cB$ for $c > 0$ sufficiently small (and depending only on fixed parameters) and $A \gg B$ if $B \ll A$. We denote $A \wedge B := \min(A, B)$ and $A \vee B := \max(A, B)$. We refer to the following inequality

$$\|D^s P_N f\|_{L^q} \lesssim N^{s+\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p}, \quad 1 \leq p \leq q \leq \infty,$$

simply as Bernstein inequality. Here P_N is the frequency projection on the annulus $\xi \sim N$.

It is useful recall that the Strichartz estimates are the main tool to study the nonlinear Schrödinger flow. We recall the periodic Strichartz estimates from [2, 6]:

$$(10) \quad \|e^{it\Delta} P_N f\|_{L_{x,t}^p(\Omega^{d+1})} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|P_N f\|_{L_x^2(\Omega^d)}, \quad p \geq 2 \left(\frac{d+2}{d} \right).$$

2. PROOF OF THEOREM 1.1

Recall that the flow of (1) is locally well defined for initial data in $f \in H^s(\mathbb{T}^2)$ for $s > 0$ [2]. The solutions are constructed via a fixed point argument in the restriction space $X_\delta^{s,b}$ for $\delta > 0$ sufficiently small (depending polynomially on the $H^s(\mathbb{T}^2)$ norm of f). We recall that

$$\|F\|_{X_\delta^{s,b}} := \inf_{G=F} \inf_{\text{on } t \in [0, \delta]} \|G\|_{X^{s,b}},$$

where

$$\|F\|_{X^{s,b}}^2 := \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}^d} \langle \tau + |n|^2 \rangle^{2b} \langle n \rangle^{2s} |\widehat{F}(n, \tau)|^2 d\tau \quad \text{if,}$$

and \widehat{F} is the space-time Fourier transform of F .

Let Φ_t^N be the flow associated to the truncated NLS equation

$$(11) \quad i\partial_t \Phi_t^N f + \Delta \Phi_t^N f = P_{\leq N} \mathcal{N}(\Phi_t^N f),$$

with initial datum $\Phi_0^N f := P_{\leq N} f$. We denote $P_{\leq N}$ the frequency projection on the ball of radius N centered in the origin. We write $\Phi_t f := \Phi_t^\infty f$ for the flow of the NLS equation with initial datum $f = P_\infty f$. We also denote $P_{>N} := P_\infty - P_{\leq N}$ and as already mentioned $P_N := P_{\leq N} - P_{\leq N/2}$.

A similar well-posedness result holds for the truncated flow, uniformly in $N \in \mathbb{N}$. Of course, at fixed N , since the equation (11) is finite dimensional, one can construct a global flow in an elementary way using the Cauchy theorem for ODE and the conservation of $\|\Phi_t^N f\|_{L^2(\mathbb{T}^2)}$ (which holds for all $N \in \mathbb{N}$). However, in the following we will need (as usual in the study of NLS) a control of $\Phi_t^N f$ uniform over N . This is not elementary and will be ensured by the local well-posedness theory in the restriction space.

As already recalled, the main tool in the study of the pointwise convergence properties of the linear Schrödinger equation is the maximal Schrödinger operator

$$t \rightarrow \sup_{0 \leq t \leq \delta} |e^{it\Delta} f(x)|, \quad \delta > 0.$$

Assume indeed that for some $\delta \in (0, 1]$ one has

$$(12) \quad \left\| \sup_{0 \leq t \leq \delta} |e^{it\Delta} f(x)| \right\|_{L_x^2(\mathbb{T}^2)} \lesssim \|f\|_{H_x^{\frac{1}{2}}(\mathbb{T}^2)},$$

then is not hard to see that $e^{it\Delta} f(x) \rightarrow f(x)$ as $t \rightarrow 0$ for almost every (with respect to the Lebesgue measure) $x \in \mathbb{T}^2$. The proof is a straightforward modification of the argument that we will use to prove Proposition 2.1 below.

In the nonlinear setting we need a (nonlinear) replacement of (12). A convenient replacement is the maximal estimate (13).

Proposition 2.1. *Let $f \in L^2(\mathbb{T}^2)$ be such that*

$$(13) \quad \lim_{N \rightarrow \infty} \left\| \sup_{0 \leq t \leq \delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L_x^2(\mathbb{T}^2)} = 0.$$

Then $\Phi_t f(x) \rightarrow f(x)$ as $t \rightarrow 0$ for almost every $x \in \mathbb{T}^2$.

From the proof it will be clear that in (13) we can replace the L^2 norm with a weak L^1 norm. However, it is usually convenient to work in the L^2 setting.

Proof. To prove Proposition 2.1 we decompose the difference as follows:

$$(14) \quad |\Phi_t f(x) - f(x)| \leq |\Phi_t f(x) - \Phi_t^N f(x)| + |\Phi_t^N f(x) - P_{\leq N} f(x)| + |P_{>N} f(x)|$$

and pass to the limit $t \rightarrow 0$. It is elementary to show that the second term on the right hand side is zero, namely

$$\lim_{t \rightarrow 0} \Phi_t^N f(x) = P_{\leq N} f(x),$$

for all $x \in \mathbb{T}^2$. So we arrive at²

$$\limsup_{t \rightarrow 0} |\Phi_t f - f| \leq \limsup_{t \rightarrow 0} |\Phi_t f - \Phi_t^N f| + |P_{>N} f|.$$

Let $\lambda > 0$. Using the Chebyshev inequality

$$\begin{aligned} |\{x \in \mathbb{T}^2 : \limsup_{t \rightarrow 0} |\Phi_t f - f| > \lambda\}| &\leq |\{x \in \mathbb{T}^2 : \sup_{0 \leq t \leq \delta} |\Phi_t f - \Phi_t^N f| > \lambda/2\}| \\ &\quad + |\{x \in \mathbb{T}^2 : |P_{>N} f| > \lambda/2\}| \\ &\lesssim \lambda^{-2} \left(\left\| \sup_{0 \leq t \leq \delta} |\Phi_t f - \Phi_t^N f| \right\|_{L^2(\mathbb{T}^2)}^2 + \|P_{>N} f\|_{L^2(\mathbb{T}^2)}^2 \right), \end{aligned}$$

where $|\cdot|$ is the Lebesgue measure. On the other hand we have $\|P_{>N} f\|_{L^2(\mathbb{T}^2)} \rightarrow 0$ as $N \rightarrow \infty$ (since $f \in L^2(\mathbb{T}^2)$) and

$$\lim_{N \rightarrow \infty} \left\| \sup_{0 \leq t \leq \delta} |\Phi_t f - \Phi_t^N f| \right\|_{L^2(\mathbb{T}^2)} = 0$$

²Hereafter we remove the x variable in the argument of decompositions like (14) to simplify the notation.

by assumption (13). Thus we arrive to

$$|\{x \in \mathbb{T}^2 : \limsup_{t \rightarrow 0} |\Phi_t f - f| > \lambda\}| = 0$$

and the statement follows taking the union over $\lambda > 0$. \square

It is not easy to verify the condition (13) directly. However, we can take advantage a simple lemma which allows to embed a suitable restriction space into the relevant maximal space, namely the space induced by the norm

$$\left\| \sup_{t \in [0, \delta]} |F(x, t)| \right\|_{L_x^2(\mathbb{T}^2)}, \quad F : (x, t) \in \mathbb{T}^2 \times \mathbb{R} \rightarrow F(x, t) \in \mathbb{C}.$$

In other words, we can bound the $L_x^2(\mathbb{T}^2)$ norm of the associated maximal function

$$x \rightarrow \sup_{0 \leq t \leq \delta} |F(x, t)|$$

with an appropriate $X_\delta^{s, b}$ norm of F . In fact, this is a rather general property of the restriction spaces $X_\delta^{s, b}$ with $b > \frac{1}{2}$. The proof can be found in [30, Lemma 2.9], in the non periodic case. The argument adapts to the periodic case as well.

Lemma 2.2. *Let $b > \frac{1}{2}$ and let Y be a Banach space of functions*

$$F : (x, t) \in \Omega^d \times \mathbb{R} \rightarrow F(x, t) \in \mathbb{C}.$$

Let $\alpha \in \mathbb{R}$. Assume

$$(15) \quad \|e^{i\alpha t} e^{it\Delta} f(x)\|_Y \leq C \|f\|_{H^s(\Omega^d)},$$

with a constant $C > 0$ uniform over $\alpha \in \mathbb{R}$. Then

$$\|F\|_Y \leq C \|F\|_{X^{s, b}}.$$

Using Lemma 2.2 with

$$\|F\|_Y = \left\| \sup_{0 \leq t \leq \delta} |F(x, t)| \right\|_{L_x^2(\mathbb{T}^2)}$$

and the fact that the maximal estimate (12) hold for $s > 1/2$ we have the following

Lemma 2.3. *Let $b > \frac{1}{2}$ and $s > 1/2$. We have*

$$(16) \quad \left\| \sup_{0 \leq t \leq \delta} |F(x, t)| \right\|_{L_x^2(\mathbb{T}^2)} \lesssim \|F\|_{X_\delta^{s, b}}.$$

We will combine the following lemma with the embedding from Lemma 2.3 to verify the maximal estimate hypothesis of Proposition 2.1 for the cubic NLS on \mathbb{T}^2 .

Lemma 2.4. *Let $d = 2$ and $s > 0$. Then*

$$(17) \quad \|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{s, -\frac{1}{2}++}} \lesssim \left(\|u\|_{X^{s, \frac{1}{2}+}}^2 + \|v\|_{X^{s, \frac{1}{2}+}}^2 \right) \|u - v\|_{X^{s, \frac{1}{2}+}}$$

In fact Lemma 2.4 is a consequence of the following slightly more general statement (that will be useful later) due to Bourgain [3].

Lemma 2.5. *Let $d = 2$ and $s > 0$. Let $M_1 \geq M_2 \geq M_3$ be dyadic scales. Then*

$$(18) \quad \begin{aligned} & \|(\mathbb{P}_{M_1} F)(\mathbb{P}_{M_2} G)(\mathbb{P}_{M_3} H)\|_{X^{s, -\frac{1}{2}++}} \\ & \lesssim \|\mathbb{P}_{M_1} F\|_{X^{s, \frac{1}{2}+}} \|\mathbb{P}_{M_2} G\|_{X^{0+, \frac{1}{2}+}} \|\mathbb{P}_{M_3} H\|_{X^{0, \frac{1}{2}+}}. \end{aligned}$$

We denote $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Hereafter η will be a smooth cut-off of $[0, 1]$. Taking $\delta = \delta(R_0) < 1$ sufficiently small and combining (25), (26), (27) and Lemma 2.4 one can show that the map

$$(19) \quad \Gamma(u(x, t)) = \eta(t)e^{it\Delta} \mathbb{P}_{\leq N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} \mathbb{P}_{\leq N} \mathcal{N}(u(x, t')) dt'$$

is a contraction on the ball $\{u : \|u\|_{X_\delta^{s, \frac{1}{2}+}} \leq 2R_0\}$, for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. This is a standard argument, so we omit the proof (see for instance [18, Section 3.5.1]). Moreover, a similar computation is part of the proof of Theorem 1.1. However, we stress that the value of δ is uniform in $N \in 2^{\mathbb{N}} \cup \{\infty\}$. In particular we have

$$(20) \quad \|\Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}} \leq 2R_0, \quad \text{for all } N \in 2^{\mathbb{N}} \cup \{\infty\}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3 we have

$$\left\| \sup_{0 \leq t \leq \delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L_x^2(\mathbb{T}^2)} \lesssim \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}}.$$

Thus using Proposition 2.1 it suffices to show that the right hand side goes to zero as $N \rightarrow \infty$. For $t \in [0, \delta]$ we have (see (19))

$$\begin{aligned} & \Phi_t f(x) - \Phi_t^N f(x) \\ &= \eta(t)e^{it\Delta} \mathbb{P}_{>N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} (\mathcal{N}(\Phi_{t'} f(x)) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_{t'}^N f(x))) dt'. \end{aligned}$$

Then using (25) and (26) we have

$$(21) \quad \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}} \lesssim \|\mathbb{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathcal{N}(\Phi_t f) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_t^N f)\|_{X_\delta^{s, -\frac{1}{2}+}}.$$

To handle the nonlinear contribution we further decompose

$$\mathcal{N}(\Phi_t f) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_t^N f) = \mathbb{P}_{\leq N} (\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)) + \mathbb{P}_{>N} \mathcal{N}(\Phi_t f)$$

so that

$$(22) \quad \begin{aligned} \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}} &\lesssim \|\mathbb{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathbb{P}_{>N} \mathcal{N}(\Phi_t f)\|_{X_\delta^{s, -\frac{1}{2}+}} \\ &\quad + \|\mathbb{P}_{\leq N} (\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f))\|_{X_\delta^{s, -\frac{1}{2}+}}. \end{aligned}$$

Then by (27), Lemma 2.4, and (20), we get

$$(23) \quad \|\mathbb{P}_{\leq N} (\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f))\|_{X_\delta^{s, -\frac{1}{2}+}} \lesssim \delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}},$$

where we recall $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Plugging (23) into (22), taking $\delta = \delta(R_0)$ small enough and absorbing

$$\delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}} \leq \frac{1}{2} \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}}$$

into the left hand side, we arrive to

$$(24) \quad \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s, \frac{1}{2}+}} \lesssim \|\mathbb{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathbb{P}_{>N} \mathcal{N}(\Phi_t f)\|_{X_\delta^{s, -\frac{1}{2}+}}$$

The right hand side of (24) goes to zero as $N \rightarrow \infty$ since $f \in H^s(\mathbb{T}^2)$ and $\mathcal{N}(\Phi_t f) \in X_\delta^{s, -\frac{1}{2}+}$; in fact applying Lemma 2.4 with $v = 0$ and recalling (20) we have

$$\|\mathcal{N}(\Phi_t f)\|_{X_\delta^{s, -\frac{1}{2}+}} \lesssim \|\Phi_t f\|_{X_\delta^{s, \frac{1}{2}+}}^3 \lesssim R_0^3.$$

This concludes the proof of (3).

We conclude this section recalling some well known properties of restriction spaces that we have used (and that we will use in the rest of the paper). Recall that η is a smooth cut-off of the unit interval.

Lemma 2.6. *Let $s \in \mathbb{R}$. Then*

$$(25) \quad \|\eta(t)e^{it\Delta}f(x)\|_{X^{s, \frac{1}{2}+}} \lesssim \|f\|_{H^s(\Omega^d)},$$

$$(26) \quad \left\| \eta(t) \int_0^t e^{i(t-t')\Delta} F(\cdot, t') dt' \right\|_{X^{s, \frac{1}{2}+}} \lesssim \|F\|_{X^{s, -\frac{1}{2}+}},$$

$$(27) \quad \|F\|_{X_\delta^{s, -\frac{1}{2}+}} \lesssim \delta^{0+} \|F\|_{X_\delta^{s, -\frac{1}{2}++}}.$$

3. PROOF OF PROPOSITION 1.3

Here we prove almost surely uniform convergence of the randomized Schrödinger flow to the initial datum, at the H^{0+} level, namely Proposition 1.3. Thus our goal is to show that $e^{it\Delta}f^\omega \rightarrow f^\omega$ as $t \rightarrow 0$ uniformly over $x \in \mathbb{T}^d$ and \mathbb{P} -almost surely for data f^ω defined as

$$(28) \quad f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2} + \alpha}} e^{in \cdot x}, \quad x \in \mathbb{T}^d,$$

where $\alpha > 0$ and each g_n^ω is complex and independently drawn from a standard normal distribution. In fact, the argument we present works for independent g_n^ω drawn from any distribution with sufficiently decay of the tails (for instance sub-Gaussian is enough). This will not be the case in Theorem 1.4, where we will need to take advantage of the hypercontractivity of (multilinear forms of) normal distributions. However, we only present the standard normal case for definiteness, also in this section.

Fix $t \in \mathbb{R}$. We have that \mathbb{P} -almost surely

$$e^{it\Delta}f^\omega \in \bigcap_{s < \alpha} H^s(\mathbb{T}^d).$$

This is an immediate consequence of (44) below, taking the union over $\varepsilon > 0$. In fact, for all $t \in \mathbb{R}$ we have \mathbb{P} -almost surely

$$e^{it\Delta}f^\omega \in \bigcap_{s < \alpha} C^s(\mathbb{T}^d),$$

thus in particular $e^{it\Delta}f^\omega$ are \mathbb{P} -almost surely continuous functions of the x variable. This is a consequence of the higher integrability property (34) below, from which one can easily deduce uniform convergence as $N \rightarrow \infty$ of the sequence $P_{\leq N}f^\omega$, with probability larger than $1 - \varepsilon$. So the limit f^ω is continuous with the same probability, and the almost sure continuity follows taking the union over $\varepsilon > 0$.

Before completing the proof of Proposition 1.3, we recall few lemmata. We start recalling the following well-known concentration bound:

Lemma 3.1 ([7, Lemma 3.1]). *There exists a constant C such that*

$$(29) \quad \left\| \sum_{n \in \mathbb{Z}^d} g_n^\omega a_n \right\|_{L_\omega^r} \leq Cr^{\frac{1}{2}} \|a_n\|_{\ell_n^2(\mathbb{Z}^d)}$$

for all $r \geq 2$ and $\{a_n\} \in \ell^2(\mathbb{Z}^d)$.

Using (29) with $a_n = e^{in \cdot x - i|n|^2 t} \langle n \rangle^{-\frac{d}{2} - \alpha}$ we obtain for $r \geq 2$ that for f^ω an in (28)

$$(30) \quad \|P_N e^{it\Delta} f^\omega\|_{L_\omega^r} \leq Cr^{\frac{1}{2}} N^{-\alpha},$$

with a constant uniform in $t \in \mathbb{R}$. From this, we also have improved L_x^p estimates for randomized data.

Lemma 3.2. *Let $p \in [2, \infty)$. Assume f^ω is as in (28). There exists constants C and c , independent of $t \in \mathbb{R}$, such that*

$$(31) \quad \mathbb{P}(\|P_N e^{it\Delta} f^\omega\|_{L_x^p(\mathbb{T}^d)} > \lambda) \leq Ce^{-cN^{2\alpha}\lambda^2}.$$

Thus

$$(32) \quad \mathbb{P}(\|P_N e^{it\Delta} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} > \lambda) \leq Ce^{-cN^{2\alpha} - \lambda^2}.$$

In particular, for any $\varepsilon > 0$ sufficiently small, we have

$$(33) \quad \|P_N e^{it\Delta} f^\omega\|_{L_x^p(\mathbb{T}^d)} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}$$

and

$$(34) \quad \|P_N e^{it\Delta} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}},$$

with probability at least $1 - \varepsilon$.

Proof. We prove (31), then (32) follows by Bernstein inequality. By Minkowski's inequality and Lemma 3.1 above, we have for any $r \geq p \geq 2$

$$\left(\int \|P_N e^{it\Delta} f^\omega\|_{L_x^p(\mathbb{T}^d)}^r d\mathbb{P}(\omega) \right)^{\frac{1}{r}} \leq \left\| \|P_N e^{it\Delta} f^\omega\|_{L_\omega^r} \right\|_{L_x^p(\mathbb{T}^d)} \leq CN^{-\alpha} r^{\frac{1}{2}},$$

which is enough to conclude that $\|P_N e^{it\Delta} f^\omega\|_{L_x^p(\mathbb{T}^d)}$ is a sub-Gaussian random variable satisfying the tail bound (31). \square

Note that using (31)-(32), the triangle inequality

$$\|P_{>N}(\cdot)\| \leq \sum_{M \in 2^{\mathbb{N}}: M > N} \|P_M(\cdot)\|,$$

the union bound and the fact that

$$\sum_{M \in 2^{\mathbb{N}}: M > N} e^{-cM^{2\alpha}k^{-2}} \lesssim e^{-cN^{2\alpha}k^2},$$

we see that, for all $t \in \mathbb{R}$ and $\alpha > 0$ we have ($p < \infty$)

$$(35) \quad \mathbb{P}(\|e^{it\Delta} P_{>N} f^\omega\|_{L_x^p(\mathbb{T}^d)} > \lambda) \lesssim e^{-cN^{2\alpha}\lambda^2}$$

$$(36) \quad \mathbb{P} \left(\|e^{it\Delta} \mathbf{P}_{>N} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} > \lambda \right) \lesssim e^{-cN^{2\alpha-\lambda^2}}$$

Remark 3.3. *Proceeding as we did to prove (35)-(36) we also easily see that the exceptional set where (33)-(34) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right hand side of the estimates.*

Proceeding as in the proof of Lemma 3.2 we also obtain improved Strichartz estimates for randomized data.

Lemma 3.4. *Let $p \in [2, \infty)$. Assume f^ω is as in (28). Then we have*

$$(37) \quad \mathbb{P} \left(\|e^{it\Delta} \mathbf{P}_N f^\omega\|_{L_{x,t}^p(\mathbb{T}^{d+1})} > \lambda \right) \leq C e^{-cN^{2\alpha}\lambda^2}.$$

Thus

$$(38) \quad \mathbb{P} \left(\|e^{it\Delta} \mathbf{P}_N f^\omega\|_{L_{x,t}^\infty(\mathbb{T}^{d+1})} > \lambda \right) \leq C e^{-cN^{2\alpha-\lambda^2}}.$$

In particular, for any $\varepsilon > 0$ sufficiently small, we have

$$(39) \quad \|e^{it\Delta} \mathbf{P}_N f^\omega\|_{L_{x,t}^p(\mathbb{T}^{d+1})} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}$$

and

$$(40) \quad \|e^{it\Delta} \mathbf{P}_N f^\omega\|_{L_{x,t}^\infty(\mathbb{T}^{d+1})} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}},$$

with probability at least $1 - \varepsilon$.

The bounds (37)-(38) implies

$$(41) \quad \mathbb{P} \left(\|e^{it\Delta} \mathbf{P}_{>N} f^\omega\|_{L_{x,t}^p(\mathbb{T}^{d+1})} > \lambda \right) \lesssim e^{-cN^{2\alpha}\lambda^2}$$

$$(42) \quad \mathbb{P} \left(\|e^{it\Delta} \mathbf{P}_{>N} f^\omega\|_{L_{x,t}^\infty(\mathbb{T}^{d+1})} > \lambda \right) \lesssim e^{-cN^{2\alpha-\lambda^2}}$$

exactly as (31)-(32) imply (35)-(36). Also we have an analogous of Remark (3.3):

Remark 3.5. *The exceptional set where (39)-(40) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right hand side of the estimates.*

Fix $t \in \mathbb{R}$. Later we will also need the following bound for the H^s norm of $e^{it\Delta} f^\omega$ with $s < \alpha$. This is a well know fact that we recall applying again (29) with $a_n = e^{in \cdot x - |n|^2 t} \langle n \rangle^{-\frac{d}{2} - \alpha + s}$, so that we get for $r \geq 2$

$$\| \mathbf{P}_N \langle D \rangle^s e^{it\Delta} f^\omega \|_{L_w^r} \leq C r^{\frac{1}{2}} N^{s-\alpha}, \quad s < \alpha.$$

Here $\langle D \rangle$ denotes the Fourier multiplier operator $\langle n \rangle$. Proceeding as in the proof of Lemma 3.2 we also obtain

$$(43) \quad \mathbb{P} \left(\| \langle D \rangle^s \mathbf{P}_N e^{it\Delta} f^\omega \|_{L_x^2(\mathbb{T}^d)} > \lambda \right) \leq C e^{-cN^{2(\alpha-s)}\lambda^2}, \quad s < \alpha$$

and in particular, for any $\varepsilon > 0$ sufficiently small

$$(44) \quad \|e^{it\Delta} f^\omega\|_{H_x^s(\mathbb{T}^d)} \lesssim (-\ln \varepsilon)^{1/2} \quad s < \alpha, \quad t \in \mathbb{R},$$

with probability at least $1 - \varepsilon$. Again the constant is uniform on $t \in \mathbb{R}$.

We are now ready to complete the proof of Proposition 1.3.

Proof of Proposition 1.3. Invoking the Borel-Cantelli lemma it is enough to show that

$$(45) \quad \mathbb{P} \left(\limsup_{t \rightarrow 0} \|e^{it\Delta} f^\omega - f^\omega\|_{L_x^\infty(\mathbb{T}^d)} > 1/k \right) \lesssim \gamma_k,$$

for a summable sequence $\{\gamma_k\}_{k \in \mathbb{N}}$. Let us decompose

$$(46) \quad |e^{it\Delta} f^\omega - f^\omega| \leq |e^{it\Delta} P_{>N} f^\omega| + |e^{it\Delta} P_{\leq N} f^\omega - P_{\leq N} f^\omega| + |P_{>N} f^\omega|.$$

Using (36) (with $t = 0$) and (42) we see that

$$(47) \quad \|e^{it\Delta} P_{>N} f^\omega\|_{L_{x,t}^\infty(\mathbb{T}^{d+1})} + \|P_{>N} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} \leq \frac{1}{2k}$$

holds for all ω outside an exceptional set of measure $\lesssim e^{-cN^{2\alpha}k^{-2}}$. We choose $N = N_k$ via the identity $N_k^{2\alpha} = k^3$, in such a way that $e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$ is summable (over $k \in \mathbb{N}$). Let $s^* > d/2$. Since

$$e^{it\Delta} P_{\leq N_k} f^\omega - P_{\leq N_k} f^\omega = \sum_{|n| \leq N_k} (e^{-it|n|^2} - 1) e^{in \cdot x} \widehat{f^\omega}(n),$$

using Cauchy-Schwartz, the summability of $\langle n \rangle^{-2s^*}$ (over $n \in \mathbb{Z}^d$) and (44) with $s = 0$, $t = 0$ (in the last inequality) we get

$$(48) \quad \begin{aligned} \|e^{it\Delta} P_{\leq N_k} f^\omega - P_{\leq N_k} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} &\lesssim \sup_{|n| \leq N_k} |e^{-it|n|^2} - 1| \left(\sum_{|n| \leq N_k} \langle n \rangle^{2s^*} |\widehat{f^\omega}(n)|^2 \right)^{1/2} \\ &\lesssim |t|(N_k)^{s^*+2} \|f^\omega\|_{L^2} \leq |t|(N_k)^{s^*+2} \frac{1}{k}, \end{aligned}$$

for ω outside an exceptional set of probability $\lesssim e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$. From the previous inequality, looking at t so small that $|t|(N_k)^{s^*+2} \leq 1/2$, we have

$$(49) \quad \mathbb{P} \left(\limsup_{t \rightarrow 0} \|e^{it\Delta} P_{\leq N^*} f^\omega - P_{\leq N^*} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} > 1/k \right) \lesssim e^{-ck}.$$

Combining (36)-(36) and recalling the decomposition (46) the proof is concluded. □

4. PROOF OF THEOREM 1.4

In this section, we consider the cubic Wick-ordered NLS (8) on \mathbb{T}^d ($d = 1, 2$) as in the work of Bourgain in [3]. Namely, we look at the nonlinearity

$$\mathcal{N}(u) := \pm u (|u|^2 - 2\mu), \quad \mu := \int_{\mathbb{T}^d} |u(x, t)|^2 dx.$$

We are interested again in randomized initial data, i.e. f^ω is taken to be of the form (28). Recall (see (44)) that such data is \mathbb{P} -almost surely in H^s for all $s < \alpha$ and

$$(50) \quad \|f^\omega\|_{H^s} \lesssim (-\ln \varepsilon)^{1/2}, \quad s < \alpha,$$

with probability at least $1 - \varepsilon$, for all $\varepsilon \in (0, 1)$ sufficiently small. Since we work with any $\alpha > 0$, we are considering initial data in H^{0+} . We approximate equation

(8) as in (11), for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Recall that $\Phi_t^N f^\omega$ denotes the associated flow, with initial datum

$$\Phi_0^N f^\omega := P_{\leq N} f^\omega = \sum_{|n| \leq N} \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2} + \alpha}} e^{in \cdot x}.$$

We write $\Phi_t f^\omega = \Phi_t^\infty f^\omega$ for the flow of (8) with datum $f^\omega = P_\infty f^\omega$.

The relevant choice of σ in the following statement is $\sigma = \frac{1}{2}-$ (we will use this to prove Theorem 1.4).

Proposition 4.1. *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume*

$$(51) \quad u = u(I) + u(II), \quad u(I) = e^{it\Delta} P_{\leq N} f^\omega, \quad \|u(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}} < 1$$

and the same for v . Then

$$(52) \quad \|\mathcal{N}(u)\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon)^{3/2}$$

$$(53) \quad \|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon) \|u - v\|_{X^{\alpha+\sigma, \frac{1}{2}+}}$$

for initial data of the form (28), with probability at least $1 - \varepsilon$, for all $\varepsilon \in (0, 1)$ sufficiently small. If we take u as in (51) and we instead assume

$$v = v(I) + u(II), \quad v(I) = e^{it\Delta} f^\omega, \quad \|u(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}} < 1,$$

we have

$$(54) \quad \|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim N^{-\alpha}.$$

Remark 4.2. *Recall that α indicates the regularity of the initial datum. We are denoting by σ the amount of smoothing one can prove for the Wick-ordered cubic nonlinearity \mathcal{N} . More precisely, since the initial data (28) belongs to $H^{\alpha-}$, one can interpret this statement as saying that, with arbitrarily large probability, \mathcal{N} is $\sigma+$ smoother than f^ω . Since $\sigma < \frac{1}{2}$ is permissible, we reach $\frac{1}{2}-$ smoothing for \mathcal{N} and, combining with (26), also for the Duhamel contribution $\Phi_t^N f^\omega - e^{it\Delta} P_{\leq N} f^\omega$.*

In fact a stronger statement than 4.1 has been proved in [13]. Namely that the reminder can be further decomposed in as a sum of two terms. The first one, to which one we refer as paracontrolled, lies in $X^{\frac{1}{2}-, \frac{1}{2}+}$ but has a precise random structure. The second one is a smoother deterministic reminder that lies in $X^{1-, \frac{1}{2}+}$.

Here we only explain how to get Proposition 4.1 for the first Picard iteration, namely when 4. Recall that η is a smooth cut-off of the unit interval. Let us fix $\alpha > 0$. Using (26), (27) and Proposition 4.1 one can show that for all $\delta > 0$ sufficiently small the following holds. For all $N \in 2^{\mathbb{N}} \cup \{\infty\}$, the map

$$(55) \quad \Gamma^N(u) := \eta(t) e^{it\Delta} P_{\leq N} f^\omega - i\eta(t) \int_0^t e^{i(t-s)\Delta} P_{\leq N} \mathcal{N}(u(\cdot, s)) ds$$

is a contraction on the set

$$(56) \quad \left\{ e^{it\Delta} P_{\leq N} f^\omega + g, \quad \|g\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} < 1 \right\}$$

equipped with the $X_\delta^{\alpha+\sigma, \frac{1}{2}+}$ norm, outside an exceptional set (we call it a δ -exceptional set) of initial data of probability smaller than $e^{-\delta^{-\gamma}}$, with $\gamma > 0$ a

given small constant. Notice that this holds uniformly over $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Again, this is a standard routine calculation that we omit (see for instance [18, Section 3.5.1]). We only explain how to find the relation between the local existence time δ and the size of the exceptional set. Given any $\varepsilon \in (0, 1)$ sufficiently small, using (26), (27) and Proposition 4.1, we have

$$\|\Gamma^N(u) - \eta(t)e^{it\Delta} \mathbf{P}_{\leq N} f^\omega\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} \lesssim \delta^{0+} (-\ln \varepsilon)^{3/2},$$

for all f^ω outside an exceptional set of probability smaller than ε . Letting δ such that $\varepsilon = e^{-\delta^{-\gamma}}$ with $\gamma > 0$ a fixed small constant, we have $C\delta^{0+} (-\ln \varepsilon)^{3/2} < 1$ for all $\delta > 0$ sufficiently small. Note that the measure $e^{-\delta^{-\gamma}}$ of the δ -exceptional set converges to zero as $\delta \rightarrow 0$. In particular, for ω outside the δ -exceptional set, the fixed point $\Phi_t^N f^\omega$ of the map (55) belongs to the set (56), namely

$$(57) \quad \|\Phi_t^N f^\omega - e^{it\Delta} \mathbf{P}_{\leq N} f^\omega\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} < 1, \quad N \in 2^{\mathbb{N}} \cup \{\infty\}.$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. It suffices to show that

$$(58) \quad \lim_{N \rightarrow \infty} \left\| \sup_{0 \leq t \leq \delta} |\Phi_t f^\omega(x) - \Phi_t^N f^\omega(x)| \right\|_{L_x^2(\mathbb{T}^2)} = 0$$

for all f^ω outside a δ -exceptional set A_δ . Note indeed that (58) imply that given f^ω we can find, \mathbb{P} -almost surely, a δ_ω (which depends on ω) such that (58) is satisfied. Indeed, if we could not do so, this would mean that $f^\omega \in \bigcap_{\delta > 0} A_\delta$, and the probability of this event is zero, since $\mathbb{P}(A_\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Once we have (58) with $\delta = \delta_\omega$, we have \mathbb{P} -almost surely

$$\lim_{t \rightarrow 0} \Phi_t^\omega f^\omega(x) - f^\omega(x) = 0, \quad \text{for a.e. } x \in \mathbb{T}^2,$$

as claimed, simply invoking Proposition 2.1.

In order to prove (58) we decompose

$$|\Phi_t f^\omega - \Phi_t^N f^\omega| \leq |e^{it\Delta} \mathbf{P}_{> N} f^\omega| + |\Phi_t f^\omega - e^{it\Delta} f^\omega - (\Phi_t^N f^\omega - e^{it\Delta} \mathbf{P}_{\leq N} f^\omega)|,$$

Thus, recalling the decay of the high frequency linear term given by (36), it remains to show that

$$(59) \quad \lim_{N \rightarrow \infty} \left\| \sup_{0 \leq t \leq \delta} |\Phi_t f^\omega - e^{it\Delta} f^\omega - (\Phi_t^N f^\omega - e^{it\Delta} \mathbf{P}_{\leq N} f^\omega)| \right\|_{L^2(\mathbb{T}^2)} = 0,$$

for all f^ω outside a δ -exceptional set.

For any $\alpha > 0$, we can choose σ sufficiently close to $\frac{1}{2}$ that

$$(60) \quad \frac{1}{2} < \alpha + \sigma.$$

Thus, using the $X^{s,b}$ space embedding from Lemma 2.3, it suffices to prove

$$(61) \quad \lim_{N \rightarrow \infty} \|w - w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} = 0,$$

where

$$w^N := \Phi_t^N f - e^{it\Delta} \mathbf{P}_{\leq N} f^\omega, \quad w := w^\infty.$$

Notice that by (57) we have

$$\|w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} < 1, \quad N \in 2^{\mathbb{N}} \cup \{\infty\}.$$

Since for $t \in [0, \delta]$ we have

$$(62) \quad w - w^N = -i\eta(t) \int_0^t e^{i(t-t')\Delta} (\mathcal{N}(\Phi_{t'} f^\omega) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_{t'}^N f^\omega)) dt',$$

using (26), (27), we get

$$(63) \quad \|w - w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} \lesssim \delta^{0+} \|\mathcal{N}(\Phi_t f) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_t^N f)\|_{X_\delta^{\alpha+\sigma, -\frac{1}{2}++}}.$$

We decompose

$$(64) \quad \begin{aligned} \mathcal{N}(\Phi_t f) - \mathbb{P}_{\leq N} \mathcal{N}(\Phi_t^N f) = \\ \mathbb{P}_{\leq N} (\mathcal{N}(e^{it\Delta} \mathbb{P}_{\leq N} f^\omega + w) - \mathcal{N}(e^{it\Delta} \mathbb{P}_{\leq N} f^\omega + w^N)) + \text{Remainders}, \end{aligned}$$

where

$$\text{Remainders} := \mathbb{P}_{\leq N} (\mathcal{N}(e^{it\Delta} f^\omega + w) - \mathcal{N}(e^{it\Delta} \mathbb{P}_{\leq N} f^\omega + w)) + \mathbb{P}_{> N} \mathcal{N}(\Phi_t f).$$

Notice that by (52), (54) we have

$$(65) \quad \|\text{Remainders}\|_{X_\delta^{\alpha+\sigma, -\frac{1}{2}++}} \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

with probability at least $1 - \varepsilon$. Using (53) we can estimate

$$(66) \quad \begin{aligned} \|\mathbb{P}_{\leq N} (\mathcal{N}(e^{it\Delta} \mathbb{P}_{\leq N} f^\omega + w) - \mathcal{N}(e^{it\Delta} \mathbb{P}_{\leq N} f^\omega + w^N))\|_{X_\delta^{\alpha+\sigma, -\frac{1}{2}++}} \\ \lesssim (-\ln \varepsilon) \|w - w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} \end{aligned}$$

and (63), (64), (66) give

$$(67) \quad \|w - w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} \lesssim \delta^{0+} (-\ln \varepsilon) \|w - w^N\|_{X_\delta^{\alpha+\sigma, \frac{1}{2}+}} + \|\text{Remainders}\|_{X_\delta^{\alpha+\sigma, -\frac{1}{2}++}}$$

with probability at least $1 - \varepsilon$. Since with our choice of $\varepsilon = e^{-\delta^{-\gamma}}$ we have $C\delta^{0+} (-\ln \varepsilon)^{3/2} < 1$, we can absorb the first term on the right hand side into the left hand side and we still have that (65) holds outside a δ -exceptional set. Thus letting $N \rightarrow \infty$ the proof of (9) is complete. \square

Remark 4.3. *It is worthy to remark that, comparing with for instance [3], the procedure which allows to promote a statement valid on a δ -exceptional set A_δ for arbitrarily small $\delta > 0$ to a statement which is valid with probability = 1 is far easier. In particular it does not involve any control on the evolution of the (Gaussian) measure induced by the random Fourier series. This is because we are considering a property which has to be verified only at time $t = 0$ a.s., instead that in a time interval containing $t = 0$, as in [3].*

We now give some hints on the proof of the smoothing estimates given in Proposition 4.1.

Proof of Proposition 4.1. Notice that the Wick-ordered nonlinearity can be written as

$$(68) \quad \mathcal{N}(u(x, \cdot)) = \sum_{n_2 \neq n_1, n_3} \widehat{u}(n_1) \widehat{\bar{u}}(n_2) \widehat{u}(n_3) e^{i(n_1 - n_2 + n_3) \cdot x} - \sum_n \widehat{u}(n) |\widehat{u}(n)|^2 e^{in \cdot x}$$

where we are looking at the nonlinear term for fixed time and $\widehat{u}(\cdot)$ denotes the space Fourier coefficients. Looking at a similar expansion for the difference $\mathcal{N}(u) - \mathcal{N}(v)$ it is easy to see that we can deduce (4.1) from a slightly more general Lemma 4.4 given below. It implies the desired statement

$$u_j(n_j) = u(n_j), \quad v(n_j), \quad \text{or} \quad u(n_j) - v(n_j).$$

□

We will give a proof of the following Lemma in the fully random case $J_j = I$ for $j = 1, 2, 3$, which correspond to the study of the first Picard iterate. Comparing with 4.4 (and [13]) there is a simplification coming from the fact that our f^ω is slightly more regular, namely we consider $\alpha > 0$ instead of $\alpha = 0$.

Lemma 4.4. *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$ the following holds. Assume for $j = 1, 2, 3$*

$$(69) \quad u_j(I) = e^{it\Delta} \mathbf{P}_{\leq N} f^\omega, \quad \|u_j(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}} < 1.$$

Let $J_j \in \{I, II\}$, $j = 1, 2, 3$. Then, for all $\varepsilon \in (0, 1)$ sufficiently small we have the following

$$(70) \quad \|\mathcal{N}(u_1(J_1), \bar{u}_2(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon)^{3/2},$$

and more precisely

$$(71) \quad \|\mathcal{N}(u_1(II), \bar{u}_2(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon) \|u_1(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}},$$

$$(72) \quad \|\mathcal{N}(u_1(J_1), \bar{u}_2(II), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon) \|u_2(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}},$$

with probability at least $1 - \varepsilon$. Moreover, if in (69) we replace for some $j = j^$ the projection operator $\mathbf{P}_{\leq N}$ by $\mathbf{P}_{> N}$, then the estimate (70) with $J_{j^*} = I$ holds with an extra factor $N^{-\alpha}$ on the right hand side.*

Remark 4.5. *Saying that this estimates holds with probability at least $1 - \varepsilon$ means, more precisely, that they hold for all ω outside an exceptional set of probability $\leq \varepsilon$. Moreover, this set can be chosen to be independent on $N \in 2^{\mathbb{N}} \cup \{\infty\}$.*

Remark 4.6. *Notice that by the symmetry $n_1 \leftrightarrow n_3$ the estimate (71) implies an analogous estimate for $u_3(II)$.*

Here we only consider the case $J_j = I$ for $j = 1, 2, 3$, namely the case in which all the contributions are a linear random evolution. We prove the bound (70) relative to this case and to $N = \infty$. Moreover we split the nonlinearity as a difference of two terms (see (68))

$$\begin{aligned} \mathcal{N}_1(u_1(J_1), \bar{u}_2(J_2), u_3(J_3)) &= \sum_{n_2 \neq n_1, n_3} \widehat{u_1(J_1)}(n_1) \widehat{\bar{u}_2(J_2)}(n_2) \widehat{u_3(J_3)}(n_3) e^{i(n_1 - n_2 + n_3) \cdot x}, \\ \mathcal{N}_2(u_1(J_1), \bar{u}_2(J_2), u_3(J_3)) &= \sum_n \widehat{u_1(J_1)}(n) \widehat{\bar{u}_2(J_2)}(n) \widehat{u_3(J_2)}(n) e^{in \cdot x}. \end{aligned}$$

and we prove (70) only for \mathcal{N}_1 , which is the most challenging contribution. The proof for \mathcal{N}_2 is indeed elementary.

To prove (70) will be useful to recall that the space-time Fourier transform of $e^{it\Delta}f^\omega$ is

$$\widehat{e^{it\Delta}f^\omega}(n, \tau) = \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2} + \alpha}} \delta(\tau + |n|^2),$$

where δ is the delta function. So a direct computation gives

$$\|e^{it\Delta}f^\omega\|_{X^{0+, \frac{1}{2}+}}^2 = \sum_n \frac{|g_n^\omega|^2}{\langle n \rangle^{d+2\alpha-}},$$

which, recalling $\int |g_n^\omega|^2 d\omega = 1$, immediately implies

$$\| \|e^{it\Delta}f^\omega\|_{X^{0+, \frac{1}{2}+}} \|_{L_\omega^2}^2 = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < \infty.$$

Since we can expand the LHS as a bilinear form in the Gaussian variables g_n^ω , we get by Gaussian hypercontractivity

$$\| \|e^{it\Delta}f^\omega\|_{X^{0+, \frac{1}{2}+}} \|_{L_\omega^q}^2 = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < C_q < \infty.$$

Proceeding essentially as in the Proof of Lemmas 3.2-3.4 (recall also the Remarks 3.3-RemarkUniform1Bis) this allows to prove a pointwise bound

$$(73) \quad \|e^{it\Delta}f^\omega\|_{X^{0+, \frac{1}{2}+}} \lesssim \sqrt{\ln\left(\frac{1}{\varepsilon}\right)},$$

with probability larger than $1 - C\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Let N, N_1, N_2, N_3 be dyadic scales. We denote with \tilde{N} the maximum between N_1, N_2, N_3 . First we perform a reduction to remove frequencies which are far from the paraboloid. More precisely, we denote with P_A the space-time Fourier projection into the set A and our goal is to reduce

$$(74) \quad \begin{aligned} & \sum_{N_1, N_2, N_3} \|\mathcal{N}_1(P_{N_1} u_1(I), P_{N_2} \bar{u}_2(I), P_{N_3} u_3(I))\|_{X^{\alpha+\sigma, -\frac{1}{2}++}}^2 \\ &= \sum_{N, N_1, N_2, N_3} N^{2\alpha+2\sigma} \|P_N \mathcal{N}_1(P_{N_1} u_1(I), P_{N_2} \bar{u}_2(I), P_{N_3} u_3(I))\|_{X^{0, -\frac{1}{2}++}}^2 \end{aligned}$$

to

$$(75) \quad \sum_{N, N_1, N_2, N_3} N^{2\alpha+2\sigma} \|P_N P_{\{\langle \tau + |n|^2 \rangle \leq \tilde{N}^{1+\frac{1}{10}}\}} \mathcal{N}_1(P_{N_1} u_1(I), P_{N_2} \bar{u}_2(I), P_{N_3} u_3(I))\|_{X^{0, -\frac{1}{2}++}}^2$$

To obtain this reduction, it is sufficient to show that projection of the nonlinearity onto the complementary set is appropriately bounded; i.e. that

$$(76) \quad \begin{aligned} & \sum_{N, N_1, N_2, N_3} N^{2\alpha+2\sigma} \|P_N P_{\{\langle \tau + |n|^2 \rangle > \tilde{N}^{\frac{11}{10}}\}} \mathcal{N}_1(P_{N_1} u_1(I), P_{N_2} \bar{u}_2(I), P_{N_3} u_3(I))\|_{X^{0, -\frac{1}{2}++}}^2 \\ & \lesssim (-\ln \varepsilon)^3 \end{aligned}$$

with probability at least $1 - \varepsilon$. To do so we abbreviate

$$\mathcal{N}_1^{N_1, N_2, N_3}(\cdot) := \mathcal{N}_1(P_{N_1} u_1(I), P_{N_2} \bar{u}_2(I), P_{N_3} u_3(I)).$$

and we bound

$$\begin{aligned}
(77) \quad & \sum_{N_1, N_2, N_3} N^{2\alpha+2\sigma} \|\mathbb{P}_N \mathbb{P}_{\{\langle \tau+|n|^2 \rangle > \tilde{N}^{\frac{11}{10}}\}} \mathcal{N}_1^{N_1, N_2, N_3}\|_{X^{0, -\frac{1}{2}++}}^2 \\
& \sim N^{2\alpha+2\sigma} \sum_{\substack{N_1, N_2, N_3 \\ n \sim N}} \int \frac{\chi_{\{\langle \tau+|n|^2 \rangle > \tilde{N}^{\frac{11}{10}}\}}}{\langle \tau+|n|^2 \rangle^{1--}} \left| \overline{\mathcal{N}_1^{N_1, N_2, N_3}(\cdot)}(n, \tau) \right|^2 d\tau \\
& \lesssim N^{2\alpha+2\sigma-1-\frac{1}{10}+3(0+)} \sum_{\substack{N_1, N_2, N_3 \\ n \sim N}} \int \left| \overline{\mathcal{N}_1^{N_1, N_2, N_3}(\cdot)}(n, \tau) \right|^2 d\tau \\
& \sim N^{2\alpha-\frac{1}{20}} \sum_{N_1, N_2, N_3} \|\mathbb{P}_N \mathcal{N}_1^{N_1, N_2, N_3}\|_{L_{x,t}^2}^2,
\end{aligned}$$

recalling that $\sigma < 1/2$ (here in fact we may have more smoothing than $\frac{1}{2}-$). We have used the fact that at least one of the frequency scales N_j has to be comparable to N , otherwise the contribution is zero by orthogonality, and so particular we have $N \lesssim \tilde{N}$ (recall that $\tilde{N} = \max(N_1, N_2, N_3)$). In order to continue the estimate we assume for definiteness that $N_1 \sim N$. The other possible case is $N_2 \sim N$ (since everything is symmetric under $n_1 \leftrightarrow n_3$) and one can indeed immediately check that the estimate (78) below is still valid in this case, with obvious changes. Thus we have using Hölder's inequality, the improved Strichartz inequality (40) for randomized functions (for the L^∞ norm of $u_1(I)$), and the Strichartz inequality (10) (for the L^4 norms of $u_2(I)$ and $u_3(I)$), we obtain

$$\begin{aligned}
(78) \quad & \|\mathbb{P}_N \mathcal{N}_1^{N_1, N_2, N_3}\|_{L_{x,t}^2}^2 \\
& \leq \|\mathbb{P}_{N_1} u_1(I)\|_{L_{x,t}^\infty}^2 \|\mathbb{P}_{N_2} \overline{u_2}(I)\|_{L_{x,t}^4}^2 \|\mathbb{P}_{N_3} u_3(I)\|_{L_{x,t}^4}^2 \cdot \\
& \lesssim (-\ln \varepsilon) N_1^{-2\alpha} \|\mathbb{P}_{N_2} \overline{u_2}(I)\|_{L_{x,t}^4}^2 \|\mathbb{P}_{N_3} u_3(I)\|_{L_{x,t}^4}^2, \\
& \lesssim (-\ln \varepsilon) N^{-2\alpha} \|\mathbb{P}_{N_2} \overline{u_2}(I)\|_{X^{0+, \frac{1}{2}+}}^2 \|\mathbb{P}_{N_3} u_3(I)\|_{X^{0+, \frac{1}{2}+}}^2
\end{aligned}$$

this holds on a set of probability larger than $1 - \varepsilon$ and this set may be chosen to be independent on $N_1 \in \mathbb{N} \cup \{\infty\}$ (see Remark 3.5) and thus on $N \in \mathbb{N} \cup \{\infty\}$. Plugging (78) into (77), summing over the N_j and using (73) we arrive to the needed bound

$$\begin{aligned}
\text{LHS of (76)} & \lesssim (-\ln \varepsilon) \sum_{N, N_1} N^{-\frac{1}{20}} \|u_2(I)\|_{X^{0+, \frac{1}{2}+}}^2 \|u_3(I)\|_{X^{0+, \frac{1}{2}+}}^2 \\
& \lesssim (-\ln \varepsilon)^3 \sum_{N, N_1} N^{-\frac{1}{40}} N_1^{-\frac{1}{40}} \lesssim (-\ln \varepsilon)^3
\end{aligned}$$

Note that in (78) we could also use a weaker bound replacing the L^4 norm with the L^∞ , that in the fully random case $J_j = I$ for all j is controlled invoking (40) for all $j = 1, 2, 3$. However the L^4 bound is more robust since it works also in the other cases, where the contributions are not all random (namely if some J_j is of the form II).

So we have reduced to (75). We have

$$(79) \quad \mathbb{P}_N \mathbb{P}_{\{\langle \tau+|n|^2 \rangle \leq \tilde{N}^{\frac{11}{10}}\}} \mathcal{N}_1^{N_1, N_2, N_3}(\cdot)$$

$$\begin{aligned}
&= \mathbb{P}_N \mathbb{P} \left\{ \langle \tau + |n|^2 \rangle \leq \tilde{N}^{\frac{11}{10}} \right\} \left(\sum_{|n_j| \sim N_j} e^{ix \cdot (n_1 - n_2 + n_3)} e^{-it(|n_1|^2 - |n_2|^2 + |n_3|^2)} \right) \\
&\quad \times \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}}
\end{aligned}$$

Thus we see that (75) satisfies the desired inequalities (70) as long as we can bound (80)

$$\begin{aligned}
N^{2\alpha+2\sigma} \left\| \sum_{N_1, N_2, N_3} \mathbb{P}_N \mathbb{P} \left\{ \langle \tau + |n|^2 \rangle \leq \tilde{N}^{\frac{11}{10}} \right\} \left(\sum_{|n_j| \sim N_j} e^{ix \cdot (n_1 - n_2 + n_3)} e^{-it(|n_1|^2 - |n_2|^2 + |n_3|^2)} \right) \right. \\
\left. \times \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right\|_{X^{0, -\frac{1}{2}++}}^2 \lesssim (-\ln \varepsilon)^3 N^{0-},
\end{aligned}$$

on a set of probability larger than $1 - \varepsilon$.

Since

$$\begin{aligned}
(81) \quad \mathcal{F} \left(e^{ix \cdot (n_1 - n_2 + n_3)} e^{-it(|n_1|^2 - |n_2|^2 + |n_3|^2)} \right) (n, \tau) \\
= \sum_{n_1 - n_2 + n_3 = n} \delta(\tau + |n_1|^2 - |n_2|^2 + |n_3|^2),
\end{aligned}$$

where \mathcal{F} is the space-time Fourier transform and δ is the delta function, we reduce (80) to showing that

$$\begin{aligned}
(82) \quad N^{2\alpha+2\sigma} \sum_{N_1, N_2, N_3} \sum_{|n| \sim N} \int \frac{\chi_{\{\langle \tau + |n|^2 \rangle \leq \tilde{N}^{\frac{11}{10}}\}}}{\langle \tau + |n|^2 \rangle^{1--}} \\
\times \left| \sum_{\substack{|n_j| \sim N_j, n_2 \neq n_1, n_3 \\ n = n_1 - n_2 + n_3 \\ \tau + |n_1|^2 - |n_2|^2 + |n_3|^2 = 0}} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right| d\tau \lesssim (-\ln \varepsilon)^3 N^{0-},
\end{aligned}$$

with probability at least $1 - \varepsilon$. Letting

$$\mu := |n|^2 + \tau = |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2,$$

(the second identity holds over the integration set, since we have a factor

$$\delta(\tau + |n_1|^2 - |n_2|^2 + |n_3|^2)$$

in the integrand) and recalling that $N \lesssim \tilde{N}$, this follows by

$$\begin{aligned}
(83) \quad N^{2\alpha+2\sigma} \sum_{N_1, N_2, N_3} \sum_{|n| \sim N} \int \frac{\chi_{\{\langle \mu \rangle \leq \tilde{N}^{\frac{11}{10}}\}}}{\langle \mu \rangle^{1--}} \\
\times \left| \sum_{\substack{|n_j| \sim N_j, n_2 \neq n_1, n_3 \\ n = n_1 - n_2 + n_3 \\ -|n|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2 = \mu}} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right| d\tau \lesssim (-\ln \varepsilon)^3 N^{0-},
\end{aligned}$$

with probability at least $1 - \varepsilon$. Using Hölder inequality in $d\mu$ we reduce to prove (here we use the symmetry $\mu \leftrightarrow -\mu$)

$$(84) \quad N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1, N_2, N_3} \sup_{|\mu| \lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n| \sim N} \left| \sum_{\substack{|n_j| \sim N_j, n_2 \neq n_1, n_3 \\ n = n_1 - n_2 + n_3 \\ \mu = |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2}} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right|^2 \lesssim (-\ln \varepsilon)^3 N^{0-},$$

with probability at least $1 - \varepsilon$. We rewrite (84) as

$$(85) \quad N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1, N_2, N_3} \sup_{|\mu| \lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n| \sim N} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right|^2 \lesssim (-\ln \varepsilon)^3 N^{0-},$$

where for fixed n, μ we have denoted

$$(86) \quad R_n(n_1, n_2, n_3) := \left\{ (n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 : |n_j| \sim N_j, j = 1, 2, 3, \right. \\ \left. n_2 \neq n_1, n_3, n_1 - n_2 + n_3 = n, \mu = |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 \right\}.$$

The set $R_n(\cdot)$ depends on μ also (like all the sets we will define below). However we omit this dependence to simplify the notation. Notice that in the definition of $R_n(\cdot)$ the condition

$$|n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = \mu$$

can be equivalently replaced by

$$2(n_1 - n_2) \cdot (n_3 - n_2) = \mu.$$

We also note that we have reduced to a case in which at least one of the frequencies N_1, N_3 is comparable to \tilde{N} . Indeed, if both $N_1 \ll \tilde{N}$ and $N_3 \ll \tilde{N}$ we must have $N_2 = \tilde{N}$ and $\mu \sim N^2$, which contradicts the fact that $\mu \lesssim N^{\frac{11}{10}}$. Since the roles of N_1 and N_3 are symmetric (they are always the size of the indices of the Fourier coefficients of u_1, u_3), hereafter we assume that

$$N_1 \sim \tilde{N} \gtrsim N.$$

To estimate (85) will be also useful to introduce the set

$$(87) \quad S(n_1, n_2, n_3) := \left\{ (n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 : |n_j| \sim N_j, j = 1, 2, 3, \right. \\ \left. n_2 \neq n_1, n_3, \mu = 2(n_1 - n_2) \cdot (n_3 - n_2) \right\}.$$

We recall that the Gaussian variables contracts in the following way

$$(88) \quad \int g_n^\omega \overline{g_{n'}^\omega} d\mathbb{P}(\omega) = 0, \quad \int g_n^\omega \overline{g_{n'}^\omega} d\mathbb{P}(\omega) = \begin{cases} 0 & \text{if } n \neq n' \\ 1 & \text{if } n = n' \end{cases}.$$

along with the fact that the sum is restricted over $n_1, n_3 \neq n_2$ and symmetric under $n_1 \leftrightarrow n_3$, we get

$$(89) \quad \int \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right|^2 d\mathbb{P}(\omega) \\ = 2 \sum_{R_n(n_1, n_2, n_3)} \frac{1}{\langle n_1 \rangle^{2\alpha+2}} \frac{1}{\langle n_2 \rangle^{2\alpha+2}} \frac{1}{\langle n_3 \rangle^{2\alpha+2}} = 2 \sum_{R_n(n_1, n_2, n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2}.$$

In other words, the $L^2(d\omega)$ norm of the gaussian trilinear form is controlled by square root of the right hand side of (89). Using the hypercontractivity of the Gaussians (see [33, 19]), we can promote this to an $L^q(d\omega)$ bound, with a multiplicative factor which is factor $q^{3/2}$. Then using Bernstein inequality (as we did in Section 3), this also gives to us a (uniform) pointwise bound

$$(90) \quad \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right|^2 \\ \lesssim (-\ln \varepsilon)^3 N_1^{0+} \sum_{R_n(n_1, n_2, n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2},$$

with an extra N_1^{0+} loss, that is valid for ω outside an exceptional set of probability $\leq \varepsilon$ (again, proceeding as in Section 3, we see that this exceptional set can be chosen to be independent on N , as required).

We finally distinguish two last possibilities. First restrict the summation over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 \neq n_3$ (with a small abuse of notation we do not introduce additional notation for this restriction). In this case we get, with probability $> 1 - \varepsilon$, the following estimate

$$(91) \quad \sum_{|n| \sim N} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \right|^2 \\ \lesssim (-\ln \varepsilon)^3 \sum_{|n| \sim N} \sum_{R_n(n_1, n_2, n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ \lesssim (-\ln \varepsilon)^3 \sum_{S(n_1, n_2, n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ \sim (-\ln \varepsilon)^3 \sum_{S(n_1, n_2, n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ \lesssim (-\ln \varepsilon)^3 N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \#S(n_1, n_2, n_3) \\ \lesssim (-\ln \varepsilon)^3 N_1^{-2\alpha-1} N_2^{-2\alpha} N_3^{-2\alpha},$$

where we used that if $n_1 \neq n_3$, then

$$\#S(n_1, n_2, n_3) \lesssim N_1 N_2^2 N_3^2;$$

this is because once we have fixed n_2, n_3 in $N_2^2 N_3^2$ possible ways, we remain with at most N_1 choices for n_1 by the relation $\mu = 2(n_1 - n_2) \cdot (n_3 - n_2)$. This fact has a clear geometric interpretation, namely that this relation force the (two dimensional)

lattice point n_1 to belong to the portion of a line which lie inside a ball of radius $\lesssim N_1$ (and there are $\lesssim N_1$ such lattice points n_1).

The second possibility is that we sum over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 = n_3$. In this case restriction $\mu = 2|n_1 - n_2|^2$ implies that once we have chosen n_2 in N_2^2 possible ways, we remain with $\lesssim \mu^{0+} \lesssim N_1^{0++}$ choices for $n_1 = n_3$ (since a circle of radius μ contains $\lesssim \mu^{0+}$ lattice points). This gives an even better bound than the one above.

Thus, summing the (91) over N_2, N_3 and recalling that $N_1 \sim \tilde{N} \gtrsim N$, we have bounded, with probability $> 1 - \varepsilon$, the expression (85) by

$$N^{2\alpha+2\sigma} N_1^{0+} \sum_{N_1} N_1^{-2\alpha-1} \lesssim (-\ln \varepsilon)^3 N_1^{2\sigma-1+0+} \lesssim (-\ln \varepsilon)^3 N^{0-},$$

where we used $\sigma < \frac{1}{2}$.

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