

Effective surface energies in nematic liquid crystals as homogenised rugosity effects

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Abstract

We study the effect of boundary rugosity in nematic liquid crystalline systems. We consider a highly general formulation of the problem, able to simultaneously deal with several liquid crystal theories. We use techniques of Gamma convergence and demonstrate that the effect of fine-scale surface oscillations may be replaced by an effective homogenised surface energy on a simpler domain. The homogenisation limit is then quantitatively studied in a simplified setting, obtaining convergence rates.

1 Introduction

Nematic liquid crystals are anisotropic states of matter formed from elongated molecules that share attributes with both crystals and isotropic liquids. Whilst they appear as “classical” fluids to the naked eye, the constituent rod-like molecules of a nematic admit long-range orientational ordering which leads to a macroscopic anisotropy of the system and optical birefringence, a property more typical in solid crystals. More so, as these anisotropic structures often admit characteristic length scales comparable to the wavelength of visible light, this allows them to play key roles in a variety of optics applications, the most prominent of which is the liquid crystal display (LCD) [34, 60].

While anisotropy is found in many systems and may be exploited in technological applications, the prevalence of liquid crystals arises not only from the interactions it is capable of having with electromagnetic fields and light, but from the fact that they possess a broken continuous symmetry. The consequence of this is that it is possible to reorient or reorganise the structure of a nematic via the imposition of modest external fields. That is to say, the material is *soft*. As well as their long history of technological use, in more recent history there has been a sustained interest in liquid crystalline systems in biological applications, where many naturally occurring biological structures may be identified with liquid crystalline phases (such as the structure of a bi-lipid membrane, or the collective dynamics of active systems from the scale of swimming bacteria to flocks of birds [23, 31, 58]), and complex materials with a liquid crystalline nature have presented themselves as candidate for soft prosthesis [32, 57].

In both biological and technological contexts, the behaviour of liquid crystalline materials is highly dependent on the environment in which they are found. Owing to the dielectric

and diamagnetic anisotropy of mesogenic (rod-like) molecules, the imposition of electromagnetic fields has become the standard method to control nematics in technological applications. Another important component of the system which determines the ground states, however, is the interactions that the material has with its bounding container, that is, surface effects. It should be noted that liquid crystals may have interfaces with air, other liquids, solids, or even have situations such as wetting where nematic-air and nematic-solid interfaces each play a role, and each can provide a variety of behaviours depending on the particular materials involved [8, 41, 45, 46, 52]. In this work we will consider the case which is most apparent in technological applications, that of the nematic-solid interface.

The classical approach to modelling nematic-solid interfaces in a nematic system is to presume that the surface is somewhat smooth, and molecules generally have a preference to sit at a given angle to the surface, with the most common cases being that they prefer to be parallel (planar degenerate anchoring) or perpendicular (homeotropic anchoring) to the surface. A surface anchoring energy represents this preference in mathematical modelling, although at physically meaningful length scales it is often reasonable to replace the surface energy with a Dirichlet boundary condition, which may be viewed as a case in which the effective surface energy becomes infinitely large away from its minimiser. Given that the geometry of the surface must therefore cause fluctuations in the nematic, at least near the surface itself, it is natural to wish to quantify these effects, and more so ask ourselves if we can exploit them. Additive manufacturing processes and photolithography open the possibility of producing well-designed surfaces which may improve control of nematic systems [19, 43, 61]. More so, the ZBD (Zenithal Bistable Device) is a successful commercial bistable display that exploits surface-induced distortions of the nematic, showing a glimpse of the future possibilities [15, 59].

As often happens in soft-matter physics, the presence of multiple length scales in nematics leads to a variety of models that one may use to describe them, depending on the types of features and mechanisms that one wishes to capture and describe. Previous research into the behaviour of nematic-solid interfaces is no different, with molecular dynamics simulations, including simulations up to atom-level resolution [49, 50, 63], mean-field approaches [44, 56, 62], and the continuum mechanics language of PDEs and the calculus of variations [48, 51]. In particular there has been a wealth of research into understanding the effects of one-dimensional periodic surfaces, with a particular focus on understanding how these provoke the presence of defects and create “effective” surface energies that depend on the structure itself [30, 35].

Within this work we will consider the continuum mechanics approach, and aim to understand via rigorous asymptotic analysis the way in which undulated surfaces, in which the wavelength is of comparable size to the amplitude, can lead to effective surface energies in the limit as the amplitude converges to zero. Problems of this flavour have been considered in the language of homogenisation of PDEs in a domain with an oscillating boundary, where certain scalar, linear, *rugose* systems may be rigorously proven to have certain effective behaviours in the limit. These have been considered, for example, in the context of [2, 3, 4, 11, 18] or [25], but the list is not by any means exhaustive. A contemporary overview of the literature from this direction can be found, for example, in the introduction of [3]. The nature of physically meaningful surface energies in the context nematic liquid crystals however provides models that have yet to be considered in the literature within this homogenisation framework.

Our approach will be two-pronged. Firstly, we will consider a large, general class of models, which include the “classical” non-linear models for nematic liquid crystals in arbitrary dimension, as well as the mathematically similar Ginzburg-Landau model of superconductivity [12], and demonstrate through techniques of Γ -convergence that, under appropriate conditions and within a large class of energy functionals, in the limit as the rugosity converges to zero, we can obtain an effective surface energy, whilst the interior energy of the nematic is unchanged.

Our second direction considers a simplified setting of a two-dimensional slab with periodic rugosity and a quadratic free energy, which provides a toy model of a *paranematic*. That is, a high-temperature system of mesogenic molecules which has melted into an isotropic state, but

still admits some local nematic ordering induced by the surface. In this case, the simplicity of the system compared to the general case allows us to make much stronger conclusions, and we are able to provide quantitative estimates on how ground states behave in the homogenised limit.

1.1 Models of liquid crystals

Owing to the presence of multiple length scales in a liquid crystal, a large number of models exist, coming in many distinct flavours, which aim to describe particular mechanisms and predict or explain particular behaviours. Within this work we will only consider the perspective of continuum mechanics, in which the language used is that of partial differential equations and the calculus of variations. However, even within this class of models there are still various descriptions that capture different aspects of the material.

1.1.1 Landau-de Gennes

The Landau-de Gennes Q-tensor theory is a popular model for describing nematic liquid crystals, owing to its ability to describe biaxiality and defects [22, 42]. If we have a domain $\Omega \subset \mathbb{R}^3$ which contains nematic liquid crystal, we presume that the (local) distribution of orientations near a point $x \in \Omega$ can be described by a probability distribution $f_x : \mathbb{S}^2 \rightarrow [0, \infty)$, where we view $p \in \mathbb{S}^2$ to represent the orientation of the long axis of a molecule. Molecules are presumed to be (statistically) head-to-tail symmetric, which gives that $f_x(p) = f_x(-p)$, and consequently the first order moment of f_x always vanishes. To this end, in the process of coarse-graining, we consider the second normalised moment Q , which is a traceless symmetric 3×3 matrix known as the *Q-tensor*, defined by

$$Q(x) = \int_{\mathbb{S}^2} f_x(p) \left(p \otimes p - \frac{1}{3} I \right) dp \in \text{Sym}_0(3) = \left\{ A \in \mathbb{R}^{3 \times 3} : \text{Tr}(A) = 0, A = A^T \right\}. \quad (1)$$

We highlight several characteristics of the Q-tensor:

- If $f_x(p) = \frac{1}{4\pi}$, that is, the system is isotropic (disordered), then $Q(x) = 0$.
- If f_x is axisymmetric, that is, $f_x(p) = g_x(p \cdot e)$ for a distinguished direction $e \in \mathbb{S}^2$ and some $g_x : [-1, 1] \rightarrow \mathbb{R}$, then $Q(x) = s \left(e \otimes e - \frac{1}{3} I \right)$ for some scalar s . We say that such a Q-tensor is *uniaxial*, with *director* e and *scalar order parameter* s .
- Any $Q(x)$ which can be defined via (1) for $f_x \in L^1(\mathbb{S}^2)$ must satisfy the eigenvalue constraint $\lambda_{\min}(Q) > -\frac{1}{3}$, often referred to as *physicality* [6].

For simplicity, we neglect the “microscopic” definition of Q as in (1) and consider $Q : \Omega \rightarrow \text{Sym}_0(3)$ to be the order parameter of interest, and interpret stable equilibria of the nematic as minimisers of the free energy

$$\int_{\Omega} F_{el}(\nabla Q(x), Q(x)) + F_b(Q(x)) dx + \int_{\partial\Omega} w(\nu(x), Q(x)) dS(x). \quad (2)$$

Here $F_{el} : (\text{Sym}_0(3) \times \mathbb{R}^3) \times \text{Sym}_0(3) \rightarrow [0, \infty)$ is a frame indifferent elastic energy which penalises spatial distortions. The elastic energy is typically quadratic in ∇Q , which gives $W^{1,2}$ as the natural function space for the problem. A commonly considered form for F_{el} is

$$F_{el}(D, Q) = \frac{L_1}{2} D_{ijk} D_{ijk} + \frac{L_2}{2} D_{ijk} D_{ikj} + \frac{L_3}{2} D_{ijj} D_{ikk}, \quad (3)$$

where Einstein notation is used, and L_1, L_2, L_3 are constants. More general forms are possible, however, such as in [39].

F_b is a frame indifferent bulk energy, which is minimised either at isotropic Q-tensors ($Q = 0$), or uniaxial Q-tensors with a certain fixed, positive scalar order parameter ($Q = s_0 \left(n \otimes n - \frac{1}{3} I \right)$),

for $s_0 > 0$ fixed and any $n \in \mathbb{S}^2$). The typical expressions for Q are either quartic or sixth-order polynomials in Q , the most used being the quartic one

$$F_b(Q) = \frac{a}{2}\text{Tr}(Q^2) - \frac{b}{3}\text{Tr}(Q^3) + \frac{c}{4}\text{Tr}(Q^4). \quad (4)$$

for material constants $b > 0, c > 0$, while a depends on both the material and temperature, and may either be positive or negative. Furthermore Katriel *et. al.* [33] and Ball and Majumdar [6] have provided examples of a bulk energy which is singular as $\lambda_{\min}(Q) \rightarrow -\frac{1}{3}$, which is employed to ensure the physicality of solutions. In the case of *paranematics*, where the temperature is above the nematic-isotropic transition temperature and $Q = 0$ is the ground state, it is possible to obtain some degree of nematic order, that is, non-zero Q , by the imposition of external fields or interactions with surfaces or colloids. In this case, one may replace the bulk energy with a quadratic, penalising the deviation from this ground state, $F_b(Q) = \frac{c}{2}|Q|^2$ for $c > 0$, as has been done in [26, 27].

The surface energy w depends not only on Q , but the outer unit normal $\nu : \partial\Omega \rightarrow \mathbb{S}^2$, which is necessary to capture the behaviour that molecules tend to prefer a specific angle with respect to the surface, and will depend on the particular materials considered. There are a wide variety of surface energies considered in the literature, but via frame indifference, it is known that w must be of the form

$$w(\nu, Q) = \tilde{w}(\text{Tr}(Q^2), \text{Tr}(Q^3), Q\nu \cdot \nu, Q^2\nu \cdot \nu) \quad (5)$$

for some function \tilde{w} , which depends only on the scalar invariants of Q, ν [17, 53]. The simplest of all such models comes from the presumption of homeotropic alignment, and impose the surface energy

$$w(\nu, Q) = \frac{w_0}{2} \left| Q - s_0 \left(\nu \otimes \nu - \frac{1}{3}I \right) \right|. \quad (6)$$

It should be noted that, for mathematical simplicity, analogues of the Q-tensor model can be defined for a two-dimensional universe, in which case we define $Q \in \text{Sym}_0(2)$ to be a 2×2 symmetric traceless matrix, defined as the average of $(p \otimes p - \frac{1}{2}I)$ for $p \in \mathbb{S}^1$.

1.1.2 Oseen-Frank

We recall that in a nematic described by the Landau-de Gennes model, the ‘‘bulk’’ energy F_b is minimised at Q of the form $Q = s_0 (n \otimes n - \frac{1}{3}I)$, where $s_0 > 0$ is fixed and n can be any unit vector. It is known that in domains much larger than the characteristic correlation length of the molecules the penalisation of the bulk energy dominates, and it is appropriate to impose the constraint that Q minimises the bulk energy pointwise [9, 16, 28, 38, 40, 55]. In certain cases, at least, this becomes equivalent to the Oseen-Frank energy, in which we use only the director $n : \Omega \rightarrow \mathbb{S}^2$ to describe the nematic [7]. The typical free energy we consider is then

$$\int_{\Omega} F_{el}(n(x), \nabla n(x)) dx + \int_{\partial\Omega} w(\nu(x), n(x)) dS(x). \quad (7)$$

The elastic energy is typically given of the form

$$F_{el}(n, \nabla n) = \frac{K_1}{2}(\nabla \cdot n)^2 + \frac{K_2}{2}(n \cdot \nabla \times n - \tau)^2 + \frac{K_3}{2}|n \times \nabla \times n|^2, \quad (8)$$

where K_1, K_2, K_3, τ are material constants. Note that if $\tau = 0$, the energy is minimised at the undistorted state, $\nabla n = 0$. However if $\tau \neq 0$, then the energy is minimised at a *cholesteric* state, where the ground state admits a helical structure [10].

For the surface energy, frame invariance implies that w may only depend on $(\nu \cdot n)^2$. The simplest possible such surface energy is the Rapini-Papoular surface energy [47]

$$w(\nu, n) = \frac{w_0}{2} |\nu \cdot n|^2, \quad (9)$$

where $w_0 > 0$ implies that the director wishes to be in the tangent plane to the surface (planar degenerate anchoring), and $w_0 < 0$ implies that director prefers to be orthogonal to the surface (homeotropic anchoring).

Remark 1.1. An equivalent formulation of the Oseen-Frank model (7) is to consider director fields $n \in W^{1,2}(\Omega, \mathbb{R}^3)$ with the free energy

$$\int_{\Omega} F_{el}(n(x), \nabla n(x)) + F_b(n(x)) dx + \int_{\partial\Omega} w(\nu(x), n(x)) dS(x), \quad (10)$$

where F_{el}, w are as in (7) and the bulk energy is defined by $F_b(n) = 0$ if $|n| = 1$, and $F_b(n) = +\infty$ otherwise. This particular formulation allows us to consider n as living in a vector space, rather than a Sobolev space of manifold-valued maps, albeit with a highly singular energy functional.

1.2 Outline of the paper and main results

The paper is broadly organised into three sections. In Section 2, we consider a general problem that encompasses the main theories of nematic liquid crystals, and we consider the asymptotic limit as the surface oscillations converge to zero with fine structure. In Section 3, we interpret these general results in the context of the Landau-de Gennes and Oseen Frank models of nematic liquid crystals. Finally, in Section 4, we consider a simplified model in which we are able to prove more precise estimates on the asymptotic behaviour of the model, including error estimates in L^2 . We now outline the key aspects of our analysis.

1.2.1 Summary of Section 2

Within this section we consider integral functionals of the form

$$\mathcal{F}_{\varepsilon}(u) = \int_{\Omega_{\varepsilon}} F(\nabla u(x), u(x), x) dx + \int_{\Gamma_{\varepsilon}} w(P_{\Gamma}x, \nu(x), n(x)) dS(x). \quad (11)$$

for $u \in W^{1,p}(\Omega_{\varepsilon}, V)$ for a finite dimensional vector space V , and P_{Γ} the projection onto a particular manifold Γ . The precise assumptions necessary for our analysis are presented in Section 2.1. One of the key assumptions is that Ω_{ε} is a graph of a function defined on a limiting domain Ω , encoded by a function $\varphi_{\varepsilon} : \Gamma \rightarrow \mathbb{R}$, satisfying $\|\varphi_{\varepsilon}\|_{\infty} = O(\varepsilon)$, $\|D\varphi_{\varepsilon}\|_{\infty} = O(1)$, which generates a Young measure in a particular sense, as outlined in Assumption 2. The assumptions on F, w account for many typical models encountered in the modelling of liquid crystals. Furthermore, as the domains Ω_{ε} are ε dependent, we need to define a notion of convergence of functions $u_{\varepsilon} \in W^{1,p}(\Omega_{\varepsilon}, V)$ to functions in $u \in W^{1,p}(\Omega, V)$, which we call *rugose convergence*, denoted $u_{\varepsilon} \xrightarrow{R} u$, and is defined in Definition 2.5. Once we have established our technical assumptions, we proceed with identifying the Γ -limit of $\mathcal{F}_{\varepsilon}$ with respect to rugose convergence.

First we prove our equicoercivity result, showing that if $\mathcal{F}_{\varepsilon}(u_{\varepsilon})$ is uniformly bounded, then we may extract a subsequence u_{ε_j} and $u \in W^{1,p}(\Omega, V)$ so that $u_{\varepsilon_j} \xrightarrow{R} u$ (Proposition 2.1). The argument is standard, and reflects that given in, for instance, [11],[18] and [25], whereby the only technicality is to ensure that the rugose limit is weakly differentiable with the correct derivative. Generally, we see that under rugose convergence, the interior term, that is,

$$\int_{\Omega_{\varepsilon}} F(\nabla u(x), u(x), x) dx,$$

is “well-behaved”, and does not require much technical consideration. For the surface term however, first we note that, up to an small error, the surface term may be replaced by a simpler function that, rather than depending on the full geometry, only sees the rugosity via $\hat{D}\varphi_{\varepsilon}$, according to Lemma 2.1. In particular, the argument shows that curvature of Ω has a vanishing

contribution in the limit. This reduction in complexity of the model also allows us to relate the surface energy in the limit with the Young measures described in Assumption 2. Once this has been done, we show that rugose convergence $u_\varepsilon \xrightarrow{R} u$ also implies the weak- $W^{1,p}$ convergence of u_ε composed with a change of variables $\Phi_\varepsilon : \Omega \rightarrow \Omega_\varepsilon$ Lemma 2.3, which gives us a well-behaved notion of a type of “trace” onto Γ for functions in $W^{1,p}(\Omega_\varepsilon, V)$, and is used to show that for $u_\varepsilon \xrightarrow{R} u$,

$$\int_{\Gamma_\varepsilon} w(x, \nu_\varepsilon(x), u_\varepsilon(x)) dS(x) \rightarrow \int_{\Gamma} w_h(x, u(x)) dS(x) \quad (12)$$

in Proposition 2.6, where w_h admits an explicit representation in (47). Combining this “continuity” of the surface term with classical lower semicontinuity results on the interior gives the liminf inequality necessary for the Γ -convergence (Proposition 2.7). To show the limsup inequality, we construct recovery sequences by extending our candidate u via reflections to a larger domain, and considering its restriction to Ω_ε (Proposition 2.9).

Finally, we note that Γ convergence, strictly speaking, requires that the domain of definition of \mathcal{F}_ε is independent of $\varepsilon > 0$, which is not the case in the problem we consider. However, this will not be problematic, as our conclusion is to employ the fundamental theorem of Γ -convergence, which holds nonetheless via the exact same argument as the classical case (Proposition 2.10).

With these ingredients, we are able to prove the main theorem of this section.

Theorem 1.1. Let Ω_ε , for $\varepsilon_0 > \varepsilon > 0$, Ω denote sets satisfying Assumption 1 and Assumption 2, the interior energy F satisfy Assumption 4, and the surface energy w satisfy Assumption 5. Then there exists a local energy functional $w_h : \Gamma \times V \rightarrow \mathbb{R}$ so that for every $u \in L^q$, $\lim_{\varepsilon \rightarrow 0} w(\cdot, \tilde{\nu}_\varepsilon, u) \gamma_\varepsilon \xrightarrow{L^1(\Gamma)} w_h(\cdot, u)$, and $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ with respect to $W^{1,p}$ -rugose convergence, where $\mathcal{F} : W^{1,p}(\Omega, V) \rightarrow [0, \infty)$ is given by

$$\mathcal{F}(u) = \int_{\Omega} F(\nabla u(x), u(x), x) dx + \int_{\Gamma} w_h(x, u(x)) dS(x). \quad (13)$$

The homogenised surface energy $w_h : \Gamma \times V \rightarrow \mathbb{R}$ is given by

$$w_h(x, u) = \int_{\nu + T_x \Gamma} w\left(x, \frac{1}{|\nu|} \nu, u\right) |\nu| d\mu_x(\nu), \quad (14)$$

where $(\mu_x)_{x \in \Gamma}$ is the Young measure generated by $\hat{D}\varphi_\varepsilon : \Gamma \rightarrow \mathbb{R}^N$ as given in Assumption 3. Furthermore the energies \mathcal{F}_ε are equicoercive with respect to $W^{1,p}$ -rugose convergence, and if u_ε are minimisers of \mathcal{F}_ε for every $\varepsilon_0 > \varepsilon > 0$, there exists a subsequence $\varepsilon_j \rightarrow 0$ and minimiser u of \mathcal{F} such that $u_{\varepsilon_j} \xrightarrow{R} u$.

1.2.2 Summary of Section 3

We consider the Oseen-Frank and the Landau-de Gennes models for nematic liquid crystals and we illustrate in both cases an example of homogenised surface energy for the limit problem, depending on the choice of surface energy for the rugose problem.

For the Oseen-Frank model, we consider the Rapini-Papoular surface energy, which is given by

$$w(\nu, n) = \frac{w_0}{2} (n \cdot \nu)^2,$$

where $n \in W^{1,2}(\Omega, \mathbb{R}^3)$, and we prove that the homogenised surface energy in this case is of the form

$$w_h(x, n) = \frac{1}{2} A_{ef} n \cdot n,$$

where $A_{ef} : \Gamma \rightarrow \mathbb{R}^{3 \times 3}$ is a 3×3 symmetric tensor given by

$$A_{ef} = w_0 \int_{\nu(x) + T_x \Gamma} \frac{1}{|v|} v \otimes v \, d\mu_x(v).$$

For the Landau-de Gennes model, we consider the following form for the surface energy:

$$w(x, \nu, Q) = \frac{w_0}{2} \left| Q - s_0 \left(\nu \otimes \nu - \frac{1}{3} I \right) \right|^2$$

and we prove that, in this case, that the homogenised surface energy is of the following form:

$$w_h(x, Q) = \frac{w_{ef}}{2} |Q - Q_{ef}|^2 + R,$$

where R is a remainder that depends only on x and not on Q , hence it does not affect any minimisers, and the terms w_{ef} and Q_{ef} are given by:

$$\begin{aligned} w_{ef} &= w_0 \int_{\nu(x) + T_x \Gamma} |v| \, d\mu_x, \\ Q_{ef} &= \frac{s_0}{w_{ef}} \int_{\nu(x) + T_x \Gamma} \left(\frac{1}{|v|^2} v \otimes v - \frac{1}{3} I \right) |v| \, d\mu_x(v). \end{aligned}$$

1.2.3 Summary of Section 4

In physical situations one has, after suitable non-dimensionalisations, a small but finite ε parameter. Thus in order to understand the relevance of the theories previously developed, that study the limit problem, one needs to obtain error estimates. Moreover one aims to obtain error estimates with a convergence as good as possible and it will turn out that our main result shows that by using weaker norms one can improve the convergence rate, without needing to resort to correctors. In order to understand these issues we analyze the most simplified setting possible that still has a certain physical relevance.

We consider the situation of a two-dimensional slab with periodic rugosity and the Landau-de Gennes model for the description of the nematic liquid crystal used. More specifically, the limiting domain is of the form $\Omega_0 = \{(x, y) \mid x \in [0, 2\pi), y \in (0, R)\}$, where $R > 0$ is a constant, and the rugose domain is of the form $\Omega_\varepsilon = \{(x, y) \mid x \in [0, 2\pi), y \in (\varphi_\varepsilon(x), R)\}$, where $\varphi_\varepsilon(x) = \varepsilon \varphi(x/\varepsilon)$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 2π -periodic function with $\varphi \geq 0$. We denote with $\Gamma_\varepsilon = \{(x, \varepsilon \varphi(x/\varepsilon)) \mid x \in [0, 2\pi)\}$ the rugose boundary and with $\Gamma_R = \{(x, R) \mid x \in [0, 2\pi)\}$ the fixed upper boundary of the domains. We consider a quadratic free energy of the following form:

$$\mathcal{F}_\varepsilon[Q] = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 \, d(x, y) + \int_{\Gamma_\varepsilon} \frac{w_0}{2} |Q - Q_\varepsilon^0|^2 \, d\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2} |Q - Q_R|^2 \, d\sigma_R,$$

where $c > 0$ is constant, $w_0 > 0$ is the anchoring strength, $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2} I$ and $Q_R = \nu_R \otimes \nu_R - \frac{1}{2} I$ (ν_ε and ν_R are the outward normals to Γ_ε and Γ_R).

In this simplified model, using Proposition 4.1, we are able to identify the homogenised surface energy as $\frac{w_{ef}}{2} |Q - Q_{ef}|^2$, with $w_{ef} = w_0 \gamma$ and $Q_{ef} = \frac{1}{\gamma} \begin{pmatrix} G_1 & G_2 \\ G_2 & -G_1 \end{pmatrix}$, where γ , G_1 and G_2 are defined in Definition 4.1. The homogenised free energy functional is then of the form

$$\mathcal{F}_0[Q] = \int_{\Omega_0} |\nabla Q|^2 + c|Q|^2 \, d(x, y) + \int_{\Gamma_0} \frac{w_{ef}}{2} |Q - Q_{ef}|^2 \, d\sigma_0 + \int_{\Gamma_R} \frac{w_0}{2} |Q - Q_R|^2 \, d\sigma_R,$$

where ν_0 is the outward normal to $\Gamma_0 = \{(x, 0) \mid x \in [0, 2\pi)\}$.

Let Q_ε be the minimiser of \mathcal{F}_ε and Q_0 the minimiser of \mathcal{F}_0 . In [11] and [25], the authors are able to prove that $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$. According to [18], our simplified model is under the case $0 = \beta = \alpha - 1$, in which they prove that $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq K_2(\sqrt{\varepsilon} + 1)$. Both in [?] and [3], it is proved that $(Q_\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2(\Omega_\varepsilon)$ to Q_0 , under various assumptions for the domains. Using boundary layers, in [2] the authors are able to prove that $\|Q_\varepsilon - Q_0 - \varepsilon Q_1\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$, where Q_1 is a first-order boundary term. In this work, we are able to prove the following L^2 error estimate:

Theorem 1.2. For any $p \in (2, +\infty)$, there exists an ε -independent constant C such that:

$$\|Q_0 - Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \cdot \varepsilon^{\frac{p-1}{p}},$$

where the constant C depends on $c, w_0, p, \|\varphi\|_{L^\infty([0, 2\pi])}, \|\varphi'\|_{L^\infty([0, 2\pi])}, \Omega_0$ and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

It is easy to observe that the fraction $\frac{p-1}{p}$, with $p \in (2, +\infty)$, allows us to obtain any desired exponent from the interval $(1/2, 1)$. In order to prove this theorem, we first show in Section 4.3 that Q_ε and Q_0 exist and admit $W^{2,p}$ regularity, for any $p \in (2, +\infty)$. Then, we adapt in Section 4.4 the proofs from [11] and [25] to the case of $W^{1,p}$ functions in order to obtain Proposition 4.9. The result that dictates the exponent of ε from our error estimate is Lemma 4.2. A similar estimate to this lemma represents [3, Lemma 5.1, (15)], where the exponent obtained is $\frac{d+2}{2d}$ for $L^{\frac{2d}{d-2}}$ estimates, for any $d > 2$. The proof of our error estimate is also based on the construction of an extension operator, from $W^{1,p}(\Omega_\varepsilon)$ to $W^{1,p}(\Omega_0)$, which is defined in Definition 4.13 and has ε -independent bounds. With all of these ingredients, we are able then to prove the main result of this part, in Section 4.5.

2 Γ -convergence in the general case

Within this section we aim to describe the asymptotic behaviour of minimisers of a general class of nonlinear energy functionals. It is well known that pointwise convergence of a sequence of energy functionals is not sufficient to guarantee convergence of their minimisers to a minimiser of the limiting energy. In response to this challenge, many techniques have been proposed, and within this section we will use the techniques of Γ -convergence, of which we recall the definition below, following [14, Theorem 2.1]

Definition 2.1. Let X be a topological space and $\mathcal{F}_\varepsilon : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be energy functionals for every $\varepsilon > 0$. Then we say that \mathcal{F}_ε Γ -converges to $\mathcal{F} : X \rightarrow \mathbb{R} \cup \{+\infty\}$ if the following hold:

- (Liminf inequality) For every sequence $x_\varepsilon \rightarrow x$ in X , we have that $\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x)$.
- (Limsup inequality) For every $x \in X$, there exists a sequence $x_\varepsilon \rightarrow x$ such that $\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \mathcal{F}(x)$.

Furthermore, we say that the functionals are *equicoercive* if for any sequence $(x_\varepsilon)_{\varepsilon>0}$ such that $\mathcal{F}_\varepsilon(x_\varepsilon)$ is uniformly bounded, there exists a subsequence $\varepsilon_j \rightarrow 0$ such that $x_{\varepsilon_j} \rightarrow x$ for some $x \in X$.

The relevance of Γ -convergence in minimisation problems is made apparent by the fundamental theorem of Γ -convergence, which states that if \mathcal{F}_ε are a series of equicoercive functionals and \mathcal{F}_ε Γ -converges to \mathcal{F} , then for any sequence x_ε such that $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(x_\varepsilon) - \inf_{x'} \mathcal{F}_\varepsilon(x') = 0$, there exists a subsequence $\varepsilon_j \rightarrow 0$ and $x \in X$, such that $x_{\varepsilon_j} \rightarrow x$ and $\mathcal{F}(x) = \min_{x'} \mathcal{F}(x')$ (see, e.g., [14, Theorem 2.10]). In words, a sequence of almost minimisers of \mathcal{F}_ε , up to a subsequence, converges to a minimiser of the Γ -limit. Furthermore, $\min_{x'} \mathcal{F}(x') = \lim_{\varepsilon \rightarrow 0} \inf_{x' \in X} \mathcal{F}_\varepsilon(x')$.

2.1 Technical assumptions and statement of the main results

2.1.1 Geometric assumptions

Assumption 1 (Properties of Ω). $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary Γ . Γ is a C^2 compact manifold without boundary of dimension $N - 1$. Throughout, the function $\nu : \Gamma \rightarrow \mathbb{S}^{N-1}$ will denote the outward unit normal vector to Γ .

Assumption 2. Ω_ε will denote uniformly bounded open subsets of \mathbb{R}^N such that:

1. For every $U \subset\subset \Omega$, there exists $\varepsilon_0 > 0$ so that if $\varepsilon < \varepsilon_0$, $U \subset\subset \Omega_\varepsilon$.
2. For every $U \subset\subset \mathbb{R}^N \setminus \Omega$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, $U \cap \Omega_\varepsilon = \emptyset$.

We denote $\Gamma_\varepsilon = \partial\Omega_\varepsilon$. We require that there exists $\varphi_\varepsilon \in W^{1,\infty}(\Gamma, \mathbb{R})$ with the following properties:

1. $\Gamma \ni x \mapsto x + \varphi_\varepsilon(x)\nu(x) \in \Gamma_\varepsilon$ defines a bijection.
2. $\|\varphi_\varepsilon\|_{L^\infty(\Gamma)} = O(\varepsilon)$.
3. $\|D\varphi_\varepsilon\|_{L^\infty(\Gamma)} = O(1)$.

Furthermore, we denote $\nu_\varepsilon : \Gamma_\varepsilon \rightarrow \mathbb{S}^{N-1}$ the outer unit normal to Γ_ε , and the surface element $\gamma_\varepsilon : \Gamma \rightarrow (0, \infty)$ such that

$$\int_{\Gamma_\varepsilon} g(x) dS(x) = \int_{\Gamma} g(x + \varphi_\varepsilon(x)\nu(x)) \gamma_\varepsilon(x) dS(x) \quad (15)$$

for all integrable $g : \Gamma_\varepsilon \rightarrow \mathbb{R}$. Finally, we define $\tilde{\nu}_\varepsilon : \Gamma \rightarrow \mathbb{S}^{N-1}$ to be

$$\tilde{\nu}_\varepsilon(x) = \nu_\varepsilon(x + \varphi_\varepsilon(x)\nu(x)). \quad (16)$$

Remark 2.1. We note that D refers to the derivative on Γ , $D\varphi(x) \in (T_x\Gamma)^*$. Given a C^1 curve $\xi : [-1, 1] \rightarrow \Gamma$ with $\xi(0) = x$ and $\xi'(0) = \tau \in T_x\Gamma$,

$$\langle D\varphi_\varepsilon(x), \tau \rangle = \frac{\partial \varphi_\varepsilon \circ \xi}{\partial t}(0). \quad (17)$$

Furthermore, for an orthonormal basis $(\tau_i(x))_{i=1}^{N-1}$ of $T_x\Gamma$, we have that

$$|D\varphi_\varepsilon(x)|_\Gamma^2 = \sum_{i=1}^{N-1} |\langle D\varphi_\varepsilon(x), \tau_i(x) \rangle|^2, \quad (18)$$

where $|\cdot|_\Gamma$ is the norm on $T_x\Gamma$ induced by the metric on Γ .

Definition 2.2. We denote $P_\Gamma : \mathbb{R}^N \rightarrow \Gamma$ the nearest point projection to Γ . We presume that δ is sufficiently small so that

$$\{x \in \Omega : d(x, \partial\Omega) = \delta\} \quad (19)$$

is a C^1 domain, parametrised by

$$\{x \in \Omega : d(x, \Gamma) = \delta\} = \{x - \delta\nu(x) : x \in \partial\Omega\}. \quad (20)$$

Furthermore, we presume δ is sufficiently small so that $P_\Gamma|_{\Gamma+B_\delta}$ is a C^1 function. By [37, Lemma 4], as Γ is compact, such a δ exists.

Remark 2.2. Let X, V be metric spaces where X is a positive, complete measure space. We say that a sequence of functions $f_i : X \rightarrow V$ generates a Young measure $(\mu_x)_{x \in X}$, where for a.e. $x \in X$, μ_x defines a positive measure on V , if for every continuous $F : V \rightarrow \mathbb{R}$ such that $F(f_i)$ is uniformly integrable,

$$\int_X F(f_i) dx \rightarrow \int_{X \times V} F(v) d\mu_x(v) dx. \quad (21)$$

In fact, if this holds and X is an open domain in \mathbb{R}^M and $V = \mathbb{R}^N$, then following [5], for any function $F : X \times V \rightarrow \mathbb{R}$ which is X -measurable in its first variable and continuous in its second, if the functions $x \mapsto F(x, f_i(x))$ are uniformly integrable, then

$$\int_X F(x, f_i) dx \rightarrow \int_{X \times V} F(x, v) d\mu_x(v) dx. \quad (22)$$

It is straightforward to extend this result to the case when $X = \Gamma$, which satisfies Assumption 1, by considering a finite open covering and local coordinate systems.

To illustrate Young measures, we consider a simple example

Example 2.1. Let $\varphi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}$ be a smooth function. We will freely identify \mathbb{S}^1 with $[0, 2\pi]$, and φ with a 2π -doubly periodic function on \mathbb{R}^2 , throughout this example. That is, $\varphi(x, y) = \varphi(x + 2k_1\pi, y + 2k_2\pi)$ for all $k_1, k_2 \in \mathbb{Z}$. Let $f_i(x) = \varphi(x, ix)$ for integer i . Then, given any continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$, we have that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\mathbb{S}^1} F(f_i(x)) dx &= \lim_{i \rightarrow \infty} \int_0^{2\pi} F(\varphi(x, ix)) dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} F(\varphi(x, x')) dx' dx \end{aligned} \quad (23)$$

by, for example, [20, Lemma 9.1]. Now if, for every x , we define the measure

$$\mu_x(A) = \frac{1}{2\pi} |\{z \in [0, 2\pi] : \varphi(x, z) \in A\}|, \quad (24)$$

we may re-write the integral as

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} F(\varphi(x, x')) dx' = \int_{\mathbb{R}} F(v) d\mu_x(v) dx \quad (25)$$

for every $x \in [0, 2\pi]$. Thus for every continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\int_{\mathbb{S}^1} F(f_i(x)) dx \rightarrow \int_{\mathbb{S}^1} \int_{\mathbb{R}} F(v) d\mu_x(v) dx \quad (26)$$

by Fubini's theorem. Thus $(\mu_x)_{x \in \mathbb{S}^1}$ is the family of Young measures generated by f_i .

Furthermore, it is immediate from (24) that $\mu_x(\mathbb{R}) = 1$. This property does not hold generally for Young measures, rather that $\mu_x(\mathbb{R}) \leq 1$.

We further note that f_i in our example has weak- L^2 limit given by

$$\int_0^{2\pi} \varphi(x, x') dx' = \int_{\mathbb{R}} v d\mu_x(v), \quad (27)$$

which can be seen by taking $F(v) = v$ in (23). The property that the weak limit of f_i is, pointwise, given by the average of μ_x is general. This highlights the fact that Young measures contain information about the weak limit via its average, but also “retain” information about oscillations that the weak limit that would otherwise lose. It is thus a natural language for the problem we consider, where we wish to keep track of how fine-scale oscillations contribute to our surface energy, and will later be quantified in Proposition 2.3.

Assumption 3. Given φ_ε satisfying Assumption 2, we define the map $\hat{D}\varphi_\varepsilon : \Gamma \rightarrow \mathbb{R}^N$ as follows. The derivative operator $D\varphi_\varepsilon : \Gamma \rightarrow (T_x\Gamma)^*$ acts on tangent vectors $\tau \in T_x\Gamma$ via the duality pairing $\langle D\varphi_\varepsilon(x), \tau \rangle$. At each point $x \in \Gamma$, we take an orthonormal basis $(\tau_i(x))_{i=1}^{N-1}$ of $T_x\Gamma$, and define $\hat{D}\varphi_\varepsilon(x)$ via

$$\hat{D}\varphi_\varepsilon(x) = \nu(x) - \sum_{i=1}^{N-1} \langle D\varphi_\varepsilon(x), \tau_i(x) \rangle \tau_i(x), \quad (28)$$

Furthermore, we presume that $\hat{D}\varphi_\varepsilon$ generates a Young measure $(\mu_x)_{x \in \Gamma}$.

Remark 2.3. As the gradient $D\varphi_\varepsilon(x)$ is defined as a linear operator on the tangent space $T_x\Gamma$, we will employ $\hat{D}\varphi_\varepsilon$ as an analogue of the gradient of φ_ε which is \mathbb{R}^N valued, and will be necessary to simplify computations involving the surface normal of Γ_ε in the sequel (see Lemma 2.1). We note that $\hat{D}\varphi_\varepsilon$ does not depend on the (pointwise) orthonormal bases taken to define τ_i .

We will provide some preliminary results necessary for dealing with Young measures in Proposition 2.2

2.1.2 Functional assumptions

Assumption 4 (Properties of F). Let V be a finite dimensional vector space. We take D to be a bounded set such that $\Omega \cup \bigcup_{\varepsilon>0} \Omega_\varepsilon \subset D$. Then $F : (V \times \mathbb{R}^N) \times V \times D \rightarrow [0, +\infty]$ denotes a Carathéodory function, convex in its first variable, satisfying the following p -growth lower bound, that there exists some $C > 0$ so that

$$F(A, u, x) \geq C(|A|^p + |u|^p - 1) \quad (29)$$

for all $A \in V \times \mathbb{R}^N$, $u \in V$, $x \in D$. Furthermore we presume there exists a homogeneous functional $F_h : (V \times \mathbb{R}^N) \times V \rightarrow [0, \infty]$ and $C_1 > 1$ such that

$$F_h(A, u) \leq F(A, u, x) \leq C_1 F_h(A, u) \quad (30)$$

for all $A \in V \times \mathbb{R}^N$, $u \in V$, $x \in D$. Furthermore, we presume that for every $L_0 > 1$, there exists some $C > 0$ so that for every $L \in \mathbb{R}^{N \times N}$ with $L_0^{-1}I < L^T L < L_0 I$ (where the inequality corresponds to that as bilinear forms),

$$F_h(AL, u) \leq C(F_h(A, u) + 1). \quad (31)$$

for all $A \in V \times \mathbb{R}^N$, $u \in V$. Finally, we presume that there exists some $u \in W^{1,p}(\Omega, V)$, where Ω is as in Assumption 1, such that

$$\int_{\Omega} F(\nabla u(x), u(x), x) dx < +\infty. \quad (32)$$

Remark 2.4. The assumptions on F are somewhat non-standard, and are designed to account simultaneously for various energies relevant in the modelling of liquid crystals, where each has its own non-standard aspects. Generally, such energies find themselves of the form

$$F(A, u, x) = F_{el}(A, u) + F_b(u), \quad (33)$$

where the elastic energy F_{el} is p -positively homogeneous in its first argument and F_b may have different growth to F_{el} . In particular, F_b may only be finite on a compact or bounded set. These are typical in models of nematics, such as Landau-de Gennes, where F_{el} is quadratic, and F_b is either a polynomial or singular bulk potential which lacks an upper bound in terms of $|Q|^2$. A more extreme case is Oseen-Frank, in which one may interpret the unit vector constraint $|u| = 1$ almost everywhere as one of the form (33) with $F_b(u) = 0$ if $|u| = 1$ and $F_b(u) = \infty$ otherwise, and thus no power of u can bound F from above, as discussed in Remark 1.1.

Furthermore, the presence of a magnetic field would lead to x -dependence in F , via the addition of a term $\chi(H \cdot n)^2$ for Oseen-Frank or $\chi QH \cdot H$, where $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the externally imposed magnetic field.

It is immediate that these models all comply with Assumption 4, with the restriction that H , if present, be bounded.

The reason for (31) is such that if $u \in W^{1,p}(U)$ and $g : U \rightarrow g(U)$ is a bi-lipschitz map, then for $\tilde{u} = u \circ g^{-1}$

$$\int_{g(U)} F_h(\nabla \tilde{u}(x), \tilde{u}(x)) dx \leq CL_0 \int_U F_h(\nabla u(x), u(x)) + 1 dx, \quad (34)$$

where L_0 depends on the bi-Lipschitz constants of g . This inequality will allow us to define finite-energy extensions of $W^{1,p}(\Omega, V)$ functions via reflections in the sequel.

Furthermore, we note that whilst we are considering F to be convex in the gradient of u this particular assumption may be relaxed to F which defines a weakly lower semicontinuous energy functional in $W^{1,p}$. For example quasiconvex, non-convex energies, such as forms considered in an elasticity like $F(A, u, x) = |A|^p + \det(A)^{-1}$ with $\det(A) > 0$ and $+\infty$ otherwise, with $p > N$, may be considered, provided the growth bounds we impose are satisfied. Finally, we note that the existence of $u \in W^{1,p}(\Omega, V)$ such that (32) holds suffices in order to ensure that the Γ -limit functional that will be obtained is non infinite everywhere.

Assumption 5 (Assumptions on w). Let $1 < p < \infty$ be as in Assumption 4 and $1 \leq q < p^*$, where $p^* = \frac{(N-1)p}{N-p}$ if $p < N$ or $p^* = \infty$ if $p \geq N$. We presume that $w : \Gamma \times \mathbb{S}^{N-1} \times V \rightarrow [0, \infty)$ satisfies the following.

- There exists $C > 0$ such that for a.e. $x \in \Gamma$, every $\nu \in \mathbb{S}^{N-1}$, $u_1, u_2 \in V$, $|w(x, \nu, u_1) - w(x, \nu, u_2)| \leq C(|u_1|^{q-1} + |u_2|^{q-1} + 1)|u_1 - u_2|$.
- There exists $C > 0$ such that for a.e. $x \in \Gamma$, every $\nu \in \mathbb{S}^{N-1}$, $u \in V$, $|w(x, \nu, u)| < C(|u|^q + 1)$
- There exists $C > 0$ such that for a.e. $x \in \Gamma$, every $\nu_1, \nu_2 \in \mathbb{S}^{N-1}$, $|w(x, \nu_1, u) - w(x, \nu_2, u)| < C(|u|^q + 1)|\nu_1 - \nu_2|$.

Remark 2.5. We have in mind the case where

$$w(x, \nu, u) = \sum_{i=0}^m a_i(\nu)[u, \dots, u], \quad (35)$$

where for every $\nu \in \mathbb{S}^{N-1}$, $a_i(\nu)$ is an i -linear map on V , and $m < q$. That is, w is of polynomial type in u with degree m , less than q , with coefficients that depend on ν . We will take a_i to be continuous in ν in such cases, and as we will see in Proposition 2.4, functions of this form admit a greatly simplified homogenised surface energy. In particular, the homogenised surface energy will also be of polynomial type with degree m .

Furthermore, the choice of $q < p^*$ in the growth bounds for w is chosen as a technical assumption, as this implies that the trace operator from $W^{1,p}(\Omega_\varepsilon)$ is a compact operator into $L^q(\Gamma_\varepsilon)$.

Definition 2.3. We define the functionals $\mathcal{F}_\varepsilon : W^{1,p}(\Omega_\varepsilon, V) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{F}_\varepsilon(u) = \int_{\Omega_\varepsilon} F(\nabla u(x), u(x), x) dx + \int_{\Gamma_\varepsilon} w(P_\Gamma x, \nu_\varepsilon(x), u(x)) dS(x). \quad (36)$$

2.1.3 Function space assumptions

Definition 2.4. Let $U \subset \mathbb{R}^N$ be a measurable set. Let V be a finite dimensional vector space, and $1 < p < \infty$. We define the extension by zero operator $E_U : \bigcup_{U' \subset U} L^p(U', V) \rightarrow L^p(U, V)$, where for $u \in L^p(U', V)$, $E_U u = u$ on U' , $E_U u = 0$ otherwise. Similarly, we define the restriction operator $R_U : \bigcup_{U \subset U'} L^p(U', V) \rightarrow L^p(U, V)$ by $R_U u = u|_U$.

Definition 2.5 (Rugose convergence of functions). Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1 and 2, and V be a finite dimensional vector space. Let $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ for every $\varepsilon > 0$. Let D be a bounded set, such that $\Omega_\varepsilon \subset\subset D$ for every $\varepsilon > 0$. Then we say that u_ε converges to $u \in W^{1,p}(\Omega, V)$ in the rugose sense, denoted $u_\varepsilon \xrightarrow{R} u$, if the following hold:

- For every $U \subset\subset \Omega$, $R_U u_\varepsilon \xrightarrow{W^{1,p}(U, V)} R_U u$.
- $E_D u_\varepsilon \xrightarrow{L^p(D, V)} E_D u$.

- $E_D \nabla u_\varepsilon \xrightarrow{L^p(D, V)} E_D \nabla u$.

Remark 2.6. We note that Definition 2.5 is independent of the choice of $D \subset \mathbb{R}^N$, provided that $\Omega_\varepsilon \subset\subset D$ for all $\varepsilon > 0$.

Theorem 2.1. Let Ω_ε , for $\varepsilon_0 > \varepsilon > 0$, Ω denote sets satisfying Assumption 1 and Assumption 2, the interior energy F satisfy Assumption 4, and the surface energy w satisfy Assumption 5. Then there exists an effective surface energy density function $w_h : \Gamma \times V \rightarrow \mathbb{R}$ so that for every $u \in L^q(\Gamma)$, $\lim_{\varepsilon \rightarrow 0} w(\cdot, \tilde{v}_\varepsilon, u) \gamma_\varepsilon \xrightarrow{L^1(\Gamma)} w_h(\cdot, u)$, and $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ with respect to $W^{1,p}$ -rugose convergence, where $\mathcal{F} : W^{1,p}(\Omega, V) \rightarrow [0, \infty)$ is given by

$$\mathcal{F}(u) = \int_{\Omega} F(\nabla u(x), u(x), x) dx + \int_{\Gamma} w_h(x, u(x)) dS(x). \quad (37)$$

The homogenised surface energy $w_h : \Gamma \times V \rightarrow \mathbb{R}$ is given by

$$w_h(x, u) = \int_{\nu + T_x \Gamma} w\left(x, \frac{1}{|v|} v, u\right) |v| d\mu_x(v), \quad (38)$$

where $(\mu_x)_{x \in \Gamma}$ is the Young measure generated by $\hat{D}\varphi_\varepsilon : \Gamma \rightarrow \mathbb{R}^N$ as given in Assumption 3. Furthermore the energies \mathcal{F}_ε are equicoercive with respect to $W^{1,p}$ -rugose convergence, and if u_ε are minimisers of \mathcal{F}_ε for every $\varepsilon_0 > \varepsilon > 0$, there exists a subsequence $\varepsilon_j \rightarrow 0$ and minimiser u of \mathcal{F} such that $u_{\varepsilon_j} \xrightarrow{R} u$.

2.2 Proof of Γ -convergence

2.2.1 Compactness

Proposition 2.1. Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1 and 2, w satisfy Assumptions 5 and 3, and F satisfy Assumption 4. Let $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ for every $\varepsilon > 0$ be such that

$$\mathcal{F}_\varepsilon(u_\varepsilon) < M. \quad (39)$$

Then there exists a subsequence $\varepsilon_j \rightarrow 0$ and $u \in W^{1,p}(\Omega, V)$ such that $u_j := u_{\varepsilon_j} \xrightarrow{R} u$.

Proof. Take D to be a bounded, open set such that $\Omega_\varepsilon \subset\subset D$ for all $\varepsilon > 0$. Using the non-negativity of the surface term, we immediately obtain the estimate

$$\begin{aligned} M &\geq \mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_{\Omega_\varepsilon} F(\nabla u_\varepsilon(x), u_\varepsilon(x), x) dx \\ &\geq \int_{\Omega_\varepsilon} C(|\nabla u_\varepsilon(x)|^p + |u_\varepsilon(x)|^p - 1) dx \\ &= -C|\Omega_\varepsilon| + C \int_D |E_D \nabla u_\varepsilon(x)|^p + |E_D u_\varepsilon(x)|^p dx. \end{aligned} \quad (40)$$

As $|\Omega_\varepsilon|$ is uniformly bounded, and thus both $E_D u_\varepsilon, E_D \nabla u_\varepsilon$ are uniformly bounded in $L^p(D)$, we may thus extract a subsequence such that $E_D u_j \rightharpoonup u_0$ and $E_D \nabla u_j \xrightarrow{L^p} A$ for some $u_0 \in L^p(D, V)$ and $A \in L^p(D, V \times \mathbb{R}^N)$. Furthermore, for any $U \subset D$ with $U \subset\subset \mathbb{R}^N \setminus \Omega$, $U \cap \Omega_\varepsilon = \emptyset$ for small enough ε , therefore $E_D u_j, E_D \nabla u_j$ will both be zero on U for sufficiently small epsilon. This implies $\text{supp}(u_0) \cap U = \text{supp}(A) \cap U = \emptyset$ for all such U . Taking the union of all such U implies that $\text{supp}(u_0), \text{supp}(A) \subset \bar{\Omega}$.

We now define $u = R_\Omega u_0$. We aim to show that $R_\Omega A = \nabla u$. To do so we take any $\eta \in \mathcal{D}(\Omega, V)$. In particular, $\text{supp}(\eta) \subset\subset \Omega$, which in turn implies that for all sufficiently small $\varepsilon > 0$, $\text{supp}(\eta) \subset \Omega_\varepsilon$. Therefore we know that

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} u_{\varepsilon_j} \cdot \nabla \eta + \nabla u_{\varepsilon_j} \cdot \eta \, dx = \int_{\text{supp}(\eta)} u_{\varepsilon_j} \cdot \nabla \eta + \nabla u_{\varepsilon_j} \cdot \eta \, dx \\ &\rightarrow \int_{\text{supp}(\eta)} u_0 \cdot \nabla \eta + A \cdot \eta \, dx \\ &= \int_{\Omega} u \cdot \nabla \eta + R_\Omega A \cdot \eta \, dx \end{aligned} \quad (41)$$

Therefore $R_\Omega A = \nabla u$ in the sense of distributions, however as A is in $L^p(\Omega, V)$, this implies that $u \in W^{1,p}(\Omega, V)$.

It only remains to show that $R_U u_j \xrightarrow{W^{1,p}} R_U u$ on $U \subset\subset \Omega$. However, as $\|R_U u_j\|_{W^{1,p}}$ is uniformly bounded and $R_U u_j \xrightarrow{L^p} R_U u$, $R_U \nabla u_j \xrightarrow{L^p} R_U \nabla u$, as all sub-sequences must have a $W^{1,p}$ -weakly converging sub-sub-sequence, whose limit may only be $R_U u$. \square

2.2.2 Surface terms

We first provide some preliminary results on the family of Young measures $(\mu_x)_{x \in \Gamma}$ generated by $\hat{D}\varphi_\varepsilon$. In particular, we highlight that the assumption that $\hat{D}\varphi_\varepsilon$ generates a family of Young measures may always be taken up to a subsequence, provided φ_ε satisfies the appropriate Lipschitz-type bounds.

Proposition 2.2. Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1 and 2. Then $\mu_x(V) = 1$, and $\text{supp}(\mu_x) \subset \nu(x) + T_x \Gamma$.

Furthermore, we have that for any sequence $\varphi_\varepsilon : \Gamma \rightarrow \mathbb{R}$ satisfying the bounds $\|\varphi_\varepsilon\|_\infty < C\varepsilon$, $\|D\varphi_\varepsilon\|_\infty < C$, we may always extract a subsequence $\varepsilon_j \rightarrow 0$ such that $\hat{D}\varphi_{\varepsilon_j}$ generates a family of Young measures.

Proof. First we show the compactness statement. If Γ were a measurable, positive measure subset of \mathbb{R}^{N-1} , then as $|\hat{D}\varphi_\varepsilon(x)|$ is uniformly bounded, the result would be a direct consequence of [5, Theorem 1]. It is however straightforward to generalise this to the case where Γ is a C^2 manifold, by taking a partition of unity $(\xi_i)_{i=1}^M$, so that $\xi_i \geq 0$ for all i , $\sum_{i=1}^M \xi_i = 1$, and there

exists domains $(U_i)_{i=1}^M$ with $\bigcup_{i=1}^M U_i = \Gamma$, $\text{supp}(\xi_i) \subset U_i$. Take U_i such that there exists a C^1 map $\eta_i : \tilde{U}_i \rightarrow U_i$, where $\tilde{U}_i \subset \mathbb{R}^{N-1}$. Then, given any continuous function $F : \mathbb{R}^N \rightarrow \mathbb{R}$, we have that

$$\int_{\Gamma} F(x, \hat{D}\varphi_\varepsilon(x)) \, dx = \sum_{i=1}^M \int_{\tilde{U}_i} F(\eta_i(x'), \hat{D}\varphi_\varepsilon(\eta_i(x'))) \xi_i(\eta_i(x')) J_i(x') \, dx', \quad (42)$$

where J_i are the corresponding volume elements. From this, it is immediate that the results for domains in \mathbb{R}^{N-1} may be lifted from the result of Ball applied to the functions $f_\varepsilon^i = \hat{D}\varphi_\varepsilon \circ \eta_i$, and by inverting the integral decomposition (42), the compactness result holds. Explicitly, as a

subsequence of $f_\varepsilon^i, f_{\varepsilon_j}^i$, generates Young measures $\mu_{\eta^i(x)}^i = \mu_{x'}^i$ for $x \in U^i$, we have that

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \sum_{i=1}^M \int_{\tilde{U}_i} F(\eta_i(x'), \hat{D}\varphi_\varepsilon(\eta_i(x'))) \xi_i(\eta_i(x')) J_i(x') dx' \\
&= \sum_{i=1}^M \int_{\tilde{U}_i} \int_{\mathbb{R}^N} F(\eta_i(x'), v) \xi_i(x') J_i(x') d\mu_{x'}^i dx' \\
&= \sum_{i=1}^M \int_{\tilde{U}_i} \int_{\mathbb{R}^N} F(\eta_i(x'), v) d\mu_{x'}^i \xi_i(\eta_i(x')) J_i(x') dx' \tag{43} \\
&= \sum_{i=1}^M \int_{U_i} \int_{\mathbb{R}^N} F(x, v) d\mu_x^i \xi_i(x) dx \\
&= \int_{\Gamma} \int_{\mathbb{R}^N} F(x, v) \sum_{i=1}^M \xi_i(x) d\mu_x^i(v) dx.
\end{aligned}$$

Thus we see that the Young measure $\mu_x = \sum_{i=1}^M \xi_i(x) d\mu_x^i$ is the family of Young measures corresponding to $\hat{D}\varphi_\varepsilon$.

To show that $\mu_x(\mathbb{R}^N) = 1$, via the decomposition (42), again we apply the result of Ball on each domain, which implies that $\mu_x^i(\mathbb{R}^N) = 1$ for almost every x as the integrand is uniformly bounded from above, and since $\sum_{i=1}^M \xi_i = 1$, we have that $\mu_x(\mathbb{R}^N) = 1$ almost everywhere.

Finally, to show that $\text{supp}(\mu_x) \subset \nu(x) + T_x\Gamma$ almost everywhere, we consider the function

$$F(x, v) = |(I - \nu(x) \otimes \nu(x))(v - \nu(x))|^2. \tag{44}$$

We note that this is a C^1 , non-negative function, which is zero if and only if $v \in \nu(x) + T_x\Gamma$. As $\hat{D}\varphi_\varepsilon$ is uniformly bounded, it is thus immediate that $F(x, \hat{D}\varphi_\varepsilon(x))$ is a uniformly integrable sequence of functions, and therefore as $\hat{D}\varphi_\varepsilon(x) \in \nu(x) + T_x\Gamma$,

$$0 = \int_{\Gamma} F(x, \hat{D}\varphi_\varepsilon(x)) dx \rightarrow \int_{\Gamma} \int_V F(x, v) d\mu_x(v) dx. \tag{45}$$

Therefore, we must have that $\int_V F(x, v) d\mu_x(v) = 0$ for almost every x , which thus implies that $\text{supp}(\mu_x) \subset \nu(x) + T_x\Gamma$. \square

We now provide a lemma that allows us to greatly simplify the surface terms

Lemma 2.1. Let the surface energy w and φ_ε satisfy Assumption 5, Assumption 3, and $1 \leq q \leq p^*$ be as in Assumption 5. Then for every $u \in L^q(\Gamma, V)$, there exists $c > 0$ such that for ε sufficiently small,

$$\int_{\Gamma} \left| w(x, \nu_\varepsilon(x), u(x)) \gamma_\varepsilon(x) - w\left(x, \frac{1}{|\hat{D}\varphi_\varepsilon(x)|} \hat{D}\varphi_\varepsilon(x), u(x)\right) |\hat{D}\varphi_\varepsilon(x)| \right| dS(x) < c\varepsilon. \tag{46}$$

Proof. We defer the proof to Appendix A \square

Proposition 2.3. Let $\Omega, \Omega_\varepsilon, \varphi_\varepsilon$ satisfy Assumptions 1 and 2, and w_h satisfy Assumptions 3 and 5, with $p^* > q > 1$ as in Assumption 5. There exists a homogenised surface energy $w_h : \Gamma \times V \rightarrow \mathbb{R}$ given by

$$w_h(x, u) = \int_{\nu(x) + T_x\Gamma} w\left(x, \frac{1}{|v|}v, u\right) |v| d\mu_x(v), \tag{47}$$

where $(\mu_x)_{x \in \Gamma}$ is the Young measure generated by $\hat{D}\varphi_\varepsilon$, as in Assumption 3, such that for every $u \in L^q(\Gamma, V)$,

$$\int_{\Gamma} w(x, \tilde{\nu}_\varepsilon(x), u(x)) \gamma_\varepsilon(x) dx \rightarrow \int_{\Gamma} w_h(x, u(x)) dx, \quad (48)$$

with $\tilde{\nu}_\varepsilon, \gamma_\varepsilon$ as defined in Assumption 2.

Proof. By Lemma 2.1, we see that it suffices to prove that

$$\begin{aligned} & \int_{\Gamma} w \left(x, \frac{1}{|\hat{D}\varphi_\varepsilon(x)|} \hat{D}\varphi_\varepsilon(x), u(x) \right) |\hat{D}\varphi_\varepsilon(x)| dS(x) \\ & \rightarrow \int_{\Gamma} \int_{\nu(x) + T_x \Gamma} w \left(x, \frac{1}{|v|} v, u(x) \right) |v| d\mu_x(v) dS(x). \end{aligned} \quad (49)$$

For fixed $u \in L^q$, we define the auxiliary function $\tilde{w} : \Gamma \times \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\tilde{w}(x, v) = w \left(x, \frac{1}{|v|} v, u(x) \right) |v| \eta(v), \quad (50)$$

where η is a smooth cutoff function, zero in a neighbourhood of the origin, and $\eta(v) = 1$ for $|v| > \frac{1}{2}$. This defines a Carathéodory function for $u \in L^q$, and as we only consider $v = \hat{D}\varphi_\varepsilon$, which satisfies $|v| \geq 1$, we see that

$$\tilde{w}(x, \hat{D}\varphi_\varepsilon(x)) = w \left(x, \frac{1}{|\hat{D}\varphi_\varepsilon(x)|} \hat{D}\varphi_\varepsilon(x), u(x) \right) |\hat{D}\varphi_\varepsilon(x)| \quad (51)$$

almost everywhere. We also see that for every u , $\tilde{w}(\cdot, \hat{D}\varphi_\varepsilon)$ is uniformly integrable via the definition of \tilde{w} and Assumption 5, as

$$|\tilde{w}(x, \hat{D}\varphi_\varepsilon)| \leq C |\hat{D}\varphi_\varepsilon| (1 + |u|^q), \quad (52)$$

using that $|u|^q$ is integrable and $\hat{D}\varphi_\varepsilon$ admits uniform L^∞ bounds. In particular, this implies that

$$\int_{\Gamma} \tilde{w}(x, \hat{D}\varphi_\varepsilon(x)) dS(x) \rightarrow \int_{\Gamma} \int_{\nu(x) + T_x \Gamma} \tilde{w}(x, v) d\mu_x(v) dS(x), \quad (53)$$

by [5], at which point re-writing \tilde{w} in terms of w and u gives the necessary result. \square

Corollary 2.1. The function w_h as given by (47) satisfies the same growth conditions in u as w in Assumption 5. That is,

- There exists $C > 0$ such that for a.e. $x \in \Gamma$ and $u \in V$, $|w_h(x, u)| \leq C(|u|^q + 1)$.
- $|w_h(x, u_1) - w_h(x, u_2)| \leq C|u_1 - u_2| (|u_1|^{q-1} + |u_2|^{q-1} + 1)$ for a.e. $x \in \Gamma$ and all $u_1, u_2 \in V$.

Proof. The result is a straightforward consequence of the integral representation of w_h in (47), using that $\text{supp}(\mu_x) \subset \nu(x) + T_x \Gamma$ and $\mu_x(\mathbb{R}^N) = 1$, as proven in Proposition 2.2 \square

In the case of simpler surface energies, we are able to provide w_h in a more explicit sense.

Proposition 2.4. Let $p^* > q > 1$, as in Assumption 5. Let w be of the form

$$w(x, \nu, u) = \sum_{i=0}^m a_i(\nu)[u, u, \dots, u], \quad (54)$$

where $a_i(\nu)$ is an i -linear map on a finite dimensional vector space V , identified with a member of $(V^*)^i$, m is an integer satisfying $m < q$, and a_i is continuous in ν . Then

$$a_i(\nu_\varepsilon)\gamma_\varepsilon \xrightarrow{*} A_i = \int_{\nu+T_x\Gamma} a_i(v)|v| d\mu_x(v), \quad (55)$$

where the convergence, explicitly, is weak- $*$ L^∞ convergence, and

$$w_h(x, u) = \sum_{i=0}^m A_i(x)[u, u, \dots, u]. \quad (56)$$

Proof. First we note that as a_i is continuous and ν_ε is bounded, $a_i(\nu_\varepsilon)\gamma_\varepsilon$ is certainly bounded in L^∞ . Furthermore, using Lemma 2.1, we may estimate uniformly in ε that $a_i(\nu_\varepsilon)\gamma_\varepsilon = a_i\left(\frac{1}{|\hat{D}\varphi_\varepsilon|}\hat{D}\varphi_\varepsilon\right)|\hat{D}\varphi_\varepsilon| + O(\varepsilon)$. Therefore, by [5], we must have that

$$a_i(\nu_\varepsilon)\gamma_\varepsilon \xrightarrow{L^1} \int_{\nu(x)+T_x\Gamma} a_i(v)|v| d\mu_x(v) = A_i. \quad (57)$$

We note that $a_i(\nu_\varepsilon)$ also admits uniform L^∞ bounds, and thus is pre-compact with weak- $*$ L^∞ convergence. As weak- $*$ L^∞ limits and weak- L^1 limits must be equal if both exist, the pre-compactness with respect to weak- $*$ L^∞ convergence and weak L^1 convergence of $a_i(\nu_\varepsilon)$ imply that $a_i(\nu_\varepsilon) \xrightarrow{*L^\infty} A_i$ also. \square

In order to understand how the surface energy behaves in the limit, our aim will be to define a “trace” of $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ onto Γ , which is continuous under rugose convergence. That is, we want a way of understanding how $u_\varepsilon|_{\Gamma_\varepsilon}$ “approaches” $u|_\Gamma$. To do so, we will define a bi-Lipschitz change of variables $\Phi_\varepsilon : \Omega \rightarrow \Omega_\varepsilon$, which is such that if $u_\varepsilon \xrightarrow{R} u$, then $u_\varepsilon \circ \Phi_\varepsilon \xrightarrow{W^{1,p}} u$. We may then understand the trace of $u_\varepsilon \circ \Phi_\varepsilon$ onto Γ .

Definition 2.6. We define the transformation $\Phi_\varepsilon : \Omega \rightarrow \Omega_\varepsilon$ as follows. Let $\delta > 0$ be fixed, and sufficiently small so that the projection P_Γ , as defined in Definition 2.6, is well-defined and C^1 . We then define $\Omega^\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$, and consider only ε sufficiently small so that $\|\varphi_\varepsilon\|_\infty < \frac{1}{2}\delta$, so that the following are well defined. If $x \in \Omega^\delta$, $\Phi_\varepsilon x := x$. If $x \in \Omega \setminus \Omega^\delta$, then we may write that $x = P_\Gamma x + \delta(t_x - 1)\nu(P_\Gamma x)$ for some $t_x \in [0, 1]$, which is readily evaluated as

$$t_x = 1 - \frac{|x - P_\Gamma x|}{\delta}. \quad (58)$$

Then we define

$$\Phi_\varepsilon x = P_\Gamma x + ((\varphi_\varepsilon(P_\Gamma x) + \delta)t_x - \delta)\nu(P_\Gamma x) \quad (59)$$

We note that the transformation restricted to lines of the form $s \mapsto x + s\nu(x)$ for $x \in \Gamma$ and s sufficiently small is merely a linear rescaling, so that $\Phi_\varepsilon x = x$ for $x \in \partial\Omega^\delta$, and $\Phi_\varepsilon x = x + \varphi_\varepsilon(x)\nu(x)$ for $x \in \Gamma$.

Proposition 2.5. For $\varepsilon > 0$ sufficiently small and Φ_ε defined as in (59), Φ_ε is invertible, and $\Phi_\varepsilon^{-1} : \Omega_\varepsilon \rightarrow \Omega$ is given by $\Phi_\varepsilon^{-1}x = x$ on Ω^δ , and

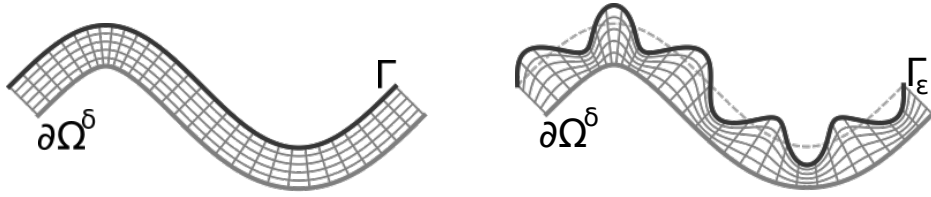
$$\Phi_\varepsilon^{-1}x = P_\Gamma x + \delta \frac{s_x - \varphi_\varepsilon(P_\Gamma x) - \delta}{\delta + \varphi_\varepsilon(P_\Gamma x)} \nu(P_\Gamma x), \quad (60)$$

where

$$s_x = |x - P_\Gamma x - \delta\nu(P_\Gamma x)|. \quad (61)$$

In particular, Φ_ε defines a family of uniformly bi-Lipschitz maps between Ω and Ω_ε . That is, there exists a constant $L > 0$ such that for all $\varepsilon, x, y \in \Omega$

$$L^{-1}|x - y| < |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)| < L|x - y|. \quad (62)$$



(a) $\Omega \setminus \Omega^\delta$ with corresponding (x, t) coordinate lines. Γ is the upper thick line, and $\partial\Omega^\delta$ the lower thick line.

(b) $\Phi_\varepsilon(\Omega \setminus \Omega^\delta)$ with corresponding (x, t) coordinate lines post-deformation. Γ is visible as a dashed line, Γ_ε is the upper thick line, and $\partial\Omega^\delta$ the lower thick line.

Figure 1: The (x, t) coordinate system before and after application of Φ_ε on $\Omega \setminus \Omega^\delta$.

Proof. To prove that the expression is the inverse on $\Omega_\varepsilon \setminus \Omega^\delta$, first we notice that for such x , $P_\Gamma \Phi_\varepsilon(x) = P_\Gamma x$, so that $x = P_\Gamma(x) + \delta(s_x - 1)\nu(P_\Gamma(x))$, where it is straightforward to verify that s_x is as in (61). Thus it remains to find $y \in \Omega \setminus \Omega^\delta$ so that

$$P_\Gamma x + \delta(s_x - 1)\nu(P_\Gamma x) = \Phi_\varepsilon y = P_\Gamma y + ((\varphi_\varepsilon(P_\Gamma y) + \delta)t_y - \delta)\nu(P_\Gamma y), \quad (63)$$

where $P_\Gamma y = P_\Gamma x$. By subtracting $P_\Gamma x$ from both sides and taking norms, this reduces to the simple scalar equation

$$\delta(s_x - 1) = (\varphi_\varepsilon(P_\Gamma x) + \delta)t_y - \delta, \quad (64)$$

which is easily inverted to give

$$t_y = \frac{\delta s_x}{\varphi_\varepsilon(P_\Gamma x) + \delta}, \quad (65)$$

which is readily substituted into $x = P_\Gamma x + \delta(t_x - 1)\nu(P_\Gamma x)$ to give the form in (60).

To observe that these maps are bi-Lipschitz, it suffices to notice that both $\Phi_\varepsilon, \Phi_\varepsilon^{-1}$ are continuous functions, and their derivatives can be uniformly bounded from above via their algebraic expressions, using that $P_\Gamma, \varphi_\varepsilon, t_x, s_x$ are all Lipschitz, with Lipschitz bounds that are independent of ε , and the algebraic expressions for $\Phi_\varepsilon, \Phi_\varepsilon^{-1}$ are non-singular on their domains of definition. \square

Lemma 2.2. Φ_ε , as defined in (59), converges uniformly to the identity. Furthermore, for every $U \subset\subset \Omega$, there exists $\varepsilon_0 > 0$ so that for all $\varepsilon > \varepsilon_0$, $\Phi_\varepsilon(U) \subset\subset \Omega$.

Proof. Clearly on Ω^δ there is nothing to prove as $\Phi_\varepsilon|_{\Omega^\delta}$ is the identity, so we consider the case with $x \in \Omega \setminus \Omega^\delta$. In this case we have that for $x = P_\Gamma x + \delta(t_x - 1)\nu(P_\Gamma x)$,

$$\begin{aligned} |x - \Phi_\varepsilon(x)| &= |P_\Gamma x + \delta(t_x - 1)\nu(P_\Gamma x) - P_\Gamma x - ((\varphi_\varepsilon(P_\Gamma x) + \delta)t_x - \delta)\nu(P_\Gamma x)| \\ &= |\delta(t_x - 1) - ((\varphi_\varepsilon(P_\Gamma x) + \delta)t_x - \delta)| \\ &= |\varphi_\varepsilon(P_\Gamma x)t_x| \end{aligned} \quad (66)$$

which, as $t_x \in [0, 1]$ and $\|\varphi_\varepsilon\|_\infty \rightarrow 0$, converges to zero uniformly in x as $\varepsilon \rightarrow 0$.

It is then immediate that for $U \subset\subset \Omega$ we must have that $\Phi_\varepsilon \subset \Omega$ for sufficiently small ε , since for any η sufficiently small so that $U + B_\eta \subset\subset \Omega$, we can take ε sufficiently small so that $|\Phi_\varepsilon(x) - x| < \eta$, and thus $\Phi_\varepsilon(U) \subset U + B_\eta$. \square

Lemma 2.3. Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1, 2. Let $1 < p < \infty$, $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ be such that $u_\varepsilon \xrightarrow{R} u_0 \in W^{1,p}(\Omega, V)$. Then $u_\varepsilon \circ \Phi_\varepsilon \in W^{1,p}(\Omega, V)$ satisfies $u_\varepsilon \circ \Phi_\varepsilon \xrightarrow{W^{1,p}(\Omega)} u_0$ also.

Proof. Our first observation is that it suffices to show that for any subsequence $\varepsilon_j \rightarrow 0$, there exists a further subsequence $\varepsilon_{j_k} \rightarrow 0$ such that $u_{\varepsilon_{j_k}} \circ \Phi_{\varepsilon_{j_k}} \xrightarrow{W^{1,p}} u_0$. If this were true, then as all subsequences of $u_\varepsilon \circ \Phi_\varepsilon$ have a $W^{1,p}$ -weakly converging subsequence, all admitting the same limit, we must have that $u_\varepsilon \circ \Phi_\varepsilon \xrightarrow{W^{1,p}} u_0$. Then we observe that as Φ_ε is bi-Lipschitz with bi-Lipschitz constants bounded away from 0 and ∞ independently of ε , and $\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon, V)}$ is bounded, we therefore have that $u_\varepsilon \circ \Phi_\varepsilon$ is bounded in $W^{1,p}(\Omega, V)$ [64, Theorem 2.2.2]. Therefore, given any subsequence $\varepsilon_j \rightarrow 0$, we may take a subsequence $\varepsilon_{j_k} \rightarrow 0$ such that $u_{\varepsilon_{j_k}} \circ \Phi_{\varepsilon_{j_k}}$ converges weakly in $W^{1,p}(\Omega, V)$, and strongly in $L^p(\Omega, V)$ to some \tilde{u}_0 . As such, it remains only to show that $\tilde{u}_0 = u_0$.

For brevity of notation, we denote $u_k = u_{\varepsilon_{j_k}}$, $\Phi_k = \Phi_{\varepsilon_{j_k}}$. To show that $\tilde{u}_0 = u_0$, it suffices to show that for any $U \subset\subset \Omega$,

$$\lim_{j \rightarrow \infty} \int_U |u_0(x) - u_k \circ \Phi_k(x)|^p dx = 0. \quad (67)$$

First, we let \hat{u}_0 denote a Lipschitz approximant of u_0 on an open set U' which satisfies

$$\bigcap_{\varepsilon < \varepsilon_0} \Phi_\varepsilon(U) \subset\subset U' \subset\subset \Omega. \quad (68)$$

An $\varepsilon_0 > 0$ that permits this chain of inclusions is guaranteed by Lemma 2.2. Then we estimate

$$\|u_0 - u_k \circ \Phi_k\|_{L^p(U)} \leq \|u_0 - \hat{u}_0\|_{L^p(U)} + \|\hat{u}_0 - \hat{u}_0 \circ \Phi_k\|_{L^p(U)} + \|\hat{u}_0 \circ \Phi_k - u_k \circ \Phi_k\|_{L^p(U)}. \quad (69)$$

Using Lemma 2.2, as \hat{u}_0 is Lipschitz on U' , it is clear that $\lim_{j \rightarrow \infty} \|\hat{u}_0 - \hat{u}_0 \circ \Phi_k\|_{L^p(U, V)} = 0$. Then, as Φ_k is uniformly bi-Lipschitz, we must have that $\|\hat{u}_0 \circ \Phi_k - u_k \circ \Phi_k\|_{L^p(U, V)} \leq C \|\hat{u}_0 - u_k\|_{L^p(U', V)}$ for some $C > 0$. Thus we may estimate

$$\begin{aligned} \limsup_{j \rightarrow \infty} \|u_0 - u_k \circ \Phi_k\|_{L^p(U, V)} &\leq \|u_0 - \hat{u}_0\|_{L^p(U, V)} + C \limsup_{j \rightarrow \infty} \|\hat{u}_0 - u_k\|_{L^p(U', V)} \\ &= \|u_0 - \hat{u}_0\|_{L^p(U, V)} + C \limsup_{j \rightarrow \infty} \|\hat{u}_0 - u_0\|_{L^p(U', V)}, \end{aligned} \quad (70)$$

as, by assumption, $u_k \xrightarrow{R} u$ and thus $u_k \xrightarrow{L^p(U', V)} u_0$. Finally, we note that as \hat{u}_0 was an arbitrary approximation, by taking \hat{u}_0 to approximate u_0 in $L^p(U, V)$, which is permitted by the density of Lipschitz functions in L^p , we obtain that $\limsup_{j \rightarrow \infty} \|u_0 - u_k \circ \Phi_k\|_{L^p(U, V)} = 0$. As U was arbitrary, $u_0 = \tilde{u}_0$. □

Proposition 2.6. Let $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ such that $u_\varepsilon \xrightarrow{R} u$. Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} w(P_\Gamma x, \nu_\varepsilon(x), u_\varepsilon(x)) dS(x) = \int_\Gamma w_h(x, u(x)) dS(x) \quad (71)$$

Proof. From the Lemma 2.3, we must have that $u_\varepsilon \circ \Phi_\varepsilon \xrightarrow{W^{1,p}(\Omega)} u$. Then, for q as in Assumption 5, $(u \circ \Phi_\varepsilon)|_\Gamma \xrightarrow{L^q(\Gamma)} u|_\Gamma$ as the embedding $W^{1,p}(\Omega, V) \hookrightarrow L^q(\Gamma, V)$ is compact.

We may then perform a change of variables on the integral to give

$$\int_{\Gamma_\varepsilon} w(P_\Gamma x, \nu_\varepsilon(x), u_\varepsilon(x)) dS(x) = \int_\Gamma w(x, \tilde{\nu}_\varepsilon(x), u_\varepsilon \circ \Phi_\varepsilon(x)) \gamma_\varepsilon(x) dS(x), \quad (72)$$

which combined with Proposition 2.3 and the strong $L^q(\Gamma, V)$ convergence of $u_\varepsilon \circ \Phi_\varepsilon|_\Gamma$ implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} w(x, \nu_\varepsilon(x), u_\varepsilon(x)) dS(x) = \int_\Gamma w_h(x, u(x)) dS(x). \quad (73)$$

□

2.2.3 Proof of Theorem 2.1

Definition 2.7. Define $\mathcal{F} : W^{1,p}(\Omega, V) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{F}(u) = \int_{\Omega} F(\nabla u(x), u(x), x) dx + \int_{\Gamma} w_h(x, u(x)) dS(x), \quad (74)$$

where F is as in Assumption 4, and w_h as in (47).

Proposition 2.7 (Liminf). Let $u_{\varepsilon} \in W^{1,p}(\Omega_{\varepsilon}, V)$ for $\varepsilon > 0$, such that $u_{\varepsilon} \xrightarrow{R} u \in W^{1,p}(\Omega, V)$. Then

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_{\varepsilon}(u_{\varepsilon}) \geq \mathcal{F}(u). \quad (75)$$

Proof. Take a subsequence $\varepsilon_j \rightarrow 0$, $u_j = u_{\varepsilon_j}$, such that $\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_{\varepsilon}(u_{\varepsilon}) = \lim_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j)$. If the limit is $+\infty$ the result is trivial, so assume otherwise. In this case, by Proposition 2.1, we can take a further subsequence (without relabelling) such that $u_j \xrightarrow{R} u$ for some $u \in W^{1,p}(\Omega, V)$. In particular the surface term, following Proposition 2.6, satisfies

$$\lim_{j \rightarrow \infty} \int_{\Gamma_{\varepsilon_j}} w(P_{\Gamma}x, \nu_{\varepsilon_j}(x), u_j(x)) dS(x) = \int_{\Gamma} w_h(x, u(x)) dS(x). \quad (76)$$

We thus consider the interior term. Take $U \subset \subset \Omega$. For j sufficiently large, $U \subset \Omega_{\varepsilon_j}$, therefore, by the non-negativity of F , we have that

$$\liminf_{j \rightarrow \infty} \int_{\Omega_{\varepsilon_j}} F(\nabla u_j(x), u_j(x), x) dx \geq \liminf_{j \rightarrow \infty} \int_U F(\nabla u_j(x), u_j(x), x) dx \geq \int_U F(\nabla u(x), u(x), x) dx \quad (77)$$

by standard semicontinuity results for integral functionals [21]. As U is arbitrary, and the right-hand side is increasing in U by the non-negativity of F , we therefore have that

$$\liminf_{j \rightarrow \infty} \int_{\Omega_{\varepsilon_j}} F(\nabla u_j(x), u_j(x), x) dx \geq \int_{\Omega} F(\nabla u(x), u(x), x) dx, \quad (78)$$

completing the proof. \square

Proposition 2.8. Let Ω satisfy Assumption 1, F satisfy 4, w satisfy 3. Then for $\delta > 0$ as in 2.6, and $\Omega' = \Omega + B_{\delta} = \{x + y : x \in \Omega, |y| < \delta\}$. Then, for every $u \in W^{1,p}(\Omega, V)$ such that $\mathcal{F}(u) < +\infty$, there exists an extension $Tu \in W^{1,p}(\Omega', V)$ such that $u'|_{\Omega} = u$, and

$$\int_{\Omega'} F(\nabla(Tu)(x), Tu(x), x) dx < +\infty. \quad (79)$$

Proof. The proof strategy is to show first that there is a bounded linear extension operator, defined by reflections, $T : W^{1,p}(\Omega, V) \rightarrow W^{1,p}(\Omega', V)$. Then, we will show that this reflection operator is such that if $\mathcal{F}(u)$ has finite energy, then (79) holds.

First, we consider the simpler case when $u \in C^1(\bar{\Omega}, V)$, which we will later extend via density. We will exploit a reflection argument, of a similar flavour to that of [1, Theorem 5.22], albeit simplified for the case at hand. First we define the reflection, in analogy to the argument presented in Proposition 2.5. For $x \in \Omega' \setminus \Omega$, we may write

$$x = P_{\Gamma}x + t_x \delta \nu(P_{\Gamma}x), \quad (80)$$

where $t_x \in [0, 1]$ is given explicitly via $t_x = \frac{1}{\delta} |x - P_{\Gamma}x|$. We thus define

$$Tu(x) = u(x - t_x \delta \nu(P_{\Gamma}x)) \quad (81)$$

for $x \in \Omega' \setminus \Omega$. As $P_\Gamma x$ and t_x are Lipschitz functions, and ν is C^1 , we thus have that the map $x \mapsto x - t_x \delta \nu(P_\Gamma x)$ is a Lipschitz map, and in fact bi-Lipschitz, by noting that its inverse is given by the simple formula $x' \mapsto x' + |x' - P_\Gamma x'| \nu(P_\Gamma x')$. Thus we see that Tu defined by $Tu(x) = u(x)$ on $\bar{\Omega}$, and by (81) on $\Omega' \setminus \Omega$ defines a Lipschitz function, as $u \in C^1(\bar{\Omega}, V)$. Furthermore, as the reflection mapping $x \mapsto x - t_x \delta \nu(P_\Gamma x)$ is bi-Lipschitz as a map between $\Omega \setminus \Omega^\delta$ (as defined in Definition 2.6) and $\Omega' \setminus \Omega$, we have that

$$\begin{aligned} \int_{\Omega'} |\nabla(Tu)(x)|^p dx &= \int_{\Omega} |\nabla(Tu)(x)|^p dx + \int_{\Omega' \setminus \Omega} |\nabla(Tu)(x)|^p dx \\ &\leq \int_{\Omega} |\nabla(Tu)(x)|^p dx + C \int_{\Omega \setminus \Omega^\delta} |\nabla u(x)|^p dx \\ &\leq (C+1) \int_{\Omega} |\nabla u(x)|^p dx, \end{aligned} \quad (82)$$

where C is independent of u , and only depends on the bi-Lipschitz constants of the reflection map. Similarly, and more trivially, we have that

$$\int_{\Omega'} |Tu(x)|^p dx \leq C \int_{\Omega} |u(x)|^p dx, \quad (83)$$

so that $Tu \in W^{1,p}(\Omega', V)$, and $\|Tu\|_{W^{1,p}(\Omega', V)} \leq C \|u\|_{W^{1,p}(\Omega, V)}$. Finally, we note that T is a linear map, and by the previous argument it is continuous, therefore by the density of $C^1(\bar{\Omega}, V)$ in $W^{1,p}(\Omega, V)$, we may extend T continuously to a bounded linear extension map $T : W^{1,p}(\Omega, V) \rightarrow W^{1,p}(\Omega', V)$.

It now remains to show that Tu satisfies (79) if $\mathcal{F}(u) < +\infty$. Again, we exploit that the mapping $x \mapsto x - t_x \delta \nu(P_\Gamma x)$ is bi-Lipschitz, in tandem with Assumption 4, to give that

$$\begin{aligned} &\int_{\Omega'} F(T\nabla u(x), Tu(x), x) dx \\ &= \int_{\Omega} F(\nabla u(x), u(x), x) dx + \int_{\Omega' \setminus \Omega} F(\nabla Tu(x), Tu(x), x) dx \\ &\leq \int_{\Omega} F(\nabla u(x), u(x), x) dx + C \int_{\Omega \setminus \Omega^\delta} F(\nabla u(x), u(x), x) + 1 dx \\ &\leq (C+1) \left(\int_{\Omega} F(\nabla u(x), u(x), x) dx + |\Omega| \right) \end{aligned} \quad (84)$$

so that

$$\int_{\Omega'} F(\nabla(Tu)(x), Tu(x), x) dx < \infty. \quad (85)$$

□

Proposition 2.9 (Limsup). Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1, 2, F, w satisfy Assumptions 4, 3, and 5. Let $u \in W^{1,p}(\Omega, V)$. Then there exists a sequence $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ such that $u_\varepsilon \xrightarrow{R} u$ and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}(u). \quad (86)$$

Proof. Let $T : W^{1,p}(\Omega, V) \rightarrow W^{1,p}(\Omega', V)$, as in Proposition 2.8. We define $u_\varepsilon = R_{\Omega_\varepsilon} Tu$, and note that this implies that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$. First we must show that $u_\varepsilon \xrightarrow{R} u$. It is immediate that $u_\varepsilon|_U = u|_U$ for all $U \subset \Omega \cap \Omega_\varepsilon$, which consequently implies that $u_\varepsilon|_U, \nabla u_\varepsilon|_U$ converge strongly in $L^p(U)$ to $u, \nabla u$ respectively. As the extensions by zero of $u_\varepsilon, \nabla u_\varepsilon$ may only disagree with $u, \nabla u$ on $\Omega_\varepsilon \setminus \Omega$, we see that

$$\|E_D u_\varepsilon - E_D u\|_{L^p(D, V)} = \left(\int_{(\Omega_\varepsilon \setminus \Omega) \cup (\Omega \setminus \Omega_\varepsilon)} |Tu(x)|^p dx \right)^{\frac{1}{p}}, \quad (87)$$

which as $|\Omega_\varepsilon \setminus \Omega| + |\Omega \setminus \Omega_\varepsilon| \rightarrow 0$ and $\|Tu\|_{L^p(\Omega', V)} < \infty$,

$$\lim_{\varepsilon \rightarrow 0} \int_{(\Omega_\varepsilon \setminus \Omega) \cup (\Omega \setminus \Omega_\varepsilon)} |Tu(x)|^p dx = 0. \quad (88)$$

The same argument applies to show that $E_D \nabla u_\varepsilon \rightarrow E_D \nabla u$. Therefore $u_\varepsilon \xrightarrow{R} u$

Moreso, $x \mapsto F(\nabla Tu(x), Tu(x), x)$ is an integrable function on Ω' . As $\mathcal{L}^N((\Omega_\varepsilon \setminus \Omega) \cup (\Omega \setminus \Omega_\varepsilon)) \rightarrow 0$, this therefore implies that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} F(\nabla u_\varepsilon(x), u_\varepsilon(x), x) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} F(\nabla Tu(x), Tu(x), x) dx \\ &= \int_{\Omega} F(\nabla Tu(x), Tu(x), x) dx \\ &= \int_{\Omega} F(\nabla u(x), u(x), x) dx. \end{aligned} \quad (89)$$

Finally, as $u_\varepsilon \xrightarrow{R} u$, by Proposition 2.6 we have that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} w(P_\Gamma x, \nu_\varepsilon(x), u_\varepsilon(x)) dS(x) = \int_{\Gamma} w_h(x, u(x)) dS(x). \quad (90)$$

□

Proposition 2.10 (Fundamental of Γ -convergence on varying domains). Let $\Omega, \Omega_\varepsilon$ satisfy Assumptions 1, 2, F, w satisfy Assumptions 4, 3, and 5. Then if $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ are minimisers \mathcal{F}_ε for every $\varepsilon > 0$, there exists a subsequence $\varepsilon_j \rightarrow 0$ and minimiser $u \in W^{1,p}(\Omega, V)$ of \mathcal{F} such that $u_{\varepsilon_j} \xrightarrow{R} u$.

Proof. The proof follows exactly according to that of [14].

We begin by noting that there exists $u \in W^{1,p}(\Omega, V)$ such that $\mathcal{F}(u) < +\infty$. This holds as by Assumption 4, there exists u such that

$$\int_{\Omega} F(\nabla u(x), u(x), x) dx < +\infty, \quad (91)$$

and we note that by the estimates in Corollary 2.1 and the compact embedding $W^{1,p}(\Omega, V) \hookrightarrow L^q(\Gamma, V)$, we must also have that

$$\int_{\Gamma} w_h(x, u(x)) dx < +\infty, \quad (92)$$

so that $\mathcal{F}(u) < +\infty$. By taking a recovery sequence $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$ so that $u_\varepsilon \xrightarrow{R} u$, which exists via Proposition 2.9, we have that $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \rightarrow \mathcal{F}(u) < +\infty$, and consequently it must hold that $\limsup_{\varepsilon \rightarrow 0} \inf_{W^{1,p}(\Omega_\varepsilon, V)} \mathcal{F}_\varepsilon < +\infty$ also. Therefore minimisers u_ε^* of \mathcal{F}_ε must satisfy the bound that $\mathcal{F}_\varepsilon(u_\varepsilon^*) \leq M$ for all $\varepsilon_0 > \varepsilon > 0$ and some $M > 0$. Thus by 2.1, there exists a subsequence $\varepsilon_j \rightarrow 0$ and $u^* \in W^{1,p}(\Omega, V)$ so that $u_{\varepsilon_j}^* \xrightarrow{R} u^*$. By 2.7, thus

$$\liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}^*) \geq \mathcal{F}(u^*). \quad (93)$$

Now, take any $u \in W^{1,p}(\Omega, V)$. By 2.9, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ so that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, V)$, $u_\varepsilon \xrightarrow{R} u$, and $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}(u)$. Then, as u_ε^* are minimisers of \mathcal{F}_ε by assumption, we have that

$$\mathcal{F}(u) = \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \limsup_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) \geq \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}^*) \geq \mathcal{F}(u^*). \quad (94)$$

Therefore u^* is a minimiser of \mathcal{F} . \square

3 Applications to liquid crystals models

3.1 Oseen-Frank

We now turn to the cases of surface energies in models for nematic liquid crystals. First we consider the Oseen-Frank model, where $V = \mathbb{R}^3$, although finite-energy states $n \in W^{1,2}(\Omega, V)$ satisfy $|n(x)| = 1$ almost everywhere, as described in Remark 1.1. The Rapini-Papoular surface energy is given by

$$w(\nu, n) = \frac{w_0}{2} (n \cdot \nu)^2. \quad (95)$$

The sign of w_0 produces different behaviour of the surface energy, although it does not change the existence of a lower bound, as $|n(x)| = 1$ almost everywhere for finite energy n . For $w_0 < 0$, the energy corresponds to *homeotropic anchoring*, and is minimised when $n \cdot \nu = \pm 1$, whereby molecules prefer to be aligned perpendicularly to the surface. On the other hand, $w_0 > 0$ corresponds to *planar degenerate anchoring*, in which n prefers to lie in the plane such that $\nu \cdot n = 0$. In either case, we may write the energy, ignoring additive constants, as

$$w(\nu, n) = \frac{w_0}{2} (n \otimes n) \cdot (\nu \otimes \nu). \quad (96)$$

In particular, following Proposition 2.3 we see that the homogenised surface energy is described entirely by a single 3×3 symmetric tensor $A_{ef} : \Gamma \rightarrow \mathbb{R}^{3 \times 3}$ given by,

$$A_{ef}(x) = w_0 \int_{\nu(x)+T_x\Gamma} \frac{1}{|v|} v \otimes v d\mu_x(v), \quad (97)$$

from which the homogenised surface energy can be written as

$$w_h(x, n) = \frac{1}{2} A_{ef} n \cdot n. \quad (98)$$

In particular, from its definition we see that A_{ef} is a symmetric 3×3 matrix and thus admits a spectral decomposition $A_{ef} = \sum_{i=1}^3 \lambda_i e_i \otimes e_i$ at each point $x \in \Gamma$. As the trace is a linear function, we may take the trace before or after the integral to obtain

$$\frac{1}{w_0} \text{Tr}(A_{ef}) = \int_{\nu(x)+T_x\Gamma} |v| d\mu_x(v) \geq 1, \quad (99)$$

and similarly for any $e \in \mathbb{S}^2$,

$$\frac{1}{w_0} A_{ef} e \cdot e = \int_{\nu(x)+T_x\Gamma} (e \cdot v)^2 |v|^{-1} d\mu_x(v) \in [0, \frac{1}{w_0} \text{Tr}(A_{ef})]. \quad (100)$$

We see that there are four possibilities for A_{ef} , qualitatively speaking. We order the eigenvalues such that $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

1. $\lambda_1 = \lambda_2 < \lambda_3$. In this case, the energy behaves as a degenerate anchoring in the e_1, e_2 plane with weight $\lambda_3 - \lambda_2$. We may write the limiting surface energy as $A_{ef}n \cdot n = (\lambda_3 - \lambda_2)(e_3 \cdot n)^2 + \lambda_2$. We note that the latter term is n independent, and $(\lambda_3 - \lambda_2)$ is positive.
2. $\lambda_1 < \lambda_2 = \lambda_3$. In this case, similarly to the previous case, we have a simple preferred axis e_1 giving weak homeotropic anchoring with weight $\lambda_1 - \lambda_3$, and the energy may be written as $A_{ef}n \cdot n = (\lambda_1 - \lambda_3)(e_1 \cdot n)^2 + \lambda_3$, where we note that the coefficient $\lambda_1 - \lambda_3$ is negative.
3. $\lambda_1 = \lambda_2 = \lambda_3$. This reduces to no anchoring condition on n , as $A_{ef}n \cdot n = \lambda_1$ for all n .
4. $\lambda_1 < \lambda_2 < \lambda_3$. Here we have a preferred axis e_1 , however the energy is not rotationally symmetric around e_1 , with deviations in the e_2 being preferable to those in the e_3 direction. Finally, we note that up to a constant w_h is uniquely defined only by to the traceless part of A_{ef} , by writing

$$A_{ef} = \left(A - \frac{\text{Tr}(A_{ef})}{3} I \right) + \frac{\text{Tr}(A_{ef})}{3} I = A_{ef}^\circ + \frac{\text{Tr}(A_{ef})}{3} I, \quad (101)$$

we see that

$$w_h(x, n) = A_{ef}^\circ n \cdot n + \frac{1}{3} \text{Tr}(A_{ef}), \quad (102)$$

where the latter term is independent of n and thus does not affect minimisers.

3.2 Landau-de Gennes

The simplest surface energy for Landau-de Gennes, in which $V = \text{Sym}_0(3)$, the traceless symmetric matrices, is given as

$$w(x, \nu, Q) = \frac{w_0}{2} \left| Q - s_0 \left(\nu \otimes \nu - \frac{1}{3} I \right) \right|^2. \quad (103)$$

The constant $w_0 > 0$ is the anchoring strength, clearly a larger value of w_0 corresponds to a higher penalisation for deviation from the preferred state. This energy is minimised at $Q = s_0 \left(\nu \otimes \nu - \frac{1}{3} I \right)$, meaning that molecules generally want to lie parallel to the surface if $s_0 > 0$ or perpendicularly if $s_0 < 0$. Similarly to the Oseen-Frank model, we can investigate the behaviour of the model in the limit using Proposition 2.3, first by expanding the energy as

$$w(x, \nu, Q) = \frac{w_0}{2} |Q|^2 - w_0 s_0 Q \cdot \left(\nu \otimes \nu - \frac{1}{3} I \right) + \frac{w_0 s_0^2}{3}. \quad (104)$$

Then we define the effective anchoring strength w_{ef} and the preferred direction Q_{ef} as

$$\begin{aligned} w_{ef}(x) &= w_0 \int_{\nu(x) + T_x \Gamma} |v| d\mu_x, \\ Q_{ef}(x) &= \frac{s_0}{w_{ef}(x)} \int_{\nu(x) + T_x \Gamma} \left(\frac{1}{|v|^2} v \otimes v - \frac{1}{3} I \right) |v| d\mu_x(v). \end{aligned} \quad (105)$$

Then we may write the homogenised surface energy as

$$\begin{aligned} w_h(x, Q) &= \int_{\nu(x) + T_x \Gamma} \left(\frac{w_0}{2} |Q|^2 - w_0 s_0 Q \cdot \left(\frac{1}{|v|^2} v \otimes v - \frac{1}{3} I \right) + \frac{2w_0 s_0^2}{3} \right) |v| d\mu_x(v) \\ &= \frac{w_{ef}}{2} |Q - Q_{ef}|^2 + R, \end{aligned} \quad (106)$$

where R is a remainder term that depends only on x and not Q , and thus does not affect minimisers. We observe that the homogenised surface energy is thus of the same quadratic

form, but Q_{ef} plays the role of the preferred direction, and w_{ef} as the new anchoring strength. As $|v| \geq 1$ for all $v \in \nu(x) + T_x \Gamma$, by the integral form of w_{ef} we see that $w_{ef} > w_0$. That is, the anchoring strength can only increase. Similarly, if f is any frame indifferent convex function, we have that for any $e \in \mathbb{S}^2$,

$$f\left(s_0 \left(e_1 \otimes e_1 - \frac{1}{3}I\right)\right) = g_f(s_0) \quad (107)$$

for some function $g_f : \mathbb{R} \rightarrow \mathbb{R}$. By applying Jensen's inequality, we see that

$$\begin{aligned} f(Q_{ef}(x)) &= f\left(\frac{s_0}{\int_{\nu(x)+T_x\Gamma} |v| d\mu_x(v)} \int_{\nu(x)+T_x\Gamma} \left(\frac{1}{|v|^2} v \otimes v - \frac{1}{3}I\right) |v| d\mu_x(v)\right) \\ &\leq \frac{1}{\int_{\nu(x)+T_x\Gamma} |v| d\mu_x(v)} \int_{\nu(x)+T_x\Gamma} f\left(s_0 \left(\frac{1}{|v|^2} v \otimes v - \frac{1}{3}I\right)\right) |v| d\mu_x(v) \\ &\leq \frac{1}{\int_{\nu(x)+T_x\Gamma} |v| d\mu_x(v)} \int_{\nu(x)+T_x\Gamma} g_f(s_0) |v| d\mu_x(v) = g_f(s_0). \end{aligned} \quad (108)$$

In particular, taking $f(Q) = \lambda_{\max}(Q)$, $f(Q) = -\lambda_{\min}(Q)$, and $f(Q) = |Q|$, we see that $\lambda_{\max}(Q_{ef}) \leq \frac{2s_0}{3}$, $\lambda_{\min}(Q_{ef}) \geq -\frac{s_0}{3}$, and $|Q| \leq s_0 \sqrt{\frac{2}{3}}$.

4 Error estimates in periodic media

4.1 Introduction of the problem

Consider an ℓ -periodic slab domain in two-dimensions, which represents the typical geometry of liquid crystal experiments, given explicitly as

$$\Omega'_\varepsilon = \{(x', y') \in \mathbb{R}^2 : \varphi'_\varepsilon(x') < y' < R'\},$$

in which $\varphi'_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is an ℓ -periodic function and let $\partial\Omega'_\varepsilon = \Gamma'_\varepsilon \cup \Gamma'_R$, where we denote $\Gamma'_\varepsilon = \{(x', \varphi'_\varepsilon(x')) : x' \in \mathbb{R}\}$ and $\Gamma'_R = \{(x', R') : x' \in \mathbb{R}\}$. We consider a toy model, representative of paranematic systems as in [13, 26, 54], over $Q' \in \text{Sym}_0(2) = \{A \in \mathbb{R}^{2 \times 2} : A^T = A, \text{Tr}(A) = 0\}$. We consider solutions that respect the symmetry of the domain, that is, $Q' \in W_{loc}^{1,2}(\Omega'_\varepsilon, \text{Sym}_0(2))$ such that $Q'(x' + \ell, y') = Q'(x', y')$ for almost every x', y' . The free energy per periodic cell, $C\Omega'_\varepsilon$, is to be given as

$$\mathcal{F}'_\varepsilon(Q') = \int_{C\Omega'_\varepsilon} \frac{L'}{2} |\nabla Q'|^2 + \frac{c'}{2} |Q'|^2 dx' dy' + \int_{C\partial\Omega'_\varepsilon} \frac{w'_0}{2} \left| Q' - s'_0 \left(\nu \otimes \nu - \frac{1}{2}I \right) \right|^2 d\sigma(x').$$

Here $C\Omega'_\varepsilon = \{(x', y') \in \mathbb{R}^2 : \varphi'_\varepsilon(x') < y' < R', 0 \leq x' < \ell\}$, $c' > 0$, $C\partial\Omega'_\varepsilon = C\Gamma'_\varepsilon \cup C\Gamma'_R$, with $C\Gamma'_\varepsilon = \{(x', \varphi'_\varepsilon(x')) : x' \in [0, \ell)\}$ and $C\Gamma'_R = \{(x', R') : x' \in [0, \ell)\}$, $w'_0 > 0$, $s'_0 \in \mathbb{R}$ and ν is the exterior normal. We may non-dimensionalise the system by considering variables $(x, y) = \frac{2\pi}{\ell}(x', y')$, $Q(x, y) = \frac{1}{s'_0} Q'(x', y')$, $\mathcal{F}_\varepsilon = \frac{1}{L'(s'_0)^2} \mathcal{F}'_\varepsilon$, $c = \frac{c'\ell^2}{4L'\pi^2}$, $w_0 = \frac{w'_0\ell}{2L'\pi}$, $R = \frac{2\pi}{\ell} R'$, $\varphi_\varepsilon(x) = \varphi'_\varepsilon(x')$ to give

$$\mathcal{F}_\varepsilon(Q) = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 dx dy + \int_{\partial\Omega_\varepsilon} \frac{w_0}{2} \left| Q - \left(\nu \otimes \nu - \frac{1}{2}I \right) \right|^2 d\sigma(x), \quad (109)$$

with rescaled domain

$$\Omega_\varepsilon = \{(x, y) : 0 \leq x < 2\pi, \varphi_\varepsilon(x) < y < R\}.$$

Moreover, we can write $\partial\Omega_\varepsilon = \Gamma_\varepsilon \cup \Gamma_R$, where $\Gamma_\varepsilon = \{(x, y) : x \in [0, 2\pi), y = \varphi_\varepsilon(x)\}$ and $\Gamma_R = \{(x, R) : x \in [0, 2\pi)\}$, so that we have:

$$\mathcal{F}_\varepsilon(Q) = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 \, dx \, dy + \int_{\Gamma_\varepsilon} \frac{w_0}{2} |Q - Q_\varepsilon^0|^2 \, d\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2} |Q - Q_R|^2 \, d\sigma_R, \quad (110)$$

with $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$ and $Q_R = \nu_R \otimes \nu_R - \frac{1}{2}I$, where ν_ε and ν_R are the outward normals to Γ_ε and Γ_R .

We also consider the limit domain

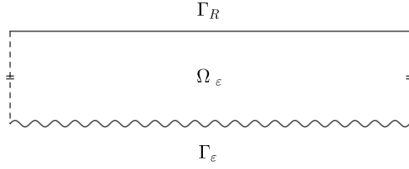
$$\Omega_0 = \{(x, y) : x \in [0, 2\pi), 0 < y < R\}$$

with $\partial\Omega_0 = \Gamma_0 \cup \Gamma_R$, where $\Gamma_0 = \{(x, 0) : x \in [0, 2\pi)\}$.

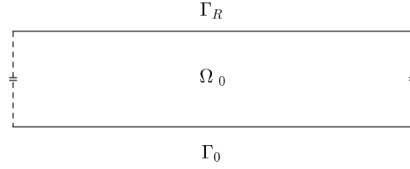
4.2 Technical assumptions and main result

Assumption 6. Let $\varepsilon > 0$. We assume that $\varphi_\varepsilon(x) = \varepsilon \cdot \varphi(x/\varepsilon)$, where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 2π -periodic function with $\varphi \geq 0$.

Remark 4.1. Using Assumption 6, we obtain that $\Omega_\varepsilon \subset \Omega_0$ for all $\varepsilon > 0$ and that $\Omega_\varepsilon \rightarrow \Omega_0$ as $\varepsilon \rightarrow 0$.



(a) The oscillating domain Ω_ε .



(b) The limit domain Ω_0 .

Remark 4.2. In the next subsections, we write the first derivative of φ as φ' . Moreover, we write $\|\varphi\|_\infty$ and $\|\varphi'\|_\infty$ instead of $\|\varphi\|_{L^\infty([0, 2\pi])}$ and $\|\varphi'\|_{L^\infty([0, 2\pi])}$.

Assumption 7. We assume that:

$$0 < \varepsilon = \frac{1}{2k} < \|\varphi\|_\infty^{-1} \cdot \frac{R}{2}, \text{ with } k \in \mathbb{N}^*, k > \|\varphi\|_\infty \cdot \frac{1}{R}.$$

Remark 4.3. Using Assumption 7, we obtain that $\{(x, y) \mid x \in [0, 2\pi), y \in (R/2, R)\} \subset \Omega_\varepsilon$, which tells us that the oscillations of Γ_ε have an amplitude lower than half of the height of the domain Ω_0 .

Remark 4.4. Using Assumption 6, the arclength parameter of the curve Γ_ε can be described as the function $\gamma_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$, defined as:

$$\gamma_\varepsilon(t) = \sqrt{1 + (\varphi'(t))^2}, \quad \forall t \in \mathbb{R} \quad (111)$$

and we obtain that:

$$1 \leq \gamma_\varepsilon(t) < \sqrt{1 + \|\varphi'\|_\infty^2}, \quad \forall t \in \mathbb{R}. \quad (112)$$

Moreover, the outward normal ν_ε to Γ_ε has the following form:

$$\nu_\varepsilon := \nu_\varepsilon(x/\varepsilon) = \frac{1}{\gamma_\varepsilon(x/\varepsilon)} (\varphi'(x/\varepsilon), -1), \quad (113)$$

for all $x \in [0, 2\pi)$.

Definition 4.1. Let $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ be two real functions defined as

$$g_1(t) := \frac{(\varphi'(t))^2 - 1}{2(1 + (\varphi'(t))^2)} \quad \text{and} \quad g_2(t) := \frac{-2\varphi'(t)}{2(1 + (\varphi'(t))^2)}$$

for all $t \in \mathbb{R}$ and let $\gamma, G_1, G_2 \in \mathbb{R}$ be defined as:

$$\gamma := \frac{1}{2\pi} \int_0^{2\pi} \gamma_\varepsilon(t) dt, \quad G_1 := \frac{1}{2\pi} \int_0^{2\pi} g_1(t) \cdot \gamma_\varepsilon(t) dt, \quad G_2 := \frac{1}{2\pi} \int_0^{2\pi} g_2(t) \cdot \gamma_\varepsilon(t) dt.$$

Remark 4.5. Since $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$ and $\nu_\varepsilon = \nu_\varepsilon(x/\varepsilon)$, then $Q_\varepsilon^0(x/\varepsilon) = \begin{pmatrix} g_1(x/\varepsilon) & g_2(x/\varepsilon) \\ g_2(x/\varepsilon) & -g_1(x/\varepsilon) \end{pmatrix}$, for all $x \in [0, 2\pi)$. Moreover, γ, G_1 and G_2 are constants and $\gamma \geq 1$.

Definition 4.2. Let $w_{ef} := \gamma w_0$ and $Q_{ef} := \frac{1}{\gamma} \begin{pmatrix} G_1 & G_2 \\ G_2 & -G_1 \end{pmatrix}$.

Remark 4.6. In this simplified model, $w_{ef} \in \mathbb{R}$ is constant and, since $\gamma \geq 1$, we have $w_{ef} \geq w_0$ (as previously observed in Section 3.2. Moreover, $Q_{ef} \in \text{Sym}_0(2)$ is a constant Q -tensor.

Proposition 4.1. We have $\gamma_\varepsilon(\cdot) \rightarrow \gamma$ and $\gamma_\varepsilon(\cdot/\varepsilon)Q_\varepsilon^0(\cdot/\varepsilon) \rightarrow \gamma Q_{ef}$ in $L^2([0, 2\pi))$, as $\varepsilon \rightarrow 0$.

Proof. Since φ and φ' are 2π -periodic, according to Assumption 6, and $\varepsilon^{-1} \in \mathbb{N}^*$, according to Assumption 7, then the functions $\gamma_\varepsilon, g_1\gamma_\varepsilon$ and $g_2\gamma_\varepsilon$ are also 2π -periodic and they tend to their mean as $\varepsilon \rightarrow 0$ (see for instance [20, Lemma 9.1]). This implies the conclusion. \square

Definition 4.3. Let Q_ε be a minimiser of the functional \mathcal{F}_ε . This implies that Q_ε verifies the following Euler-Lagrange equations:

$$\begin{cases} -\Delta Q_\varepsilon + cQ_\varepsilon = 0 & \text{in } \Omega_\varepsilon; \\ \frac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} + \frac{w_0}{2} Q_\varepsilon = \frac{w_0}{2} Q_\varepsilon^0 & \text{on } \Gamma_\varepsilon; \\ \frac{\partial Q_\varepsilon}{\partial \nu_R} + \frac{w_0}{2} Q_\varepsilon = \frac{w_0}{2} Q_R & \text{on } \Gamma_R. \end{cases} \quad (114)$$

Remark 4.7. We prove in Section 4.3 that there exists a unique $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ solution of (114), for any $1 < p < +\infty$.

Definition 4.4. Let $\nu_0(\cdot, 0) = (0, -1)$ be the outward normal of Γ_0 . We consider the following PDE:

$$\begin{cases} -\Delta Q_0 + cQ_0 = 0 & \text{in } \Omega_0; \\ \frac{\partial Q_0}{\partial \nu_0} + \frac{w_{ef}}{2} Q_0 = \frac{w_{ef}}{2} Q_{ef} & \text{on } \Gamma_0; \\ \frac{\partial Q_0}{\partial \nu_R} + \frac{w_0}{2} Q_0 = \frac{w_0}{2} Q_R & \text{on } \Gamma_R. \end{cases} \quad (115)$$

Remark 4.8. In Section 4.3, we prove that we have a unique $Q_0 \in W^{2,p}(\Omega_0)$ solution of (115), for any $p \in (1, +\infty)$.

Remark 4.9. It is easy to see that under these assumptions, we have $Q_\varepsilon \xrightarrow{R} Q_0$ in the sense from Definition 2.5. However, in this simplified case, we can obtain *quantitative estimates*, which are presented in the main theorem of this section:

Theorem 4.1. For any $p \in (2, +\infty)$, there exists an ε -independent constant C such that:

$$\|Q_0 - Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \cdot \varepsilon^{\frac{p-1}{p}}, \quad (116)$$

where the constant C depends on $c, w_0, p, \|\varphi\|_\infty, \|\varphi'\|_\infty, \Omega_0$ and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

Remark 4.10. The constant C from Theorem 4.1 can actually be chosen of the following form:

$$C = \max\{1, \|\varphi\|_\infty^{(p-1)/p}\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C(w_0, c, p, \Omega_0, Q_0),$$

where $C(w_0, c, p, \Omega_0, Q_0)$ is an ε -independent constant depending only on w_0, c, p, Ω_0 and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

4.3 Regularity of Q_ε and Q_0

In this subsection, we prove that there exists a unique solution Q_ε of (114) and a unique solution Q_0 of (115), with $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ and $Q_0 \in W^{2,p}(\Omega_0)$, for any $p \in (1, +\infty)$.

Remark 4.11. It is easy to see that problems (114) and (115) admit solutions from $W^{1,2}(\Omega_0)$ and $W^{1,2}(\Omega_\varepsilon)$ by using, for example, direct methods, such as in [21], or the approach via Lax-Milgram theorem for elliptic problems, such as in [24] or [29].

Definition 4.5. We denote by Φ_{polar} the polar coordinates transform:

$$\Phi_{polar}(x, y) = (y \cos x, y \sin x), \quad \forall (x, y) \in [0, 2\pi) \times (0, R),$$

by Φ_{tl} the following translation:

$$\Phi_{tl}(x, y) = (x, y + 2R), \quad \forall (x, y) \in [0, 2\pi) \times (0, R)$$

and let $\Phi : \Omega_0 \rightarrow \Phi(\Omega_0)$ be defined as $\Phi = \Phi_{polar} \circ \Phi_{tl}$. We define

$$\mathcal{U}_\varepsilon = \Phi(\Omega_\varepsilon) \quad \text{and} \quad \mathcal{U}_0 = \Phi(\Omega_0).$$

Definition 4.6. For any $a, b \in \mathbb{R}$ with $a, b > 0$, we denote

$$A_{a,b} = \{(y \cos x, y \sin x) \mid x \in [0, 2\pi), y \in (a, b)\}.$$

Remark 4.12. The transformation $\Phi : \Omega_0 \rightarrow \mathcal{U}_0$ is smooth and bi-Lipschitz. Moreover, using Assumption 6, we have that \mathcal{U}_ε is a bounded open domain from \mathbb{R}^2 with a C^2 boundary and, using Assumption 7, we have that $A_{5R/2, 3R} \subset \mathcal{U}_\varepsilon \subset \mathcal{U}_0 = A_{2R, 3R}$.

In order to prove that Q_ε and Q_0 admit $W^{2,p}$ regularity, we use [29, Theorem 2.4.2.6]:

Theorem 4.2. Let Ω be a bounded open subset of \mathbb{R}^n , with a $C^{1,1}$ boundary. Let $a_{i,j}$ and b_i be uniformly Lipschitz functions in $\bar{\Omega}$ and let a_i be bounded measurable functions in $\bar{\Omega}$. Assume that $a_{i,j} = a_{j,i}$, $1 \leq i, j \leq n$ and that there exists $\alpha > 0$ with

$$\sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \leq -\alpha |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \bar{\Omega}$. Assume in addition that $a_0 \geq \beta > 0$ a.e. in Ω and that

$$b_0 b_\nu = b_0 \sum_{j=1}^n b_j \nu^j \geq 0, \quad b_\nu \neq 0$$

on $\Gamma = \partial\Omega$. Then for every $f \in L^p(\Omega)$ and every $g \in W^{1-1/p,p}(\Gamma)$, there exists a unique $u \in W^{2,p}(\Omega)$, which is a solution of

$$\begin{cases} \sum_{i,j=1}^n D_i(a_{i,j}D_j u) + \sum_{i=1}^n a_i D_i u + a_0 u = f & \text{in } \Omega \\ \text{Tr} \left(\sum_{j=1}^n b_j D_j u + b_0 u \right) = g & \text{on } \Gamma \end{cases}$$

where Tr is the trace operator.

Corollary 4.1. For any $p \in (1, +\infty)$, there exists a unique $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ which solves the problem (114).

The proof of this corollary can be found in Appendix B. In a similar way, we can show that:

Corollary 4.2. For any $p \in (1, +\infty)$, there exists a unique $Q_0 \in W^{2,p}(\Omega_0)$ which solves the problem (115).

Remark 4.13. Using the method of separation of variables, one can find that Q_0 is of the form:

$$Q_0(x, y) = c_1 \cdot e^{y\sqrt{c}} + c_2 \cdot e^{-y\sqrt{c}},$$

where c_1 and c_2 are two constant Q -tensors that can be found from:

$$\begin{cases} c_1 \cdot \left(\frac{w_{ef}}{2} - \sqrt{c} \right) + c_2 \cdot \left(\frac{w_{ef}}{2} + \sqrt{c} \right) = \frac{w_{ef}}{2} \cdot Q_{ef}; \\ c_1 \cdot e^{R\sqrt{c}} \cdot \left(\frac{w_0}{2} + \sqrt{c} \right) + c_2 \cdot e^{-R\sqrt{c}} \cdot \left(\frac{w_0}{2} - \sqrt{c} \right) = \frac{w_0}{2} \cdot Q_R. \end{cases}$$

4.4 Some integral inequalities

Definition 4.7. Let $p \in (1, +\infty)$. Let us consider the trace operator $\text{Tr} : W^{1,p}(\Omega_0) \rightarrow W^{1-1/p,p}(\partial\Omega_0)$. We denote $C_{tr}(p, \Omega_0)$ the constant given by the trace inequality, that is:

$$\|\text{Tr}(\omega)\|_{W^{1-1/p,p}(\partial\Omega_0)} \leq C_{tr}(p, \Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_0)}, \quad \forall \omega \in W^{1,p}(\Omega_0).$$

Remark 4.14. For notation simplicity, we choose to write $v(\cdot, 0)$ instead of $[\text{Tr}(v)]|_{\Gamma_0}(\cdot, 0)$, whenever $v \in W^{1,p}(\Omega_0)$.

Definition 4.8. We consider the following bilinear functional on $W^{1,2}(\Omega_\varepsilon) \times W^{1,2}(\Omega_\varepsilon)$:

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + c \cdot u \cdot v) dx + \frac{w_0}{2} \left(\int_{\Gamma_\varepsilon} u \cdot v d\sigma_\varepsilon + \int_{\Gamma_R} u \cdot v d\sigma_R \right),$$

for any $u, v \in W^{1,2}(\Omega_\varepsilon)$.

Remark 4.15. For notation simplicity, whenever we choose $v \in W^{1,2}(\Omega_0)$, we write $a_\varepsilon(\cdot, v)$ instead of $a_\varepsilon(\cdot, v|_{\Omega_\varepsilon})$.

The goal of this subsection is to prove the following proposition:

Proposition 4.2. Let $p \in (2, +\infty)$. Then there exists an ε -independent constant C_I such that:

$$|a_\varepsilon(Q_0 - Q_\varepsilon, v)| \leq C_I \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)}, \quad \forall v \in W^{1,p}(\Omega_0).$$

Remark 4.16. In Appendix B, we prove that $a_\varepsilon(\cdot, v)$ is well-defined for all $v \in W^{1,p}(\Omega_0)$. Moreover, in order to obtain Proposition 4.2, we split $a_\varepsilon(Q_0 - Q_\varepsilon, v)$ into several parts, which are presented in the following definition.

Definition 4.9. Let $v \in W^{1,p}(\Omega_0)$. We denote:

$$\begin{aligned}
I_1 &= - \int_{\Omega_0 \setminus \Omega_\varepsilon} (\nabla Q_0 \cdot \nabla v + c \cdot Q_0 \cdot v) d(x, y), \\
I_{21} &= - \frac{w_0}{2} \int_0^{2\pi} (v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)) \cdot Q_\varepsilon^0(x/\varepsilon) \cdot \gamma_\varepsilon(x/\varepsilon) dx, \\
I_{31} &= \frac{w_0}{2} \int_0^{2\pi} (Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0) \cdot v(x, 0)) \cdot \gamma_\varepsilon(x/\varepsilon) dx, \\
I_{22} &= \frac{w_0}{2} \int_0^{2\pi} v(x, 0) \cdot \left(\gamma Q_{ef} - \gamma_\varepsilon(x/\varepsilon) Q_\varepsilon^0(x/\varepsilon) \right) dx, \\
I_{32} &= - \frac{w_0}{2} \int_0^{2\pi} Q_0(x, 0) \cdot v(x, 0) \cdot \left(\gamma - \gamma_\varepsilon(x/\varepsilon) \right) dx.
\end{aligned}$$

Proposition 4.3. We have

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 + I_{21} + I_{22} + I_{31} + I_{32}, \quad \forall v \in W^{1,p}(\Omega_0). \quad (117)$$

Remark 4.17. The proof of Proposition 4.3 can be found in Appendix B. Before proving Proposition 4.2, we obtain first estimates for each of the integrals from Definition 4.9.

Proposition 4.4. Let $p \in (1, +\infty)$. Then for any $v \in W^{1,p}(\Omega_0)$, we have:

$$|I_1| \leq C_1 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0 \setminus \Omega_\varepsilon)}, \quad \forall v \in W^{1,p}(\Omega_0),$$

where

$$I_1 = - \int_{\Omega_0 \setminus \Omega_\varepsilon} (\nabla Q_0 \cdot \nabla v + c \cdot Q_0 \cdot v) d(x, y).$$

and

$$C_1 = (2\pi \|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \max\{c, 1\}. \quad (118)$$

Proof. We apply Hölder inequality with coefficients p and $p' = \frac{p}{p-1}$:

$$\begin{aligned}
|I_1| &\leq \int_{\Omega_0 \setminus \Omega_\varepsilon} |\nabla Q_0 \cdot \nabla v| + c |Q_0 \cdot v| d(x, y) \\
&\leq \left(\int_{\Omega_0 \setminus \Omega_\varepsilon} |\nabla Q_0|^{p'} d(x, y) \right)^{1/p'} \cdot \left(\int_{\Omega_0 \setminus \Omega_\varepsilon} |\nabla v|^p d(x, y) \right)^{1/p} + \\
&\quad + c \cdot \left(\int_{\Omega_0 \setminus \Omega_\varepsilon} |Q_0|^{p'} d(x, y) \right)^{1/p'} \cdot \left(\int_{\Omega_0 \setminus \Omega_\varepsilon} |v|^p d(x, y) \right)^{1/p} \\
&\leq |\Omega_0 \setminus \Omega_\varepsilon|^{1/p'} \left(\|\nabla Q_0\|_{L^\infty(\Omega_0 \setminus \Omega_\varepsilon)} \cdot \|\nabla v\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)} + c \cdot \|Q_0\|_{L^\infty(\Omega_0 \setminus \Omega_\varepsilon)} \cdot \|v\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)} \right).
\end{aligned}$$

We have:

$$\begin{aligned}
|\Omega_0 \setminus \Omega_\varepsilon| &= \int_{\Omega_0 \setminus \Omega_\varepsilon} 1 dx = \int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} 1 dy dx \\
&= \int_0^{2\pi} \varepsilon\varphi(x/\varepsilon) dx \leq \varepsilon \cdot 2\pi \|\varphi\|_\infty.
\end{aligned}$$

In the end, we obtain that:

$$|I_1| \leq C_1 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0 \setminus \Omega_\varepsilon)},$$

where

$$C_1 = (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \max\{c, 1\}.$$

□

For the integrals I_{21} and I_{31} , from Definition 4.9, we prove first the following lemma:

Lemma 4.1. Let $1 < q < p < +\infty$ and $\omega \in W^{1,p}(\Omega_0)$. Then:

$$\left(\int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q dx \right)^{1/q} \leq C_{2,q} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}, \quad (119)$$

with

$$C_{2,q} = (2\pi)^{\frac{p-q}{pq}} \|\varphi\|_\infty^{\frac{p-1}{p}}. \quad (120)$$

Proof. We prove the result first for $C^1(\overline{\Omega_0})$ functions.

Let $\omega \in C^1(\overline{\Omega_0})$. Then we have the following inequality:

$$|\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)| \leq \int_0^{\varepsilon\varphi(x/\varepsilon)} \left| \frac{\partial\omega}{\partial y}(x, t) \right| dt,$$

for any $x \in [0, 2\pi]$.

We apply Hölder inequality with exponents q and $q' = \frac{q}{q-1}$:

$$\begin{aligned} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)| &\leq \left(\int_0^{\varepsilon\varphi(x/\varepsilon)} \left| \frac{\partial\omega}{\partial y}(x, t) \right|^q dt \right)^{1/q} \left(\int_0^{\varepsilon\varphi(x/\varepsilon)} 1^{q/(q-1)} dt \right)^{(q-1)/q} \\ \Rightarrow |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q &\leq \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1)} \cdot \left| \frac{\partial\omega}{\partial y}(x, t) \right|^q dt \\ \Rightarrow \int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q dx &\leq \int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1)} \cdot \left| \frac{\partial\omega}{\partial y}(x, t) \right|^q dt dx \end{aligned}$$

Let now $k = \frac{p}{q} > 1$. We apply Hölder inequality with exponents k and $k' = \frac{k}{k-1} = \frac{p}{p-q}$ to obtain:

$$\begin{aligned} \int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q dx &\leq \left(\int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1) \cdot \frac{p}{p-q}} dt dx \right)^{\frac{p-q}{p}} \\ &\quad \cdot \left(\int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} \left| \frac{\partial\omega}{\partial y}(x, t) \right|^{q \cdot \frac{p}{q}} dt dx \right)^{\frac{q}{p}} \\ &\leq \varepsilon^{(q-1) + \frac{p-q}{p}} \left(\int_0^{2\pi} (\varphi(x/\varepsilon))^{1 + \frac{pq-p}{p-q}} dx \right)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}^q \\ &\leq \varepsilon^{\frac{q(p-1)}{p}} \left(\int_0^{2\pi} (\varphi(x/\varepsilon))^{\frac{pq-q}{p-q}} dx \right)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}^q \\ &\leq \varepsilon^{\frac{q(p-1)}{p}} \cdot \|\varphi\|_\infty^{\frac{q(p-1)}{p}} \cdot (2\pi)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}^q. \end{aligned}$$

In the end, we get

$$\left(\int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q dx \right)^{1/q} \leq C_{2,q} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)},$$

with $C_{2,q}$ from (120). We conclude the proof by a classical density argument, using the embeddings $C^1(\overline{\Omega_0}) \hookrightarrow W^{1,p}(\Omega_0) \hookrightarrow L^q(\partial\Omega_0)$. □

Proposition 4.5. Let $p \in (1, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$, we have

$$|I_{21}| \leq C_{21} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0 \setminus \Omega_\varepsilon)},$$

where

$$I_{21} = -\frac{w_0}{2} \int_0^{2\pi} (v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)) \cdot Q_\varepsilon^0(x/\varepsilon) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx$$

and

$$C_{21} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2}. \quad (121)$$

Proof. Let $1 < q < p$ and $q' = \frac{q}{q-1}$. Then:

$$\begin{aligned} \frac{2}{|w_0|} |I_{21}| &\leq \int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)| \cdot \left| Q_\varepsilon^0(x/\varepsilon) \cdot \sqrt{1 + (\varphi'(x/\varepsilon))^2} \right| \, dx \\ &\leq \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q} \\ &\quad \cdot \left(\int_0^{2\pi} \left| Q_\varepsilon^0(x/\varepsilon) \cdot \sqrt{1 + (\varphi'(x/\varepsilon))^2} \right|^{q'} \, dx \right)^{1/q'} \\ &\leq \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q} \\ &\quad \cdot \left(\int_0^{2\pi} \left(\frac{\sqrt{2}}{2} \right)^{q'} \cdot \left(\sqrt{1 + (\varphi'(x/\varepsilon))^2} \right)^{q'} \, dx \right)^{1/q'} \\ &\leq \frac{\sqrt{2}}{2} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q} \cdot \left(\int_0^{2\pi} (\sqrt{1 + \|\varphi'\|_\infty^2})^{q'} \, dx \right)^{1/q'} \\ &\leq \frac{\sqrt{2}}{2} \cdot (2\pi)^{1/q'} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q}, \end{aligned}$$

where we have used (112) and that $|Q_\varepsilon^0(x/\varepsilon)| = \frac{\sqrt{2}}{2}$, for all $x \in [0, 2\pi)$.

We apply now Lemma 4.1, with the constant $C_{2,q}$ from (120), in order to obtain:

$$|I_{21}| \leq C_{21} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla v\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}.$$

with

$$C_{21} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi)^{\frac{q-1}{q}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C_{2,q} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2}.$$

□

Proposition 4.6. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$, we have:

$$|I_{31}| \leq C_{31} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$I_{31} = \frac{w_0}{2} \int_0^{2\pi} (Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0) \cdot v(x, 0)) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx$$

and

$$C_{31} = \frac{|w_0|}{2} \sqrt{1 + \|\varphi'\|_\infty^2} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot (|\Omega_0|^{1/p} \cdot C_{tr}(p, \Omega_0) + 1). \quad (122)$$

Proof. Let $1 < q < p$ and $q' = \frac{q}{q-1}$.

Using (112), we have:

$$\begin{aligned}
\frac{2}{|w_0|} |I_{31}| &\leq \sqrt{1 + \|\varphi'\|_\infty^2} \int_0^{2\pi} |(Q_0 \cdot v)(x, \varepsilon\varphi(x/\varepsilon)) - (Q_0 \cdot v)(x, 0)| \, dx \\
&\leq \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon))| \cdot |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)| \, dx \right) + \\
&\quad + \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |v(x, 0)| \cdot |Q_0(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0)| \, dx \right) \\
&\leq \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon))|^{q'} \, dx \right)^{1/q'} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q} + \\
&\quad + \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |v(x, 0)|^p \, dx \right)^{1/p} \cdot \left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0)|^{p'} \, dx \right)^{1/p'}
\end{aligned}$$

where we have applied Hölder inequality with exponents q and q' for the first term and with exponents p and p' for the second one.

For the first term, we apply Lemma 4.1 and use the $L^\infty(\Omega_0)$ bounds for Q_0 in order to obtain that:

$$\begin{aligned}
&\left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon))|^{q'} \, dx \right)^{1/q'} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)|^q \, dx \right)^{1/q} \leq \\
&\leq (2\pi)^{1/q'} \cdot \|Q_0\|_{L^\infty(\Omega_0)} \cdot (2\pi)^{\frac{p-q}{pq}} \cdot \|\varphi\|_\infty^{\frac{p-1}{p}} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla v\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)} \\
&\leq ((2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{L^\infty(\Omega_0)}) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)}
\end{aligned}$$

since $\Omega_0 \setminus \Omega_\varepsilon \subset \Omega_0$.

For the other term, we see that we can apply Lemma 4.1 with exponents p' and p , since $p > 2$ implies that $1 < p' < p$, in order to obtain:

$$\begin{aligned}
&\left(\int_0^{2\pi} |v(x, 0)|^p \, dx \right)^{1/p} \cdot \left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0)|^{p'} \, dx \right)^{1/p'} \leq \\
&\leq \left(\int_{\Gamma_0} |v|^p \, d\sigma_0 \right)^{1/p} \cdot (2\pi)^{\frac{p-p'}{pp'}} \cdot \|\varphi\|_\infty^{\frac{p-1}{p}} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla Q_0\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)} \\
&\leq ((2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|\nabla Q_0\|_{L^\infty(\Omega_0)} \cdot |\Omega_0|^{1/p} \cdot C_{tr}(p, \Omega_0)) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},
\end{aligned}$$

where we use Definition 4.7 and the fact that, if $1 < p' < p$, then $\frac{p-p'}{pp'} < \frac{p}{pp'} = \frac{1}{p'} = \frac{p-1}{p}$.

In the end, we obtain that:

$$|I_{31}| \leq C_{31} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$C_{31} = \frac{|w_0|}{2} \sqrt{1 + \|\varphi'\|_\infty^2} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot (|\Omega_0|^{1/p} \cdot C_{tr}(p, \Omega_0) + 1).$$

□

Let us now prove the following lemma.

Lemma 4.2. Let us consider the case in which Assumption 7 holds. Let $p \in (2, +\infty)$, $\omega \in W^{1,p}(\Omega_0)$, V be a Banach space and $b : \mathbb{R} \rightarrow V$ a 2π -periodic function such that $b \in L^\infty([0, 2\pi))$, for which we write $\|b\|_\infty$ instead of $\|b\|_{L^\infty([0, 2\pi))}$. Then:

$$\left| \int_0^{2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx \right| \leq C_3 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1,p}(\Omega_0)},$$

where $B = \frac{1}{2\pi} \int_0^{2\pi} b(t) \, dt$ and $C_3 = (2\pi)^{\frac{2p-2}{p}} \cdot \|b\|_\infty \cdot C_{tr}(p, \Omega_0)$.

Proof. From Assumption 7, we have that $\varepsilon^{-1} = 2k \in \mathbb{N}^*$. We write then:

$$\begin{aligned} \int_0^{2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx &= \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx \\ &= \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} \omega(x, 0)b(t) \, dt \, dx - \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \omega(x, 0)b(x/\varepsilon) \, dx. \end{aligned} \quad (123)$$

Using the change of variables $x = x' + j\varepsilon \cdot 2\pi$, we obtain:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \omega(x, 0)b(x/\varepsilon) \, dx = \sum_{j=0}^{2k-1} \int_0^{\varepsilon \cdot 2\pi} \omega(x' + j\varepsilon \cdot 2\pi, 0)b\left(\frac{x' + j\varepsilon \cdot 2\pi}{\varepsilon}\right) \, dx'$$

and, since the function b is 2π -periodic and $j \in \mathbb{N}$, we can rewrite the last equality as:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \omega(x, 0)b(x/\varepsilon) \, dx = \sum_{j=0}^{2k-1} \int_0^{\varepsilon \cdot 2\pi} \omega(x' + j\varepsilon \cdot 2\pi, 0)b(x'/\varepsilon) \, dx'.$$

Using now the change of variables $x' = \varepsilon t$, we get:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \omega(x, 0)b(x/\varepsilon) \, dx = \sum_{j=0}^{2k-1} \int_0^{2\pi} \varepsilon \cdot \omega(\varepsilon t + j\varepsilon \cdot 2\pi, 0)b(t) \, dt.$$

Since

$$\varepsilon = \frac{1}{2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} 1 \, dx,$$

then (123) becomes:

$$\begin{aligned} \int_0^{2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx &= \\ &= \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} \omega(x, 0)b(t) \, dt \, dx - \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} \omega(\varepsilon t + j\varepsilon \cdot 2\pi, 0)b(t) \, dt \, dx \\ &= \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} (\omega(x, 0) - \omega(\varepsilon t + j\varepsilon \cdot 2\pi, 0)) \cdot b(t) \, dt \, dx. \end{aligned}$$

Then

$$\begin{aligned} \left| \int_0^{2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx \right| &\leq \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} |\omega(x, 0) - \omega(\varepsilon t + j\varepsilon \cdot 2\pi, 0)| \cdot |b(t)| \, dt \, dx \\ &\leq \frac{\|b\|_\infty}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} |\omega(x, 0) - \omega(\varepsilon t + j\varepsilon \cdot 2\pi, 0)| \, dt \, dx. \end{aligned}$$

We apply now the change of variables $t' = \varepsilon t + j\varepsilon \cdot 2\pi$ and drop the primes to obtain:

$$\left| \int_0^{2\pi} \omega(x, 0)(B - b(x/\varepsilon)) \, dx \right| \leq \frac{\|b\|_\infty}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \frac{|\omega(x, 0) - \omega(t, 0)|}{\varepsilon} \, dt \, dx.$$

Then:

$$\begin{aligned} & \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \varepsilon^{-1} \cdot |\omega(x, 0) - \omega(t, 0)| \, dt \, dx = \\ &= \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \frac{|\omega(x, 0) - \omega(t, 0)|}{|(x, 0) - (t, 0)|} \cdot \frac{|(x, 0) - (t, 0)|}{\varepsilon} \, dt \, dx \\ &\leq \sum_{j=0}^{2k-1} \left[\left(\int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \frac{|\omega(x, 0) - \omega(t, 0)|^p}{|(x, 0) - (t, 0)|^p} \, dt \, dx \right)^{1/p} \cdot \right. \\ &\quad \left. \cdot \left(\int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \frac{|x - t|^{p'}}{\varepsilon^{p'}} \, dt \, dx \right)^{1/p'} \right], \end{aligned}$$

where we have applied Hölder inequality with exponents p and p' . Since $|x - t| \leq \varepsilon \cdot 2\pi$, for any $x, t \in [j\varepsilon \cdot 2\pi, (j+1)\varepsilon \cdot 2\pi]$, we obtain:

$$\begin{aligned} & \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \varepsilon^{-1} \cdot |w(x, 0) - w(t, 0)| \, dt \, dx \leq \\ &\leq (2\pi)^{1+2/p'} \cdot \varepsilon^{2/p'} \cdot \sum_{j=0}^{2k-1} \left(\int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \frac{|w(x, 0) - w(t, 0)|^p}{|(x, 0) - (t, 0)|^p} \, dt \, dx \right)^{1/p} \\ &\leq (2\pi)^{1+2/p'} \cdot \varepsilon^{2/p'} \cdot \sum_{j=0}^{2k-1} \left(\int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} \frac{|w(x, 0) - w(t, 0)|^p}{|(x, 0) - (t, 0)|^p} \, dt \, dx \right)^{1/p}. \end{aligned}$$

Let us denote now:

$$r_j = \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_0^{2\pi} \frac{|w(x, 0) - w(t, 0)|^p}{|(x, 0) - (t, 0)|^p} \, dt \, dx, \quad \forall j \in \{0, 1, \dots, 2k-1\}.$$

We have that $r_j \geq 0$, for all $j \in \{0, 1, \dots, 2k-1\}$. The function $\mathbb{R} \ni x \rightarrow x^{1/p}$ is concave since $p \in (2, +\infty)$, therefore we have the Jensen inequality:

$$\sum_{j=0}^{2k-1} r_j^{1/p} \leq 2k \cdot \left(\frac{1}{2k} \sum_{j=0}^{2k-1} r_j \right)^{1/p}.$$

Since $2k = \varepsilon^{-1}$ and

$$\left(\sum_{j=0}^{2k-1} r_j \right)^{1/p} = \left(\int_0^{2\pi} \int_0^{2\pi} \frac{|\omega(x, 0) - \omega(t, 0)|^p}{|x - t|^p} \, dt \, dx \right)^{1/p} \leq \|\omega\|_{W^{1-1/p, p}(\Gamma_1)},$$

due to the fact that the left hand side of the last inequality represents the Gagliardo seminorm

defined on the space $W^{1-1/p,p}(\Gamma_1)$, we obtain that

$$\begin{aligned}
& \sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \varepsilon^{-1} \cdot |w(x,0) - w(t,0)| \, dt \, dx \leq \\
& \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{2p-2}{p}} \cdot (2k)^{1-1/p} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)} \\
& \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{2p-2}{p} - 1 + \frac{1}{p}} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)} \\
& \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)}.
\end{aligned}$$

Using Definition 4.7, we obtain that:

$$\left| \int_0^{2\pi} \omega(x,0)(B - b(x/\varepsilon)) \, dx \right| \leq C_3 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1,p}(\Omega_0)}$$

with

$$C_3 = (2\pi)^{\frac{2p-2}{p}} \cdot \|b\|_{\infty} \cdot C_{tr}(p, \Omega_0).$$

□

Proposition 4.7. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$ we have:

$$|I_{32}| \leq C_{32} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$I_{32} = -\frac{w_0}{2} \int_0^{2\pi} Q_0(x,0) \cdot v(x,0) \cdot \left(\gamma - \gamma_{\varepsilon}(x/\varepsilon) \right) \, dx$$

and

$$C_{32} = \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \sqrt{1 + \|\varphi'\|_{\infty}^2} \cdot C_{tr}(p, \Omega_0) \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)}. \quad (124)$$

Proof. Let $v \in W^{1,p}(\Omega_0)$. Since $Q_0 \in W^{1,\infty}(\Omega_0)$, we have that $Q_0 \cdot v \in W^{1,p}(\Omega_0)$. We apply Lemma 4.2 for $\omega = Q_0 \cdot v$ and $b = \gamma_{\varepsilon}$, with $V = \mathbb{R}$. In this way, we obtain that:

$$|I_{32}| \leq \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \|\gamma_{\varepsilon}\|_{\infty} \cdot C_{tr}(p, \Omega_0) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|Q_0 \cdot v\|_{W^{1,p}(\Omega_0)}.$$

Using now that $\|\gamma_{\varepsilon}\|_{\infty} = \sqrt{1 + \|\varphi'\|_{\infty}^2}$ and that $\|Q_0 \cdot v\|_{W^{1,p}(\Omega_0)} \leq \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \|v\|_{W^{1,p}(\Omega_0)}$, we obtain the conclusion. □

Proposition 4.8. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$ we have:

$$|I_{22}| \leq C_{22} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$I_{22} = \frac{w_0}{2} \int_0^{2\pi} v(x,0) \cdot \left(\gamma Q_{ef} - \gamma_{\varepsilon}(x/\varepsilon) Q_{\varepsilon}^0(x/\varepsilon) \right) \, dx$$

and

$$C_{22} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \sqrt{1 + \|\varphi'\|_{\infty}^2} \cdot C_{tr}(p, \Omega_0). \quad (125)$$

Proof. Let $v \in W^{1,p}(\Omega_0)$. We apply Lemma 4.2 for $\omega = v$ and $b = \gamma_\varepsilon Q_\varepsilon^0$, with $V = \text{Sym}_0(2)$. In this way, we have:

$$|I_{22}| \leq \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \|Q_\varepsilon^0 \cdot \gamma_\varepsilon\|_\infty \cdot C_{tr}(p, \Omega_0) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)}.$$

Since $|Q_\varepsilon^0| = \frac{\sqrt{2}}{2}$ and $\|\gamma_\varepsilon\|_\infty = \sqrt{1 + \|\varphi'\|_\infty^2}$, we obtain the conclusion. \square

We are now able to prove Proposition 4.2.

Proof of Proposition 4.2. We combine (117) with Propositions 4.4 to 4.8 to obtain the conclusion, with, for example,

$$C_I = C_1 + C_{21} + C_{31} + C_{22} + C_{32},$$

where these constants are defined in (118), (121), (122), (124) and (125). \square

Remark 4.18. We can actually choose C_I of the following form:

$$C_I = \max\{1, \|\varphi\|_\infty^{(p-1)/p}\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C(w_0, c, p, \Omega_0, Q_0),$$

where $C(w_0, c, p, \Omega_0, Q_0)$ is an ε -independent constant depending only on w_0, c, p, Ω_0 and Q_0 .

4.5 Proof of the error estimate

The goal in this subsection would be to place instead of v in the right-hand side of the inequality from Proposition 4.2 something that depends on $(Q_0 - Q_\varepsilon)$, such that we can obtain Theorem 4.1.

Throughout this subsection, we fix $p \in (2, +\infty)$.

Definition 4.10. Let $u_\varepsilon := Q_0 - Q_\varepsilon$.

Remark 4.19. The function u_ε solves the following PDE:

$$\begin{cases} -\Delta u_\varepsilon + cu_\varepsilon = 0, & \text{in } \Omega_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \frac{w_0}{2} u_\varepsilon = g_\varepsilon, & \text{on } \Gamma_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial \nu_R} + \frac{w_0}{2} u_\varepsilon = 0, & \text{on } \Gamma_R \end{cases}$$

where $g_\varepsilon = \frac{\partial Q_0}{\partial \nu_\varepsilon} + \frac{w_0}{2}(Q_0 - Q_\varepsilon^0)$. Since $\Gamma_\varepsilon \subset \Omega_0$, we have no problems with defining g_ε .

By Corollaries 4.1 and 4.2, we also have that $u_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ and $g_\varepsilon \in W^{1-1/p,p}(\Gamma_\varepsilon)$, for any $1 < p < +\infty$.

We would like now to prove the following proposition (to be compared with Proposition 4.2):

Proposition 4.9. There exists an ε -independent constant C_0 such that:

$$|a_\varepsilon(Q_0 - Q_\varepsilon, v)| \leq C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_\varepsilon)}, \quad \forall v \in W^{1,p}(\Omega_\varepsilon).$$

In order to do so, we need to construct an extension operator $E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\Omega_0)$ such that $E_\varepsilon \omega \equiv \omega$ in Ω_ε for any $\omega \in W^{1,p}(\Omega_\varepsilon)$, with the operator E_ε bounded uniformly in ε . For this, we adapt the bi-Lipschitz maps Φ_ε from (59) and Φ_ε^{-1} from (60) to this simplified model. In (59) and (60), the transformations Φ_ε and Φ_ε^{-1} are only between Ω_ε and Ω_0 . In order to construct the desired extension, we introduce the following notation.

Definition 4.11. Let $\Omega_1^\varepsilon = \left\{ (x, y) \mid x \in [0, 2\pi), y \in \left(-\frac{R\varepsilon\varphi(x/\varepsilon)}{R - \varepsilon\varphi(x/\varepsilon)}, R \right) \right\}$.

Remark 4.20. Using Assumption 7, it is easy to see that Ω_1^ε is well defined, since $R - \varepsilon\varphi(x/\varepsilon) > 0$ for all $x \in [0, 2\pi)$.

Definition 4.12. We define $\Phi_\varepsilon : \Omega_1^\varepsilon \rightarrow \Omega_0$ as

$$\Phi_\varepsilon(x, y) = \left(x, y \cdot \frac{R - \varepsilon\varphi(x/\varepsilon)}{R} + \varepsilon\varphi(x/\varepsilon) \right), \forall (x, y) \in \Omega_1^\varepsilon$$

and $\Phi_\varepsilon^{-1} : \Omega_0 \rightarrow \Omega_1^\varepsilon$ as

$$\Phi_\varepsilon^{-1}(x, y) = \left(x, \frac{R(y - \varepsilon\varphi(x/\varepsilon))}{R - \varepsilon\varphi(x/\varepsilon)} \right), \forall (x, y) \in \Omega_0.$$

Remark 4.21. We have $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$ and, using Assumption 7, we can prove the following sequence of inclusions:

$$\{(x, y) \mid x \in [0, 2\pi), y \in (R/2, R)\} \subset \Omega_\varepsilon \subset \Omega_0 \subset \Omega_1^\varepsilon \subset \{(x, y) \mid x \in [0, 2\pi), y \in (-R, R)\}.$$

Proposition 4.10. Φ_ε defines a family of C^2 uniformly bi-Lipschitz maps between Ω_1 and Ω_0 . Moreover, there exists an ε -independent constant C_Φ such that:

$$C_\Phi^{-1} \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)} \leq \|\tilde{\omega}\|_{W^{1,p}(\Omega_0)} \leq C_\Phi \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)}, \quad (126)$$

for all $\omega \in W^{1,p}(\Omega_\varepsilon)$, where $\tilde{\omega} = \omega \circ \Phi_\varepsilon|_{\Omega_0}$, and

$$C_\Phi^{-1} \cdot \|\omega\|_{W^{1,p}(\Omega_0)} \leq \|\tilde{\omega}\|_{W^{1,p}(\Omega_1^\varepsilon)} \leq C_\Phi \cdot \|\omega\|_{W^{1,p}(\Omega_0)}, \quad (127)$$

for all $\omega \in W^{1,p}(\Omega_0)$, where $\tilde{\omega} = \omega \circ \Phi_\varepsilon$.

Proof. Since $\varphi \in C^2$, then $\Phi_\varepsilon \in C^2$. Moreover, since the definition of Φ_ε from Definition 4.12 is based on (59), then one can argue similarly as in Proposition 2.5 to obtain that Φ_ε and its inverse are Lipschitz with Lipschitz constant independent of ε . More specifically, using the bound

$$0 < \varepsilon \cdot \|\varphi\|_\infty < \frac{R}{2}$$

from Assumption 6, we can obtain that

$$\left| \frac{\partial \Phi_\varepsilon}{\partial x}(x, y) \right| \leq \max\{2R, 1\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \quad \text{and} \quad \left| \frac{\partial \Phi_\varepsilon}{\partial y}(x, y) \right| \leq 1, \quad \forall (x, y) \in \Omega_1^\varepsilon,$$

which implies that Φ_ε is Lipschitz with its Lipschitz constant bounded ε -independent. In the same way, the first order derivatives of Φ_ε^{-1} can be bounded ε -independent (using the same bound as above for ε). To obtain the constant C_Φ , we use the same ε -independent bounds for the first order derivatives of Φ_ε and Φ_ε^{-1} when we apply chain rule in $\tilde{\omega} = \omega \circ \Phi_\varepsilon$ with $\omega \in W^{1,p}(\Omega_0)$. \square

Remark 4.22. If we were to define $E_\varepsilon\omega = \omega \circ \Phi_\varepsilon|_{\Omega_0}$, for any $\omega \in W^{1,p}(\Omega_\varepsilon)$, then we would not have had $E_\varepsilon\omega \equiv \omega$ inside of Ω_ε , so we need the more sophisticated extension that will be provided in Definition 4.13 below.

Corollary 4.3. Let $\Omega_2 := \{(x, y) \mid x \in [0, 2\pi), y \in (-R, R)\}$ and let $T : W^{1,p}(\Omega_0) \rightarrow W^{1,p}(\Omega_2)$ the following extension operator:

$$T\omega(x, y) = \begin{cases} \omega(x, y), & \text{if } y \in (0, R); \\ \omega(x, -y), & \text{if } y \in (-R, 0); \end{cases}$$

for any $\omega \in W^{1,p}(\Omega_0)$. Then there exists an ε -independent constant $C_{ext}(p, \Omega_0) > 0$ such that:

$$\|T\omega\|_{W^{1,p}(\Omega_2)} \leq C_{ext}(p, \Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_0)}, \quad \forall \omega \in W^{1,p}(\Omega_0).$$

Remark 4.23. The proof is a simple exercise which consists of applying the method of extending a Sobolev function by reflection against a flat boundary, illustrated in [24] for $W^{1,p}(\Omega_0)$ functions.

Definition 4.13. Let $E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\Omega_0)$ defined as

$$E_\varepsilon\omega := \left((T(\omega \circ \Phi_\varepsilon|_{\Omega_0})) \Big|_{\Omega_\varepsilon^{\ddagger}} \circ \Phi_\varepsilon^{-1} \right),$$

for any $\omega \in W^{1,p}(\Omega_\varepsilon)$. In this way, $E_\varepsilon\omega \equiv \omega$ in Ω_ε , for any $\omega \in W^{1,p}(\Omega_\varepsilon)$.

Proposition 4.11. There exists an ε -independent constant C_{ext} such that:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \leq C_{ext} \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)}, \quad \forall \omega \in W^{1,p}(\Omega_\varepsilon).$$

Proof. Let $\omega \in W^{1,p}(\Omega_\varepsilon)$ and $\tilde{\omega} = \omega \circ \Phi_\varepsilon|_{\Omega_0}$. Since the transformation Φ_ε^{-1} is bi-Lipschitz with its constants bounded ε -independent, then, using (127):

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} = \left\| \left((T\tilde{\omega}) \Big|_{\Omega_\varepsilon^{\ddagger}} \circ \Phi_\varepsilon^{-1} \right) \right\|_{W^{1,p}(\Omega_0)} \leq C_\Phi^{-1} \left\| (T\tilde{\omega}) \Big|_{\Omega_\varepsilon^{\ddagger}} \right\|_{W^{1,p}(\Omega_\varepsilon^{\ddagger})} \leq C_\Phi^{-1} \|T\tilde{\omega}\|_{W^{1,p}(\Omega_2)},$$

where in the last inequality we use that $T\tilde{\omega}|_{\Omega_\varepsilon^{\ddagger}}$ is a restriction of $T\tilde{\omega}$ from Ω_2 . Then:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \leq C_\Phi^{-1} \|T\tilde{\omega}\|_{W^{1,p}(\Omega_2)} \leq C_\Phi^{-1} \cdot C_{ext}(p, \Omega_0) \cdot \|\tilde{\omega}\|_{W^{1,p}(\Omega_0)},$$

where we have used Corollary 4.3. Since $\tilde{\omega} = \omega \circ \Phi_\varepsilon|_{\Omega_0}$, then, using (126), we obtain:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \leq C_{ext}(p, \Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)}.$$

Therefore, we can actually choose $C_{ext} = C_{ext}(p, \Omega_0)$ given by Corollary 4.3. \square

Proof of Proposition 4.9. Let $v \in W^{1,p}(\Omega_\varepsilon)$. Then $E_\varepsilon v \in W^{1,p}(\Omega_0)$ and we can apply Proposition 4.2 to obtain:

$$|a_\varepsilon(u_\varepsilon, E_\varepsilon v)| \leq C_I \cdot \varepsilon^{\frac{p-2}{p}} \cdot \|E_\varepsilon v\|_{W^{1,p}(\Omega_0)}.$$

Using Definition 4.13 and Proposition 4.11, we obtain:

$$|a_\varepsilon(u_\varepsilon, v)| \leq (C_I \cdot C_{ext}) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_\varepsilon)}.$$

\square

Corollary 4.4. There exists a unique solution $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ that solves the following PDE:

$$\begin{cases} -\Delta v_\varepsilon + cv_\varepsilon = u_\varepsilon, & \text{in } \Omega_\varepsilon \\ \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} + \frac{w_0}{2}v_\varepsilon = 0, & \text{on } \Gamma_\varepsilon \\ \frac{\partial v_\varepsilon}{\partial \nu_R} + \frac{w_0}{2}v_\varepsilon = 0, & \text{on } \Gamma_R \end{cases} \quad (128)$$

where $u_\varepsilon = Q_0 - Q_\varepsilon$.

Proof. The proof follows the same steps as in Corollary 4.1. \square

Proposition 4.12. The function v_ε satisfies the following inequality:

$$\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \min\{c^{-1/2}, c^{-1}\} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad (129)$$

where c is the positive constant from the bulk energy defined in (109).

Proof. Let $w \in W^{1,p}(\Omega_\varepsilon)$. Since $-\Delta v_\varepsilon + cv_\varepsilon = u_\varepsilon$ in Ω_ε , we have:

$$\begin{aligned} \int_{\Omega_\varepsilon} u_\varepsilon \cdot w \, d(x, y) &= \int_{\Omega_\varepsilon} (-\Delta v_\varepsilon + cv_\varepsilon) \cdot w \, d(x, y) \\ &= \int_{\Omega_\varepsilon} (\nabla v_\varepsilon \cdot \nabla w + cv_\varepsilon \cdot w) \, d(x, y) - \int_{\partial\Omega_\varepsilon} \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \cdot w \, d\sigma_\varepsilon \\ &= \int_{\Omega_\varepsilon} (cv_\varepsilon \cdot w + \nabla v_\varepsilon \cdot \nabla w) \, d(x, y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} v_\varepsilon \cdot w \, d\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} v_\varepsilon \cdot w \, d\sigma_R \end{aligned}$$

Taking $w = u_\varepsilon$, we obtain:

$$\begin{aligned} a_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_{\Omega_\varepsilon} (cu_\varepsilon \cdot v_\varepsilon + \nabla u_\varepsilon \cdot \nabla v_\varepsilon) \, d(x, y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} u_\varepsilon \cdot v_\varepsilon \, d\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} u_\varepsilon \cdot v_\varepsilon \, d\sigma_R \\ &= \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2. \end{aligned} \quad (130)$$

Taking $w = v_\varepsilon$, we obtain:

$$c\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{w_0}{2}\|v_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2 + \frac{w_0}{2}\|v_\varepsilon\|_{L^2(\Gamma_R)}^2 = \int_{\Omega_\varepsilon} u_\varepsilon \cdot v_\varepsilon \, dx.$$

Now we can see that

$$c\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{w_0}{2}\|v_\varepsilon\|_{L^2(\partial\Omega_\varepsilon)}^2 \leq \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

which implies that $\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. In the same way,

$$\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{w_0}{2}\|v_\varepsilon\|_{L^2(\partial\Omega_\varepsilon)}^2 \leq \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2$$

using the last inequality proved. This implies that $\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1/2} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. In the end, we obtain that:

$$\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \min\{c^{-1/2}, c^{-1}\} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

\square

Definition 4.14. Let $\bar{v}_\varepsilon := v_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0} \circ \Phi^{-1}$, where Φ_ε is introduced in Definition 4.12 and Φ in Definition 4.5. Since $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$, then $\Omega_\varepsilon = (\Phi_\varepsilon|_{\Omega_0} \circ \Phi^{-1})(\mathcal{U}_0)$.

Corollary 4.5. We have that $\bar{v}_\varepsilon \in W^{2,p}(\mathcal{U}_0)$, for any $p > 2$.

Proof. Since $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$, Φ_ε and Φ are bi-Lipschitz with $\Phi \in C^2(\Omega_0)$ and Φ^{-1} smooth, we obtain the conclusion. \square

Remark 4.24. Using Definition 4.14, we can see that \bar{v}_ε solves a PDE of the form:

$$\begin{cases} \mathcal{L}\bar{v}_\varepsilon + c\bar{v}_\varepsilon = \bar{u}_\varepsilon, & \text{in } \mathcal{U}_0 \\ \nabla\bar{v}_\varepsilon \cdot \bar{l}_1 + \frac{w_0}{2}\bar{v}_\varepsilon = 0, & \text{on } \Phi(\Gamma_0) \\ \nabla\bar{v}_\varepsilon \cdot \bar{l}_2 + \frac{w_0}{2}\bar{v}_\varepsilon = 0, & \text{on } \Phi(\Gamma_R) \end{cases} \quad (131)$$

where \mathcal{L} is an uniformly elliptic operator, $\bar{l}_1 \in C^1(\Phi(\Gamma_0))$, $\bar{l}_2 \in C^1(\Phi(\Gamma_R))$ and $\bar{u}_\varepsilon := u_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0} \circ \Phi^{-1} \in W^{2,p}(\mathcal{U}_0)$.

Proposition 4.13. There exists an ε -independent constant $C_{reg}(\mathcal{U}_0)$ such that \bar{v}_ε satisfies the following inequality:

$$\|\bar{v}_\varepsilon\|_{H^2(\mathcal{U}_0)} \leq C_{reg}(\mathcal{U}_0) \cdot (\|\mathcal{L}\bar{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + \|\bar{v}_\varepsilon\|_{H^{1/2}(\partial\mathcal{U}_0)} + \|\bar{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}).$$

Proof. We apply [29, Theorem 2.3.3.2], since all the required conditions are satisfied. \square

We can now prove the main result of this section.

Proof of Theorem 4.1. We apply first Proposition 4.9 with $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = a_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)},$$

where we also use (130). We apply now Proposition 4.10:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} \leq (C_0 \cdot C_\Phi) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0}\|_{W^{1,p}(\Omega_0)}.$$

We now use the compact embedding $W^{2,2}(\Omega_0) \hookrightarrow W^{1,p}(\Omega_0)$ to obtain:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq (C_0 \cdot C_\Phi \cdot C_{emb}(\Omega_0)) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0}\|_{W^{2,2}(\Omega_0)},$$

where $C_{emb}(\Omega_0)$ is the constant given by the compact embedding used.

Recalling Definition 4.5, it is easy to see that there exists $C_{polar} > 0$, which is ε -independent, such that:

$$C_{polar}^{-1} \|\omega\|_{W^{2,2}(\Omega_0)} \leq \|\omega \circ \Phi^{-1}\|_{W^{2,2}(\mathcal{U}_0)} \leq C_{polar} \|\omega\|_{W^{2,2}(\Omega_0)}, \quad \forall \omega \in W^{2,2}(\Omega_0). \quad (132)$$

In this way, we have:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq C'_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\bar{v}_\varepsilon\|_{W^{2,2}(\mathcal{U}_0)},$$

where $C'_0 = C_0 \cdot C_\Phi \cdot C_{emb}(\Omega_0) \cdot C_{polar}$.

We apply now Proposition 4.13:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq (C'_0 \cdot C_{reg}(\mathcal{U}_0)) \cdot \varepsilon^{\frac{p-1}{p}} \cdot (\|\mathcal{L}\bar{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + \|\bar{v}_\varepsilon\|_{H^{1/2}(\partial\mathcal{U}_0)} + \|\bar{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}).$$

Using (131) and the trace inequality for the trace operator $\text{Tr} : H^1(\mathcal{U}_0) \rightarrow H^{1/2}(\partial\mathcal{U}_0)$, we get:

$$\begin{aligned} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 &\leq (C'_0 \cdot C_{reg}(\mathcal{U}_0)) \cdot \varepsilon^{\frac{p-1}{p}} \cdot (\|\bar{u}_\varepsilon - c\bar{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + (1 + C_{tr}(\mathcal{U}_0))\|\bar{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}) \\ &\leq (C'_0 \cdot C_{reg}(\mathcal{U}_0)) \cdot \varepsilon^{\frac{p-1}{p}} \cdot (\|\bar{u}_\varepsilon\|_{L^2(\mathcal{U}_0)} + c \cdot \|\bar{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \|\bar{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}) \end{aligned}$$

and, using (132), we obtain:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq (C'_0 \cdot C_{reg}(\mathcal{U}_0) \cdot C_{polar}^{-1}) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left(\|u_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0}\|_{L^2(\Omega_0)} + c \cdot \|v_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0}\|_{L^2(\Omega_0)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \|v_\varepsilon \circ \Phi_\varepsilon|_{\Omega_0}\|_{H^1(\Omega_0)} \right).$$

Using Proposition 4.10, we get:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq (C'_0 \cdot C_{reg}(\mathcal{U}_0) \cdot C_{polar}^{-1} \cdot C_\Phi^{-1}) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left(\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + c \cdot \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \|v_\varepsilon\|_{H^1(\Omega_\varepsilon)} \right).$$

and then, using Proposition 4.12, we obtain:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq (C_0 \cdot C_{emb}(\Omega_0) \cdot C_{reg}(\mathcal{U}_0)) \cdot \left(2 + (1 + C_{tr}(\mathcal{U}_0)) \cdot \min\{c^{-1/2}, c^{-1}\} \right) \cdot \varepsilon^{\frac{p-1}{p}},$$

for any $p > 2$. □

5 Concluding remarks

Within this work we have considered the effect of rugosity in liquid crystalline systems in the limit where the wavelength and amplitude of the surface oscillations go to zero at a comparable rate.

We have considered two different aspects of the problem:

- firstly a highly general formulation of the problem that is able to simultaneously treat many different common models of liquid crystals, including Oseen-Frank, and Landau-de Gennes with either polynomial or singular bulk potential. The fact that we are able to consider a broad set of models in one framework is due to the fact that their specific peculiarities are more apparent in the “bulk” of the domain, and have little effect on the surface energy.
- Secondly we have considered a more simple linear model in a periodic two-dimensional geometry, which we are able to interpret as a toy model for paranematic systems. Within this simpler situation, we are able to make far more precise estimates on the convergence towards the homogenised limit, and near-explicit calculation of the relevant coefficients in the homogenised surface energy.

These two threads both show the same key feature, that the influence of the geometry of the oscillatory surfaces we consider may be replaced by effective surface energies on a simpler limiting domain, which may be represented using particular types of weak limits; that of families of Young measures in the general case, and weak- L^2 limits in the second.

The gap between the two cases leaves an obvious question, as to whether the more precise estimates for the linear two-dimensional system may be extended to more realistic non-linear models, both in higher dimensions and more complex geometries than that of a periodic cell.

Furthermore, the inexplicit nature of the effective surface energies, described only as weak limits defined on potentially complex geometries, raises the question as to whether a more detailed analysis of such surface oscillations could give rise to more constructive methods for evaluating the effective surface energies, or more ambitiously, towards an inverse problem of designing them.

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A Arithmetic on surface terms

To analyse the surface terms, we will need a quantity that is morally the wedge product of $N-1$ vectors in \mathbb{R}^N . We define precisely what we mean by this as follows.

Definition A.1. Let $(v^i)_{i=1}^{N-1}$ be a collection of $N-1$ vectors in \mathbb{R}^N . We define, in a canonical orthonormal basis $(e_i)_{i=1}^N$ with $v_j^i = v^i \cdot e_j$, the $N \times N-1$ matrix M to be the matrix with row i equal to $(v_j^i)_{j=1}^N$. That is,

$$M = \begin{pmatrix} v_1^1 & v_2^1 & \dots & v_N^1 \\ v_1^2 & v_2^2 & \dots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_2^{N-1} & \dots & v_N^{N-1} \end{pmatrix} \quad (133)$$

Then for $1 < i < N$, we define M_i to be the $(N-1) \times (N-1)$ matrix which corresponds to M with the i -th column removed. We define the product

$$\bigwedge_{i=1}^{N-1} v^i = \sum_{i=1}^N \det(M_i) (-1)^i e_i. \quad (134)$$

Remark A.1. We present this definition in a self-contained manner, however it is simply a specific case of the exterior product [36, Chapter 9]. Using the language of differential geometry, we identify vectors $v_i \in \mathbb{R}^N$ with 1-vectors, denoted $v_i \in \Lambda^1(\mathbb{R}^N)$. The exterior product of $(N-1)$ 1-vectors gives an $(N-1)$ -vector, $\bigwedge_{i=1}^{N-1} v_i = v_1 \wedge v_2 \wedge \dots \wedge v_{N-1} \in \Lambda^{N-1}(\mathbb{R}^N)$. The space $\Lambda^{N-1}(\mathbb{R}^N)$ may be canonically identified with the space of 1-vectors $\Lambda^1(\mathbb{R}^N)$ via the Hodge star operator, so that $\mathbb{R}^N \sim \Lambda^1(\mathbb{R}^N) \sim \Lambda^{N-1}(\mathbb{R}^N)$. In this sense, we may interpret $\bigwedge_{i=1}^{N-1} v_i$ for $v_i \in \mathbb{R}^N$ as a vector in \mathbb{R}^N in a canonical way.

Remark A.2. We abuse notation for brevity, and interpret the ‘‘determinant’’ of the $N \times N$ matrix with first column corresponding to the canonical basis of \mathbb{R}^N as

$$\begin{aligned} & \det \begin{pmatrix} e_1 & e_2 & \dots & e_N \\ v_1^1 & v_2^1 & \dots & v_N^1 \\ v_1^2 & v_2^2 & \dots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_2^{N-1} & \dots & v_N^{N-1} \end{pmatrix} \\ &= \det \begin{pmatrix} v_2^1 & v_3^1 & \dots & v_N^1 \\ v_2^2 & v_3^2 & \dots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_2^{N-1} & v_3^{N-1} & \dots & v_N^{N-1} \end{pmatrix} e_1 - \det \begin{pmatrix} v_1^1 & v_3^1 & \dots & v_N^1 \\ v_1^2 & v_3^2 & \dots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_3^{N-1} & \dots & v_N^{N-1} \end{pmatrix} e_2 + \dots \\ &= \sum_{i=1}^N \det(M_i) (-1)^i e_i = \bigwedge_{i=1}^{N-1} v^i. \end{aligned} \quad (135)$$

Remark A.3. It is immediate from the expression (135) that if $1 \leq j \leq N-1$,

$$v^j \cdot \bigwedge_{i=1}^{N-1} v^i = \det \begin{pmatrix} v_1^j & v_2^j & \cdots & v_N^j \\ v_1^1 & v_2^1 & \cdots & v_N^1 \\ v_1^2 & v_2^2 & \cdots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_2^{N-1} & \cdots & v_N^{N-1} \end{pmatrix} = 0, \quad (136)$$

thus we are able to write the outer unit normal to Γ_ε , with respect to a correctly oriented orthonormal basis $(\tau_i)_{i=1}^{N-1}$ of $T_x\Gamma$ as

$$\begin{aligned} \Lambda_\varepsilon(x) &= \bigwedge_{i=1}^{N-1} \langle D(x + \varphi_\varepsilon(x))\nu(x), \tau_i \rangle, \\ \tilde{\nu}_\varepsilon &= \frac{1}{|\Lambda_\varepsilon|} \Lambda_\varepsilon. \end{aligned} \quad (137)$$

Furthermore, the surface element γ_ε can be written in terms of Λ_ε as

$$\gamma_\varepsilon = |\Lambda_\varepsilon|. \quad (138)$$

These representations involving the operator \bigwedge thus permit us to simplify the surface energy terms sufficiently to produce more explicit results.

We also note the further identity, that for any scalars $(c_i)_{i=1}^{N-1}$, vectors $v, (v^i)_{i=1}^{N-1}$, we have that

$$\begin{aligned} v \cdot \bigwedge_{i=1}^{N-1} (v^i + c_i v) &= \det \begin{pmatrix} v_1 & v_2 & \cdots & v_N \\ v_1^1 + c_1 v_1 & v_2^1 + c_1 v_2 & \cdots & v_N^1 + c_1 v_N \\ v_1^2 + c_2 v_1 & v_2^2 + c_2 v_2 & \cdots & v_N^2 + c_2 v_N \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} + c_{N-1} v_1 & v_2^{N-1} + c_{N-1} v_2 & \cdots & v_N^{N-1} + c_{N-1} v_N \end{pmatrix} \\ &= \det \begin{pmatrix} v_1 & v_2 & \cdots & v_N \\ v_1^1 & v_2^1 & \cdots & v_N^1 \\ v_1^2 & v_2^2 & \cdots & v_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{N-1} & v_2^{N-1} & \cdots & v_N^{N-1} \end{pmatrix} = v \cdot \bigwedge_{i=1}^{N-1} v^i. \end{aligned} \quad (139)$$

Proposition A.1. Let φ_ε and domains $\Omega_\varepsilon, \Omega$ satisfy Assumptions 1, 2 and $\hat{D}\varphi_\varepsilon$ as given in Assumption 3. Then, for ε sufficiently small, we may estimate $\gamma_\varepsilon(x), \nu_\varepsilon(x)$ in terms of $\hat{D}\varphi_\varepsilon$ up to a quantity of order ε as

$$\begin{aligned} \gamma_\varepsilon &= |\hat{D}\varphi_\varepsilon| + O(\varepsilon), \\ \tilde{\nu}_\varepsilon &= \frac{1}{|\hat{D}\varphi_\varepsilon|} \hat{D}\varphi_\varepsilon + O(\varepsilon) \end{aligned} \quad (140)$$

where $h(x) = O(\varepsilon)$ is taken to mean that $|h(x)| \leq C\varepsilon$ for some C which is independent of x, ε .

Proof. First, we note that the map $\Gamma \ni x \mapsto x + \varphi_\varepsilon(x)\nu(x) \in \Gamma_\varepsilon$ is differentiable, and its derivative in the direction $\tau_i \in T_x\Gamma$ satisfies

$$\langle D(x + \varphi_\varepsilon(x)\nu(x)), \tau_i \rangle = \tau_i + \langle D\varphi_\varepsilon(x), \tau_i \rangle \nu(x) + \varphi_\varepsilon(x) \langle D\nu(x), \tau_i \rangle. \quad (141)$$

Now, as Γ is a C^2 compact manifold, this implies that ν is C^1 with bounded C^1 norm, and since $|\tau_i| = 1$, we have that $|\varphi_\varepsilon(x)\langle D\nu(x), \tau_i \rangle| \leq \|\varphi_\varepsilon\|_\infty \|D\nu\|_\infty = O(\varepsilon)$. Now we recall that the quantity Λ_ε defined by

$$\Lambda_\varepsilon(x) = \bigwedge_{i=1}^{N-1} \langle D(x + \varphi_\varepsilon(x)\nu(x)), \tau_i \rangle \quad (142)$$

provides expressions for $\tilde{\nu}_\varepsilon, \gamma_\varepsilon$, and turn our attention to Λ_ε . We may then write, as all quantities are uniformly bounded in x, ε , and the wedge product is locally Lipschitz continuous, that

$$\Lambda_\varepsilon(x) = O(\varepsilon) + \bigwedge_{i=1}^{N-1} \left(\tau_i + \langle D\varphi_\varepsilon(x), \tau_i \rangle \nu(x) \right). \quad (143)$$

We note that the $O(\varepsilon)$ term contains all contributions related to the curvature of Γ , i.e., $D\nu(x)[\tau_i]$. We may evaluate Λ_ε componentwise, as $(\tau_i)_{i=1}^{N-1}$ with ν form an orthonormal basis for \mathbb{R}^N , and we see that

$$\begin{aligned} \Lambda_\varepsilon(x) \cdot \nu &= \nu \cdot \bigwedge_{i=1}^{N-1} \left(\tau_i + \langle D\varphi_\varepsilon(x), \tau_i \rangle \nu(x) \right) + O(\varepsilon) \\ &= \nu \cdot \bigwedge_{i=1}^{N-1} \tau_i + O(\varepsilon) = 1 + O(\varepsilon), \end{aligned} \quad (144)$$

via the identity (139), and that $\bigwedge_{i=1}^{N-1} \tau_i = \nu$. Then we may evaluate

$$\begin{aligned} \Lambda_\varepsilon(x) \cdot \tau_i &= \Lambda_\varepsilon(x) \cdot (\tau_i + \langle D\varphi_\varepsilon(x), \tau_i \rangle \nu(x)) - \Lambda_\varepsilon(x) \cdot (\langle D\varphi_\varepsilon(x), \tau_i \rangle \nu(x)) \\ &= -\langle D\varphi_\varepsilon(x), \tau_i \rangle \Lambda_\varepsilon(x) \cdot \nu(x) = -\langle D\varphi_\varepsilon(x), \tau_i \rangle + O(\varepsilon). \end{aligned} \quad (145)$$

In particular, as ν and $(\tau_i)_{i=1}^{N-1}$ form an orthonormal basis, we must have that

$$\begin{aligned} \Lambda_\varepsilon(x) &= (\Lambda_\varepsilon(x) \cdot \nu(x)) \nu(x) + \sum_{i=1}^{N-1} (\Lambda_\varepsilon(x) \cdot \tau_i(x)) \tau_i(x) \\ &= \nu(x) - \sum_{i=1}^{N-1} \langle D\varphi_\varepsilon(x), \tau_i \rangle \tau_i(x) + O(\varepsilon) \\ &= \hat{D}\varphi_\varepsilon(x) + O(\varepsilon). \end{aligned} \quad (146)$$

Then, using that $|\hat{D}\varphi_\varepsilon| \geq 1$ and is thus certainly bounded away from zero, it is immediate that

$$\begin{aligned} \gamma_\varepsilon &= |\Lambda_\varepsilon| = |\hat{D}\varphi_\varepsilon| + O(\varepsilon) \\ \nu_\varepsilon &= \frac{1}{|\Lambda_\varepsilon|} \Lambda_\varepsilon = \frac{1}{|\hat{D}\varphi_\varepsilon|} \hat{D}\varphi_\varepsilon + O(\varepsilon) \end{aligned} \quad (147)$$

provided ε is sufficiently small. \square

With this, it is straightforward that Lemma 2.1 holds.

Proof of Lemma 2.1. As $w(x, \nu, u)$ is Lipschitz with Lipschitz constant bounded by $C(1 + |u|^q)$ and $|D\varphi_\varepsilon|$ is uniformly bounded, the result is immediate as $u \in L^q(\Gamma)$. \square

Remark A.4. The consequence of this corollary is that the effective surface energy w_h can be described as a limit of only first order quantities, that is, derivatives of φ_ε and the unit normal vector ν . In particular, curvature of the domain does not have an effect, and in the proof of Proposition A.1, we see that this is because the influence of curvature via $\langle D\nu(x), \tau_i \rangle$ in $\gamma_\varepsilon, \nu_\varepsilon$ is multiplied by φ_ε , an order ε quantity. In particular this suggests that ‘‘higher order’’ behaviour as $\varepsilon \rightarrow 0$ could be affected by curvature of the limiting domain.

B Proofs for error estimate

In this subsection, we prove Corollary 4.1, that $a_\varepsilon(Q_\varepsilon - Q_0, v)$ is well-defined for any $v \in W^{1,p}(\Omega_0)$ and then we prove Proposition 4.3.

Proof of Corollary 4.1. We consider $\tilde{Q}_\varepsilon : \mathcal{U}_\varepsilon \rightarrow \text{Sym}_0(2)$ such that:

$$Q_\varepsilon(x, y) = \tilde{Q}_\varepsilon(\Phi(x, y)), \quad \forall (x, y) \in \Omega_\varepsilon,$$

and we denote $(\tilde{x}, \tilde{y}) = \Phi(x, y)$ and $(x, y) = (\Phi_1^{-1}(\tilde{x}, \tilde{y}), \Phi_2^{-1}(\tilde{x}, \tilde{y}))$.

Then \tilde{Q}_ε solves a PDE of the following form:

$$\begin{cases} \sum_{i,j=1}^2 D_i(a_{ij}D_j\tilde{Q}_\varepsilon) + \sum_{i=1}^2 a_i D_i\tilde{Q}_\varepsilon + c\tilde{Q}_\varepsilon = 0, & \text{in } \mathcal{U}_\varepsilon; \\ \sum_{i=1}^2 b_{1i}D_i\tilde{Q}_\varepsilon + \frac{w_0}{2}\tilde{Q}_\varepsilon = \frac{w_0}{2}\tilde{Q}_\varepsilon^0, & \text{on } \Phi(\Gamma_\varepsilon); \\ \sum_{i=1}^2 b_{2i}D_i\tilde{Q}_\varepsilon + \frac{w_0}{2}\tilde{Q}_\varepsilon = \frac{w_0}{2}\tilde{Q}_R, & \text{on } \Phi(\Gamma_R). \end{cases} \quad (148)$$

where $D_1 = \frac{\partial}{\partial \tilde{x}}$ and $D_2 = \frac{\partial}{\partial \tilde{y}}$. We have that $a_{ij} \in C^\infty(\mathcal{U}_\varepsilon)$ with:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -\tilde{y}^2 - \frac{\tilde{x}^2}{\tilde{x}^2 + \tilde{y}^2} & \tilde{x}\tilde{y} - \frac{\tilde{x}\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} \\ \tilde{x}\tilde{y} - \frac{\tilde{x}\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} & -\tilde{x}^2 - \frac{\tilde{y}^2}{\tilde{x}^2 + \tilde{y}^2} \end{pmatrix}.$$

The coefficients a_i are also from $C^\infty(\mathcal{U}_\varepsilon)$ with:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \tilde{x} + D_1 a_{11} + D_2 a_{21} \\ \tilde{y} + D_1 a_{12} + D_2 a_{22} \end{pmatrix}$$

The coefficients b_{1i} are from $C^1(\Phi(\Gamma_\varepsilon))$ and can be obtained explicitly from $\nabla Q_\varepsilon \cdot \nu_\varepsilon = \sum_{i=1}^2 b_{1i}D_i\tilde{Q}_\varepsilon$. In the same way, b_{2i} are from $C^1(\Phi(\Gamma_R))$ and can be obtained explicitly from $\nabla Q_\varepsilon \cdot (0, 1) = \sum_{i=1}^2 b_{2i}D_i\tilde{Q}_\varepsilon$. Moreover, $\tilde{Q}_\varepsilon^0 = Q_\varepsilon^0 \circ \Phi^{-1}$ and $\tilde{Q}_R = Q_R \circ \Phi^{-1}$. Since $Q_\varepsilon^0 \in C^1(\Gamma_\varepsilon)$ and Φ is a smooth bi-Lipschitz transformation, then $\tilde{Q}_\varepsilon^0 \in C^1(\Phi(\Gamma_\varepsilon))$ which implies that $\tilde{Q}_\varepsilon^0 \in W^{1,1-1/p}(\Phi(\Gamma_\varepsilon))$. In the same way, $\tilde{Q}_R \in W^{1,1-1/p}(\Phi(\Gamma_R))$.

Therefore, we can apply Theorem 4.2 to obtain that there exists a unique solution $\tilde{Q}_\varepsilon^0 \in W^{2,p}(\mathcal{U}_\varepsilon)$ of the problem (148). Using now that Φ is smooth, we obtain that there exists a unique solution $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ of the problem (114). \square

Let $p \in (1, +\infty)$ and $v \in W^{1,p}(\Omega_0)$. We recall that $\Omega_\varepsilon \subset \Omega_0$, for all $\varepsilon > 0$, hence we also have that $v|_{\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon)$. Applying Corollary 4.1 and Corollary 4.2 with exponent $p' = \frac{p-1}{p} \in (1, +\infty)$, we obtain that $Q_\varepsilon \in W^{1,p'}(\Omega_\varepsilon)$ and that $Q_0 \in W^{1,p'}(\Omega_0)$. Using Definition 4.8, it is easy to see that $a_\varepsilon(Q_\varepsilon - Q_0, v)$ is well-defined.

Proof of Proposition 4.3. In the following paragraphs, we fix $p \in (1, +\infty)$ and $v \in W^{1,p}(\Omega_0)$. We recall now that Q_ε solves weakly (114), which is the following PDE:

$$\begin{cases} -\Delta Q_\varepsilon + cQ_\varepsilon = 0 & \text{in } \Omega_\varepsilon; \\ \frac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} + \frac{w_0}{2} Q_\varepsilon = \frac{w_0}{2} Q_\varepsilon^0 & \text{on } \Gamma_\varepsilon; \\ \frac{\partial Q_\varepsilon}{\partial \nu_R} + \frac{w_0}{2} Q_\varepsilon = \frac{w_0}{2} Q_R & \text{on } \Gamma_R. \end{cases}$$

Then, using the integration by parts formula, we get:

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} (-\Delta Q_\varepsilon + cQ_\varepsilon) \cdot v \, d(x, y) \\ &= \int_{\Omega_\varepsilon} (\nabla Q_\varepsilon \cdot \nabla v + cQ_\varepsilon \cdot v) \, d(x, y) - \int_{\Gamma_\varepsilon} \frac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} \, d\sigma_\varepsilon - \int_{\Gamma_R} \frac{\partial Q_\varepsilon}{\partial \nu_R} \, d\sigma_R \\ &= \int_{\Omega_\varepsilon} (\nabla Q_\varepsilon \cdot \nabla v + cQ_\varepsilon \cdot v) \, d(x, y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} (Q_\varepsilon - Q_\varepsilon^0) \cdot v \, d\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} (Q_\varepsilon - Q_R) \cdot v \, d\sigma_R \\ &= a_\varepsilon(Q_\varepsilon, v) - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, d\sigma_\varepsilon - \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \, d\sigma_R, \end{aligned}$$

according to Definition 4.8. Hence

$$a_\varepsilon(Q_\varepsilon, v) = \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, d\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \, d\sigma_R. \quad (149)$$

For Q_0 , we remind the reader that it solves weakly 115, which is the following PDE:

$$\begin{cases} -\Delta Q_0 + cQ_0 = 0 & \text{in } \Omega_0; \\ \frac{\partial Q_0}{\partial \nu_0} + \frac{w_{ef}}{2} Q_0 = \frac{w_{ef}}{2} Q_{ef} & \text{on } \Gamma_0; \\ \frac{\partial Q_0}{\partial \nu_R} + \frac{w_0}{2} Q_0 = \frac{w_0}{2} Q_R & \text{on } \Gamma_R. \end{cases}$$

Then, using the previous PDE and the integration by parts formula, we get:

$$\begin{aligned} 0 &= \int_{\Omega_0} (-\Delta Q_0 + cQ_0) \cdot v \, d(x, y) \\ &= \int_{\Omega_0} (\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v) \, d(x, y) - \int_{\Gamma_0} \frac{\partial Q_0}{\partial \nu_0} \cdot v \, d\sigma_0 - \int_{\Gamma_R} \frac{\partial Q_0}{\partial \nu_R} \cdot v \, d\sigma_R \\ &= \int_{\Omega_0} (\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v) \, d(x, y) + \frac{w_{ef}}{2} \int_{\Gamma_0} (Q_0 - Q_{ef}) \cdot v \, d\sigma_0 + \\ &\quad + \frac{w_0}{2} \int_{\Gamma_R} (Q_0 - Q_R) \cdot v \, d\sigma_R \\ &= a_\varepsilon(Q_0, v) + \int_{\Omega_0 \setminus \Omega_\varepsilon} (\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v) \, d(x, y) + \frac{w_{ef}}{2} \int_{\Gamma_0} (Q_0 - Q_{ef}) \cdot v \, d\sigma_0 - \\ &\quad - \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \, d\sigma_R - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, d\sigma_\varepsilon, \end{aligned}$$

where we have used Definition 4.8. Using (149), we obtain that:

$$a_\varepsilon(Q_0, v) - a_\varepsilon(Q_\varepsilon, v) = - \int_{\Omega_0 \setminus \Omega_\varepsilon} (\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v) \, d(x, y) - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, d\sigma_\varepsilon - \\ - \frac{w_{ef}}{2} \int_{\Gamma_0} (Q_0 - Q_{ef}) \cdot tv \, d\sigma_0 + \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, d\sigma_\varepsilon$$

Using Definition 4.9, we already notice that:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, d\sigma_\varepsilon - \frac{w_{ef}}{2} \int_{\Gamma_0} (Q_0 - Q_{ef}) \cdot v \, d\sigma_0 + \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, d\sigma_\varepsilon. \quad (150)$$

Since $\Gamma_\varepsilon = \{(x, \varepsilon\varphi(x/\varepsilon)) \mid x \in [0, 2\pi]\}$, $\Gamma_0 = \{(x, 0) \mid x \in [0, 2\pi]\}$ and $\Gamma_R = \{(x, R) \mid x \in [0, 2\pi]\}$, we obtain:

$$\frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, d\sigma_\varepsilon = \frac{w_0}{2} \int_0^{2\pi} Q_\varepsilon^0(x/\varepsilon) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx, \\ \frac{w_{ef}}{2} \int_{\Gamma_0} (Q_0 - Q_{ef}) \cdot v \, d\sigma_0 = \frac{w_0}{2} \int_0^{2\pi} (Q_0(x, 0) \cdot \gamma - Q_{ef} \cdot \gamma) \cdot v(x, 0) \, dx$$

and

$$\frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, d\sigma_\varepsilon = \frac{w_0}{2} \int_0^{2\pi} Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx,$$

where we have used (111), (113) and Remark 4.5.

Using the previous equalities in (150), we obtain that:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 - \frac{w_0}{2} \int_0^{2\pi} (v(x, \varepsilon\varphi(x/\varepsilon)) - v(x, 0)) \cdot \gamma_\varepsilon(x/\varepsilon) \cdot Q_\varepsilon^0(x/\varepsilon) \, dx - \\ - \frac{w_0}{2} \int_0^{2\pi} v(x, 0) \cdot \gamma_\varepsilon(x/\varepsilon) \cdot Q_\varepsilon^0(x/\varepsilon) \, dx + \frac{w_0}{2} \int_0^{2\pi} v(x, 0) \cdot \gamma \cdot Q_{ef} \, dx - \\ - \frac{w_0}{2} \int_0^{2\pi} Q_0(x, 0) \cdot v(x, 0) \cdot \gamma \, dx + \frac{w_0}{2} \int_0^{2\pi} Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx.$$

We see now that the second integral from the right-hand side from the last equality is I_{21} , according to Definition 4.9, and that the next two terms generate I_{22} , according to the same definition as before. Hence:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 + I_{21} + I_{22} + \\ + \frac{w_0}{2} \int_0^{2\pi} (Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x, 0) \cdot v(x, 0)) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx + \\ + \frac{w_0}{2} \int_0^{2\pi} Q_0(x, 0) \cdot v(x, 0) \cdot \gamma_\varepsilon(x/\varepsilon) \, dx - \frac{w_0}{2} \int_0^{2\pi} Q_0(x, 0) \cdot v(x, 0) \cdot \gamma \, dx.$$

The last equality proves (117). □

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