# MONODROMY CONJECTURE FOR SEMI-QUASIHOMOGENEOUS HYPERSURFACES 

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#### Abstract

We give a proof the monodromy conjecture relating the poles of motivic zeta functions with roots of $b$-functions for isolated quasihomogeneous hypersurfaces, and more generally for semi-quasihomogeneous hypersurfaces. We also give a strange generalization allowing a twist by certain differential forms.


## 1. Introduction

The strong monodromy conjecture of Igusa and Denef-Loeser predicts that the order of a pole of the motivic zeta function $Z_{f}^{m o t}(s)$ of a nonconstant polynomial $f \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ is less than or equal to its multiplicity as a root of the $b$-function $b_{f}(s)$ of $f$. The conjecture is open even in the case $f$ has isolated singularities. In this note we prove it for a subclass of isolated hypersurface singularities:

Theorem 1.1. The strong monodromy conjecture holds for semi-quasihomogeneous hypersurface singularities.

Recall that a germ of holomorphic function on a complex manifold is said to define a quasihomogeneous hypersurface singularity if it is analytically isomorphic to the germ at the origin of a weighted homogeneous polynomial. A hypersurface singularity is semi-quasihomogeneous if it is analytically isomorphic to the germ at the origin of a polynomial $f=f_{d}+f_{>d}$ where $f_{d}$ is a weighted homogeneous polynomial of degree $d$ with an isolated singularity at the origin, and $f_{>d}$ is a finite linear combination of monomials of weighted degree $>d$. We call such polynomials $f$ semi-weighted homogeneous of initial degree $d$.

Theorem 1.1 follows from the next one, which we prove using the main result of $[\mathrm{L}+20]$ allowing computations of motivic zeta functions from partial embedded resolutions:
Theorem 1.2. Let $w_{0}, \ldots, w_{n} \in \mathbb{Z}_{>0}^{n+1}$ be a weight vector. Let $f \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ be a semiweighted homogeneous polynomial of initial degree $d$ with respect to these weights. Assume $f_{d}$ is irreducible (this is automatic if $n>1$ ). Then the poles of $Z_{f}^{m o t}(s)$ are of order at most one and are contained in $\left\{-1,-\frac{w_{0}+\ldots+w_{n}}{d}\right\}$ if $w_{0}+\ldots+w_{n} \neq d$, otherwise -1 is the only pole of $Z_{f}^{\text {mot }}(s)$ and it has order at most two.

A similar result holds if $\mathbb{C}$ is replaced by a field of characteristic zero, see Remark 2.3.1. The case $n=1$ and $f_{d}$ is reducible is also easy to deal with but it depends on some classification, see Remark 2.2.1.

Both results, even for the isolated weighted homogeneous case, do not seem recorded in the literature.

The version of Theorem 1.2 for Igusa's local zeta functions is [Z01, Theorem 3.5]. It is known that motivic zeta functions specialize to Igusa's $p$-adic local zeta functions. Thus

Theorem 1.2 implies the characteristic-zero case of [Z01, Theorem 3.5], giving it a different proof. The version of Theorem 1.1 for Igusa local zeta functions of semi-weighted homogeneous polynomials with an additional non-degeneracy assumption is [Z01, Corollary 3.6].

Remark 1.1. A homogeneous polynomial with an isolated singularity does not have to be Newton nondegenerate, e.g. $(x+y)^{2}+x z+z^{2}$, from [Ko76, 1.21]. Thus the existing results on motivic zeta functions for nondegenerate polynomials do not suffice to prove any of the two theorems from above.

We also give a strange generalization of Theorem 1.1. Let $g\left(x_{0}, \ldots, x_{n}\right)$ be another polynomial. One has the twisted motivic zeta function $Z_{f, g}^{m o t}(s)$ obtained by replacing the algebraic top-form $\mathrm{d} x$ with $g \mathrm{~d} x$, see (3). One also has the twisted $b$-function $b_{f, g}(s)$ obtained by replacing $f^{s}$ with $g f^{s}$, see (5). We denote by $(\partial f)$ be the Jacobian ideal of $f$ in the ring $\mathcal{O}=\mathbb{C}\left\{x_{0}, \ldots, x_{n}\right\}$ of convergent power series, that is, the ideal generated by the first order partial derivatives of $f$.

Theorem 1.3. With the assumptions of Theorem 1.2, let $g=x^{\beta}$ with $\beta \in \mathbb{N}^{n+1}$ be a monomial. Let $l(\beta)=\sum_{i=0}^{n} w_{i}\left(\beta_{i}+1\right) / d$. Assume that
(1) $\quad f_{d}$ contains no monomial $x_{i} x_{j}^{k}$ of weighted degree d with $i \neq j$ and $k>0$ if $\beta_{i} \neq 0$
and

$$
\begin{equation*}
g+(\partial f) \not \subset \sum_{l(\gamma)>l(\beta)} \mathcal{O} x^{\gamma}+(\partial f) \tag{2}
\end{equation*}
$$

(if $f=f_{d}$ is weighted homogeneous (2) is equivalent to $g \notin(\partial f)$ ). Then the order of a pole of $Z_{f, g}^{\text {mot }}(s)$ is less than or equal to its multiplicity as a root of $b_{f, g}(s)$.

This provides the first cases for which a question of Mustaţă [M10] has a positive answer.
Theorem 1.3 follows from properties of the microlocal $V$-filtration together with the following (not strange) generalization of Theorem 1.2:
Theorem 1.4. With the assumptions of Theorem 1.2, let $g=x^{\beta}$ with $\beta \in \mathbb{N}^{n+1}$ be a monomial satisfying (1). Then the poles of $Z_{f, g}^{m o t}(s)$ are of order at most one and are contained in $\{-1,-l(\beta)\}$ if $l(\beta) \neq 1$, otherwise -1 is the only pole of $Z_{f, g}^{m o t}(s)$ and has order at most two.

Remark 1.2. We explain why Theorem 1.3 is a strange generalization of the strong monodromy conjecture. A twisted generalization of Theorem 1.1 relating the poles of $Z_{f, g}^{\text {mot }}(s)$ to the roots of $b_{f, g}(s)$ is not true. For example, Theorem 1.3 is not true for arbitrary monomials:
(i) Take $f=y^{2}-x^{3}$ with weight vector $w=(2,3)$ for $(x, y)$, and let $g=y$. Then $-l(\beta)=-\frac{8}{6}$ is a pole of $Z_{f, g}^{m o t}(s)$, but it is not a root of $b_{f, g}(s)=(s+1)\left(s+\frac{11}{6}\right)\left(s+\frac{13}{6}\right)$. Here $g \in(\partial f)$ fails (2).
(ii) Take $f=y^{3}-x^{7}+x^{5} y$ with weight vector $w=(3,7)$, and let $g=x^{6}$. Then $-l(\beta)=-\frac{28}{21}$ is a pole of $Z_{f, g}^{m o t}(s)$, but it is not a root of $b_{f, g}(s)$, since one can check that $-\frac{29}{21}$ is the biggest root of $\frac{b_{f, g}(s)}{s+1}$. Here $g$ fails (2) in a more subtle way since $x^{4} y \in g+(\partial f)$ and $l\left(x^{4} y\right)=\frac{29}{21}>\frac{28}{21}$.
Remark 1.3. A (not strange) generalization of the strong monodromy conjecture was posed in [Bu15]: for any polynomials $f$ and $g$, the poles of $Z_{f, g}(s)$ should be roots of the monic
polynomial $b(s)$ generating the specialization of the Bernstein-Sato ideal $B_{(f, g)} \subset \mathbb{C}\left[s_{1}, s_{2}\right]$ to $\left(s_{1}, s_{2}\right)=(s, 1)$. (In the example (i) from Remark 1.2, $b(s)=\prod_{k=6,8,10,11,13,14,16}(6 s+k)$ so $-\frac{8}{6}$ is a root.) In fact, it is more generally conjectured in [Bu15] that the polar locus of the multi-variable motivic zeta function $Z_{F}^{\text {mot }}\left(s_{1}, \ldots, s_{r}\right)$ is contained in the zero locus in $\mathbb{C}^{r}$ of the Bernstein-Sato ideal $B_{F} \subset \mathbb{C}\left[s_{1}, \ldots, s_{r}\right]$ for any tuple of polynomials $F=\left(f_{1}, \ldots, f_{r}\right)$.

Remark 1.4. Condition (2) on $g=x^{\beta}$ implies that $l(\beta)$ is a spectral number of $f$, see 3.3 and $[\mathrm{J}+19,1.6]$. More generally, we say that $g \in \mathbb{C}\left[x_{0}, \ldots, x_{n+1}\right]$ achieves the spectral number $\alpha>0$ of a polynomial $f$ with an isolated singularity if the class of $g \mathrm{~d} x$ is nonzero in $G r_{V}^{\alpha} \Omega_{f}^{n+1}$, see 3.3. Then one can view the above results as partial confirmation of:
Question 1.1. Let $f \in \mathbb{C}\left[x_{0}, \ldots, x_{n+1}\right]$ be a semi-weighted homogeneous polynomial. Let $\alpha$ be a spectral number of $f$ at the origin. Does there exist $g \in \mathbb{C}\left[x_{0}, \ldots, x_{n+1}\right]$ achieving the spectral number $\alpha$ such that the only non-integral pole of $Z_{f, g}^{\text {mot }}(s)$ is $-\alpha$ ?

## Remark 1.5.

(i) The eigenvalue version of the question is true for all polynomials $f$ with an isolated singularity: if $\lambda$ is an eigenvalue of the monodromy of $f$ at the origin, there exists $g \in$ $\mathbb{C}\left[x_{0}, \ldots, x_{n+1}\right]$ such that $Z_{f, g}^{\text {mot }}(s)$ with only one non-integral pole $-\alpha$ such that $e^{2 \pi i \alpha}=\lambda$ by [CV17].
(ii) The $b$-function version of the question is not true for all polynomials $f$ with an isolated singularity: $-\frac{6}{13}$ is a root of $b_{f}(s)$ if $f=x y^{5}+x^{3} y^{2}+x^{4} y$, but $-\frac{6}{13}$ not a pole of $Z_{f, g}^{m o t}(s)$ for any $g$ by [Bo13, Remark 3.1]. Here $f$ is not semi-weighted homogeneous and $\frac{6}{13}$ is not a spectral number of $f$.
(iii) The spectral version, namely Question 1.1, is not true for all polynomials $f$ with an isolated singularity: take $f=\left(y^{2}-x^{3}\right)^{2}-x^{5} y$, then $\frac{5}{12}$ is a spectral number at the origin. Here $\frac{5}{12}$ is also the $\log$ canonical threshold $\operatorname{lct}(f)$. It is known, with the same proof as for $g=1$, that the maximal pole of $Z_{f, g}^{m o t}(s)$ is the negative of

$$
l c t_{g}(f)=\min \left\{\alpha>0 \mid g \notin \mathcal{J}\left(f^{\alpha}\right)\right\}
$$

where $\mathcal{J}\left(f^{\alpha}\right)$ are the multiplier ideals of $f$, cf. [M10], [DM20]. Thus the only $g$ with $Z_{f, g}^{\text {mot }}(s)$ having $-\frac{5}{12}$ as a pole must satisfy that $\operatorname{lct}_{g}(f)=\operatorname{lct}(f)$. Therefore $g(0) \neq 0$ and hence $Z_{f, g}^{\text {mot }}(s)$ has the same poles as $Z_{f}^{\text {mot }}(s)$. Since $f$ is an irreducible plane curve with 2 Puiseux pairs, one can compute that $-\frac{5}{12}$ and $-\frac{11}{26}$ are the only non-integral poles of $Z_{f}^{m o t}(s)$.

Remark 1.6. It is known that $\frac{b_{f}(s)}{s+1}$ is the minimal polynomial of the action of $s$ on $\tilde{H}_{f}^{\prime \prime} / s \tilde{H}_{f}^{\prime \prime}$, where $\tilde{H}_{f}^{\prime \prime}$ is the saturation of the Brieskorn lattice, by a result of Malgrange and Pham, see [Sa94]. In light of the strong monodromy conjecture, a natural question is if the canonical splitting of the class of $[\mathrm{d} x]$ in $\tilde{H}_{f}^{\prime \prime} / s \tilde{H}_{f}^{\prime \prime}$ is a linear combination of classes $\omega_{\alpha} \in \tilde{H}_{f}^{\prime \prime} / s \tilde{H}_{f}^{\prime \prime}$, such that $\alpha$ are the non-trivial poles of $Z_{f}^{\text {mot }}(s)$ and $\omega_{\alpha}$ are eigenvectors for $s$ with eigenvalue $\alpha$. While this is true in the isolated weighted homogeneous case, it is not true in general: example (iii) from Remark 1.5 is a counterexample.

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## 2. Motivic zeta functions

2.1. Motivic zeta functions. Consider two regular functions $f, g: X \rightarrow \mathbb{C}$ on a smooth complex algebraic variety $X$, with $f$ non-invertible. Let $\mu: Y \rightarrow X$ be an embedded resolution of $f g$. Let $E_{i}$ with $i \in J$ be the irreducible components of the pullback $\mu^{*}(\operatorname{div}(f))$ of the divisor of $f, E_{I}^{\circ}=\cap_{i \in I} E_{i} \backslash \cup_{i \in J \backslash I} E_{i}$ for $I \subset J$, and $\mu^{*}(\operatorname{div}(f))=\sum_{i \in J} N_{i} E_{i}$. Let $K_{\mu}-\mu^{*}(\operatorname{div}(g))=\sum_{i \in J}\left(\nu_{i}-1\right) E_{i}$ where $K_{\mu}$ is the relative canonical divisor.

Define

$$
\begin{equation*}
Z_{f, g}^{m o t}(s):=\mathbb{L}^{-(n+1)} \sum_{\emptyset \neq I \subset J}\left[E_{I}^{\circ}\right] \prod_{i \in I} \frac{(\mathbb{L}-1) \mathbb{L}^{-\left(N_{i} s+\nu_{i}\right)}}{1-\mathbb{L}^{-\left(N_{i} s+\nu_{i}\right)}} \tag{3}
\end{equation*}
$$

where $\left[E_{I}^{\circ}\right]$ is the class of $E_{I}^{\circ}$ in the localization $\mathcal{M}=K_{0}\left(\operatorname{Var}_{\mathbb{C}}\right)\left[\mathbb{L}^{-1}\right]$ of the Grothendieck ring of complex varieties along the class $\mathbb{L}=\left[\mathbb{A}^{1}\right]$. The definition is independent of the choice of $\mu$.

The smallest set $\Omega$ of rational numbers $-\frac{\nu}{N}$ with multiplicities such that $Z_{f, g}^{\operatorname{mot}}(s)$ is a rational function in $1-\mathbb{L}^{-(N s+\nu)}$ with $\frac{\nu}{N} \in \Omega$ (with pole orders given by the multiplicities) over the $\operatorname{ring} \mathcal{M}\left[\mathbb{L}^{-s}\right]$, is called the set of poles of $Z_{f, g}^{\text {mot }}(s)$.

When $g=1, Z_{f, g}^{m o t}(s)$ is denoted $Z_{f}^{m o t}(s)$, the usual Denef-Loeser zeta function of $f$.
2.2. Proof of Theorem 1.2. Let $\mu: Y \rightarrow \mathbb{C}^{n+1}$ be the $w$-weighted blowup of the origin. Let $E$ be the exceptional divisor and $H$ the strict transform of $\{f=0\}$. We show first that $\mu$ is an embedded $\mathbb{Q}$-resolution of $f$, see $[\mathrm{L}+20, \S 1.4]$. By definition this means that $E \cup H$ has $\mathbb{Q}$-normal crossings, that is, it is locally analytically isomorphic to the quotient of a union of coordinate hyperplanes among those given by a local system of coordinates $t_{0}, \ldots, t_{n}$ with a diagonal action of a finite abelian subgroup of $G$ of $G L_{n+1}(\mathbb{C})$, i.e. locally

$$
f \circ \mu=t_{0}^{N_{0}} \ldots t_{n}^{N_{n}}: \mathbb{C}^{n+1} / G \rightarrow \mathbb{C}
$$

for some $N_{i} \in \mathbb{N}$.
The exceptional divisor $E$ is isomorphic to the $w$-weighted projective $n$-space

$$
\mathbb{P}_{w}^{n}=\left(\mathbb{C}^{n+1} \backslash 0\right) / \sim
$$

with $\left(u_{0}, \ldots, u_{n}\right) \sim\left(\lambda^{w_{0}} u_{0}, \ldots, \lambda^{w_{n}} u_{n}\right)$ for all $\lambda \in \mathbb{C}^{*}$. Denote by $\left[u_{0}: \ldots: u_{n}\right]_{w}$ the class of a point in $\mathbb{P}_{w}^{n}$. The chart $U_{0}=\left\{u_{0} \neq 0\right\}$ of $\mathbb{P}_{w}^{n}$ is identified with the quotient

$$
\frac{1}{w_{0}}\left(w_{1}, \ldots, w_{n}\right)=\mathbb{C}^{n} / \mu_{w_{0}}
$$

of the action of the $w_{0}$-roots of unity $\lambda$ defined by

$$
\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(\lambda^{w_{1}} x_{1}, \ldots, \lambda^{w_{n}} x_{n}\right)
$$

A similar description holds for the other charts.
Write $f=f_{d}+f_{>d}$, where $f_{d}$ is the degree $d$ term, so that the origin is an isolated singularity of $f$ and $f_{d}$. Since we assumed $f_{d}$ is irreducible, $f$ is also since their singularity is isolated. Thus the strict transform $H$ is irreducible. The intersection $E \cap H$ is the hypersurface defined by $f_{d}$ in $\mathbb{P}_{w}^{n}$. By [St77, §4], the intersection $E \cap H$ has, like $E \simeq \mathbb{P}_{w}^{n}$, only abelian quotient
singularities. That proof also implies our claim about $\mu$ being an embedded $\mathbb{Q}$-resolution, as we show next.

By definition $Y \subset \mathbb{C}^{n+1} \times \mathbb{P}_{w}^{n}$ and $\mu$ is the restriction of the projection onto the first factor. The chart $Y_{0}=Y \cap\left\{u_{0} \neq 0\right\}$ of $Y$ is identified via the surjection

$$
\begin{gathered}
\phi_{0}: \mathbb{C}^{n+1} \rightarrow Y_{0} \\
\left(x_{0}, u_{1}, \ldots, u_{n}\right) \mapsto\left(\left(x_{0}^{w_{0}}, x_{0}^{w_{1}} u_{1}, \ldots, x_{0}^{w_{n}} u_{n}\right),\left[1: u_{1}: \ldots: u_{n}\right]_{w}\right)
\end{gathered}
$$

with the quotient $\frac{1}{w_{0}}\left(-1, w_{1}, \ldots, w_{n}\right)$ of $\mathbb{C}^{n+1}$ by the group of $w_{0}$-roots of unity. A similar description holds for the other charts.

In the chart $Y_{0}$, the exceptional divisor $E$ is given by $\left\{x_{0}=0\right\}$. The pullback of $f$ is given by

$$
f\left(x_{0}^{w_{0}}, x_{0}^{w_{1}} u_{1}, \ldots, x_{0}^{w_{n}} u_{n}\right)=x_{0}^{d}\left(f_{d}\left(1, u_{1}, \ldots, u_{n}\right)+x_{0} h\left(x_{0}, u_{1}, \ldots, u_{n}\right)\right)
$$

for some polynomial $h$. Thus the $\mathbb{Q}$-normal crossings condition is satisfied in this chart if $g\left(u_{1}, \ldots, u_{n}\right):=f_{d}\left(1, u_{1}, \ldots, u_{n}\right)$ is smooth on $\mathbb{C}^{n}$. We check this with the Jacobian criterion. Since $\left(\partial g / \partial u_{i}\right)\left(u_{1}, \ldots u_{n}\right)=\left(\partial f_{d} / \partial x_{i}\right)\left(1, u_{1}, \ldots, u_{n}\right)$ for $1 \leq i \leq n$, smoothness of $g$ follows from the equation

$$
d \cdot f_{d}=\sum_{i=0}^{n} w_{i} x_{i} \frac{\partial f_{d}}{\partial x_{i}}
$$

together with the fact that the origin is the only singular point of $f_{d}$. Note that $E \cap H$ is given in $Y_{0}$ by the image under $\phi_{0}$ of the zero locus of the ideal $\left(x_{0}, g\right)$. Since a similar description holds in the other charts, $E \cap H$ has abelian quotient singularities since $\phi_{0}$ is a quotient map.

Next we note that

$$
\left(\mu_{\mid Y_{0}} \circ \phi_{0}\right)^{*}\left(\mathrm{~d} x_{0} \wedge \ldots \wedge \mathrm{~d} x_{n}\right)=w_{0} x_{0}^{|w|-1} \mathrm{~d} x_{0} \wedge \mathrm{~d} u_{1} \wedge \ldots \wedge \mathrm{~d} u_{n},
$$

where $|w|=w_{0}+\ldots+w_{n}$. A similar description holds in the other charts.
We have now all the information needed to apply the formula of $[L+20]$ computing the motivic zeta function of $f$. Since $\mu$ is an embedded $\mathbb{Q}$-resolution of $f, Y$ is a disjoint union of strata $S_{k}$ characterised by the existence of data $\left(G_{k}, \mathbf{N}_{k}, \boldsymbol{\nu}_{k}\right)$ satisfying the following conditions. Locally around a generic point of $S_{k}, Y$ is analytically isomorphic to $\mathbb{C}^{n+1} / G_{k}$ for some finite abelian group $G_{k}$, acting diagonally on the coordinates $t_{0}, \ldots, t_{n}$ of $\mathbb{C}^{n+1}$ and small (i.e. not containing rotations around the hyperplanes other than the identity); $f \circ \mu$ is given by $t_{0}^{N_{0, k}} \ldots t_{n}^{N_{n, k}}$; and, the relative canonical divisor of $\mu$ is given by $t_{0}^{\nu_{0, k}-1} \ldots t_{n}^{\nu_{n, k}-1}$. Then by $[L+20$, Theorem 4]

$$
\begin{equation*}
Z_{f}^{m o t}(s)=\mathbb{L}^{-(n+1)} \sum_{k}\left[S_{k}\right] \cdot T_{k}(s) \cdot \prod_{i=0}^{n} \frac{(\mathbb{L}-1) \mathbb{L}^{-\left(N_{i, k} s+\nu_{i, k}\right)}}{1-\mathbb{L}^{-\left(N_{i, k} s+\nu_{i, k}\right)}} \tag{4}
\end{equation*}
$$

where $T_{k}(s)$ has no poles. We have showed that candidate poles from the product term in this formula contributed by $S_{k}$ are the zeros of

$$
1, d s+|w|, s+1,(s+1)(d s+|w|)
$$

if $S_{k}$ is contained in

$$
Y \backslash(E \cup H), E \backslash(E \cap H), H \backslash(E \cap H), E \cap H
$$

respectively. This finishes the proof.

Remark 2.2.1. If $n=1$ and $f_{d}$ is not irreducible, then one has a classification up to a change of holomorphic coordinates of all possible cases for $f_{d}$ in [K00, Lemmas 3.3 and 3.4].
2.3. Proof of Theorem 1.4. In the proof of Theorem 1.2 one has
$\left(\mu_{\mid Y_{0}} \circ \phi_{0}\right)^{*}\left(x_{0}^{\beta_{0}} \ldots x_{n}^{\beta_{n}} \mathrm{~d} x_{0} \wedge \ldots \wedge \mathrm{~d} x_{n}\right)=w_{0} x_{0}^{-1+\sum_{i=0}^{n} w_{i}\left(\beta_{i}+1\right)} u_{1}^{\beta_{1}} \ldots u_{n}^{\beta_{n}} \mathrm{~d} x_{0} \wedge \mathrm{~d} u_{1} \wedge \ldots \wedge \mathrm{~d} u_{n}$.
By $\left[\mathrm{L}+20\right.$, Theorem 4], the zeta function $Z_{f, x^{\beta}}^{m o t}(s)$ is as in (4) but with the relative canonical divisor replaced by the above form. Running the proof of Theorem 1.2, we note that everything works similarly. The assumption on $x^{\beta}$ implies that $x_{0} f_{d}(1, u) u_{1}^{\beta_{1}} \ldots u_{n}^{\beta_{n}}$ is $\mathbb{Q}$-snc, since $f_{d}(1, u)$ is smooth in the variables $u$ and hence its tangent cone at $\left\{x_{0}=u_{1}=\ldots=u_{n}=0\right\}$ must be a linear combination of the $u_{i}$ with $1 \leq i \leq n$.

Remark 2.3.1. All the results from the introduction admit a slight generalization by replacing in the definition (3) of the motivic zeta function the category of $\mathbb{C}$-varieties with that of $k$-varieties. This holds since all the morphisms, including the group actions, in the above proofs concerning motivic zeta functions are defined over $k$.

## 3. Bernstein-Sato polynomials

3.1. $b$-functions. Let $f, g:(X, 0) \rightarrow(\mathbb{C}, 0)$ be germs of holomorphic functions on a complex manifold with $f(0)=0$. We set $\mathcal{O}=\mathcal{O}_{X, 0}$ and $\mathcal{D}=\mathcal{D}_{X, 0}$, the ring of germs of analytic functions, respectively analytic linear differential operators. We denote by $b_{f, g}(s)$ the nonzero monic generator $b_{f, g}(s)$ of the ideal of polynomials $b(s) \in \mathbb{C}[s]$ of minimal degree satisfying

$$
\begin{equation*}
b(s) g f^{s}=P g f^{s+1} \quad \text { for some } P \in \mathcal{D}[s] . \tag{5}
\end{equation*}
$$

It is a non-trivial well-known result that $b_{f, g}(s)$ is well-defined.
If $g / f$ is not holomorphic, then $b_{f, g}(s)$ is divisible by $s+1$ by Lemma 3.2.1. Define in this case the reduced b-function

$$
\tilde{b}_{f, g}(s)=\frac{b_{f, g}(s)}{s+1} .
$$

When $g=1, b_{f, g}(s)\left(\right.$ resp. $\left.\tilde{b}_{f, g}(s)\right)$ is denoted $b_{f}(s)$ (resp. $\left.\tilde{b}_{f}(s)\right)$, the usual $b$-function (resp. reduced $b$-function) or Bernstein-Sato polynomial of $f$.

If $f, g: X \rightarrow \mathbb{C}$ are regular functions on a smooth complex affine variety, one can apply the same definitions with $\mathcal{D}$ replaced by the ring of global algebraic linear differential operators, and the resulting $b$-function is the lowest common multiple (well-defined due to finiteness of $b$-constant strata) of the local $b$-functions defined above for germs at points along $f^{-1}(0)$.
3.2. Microlocal $b$-functions. Let $f:(X, 0) \rightarrow(\mathbb{C}, 0)$ be the germ of a holomorphic function on a complex manifold with $f(0)=0$. Let $i: X \rightarrow X \times \mathbb{C}, x \mapsto(x, f(x))$ be the graph embedding of $f$. Let $t$ be the coordinate of $\mathbb{C}$. Define the rings $\mathcal{R}=\mathcal{D}\left[t, \partial_{t}\right], \tilde{\mathcal{R}}=\mathcal{R}\left[\partial_{t}^{-1}\right]$, with $\partial_{t} t-t \partial_{t}=1$. Define the $V$-filtration on $\mathcal{R}, \tilde{\mathcal{R}}$ by

$$
V^{p} \mathcal{R}=\sum_{i-j \geq p} \mathcal{D} t^{i} \partial_{t}^{j}
$$

and similarly for $\tilde{\mathcal{R}}$.
Let $\mathcal{M}$ be a $\mathcal{D}$-module. Define

$$
\mathcal{M}_{f}=\underset{6}{\mathcal{M}} \otimes_{\mathbb{C}} \mathbb{C}\left[\partial_{t}\right]
$$

the stalk of the $\mathcal{D}$-module direct image $i_{+} \mathcal{M}$ at $(0,0)$ in $X \times \mathbb{C}$. Denoting $m \otimes \partial_{t}^{i}$ by $m \partial_{t}^{i} \delta$ for a local section $m$ of $\mathcal{M}$, the left $\mathcal{R}$-module structure on $\mathcal{M}_{f}$ is defined by

$$
\xi\left(m \partial_{t}^{i} \delta\right)=(\xi m) \partial_{t}^{i} \delta-(\xi f) m \partial_{t}^{i+1} \delta, \quad t\left(m \partial_{t}^{i} \delta\right)=f m \partial_{t}^{i} \delta-i m \partial_{t}^{i-1} \delta
$$

for $\xi$ a local vector field on $(X, 0)$. Equivalently, $\delta$ the delta function of $t-f$. Define the algebraic microlocalization

$$
\tilde{\mathcal{M}}_{f}=\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}\left[\partial_{t}, \partial_{t}^{-1}\right] .
$$

Then $\tilde{\mathcal{M}}_{f}$ is a left $\tilde{\mathcal{R}}$-module with the action defined as above.
Assume from now that $\mathcal{M}$ is holonomic. Let $u$ be a local section of $\mathcal{M}_{f}$ (resp. $\tilde{\mathcal{M}}_{f}$ ). The $b$-function $b_{u}(s)$ (resp. microlocal b-function $\left.\tilde{b}_{u}(s)\right)$ is the minimal polynomial of the action of $s:=-\partial_{t} t$ on $V^{0} \mathcal{R} \cdot u / V^{1} \mathcal{R} \cdot u$ (resp. $\left.V^{0} \tilde{\mathcal{R}} \cdot u / V^{1} \tilde{\mathcal{R}} \cdot u\right)$. This is a well-defined polynomial by [Ka78], [KK80], [L87] (with $\tilde{b}_{u}(s)$ defined in terms of the usual microlocalization $M\left\{\left\{\partial_{t}^{-1}\right\}\right\}\left[\partial_{t}\right]$, but this definition can be shown to be equivalent to the one above, see [Sa94, 1.4]). If $\mathcal{M}$ has quasi-unipotent local monodromy on subsets forming a suitable Whitney stratification, this polynomial has rational roots.

It is known that $\delta$ can be identified with $f^{s}$ and $b_{m \delta}(s)$ is the monic generator of the ideal of polynomials $b(s) \in \mathbb{C}[s]$ satisfying

$$
\begin{equation*}
b(s) m f^{s}=Q(s) m f^{s+1} \tag{6}
\end{equation*}
$$

for some $Q \in \mathcal{D}[s]$, for $m \in \mathcal{M}$. Thus if $\mathcal{M}=\mathcal{O}$ and $g \in \mathcal{O}$, then

$$
b_{g \delta}(s)=b_{f, g}(s)
$$

If $g=1$, then by [Sa94, Prop. 0.3] the microlocal $b$-function is the reduced $b$-function:

$$
\tilde{b}_{f}(s)=\tilde{b}_{f \delta}(s) .
$$

The proof can be easily adapted to yield a more general result:
Lemma 3.2.1. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module and $m \in \mathcal{M}$ a local section.
(i) If $f^{-1} m \notin \mathcal{D} m$ then $b_{m \delta}(s)$ is divisible by $s+1$.
(ii) If in addition $f$ is injective on $\mathcal{D} m$, then

$$
b_{m \delta}(s)=(s+1) \tilde{b}_{m \delta}(s)
$$

(iii) In particular, if $g \in \mathcal{O}=\mathcal{M}$ and $g / f$ is not holomorphic, then $b_{f, g}(s)$ is divisible by $s+1$ and the reduced b-function $\tilde{b}_{f, g}(s)$ is the microlocal b-function $\tilde{b}_{g \delta}(s)$.

Proof. (i) Setting $s=-1$ we have $b_{m \delta}(-1) m f^{-1}=Q(-1) m$ for $Q(s)$ as in (6). Since $m f^{-1} \notin \mathcal{D} m, b_{m \delta}(-1)$ must be zero.
(ii) The proof of [Sa94, Lemma 1.6] gives without using any of the two assumptions on $m$ that there exists $P \in \partial_{t}^{-1} V^{0} \mathcal{R}$ such that $\tilde{b}_{m \delta}(s) m \delta=P m \delta$. By multiplying by $s+1=-t \partial_{t}$ one obtains that $(s+1) \tilde{b}_{m \delta}(s)$ is divisible by $b_{m \delta}(s)$.

Conversely, we show $\tilde{b}_{m \delta}(s)$ divides by $b_{m \delta}(s) /(s+1)$. One has $Q(s) f=t Q^{\prime}(s)$ for some $Q^{\prime} \in \mathcal{D}[s]$ if $Q$ is as in (6). Then

$$
(s+1)\left(\frac{b_{m \delta}(s)}{s+1}-\partial_{t}^{-1} Q^{\prime}\right) m \delta=0
$$

since $s+1=-t \partial_{t}$, using (i). It is enough to show that $t$ is injective on the algebraic microlocalization $(\mathcal{D} m)\left[\partial_{t}, \partial_{t}^{-1}\right]$, since this implies that

$$
\left(\frac{b_{m \delta}(s)}{s+1}-\partial_{t}^{-1} Q^{\prime}\right) m \delta=0
$$

by the invertibility of $\partial_{t}$. Since $(\mathcal{D} m)\left[\partial_{t}, \partial_{t}^{-1}\right]$ is exhausted by the filtration $F_{p}=\oplus_{i \leq p} \mathcal{D} m \partial_{t}^{-i} \delta$, it is enough to show that $t$ is injective on $G r_{p}^{F}$ for all $p \in \mathbb{Z}$. This is equivalent to $f$ being injective on $\mathcal{D} m$.

Since $\mathcal{M}$ is holonomic, there exists $m \in \mathcal{M}$ with $\mathcal{D} m=\mathcal{M}$ locally. The filtration on $\mathcal{M}_{f}$ (resp. $\tilde{\mathcal{M}}_{f}$ ) defined by

$$
V^{p} \mathcal{R} \cdot m \delta \quad\left(\text { resp. } V^{p} \tilde{\mathcal{R}} \cdot m \delta\right)
$$

gives rise to the decreasing $V$-filtration $V^{\alpha} \mathcal{M}_{f}$ (resp. microlocal $V$-filtration $V^{\alpha} \tilde{\mathcal{M}}_{f}$ ) indexed discretely by $\alpha \in \mathbb{C}$ with a fixed total order on $\mathbb{C}$ (if $\mathcal{M}$ has quasi-unipotent local monodromy one can take $\alpha \in \mathbb{Q}$ ), using the decomposition of the action by $s$ on quotients $V^{p} / V^{q}$ with $p<q$, see $[S a 93, \S 1]$, [Sa94, §2]. The existence of the $V$-filtration is equivalent to the existence of $b$-functions (resp. microlocal $b$-functions). The $V$-filtration (resp. microlocal $V$-filtration) is uniquely characterized by:
(i) $V^{p} \mathcal{R} \cdot V^{\alpha} \mathcal{M}_{f} \subset V^{\alpha+p} \mathcal{M}_{f}$,
(ii) $V^{\alpha} \mathcal{M}_{f}$ are lattices of $\mathcal{M}_{f}$, i.e. finite $V^{0} \mathcal{R}$-modules generating $\mathcal{M}_{f}$ over $\mathcal{R}$, and
(iii) $s+\alpha$ is nilpotent on $G r_{V}^{\alpha} \mathcal{M}_{f}$
(resp. similar conditions with $\tilde{\mathcal{R}}, \tilde{\mathcal{M}}_{f}$ replacing $\mathcal{R}, \mathcal{M}_{f}$ ).
One has

$$
V^{\alpha} \mathcal{M}_{f}=\left\{u \in \mathcal{M}_{f} \mid b_{u}(s) \text { has all roots } \leq-\alpha\right\}
$$

by [S87], [Sa93, Cor. 1.7]. The same proof, relying on the unique characterization from above, and using that $V^{1} \tilde{\mathcal{R}}=\partial_{t}^{-1} V^{0} \tilde{\mathcal{R}}$ instead of $V^{1} \mathcal{R}=t V^{0} \mathcal{R}$ in [Sa93, (1.7.1)], gives:
Proposition 3.2.1. $V^{\alpha} \tilde{\mathcal{M}}_{f}=\left\{u \in \tilde{\mathcal{M}}_{f} \mid \tilde{b}_{u}(s)\right.$ has all roots $\left.\leq-\alpha\right\}$.
One defines the microlocal $V$-filtration $\tilde{V}^{\alpha} \mathcal{O}$ on $\mathcal{O}$ by

$$
\left(\tilde{V}^{\alpha} \mathcal{O}\right) \delta=(\mathcal{O} \delta) \cap V^{\alpha} \tilde{\mathcal{M}}_{f}
$$

for $\mathcal{M}=\mathcal{O}$. Lemma 3.2.1 and Proposition 3.2.1 imply:
Corollary 3.2.1. $\left(\tilde{V}^{\alpha} \mathcal{O}\right) \backslash f \mathcal{O}=\left\{g \in \mathcal{O} \backslash f \mathcal{O} \mid \tilde{b}_{f, g}(s)\right.$ has all roots $\left.\leq-\alpha\right\}$.
3.3. Brieskorn lattices. Let $f:(X, 0) \rightarrow(\mathbb{C}, 0)$ be the germ of a holomorphic function on a complex manifold of dimension $n+1$ with $f(0)=0$ and $f$ having an isolated singularity at 0 . There is a diagram

where the lower map is an isomorphism of $\mathbb{C}$-vector spaces, the vertical map is surjective, and the top map is an inclusion. Here $\mathcal{O}=\mathcal{O}_{X, 0},(\partial f)$ is the ideal generated by the first order partial derivatives of $f, \Omega^{p}$ consists of the germs of the holomorphic $p$-forms at the
origin, $\mathcal{O} /(\partial f)$ is called the Milnor algebra, $H_{f}^{\prime \prime}$ is called the Brieskorn lattice, and $G_{f}$ is called the Gauss-Manin system. The Brieskorn lattice is a free module of rank equal to the Milnor number $\mu_{f}=\operatorname{dim}_{\mathbb{C}} \mathcal{O} /(\partial f)$ over $\mathbb{C}\{t\}$ and also over $\mathbb{C}\left\{\left\{\partial_{t}^{-1}\right\}\right\}$, where the action of $t$ is given by multiplication by $f$ and the action of $\partial_{t}^{-1}$ is defined by $\partial_{t}^{-1}[\omega]=[\mathrm{d} f \wedge \eta]$ for $\mathrm{d} \eta=\omega$. The Gauss-Manin system is the localization of $H_{f}^{\prime \prime}$ by the action of $\partial_{t}^{-1}$. It is a free $\mathbb{C}\left\{\left\{\partial_{t}^{-1}\right\}\right\}\left[\partial_{t}\right]$-module of rank $\mu_{f}$ with an action of $t$, and it is a regular holonomic $\mathcal{D}$-module. Consequently it admits the rational $V$-filtration such that $\partial_{t} t-\alpha$ is nilpotent on $G r_{V}^{\alpha} G_{f}$. The induces a $V$-filtration on $H_{f}^{\prime \prime}$ and on the quotient $\Omega_{f}^{n+1}$. See [Br70, Seb70, Sa89]. On the other hand, the microlocal $V$-filtration $\tilde{V}^{\alpha} \mathcal{O}$ defined above induces a filtration on the quotient. These two filtrations are the same:

Proposition 3.3.1. [J+19, Proposition 1.4] The microlocal $V$-filtration on the Milnor algebra $\mathcal{O} /(\partial f)$ agrees with the $V$-filtration on $\Omega_{f}^{n+1}$.
3.4. Semi-weighted homogeneous polynomials. Assume now that $f=f_{d}+f_{>d}$ is a semi-weighted homogeneous polynomial of initial degree $d$ for the weight vector $\left(w_{0}, \ldots, w_{n}\right) \in$ $\mathbb{N}^{n+1}$. For $\beta \in \mathbb{N}^{n+1}$ define $l(\beta)=\sum_{i=0}^{n} w_{i}\left(\beta_{i}+1\right) / d$. Define a new filtration $V_{\omega}$ on $\mathcal{O}$ by setting for $\alpha \in \mathbb{Q}$

$$
V_{w}^{\alpha} \mathcal{O}=\sum_{l(\beta) \geq \alpha} \mathcal{O} x^{\beta}
$$

Proposition 3.4.1. $[J+19,1.6]$ If $f=f_{d}+f_{>d}$ is a semi-weighted homogeneous polynomial of initial degree $d$ for the weight vector $\left(w_{0}, \ldots, w_{n}\right) \in \mathbb{N}^{n+1}$, then the $V$-filtration on $H_{f}^{\prime \prime}$ is the quotient of the filtration $V_{w}$.

Corollary 3.4.1. With assumptions as in Proposition 3.4.1, let $g \in \sum_{l(\beta) \geq \alpha} \mathcal{O} x^{\beta} \subset \mathcal{O}$ for some $\alpha \in \mathbb{Q}$ such that

$$
g+(\partial f) \not \subset \sum_{l(\beta)>\alpha} \mathcal{O} x^{\beta}+(\partial f) .
$$

Then $-\alpha$ is the biggest root of $\tilde{b}_{g \delta}(s)$.
Proof. The assumption is equivalent in general with $[g \mathrm{~d} x] \neq 0$ in $G r_{V}^{\alpha} \Omega_{f}^{n+1}$, by Proposition 3.4.1. It follows from Proposition 3.3.1 that $g \delta$ is non-zero in $G r_{V}^{\alpha} \tilde{\mathcal{M}}_{f}$ for $\mathcal{M}=\mathcal{O}$ in the notation of 3.2. This implies that the maximal root of $\tilde{b}_{g \delta}(s)$ is $-\alpha$ by Proposition 3.2.1.
3.5. Proof of Theorem 1.1. Let $h \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ define a semi-quasihomogeneous hypersurface singularity locally analytically isomorphic to $f$ as in Theorem 1.2. The proof of Theorem 1.2 only uses local analytic properties of $f$, hence it also holds for $h$. Now, the local $b$-function of $h$ is an analytic invariant and thus equals that of $f$. By Corollary 3.4.1 for the monomial $g=1$, one has that $(s+1)(s+|w| / d)$ divides the local $b$-function of $f$. This finishes the proof for the case $n>1$. If $n=1$ one can analyze directly the classification of $f_{d}$ following Remark 2.2 .1 (this is also a particular case of result [Lo88] for general plane curves.)
3.6. Proof of Theorem 1.3. It follows from Theorem 1.4 together with Corollary 3.4.1 and Lemma 3.2.1 (iii). The equivalency of the assumption in the case $f=f_{d}$ follows by considering the decomposition into weighted homogeneous terms, since $(\partial f)$ is generated by weighted homogeneous polynomials in this case.

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## Erratum (September 7, 2023)

A mistake in Section 3 was pointed out by M. Saito. We thank him for pointing this out. We address it here. All results in the Introduction remain true.

The definition of the microlocal $V$-filtration on $\mathcal{O}$ before Corollary 3.2.1 should be

$$
(\mathcal{O}, \tilde{V})=\operatorname{Gr}_{F}^{0}\left(\tilde{M}_{f}, V\right)
$$

Corollary 3.2.1 is false for the microlocal $V$-filtration on $\mathcal{O}$, and counterexamples are provided by M. Saito in [Sa23, 2.2]. Corollary 3.2.1 is however not used in the rest of the paper. The mistake propagates though to Corollary 3.4.1 since Propositions 3.3.1 and 3.4.1 should be used with the correct microlocal $V$-filtration. Proposition 3.4.1 was first asserted in [J+19, 1.6], see also [Sa23, 2.2d]. We adjust the conclusion of Corollary 3.4.1:

Corollary 3.6.1. With assumptions as in Proposition 3.4.1, let $g \in \sum_{l(\beta) \geq \alpha} \mathcal{O} x^{\beta} \subset \mathcal{O}$ for some $\alpha \in \mathbb{Q}$ such that

$$
g+(\partial f) \not \subset \sum_{l(\beta)>\alpha} \mathcal{O} x^{\beta}+(\partial f)
$$

Then $-\alpha$ is a root of $b_{f, g}(s)$.
Proof. The assumption is equivalent in general with $[g \mathrm{~d} x] \neq 0$ in $G r_{V}^{\alpha} \Omega_{f}^{n+1}$, by Proposition 3.4.1. Since $V$ on $\Omega_{f}^{n+1}$ is the quotient of the $V$-filtration on $H_{f}^{\prime \prime}$, it follows that $[g \mathrm{~d} x] \neq 0$ in $G r_{V}^{\alpha} H_{f}^{\prime \prime}$. Then $-\alpha$ is a root of $b_{f, g}(s)$, cf. [Sa23, (2.2.4)]. We repeat that argument here. Let $b(s)=b_{f, g}(s)$. By assumption there is $P(s) \in \mathcal{D}[s]$ such that $b(s) g f^{s}=P(s) f g f^{s}$. Equivalently, $b(s) g \delta=P(s) f g \delta$ in $\mathcal{M}_{f}$. The filtered de Rham complex $\left(\mathrm{DR}_{X}\left(\mathcal{M}_{f}\right), V\right)$ is strict and the induced filtration is the $V$-filtration on the Gauss-Manin system $H^{n}\left(\mathrm{DR}_{X}\left(\mathcal{M}_{f}\right)\right)=G_{f}=H_{f}^{\prime \prime}\left[\partial_{t}\right]$. Then one has the relation $b(s)[g \mathrm{~d} x]=\left[P_{0}(s) f g \mathrm{~d} x\right]$ in $G_{f}$, where $\mathrm{d} x=\mathrm{d} x_{0} \wedge \ldots \wedge \mathrm{~d} x_{n}$ and $P_{0}(s) \in \mathcal{O}[s]$ coincides with $P(s)$ modulo $\sum_{i} \partial_{x_{i}} \mathcal{D}[s]$ by definition of $\mathrm{DR}_{X}$. Note that the class of $[g \mathrm{~d} x]$ is non-zero in $\mathrm{Gr}_{V}^{\alpha} G_{f}$ since it is non-zero in $\operatorname{Gr}_{V}^{\alpha} H_{f}^{\prime \prime}$ and $V$ on $H_{f}^{\prime \prime}$ is induced from $V$ on $G_{f}$. We show that $\left[P_{0}(s) f g \mathrm{~d} x\right] \in V^{>\alpha} G_{f}$. By the nilpotency property of $V$-filtrations, some power of $s+\alpha$ must then divide $b(s)$, which finishes the proof. To show that $\left[P_{0}(s) f g \mathrm{~d} x\right] \in V^{>\alpha} G_{f}$, by summing over powers of $s$, it is enough to show that $[h f g d x]$ lies in $V^{>\alpha} H_{f}^{\prime \prime}$ for every $h \in \mathcal{O}$. This now follows from Proposition 3.4.1 on the coincidence of the $V$-filtration and $V_{w}$-filtration on $H_{f}^{\prime \prime}$ in the semi-weighted homogeneous case, and the assumption on $[g d x]$.

We now adjust the proof of Theorem 1.3 from 3.6.
Proof of Theorem 1.3. Using Corollary 3.6.1 instead of Corollary 3.4.1, the theorem is proved for the case when $l(\beta) \neq 1$.

Suppose now $l(\beta)=1$. By Theorem 1.4, it is enough to show that -1 is a root of multiplicity at least 2 of $b_{f, g}(s)$. Let write $f_{d}=\sum_{\gamma} c_{\gamma} x^{\gamma}$ where $x^{\gamma}$ are monomials of weighted degree $d$, that is, $\sum_{i} w_{i} \gamma_{i} / d=1$ for each $\gamma \in \mathbb{N}^{n+1}$ and $c_{\gamma}$ are finitely many non-zero complex numbers. Then $1=l(\beta)=\sum_{i} w_{i}\left(\beta_{i}+1\right) / d$ implies that $\gamma_{i}=\beta_{i}+1$ for each $\gamma$. Hence up to a non-zero constant factor, $f_{d}=x^{\gamma}$ is monomial. Since $f_{d}$ has isolated singularities, this can only happen if $n=1, f_{d}=x_{0} x_{1}$, and $g=1$. We know that $b_{f, g}(s)=b_{f}(s)$ is divisible by the local $b$-function of $f$ at the origin. The latter equals the local $b$-function of $f_{d}$ at the origin since it is an local analytic invariant and we can choose a new set of analytic coordinates
$\tilde{x}_{0}, \tilde{x}_{1} \in \mathbb{C}\left\{x_{0}, x_{1}\right\}$ such that $f=\tilde{x}_{0} \tilde{x}_{1}$. The claim then follows from $b_{x_{0} x_{1}}(s)=(s+1)^{2}$. This finishes the proof of Theorem 1.3.

Note that Theorem 1.3 implies Theorem 1.1. The mistake pointed above does not really affect the proof of Theorem 1.1 in 3.5: one knows (without invoking the problematic Corollary 3.4.1 or the new Corollary 3.6.1) that $(s+1)(s+|w| / d)$ divides the local $b$-function of $f$ by the constancy of the minimal exponent in $\mu$-constant deformations and by the computation of $b$-functions of weighted homogeneous isolated hypersurfaces.

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