CONES WITH CONVOLUTED GEOMETRY THAT ALWAYS SCATTER OR RADIATE

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ABSTRACT. We investigate fixed energy scattering from conical potentials having an irregular cross-section. The incident wave can be any arbitrary non-trivial Herglotz wave. We show that a large number of such local conical scatterers scatter all incident waves, meaning that the far-field will always be non-zero. In essence there are no incident waves for which these potentials would seem transparent at any given energy. We show more specifically that there is a large collection of star-shaped cones whose local geometries always produce a scattered wave. In fact, except for a countable set, all cones from a family of deformations between a circular and a star-shaped cone will always scatter any non-trivial incident Herglotz wave. Our methods are based on the use of spherical harmonics and a deformation argument. We also investigate the related problem for sources. In particular if the support of the source is locally a thin cone, with an arbitrary cross-section, then it will produce a non-zero far-field.

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1. INTRODUCTION

When a wave of constant wavenumber encounters a penetrable obstruction, it typically produces a scattered wave. The scattered wave transmits information about the obstruction, which is a fact that is exploited by various imaging modalities [16, 29]. However, an incident wave does not necessarily scatter even in the presence of a scatterer [12]. In the quantum mechanical context we call the energy of such a wave function a *non-scattering energy*. When dealing with with acoustic or electromagnetic scattering, we use the term *non-scattering wavenumber* instead. It might happen that a potential or an obstruction is effectively transparent at certain fixed energies or wavenumbers, producing a zero far-field irrespective of the incident wave (see e.g. [24, 32, 37]). A striking converse to this type of phenomenon was first studied in [8], where the authors showed that an obstacle with a corner always scatters, i.e. it always produces a scattered wave that does not decay quickly, regardless of what the incident wave is or what energy it has. This type of obstacle is not transparent to any incident wave, i.e. it is always visible. The absence of non-scattering energies has in recent years been studied by a number of authors, see e.g. [4, 5, 6, 8, 14, 15, 18, 19, 27, 36]. One of the themes of this paper is to study the absence of non-scattering-energies for potentials that are obtained from irregular cones.

It should also be pointed out that non-scattering energies are closely related to the so called *interior transmission eigenvalues* [13]. These are, in the acoustic setting, eigenvalues for a certain non elliptic and non self-adjoint spectral problem on the support of an inhomogenity of the refractive index. A non-scattering-energy is a always an interior transmission eigenvalue when the scatterer has compact support. The converse does not hold. It is well known that transmission eigenfunctions can be approximated by normalized Herglotz waves [45]. Such waves will produce arbitrarily small scattering. However this does not imply that there would be an incident wave producing *no scattering*. This only happens in the case of a non-scattering energy, and only if this sequence of Herglotz waves aproximates a non-scattering incident wave. The study of interior transmission eigenvalues has it roots in the analysis of certain numerical reconstruction methods in inverse scattering, namely the linear sampling method and the factorization method (for more on these see [16, 17, 29, 30]). Energies which admit interior transmission eigenvalues cause challenges for these methods. For a survey on the topic of interior transmission eigenvalues see [13].

Another topic related to non-scattering are sources that produce no wave in the far field. A source f in acoustic scattering $(\Delta + k^2)u = f$ produces a far-field $u^\infty(\theta) = C\hat{f}(k\theta)$, for details, see [6]. Therefore understanding source scattering at a fixed wavenumber is related to understanding the Fourier restriction problem [23]. By a nonradiating source, we mean a source that does not produce a wave in the far field at some wavenumber or energy. The topic of sources that radiate at all energies was first studied in [2], where it was shown that that nonradiating sources having a convex or nonconvex corner or edge on their boundary must vanish there. Subsequent studies of sources that always radiate include [3, 6, 7]. Here we extend the results in [2] by showing that sources obtained from very thin or very wide cones will always produce a radiating wave irrespective of their cross section.

We will describe very briefly and in a non technical wave the various concepts needed to understand our theorems. This is also to fix notation. Proper mathematical definitions are given in Section 2.

Recall that the scattering of an incident particle wave of energy $\lambda > 0$ by a potential V can be modeled by the equation

$$(1.1) \qquad\qquad (-\Delta + V - \lambda)u = 0$$

in $\mathbb{R}^n$. Here u is the superposition of the wave function u_i of an incident particle and a scattered wave u_s, which is created by the interaction between u_i and the scatterer V. The scattered wave will have an asymptotic expansion, and a detectable part of that is called the scattering amplitude or far-field pattern.

By a non-scattering energy for a potential V, we mean a $\lambda > 0$ for which there is an incident wave which produces no scattering from V at energy λ. In other words, the scattering amplitude is zero even though the incident wave is not. A potential which *always scatters* is one that does not have any non-scattering energies.

We will analyze potentials whose support has a conical singularity on its boundary. We will consider cones $C \subset \mathbb{R}^n$, $n = 3$, that are determined by a compact cross section $K \subset \mathbb{R}^{n-1}$, $0 \in \operatorname{int} K$ in suitable coordinates by

$$(1.2) \qquad\qquad C = \left\{ (tx', t) \in \mathbb{R}^n \ : \ x' \in K, t \geq 0 \right\}.$$

For technical reasons, we will also need some regularity assumptions.

Definition 1.1. We call a cone C of the form (1.2) *regular* if the following conditions are satisfied.

(i) C is contained in a strictly convex closed circular cone,

(ii) C has a connected exterior, and

(iii) C has a bounded cross-section $K \subset \mathbb{R}^2$ such that $\chi_K \in H^\tau(\mathbb{R}^2)$ for some $\tau \in (1/4, 1/2)$.

We will in particular be interested in star-shaped cones, which we define as follows.

Definition 1.2. We say that a cone C of the form (1.2) is *star-shaped* if there exists a continuous $\sigma \colon [0, 2\pi] \to (\rho_0, \pi/2)$, $\rho_0 \in (0, \pi/2)$ with $\sigma(0) = \sigma(2\pi)$ and

$$(1.3) \qquad C \cap \mathbb{S}^2 := \left\{ (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta) \in \mathbb{S}^2 \ : \ \varphi \in [0, 2\pi), \ 0 \le \vartheta < \sigma(\varphi) \right\}.$$

Such a cone is denoted by C^σ and we usually take both σ and ρ_0 given by the context.

We will relate a conical potential to a cone as follows. Given a cone C with coordinates chosen as in (1.2), let V_C be the potential

$$(1.4) \qquad\qquad\qquad\qquad V_C := \varphi \chi_C + \Phi,$$

where $\varphi \in C_c^{1/4+\varepsilon}(\mathbb{R}^3)$, $\epsilon > 0$, and $\Phi \in e^{-\gamma|\cdot|} L^2(\mathbb{R}^3)$, $\gamma > 0$, and $\operatorname{supp} \Phi \subset \{x_3 > 0\}$.

Next, we state our main result for scattering from a potential. The theorem is a direct consequence of Theorem 5.3 and Proposition 7.6 (in the latter set $C_1^\sigma = C^\sigma$, to obtain the theorem).

Theorem 1.3. *Assume that C^σ is a star-shaped cone that is regular, in the sense of definition 1.1. Then for all $\delta > 0$, there exists a cone C, such that the Hausdorff distance*[1]

$$\operatorname{dist}_H \left(\partial C \cap S^2, \partial C^\sigma \cap S^2 \right) < \delta,$$

and such that all conical potentials V_C of the form (1.4) always scatter.

Remark 1.4. We remark that more can be said: If we deform C^σ continuously to a circular cone according to Definition 7.1, then all the cones in that family will have a shape that always scatters except possibly for a countable set of exceptions given by Proposition 7.6.

The main novelty of Theorem 1.3 is that it shows that there are a large number of conical potentials that always scatter, which have an irregular geometry, and the scattering happens for *any* incident Herglotz wave. Earlier results in the case of $\mathbb{R}^3$, have dealt with fairly regular geometries such as corners between hyperplanes [8] and circular cones or curvilinear polyhedra [18, 19, 36]). Where [36] uses a similar approach as we do here, utilizing complex geometrical optics solutions, and [18, 19] use techniques from the theory of boundary value problems in corner domains. More recent results apply to more complicated geometries, such as [14] using stationary phase method and [38] using the methods from the theory of free boundary value problems. These results are not applicable to a general Herglotz incident wave and instead assume that it does not vanish on the boundary. Non vanishing is a major technical simplification. The proof in [8] would be much shorter if one would assume that $u_i(0) \ne 0$ there. This is also demonstrated in this paper between Section 3 and sections 4, 5. The main difficulty is as follows: if one assumes the source problem or a non vanishing incident wave, then one only needs to prove the non vanishing of an integral involving spherical harmonics of degree 2, as in Definition 3.1. With a general incident wave one needs instead to prove the non vanishing of the integral for all degrees, as in Definition 5.1.

Let us discuss briefly the proof of Theorem 1.3. Our starting point is the use of complex geometrical optics solutions and the theory of spherical harmonics. We use the latter to derive

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[1]Here we use the *Hausdorff distance* between the sets A and B which is given by

$$\operatorname{dist}_H(A, B) := \max \left(\sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A) \right).$$

where $d(x, A) := \inf_{y \in A} d(x, y)$, and d is induced by the Euclidean metric.

a projection condition for cones which will guarantee potentials that always scatter. We call such cones *admissible medium cones*. See Definition 5.1.

The admissibility of a cone can be further verified by means of certain determinants being non zero. We use this determinant condition to prove that circular cones are admissible medium cones (and that they therefore always scatter) by computing explicit formulas for the determinants. This gives an alternative proof for the three dimensional result of [19, 36] without the issue of a countable set of unhandled cones.

Testing for the determinant condition requires us to use certain results on the associated Legendre polynomials. We derive them in Section 8. In particular, we need a modification of the classical Christoffel-Darboux formula in page 43 of [42]. Then to analyze potentials related to star-shaped cones, we perform a deformation argument, where we interpolate between the points of a circular cone and a star-shaped cone. Spherical harmonics are analytic, which can be used to show that the determinants of the deformed cones depend analytically on the deformation parameter. Analyticity then guarantees that certain critical integrals cannot vanish except in a countable set of points, and Theorem 1.3 follows.

We would like to further point out that even though our construction of complex geometrical optics solutions follows the argument of [36], we need to modify their argument to work with cones with irregular cross section. This is done in Section 4, where we also provide some explicit examples of cones with rather complex geometry to which our argument applies but previous arguments didn't.

We turn to the related problem of sources that radiate at every energy level next. Several types of source problems in wave propagation can be modeled by the Helmholtz or static Schrödinger equation. We will consider the following,

$$(1.5) \qquad\qquad (-\Delta - \lambda)u = f$$

in $\mathbb{R}^n$ together with a radiation condition at infinity, the Sommerfeld radiation condition (2.3). It selects the outgoing wave from all possible solutions to (1.5). The above problem models a wave u created by the source f that is oscillating at energy $\lambda > 0$ and is radiating out to infinity. We will consider sources of the form

$$(1.6) \qquad\qquad f = \varphi\chi_C + \Phi,$$

where $\varphi \in L^\infty(\mathbb{R}^n)$ is compactly supported, $\Phi \in e^{-\gamma|\cdot|}L^2(\mathbb{R}^n)$ with $\gamma > 0$ is supported on a half-space, and C is a cone of the form (1.2) with vertex outside the above-mentioned half-space. It can be shown that the wave u has the asymptotic expansion

$$(1.7) \qquad\qquad u(x) = \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}}\alpha_r(\theta) + O\Big(|x|^{-\frac{n+1}{2}}\Big).$$

The function $\alpha_r \in L^2(\mathbb{S}^{n-1})$ is called the *far-field pattern* radiated by f. We say that a source term f always radiates if $\alpha_r \neq 0$ for all $\lambda > 0$.

We will be interested in sources f determined by a fairly general family of cones that we call *admissible source cones*, Definition 3.1. These cones can have very irregular cross sections, e.g. with a fractal boundary. Our main result concerning these types of sources is a direct consequence of Theorem 3.9 in Section 3.

Theorem 1.5. *Assume that $C \subset \mathbb{R}^3$ is a cone of the form* (1.2), *that is also an admissible source cone in the sense of Definition* (3.1). *Suppose furthermore that f is the source term* (1.6), *where φ is Hölder continuous at 0 and $\varphi(0) \neq 0$. Then for all $\lambda > 0$, we have that $\alpha_r \neq 0$ in* (1.7). *That is, f always radiates.*

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The first results on sources that always radiate were obtained in [2], where it was shown that sources having a convex or non-convex corner or edge on their boundary always radiate. The main novelty of Theorem 1.5 is that it shows that a source determined by a cone with a singularity at the vertex, will always radiate essentially regardless of the geometry of the cross section of the conical source. This follows because of propositions 3.6 and 3.7.

The paper is structured as follows. In Section 2 we review some preliminary results and notation that we use in the other sections. In particular, we set the notation for spherical coordinates. We then study the source problem in Section 3. After this, we construct complex geometrical optics solutions for potentials constructed from irregular cones in Section 4. In Section 5, we define the concept of an admissible medium cone and show that potentials related to these always scatter. Section 6 proves that circular cones are admissible medium cones. After this, in Section 7, we prove Theorem 1.3. Finally, in Section 8, we derive various formulas for associated Legendre polynomials that were used in earlier sections.

2. Preliminaries

We will review various results that are needed in the sequel. We start by specifying some function spaces after which we give a short review of basic scattering theory based on [1, 26]. It is the same setup as in [35, 36] for static scattering theory. The reader can find further details in these references.

Following [1, 26, 35, 36] we use the spaces $B(\mathbb{R}^n)$ and $B^*(\mathbb{R}^n)$. The space $B(\mathbb{R}^n)$ consists of those $u \in L^2(\mathbb{R}^n)$ for which the norm

$$\|u\|_{B(\mathbb{R}^n)} = \sum_{j=1}^{\infty} (2^{j-1} \int_{X_j} |u|^2 \, dx)^{1/2}$$

is finite. See Section 14.1 in [26]. Here $X_1 = \{|x| < 1\}$ and $X_j = \{2^{j-2} < |x| < 2^{j-1}\}$ for $j \geq 2$. This is a Banach space whose dual $B^*(\mathbb{R}^n)$ consists of all $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|u\|_{B^*(\mathbb{R}^n)} = \sup_{R>1} \left[\frac{1}{R} \int_{|x|<R} |u|^2 \, dx \right]^{1/2} < \infty.$$

The set $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions is dense in $B(\mathbb{R}^n)$ but not in $B^*(\mathbb{R}^n)$. Their closure in $B^*(\mathbb{R}^n)$ is denoted by $\mathring{B}^*(\mathbb{R}^n)$, and $u \in B^*(\mathbb{R}^n)$ belongs to $\mathring{B}^*(\mathbb{R}^n)$ if and only if

$$\lim_{R \to \infty} \frac{1}{R} \int_{|x|<R} |u|^2 \, dx = 0.$$

We will also use B_2^* and $\mathring{B}_2^*$, which take into account the derivatives up to second order of the function. These are defined via the norm

$$\|u\|_{B_2^*(\mathbb{R}^n)} = \sum_{|\alpha| \leq 2} \|D^\alpha u\|_{B^*(\mathbb{R}^n)} = \|u\|_{\mathring{B}_2^*(\mathbb{R}^n)}.$$

Next, we give a short review of the relevant parts of scattering theory. In static or time harmonic scattering theory of energy $\lambda > 0$ one considers the equation

$$(2.1) \qquad (-\Delta + V - \lambda)u = 0$$

in $\mathbb{R}^n$ where u is a total wave function that is the superposition of an unperturbed incident wave function u_i and a wave u_s scattered by the potential function. That is, $u = u_i + u_s$ where

$$(2.2) \qquad (-\Delta - \lambda)u_i = 0, \qquad (-\Delta + V - \lambda)u_s = V u_i$$

in $\mathbb{R}^n$. One further requires that the scattered wave u_s satisfies the Sommerfeld radiation condition

$$(2.3) \qquad \lim_{r\to\infty} r^{\frac{n-1}{2}}\left(\partial_r u_s - iku_s\right) = 0$$

uniformly over $\hat{x} = x/r$ where $r = |x|$.

The potential V is a *short range* potential, by which we mean that $V \in L^\infty(\mathbb{R}^n)$ and that there are constant $C, \epsilon > 0$, such that

$$(2.4) \qquad |V(x)| \le C(1+|x|)^{-1-\epsilon}$$

almost everywhere in $\mathbb{R}^n$. Note that this allows for potentials whose support is unbounded.

Will will furthermore assume that the incident wave is given by a Herglotz function, i.e.

$$(2.5) \qquad u_i(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\theta\cdot x} g(\theta)\, dS, \qquad g \in L^2(\mathbb{S}^{n-1}).$$

It follows that $u_i \in B_2^*(\mathbb{R}^n)$. See Proposition 2.1 in [35]. They also show that the incident particle u_i has the asymptotics

$$(2.6) \qquad u_i(x) = \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}} g(\theta) + \frac{e^{-i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}} g(-\theta) + O\left(|x|^{-\frac{n+1}{2}}\right)$$

and moreover that the scattered wave has the asymptotics

$$(2.7) \qquad u_s(x) = \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}} \alpha_s(\theta) + O\left(|x|^{-\frac{n+1}{2}}\right).$$

The function α_s is called the *scattering amplitude* or the *far field pattern*. The *relative scattering operator* is the map

$$(2.8) \qquad S_\lambda \colon g \mapsto \alpha_s, \qquad S_\lambda \colon L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}).$$

For more technical details see [17] or more generally sections 7 and 8 in [1]. An $\alpha_s \equiv 0$ implies that u_s decays quickly at infinity. We see thus define that λ is a non-scattering energy if the kernel of S_λ is non-trivial. More precisely

Definition 2.1. We call $\lambda > 0$ a *non-scattering energy* for a short range potential V if there is a $g \in L^2(\mathbb{S}^{n-1})$ such that $g \ne 0$ and $S_\lambda g = 0$.

The following Lemma provides another characterization of a non-scattering-energy.

Lemma 2.2. *Let V be a short-range potential. Then $\lambda > 0$ is a* non-scattering energy *for V if there exist functions $v, w \in B_2^*(\mathbb{R}^n)$ so that $w \not\equiv 0$ and*

$$(2.9) \qquad \begin{cases} (-\Delta + V - \lambda)v = 0, \\ \quad\;\; (-\Delta - \lambda)w = 0, \\ \quad\;\;\; v - w \in \mathring{B}_2^*(\mathbb{R}^n) \end{cases}$$

in $\mathbb{R}^n$.

Proof. Suppose that v and w are as in the claim. Now define

$$u_i := w, \qquad u_s := v - w.$$

Clearly u_i and u_s solve equations (2.2). The incident wave $u_i = w$ can be written as a Herglotz wave function by Theorem 14.3.3 in [26]. In other words there exists an $g \in L^2(\mathbb{S}^{n-1})$ so that

$$u_i(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\theta\cdot x} g(\theta)\, dS, \qquad g \in L^2(\mathbb{S}^{n-1}).$$

Furthermore since $u_s = v - w \in \mathring{B}_2^*(\mathbb{R}^n)$, we see that

$$\lim_{R \to \infty} \frac{1}{R} \int_{|x| < R} |u_s|^2 \, dx = 0$$

which would be impossible by (2.7) unless $\alpha_s \equiv 0$. Hence $S_\lambda g = 0$. ∎

We will also make use of the following Rellich-type theorem which is Theorem 4 in [44].

Theorem 2.3. *Let $n \in \{2, 3, \ldots\}$, $\lambda > 0$, $\gamma > 0$, and let $f \in e^{-\gamma|\cdot|} L^2(\mathbb{R}^n)$ be such that $f|_H \equiv 0$ for some half-space $H \subset \mathbb{R}^n$. If $u \in \mathring{B}_2^*$ solves the equation*

$$(-\Delta - \lambda) u = f$$

in $\mathbb{R}^n$ then $u|_H \equiv 0$.

We use the spherical coordinate system given below. Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. Note that we will be a bit sloppy with notation, but this will not cause confusion. For example, given a function $f \colon \mathbb{S}^2 \to \mathbb{C}$ we will denote by $f(x)$ its value at a point $x \in \mathbb{S}^2$. But we might also denote this value by $f(\vartheta, \varphi)$ and implicitly assume that $x = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$. For $x \in \mathbb{R}^3$ we use the spherical coordintes

$$(2.10) \qquad \begin{cases} x_1 = r \sin \vartheta \, \cos \varphi, \\ x_2 = r \sin \vartheta \, \sin \varphi, \\ x_3 = r \cos \vartheta, \end{cases}$$

where $r \geq 0$, $0 \leq \varphi < 2\pi$, $0 \leq \vartheta \leq \pi$.

Much of our analysis will be based on the use of spherical harmonics. We denote the space of spherical harmonics of degree N by SH^N. Moreover we will use the basis functions Y_N^m, given by

$$(2.11) \qquad Y_N^m(\varphi, \vartheta) = (-1)^m \sqrt{\frac{2N+1}{4\pi} \cdot \frac{(N-m)!}{(N+m)!}} \, e^{im\varphi} \, P_N^m(\cos \vartheta)$$

for $N \in \mathbb{N}$, $m \in \{-N, -N+1, \ldots, N-1, N\}$ and where P_N^m is the usual *associated Legendre polynomial*, which are defined by the Legendre polynomials P_N in (8.1). For more information on these, see [22, 31, 40]. We shall have an opportunity to use the following classical result from the theory of spherical harmonics, known as Laplace's second integral representation for P_N^m. See Section 63 of Chapter III in [25].

Theorem 2.4. *Let $N \in \mathbb{N}$ and let m be an integer such that $|m| \leq N$. Then, for $0 \leq \vartheta < \pi/2$, we have*

$$(2.12) \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{im\psi} \, d\psi}{(\cos \vartheta + i \sin \vartheta \cos \psi)^{N+1}} = \frac{(N-m)! \, (-1)^m}{N!} P_N^m(\cos \vartheta).$$

Remark 2.5. *In fact, slightly more can be said: if m happens to be an integer with $|m| > N$, then the integral on the left-hand side vanishes for $0 \leq \vartheta < \pi/2$.*

Finally, we will be making use of the following lemma, which is essentially Lemma 3.6 in [8] or Lemma 4.3 in [36].

Lemma 2.6. *Let $n \in \{2, 3, \ldots\}$, $\beta \geq 0$, $1 \leq q \leq \infty$, $f \in L^q(\mathbb{R}^n)$, and let $R \colon \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that*

$$R(x) = O(|x|^\beta)$$

for $x \in \mathbb{R}^n$. Also, let $\rho \in \mathbb{C}^n \setminus \{0\}$, and let $C \subset \mathbb{R}^n$ be a closed cone with vertex at the origin such that

$$\inf_{x \in C \setminus \{0\}} \frac{x \cdot \operatorname{Re} \rho}{|x| \cdot |\operatorname{Re} \rho|} > 0.$$

Then

$$\int_C e^{-\rho \cdot x} R(x) f(x) \, dx \lesssim |\rho|^{n/q - \beta - n} \left\| e^{-\operatorname{Re} \rho / |\rho| \cdot x} |x|^{\beta} \chi_C \right\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)},$$

where, as usual, $1 \leq q' \leq \infty$ so that $1/q + 1/q' = 1$.

In particular, the value $q = \infty$ is acceptable, and then the exponent $n/q - \beta - n$ reduces to $-\beta - n$ as one would expect. The lemma is proven by the change of variables $x = y/|\rho|$ and an application of Hölder's inequality.

3. The Source Problem

In this section we prove Theorem 1.5. We begin with a few definitions.

Definition 3.1. We call a closed cone $C \subset \mathbb{R}^3$ with vertex at the origin an *admissible source cone* if it is contained in some closed strictly convex circular cone, if its exterior is connected, and if its spherical cross section $C \cap \mathbb{S}^2$ has the property that

$$\int_{\mathbb{S}^2 \cap C} Y_2^m \, dS \neq 0$$

for some $m \in \{-2, -1, 0, 1, 2\}$. Here Y_2^m denotes a spherical harmonic of degree two (2.11).

Remark 3.2. We point out that Definition 3.1 is invariant with respect to rotation. That is, if C satisfies all the conditions of the definition, then the rotated cone

$$O[C] = \{Ox \mid x \in C\}$$

also satisfies the conditions for any $O \in \mathrm{SO}(3, \mathbb{R})$.

We will show that smooth enough sources that have a jump on the boundary near a vertex of an admissible cone produce a non-zero far-field, see Theorem 3.9. But before that, we will give some examples of admissible source cones. In particular, cones whose vertex angle is small are admissible no matter the shape of their cross-section.

Definition 3.3. Let $\gamma \in (0, \pi/2)$. Then we define C_γ to be the closed strictly convex circular cone

$$C_\gamma = \left\{ x \in \mathbb{R}^3 \,\middle|\, |x'| \leq x_3 \tan \gamma \right\}$$

where $x' = (x_1, x_2)$. We also define a "magic angle" ϑ_0 to be the unique angle $\vartheta_0 \in (0, \pi/2)$ for which $\cos \vartheta_0 = 1/\sqrt{3}$.

Remark 3.4. In degrees, the magic angle ϑ_0 is approximately $54.74°$.

Proposition 3.5. *Let $\gamma \in (0, \pi/2)$. Then C_γ is an admissible source cone.*

Proof. The cone C_γ is contained in a closed strictly convex circular cone, namely C_γ itself. It is also clear that the exterior $\mathbb{R}^3 \setminus C_\gamma$ is connected. To prove the integral condition, we shall prove that

$$\int_{C_\gamma \cap \mathbb{S}^2} Y_2^0(x) \, dx \neq 0.$$

After moving to spherical coordinates, writing Y_2^0 in terms of the Legendre polynomial P_2 as in (2.11), and observing that the integrand is independent of the variable φ, we only have to show that

$$\int_0^\gamma \left(3 \cos^2 \vartheta - 1 \right) \sin \vartheta \, d\vartheta \neq 0.$$

But we can compute that

$$\int_0^\gamma \left(3\cos^2\vartheta - 1\right)\sin\vartheta\,d\vartheta = \cos\gamma\sin^2\gamma,$$

and clearly the product is non-zero. ∎

Proposition 3.6. *If $C \subset \mathbb{R}^3$ is a closed strictly convex cone with vertex at the origin, if C is not a set of measure zero, if C has connected exterior, and if $C \subseteq C_{\vartheta_0}$, then C is an admissible source cone.*

Proof. This follows directly from the observation that, in spherical coordinates, the function Y_2^0 is strictly positive when $\vartheta > \vartheta_0$. Thus, in the integral

$$\int_{C\cap\mathbb{S}^2} Y_2^0(x)\,dx$$

the integrand is strictly positive for all x, except possibly for x lying in a set of measure zero, and therefore the integral must also be strictly positive. ∎

Proposition 3.7. *If $C \subset \mathbb{R}^3$ is a closed strictly convex cone with vertex at the origin, if C is not a set of measure zero, if C has connected exterior, and if $C \subseteq C_\gamma \setminus \mathrm{int}\, C_{\vartheta_0}$ for some $\gamma \in (\vartheta_0, \pi/2)$, then C is an admissible source cone.*

Proof. This case can be handled in exactly the same way as the previous example, except that now Y_2^0 is strictly negative almost everywhere in $C \cap \mathbb{S}^2$. ∎

Proposition 3.8. *If $C \subset \mathbb{R}^3$ is a closed cone with vertex at the origin, if C has connected exterior, and if $C_{\gamma_1} \subseteq C \subseteq C_{\gamma_2}$ for some $\gamma_1 \in (0, \vartheta_0)$ and $\gamma_2 \in (\vartheta_0, \pi/2)$ such that*

$$\cos\gamma_1\sin^2\gamma_1 + \cos\gamma_2\sin^2\gamma_2 > \cos\vartheta_0\sin^2\vartheta_0,$$

then C is an admissible source corner.

Proof. Again, the point is that, in spherical polar coordinates, Y_2^0 is strictly positive when $\vartheta < \vartheta_0$ and strictly negative when $\vartheta_0 < \vartheta < \pi/2$, and so

$$\int_{C\cap\mathbb{S}^2} Y_2^0\,dS \geq \int_{C_{\gamma_1}\cap\mathbb{S}^2} Y_2^0\,dS + \int_{\left(C_{\gamma_2}\setminus C_{\vartheta_0}\right)\cap\mathbb{S}^2} Y_2^0\,dS.$$

After moving to spherical coordinates, writing Y_2^0 in terms of P_2 and simplifying, we see that the original integral is strictly positive if

$$\cos\gamma_1\sin^2\gamma_1 + \left(\cos\gamma_2\sin^2\gamma_2 - \cos\vartheta_0\sin^2\vartheta_0\right) > 0,$$

and this is precisely the hypothesis we made. ∎

We can prove the main theorem about always scattering sources. Note in the proof that since the cone C might have a very rough boundary, we cannot use the simple integration by parts formula from [2].

Theorem 3.9. *Let $\gamma > 0$ and let $\Phi \in e^{-\gamma|\cdot|}L^2(\mathbb{R}^3)$ be such that $\Phi|_H \equiv 0$ for some open half-space $H \subset \mathbb{R}^3$. Assume also that the origin belongs to the component of $\mathbb{R}^3 \setminus \mathrm{supp}\,\Phi$ containing H. Let $\varphi \in L_c^\infty(\mathbb{R}^3)$ be such that there exist constants $\alpha > 0$ and $\varphi_0 \in \mathbb{C}$ such that*

$$|\varphi(x) - \varphi_0| \lesssim |x|^\alpha$$

for almost all $x \in \mathbb{R}^3$. Let $C \subset \mathbb{R}^3$ be an admissible source cone with vertex at the origin. Finally, assume that $u \in \mathring{B}_2^*$ solves the equation

$$(-\Delta - \lambda)u = \varphi\,\chi_C + \Phi$$

in $\mathbb{R}^3$. Then $\varphi_0 = 0$.

Proof. Let $r > 0$ be so small that the ball $B(0, 2r)$ is contained in the component of $\mathbb{R}^3 \backslash \operatorname{supp} \Phi$ containing H. By Theorem 2.3, unique continuation and the connectedness of the exterior of C from Definition 3.1, we must have $u|_{B(0,2r) \backslash C} \equiv 0$. We also observe that, by rotating the entire configuration, if necessary, we may assume that C is oriented so that $C \backslash \{0\} \subset \mathbb{R}^2 \times \mathbb{R}_+$.

Next, let $\sqrt{\lambda} \le \tau < \infty$ and $\psi \in \mathbb{R}$. We shall employ the complex geometrical optics solution $\exp(-\rho \cdot x)$ where $\rho \in \mathbb{C}^3$ is defined by

$$\rho = \rho(\tau, \psi) = \tau(0, 0, 1) + i\sqrt{\tau^2 + \lambda}(\cos\psi, \sin\psi, 0).$$

In particular, we have $\rho \cdot \rho = -\lambda$, so that

$$(-\Delta - \lambda)e^{-\rho \cdot x} = 0.$$

We study the limit $\tau \to \infty$, and so all the implicit constants below will be independent of τ, even though they are allowed to depend on ψ, φ, α, r, φ_0, C, u, the implicit constant in the theorem statement, and the cut-off function χ chosen below. In the

We choose a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3)$ so that $\chi|_{B(0,r)} \equiv 1$ and that $\chi|_{\mathbb{R}^3 \backslash B(0,2r)} \equiv 0$, and we write A for the annulus $B(0, 2r) \backslash \overline{B}(0, r)$. Then, using our knowledge of the supports of $\chi, \nabla\chi, \Delta\chi, \Phi$ and u, we may argue that

$$0 = \int_{\mathbb{R}^3} \chi\, u(-\Delta - \lambda)e^{-\rho \cdot x}\, dx = \int_{\mathbb{R}^3} e^{-\rho \cdot x}(-\Delta - \lambda)(\chi u)\, dx$$
$$= \int_C e^{-\rho \cdot x} \chi\, \varphi\, dx - \int_{C \cap A} e^{-\rho \cdot x}(2\,\nabla\chi \cdot \nabla u + u\,\Delta\chi)\, dx.$$

In C, the factor $e^{-\rho \cdot x}$ is exponentially decaying, and we may estimate, for some constant $\delta > 0$ depending only on C and r, that $|e^{\rho \cdot x}| \lesssim e^{-\delta\tau}$ for $x \in C \cap A$. We now conclude that

$$\int_{C \cap A} e^{-\rho \cdot x}(2\,\nabla\chi \cdot \nabla u + u\,\Delta\chi)\, dx \lesssim e^{-\delta\tau} = o(\tau^{-3}),$$

since the constants are independent of τ.

We may now continue with

$$o(\tau^{-3}) = \int_C e^{-\rho \cdot x} \chi\, \varphi\, dx = \varphi_0 \int_C e^{-\rho \cdot x} \chi\, dx + \int_C e^{-\rho \cdot x} \chi(\varphi(x) - \varphi_0)\, dx.$$

We have $|\operatorname{Re}\rho|/|\rho| \ge 1/\sqrt{3}$. Thus, by Lemma 2.6,

$$\int_C e^{-\rho \cdot x} \chi(\varphi(x) - \varphi_0)\, dx \lesssim |\rho|^{-\alpha-3} \asymp \tau^{-3-\alpha} = o(\tau^{-3}).$$

We may also estimate

$$\int_C e^{-\rho \cdot x}(1 - \chi)\, dx \lesssim e^{-\delta\tau/2} \int_{C \backslash B(0,r)} e^{-\operatorname{Re}\rho \cdot x/2}\, dx \lesssim e^{-\delta\tau/2} = o(\tau^{-3}).$$

Thus, we may further continue with a change of variables to get

$$o(\tau^{-3}) = \varphi_0 \int_C e^{-\rho \cdot x}\, dx = \frac{\varphi_0}{|\rho|^3} \int_C e^{-\rho/|\rho| \cdot y}\, dy,$$

leading to

$$\varphi_0 \int_C e^{-\rho/|\rho| \cdot x}\, dx = o(1).$$

Since $e^{-\rho/|\rho| \cdot x} \lesssim e^{-x_3/\sqrt{3}}$ in C, and $\rho/|\rho| \to \rho_0/|\rho_0|$, where

$$\rho_0 = \rho_0(\tau, \psi) = \tau(0, 0, 1) + i\tau(\cos\psi, \sin\psi, 0),$$

we get by the dominated convergence theorem, after taking $\tau \to \infty$,

$$\varphi_0 \int_C e^{-\rho_0/|\rho_0| \cdot x}\, dx = 0.$$

Moving next to polar coordinates, we obtain

$$0 = \varphi_0 \int_C e^{-\rho_0/|\rho_0| \cdot x} \, dx = \varphi_0 \int_{C \cap \mathbb{S}^2} \int_0^\infty e^{-(\rho_0/|\rho_0| \cdot y)r} \, r^2 \, dr \, dy = \varphi_0 \int_{C \cap \mathbb{S}^2} \frac{2}{(\rho_0/|\rho_0| \cdot y)^3} \, dy.$$

Let $m \in \{-2, -1, 0, 1, 2\}$. Moving to spherical coordinates, multiplying both sides with $e^{im\psi}$, and integrating with respect to ψ over the interval $(0, 2\pi)$ gives

$$0 = \varphi_0 \int_0^{2\pi} e^{im\psi} \int_{C \cap \mathbb{S}^2} \frac{\sin \vartheta \, d\varphi \, d\vartheta}{(\cos \vartheta + i \sin \vartheta \cos(\psi - \varphi))^3} \, d\psi$$

$$= \varphi_0 \int_{C \cap \mathbb{S}^2} e^{im\varphi} \int_0^{2\pi} \frac{e^{im\psi'} \, d\psi'}{(\cos \vartheta + i \sin \vartheta \cos \psi')^3} \, \sin \vartheta \, d\varphi \, d\vartheta.$$

Using Laplace's second representation formula, i.e. Theorem 2.4, and multiplying by appropriate constant factors gives

$$0 = \varphi_0 \int_{C \cap \mathbb{S}^2} Y_2^m \, dS.$$

Since, by Definition 3.1, the last integral must be nonzero for some choice of m, we conclude that $\varphi_0 = 0$. ∎

4. Complex geometrical optics solutions

Here we construct so called complex geometrical optics solutions (CGO-solutions for short) for the Schrödinger equation with a cone like potential. To this end we first show that the admissible cones give continuous multiplication operators between certain Sobolev spaces. This argument closely follows the argument in [36]. We then construct the relevant CGO-solutions, as in that paper, as a consequence of the L^p-resolvent estimates of [28] and the multiplier properties of the cone like potential. We end the section by giving some example cones with fairly complicated structure that admit CGO-solutions.

We need to show that V is a pointwise multiplier. The idea is to first establish that if the cross-section of the cone is such that the characteristic function of the cross-section has suitable properties, then also a cylinder with the same cross-section has those properties. This can then be used to cut a infinite cone in to finite pieces with the relevant property, and using scaling properties of the Sobolev norms we get the relevant property for the cone itself.

Our starting point is a slightly reformulated version of Proposition 3.9 in [36].

Lemma 4.1. *Let $D \subset \mathbb{R}^2$ be bounded and such that $\chi_D \in H^{\tau, 2}(\mathbb{R}^2)$ for some $\tau \in [0, 1/2)$, and let C be the cylinder*

$$C = \{(x', x_3) \in \mathbb{R}^3 : \ x' \in D, \ |x_3| \le 1\}.$$

Then $\chi_C \in H^{\tau, p}(\mathbb{R}^3)$ for all $p \in [1, 2]$.

Proof. Let us prove the claim for $p = 2$. The claim for $p < 2$ follows from this case because χ_C has compact support. Set $x := (x', x_3)$, and $I := [-1, 1]$. Note that

$$\chi_C(x) = \chi_D(x')\chi_I(x_3),$$

so that

$$\widehat{\chi}_C(\xi) = \widehat{\chi}_D(\xi')\widehat{\chi}_I(\xi_3).$$

For χ_I one has the estimate

$$|\widehat{\chi}_I(\xi_3)| = \frac{|\sin(\xi_3)|}{|\xi_3|} \le C \langle \xi_3 \rangle^{-1}.$$

One has $\langle \xi \rangle^{2\tau} \leq C \langle \xi' \rangle^{2\tau} \langle \xi_3 \rangle^{2\tau}$. Using these facts gives then that

$$\|\chi_C\|_{H^{\tau,2}(\mathbb{R}^3)}^2 \leq C \int_{\mathbb{R}^3} \langle \xi' \rangle^{2\tau} |\hat{\chi}_D(\xi')|^2 \langle \xi_3 \rangle^{2\tau} |\hat{\chi}_I(\xi_3)|^2 \, d\xi$$
$$\leq C \|\chi_D\|_{H^{\tau,2}(\mathbb{R}^2)}^2 \int_{\mathbb{R}} \langle \xi_3 \rangle^{2\tau} \langle \xi_3 \rangle^{-2} \, d\xi_3$$
$$< \infty,$$

provided that $\tau < 1/2$. ∎

The previous lemma can be used as in [36] to deduce the following proposition.

Lemma 4.2. *Let $\tau \in [0, 1/2)$ and suppose $D \subset \mathbb{R}^2$ is bounded with $\chi_D \in H^{2,\tau}(\mathbb{R}^2)$. For $\delta > 0$ let C_δ be the cone*

$$C_\delta = \{(t\delta x', t) \in \mathbb{R}^3 \,:\, t \in [0, \infty), \, x' \in D\}.$$

Then $\langle x \rangle^{-\alpha} \chi_{C_\delta} \in H^{\tau,p}(\mathbb{R}^3)$ if $p \in (1, 2]$ and $\alpha > 3/p$.

Proof. This lemma is the analogue of Proposition 3.7 in [36]. Its proof can be easily modified to work in our case. We only need to use our Lemma 4.1 instead of Proposition 3.9 in [36]. ∎

We consider a potential V defined using the characteristic function of the previous lemma next.

Proposition 4.3. *Let C_δ be as in Lemma 4.2. Consider the potential*

$$V(x) := \varphi(x) \langle x \rangle^{-\alpha} \chi_{C_\delta},$$

where $\tau \in \left(0, \frac{1}{2}\right)$, $\alpha > 9/4$ and $\varphi \in C^{\tau+\varepsilon}(\mathbb{R}^3)$ for some $\varepsilon > 0$. Then

$$(4.1) \qquad \|Vg\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C \|g\|_{H^{\tau,4}(\mathbb{R}^3)}$$

for any $g \in H^{\tau,4}(\mathbb{R}^3)$ and

$$(4.2) \qquad \|V\|_{H^{\tau,4/3}(\mathbb{R}^3)} < \infty.$$

Proof. Use the notation

$$\widetilde{V} := \langle x \rangle^{-\alpha} \chi_{C_\delta},$$

so that $V = \varphi \widetilde{V}$. Since $\widetilde{V} \in H^{\tau,2}(\mathbb{R}^3)$ by Lemma 4.2, we get as a direct consequence of Proposition 3.5 in [36] that the multiplication operator $g \mapsto \widetilde{V} g$ is continuous $H^{\tau,4}(\mathbb{R}^3) \to H^{\tau,4/3}(\mathbb{R}^3)$, and hence the estimate

$$\|\widetilde{V} g\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C \|g\|_{H^{\tau,4}(\mathbb{R}^3)}$$

follows. In more detail, choose $\tilde{p} = 2$ in Proposition 3.5 in [36], which gives $p = 4$ and the Hölder conjugate $p' = \frac{4}{3}$. Furthermore by the results on p. 205 in [43], we see that multiplication by $\varphi \in C^{\tau+\varepsilon}(\mathbb{R}^3)$ is continuous on $H^{\tau,p}(\mathbb{R}^3)$, so that

$$\|Vg\|_{H^{\tau,4/3}(\mathbb{R}^3)} = \|\varphi \widetilde{V} g\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C \|\widetilde{V} g\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C \|g\|_{H^{\tau,4}(\mathbb{R}^3)}.$$

Lemma 4.2 with $p = 4/3$ and this same result in [43] shows (4.2) immediately. ∎

The previous proposition shows that potentials of the form $\varphi \langle x \rangle^{-\alpha} \chi_{C_\delta}$ give continuous multiplier operators in the sense of (4.1). The uniform Sobolev estimates of [28] and multiplier property (4.1) can be used to deduce the following existence result for a inhomogeneous equation. Then, the norm bound (4.2) implies the existence of complex geometrical optics solutions for the homogeneous partial differential equation $(-\Delta + V - \lambda)\psi = 0$. This is a standard argument, done for potentials supported on a circular cone in Proposition 3.3 of [36]. We nevertheless repeat parts of the proof here for the convenience of the reader.

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Proposition 4.4. *Let $\tau \in \mathbb{R}$. And suppose $V \in \mathcal{D}'(\mathbb{R}^3)$ is such that*

$$\|Vg\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C\|g\|_{H^{\tau,4}(\mathbb{R}^3)}$$

for all $g \in H^{\tau,4}(\mathbb{R}^3)$. Then there exist constants $C > 0$ and $R > 0$ such that whenever $\zeta \in \mathbb{C}^n$ satisfies $|\operatorname{Re}\zeta| \geq R$ and $f \in H^{\tau,4/3}(\mathbb{R}^3)$ the equation

(4.3)
$$(-\Delta + 2\zeta \cdot D + V)\psi = f$$

has a solution $\psi \in H^{\tau,4}(\mathbb{R}^3)$ satisfying

$$\|\psi\|_{H^{\tau,4}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|f\|_{H^{\tau,4/3}(\mathbb{R}^3)}.$$

Proof. Here we only repeat the case involving a nonzero V. The first part of the proof of Proposition 3.3 in [36], gives an solution operator G_ζ for the case $V = 0$, for which

$$G_\zeta \colon H^{\tau,4/3}(\mathbb{R}^3) \to H^{\tau,4}(\mathbb{R}^3),$$

and $(-\Delta + 2\zeta \cdot D)G_\zeta g = g$ for any $g \in H^{\tau,4/3}(\mathbb{R}^3)$. Furthermore G_ζ obeys the norm estimate

$$\|G_\zeta g\|_{H^{\tau,4}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|g\|_{H^{\tau,4/3}(\mathbb{R}^3)}$$

for all g in that space. Proposition 4.3 implies that

$$\|VG_\zeta g\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|g\|_{H^{\tau,4/3}(\mathbb{R}^3)}$$

again for all g as before. We can now choose an $R > 0$, so that $CR^{-1/2} = \frac{1}{2}$, and then assuming $|\operatorname{Re}(\zeta)| \geq R$ we have

(4.4)
$$\|VG_\zeta g\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq \tfrac{1}{2}\|g\|_{H^{\tau,4/3}(\mathbb{R}^3)}$$

for any g. Note that a solution to the equation $(-\Delta + 2\zeta \cdot D + V)G_\zeta v = f$ is given by any solution to $v + VG_\zeta v = f$. If we can solve v satisfying

$$(I + VG_\zeta)v = f,$$

then we get a solution ψ to (4.3) by setting $\psi = G_\zeta v$. Estimate (4.4) shows that $(I + VG_\zeta)^{-1}$ can be expressed as a Neumann series converging in $H^{\tau,4/3}(\mathbb{R}^3)$. By using the norm estimate for G_ζ and (4.4) we have moreover that

$$\|\psi\|_{H^{\tau,4}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|v\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|f\|_{H^{\tau,4/3}(\mathbb{R}^3)}.$$

$\blacksquare$

The solvability result gives us then finally the existence of suitable CGO solutions.

Theorem 4.5. *Let $\lambda \in \mathbb{R}_+$, let $\rho \in \mathbb{C}^3$ satisfy $\rho \cdot \rho = -\lambda$, and assume that $|\operatorname{Im}\rho|$ is large enough. Define*

$$V(x) = \varphi(x)\langle x\rangle^{-\alpha}\chi_C(x)$$

for $x \in \mathbb{R}^3$, where $C \subset \mathbb{R}^3$ is a closed cone with a bounded cross-section $D \subset \mathbb{R}^2$ satisfying $\chi_D \in H^\tau(\mathbb{R}^2)$, $\alpha \in (9/4, \infty)$, and $\varphi \in C^{\tau+\varepsilon}(\mathbb{R}^3)$ for some $\tau \in (1/4, 1/2)$ and $\varepsilon \in \mathbb{R}_+$.

Then there exists a solution $u = e^{-\rho \cdot x}(1 + \psi)$ to the equation

$$(-\Delta + V - \lambda)u = 0$$

in $\mathbb{R}^3$, where $\psi \in L^q(\mathbb{R}^3)$, $q = 12/(3 - 4\tau) \in (6, 12)$, and

$$\left\|\psi\right\|_{L^q(\mathbb{R}^3)} \lesssim |\operatorname{Im}\rho|^{-3/q-\delta}$$

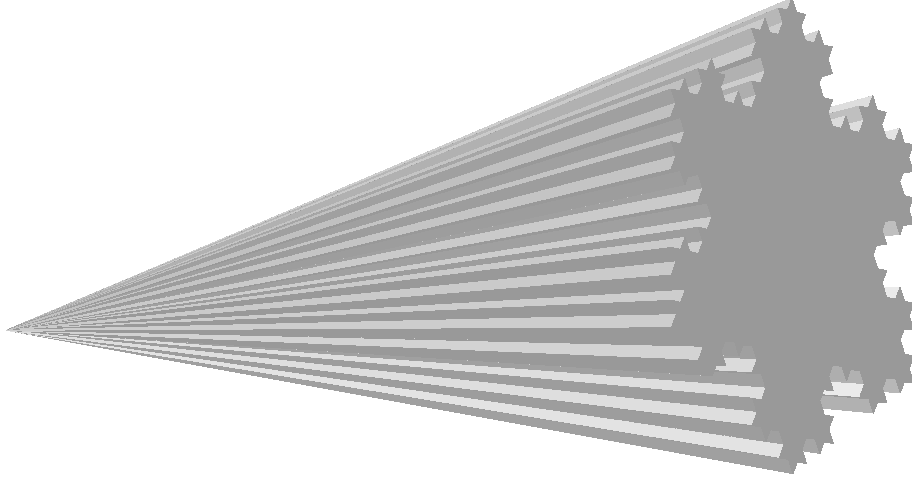*for some fixed $\delta \in \mathbb{R}_+$.*

FIGURE 1. A cone with Koch snowflake cross section

Proof. To simplify notation, let $\zeta = i\rho$. It is then required to construct a solution of the form

$$u = e^{i\zeta \cdot x}(1 + \psi), \quad \zeta \in \mathbb{C}^3, \quad \zeta \cdot \zeta = \lambda.$$

We thus need to solve the equation

$$(-\Delta + 2\zeta \cdot D + V)\psi = -V.$$

The potential V satisfies the two conditions

$$\|Vg\|_{H^{\tau,4/3}(\mathbb{R}^3)} \leq C\|g\|_{H^{\tau,4}(\mathbb{R}^3)}, \qquad \|V\|_{H^{\tau,4/3}(\mathbb{R}^3)} < \infty$$

by Proposition 4.3. The former is true for any $g \in H^{\tau,4}(\mathbb{R}^3)$ and C is independent of g. Proposition 4.4 gives us then a solution $\psi \in H^{\tau,4}(\mathbb{R}^3)$ satisfying the estimate

(4.5) $$\|\psi\|_{H^{\tau,4}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{-1/2}\|V\|_{H^{\tau,4/3}(\mathbb{R}^3)}.$$

By the Sobolev embedding Theorem we have that

$$\|\psi\|_{L^{q_\tau}(\mathbb{R}^3)} \leq C\|\psi\|_{H^{\tau,4}(\mathbb{R}^3)},$$

when $0 < \tau < n/p$ and $q_\tau = np/(n - \tau p)$ where $n = 3$ is the dimension and $p = 4$ is the Sobolev integrability. We have

$$q_\tau = \tfrac{12}{3-4\tau} \in (6, 12).$$

since $\tau \in (1/4, 1/2)$ by assumption. Now we write

$$\tfrac{1}{2} = \tfrac{3}{q_\tau} + \tfrac{q_\tau - 6}{2q_\tau} =: \tfrac{3}{q_\tau} + \delta$$

and for the estimate of ψ in the claim we want to have $\delta > 0$, whichis equivalent to having $q_\tau > 6$. This holds because $\tau > 1/4$. Combining the above with (4.5) implies that

$$\|\psi\|_{L^{q_\tau}(\mathbb{R}^3)} \leq C|\operatorname{Re}\zeta|^{(-3/q_\tau - \delta)}\|V\|_{H^{\tau,4/3}(\mathbb{R}^3)},$$

where $q_\tau \in (6, 12)$. ∎

We end this section by considering a concrete examples of the type of potential V for which the above results hold.

Example 4.6. Consider the following conic set

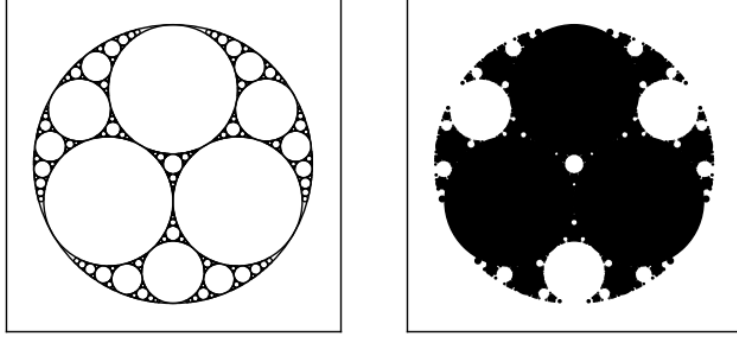$$C = \{(tx', t) \in \mathbb{R}^3 \ : \ x' \in K, \ t \in [0, \infty)\},$$

FIGURE 2. The Apollonian gasket A on the left and the set $\tilde{A}$ on the right.

where $K \subset \mathbb{R}^2$ is the interior of the famous Koch snowflake, see Figure 1. By Corollary 1.2 in [21] we know that

$$\tau < 1 - \tfrac{1}{2}\tfrac{\log 4}{\log 3} \quad \Rightarrow \quad \chi_K \in H^{\tau,2}(\mathbb{R}^2).$$

We can now choose $\tau \in (\tfrac{1}{4}, \tfrac{1}{2})$, such that $\tfrac{1}{4} < \tau < 1 - \tfrac{1}{2}\tfrac{\log 4}{\log 3} \approx 0.37$. Let $\alpha > 9/4$ and

$$V(x) = \langle x \rangle^{-\alpha} \chi_C.$$

Theorem 4.5 implies then that we can construct CGO solutions V with the desired remainder estimates.

The next example is a cone with a porous structure. We will use some further results from [21] to analyze this. We use in particular the fact that we can determine what Sobolev space the characteristic function of a set is in by looking at the box-counting dimension of the boundary of the set.

Example 4.7. Let A be the Apollonian gasket. See Figure 2. The set A can be constructed iteratively by starting with the four largest circles. One then adds all the circles that are mutually tangent to any three of the initial four circles. In this way, one obtains four new circles, or eight circles in total. One then adds the circles that are mutually tangent to the eight circles similarly, and so forth.

Now consider the set $\tilde{A} \subset \mathbb{R}^2$, which is constructed as the Apolloninan gasket, but so that $\tilde{A}$ contains the entire interior disk of the circles, but only of those added on every other iteration in the construction of A. See Figure 2.

Now we consider the conic set

$$C = \{(tx', t) \in \mathbb{R}^3 \,:\, x' \in \tilde{A},\ t \in [0, \infty)\},$$

The set $\tilde{A}$ is a porous set with non zero Lebesgue measure, which can be analyzed in a straightforward manner. Clearly we have that

$$\partial \tilde{A} \subset \partial A = A.$$

The exact Hausdorff dimension $\dim_H(A)$ of the Apolloninan Gasket is unknown, but $\dim_H(A) < 1.314534$, see e.g. [10]. The Apollonian gasket has furthermore the property that the so called box-counting dimension (or Minkowsksi dimension) of its closure coincides with the Hausdorff dimension, i.e. $\dim_M(\overline{A}) = \dim_H(\overline{A})$, see [41]. Thus we have that

$$\dim_M(\overline{A}) = \dim_H(\overline{A}) < 1.314534.$$

By the monotonicity of the box-counting dimension (see p.48 in [20]) we obtain that $\dim_M(\partial\tilde{A}) \leq \dim_M(\overline{A}) \leq 1.32$. We can now apply the results in [21] (see also [39]). Let $D \subset \mathbb{R}^n$, then in general we have that

$$(4.6) \qquad \dim_M(\partial D) < n - p\tau \quad \Rightarrow \quad \chi_D \in H^{\tau,p}(\mathbb{R}^n),$$

for $1 \leq p < \infty$. See Theorem 1.3 in [21] and the related comments. For $\partial\tilde{A}$ and $p = n = 2$, the condition becomes

$$\dim_M(\partial\tilde{A}) < 2 - 2\tau,$$

which follows if $\tau < 1 - 0.5 \cdot 1.32 = 0.34$. Again, we see that Theorem 4.5 implies that we can construct CGO solutions for the potential $V = \langle x \rangle^{-\alpha}\chi_C$ with the desirable remainder estimates if $\alpha > 9/4$, since we can choose τ so that $1/4 < \tau < 0.34$.

Remark 4.8. Condition (4.6) has a partial converse that states that for sets $D \subset \mathbb{R}^n$, we have that

$$\dim_M(\partial D) > n - p\tau \quad \Rightarrow \quad \chi_D \notin H^{\tau,p}(\mathbb{R}^n),$$

for $1 \leq p < \infty$. See [21]. This sets some limits on the applicability of the above constructions. It follows that we cannot obtain CGO-solutions with the above method for potentials supported on cones whose cross-section $D \subset \mathbb{R}^2$ has a fractal boundary with $\dim_M(\partial D)$ large enough, since the CGO construction demands that $\tau > 1/4$.

5. Admissible cones always scatter

In this section we define the notion of an admissible medium cone. Admissibility amounts essentially to an orthogonality condition and certain regularity assumptions. We then prove that admissible medium cones result in potentials that always scatter. After this we formulate a simple determinant condition for medium cones that are admissible.

Definition 5.1. We call a closed cone $C \subset \mathbb{R}^3$ an *admissible medium cone*, if

(i) C is contained in a strictly convex closed circular cone,
(ii) C has a connected exterior,
(iii) C has a bounded cross-section $D \subset \mathbb{R}^2$ such that $\chi_D \in H^\tau(\mathbb{R}^2)$ for some $\tau \in (1/4, 1/2)$, and
(iv) for any spherical harmonic $H \not\equiv 0$ of arbitrary degree N, there exists an index $m \in \{-N-2, \ldots, N+2\}$ so that

$$\int_{C \cap \mathbb{S}^2} Y_{N+2}^m H \, dS \neq 0.$$

Remark 5.2. We could equivalently formulate the last condition this way: For any spherical harmonic $H \not\equiv 0$ of arbitrary degree N, there exists a spherical harmonic Y of degree $N + 2$ so that

$$\int_{C \cap \mathbb{S}^2} Y H \, dS \neq 0.$$

Theorem 5.3. *Let $\lambda \in \mathbb{R}_+$, and let $V = \varphi\chi_C + \Phi$, where $\varphi \in C_c^{1/4+\varepsilon}(\mathbb{R}^3)$ for some $\varepsilon \in \mathbb{R}_+$, $\Phi \in e^{-\gamma|\cdot|} L^2(\mathbb{R}^3)$ for some $\gamma \in \mathbb{R}_+$ so that $\Phi|_H \equiv 0$ for some open half-space $H \subset \mathbb{R}^3$, and so that the origin belongs to the component of $\mathbb{R}^3 \setminus \operatorname{supp}\Phi$ containing H, and where C is an admissible medium cone with vertex at the origin. Finally, assume that λ is a non-scattering energy for the potential V. Then $\varphi(0) = 0$.*

Proof. By Lemma 2.2, there exist solutions $v, w \in B_2^*(\mathbb{R}^3)$ to the equations

$$\begin{cases} (-\Delta + V - \lambda) v = 0, \\ (-\Delta - \lambda) w = 0, \end{cases}$$

in $\mathbb{R}^3$, so that $u = v - w \in \mathring{B}_2^*(\mathbb{R}^3)$ and $w \not\equiv 0$. Let $r \in \mathbb{R}_+$ be so small that the ball $B(0, 2r)$ is contained in the component of $\mathbb{R}^3 \setminus \operatorname{supp} \Phi$ containing H. By Theorem 2.3, unique continuation and the connectedness of the exterior of C from Definition 5.1, we have $u|_{B(0,2r)\setminus C} \equiv 0$.

By rotating the whole setup, we may assume that $C \setminus \{0\} \subset \mathbb{R}^2 \times \mathbb{R}_+$. Next, we let $\tau \in \left[\sqrt{\lambda}, \infty \right[$ and $\psi \in \mathbb{R}$ in order to choose a complex vector $\rho \in \mathbb{C}^3$ through

$$\rho = \rho(\tau, \psi) = \tau(0, 0, 1) + i\sqrt{\tau^2 + \lambda}\,(\cos \psi, \sin \psi, 0).$$

In particular, we have $\rho \cdot \rho = -\lambda$, and, assuming that τ is large enough, Theorem 4.5 gives us a solution $u_0 = e^{-\rho \cdot x}(1 + \psi)$ to the equation

$$(-\Delta + \varphi \chi_C - \lambda) u_0 = 0$$

in $\mathbb{R}^3$ satisfying $\psi \in L^q(\mathbb{R}^3)$ for some $q \in (6, 12)$, and satisfying the estimate

$$\|\psi\|_{L^q(\mathbb{R}^3)} \lesssim \tau^{-3/q - \delta},$$

where $\delta \in \mathbb{R}_+$ is fixed. We shall study the limit $\tau \to \infty$, and so all the implicit constants will be independent of τ, but they are allowed to depend on everything else.

Since w is real-analytic and $w \not\equiv 0$, there exists a homogeneous complex polynomial $H(x)$ of some degree $N \in \{0\} \cup \mathbb{Z}_+$ such that $H(x) \not\equiv 0$ and

$$w(x) = H(x) + O(|x|^{N+1}),$$

for all $x \in B(0, 2r)$. The equation $(-\Delta - \lambda) w = 0$ implies that H is harmonic. This can be seen by applying the differential operator to the Taylor expansion of w. The argument is essentially the same as in Lemma 17 of [9].

We choose a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3)$ so that $\chi|_{B(0,r)} \equiv 1$ and $\chi|_{\mathbb{R}^3 \setminus B(0,2r)} \equiv 0$, and write A for the annular domain $B(0, 2r) \setminus \overline{B}(0, r)$. We may then argue, remembering that V and $\varphi \chi_C$ coincide in the support of χ, that

$$0 = -\int_{\mathbb{R}^3} \chi u \left(-\Delta + \varphi \chi_C - \lambda\right) u_0 \, dx = -\int_{\mathbb{R}^3} u_0 \left(-\Delta + V - \lambda\right) (\chi u) \, dx$$

$$= \int_C u_0 \chi \varphi w \, dx + \int_{C \cap A} u_0 \left(2 \nabla \chi \cdot \nabla u + u \Delta \chi\right) dx.$$

Since we may estimate $|e^{-\rho \cdot x}| \lesssim e^{-\delta' \tau}$ in the domain A for some constant $\delta' \in \mathbb{R}_+$ not depending on τ, we have

$$\int_{C \cap A} u_0 \left(2 \nabla \chi \cdot \nabla u + u \Delta \chi\right) dx \lesssim e^{-\delta' \tau} = o(\tau^{-N-3}).$$

Now, we may expand the integrand $u_0 \chi \varphi w$ in the remaining integral in steps to get

$$o(\tau^{-N-3}) = \int_C u_0 \chi \varphi w \, dx = \int_C e^{-\rho \cdot x} \psi \chi \varphi w \, dx + \int_C e^{-\rho \cdot x} \chi \left(\varphi - \varphi(0)\right) w \, dx$$

$$+ \varphi(0) \int_C e^{-\rho \cdot x} \chi \, O(|x|^{N+1}) \, dx + \varphi(0) \int_C e^{-\rho \cdot x} \chi H \, dx.$$

Using Lemma 2.6 and the remainder estimate of Theorem 4.5 for ψ, we have for the first of the four integrals on the right-hand side that

$$\int_C e^{-\rho \cdot x} \psi \chi \varphi w \, dx \lesssim |\rho|^{3/q - N - 3} \left\| e^{-\operatorname{Re} \rho / |\rho| \cdot x} |x|^N \chi_C \right\|_{L^{q'}(\mathbb{R}^n)} \|\chi \varphi \psi\|_{L^q(\mathbb{R}^n)} \lesssim \tau^{-N-3-\delta}.$$

For the second integral we use that Hölder continuity which implies that $|\varphi(x) - \varphi(0)| \leq C|x|^{1/4+\epsilon}$ and Lemma 2.6 with $q = \infty$, so that have that

$$\int_C e^{-\rho \cdot x} \chi \left(\varphi - \varphi(0)\right) w \, dx \lesssim |\rho|^{-N-3} \left\| e^{-\operatorname{Re}\rho/|\rho| \cdot x} |x|^N \chi_C \right\|_{L^1(\mathbb{R}^n)} \|\chi \varphi \psi\|_{L^\infty(\mathbb{R}^n)} \lesssim \tau^{-N-3-1/4-\epsilon}.$$

For the third integral we use Lemma 2.6 similarly and obtain that

$$\varphi(0) \int_C e^{-\rho \cdot x} \chi \, O(|x|^{N+1}) \, dx \lesssim \tau^{-N-1-3}.$$

In particular, all three are $o(\tau^{-N-3})$, and we may continue

$$o(\tau^{-N-3}) = \varphi(0) \int_C e^{-\rho \cdot x} \chi H \, dx = \varphi(0) \int_C e^{-\rho \cdot x} H \, dx - \varphi(0) \int_C e^{-\rho \cdot x} \left(1 - \chi\right) H \, dx.$$

The last integral may be estimated to be

$$\varphi(0) \int_C e^{-\rho \cdot x} \left(1 - \chi\right) H \, dx \lesssim e^{-\delta'\tau/2} \int_{C \setminus B(0,r)} e^{-\operatorname{Re}\rho \cdot x/2} |H| \, dx \lesssim e^{-\delta'\tau/2} = o(\tau^{-N-3}).$$

Now with a change of variables,

$$o(\tau^{-N-3}) = \varphi(0) \int_C e^{-\rho \cdot x} H \, dx = \frac{\varphi(0) \, 2^{(N+3)/2}}{|\rho|^{N+3}} \int_C e^{-\sqrt{2}\rho/|\rho| \cdot x} H \, dx,$$

or more simply,

$$\varphi(0) \int_C e^{-\sqrt{2}\rho/|\rho| \cdot x} H \, dx = o(1).$$

In the limit $\tau \to \infty$, we have that $\rho/|\rho| \to \rho_0/|\rho_0|$, where

$$\rho_0 = \rho_0(\tau, \psi) = \tau(0, 0, 1) + i\tau(\cos\psi, \sin\psi, 0).$$

And Since $e^{-\rho/|\rho| \cdot x} \lesssim e^{-x_3/\sqrt{3}}$ in C, the dominated convergence Theorem gives, when taking $\tau \to \infty$, that

$$\varphi(0) \int_C e^{-\sqrt{2}\rho_0/|\rho_0| \cdot x} H \, dx = 0.$$

Writing ω for the vector $\sqrt{2}\rho_0/|\rho_0| = \langle i\cos\psi, i\sin\psi, 1\rangle$, and remembering that $\omega \cdot y$ has a positive real part for $y \in C \cap \mathbb{S}^2$, we may compute through the use of polar coordinates that

$$0 = \varphi(0) \int_C e^{-\omega \cdot x} H \, dx = \varphi(0) \int_{C \cap \mathbb{S}^2} \int_0^\infty e^{-(\omega \cdot y)r} r^{N+2} \, dr \, H(y) \, dy = \frac{\varphi(0)}{(N+2)!} \int_{C \cap \mathbb{S}^2} \frac{H(y) \, dy}{(\omega \cdot y)^{N+3}}.$$

Let $m \in \{-N-2, -N-1, \ldots, N+2\}$. Moving to spherical coordinates, multiplying the last identity by $e^{im\psi}$, and integrating with respect to ψ over $[0, 2\pi]$, changing the order of integration and making the change of variables $\psi' = \psi - \varphi$ gives

$$0 = \varphi(0) \int_{C \cap \mathbb{S}^2} e^{im\varphi} \int_0^{2\pi} \frac{e^{im\psi'} \, d\psi'}{(\cos\vartheta + i\sin\vartheta \cos\psi')^{N+3}} H \sin\vartheta \, d\vartheta \, d\varphi,$$

from which Laplace's second representation theorem, i.e. Theorem 2.4, gives us

$$0 = \varphi(0) \int_{C \cap \mathbb{S}^2} e^{im\varphi} P_{N+2}(\cos\vartheta) H \sin\vartheta \, d\vartheta \, d\varphi = \varphi(0) \int_{C \cap \mathbb{S}^2} Y_{N+2}^m H \, dS.$$

Since C was assumed to be an admissible medium cone, the last integral is non-zero for some choice of m, and thus $\varphi(0) = 0$. ∎

In the rest of this section we show that the admissibility of a medium cone can be ascertained by studying certain determinants, which we will shortly define.

We are particularly interested in families of cones $\{C_\rho \; : \; \rho \in I \subset \mathbb{R}\}$ depending on the parameter ρ. Consider the $(2N+1)$-tuple Ψ of basis functions of SH^{N+2}, such that

$$\Psi = (\psi_{-N}, .., \psi_N), \quad \psi_i \neq \psi_j, \quad \psi_j \in \left\{Y_{N+2}^k \; : \; k = -(N+2), .., N+2\right\}.$$

Define the functions

$$I_N^{k,l}(\rho) := \left(\psi_k, Y_N^l\right)_{L^2(K_\rho)} = \int_{K_\rho} \psi_k \overline{Y}_N^l \, dS,$$

where $K_\rho := C_\rho \cap \mathbb{S}^2$.

These give us information of the projections of the basis functions Y_N^l on the basis functions in Ψ_N in the $L^2(K_\rho)$-space. Define the matrices

$$\mathcal{C}_{\Psi,N}(\rho) := \begin{pmatrix} I_N^{-N,-N}(\rho) & \cdots & I_N^{-N,N}(\rho) \\ \vdots & \ddots & \vdots \\ I_N^{N,-N}(\rho) & \cdots & I_N^{N,N}(\rho) \end{pmatrix}.$$

And let the corresponding determinants be

$$\mathcal{D}_{\Psi,N}(\rho) := \det \mathcal{C}_{\Psi,N}(\rho).$$

Note that we are mainly interested in a specific choice of Ψ, which is

$$\Psi_0 := \left(Y_{N+2}^{-N}, Y_{N+2}^{-N+1}, .., Y_{N+2}^N\right),$$

as this choice of Ψ is particularly useful for analyzing circular cones. Moreover we will use the abbreviations

(5.1) $$\mathcal{C}_N(\rho) := \mathcal{C}_{\Psi_0,N}(\rho), \qquad \mathcal{D}_N(\rho) := \mathcal{D}_{\Psi_0,N}(\rho).$$

The columns of the matrices $\mathcal{C}_{\Psi,N}$ gives the coefficients of the projections of the basis vectors Y^k to vectors in Ψ. The next lemma gives a simple condition for admissibility in terms of the determinants $\mathcal{D}_{\Psi,N}(\rho)$.

Lemma 5.4. *Suppose that C_ρ satisfies the regularity conditions (i)-(iii) of Defintion 5.1. Then C_ρ is an admissible medium cone if for every $N \in \mathbb{N}$, there is a Ψ_N, for which $\mathcal{D}_{\Psi_N,N}(\rho) \neq 0$.*

Proof. Assume that C_ρ is not an admissible medium cone, so that there exists an N and $H \in SH^N$, $H \neq 0$, s.t.

$$\int_{K_\rho} YH \, dS = 0,$$

for all $Y \in SH^{N+2}$. We can write H, as

$$H = \sum_{j=-N}^N a_l Y_N^l.$$

Then in particular

$$\int_{K_\rho} (a_{-N} Y_N^{-N} + \cdots + a_N Y_N^N) \overline{Y}_{N+2}^{-N-2} \, dS = 0,$$

$$\vdots$$

$$\int_{K_\rho} (a_{-N} Y_N^{-N} + \cdots + a_N Y_N^N) \overline{Y}_{N+2}^{N+2} \, dS = 0.$$

So that for any choice of Ψ, the vector $\bar{\alpha} := (\alpha_{-N}, .., \alpha_N) \neq 0$ is such that
$$\mathcal{C}_{\Psi,N}(\rho)\bar{\alpha} = 0.$$
This implies that all the matrix $\mathcal{C}_{\Psi,N}(\rho)$ is singular, for any choice of Ψ. In particular we have that
$$\mathcal{D}_{\Psi,N}(\rho) = 0,$$
for all Ψ which proves the claim. $\blacksquare$

6. Circular cones

Here we use the properties of spherical harmonics to prove that circular cones are admissible medium cones, utilizing the determinant condition of Lemma 5.4. We also use some results on associated Legendre polynomials, which we prove later in Section 8.

In this and the following sections, we use the notation below for circular cones. Compare with Definition 3.3. Recall also the notational convention mentioned before (2.10) that allows us to write $(\vartheta, \varphi) \in \mathbb{S}^2$ instead of $x \in \mathbb{S}^2$, $x = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$.

Definition 6.1. By a circular cone C_ρ we denote a cone that can be represented in a spherical coordinate system as
$$C_\rho := \Big\{ (r, \vartheta, \varphi) \in \mathbb{R}^3 \ : \ r > 0, \ \vartheta \in (0, \rho), \ 0 \leq \varphi < 2\pi \Big\},$$
where $\rho \in (0, \pi/2)$. Also we let $K_\rho := C_\rho \cap \mathbb{S}^2$. Note that a half space is not a spherical cone according to this definition.

Recall that the elements of the matrix $\mathcal{C}_N(\rho)$ are given by
$$I_N^{k,l}(\rho) = \int_{K_\rho} Y_{N+2}^k \overline{Y}_N^l \, dS,$$
and moreover that
$$\mathcal{C}_N(\rho) := \begin{pmatrix} I_N^{-N,-N}(\rho) & \cdots & I_N^{-N,N}(\rho) \\ \vdots & \ddots & \vdots \\ I_N^{N,-N}(\rho) & \cdots & I_N^{N,N}(\rho) \end{pmatrix},$$
Furthermore we set
$$\mathcal{D}_N(\rho) = \det \mathcal{C}_N(\rho).$$
The next proposition shows that circular cones are admissible medium cones, and taken together with Theorem 5.3 we see that circular cones always scatter.

Proposition 6.2. *A circular cone C_ρ is an admissible medium cone.*

Proof. By Lemma 5.4 it is clear that it is enough to show that $\mathcal{D}_N(\rho) \neq 0$. In the case of a circular cone the matrix $\mathcal{C}_N(\rho)$ is an diagonal matrix, since
$$k \neq l \quad \Rightarrow \quad I_N^{k,l}(\rho) = \int_{K_\rho} Y_{N+2}^k \overline{Y}_N^l \, dS = 0.$$
Thus we have a matrix of the form
$$\mathcal{C}_N(\rho) := \begin{pmatrix} I_N^{-N,-N}(\rho) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_N^{N,N}(\rho) \end{pmatrix}.$$

Assume that $m \in \{-N, \ldots N\}$. The diagonal elements are then of the form

$$
\begin{aligned}
I_N^{m,m}(\rho) &= \int_{K_\rho} Y_{N+2}^m \overline{Y}_N^m \, dS \\
&= C \int_0^{2\pi} \int_0^\rho e^{im\varphi} P_{N+2}^m(\cos\vartheta) e^{-im\varphi} P_N^m(\cos\vartheta) \sin\vartheta \, d\vartheta d\varphi \\
&= 2\pi C \int_0^\rho P_{N+2}^m(\cos\vartheta) P_N^m(\cos\vartheta) \sin\vartheta \, d\vartheta \\
&= 2\pi C \int_{x_0}^1 P_{N+2}^m(x) P_N^m(x) \, dx \\
&= 2\pi C' \int_{x_0}^1 P_{N+2}^{|m|}(x) P_N^{|m|}(x) \, dx,
\end{aligned}
$$

where $x_0 = \cos\rho$, and where we used the definition in (8.1) to deduce that $P_M^{-m} = c P_M^m$, for some constants c. We will evaluate the integral on the last line. Clearly we can assume that $m \geq 0$. By lemmas 8.1, 8.3, 8.4 and 8.5, we know that

$$
\int_{x_0}^1 P_{N+2}^m(x) P_N^m(x) \, dx = (1 - x_0^2) 2 a_{N+1} x_0
\begin{cases}
\displaystyle\sum_{j=0}^{(N-m)/2} c_j [P_{m+2j}^m(x_0)]^2, & N - m \text{ is even} \\
\displaystyle\sum_{j=0}^{(N-m-1)/2} \tilde{c}_j [P_{m+1+2j}^m(x_0)]^2, & N - m \text{ is odd.}
\end{cases}
$$

In particular we have that

$$
\int_{x_0}^1 P_{N+2}^m(x) P_N^m(x) \, dx \geq
\begin{cases}
C x_0 (1 - x_0^2) [P_m^m(x_0)]^2, & N - m \text{ is even} \\
C x_0 (1 - x_0^2) [P_{m+1}^m(x_0)]^2, & N - m \text{ is odd}
\end{cases}
> 0,
$$

since P_m^m and P_{m+1}^m have no zeros on the interval $x_0 \in (0, 1)$, and since $x_0 \in (0, 1)$, because $\rho \in (0, \pi/2)$. It follows that all the diagonal elements of $\mathcal{C}_N(\rho)$ are bounded away from zero, when $\rho \in (0, \pi/2)$, and hence

$$
\mathcal{D}_N(\rho) \neq 0.
$$

A circular cone is thus an admissible medium cone. ∎

7. A density argument for star-shaped cones

In this section we finish the proof of Theorem 1.3 by proving Proposition 7.6. The latter shows that all star-shaped cones have admissible medium cones arbitrarily close to them. Theorem 5.3 imples that these admissible ones always scatter.

Let us begin by recalling Definition 1.2 of a star-shaped cone, that states that a cone C^σ, with the vertex at the origin, is *star-shaped*, if

$$
C^\sigma \cap \mathbb{S}^2 := \big\{ (\vartheta, \varphi) \in \mathbb{S}^2 \ : \ 0 \leq \varphi < 2\pi, \, 0 \leq \vartheta < \sigma(\varphi) \big\},
$$

where $\sigma \colon [0, 2\pi] \to (\rho_0, \pi/2)$, $\rho_0 \in (0, \pi/2)$ is a continuous, with $\sigma(0) = \sigma(2\pi)$.

In the following, we are interested in deformations of circular cones (Definition 6.1) into a star-shaped cone, which we call star-shaped deformations.

Definition 7.1. A *star-shaped deformation* of a circular cone C_{ρ_0}, $\rho_0 \in (0, \pi/2)$, is a family of cones $\rho \mapsto C_\rho^\sigma$ with vertex at the origin and having an intersection of the form

$$
C_\rho^\sigma \cap \mathbb{S}^2 = \big\{ (\vartheta, \varphi) \in \mathbb{S}^2 \ : \ \varphi \in [0, 2\pi), \, 0 \leq \vartheta < \rho\sigma(\varphi) + (1-\rho)\rho_0 \big\}, \quad \rho \in [-\epsilon, 1],
$$

for some continuous $\sigma \colon [0, 2\pi] \to (\rho_0, \pi/2)$ with $\sigma(0) = \sigma(2\pi)$, and some $\epsilon > 0$ small enough that $C_{-\epsilon}^\sigma$ is star-shaped.

Note that C^σ_ρ is a circular cone when $\rho = 0$, in particular $C^\sigma_0 = C_{\rho_0}$. For $\rho = 1$, we have $C^\sigma_\rho = C^\sigma_1 = C^\sigma$. The star-shaped deformation is thus essentially given by an interpolation between the points in the circular cone C_{ρ_0} and the star-shaped cone C^σ.

As in the previous sections, we are interested in the functions

$$I^{k,l}_{N,\sigma}(\rho) := \int_{C^\sigma_\rho \cap \mathbb{S}^2} Y^k_{N+2} \overline{Y}^l_N \, dS,$$

this time integrated over the star-shaped deformation cap instead of a circular cap. Furthermore recall that

$$\mathcal{C}_{N,\sigma}(\rho) := \begin{pmatrix} I^{-N,-N}_{N,\sigma}(\rho) & \cdots & I^{-N,N}_{N,\sigma}(\rho) \\ \vdots & \ddots & \vdots \\ I^{N,-N}_{N,\sigma}(\rho) & \cdots & I^{N,N}_{N,\sigma}(\rho) \end{pmatrix},$$

and that

(7.1) $$\mathcal{D}_{N,\sigma}(\rho) := \det \mathcal{C}_{N,\sigma}(\rho).$$

We will also be dealing with the *associated Legendre polynomials*[2] which are defined in (8.1) in terms of the Legendre polynomials P_n, but we reproduce the equation here for convenience.

$$P^m_n := (-1)^m (1 - x^2)^{m/2} \partial^m_x P_n, \quad P^{-m}_n := (-1)^m \frac{(n-m)!}{(n+m)!} P^m_n,$$

where $m = 0, .., n$. Moreover we set $P^m_n := 0$, for $m > n$.

It will be convenient to extend the definition of the associated Legendre polynomials to $\mathbb{C}$. Note that extending the factor $(1 - x^2)^{1/2}$ can be done in several ways depending on which branch of the square root we choose.

Definition 7.2. For $z \in \mathbb{C}$ we define

$$P^m_n(z) := (-1)^m (1 - z^2)^{m/2} \partial^m_z P_n(z), \quad P^{-m}_n(z) := (-1)^m \frac{(n-m)!}{(n+m)!} P^m_n(z),$$

where we choose the square root so that it has the branch cut along $(-\infty, 0)$, that is $\sqrt{z} = \sqrt{re^{i\varphi}} = \sqrt{r} e^{i\varphi/2}$, when $\varphi \in (-\pi, \pi)$.

Lemma 7.3. *The functions P^m_n are complex analytic in $\{x + iy \in \mathbb{C} : -1 < x < 1, y \in \mathbb{R}\}$.*

Proof. Since $\partial^m_z P_n(z)$ is a polynomial and $(1 - z)^{m/2}$ is a product, it will be enough to show that $(1 - z^2)^{1/2}$ is complex analytic on $\{x + iy \in \mathbb{C} : -1 < x < 1, y \in \mathbb{R}\}$. Now

$$(1 - z^2)^{1/2} = \sqrt{z} \circ (1 - z) \circ z^2.$$

Since $\operatorname{Re} z^2 = x^2 - y^2 \in (-\infty, 1)$ when $-1 < x < 1$, we have

$$z^2 \colon \{x + iy \in \mathbb{C} : -1 < x < 1, y \in \mathbb{R}\} \to \{x + iy \in \mathbb{C} : x < 1, y \in \mathbb{R}\}$$

is analytic and

$$(1 - z) \colon \{x + iy \in \mathbb{C} : x < 1, y \in \mathbb{R}\} \to \{x + iy \in \mathbb{C} : x > 0, y \in \mathbb{R}\}$$

is analytic. Finally $\sqrt{z} \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is analytic since we choose the branch cut in Definition 7.2 to be $(-\infty, 0)$. The range of $(1 - z)$ is contained in the former's domain and so the restriction of $\sqrt{z}$ to the range of $(1 - z)$ is analytic too. Thus $(1 - z^2)^{1/2}$ is complex analytic on $\{x + iy \in \mathbb{C} : -1 < x < 1, y \in \mathbb{R}\}$, which proves the claim. ∎

---

[2]which are not always polynomials!

Next we will shows that $\mathcal{D}_{N,\sigma}$ are analytic functions in the parameter ρ. For this it is convenient to make use of the following elementary lemma. But before that, let us recall some notation. For two sets of complex numbers $A, B \subset \mathbb{C}$ we define

$$AB = \{ab \in \mathbb{C} \ : \ a \in A, \, b \in B\}, \qquad A + B = \{a + b \in \mathbb{C} \ : \ a \in A, \, b \in B\}.$$

This notation is particularly useful for talking about rectangles.

Lemma 7.4. *Let $a < b$ and $c < d$ be real numbers and δ a positive real number. Let $g, h \in L^\infty(a,b) \cap C(a,b)$ and denote*

$$\mathcal{R}' := (c,d) + i(-\delta, \delta).$$

Suppose that

(i) *a complex valued function f is complex analytic on $\mathcal{R} := \mathcal{R}'g[(a,b)]$, where $g[(a,b)]$ is the image of the interval (a,b) under g. Assume furthermore that*

(ii) *for any $z_0 \in \mathcal{R}'$, one has the estimate $\int_a^b |f(z_0 g(t))h(t)| \, dt < \infty$.*

For $z_0 \in \mathcal{R}'$, let

$$F(z_0) := \int_a^b f(z_0 g(t))h(t) \, dt.$$

Then F is complex analytic on $\mathcal{R}'$. In particular F is real analytic on (c,d).

Proof. Firstly notice that the first part of condition *(ii)* guarantees that F is well defined on the rectangle $\mathcal{R}'$. Secondly, it suffices to compute the derivative $F'(z_0)$ when $z_0 \in \mathcal{R}'$ to prove the claim. Choose a sequence (z_m), s.t. $z_m \to 0$ in $\mathbb{C}$. Define

$$F_m^D(z_0) := \int_a^b \frac{f((z_0 + z_m)g(t)) - f(z_0 g(t))}{z_m} h(t) \, dt$$

and we are going to show that F_m^D converges as $m \to \infty$ and the limit will be $F'(z_0)$. Note that $z_0 \in \mathcal{R}'$, which is an open set, and since we are interested in the limit only, we may assume that for all m the whole segment from z_0 to $z_0 + z_m$ is in a compact subset $\mathcal{K} \Subset \mathcal{R}'$, which depends only on z_0 and $\mathcal{R}'$. This implies that

(7.2) $$\xi g(t) \in \mathcal{K}g[(a,b)] \Subset \mathcal{R}'g[(a,b)] = \mathcal{R}$$

for any $t \in (a,b)$ and ξ on that segment, i.e. that $\xi g(t)$ will stay a positive distance from $\partial \mathcal{R}$. This will be used later in the proof after a mean value theorem.

To study the limit of $F_m^D(z)$ define

$$D_m(z_0, t) := \frac{f((z_0 + z_m)g(t)) - f(z_0 g(t))}{z_m}.$$

Let us study the limit of these as $m \to \infty$. Firstly, the pointwise limit. If $g(t) = 0$ then $D_m(z_0, t) = 0$ so the pointwise limit exists and is 0. If $g(t) \neq 0$ then the pointwise limit of $D_m(z_0, t)$ is $f'(z_0 g(t))g(t)$ a formula which also applies to the case $g(t) = 0$. Next, let us show that the $|D_m(z_0, \cdot)|$ have an integrable upper bound in the interval (a,b). If $g(t) \neq 0$, we have

$$|D_m(z_0, t)| = \left| \frac{f((z_0 + z_m)g(t)) - f(z_0 g(t))}{z_m g(t)} \right| |g(t)|$$
$$\lesssim |f'(\xi g(t))||g(t)| \leq \sup_{\zeta \in \mathcal{K}g(a,b)} |f'(\zeta)||g(t)|.$$

by the mean value theorem for some $\xi \in \mathbb{C}$ which lies on the segment connecting z_0 to $z_0 + z_m$. This also holds when $g(t) = 0$. The supremum above is a finite number because f is complex analytic in $\mathcal{R}$ by *(i)*, and so its derivative f' is bounded on its compact subsets. The set $\mathcal{K}g(a,b)$ is such a set according to (7.2). This means that for any fixed z_0 there is a finite

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constant C depending on a, b, c, d, δ, z_0 and f such that $|D_m(z_0, t)| \leq C|g(t)| \leq C\|g\|_\infty$ for all $t \in (a, b)$, and this is integrable.

By the dominated convergence Theorem and since $h \in L^\infty(a, b)$, we have that for each $z_0 \in \mathcal{R}'$ the following holds

$$\lim_{m \to \infty} F_m(z_0) = \lim_{m \to \infty} \int_a^b D_m(z_0, t)h(t)\, dt$$
$$= \int_a^b \lim_{m \to \infty} D_m(z_0, t)h(t)\, dt$$
$$= \int_a^b f'(z_0 g(t))g(t)h(t)\, dt$$

and that the latter is a finite complex number. This gives the existence of $F'(z_0)$. Hence F is complex analytic in $\mathcal{R}'$. ∎

Lemma 7.5. *The functions $\rho \mapsto \mathcal{D}_{N,\sigma}(\rho)$ of any star-shaped deformation $\rho \mapsto C_\rho^\sigma$, $\rho \in (-\epsilon, 1)$ are real analytic functions in $(-\epsilon_0, 1)$ for some $0 < \epsilon_0 \leq \epsilon$. They are also not identically zero.*

Proof. Using spherical coordinates and referring to (2.11) we have that

$$I_{N,\sigma}^{k,l}(\rho) = \int_{C_\rho^\sigma \cap \mathbb{S}^2} Y_{N+2}^k \overline{Y}_N^l\, dS$$
$$= \int_0^{2\pi} \int_0^{\rho_0 + \rho(\sigma(\varphi) - \rho_0)} Y_{N+2}^k(\vartheta, \varphi) \overline{Y}_N^l(\vartheta, \varphi) \sin(\vartheta)\, d\vartheta d\varphi$$
$$= C_N \int_0^{2\pi} e^{i(k-l)\varphi} \int_0^{\cos(\rho_0 + \rho(\sigma(\varphi) - \rho_0))} P_{N+2}^k(x) P_N^l(x)\, dx d\varphi$$

Changing the variable symbols in preparation for Lemma 7.4, we can write

$$I_{N,\sigma}^{k,l}(z_0) = C_N \int_0^{2\pi} f(z_0 g(t))h(t)\, dt$$

where $f = f_1 \circ f_2$ and

$$f_1(z) = \int_0^z P_{N+2}^k(w) P_N^l(w)\, dw,$$
$$f_2(z) = \cos(\rho_0 + z),$$
$$g(t) = \sigma(t) - \rho_0,$$
$$h(t) = e^{i(k-l)t}$$

and the integral in the definition of f_1 is interpreted as the complex line integral of the straight line segment from 0 to z.

We want to apply Lemma 7.4 to show that $I_{N,\sigma}^{k,l}$ is real analytic in $(-\epsilon_0, 1)$ for some $\epsilon_0 > 0$. In order to do this we need to check that the conditions *(i)* and *(ii)* of the Lemma 7.4 can be satisfied. For condition *(i)*, we need to find $\epsilon_0 > 0$ and $\delta_0 > 0$ so that f is complex analytic on the rectangle

$$(7.3) \qquad \mathcal{R} := \mathcal{R}' g[(0, 2\pi)],$$

where

$$(7.4) \qquad \mathcal{R}' := (-\epsilon_0, 1) + i(-\delta_0, \delta_0)$$

and $\epsilon_0 \leq \epsilon$ so that C_ρ^σ is defined for $\rho \in (-\epsilon_0, 1)$. Choose $\epsilon_0 > 0$ by

$$(7.5) \qquad \epsilon_0 := \min\left(\frac{\rho_0}{2\|g\|_{L^\infty}}, \epsilon\right).$$

Since $g(t) \in (0, \pi/2 - \rho_0)$ for all t, and by our choice of ϵ_0, we have for $x_0 \in (-\epsilon_0, 1)$, that

$$\rho_0/2 < \rho_0 + x_0 g(t) < \pi/2.$$

Recal also that

$$\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y),$$

where each of $\cos, \cosh, \sin, \sinh$ are real-valued when their arguments are real. Let $\delta_2 > 0$ be arbitrary and let

$$(7.6) \qquad \delta_0 := \frac{\delta_1}{\|g\|_{L^\infty}}, \qquad \delta_1 = \frac{1}{2}\min\left(\operatorname{arsinh}\delta_2, \operatorname{arcosh}\left(\frac{1}{\cos(\rho_0/2)}\right)\right)$$

both of which are positive, and they give $\delta_0 g(t) \in (0, \delta_1)$ for all t. Then for $-\epsilon_0 < x_0 < 1$ and $-\delta_0 < y_0 < \delta_0$, we have that

$$\begin{aligned}
f_2((x_0 + iy_0)g(t)) &= \cos(\rho_0 + (x_0 + iy_0)g(t)) \\
&= \cos(\rho_0 + x_0 g(t))\cosh(y_0 g(t)) - i\sin(\rho_0 + x_0 g(t))\sinh(y_0 g(t)) \\
&\subset \cos[(\rho_0/2, \pi/2)]\cosh[(0, \delta_1)] - i\sin[(\rho_0/2, \pi/2)]\sinh[(-\delta_1, \delta_1)] \\
&\subset (0, \cos(\rho_0/2)\cosh\delta_1) + i(-\sinh\delta_1, \sinh\delta_1).
\end{aligned}$$

Denote $\alpha = 0$, $\beta = \cos(\rho_0/2)\cosh\delta_1$. Then by our choice of δ_1 we see that $\alpha < \beta < 1$ and $\sinh\delta_1 < \delta_2$, and so

$$f_2((x_0 + iy_0)g(t)) \subset (\alpha, \beta) + i(-\delta_2, \delta_2) =: \mathcal{R}''$$

for $x_0 \in (-\epsilon_0, 1)$, $y_0 \in (-\delta_0, \delta_0)$ and $t \in (0, 2\pi)$. In other words,

$$(7.7) \qquad\qquad\qquad\qquad f_2 \colon \mathcal{R} \to \mathcal{R}''$$

is complex analytic. In particular the associated Legendre polynomials P_n^m are analytic in the range of f_2. In fact, $\mathcal{R}''$ is a positive distance $1 - \beta$ away from any point where we do not know them being analytic. See Lemma 7.3.

Let us prove next that f is complex analytic on the rectangle $\mathcal{R}$ given in (7.3). We know that $P_{N+2}^k P_N^l$ is analytic on the vertical strip $\{x + iy \in \mathbb{C} \ : \ -1 < x < 1, \ y \in \mathbb{R}\}$ by Lemma 7.3. Hence

$$f_1 \colon z \mapsto \int_0^z P_{N+2}^k(w)P_N^l(w)\,dw$$

is likewise analytic in the strip $\{x + iy \in \mathbb{C} \ : \ -1 < x < 1, \ y \in \mathbb{R}\}$, which happens to contain $\mathcal{R}''$. Thus $f = f_1 \circ f_2$ is analytic on $\mathcal{R}$ and so condition (i) of Lemma 7.4 is satisfied.

Next, we show that condition (ii) is satisfied, i.e. that we have $\int_0^{2\pi} |f(z_0 g(t))h(t)|dt < \infty$ for any $z_0 \in \mathcal{R}'$. Recall that $|h(t)| = 1$ in our case. Notice that f_1 is in fact uniformly bounded on $\mathcal{R}''$. This is because f_1 is analytic in the vertical strip mentioned above, and $\mathcal{R}''$ is contained in it with distance $1 - \beta > 0$ from its boundary. Since $f_2 \colon \mathcal{R} \to \mathcal{R}''$, and $f_1 \colon \mathcal{R}'' \to \mathbb{C}$ is uniformly bounded, we have $f = f_1 \circ f_2 \colon \mathcal{R} \to \mathbb{C}$ uniformly bounded. Given $z_0 \in \mathcal{R}'$ and $t \in (0, 2\pi)$ we have $z_0 g(t) \in \mathcal{R}$ and so we have a uniform bound for $f(z_0 g(t))$. Hence the integral is finite.

Lemma 7.4 imples that the maps $\rho \mapsto I_{N,\sigma}^{k,l}(\rho)$ are real analytic on $(-\epsilon_0, 1)$ where ϵ_0 is given by (7.5). The determinant functions $\mathcal{D}_{N,\sigma}$ are obtained from matrices having elements $I_{N,\sigma}^{k,l}$ as their components. The determinant is a sum of products of these elements and is therefore also real analytic in $(-\epsilon_0, 1)$.

Finally, by the proof of Proposition 6.2 we know that $\mathcal{D}_{N,\sigma}(0) \neq 0$, since C_0^σ is a circular cone. The functions $\mathcal{D}_{N,\sigma}$ are thus not identically zero in ρ. This completes the proof. ∎

We will now show that there exists a number of star-shaped cones, which are not necessarily circular, but still admissible medium cones.

Proposition 7.6. *Let $\rho \mapsto C_\rho^\sigma$, $\rho \in (-\epsilon, 1)$, be a star-shaped deformation which satisfies the regularity assumptions (i)–(iii) of Definition 5.1. Let $0 < \epsilon_0 < \epsilon$ be as in Lemma 7.5. Then there is a countable set $\mathcal{Z}$ such that all C_ρ^σ with $\rho \in (-\epsilon_0, 1) \setminus \mathcal{Z}$ are admisible.*

Proof. By Lemma 7.5 we know that for every N the map $\rho \mapsto \mathcal{D}_{N,\sigma}(\rho)$ is analytic on $(-\epsilon_0, 1)$ without being identically zero. Consider the set

$$\mathcal{Z} = \Big\{\rho \in (-\epsilon_0, 1) \ : \ \mathcal{D}_{N,\sigma}(\rho) = 0 \text{ for some } N\Big\}.$$

Lemma 5.4 implies that all of the cones C_ρ^σ with $\rho \in (-\epsilon_0, 1) \setminus \mathcal{Z}$ are admissible medium cones because $\rho \in (-\epsilon_0, 1) \setminus \mathcal{Z}$ implies that $\mathcal{D}_{N,\sigma}(\rho) \neq 0$ for all N.

The set $\mathcal{Z}$ is countable as a countable union of countable sets. This follows because $\mathcal{D}_{N,\sigma}$ has at most a countable number of zeros in $(-\epsilon_0, 1)$ since it is analytic and not identically zero there by Lemma 7.5. ∎

8. Results on associated Legendre polynomials

In this section we derive some results on associated Legendre polynomials, that are utilized in computing explicitly the determinants $\mathcal{D}_N$ in the case of circular cones, which is done in Section 6. First we extend some well known formulas for certain inner products of Legendre polynomials to the case of the associated Legendre polynomials, after which we derive a modification of the Christoffel-Darboux formula, which can be used to analyze these inner products.

Recall firstly that the associated Legendre polynomials are defined in terms of the Legendre polynomials P_n, as

$$(8.1) \qquad P_n^m := (-1)^m (1 - x^2)^{m/2} \partial_x^m P_n, \quad P_n^{-m} := (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m,$$

where $m = 0, .., n$. Moreover we set $P_n^m := 0$, for $m > n$. The Legendre polynomials are defined in [1] equation (9), p.10.

We will first derive an integral formula for the special case of the associated Legendre polynomials of the form P_n^0, i.e. in the case when they coincide with the Legendre polynomials P_n. This formula can be found in [1] equation (5), p.172. We however give the proof as a convenience to the reader.

Lemma 8.1. *Let $x_0 \in (-1, 1)$. Then we have the following formula*

$$\int_{x_0}^1 P_n^0 P_{n+2}^0 \, dx = C(1 - x_0^2)\Big(\partial_x P_{n+2}^0 P_n^0 - P_{n+2}^0 \partial_x P_n^0\Big)\Big|_{x = x_0},$$

where $C = 1/(4n + 6)$.

Proof. Note firstly that $P_n^0 = P_n$. The Legendre polynomials in the claim solve the ODEs

$$\partial_x\Big((1 - x^2)\partial_x P_n\Big) + n(n + 1)P_n = 0,$$

$$\partial_x\Big((1 - x^2)\partial_x P_{n+2}\Big) + (n + 2)(n + 3)P_{n+2} = 0.$$

Multiplying by P_n and P_{n+2}, and subtracting the resulting equations gives, then that

$$-(4n + 6)\int_{x_0}^1 P_n P_{n+2} \, dx = \int_{x_0}^1 \partial_x\Big((1 - x^2)\partial_x P_n\Big)P_{n+2} - \partial_x\Big((1 - x^2)\partial_x P_{n+2}\Big)P_n \, dx$$

$$= (1 - x^2)\Big(\partial_x P_n P_{n+2} - P_{n+2}\partial_x P_n\Big)\Big|_{x_0}^1$$

$$= (1 - x_0^2)\Big(P_{n+2}(x_0)\partial_x P_n(x_0) - \partial_x P_{n+2}(x_0)P_n(x_0)\Big),$$

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which proves the claim. ∎

Now we generalize the formula of Lemma 8.1 to the case where $m = 1, .., n - 1$. First we derive the following simple formula (which can be found in e.g. [1] p. 205).

Lemma 8.2. *For $m \geq 1$, we have the formula*

$$\partial_x\Big((1 - x^2)^m \partial_x^m P_n\Big) = -C(1 - x^2)^{m-1}\partial_x^{m-1} P_n,$$

where $C = n(n + 1) - m(m - 1)$.

Proof. By taking repeated derivatives of the Legendre ODE

$$(1 - x^2)\partial_x^2 P_n - 2x\partial_x P_n + n(n + 1)P_n = 0,$$

one arrives by an induction argument at the formula

$$(1 - x^2)\partial_x^{m+1} P_n - 2mx\partial_x^m P_n + (n(n + 1) - m(m - 1))\partial_x^{m-1} P_n = 0.$$

Multiplying by $(1 - x^2)^{m-1}$, gives then that

$$\partial_x\Big((1 - x^2)^m \partial_x^m P_n\Big) = -(n(n + 1) - m(m - 1))(1 - x^2)^{m-1}\partial_x^{m-1} P_n.$$

∎

We are now ready to generalize the result of Lemma 8.1 to the case $m = 1, .., n - 1$.

Lemma 8.3. *Let $x_0 \in (-1, 1)$ and $m = 1, .., n - 1$. Then we have the following formula*

$$\int_{x_0}^1 P_n^m P_{n+2}^m \, dx = C(1 - x_0^2)\Big(\partial_x P_{n+2}^m P_n^m - P_{n+2}^m \partial_x P_n^m\Big)\Big|_{x=x_0},$$

where $C = 1/(4n + 6)$.

Proof. Assume that $1 \leq m \leq n - 1$. By integration by parts we get that

$$\int_{x_0}^1 P_n^m P_{n+2}^m \, dx = \int_{x_0}^1 (1 - x^2)^m \partial_x^m P_{n+2} \partial_x^m P_n \, dx$$

$$= -\int_{x_0}^1 \partial_x((1 - x^2)^m \partial_x^m P_{n+2})\partial_x^{m-1} P_n \, dx + \Big|_{x_0}^1 (1 - x^2)^m \partial_x^m P_{n+2} \partial_x^{m-1} P_n.$$

Then using the formula of Lemma 8.2 we have that

$$\int_{x_0}^1 P_n^m P_{n+2}^m \, dx = C_1 \int_{x_0}^1 (1 - x^2)^{m-1} \partial_x^{m-1} P_{n+2} \partial_x^{m-1} P_n \, dx + \Big|_{x_0}^1 (1 - x^2)^m \partial_x^m P_{n+2} \partial_x^{m-1} P_n$$

$$= C_1 \int_{x_0}^1 P_{n+2}^{m-1} P_n^{m-1} \, dx + \Big|_{x_0}^1 (1 - x^2)^m \partial_x^m P_{n+2} \partial_x^{m-1} P_n,$$

where $C_1 = (n + 2)(n + 3) - m(m - 1)$. Likewise we can use integration by parts to deduce that

$$\int_{x_0}^1 P_n^m P_{n+2}^m \, dx = C_2 \int_{x_0}^1 P_{n+2}^{m-1} P_n^{m-1} \, dx + \Big|_{x_0}^1 (1 - x^2)^m \partial_x^{m-1} P_{n+2} \partial_x^m P_n.$$

where $C_2 = n(n + 1) - m(m - 1)$. Subtracting these we obtain that

$$C_3 \int_{x_0}^1 P_{n+2}^{m-1} P_n^{m-1} \, dx = \Big|_{x_0}^1 \Big((1 - x^2)^m \partial_x^{m-1} P_n \partial_x^m P_{n+2} - (1 - x^2)^m \partial_x^m P_n \partial_x^{m-1} P_{n+2}\Big),$$

where the constant is given by $C_3 := C_1 - C_2 = 4n + 6$. This can be further simplified, since by a straight forward computation

$$(1 - x^2)^m \partial_x^{m-1} P_{n+2} \partial_x^m P_n = (1 - x^2)P_{n+2}^{m-1} \partial_x P_n^{m-1} + (m - 1)x P_{n+2}^{m-1} P_n^{m-1},$$

and likewise

$$(1-x^2)^m \partial_x^m P_{n+2} \partial_x^{m-1} P_n = (1-x^2)\partial_x P_{n+2}^{m-1} P_n^{m-1} + (m-1)x P_{n+2}^{m-1} P_n^{m-1}.$$

Putting this together gives then that

$$C_3 \int_{x_0}^1 P_{n+2}^{m-1} P_n^{m-1} \, dx = \Big|_{x_0}^1 (1-x^2)\Big(P_n^{m-1}\partial_x P_{n+2}^{m-1} - \partial_x P_n^{m-1} P_{n+2}^{m-1} \Big).$$

This proves the claim for the cases $m = 1, .., n-1$. ∎

Finally we prove the results of lemmas 8.1 and 8.3, for the case $m = n$.

Lemma 8.4. *Let $x_0 \in (-1, 1)$. Then we have the following formula*

$$\int_{x_0}^1 P_n^n P_{n+2}^n \, dx = C(1-x_0^2)\Big(\partial_x P_{n+2}^n P_n^n - P_{n+2}^n \partial_x P_n^n \Big)\Big|_{x=x_0},$$

where $C = 1/(4n+6)$.

Proof. Note that $\partial_x^{n+1} P_n = 0$. By integration by parts we have that

(8.2)
$$
\begin{aligned}
0 &= \int_{x_0}^1 (1-x^2)^{n+1} \partial_x^{n+1} P_{n+2} \partial_x^{n+1} P_n \, dx \\
&= -\int_{x_0}^1 \partial_x((1-x^2)^{n+1}\partial_x^{n+1}P_{n+2})\partial_x^n P_n \, dx + \Big|_{x_0}^1 (1-x^2)^{n+1}\partial_x^{n+1}P_{n+2}\partial_x^n P_n.
\end{aligned}
$$

Let's rewrite the first term on the r.h.s. of the last line. Using Lemma (8.2) we have that

$$\partial_x((1-x^2)^{n+1}\partial_x^{n+1}P_{n+2}) = -(4n+6)(1-x^2)^n \partial_x^n P_{n+2}.$$

Notice that by inserting this back into to (8.2) we get the term on the l.h.s. of the claim.

It is thus enough to show that the boundary term in (8.2) gives the expression on the r.h.s. of the claim. For this we use the fact that

$$
\begin{aligned}
(1-x^2)^{n+1}\partial_x^{n+1}P_{n+2}\partial_x^n P_n &= (1-x^2)P_n^n\Big((1-x^2)^{n/2}\partial_x^{n+1}P_{n+2}\Big) \\
&= (1-x^2)P_n^n\Big(\partial_x P_{n+2}^n - \partial_x(1-x^2)^{n/2}\partial_x^n P_{n+2}\Big).
\end{aligned}
$$

Now we have that

$$\partial_x P_n^n = \partial_x(1-x^2)^{n/2}\partial_x^n P_n,$$

so the previous equations yields by a short computation that

$$(1-x^2)^{n+1}\partial_x^{n+1}P_{n+2}\partial_x^n P_n = (1-x^2)\Big(P_n^n \partial_x P_{n+2}^n - P_{n+2}^n \partial_x P_n^n \Big).$$

Going back to (8.2) we see that

$$(4n+6)\int_{x_0}^1 (1-x^2)^n \partial_x^n P_{n+2} \partial_x^n P_n \, dx = -\Big|_{x_0}^1 (1-x^2)\Big(P_n^n \partial_x P_{n+2}^n - P_{n+2}^n \partial_x P_n^n \Big),$$

and that the claim holds. ∎

Next we prove the following Christoffel-Darboux type formula, which is a slight modification of the usual formula (see p.43 in [42]). Note that the r.h.s. of the claim has a sign determined by the sign of x.

Lemma 8.5. *Let $0 \leq m \leq n$. We have the following identity*

$$P_n^m \partial_x P_{n+2}^m - \partial_x P_n^m P_{n+2}^m = 2a_{n+1}x \begin{cases} \displaystyle\sum_{j=0}^{(n-m)/2} c_j [P_{m+2j}^m]^2, & n-m \text{ is even} \\ \displaystyle\sum_{j=0}^{(n-m-1)/2} \tilde{c}_j [P_{m+1+2j}^m]^2, & n-m \text{ is odd} \end{cases}$$

where the coefficients c_j are given by

$$c_j := \begin{cases} a_n, & j = (n-m)/2, \\ a_{m+2j}b_{n-1}b_{n-2}\ldots b_{m+2j}, & j < (n-m)/2. \end{cases}$$

And the coefficients $\tilde{c}_j$ are given by

$$\tilde{c}_j := \begin{cases} a_n, & j = (n-m-1)/2, \\ a_{m+1+2j}b_{n-1}b_{n-2}\ldots b_{m+1+2j}, & j < (n-m-1)/2. \end{cases}$$

And where coefficients a_k and b_k are given by

$$a_k := \frac{2k+1}{k-m+1}, \quad b_k := \frac{k+m+1}{k-m+2}, \quad \text{for} \quad m \leq k,$$

and $a_k = 0 = b_k$, when $m > k$.

Proof. The recurrence relation

$$(n-m+1)P_{n+1}^m = (2n+1)xP_n^m - (n+m)P_{n-1}^m$$

implies that

(8.3)
$$P_{n+1}^m = a_n x P_n^m - b_{n-1}P_{n-1}^m,$$
$$P_{n+2}^m = a_{n+1}x P_{n+1}^m - b_n P_n^m.$$

Using the second of these equalities gives that

(8.4) $\quad P_n^m(x)P_{n+2}^m(y) - P_n^m(y)P_{n+2}^m(x) = a_{n+1}\Big(yP_{n+1}^m(y)P_n^m(x) - xP_{n+1}^m(x)P_n^m(y)\Big).$

Lets evaluate the two terms on the r.h.s. of the equality. By successively applying the recurrence relation in (8.3) we get the equations

$$P_{n+1}^m(y)P_n^m(x) = a_n y P_n^m(y)P_n^m(x) - b_{n-1}P_{n-1}^m(y)P_n^m(x),$$
$$P_{n-1}^m(y)P_n^m(x) = a_{n-1}x P_{n-1}^m(x)P_{n-1}^m(y) - b_{n-2}P_{n-2}^m(x)P_{n-1}^m(y),$$
$$P_{n-1}^m(y)P_{n-2}^m(x) = a_{n-2}y P_{n-2}^m(x)P_{n-2}^m(y) - b_{n-3}P_{n-2}^m(x)P_{n-3}^m(y),$$
$$P_{n-3}^m(y)P_{n-2}^m(x) = a_{n-3}x P_{n-3}^m(x)P_{n-3}^m(y) - b_{n-4}P_{n-4}^m(x)P_{n-3}^m(y),$$
$$\vdots$$

where the last non-zero line is

$$\begin{cases} P_{m+1}^m(y)P_m^m(x) = a_m y P_m^m(x)P_m^m(y), & \text{if } n-m \text{ is even}, \\ P_m^m(y)P_{m+1}^m(x) = a_m x P_m^m(x)P_m^m(y), & \text{if } n-m \text{ is odd}, \end{cases}$$

since we defined $P_l^k = 0$, when $k > l$. It will be convenient to define

$$z_1 := \begin{cases} -y, & \text{if } n-m \text{ is even}, \\ x, & \text{if } n-m \text{ is odd}. \end{cases}$$

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Using these equations to rewrite the first term on the r.h.s of (8.4), we get that

$$
\begin{aligned}
yP_{n+1}^m(y)P_n^m(x) =&a_ny^2P_n^m(y)P_n^m(x)\\
&-a_{n-1}b_{n-1}xyP_{n-1}^m(x)P_{n-1}^m(y)\\
&+a_{n-2}b_{n-1}b_{n-2}y^2P_{n-2}^m(x)P_{n-2}^m(y)\\
&-a_{n-3}b_{n-1}b_{n-2}b_{n-3}xyP_{n-3}^m(x)P_{n-3}^m(y)\\
&\ \vdots\\
&-a_mb_{n-1}\ldots b_mz_1yP_m^m(y)P_m^m(x).
\end{aligned}
$$

For the second term on the r.h.s of (8.4), we get similarly the expression

$$
\begin{aligned}
xP_{n+1}^m(x)P_n^m(y) =&a_nx^2P_n^m(x)P_n^m(y)\\
&-a_{n-1}b_{n-1}yxP_{n-1}^m(y)P_{n-1}^m(x)\\
&+a_{n-2}b_{n-1}b_{n-2}x^2P_{n-2}^m(y)P_{n-2}^m(x)\\
&-a_{n-3}b_{n-1}b_{n-2}b_{n-3}yxP_{n-3}^m(y)P_{n-3}^m(x)\\
&\ \vdots\\
&-a_mb_{n-1}\ldots b_mz_2xP_m^m(x)P_m^m(y).
\end{aligned}
$$

where the z_2 in the last term is defined as

$$
z_2 := \begin{cases} -x, & \text{if } n-m \text{ is even,}\\ \ y, & \text{if } n-m \text{ is odd.} \end{cases}
$$

Subtracting the two equalities gives then that

$$
\begin{aligned}
yP_{n+1}^m(y)P_n^m(x)-xP_{n+1}^m(x)P_n^m(y) =&a_n(y^2-x^2)P_n^m(y)P_n^m(x)\\
&+a_{n-2}b_{n-1}b_{n-2}(y^2-x^2)P_{n-2}^m(x)P_{n-2}^m(y)\\
&+\ldots\\
&+a_mb_{n-1}\ldots b_m(xz_2-yz_1)P_m^m(x)P_m^m(y).
\end{aligned}
$$

Notice that the last term is non zero only if $n-m$ is even, and in this case it equals to y^2-x^2. Going back to (8.4), we get that

$$
\begin{aligned}
\frac{P_n^m(x)P_{n+2}^m(y)-P_n^m(y)P_{n+2}^m(x)}{x-y} =&-a_{n+1}\Big(a_n(y+x)P_n^m(y)P_n^m(x)\\
&+a_{n-2}b_{n-1}b_{n-2}(y+x)P_{n-2}^m(x)P_{n-2}^m(y)+\ldots\Big)
\end{aligned}
$$

Adding $\pm P_{n+2}^m(y)P_n^m(y)$ in the denominator and taking the limit $y\to x$, gives that

$$
P_n^m\partial_xP_{n+2}^m-\partial_xP_n^mP_{n+2}^m = 2a_{n+1}x\begin{cases}\displaystyle\sum_{j=0}^{(n-m)/2}c_j[P_{m+2j}^m]^2, & \text{if } n-m \text{ is even}\\[2mm]\displaystyle\sum_{j=0}^{(n-m-1)/2}\tilde{c}_j[P_{m+1+2j}^m]^2, & \text{if } n-m \text{ is odd}\end{cases}
$$

where the coefficients c_j are given by

$$
c_j := \begin{cases} a_n, & \text{if } j=(n-m)/2,\\ a_{m+2j}b_{n-1}b_{n-2}\ldots b_{m+2j}, & \text{if } j<(n-m)/2 \ . \end{cases}
$$

and the coefficients $\tilde{c}_j$ are given by

$$\tilde{c}_j := \begin{cases} a_n, & \text{if } j = (n-m-1)/2, \\ a_{m+1+2j}b_{n-1}b_{n-2}\ldots b_{m+1+2j}, & \text{if } j < (n-m-1)/2 \ . \end{cases}$$

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