Geometric Hermite interpolation by rational curves of constant width*

A. Arnal[a], J.V. Beltran[b], J. Monterde[b], D. Rochera[c,*]

[a]*Dept. de Matemàtiques, Universitat Jaume I, Av. Vicent Sos Baynat, s/n 12071 Castelló, Spain*
[b]*Dept. de Matemàtiques, Universitat de València, Av. Vicent Andrés Estellés, 1, E-46100-Burjassot (València), Spain*
[c]*BCAM – Basque Center for Applied Mathematics, Mazarredo 14, 48009 Bilbao (Basque Country), Spain*

Abstract

A constructive characterization of the support function for a rationally parameterized curve of constant width is given. In addition, a Hermite interpolation problem for such kind of curves is solved, which yields a method to determine a rational curve of constant width that passes through a set of free points with the corresponding tangent directions. Finally, the case of piecewise rational support functions is considered, which increases the design freedom. The procedure is presented in the general case of hedgehogs of constant width taking the advantage of projective hedgehogs, so that some constraints must be taken to ensure convexity of the desired curve.

Keywords: Curve of constant width, Projective hedgehog, Geometric Hermite interpolation, Rational parameterization, Support function.

1. Introduction

The width of a planar closed strictly convex curve $\boldsymbol{\alpha}$ in a direction $\mathbf{v}$ is defined as the distance between pairs of parallel supporting lines to $\boldsymbol{\alpha}$ (i.e., lines that envelope the curve $\boldsymbol{\alpha}$ smoothly) which are orthogonal to $\mathbf{v}$ (see Figure 1). The curve $\boldsymbol{\alpha}$ is said to be *of constant width* if the width is the same for all directions.

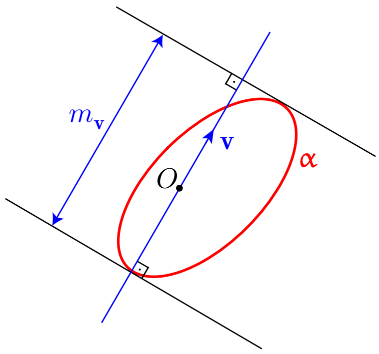

Figure 1: The width of a curve $\boldsymbol{\alpha}$ measured by the distance $m_{\mathbf{v}}$ between two parallel supporting lines.

*This is a preprint of an article published in the Journal of Computational and Applied Mathematics. The final authenticated version is available online at: `https://doi.org/10.1016/j.cam.2023.115598`.

*Corresponding author

Email addresses: `parnal@uji.es` (A. Arnal), `beltranv@uv.es` (J.V. Beltran), `monterde@uv.es` (J. Monterde), `drochera@bcamath.org` (D. Rochera)

The most famous non-circular curves of constant width are given by the family of Reuleaux polygons. In particular, a Reuleaux triangle is defined as the boundary of the intersection of three circles whose centers are the vertices of an equilateral triangle and whose radii are equal to the side of the triangle.

There have been interesting applications of curves of constant width throughout history and nowadays (see Chapter 18 of [15] and its references therein). For example, what prevents manhole covers from falling through the hole is the property of having a constant width, so in addition to a disk shape, any shape of constant width is allowed for this purpose. Some other popular applications of curves of constant width are their use as profiles of cams, that convert rotary motion into linear motion, or their use in coinage.

Curves of constant width can be generalized to certain non-convex curves called *hedgehogs of constant width* [11, 21]. These curves have two tangent lines in each direction, maintaining the same separation between any of these two lines. As a limiting case, when the width is zero, we have projective hedgehogs [13], which are curves with one tangent line in each direction.

The usual way of working with curves of constant width is thanks to a special parameterization which uses the notion of a support function [15] or a similar approach [10]. The support function is one of the most central basic concepts in convex geometry and it is widely used in other areas such as stereology [3, 4] or geometric tomography [8].

In the field of constant width curves, their parameterization by a support function is usually posed by trigonometric functions. In fact, motivated by Rabinowitz in [19], the algebraic equations of some families of constant width curves defined by trigonometric support functions have been studied in several papers (see [13], [17], [20] and [2]).

Rational expressions are preferred in the field of computer-aided geometric design for exact curve representation. The aim of this paper is to work with rational constant width curves parameterized by rational functions. With this, curves of constant width can also be integrated into the scheme of rational Bézier curves. There are some previous works in this direction, such as [1] or [22]. Other related works are [14], [9] and [24].

First, in Section 2 an introduction to support functions and rationally supported curves is presented. In particular, hedgehogs of constant width (the natural generalization of convex curves of constant width) are defined and it is recalled that any hedgehog of constant width can be seen as an offset of a projective hedgehog (seen as a curve of constant width 0).

In Section 3 we focus on rationally parameterized curves of constant width. The constant width condition can be written in terms of the support function (Proposition 2). The first objective is to characterize the rational support functions for rational curves of constant width and for this task it is enough to characterize the rational support functions of projective hedgehogs. A denominator for the rational support function that can provide a rational projective hedgehog is said to be *admissible* (Definition 3). Theorem 1 characterizes all possible admissible denominators and provides a way to construct them by giving their roots. The same result, in fact, provides a method to construct rational support functions for curves of constant width (Algorithm 1) dependent on some free parameters.

The second goal of the paper is to translate the free parameters for the construction of constant width curves into parameters with geometric meaning. In particular, in Section 4 a geometric Hermite interpolation problem is solved (Theorem 2): given an admissible denominator and a value for the width, a unique rationally parameterized curve of constant width is determined by its passage through certain user-controlled points and tangents. The result is constructive and offers an explicit expression for such a parameterization (Algorithm 2). Thus, a dynamic and interactive design of these curves is possible by choosing a set of points with their corresponding tangents. Some examples are illustrated in Section 5.

Finally, in Section 5.3 we show that it is possible to join, G^1-continuously, pieces of rational curves of different degrees in such a way that the resulting piecewise rational curve is of constant width. Thus, the use of piecewise curves increases the design freedom to construct curves of constant width and provides a way to avoid singularities in the constructed curves.

2. Support functions and constant width

Given a 2π-periodic $\mathcal{C}^2$-function $h : \mathbb{R} \to \mathbb{R}$, the envelope C of a family of supporting lines

$$X\cos\theta + Y\sin\theta = h(\theta),$$

is said to be a *hedgehog* [12]. The function h is called a *support function* and represents the signed distance from the origin to the corresponding supporting line to C that has normal vector $(\cos\theta, \sin\theta)$. See Figure 2 for a visualization. The explicit parametric expression of C is

$$\boldsymbol{\alpha}(\theta) = h(\theta)\,(\cos\theta, \sin\theta) + h'(\theta)\,(-\sin\theta, \cos\theta), \qquad \theta \in [0, 2\pi[. \tag{1}$$

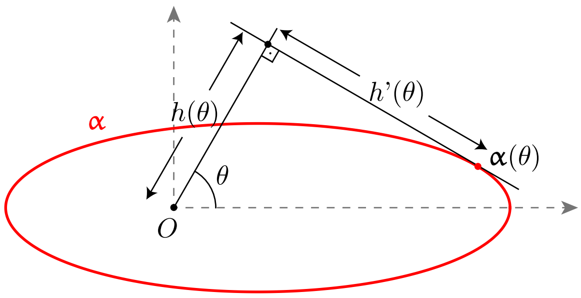

Figure 2: A curve $\boldsymbol{\alpha}$ parameterized by a support function $h(\theta)$. The parameter θ represents an angle.

Any $\mathcal{C}^2$-convex curve of positive curvature is a hedgehog and thus the definition includes any strictly convex $\mathcal{C}^2$-curve of constant width. The curve $\boldsymbol{\alpha}$ is regular if $\left\|\boldsymbol{\alpha}'(\theta)\right\| = \left|h(\theta) + h''(\theta)\right| \neq 0$, for all $\theta \in [0, 2\pi[$. In this case, the curvature function of $\boldsymbol{\alpha}$ is

$$\kappa = \frac{1}{\left|h + h''\right|}.$$

Thus, $\boldsymbol{\alpha}$ is convex and regular if and only if $h + h''$ has no zero.

Note that, given $\theta \in [0, 2\pi[$, the supporting lines to $\boldsymbol{\alpha}(\theta)$ and $\boldsymbol{\alpha}(\theta+\pi)$ are parallel and that $h(\theta)+h(\theta+\pi)$ equals the width of C in a direction $\mathbf{v}(\theta) = (\cos\theta, \sin\theta)$, see Figure 3 (left). This gives rise to the following definition of constant width.

Definition 1. A hedgehog parameterized by a support function h as in (1) is said to be *of constant width* $m \geq 0$ if

$$h(\theta) + h(\theta + \pi) = m,$$

for all $\theta \in [0, \pi[$. A hedgehog is said to be *projective* if it is of constant width 0.

Some examples of hedgehogs of constant width are in Figure 3. The limiting case when the support function satisfies $h(\theta) = -h(\theta + \pi)$ corresponds to a projective hedgehog, which is double traced and of constant width 0.

Remark 1. The importance of projective hedgehogs parameterized as in (1) is that one can use them to construct hedgehogs of constant width as their offset curves. If h is a support function of a projective hedgehog and $m > 0$ then

$$H = h + \frac{m}{2}$$

is a support function of a hedgehog of constant width m, which is its continuous offset at a distance $\frac{m}{2}$. And reciprocally, given any hedgehog of constant width, there is an associated projective hedgehog that can be computed as the locus of midpoints of the chords which measure the constant width [21].

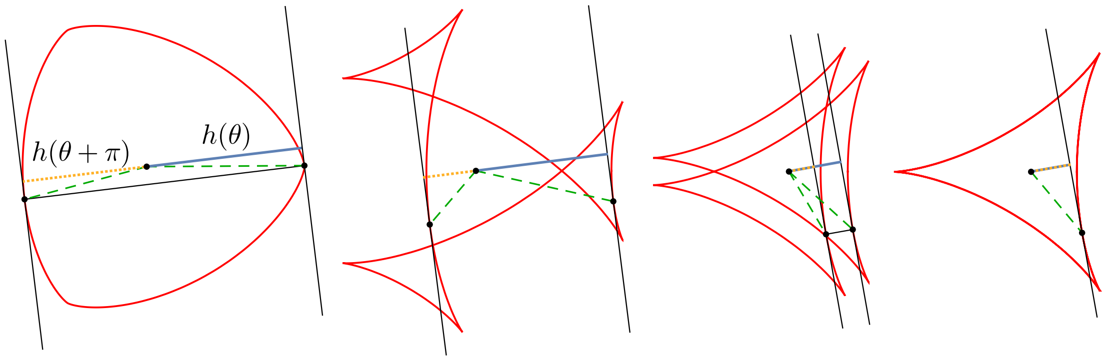

Figure 3: Some hedgehogs of constant width. The curve on the left is a convex curve of constant width. The limiting case on the right corresponds to a projective hedgehog.

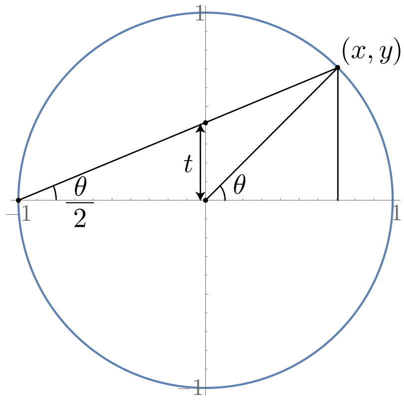

Figure 4: Geometric meaning of the parameters t and θ, related by a stereographic projection from $(-1, 0)$ in the plane.

Our aim is to study constant width curves from the point of view of computer-aided geometric design. Since the curves in this field are mainly polynomial or rational, we will parameterize the curve using a support function but in a rational way. We will work in terms of the parameter t of the usual rational parameterization of the circle (Figure 4).

There are some previous works, such as [9] and [24], that treated this kind of rational parameterizations, not only for planar curves but also for hypersurfaces in $\mathbb{R}^n$. In particular, the authors proved that any hedgehog hypersurface (envelope of a family of supporting hyperplanes [11]) parameterized by a rational support function can be rationally parameterized. A detailed description of the planar case can be found in [22], which leads to the following definition.

Definition 2. A curve $\boldsymbol{\beta}$ is said to be *rationally parameterized by a support function* $f : \mathbb{R} \to \mathbb{R}$ if it can be written as

$$\boldsymbol{\beta}(t) = f(t)\left(\frac{1-t^2}{1+t^2}, \frac{2\,t}{1+t^2}\right) + f'(t)\left(-t, \frac{1-t^2}{2}\right), \qquad t \in \mathbb{R}, \tag{2}$$

where f is a rational function.

Proposition 1. *Any hedgehog parameterized by a trigonometric support function h as in (1) can be rationally parameterized by a support function f as in (2), where the rational support function is $f(t) = h(2\arctan t)$.*

Proof. Instead of expressing all possible normal directions through the usual parameterization of a circle, i.e. $(\cos\theta, \sin\theta)$, we will consider its rational parameterization, obtained from the stereographic projection from $(-1,0)$,

$$\left(\frac{1-t^2}{1+t^2}, \frac{2\,t}{1+t^2}\right), \qquad t \in \mathbb{R}.$$

This reparameterization corresponds to the parameter change $\theta = r(t) = 2\arctan t$, that is,

$$t = \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}, \tag{3}$$

as it can be seen in Figure 4. Hence, if $\boldsymbol{\alpha}$ is a hedgehog parameterized by a support function h as in (1), then its reparameterization through r is

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}\big(r(t)\big) = h\big(r(t)\big)\,\big(\cos r(t), \sin r(t)\big) + h'\big(r(t)\big)\,\big(-\sin r(t), \cos r(t)\big).$$

Since h is trigonometric, $f(t) = h\big(r(t)\big) = h(2\arctan t)$ is a rational function by construction. Finally, using trigonometric identities and

$$f'(t) = h'\big(r(t)\big)\,r'(t) = h'\big(r(t)\big)\,\frac{2}{1+t^2},$$

we obtain (2). □

In general, not every hedgehog parameterized by a support function can be rationally reparameterized, as we show in the following example.

Example 1. Consider a plateau-type function on $[0, 2\pi[$ given as

$$h(t) = \begin{cases} g(t)\,g(\pi - t), & \text{if } t \leq \pi, \\ -g(t-\pi)\,g(2\pi - t), & \text{if } t > \pi, \end{cases}$$

where $g(t) = \frac{u(t)}{u(t)+u(1-t)}$, with

$$u(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The graph of h is in Figure 5 (left). Consider h being extended to $\mathbb{R}$ by periodicity, so as to make it 2π-periodic.

By construction h is $\mathcal{C}^\infty$ and, in addition, $h(t) + h(t+\pi) = 0$, so that the hedgehog defined by the support function h is projective, see Figure 5 (right).

However, this hedgehog (that is not trigonometric) cannot be rationally reparameterized by a rational support function. Any offset at a distance m to this projective hedgehog can be parameterized by a support function $h + \frac{m}{2}$, and corresponds to a hedgehog of constant width m. These hedgehogs of constant width cannot be rationally reparameterized either. This shows that the class of rationally parameterized hedgehogs of constant width is smaller than the class of constant width hedgehogs.

Remark 2. As noticed in [22], a rationally parameterized curve by a support function h as in (2) is convex and regular if and only if

$$4\,h(t) + 2\,t\,(1+t^2)\,h'(t) + (1+t^2)^2\,h''(t) \tag{4}$$

has no zero.

3. Rationally supported curves of constant width

In this section, we will study the conditions that the support function of a rationally parameterized curve of constant width must satisfy. First, we characterize the property of having constant width (see also [22]).

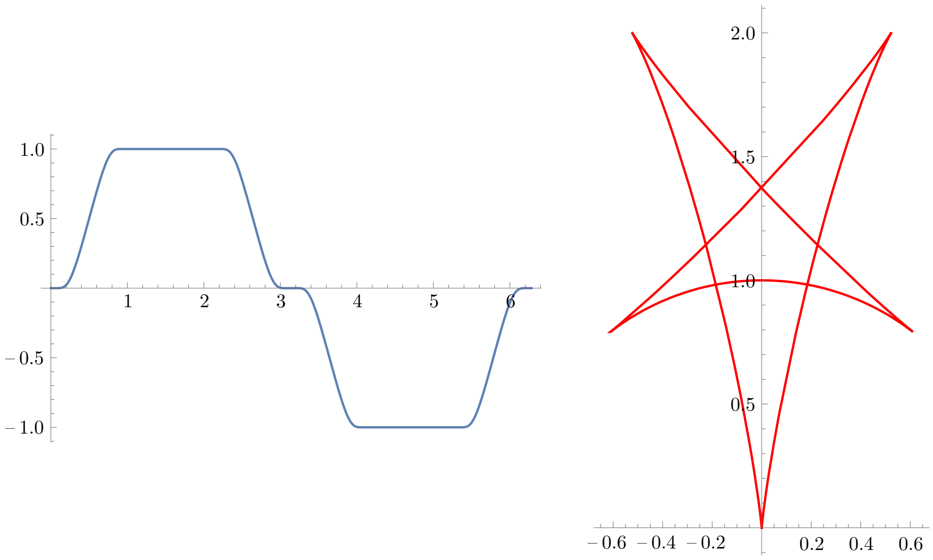

Figure 5: A plateau-type support function on $[0, 2\pi[$ (left) and its corresponding hedgehog, which is projective (right).

Proposition 2. *A rationally parameterized hedgehog by a support function h is of constant width $m \geq 0$ if and only if*

$$h(t) + h\left(-\frac{1}{t}\right) = m, \tag{5}$$

for all $t \in \mathbb{R}^+$, considering in addition that

$$h(0) + \lim_{t \to +\infty} h(t) = m.$$

Proof. We must translate the condition $h(\theta) + h(\theta + \pi) = m$, for $\theta \in [0, \pi[$, from the angle parameter θ to the parameter t of the rational parameterization. The relation between these two parameters is given by $t = \tan \frac{\theta}{2}$ (see Figure 4). Therefore, if $\theta \in [0, \pi[$, then $t \in [0, +\infty[= \mathbb{R}^+$ and the parameter value t_0 corresponding to an angle $\theta + \pi$ is

$$t_0 = \tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right) = -\frac{1}{\tan(\frac{\theta}{2})} = -\frac{1}{t}.$$

Finally, note that the limit $\lim_{t \to 0} h\left(-\frac{1}{t}\right)$ exists because the corresponding trigonometric curve of constant width via the parameter change is defined by a 2π-periodic function.

□

From Proposition 2 it follows that a rationally parameterized hedgehog is projective if and only if

$$h(t) + h\left(-\frac{1}{t}\right) = 0.$$

The authors of [9] showed elegantly that odd rational support functions correspond to those rational surfaces which can be equipped with a linear field of normal vectors (LN surfaces), and then that their offsets are rational constant width surfaces. The planar version can also be deduced from [22], where it is noted that any rationally parameterized curve is Pythagorean-hodograph, and thus, it has a rational offset. For this reason and by Remark 1, we can focus without loss of generality on discussing what a support function of a rational projective hedgehog must be. Let us suppose that

$$h(t) := \frac{p(t)}{q(t)},$$

where $p, q \in \mathbb{R}[t]$ are non-zero polynomials without any common root, that is, $\gcd(p,q) = 1$. In such a case the support function

$$H = h + \frac{m}{2} = \frac{p + \frac{m}{2}\, q}{q}$$

satisfies $\gcd\left(p + \frac{m}{2}\, q,\, q\right) = \gcd(p,q) = 1$ and it is the support function of a rational hedgehog of constant width m.

We want to study bounded constant width curves (and so the associated projective hedgehogs). Nevertheless, one can easily find unbounded curves satisfying the condition we gave in Proposition 2 if the denominator of the support function has real roots. For example, in Figure 6 we see the curve obtained from the support function $h(t) = \frac{1+t^2}{2+3t-2t^2} + \frac{m}{2}$ with $m = 20$, that satisfies the constant width condition and has 2 real roots.

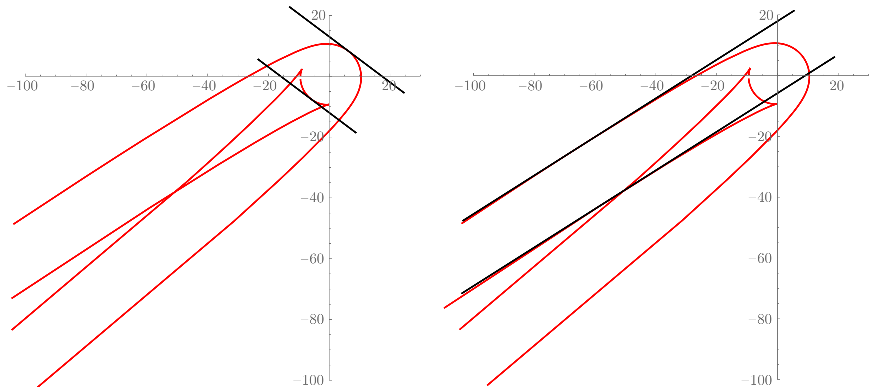

Figure 6: Unbounded rational curve satisfying the constant width condition.

In order to avoid unbounded curves, which are due to infinity limits, we will take the following assumptions on the rational function $h = \frac{p}{q}$:

1. The denominator q has only non-real complex roots. That is, $q(t) \neq 0$ for all $t \in \mathbb{R}$. As a consequence, q must be of even degree.
2. The degree of the numerator p is less than or equal to the degree of the denominator q.

What it is left is to impose the condition of being the support function of a rational projective hedgehog. If n is the degree of the polynomial q, define two polynomials

$$\widehat{p}(t) := t^n\, p\left(-\frac{1}{t}\right), \qquad \widehat{q}(t) := t^n\, q\left(-\frac{1}{t}\right).$$

Thus, we can rewrite the condition (5) for $m = 0$ as

$$p(t)\,\widehat{q}(t) + \widehat{p}(t)\,q(t) = 0. \tag{6}$$

Note that the definition of polynomials $\widehat{p}$ and $\widehat{q}$ is similar to the definition of *reciprocal polynomials* (see [18], pp. 58–59). We will see later in the proof of Theorem 1 that our polynomials satisfy a similar property for their roots.

Our purpose is to find all pairs of polynomials p and q with the assumptions above and such that (6) is satisfied. The idea is that we will characterize all the possible denominators q and provide a method to compute the numerators p accordingly. If a denominator q is allowed for our purposes we will call it admissible, as we define below.

Definition 3. We say that a polynomial $q \in \mathbb{R}[t]$ of (even) degree n is *admissible* if it is monic, it has only non-real complex roots and there exists a polynomial $p \in \mathbb{R}[t]$ of degree less than or equal to n, with $\gcd(p,q) = 1$, such that $\frac{p}{q}$ is a support function of a rationally parameterized projective hedgehog.

In the following theorem we characterize the admissible polynomials.

Theorem 1. *Let $q \in \mathbb{R}[t]$ be a polynomial of (even) degree n with only non-real complex roots. Then the following statements are equivalent:*

(i) *The polynomial q is admissible.*

(ii) *The polynomial q can be written as*

$$q(t) = \left(1+t^2\right)^{r_0} \prod_{j=1}^{s} (t-z_j)^{r_j} (t-\bar{z}_j)^{r_j} \left(t+\frac{1}{z_j}\right)^{r_j} \left(t+\frac{1}{\bar{z}_j}\right)^{r_j}, \tag{7}$$

where $z_j \in \mathbb{C} \setminus (\mathbb{R} \cup \{\mathbf{i}, -\mathbf{i}\})$, $r_j, s \in \mathbb{N}$ and the multiplicity r_0 is an odd number or zero.

Moreover, if (i) or (ii) holds, then $\widehat{q} = q$.

Proof. Let us show that (i) implies (ii). Since q is admissible, there exists a polynomial p of degree less than or equal to n, with $\gcd(p, q) = 1$, such that $\frac{p}{q}$ is a support function of a rationally parameterized projective hedgehog, that is, it satisfies (6):

$$p(t)\, \widehat{q}(t) + \widehat{p}(t)\, q(t) = 0.$$

Now, if $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is a root of q, it follows that $p(z_0)\, \widehat{q}(z_0) = 0$. Since $\gcd(p, q) = 1$, we have $\widehat{q}(z_0) = 0$, that is, z_0 is also a root of $\widehat{q}$. Moreover, if z_0 is a root of multiplicity $k \geq 2$ of q, so $q'(z_0) = 0$, then differentiating the previous expression we get

$$p'(t)\, \widehat{q}(t) + p(t)\, \widehat{q}\,'(t) + \widehat{p}\,'(t)\, q(t) + \widehat{p}(t)\, q'(t) = 0. \tag{8}$$

If we evaluate this expression at $t = z_0$ we find that $p(z_0)\, \widehat{q}\,'(z_0) = 0$, so $\widehat{q}\,'(z_0) = 0$, that means z_0 is also a root of $\widehat{q}$ with multiplicity at least 2. We can proceed recursively to show that z_0 is also a root of multiplicity k of $\widehat{q}$. Therefore, we have that q and $\widehat{q}$ are two polynomials with the same roots and same multiplicities. Notice that this happens if and only if $\widehat{q} = b\, q$, for some $b \in \mathbb{R} \setminus \{0\}$. We will see later that $b = 1$, which is the second part of the statement.

Suppose now that $z_j \in \mathbb{C} \setminus (\mathbb{R} \cup \{\mathbf{i}, -\mathbf{i}\})$ is a root of multiplicity r_j of q. We have proved that z_j is also a root of $\widehat{q}$ with multiplicity r_j, so

$$0 = \widehat{q}(z_j) = z_j^n\, q\left(-\frac{1}{z_j}\right), \tag{9}$$

and therefore $-\frac{1}{z_j}$ is a root of q too (and of $\widehat{q}$) with multiplicity r_j. To sum up, z_j, $\bar{z}_j$, $-\frac{1}{z_j}$ and $-\frac{1}{\bar{z}_j}$ are roots of q with the same multiplicity. Note that the imaginary unit must be treated separately because if $\mathbf{i}$ is a root of q then $-\frac{1}{\mathbf{i}} = \mathbf{i}$. Thus, the polynomial q can be written as

$$q(t) = \left(1+t^2\right)^{r_0} \prod_{j=1}^{s} (t-z_j)^{r_j} (t-\bar{z}_j)^{r_j} \left(t+\frac{1}{z_j}\right)^{r_j} \left(t+\frac{1}{\bar{z}_j}\right)^{r_j},$$

where $r_0 \in \mathbb{N}$. Now, observe that

$$\begin{aligned} \widehat{q}(t) = t^n\, q\left(-\frac{1}{t}\right) &= \left(1+t^2\right)^{r_0} \prod_{j=1}^{s} (-1-z_j\, t)^{r_j} (-1-\bar{z}_j\, t)^{r_j} \left(-1+t\,\frac{1}{z_j}\right)^{r_j} \left(-1+t\,\frac{1}{\bar{z}_j}\right)^{j} \\ &= \left(1+t^2\right)^{r_0} \prod_{j=1}^{s} \left(-\frac{1}{z_j}-t\right)^{r_j} \left(-\frac{1}{\bar{z}_j}-t\right)^{r_j} (-z_j+t)^{r_j} (-\bar{z}_j+t)^{r_j} = q(t), \end{aligned}$$

which proves the second part of the statement (i.e., $b = 1$). Note that the degree of q is $n = 2r_0 + 4r$, with r being the sum of other multiplicities.

Finally, we must prove that the multiplicity r_0, corresponding to the complex roots $\pm\mathbf{i}$, is either zero or odd. Suppose that $r_0 \neq 0$. From (6), as $q = \widehat{q}$, we deduce that

$$p + \widehat{p} = 0. \tag{10}$$

Thus, if we evaluate at $t = \mathbf{i}$,

$$0 = p(\mathbf{i}) + \widehat{p}(\mathbf{i}) = p(\mathbf{i}) + \mathbf{i}^n\, p\left(-\frac{1}{\mathbf{i}}\right) = p(\mathbf{i})\left(1 + \mathbf{i}^{2r_0+4j}\right) = p(\mathbf{i})\left(1 + (-1)^{r_0}\right).$$

But $p(\mathbf{i}) \neq 0$, because $q(\mathbf{i}) = 0$ and $\gcd(p, q) = 1$. Therefore, r_0 must be odd and we have proved (ii).

Now let us prove that (ii) implies (i). If q is of the form of (7), we have already seen that $q = \widehat{q}$. It is only left to prove that there exists a polynomial p of degree less than or equal to n, with $\gcd(p, q) = 1$, such that (6) or, equivalently, (10) is satisfied. By hypothesis, we know that r_0 is an odd number or zero. This means that $\pm\mathbf{i}$ are roots of q if and only if $\frac{n}{2}$ is odd.

If $\frac{n}{2}$ is odd then we can write $n = 2\,(2k - 1)$, for some $k \in \mathbb{N}$, and we can choose the polynomial

$$p(t) = t^{2k-2}\left(1 - t^2\right)$$

that has only real roots, so that $\gcd(p, q) = 1$ and, in addition, satisfies (10) because

$$\widehat{p}(t) = t^n\, p\left(-\frac{1}{t}\right) = t^n\, \frac{(-1)^{2k-2}}{t^{2k-2}}\left(1 - \frac{1}{t^2}\right) = -t^{2k-2}\left(1 - t^2\right) = -p(t).$$

If $\frac{n}{2}$ is even, then $n = 4\,k$, for some $k \in \mathbb{N}$, and moreover $r_0 = 0$. In this case we can choose the polynomial

$$p(t) = t^{2k-1}\left(1 + t^2\right)$$

that has only real roots and the complex roots $\pm\mathbf{i}$, which are not roots of q, so that $\gcd(p, q) = 1$. The condition (10) is satisfied as well because

$$\widehat{p}(t) = t^n\, p\left(-\frac{1}{t}\right) = t^n\, \frac{(-1)^{2k-1}}{t^{2k-1}}\left(1 + \frac{1}{t^2}\right) = -t^{2k-1}\left(1 + t^2\right) = -p(t).$$

This shows that the polynomial q is admissible. □

The proof of the theorem above also leads to the following results.

Corollary 1. *Let $q \in \mathbb{R}[t]$ be a monic polynomial of (even) degree with only non-real complex roots such that $q = \widehat{q}$. Then:*

1. *If $\pm\mathbf{i}$ are not roots of q, then q is admissible.*
2. *If $\pm\mathbf{i}$ are roots of q with odd multiplicity, then q is admissible.*

Remark 3. If q is admissible, note that the condition $q = \widehat{q}$ is necessary (as stated in Theorem 1), but it is not sufficient by itself as it is shown by the polynomials $q(t) = (1 + t^2)^{2k}$, $k \in \mathbb{N}$ (see actual sufficient conditions in Corollary 1).

The theorem also gives us a relation between the coefficients of the polynomials of the rational support function.

Corollary 2. *Let q be an admissible polynomial of (even) degree n and let $h = \frac{p}{q}$ be a rational support function of a rationally parameterized projective hedgehog. If $q(t) = \sum_{i=0}^{n} q_i t^i$ and $p(t) = \sum_{i=0}^{n} p_i t^i$ then*

- *$q = \widehat{q}$ and $q_{n-i} = (-1)^i\, q_i$, for $i = 0, 1, \ldots, \frac{n}{2}$,*
- *$p = -\widehat{p}$ and $p_{n-i} = (-1)^{i+1}\, p_i$, for $i = 0, 1, \ldots, \frac{n}{2}$.*

The central term of q and p depends on the parity of $\frac{n}{2}$. More precisely,

$$p_{\frac{n}{2}} = \begin{cases} \textit{free} & \textit{if } \frac{n}{2} \textit{ is odd,} \\ 0 & \textit{if } \frac{n}{2} \textit{ is even,} \end{cases} \qquad q_{\frac{n}{2}} = \begin{cases} 0 & \textit{if } \frac{n}{2} \textit{ is odd,} \\ \textit{free} & \textit{if } \frac{n}{2} \textit{ is even.} \end{cases}$$

Proof. Since q is admissible, by Theorem 1 we have that $\widehat{q} = q$. Thus, the polynomial

$$\widehat{q}(t) = t^n q\left(-\frac{1}{t}\right) = (-1)^n q_n + (-1)^{n-1} q_{n-1} t + \cdots + q_2 t^{n-2} - q_1 t^{n-1} + q_0 t^n,$$

equals q if and only if their coefficients are the same. This implies $q_{n-i} = (-1)^i q_i$, $i = 0, 1, \ldots, \frac{n}{2}$.

Similarly, once we have that $\widehat{q} = q$, substituting it in (6) we get $p + \widehat{p} = 0$ and, analogously, we obtain the relations $p_{n-i} = (-1)^{i+1} p_i$, for $i = 0, 1, \ldots, \frac{n}{2}$.

The discussion about the parity of $\frac{n}{2}$ follows directly from these relations. □

Remark 4. Let q be an admissible polynomial of (even) degree n. We have seen in Corollary 1 the importance of whether or not $\mathbf{i}$ is a root of q. In fact, from the previous results we can deduce the following claims.

- If $n = 4k - 2$, for some $k \in \mathbb{N}$, then $\mathbf{i}$ is a root of q, and therefore it is not a root of p.
- If $n = 4k$, for some $k \in \mathbb{N}$, then $\mathbf{i}$ is a root of p, and therefore it is not a root of q.

Indeed, if $n = 4k$, from the relations between the coefficients of p from Corollary 2 we deduce that

$$p(t) = \sum_{j=0}^{2k-1} p_j t^j \left(1 - (-1)^j t^{4k-2j}\right). \tag{11}$$

In this case, the complex number $z = \mathbf{i}$ is not a root of q, see Theorem 1, but it is a root of p, as it follows from expression (11) that

$$p(\mathbf{i}) = \sum_{j=0}^{2k-1} p_j\, \mathbf{i}^j \left(1 - (-1)^j\, \mathbf{i}^{4k-2j}\right) = \sum_{j=0}^{2k-1} p_j\, \mathbf{i}^j \left(1 - (-1)^j\, (-1)^j\right) = 0.$$

Example 2. Following Theorem 1 we can construct infinitely many examples of admissible polynomials that will lead us to a rational curve of constant width.

Given an odd natural number r, polynomials such as

$$q(t) = (1 + t^2)^r,$$

are admissible, and they have $\pm\mathbf{i}$ and $-\frac{1}{\pm\mathbf{i}} = \pm\mathbf{i}$ as roots. Notice that these are the denominators of the rational expressions of $\cos\big((2k+1)\,t\big)$ and $\sin\big((2k+1)\,t\big)$ used in [2] and [22] to find examples of rational curves of constant width.

Given $\theta \in \mathbb{R}$, we could also consider $z = \pm e^{\pm\mathbf{i}\theta}$ and $\frac{-1}{z} = \pm e^{\pm\mathbf{i}(\pi-\theta)} = \mp e^{\mp\mathbf{i}\theta}$, so

$$q(t) = \left(1 - 2\cos(\theta)\, t + t^2\right)^r \left(1 + 2\cos(\theta)\, t + t^2\right)^r = \left(1 - 2\cos(2\theta)\, t^2 + t^4\right)^r,$$

which is admissible.

Another way to find examples of admissible polynomials is to choose the coefficients q_i as in Corollary 2. For example, if we take the free coefficients $q_0 = 1$, $q_1 = 1$ and $q_2 = 1$, then $q_3 = -q_1 = -1$, $q_4 = q_0 = 1$ and the resulting polynomial is

$$q(t) = 1 + t + t^2 - t^3 + t^4,$$

which has not $\pm\mathbf{i}$ among its roots. Therefore, q is admissible (see the sufficient condition of Corollary 1).

Algorithm 1 (Construction of a rational curve of constant width)**.**

1. Choose a polynomial q of even degree n with only non-real complex roots. It can be done by providing its roots as in (7) of Theorem 1.
2. Choose the coefficients of $p(t) = \sum_{i=0}^{n} p_i \, t^i$ such that
$$p_{n-i} = (-1)^{i+1} \, p_i, \qquad i = 0, 1, \ldots, \frac{n}{2} - 1,$$
with the central term
$$p_{\frac{n}{2}} = \begin{cases} 0, & \text{if } \frac{n}{2} \text{ is even,} \\ \text{free}, & \text{if } \frac{n}{2} \text{ is odd,} \end{cases}$$
and such that $\gcd(p, q) = 1$ (i.e., p and q have no common roots).
3. A rationally parameterized projective hedgehog can be constructed with the rational support function $h = \frac{p}{q}$ as
$$\boldsymbol{\alpha}(t) = h(t) \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) + h'(t) \left(-t, \frac{1-t^2}{2} \right), \qquad t \in \mathbb{R}.$$
4. Any offset to $\boldsymbol{\alpha}$ at a distance $\frac{m}{2}$ is a rational hedgehog of constant width m that can be constructed as
$$\boldsymbol{\beta}(t) = \left(h(t) + \frac{m}{2} \right) \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) + h'(t) \left(-t, \frac{1-t^2}{2} \right), \qquad t \in \mathbb{R}.$$
The curve $\boldsymbol{\beta}$ is of degree less than or equal to $2n + 2$.

Number of degrees of freedom:

- If $n = 4k - 2$, for some $k \in \mathbb{N}$, then $\frac{n}{2} = 2k - 1$ is odd. The number of degrees of freedom is $2k + 1$, which corresponds to the width, the central coefficient of p and the $\frac{n}{2}$ first/last coefficients of p.
- If $n = 4k$, for some $k \in \mathbb{N}$, then $\frac{n}{2} = 2k$ is even. The number of degrees of freedom is $2k + 1$, which corresponds to the width and the $\frac{n}{2}$ first/last coefficients of p.

In both cases the number of free parameters equals to $2k+1$, with the width m being one of them. This will be important in the next section.

Of course, once a rational parameterization of a curve of constant width has been found, its formulation as a rational Bézier curve is possible by changing the power monomial basis to the Bernstein basis [7]. However, the number of free control points on the control polygon will depend on the degree n of the chosen admissible polynomial.

4. The Hermite interpolation problem

Once we know how the rational support function of a hedgehog of constant width is, our next step is to replace the free parameters it depends on (its degrees of freedom), by some other parameters having a geometric meaning instead. Particularly, in this section we will consider a Hermite interpolation problem.

The two-point geometric Hermite interpolation problem is to find a curve $\boldsymbol{\beta}$ that passes through two points, $\mathbf{P}_0$ and $\mathbf{P}_f$, and matches tangent vectors at those points with predefined vectors [16, 6]. Here, we want to show that given a set of points $\{\mathbf{P}_i\}_{i=1}^k$ and its corresponding set of vectors, $\{\mathbf{v}_i\}_{i=1}^k$ we can obtain rational parameterizations of hedgehogs of constant width, $\boldsymbol{\beta}$, that interpolate the points and have tangent lines at those points determined by such vectors:

$$\boldsymbol{\beta}(t_i) = \mathbf{P}_i, \quad \text{and} \quad \boldsymbol{\beta}'(t_i) \parallel \mathbf{v}_i, \quad i = 1, \ldots, k. \tag{12}$$

The number of points determines the degree of the denominator of the support function: given k points, the degree of q has to be $4k - 2$ or $4k$ by the discussion above. Once we choose an admissible polynomial

q and the width m we are going to show that the solution of the Hermite interpolation problem is unique and, in fact, we will give its explicit expression.

We will address the problem after a transformation of the initial data, that is, the given set of points $\{\mathbf{P}_i\}_{i=1}^k$ and the corresponding set of vectors $\{\mathbf{v}_i\}_{i=1}^k$ into three sets of numbers, namely, a set of parameter values, $\{t_i\}_{i=1}^k$, that determines the tangent directions, and two sets of signed distances, $\{d_i\}_{i=1}^k$ and $\{e_i\}_{i=1}^k$, that determine the points.

More specifically, given a point $\mathbf{P}$ and a vector $\mathbf{v} = \overrightarrow{\mathbf{PQ}} = (v_x, v_y)$, where $\mathbf{Q} \neq \mathbf{P}$, we can consider an adapted positively oriented orthonormal basis $\{\mathbf{t}_v,\, \mathbf{n}_v\}$ of $\mathbb{R}^2$ as

$$\mathbf{t}_v = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{n}_v = J\,\mathbf{t}_v,$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus, we obtain the following data (see Figure 7):

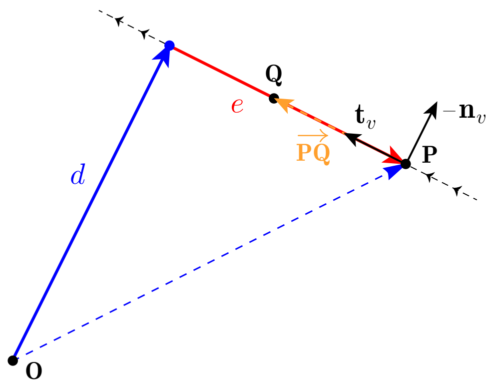

Figure 7: Given a point $\mathbf{P}$ and a vector $\mathbf{v} = \overrightarrow{\mathbf{PQ}}$, we compute three numbers, the parameter value t and the signed distances d and e.

1. The **parameter value** t**.** We will assume that the orientation of the supporting line is the same as that induced by the vector $\mathbf{v}$. In this case, $-\mathbf{n}_v = (\cos\theta, \sin\theta)$ defines the normal to the supporting line. Thus, from the relation between the parameters θ and t given by (3), we have

$$t = \frac{\sin\theta}{1+\cos\theta} = \frac{-v_x}{v_y + \sqrt{v_x^2 + v_y^2}}. \tag{13}$$

2. The **signed distance** d from the origin $\mathbf{O}$ to the line defined by the point $\mathbf{P}$ and the vector $\mathbf{v}$ is given by

$$d = \langle \overrightarrow{\mathbf{OP}}, -\mathbf{n}_v \rangle = \frac{\det\big(\overrightarrow{\mathbf{OP}}, \mathbf{v}\big)}{\|\mathbf{v}\|}. \tag{14}$$

3. The **signed distance** e from $\mathbf{P}$ to the perpendicular foot from the origin $\mathbf{O}$ is

$$e = \langle \overrightarrow{\mathbf{OP}}, \mathbf{t}_v \rangle = \frac{\big\langle \overrightarrow{\mathbf{OP}}, \mathbf{v} \big\rangle}{\|\mathbf{v}\|}. \tag{15}$$

With the new set of data, $\{t_i\}_{i=1}^k$, $\{d_i\}_{i=1}^k$ and $\{e_i\}_{i=1}^k$, the problem (12) can be reformulated as follows: given an admissible polynomial q, we must find a rational support function h, with denominator q, of a rational **projective** hedgehog such that

$$\begin{cases} h(t_i) = d_i - \frac{m}{2}, \\ h'(t_i) = e_i\, \frac{2}{1+t_i^2}, \end{cases} \tag{16}$$

for $i=1,\ldots,k$. Note that we consider $d_i - \frac{m}{2}$ as the new signed distance because we are looking for the support function h of the associated projective hedgehog, and that the term $\frac{2}{1+t^2}$ arises from the relation $\theta = 2\arctan(t)$.

The reason for finding the support function of the associated projective hedgehog lies in the following definitions and properties, which will let us find the explicit solution of the problem (16). As we will see, we will need to distinguish two cases depending on the (even) degree of q, which can be written either as $4\,k-2$ or $4\,k$.

Notice that if we have a rational parameterization of a hedgehog of constant width (or a projective hedgehog), the parameter values t and $-\frac{1}{t}$ are linked as stated in Proposition 2. Therefore, the choice of these two parameter values will lead to redundant data. Also, note that the value 0 is not allowed.

Definition 4. Given $k \in \mathbb{N}$, we say that a set $\{t_i\}_{i=1}^k \subset \mathbb{R}\setminus\{0\}$ is an *allowed set of parameters* if $t_i \neq t_j$ and $t_i \neq -\frac{1}{t_j}$ for all $i,j=1,\ldots,k$ such that $i\neq j$. We say that a set of vectors $\{\mathbf{v}_i\}_{i=1}^n \subset \mathbb{R}^2$ is *allowed* if they correspond to an allowed set of parameters $\{t_i\}_{i=1}^k$.

Next, we define two polynomials that will constitute a basis of interpolating polynomials which is adapted to our problem (see [5] or [23] as references for the solution of a generic Hermite interpolation problem).

Definition 5. Given an allowed set of parameters $\{t_i\}_{i=1}^k$, define the degree $4k-2$ polynomials

$$M_i(t) := \frac{(t-t_i)\left(t+\frac{1}{t_i}\right)}{\left(t_i+\frac{1}{t_i}\right)} \frac{\displaystyle\prod_{j=1,\, j\neq i}^{k} (t-t_j)^2\left(t+\frac{1}{t_j}\right)^2}{\displaystyle\prod_{j=1,\, j\neq i}^{k} (t_i-t_j)^2\left(t_i+\frac{1}{t_j}\right)^2}, \quad i=1,\ldots,k,$$

and

$$L_i(t) := \left(t\left(t-t_i+\frac{2}{t_i}\right)-1\right)\frac{\displaystyle\prod_{j=1,\, j\neq i}^{k} (t-t_j)^2\left(t+\frac{1}{t_j}\right)^2}{\displaystyle\prod_{j=1,\, j\neq i}^{k} (t_i-t_j)^2\left(t_i+\frac{1}{t_j}\right)^2}$$
$$+\left(\left(-t_i-\frac{2}{t_i}\right)-2\sum_{j=1,\, j\neq i}^{k}\left(\frac{1}{t_i-t_j}+\frac{1}{t_i+\frac{1}{t_j}}\right)\right)M_i(t), \quad i=1,\ldots,k.$$

Now, let us denote by $\mathcal{H}_n$ the set of polynomials

$$\mathcal{H}_n := \left\{p\in\mathbb{R}[t]\,:\,\deg(p)\le n \;\text{ and }\; p(t)+t^n\,p\left(-\frac{1}{t}\right)=0\right\}$$

which is a vector space. If p is a polynomial of degree $4k-2$ such that $p+\widehat{p}=0$, we have seen in Corollary 2 that the $2k$ first/last coefficients of p determine the entire polynomial. Therefore, $\mathcal{H}_{4k-2}$ is a vector space of dimension $2k$.

A straightforward computation shows the following properties (see an example in Figure 8 for the case $k=2$).

Lemma 1. *Given an allowed set of parameters $\{t_i\}_{i=1}^k$, the polynomials $\{L_i,\, M_i\}_{i=1}^k$ satisfy*

$$\begin{aligned} L_i(t_j) &= \delta_{ij}, & M_i(t_j) &= 0, & \\ L_i'(t_j) &= 0, & M_i'(t_j) &= \delta_{ij}, & i,j=1,\ldots,k. \end{aligned} \tag{17}$$

Moreover, L_i and M_i belong to $\mathcal{H}_{4k-2}$ for $i=1,\ldots,k$.

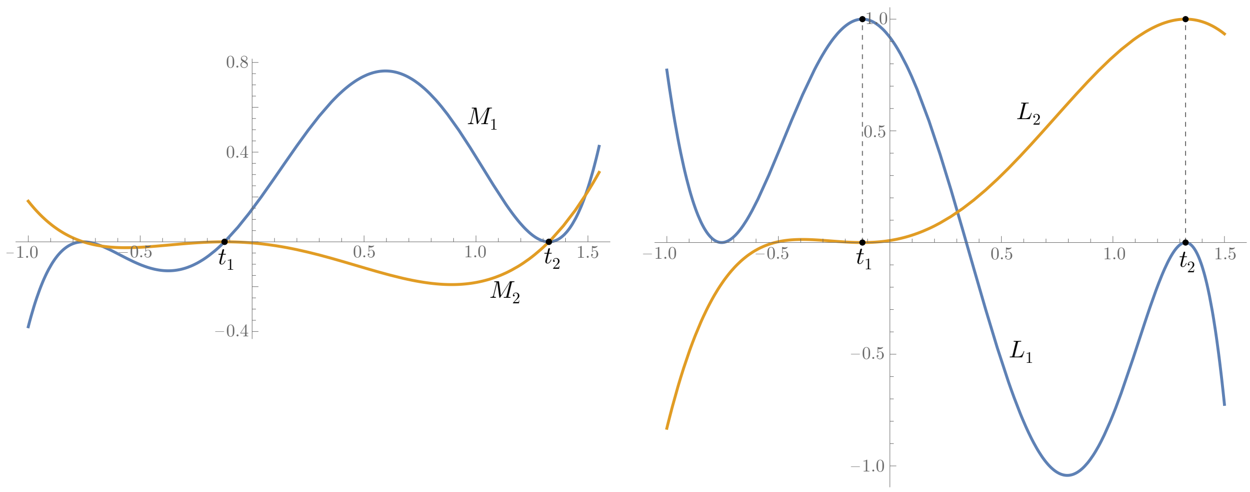

Figure 8: For $k = 2$, given a set of parameters $\{t_1, t_2\}$, where $t_1 = 4 - \sqrt{17}$ and $t_2 = \frac{1}{7}\left(\sqrt{53} + 2\right)$, a plot of the functions M_i and L_i, $i = 1, 2$, from Lemma 1.

Lemma 2. *Given an allowed set of parameters $\{t_i\}_{i=1}^k$, the polynomials $\{L_i,\, M_i\}_{i=1}^k$ constitute a basis of the vector space $\mathcal{H}_{4k-2}$.*

Proof. As a consequence of Lemma 1, the set $\{L_i,\, M_i\}_{i=1}^k$ is linearly independent. Moreover, the dimension of the vector space $\mathcal{H}_{4k-2}$ is $2k$. Therefore, $\{L_i,\, M_i\}_{i=1}^k$ is a basis of $\mathcal{H}_{4k-2}$. □

As a consequence of the previous lemmas we have the following result.

Proposition 3. *Given an allowed set of parameters $\{t_i\}_{i=1}^k$ and two sets of real numbers $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$, the polynomial*

$$p(t) := \sum_{i=1}^{k} a_i\, L_i(t) + \sum_{i=1}^{k} b_i\, M_i(t)$$

is the unique polynomial of degree $\leq 4k-2$ that satisfies $p(t_i) = a_i$, $p'(t_i) = b_i$ and

$$p(t) + t^{4k-2} p\left(-\frac{1}{t}\right) = 0. \tag{18}$$

Given an admissible denominator q of degree $4k-2$, the computation of the numerator as in Proposition 3 will provide us the polynomial of degree $\leq 4k-2$ that satisfies the property we were looking for and solves, in addition, a Hermite problem.

However, if q is of degree $4k$, the computed polynomial p satisfies (18) although we want it to satisfy the same relation but having a term t^{4k} instead of t^{4k-2}. In this case we define $\widetilde{p}(t) := \left(1 + t^2\right) p(t)$, and one can check that indeed

$$\widetilde{p}(t) + t^{4k}\, \widetilde{p}\left(-\frac{1}{t}\right) = 0,$$

so that $\widetilde{p}(t)$ satisfies the property we were looking for and its unicity follows from the unicity of p as a solution of a Hermite problem.

Recall that since we know that $\mathbf{i}$ is not a root of q, it must be a root of the numerator by Remark 4.

In the following result we provide the explicit solution of our Hermite problem in a constructive manner.

Theorem 2. *Given a set of points* $\{\mathbf{P}_i\}_{i=1}^k$*, a set of allowed vectors* $\{\mathbf{v}_i\}_{i=1}^k$*, a width* $m \geq 0$ *and an admissible polynomial* q *of degree* $4k-2$ *or* $4k$*, for some* $k \in \mathbb{N}$*, there exists a unique rationally supported hedgehog* $\boldsymbol{\beta}$ *of constant width* m *satisfying*

$$\boldsymbol{\beta}(t_i) = \mathbf{P}_i, \quad \textit{and} \quad \boldsymbol{\beta}'(t_i) \parallel \mathbf{v}_i, \quad i = 1, \ldots, k,$$

where $\{t_i\}_{i=1}^k$ *is the allowed set of parameters associated with* $\{\mathbf{v}_i\}_{i=1}^k$ *from (13). More specifically, let* H *be its rational support function and let*

$$p(t) := \sum_{i=1}^k a_i\, L_i(t) + \sum_{i=1}^k b_i\, M_i(t).$$

1. *If* q *is of degree* $4k-2$*, then* $H(t) := \frac{p(t)}{q(t)} + \frac{m}{2}$*, with*

$$a_i = \left(d_i - \frac{m}{2}\right) q(t_i) \quad \textit{and} \quad b_i = 2\, e_i\, \frac{q(t_i)}{1+t_i^2} + \left(d_i - \frac{m}{2}\right) q'(t_i). \tag{19}$$

2. *If* q *is of degree* $4k$*, then* $H(t) := \frac{(1+t^2)\, p(t)}{q(t)} + \frac{m}{2}$*, with*

$$a_i = \left(d_i - \frac{m}{2}\right) \frac{q(t_i)}{1+t_i^2}, \quad \textit{and} \quad b_i = \frac{2\, e_i\, q(t_i) + \left(d_i - \frac{m}{2}\right)\left(q'(t_i)\,(1+t_i^2) - 2\, t_i\, q(t_i)\right)}{\left(1+t_i^2\right)^2}. \tag{20}$$

Here d_i *and* e_i *are the signed distances obtained from the points* $\mathbf{P}_i$ *and vectors* $\mathbf{v}_i$ *by the expressions (14) and (15).*

Proof. The first step is to obtain the values of the parameters t_i and the signed distances d_i and e_i from the points $\mathbf{P}_i$ and vectors $\mathbf{v}_i$ using expressions (13), (14) and (15). Notice that once an admissible denominator q is chosen, we know how to solve a Hermite problem for the numerator, being this numerator either equal to $p(t)$ or equal to $p(t)\left(1+t^2\right)$ depending on the degree of q. Therefore, it is just a matter of finding appropriate values a_i and b_i such that the resulting rational support function $H = h + \frac{m}{2}$ will satisfy, indeed, the constant width condition: $H(t) + H(-1/t) = m$, for all $t \in \mathbb{R}$.

The problem we must solve is

$$\begin{cases} H(t_i) \;=\; d_i, \\ H'(t_i) \;=\; e_i\, \frac{2}{1+t_i^2}, \qquad i = 1, \ldots, k, \end{cases} \tag{21}$$

where $H = h + \frac{m}{2}$, with

1. $h(t) := \frac{p(t)}{q(t)}$, if q is of degree $4k-2$, or
2. $h(t) := \frac{(1+t^2)\, p(t)}{q(t)}$, if q is of degree $4k$.

In the first case, from the equations (21) we get

$$\begin{cases} p(t_i) \;=\; \left(d_i - \frac{m}{2}\right) q(t_i), \\ p'(t_i)\, q(t_i) - p(t_i)\, q'(t_i) \;=\; e_i\, \frac{2}{1+t_i^2}\, q(t_i)^2. \end{cases}$$

Using the first equation, we can write

$$\begin{cases} p(t_i) \;=\; \left(d_i - \frac{m}{2}\right) q(t_i), \\ p'(t_i) \;=\; 2\, e_i\, \frac{q(t_i)}{1+t_i^2} + \left(d_i - \frac{m}{2}\right) q'(t_i). \end{cases}$$

The right-hand side of these two equations are the values a_i and b_i, respectively, as given in Proposition 3 that we must set in the Hermite interpolation problem that finds p.

In the second case, the equations (21) turn into

$$\begin{cases} p(t_i) = \left(d_i - \frac{m}{2}\right) \frac{q(t_i)}{1+t_i^2}, \\ \left(p'(t_i)\left(1+t_i^2\right) + p(t_i)\, 2\, t_i\right) q(t_i) - p(t_i)\left(1+t_i^2\right) q'(t_i) = e_i \frac{2}{1+t_i^2}\, q(t_i)^2, \end{cases}$$

and again, using the first equation, these can be rewritten as

$$\begin{cases} p(t_i) = \left(d_i - \frac{m}{2}\right) \frac{q(t_i)}{1+t_i^2}, \\ p'(t_i) = \frac{2\, e_i\, q(t_i) + \left(d_i - \frac{m}{2}\right)\left(q'(t_i)\,(1+t_i^2) - 2\, t_i\, q(t_i)\right)}{\left(1+t_i^2\right)^2}. \end{cases}$$

The right-hand side of these equations provide the values a_i and b_i, respectively, in this case. □

Algorithm 2 (Construction of a rational curve of constant width interpolating a set of points with a set of tangent directions)**.**

1. Choose an even degree polynomial $q(t) = \sum_{i=0}^{n} q_i\, t^i$ with only non-real complex roots, where $n = 4k - 2$ or $n = 4k$, for some $k \in \mathbb{N}$. It can be done by providing its roots as in (7) of Theorem 1.
2. Compute the real numbers $\{t_i\}_{i=1}^k$, $\{d_i\}_{i=1}^k$ and $\{e_i\}_{i=1}^k$ corresponding to the prescribed points and tangent directions, $\{\mathbf{P}_i\}_{i=1}^k$ and $\{\mathbf{v}_i\}_{i=1}^k$, using the relations (13), (14) and (15), respectively.
3. According to Theorem 2, construct p and the support function H, for some $m \geq 0$.
4. Construct the rationally parameterized hedgehog by the rational support function H as

$$\boldsymbol{\beta}(t) = H(t)\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) + H'(t)\left(-t, \frac{1-t^2}{2}\right), \qquad t \in \mathbb{R},$$

which interpolates the points $\{\mathbf{P}_i\}_{i=1}^k$ and the tangent directions $\{\mathbf{v}_i\}_{i=1}^k$.

5. Examples with admissible polynomials of low degree

In this section we will show some examples and we will give a guide for the choice of admissible polynomials for some low degree cases. For example, for degree $n = 2$, by Theorem 1, the only admissible polynomial is $q(t) = 1 + t^2$. Analogously, for degree $n = 4$, since the multiplicity of the roots $\pm\mathbf{i}$ must be odd or zero, the only way of getting an admissible polynomial is

$$q(t) = (t - z_1)\,(t - \bar{z}_1)\left(t + \frac{1}{z_1}\right)\left(t + \frac{1}{\bar{z}_1}\right),$$

where z_1 is a non-real complex root different from $\pm\mathbf{i}$ that can be freely chosen.

In Table 1 we show for degrees up to $n = 12$ the possible choices of the admissible polynomial q. After choosing an admissible polynomial, q, of (even) degree n, the number of free parameters in p for determining the associated constant width hedgehog, depends on the parity of $\frac{n}{2}$. If $\frac{n}{2}$ is even, the number of degrees of freedom is $\frac{n}{2}$ and otherwise is $\frac{n}{2} + 1$. Thus the number of points and tangents to be interpolated is half the number of degrees of freedom.

From the designer's workflow point of view, things are done conversely. What a designer would probably decide first is how many points wants to interpolate and then, consequently, would choose the degree of the admissible polynomial q. If the user wants to generate a curve interpolating k points and tangents, then $2k$ degrees of freedom are needed for p, so the admissible polynomial could be of degree $4k - 2$ or of degree $4k$ as well. In Table 1 we also show the number of free parameters in p and the corresponding number of interpolating points for each case. Note that, additionally, we will have the width m as a free parameter.

Degree of q	Free parameters in p: $\frac{n}{2}$ if $\frac{n}{2}$ is even, $\frac{n}{2}+1$ if $\frac{n}{2}$ is odd	Admissible q	Number of interpolation points
$n=2$	2	$r_0=1$, i.e., $q(t)=1+t^2$	1
$n=4$	2	$r_0=0$, one root, z_1, with $r_1=1$, i.e., $q(t)=(t-z_1)(t-\bar{z}_1)\left(t+\frac{1}{z_1}\right)\left(t+\frac{1}{\bar{z}_1}\right)$	1
$n=6$	4	$r_0=1$ and one root, z_1, with $r_1=1$, or $r_0=3$	2
$n=8$	4	$r_0=0$ and two roots z_1,z_2 with $r_1=r_2=1$, or $r_0=0$ and one root z_1 with $r_1=2$	2
$n=10$	6	$r_0=1$ and two roots z_1,z_2 with $r_1=r_2=1$, $r_0=1$ and a root z_1 with $r_1=2$, $r_0=3$ and a root z_1 with $r_1=1$, or $r_0=5$	3
$n=12$	6	$r_0=0$ and one root z_1 with $r_1=3$, $r_0=0$ and two roots z_1,z_2 with $r_1=2$, $r_2=1$, or $r_0=0$ and three roots z_1,z_2,z_3 with $r_1=r_2=r_3=1$	3

Table 1: Possible choices of an admissible polynomial q.

Recall that given an admissible polynomial q of (even) degree n, the hedgehogs of constant width that we can construct from q are of degree less than or equal to $2n+2$ (see Algorithm 1). As a consequence, we have that the degree of the curve rises quite quickly, which can be seen as a drawback, see Table 1.

Now we will show some examples in order to illustrate which are the possibilities when we want to interpolate k points and the corresponding tangents.

5.1. Two points and two vectors

Consider the two-point geometric Hermite interpolation problem, so admissible polynomials of degrees $n=6$ and $n=8$ are needed.

In Figure 9, with two different admissible polynomials of degree $n=6$, we interpolate the same points and tangents with two different curves of the same constant width, $m=18$. Particularly, we choose the points $\mathbf{P}_1=(10,0),\mathbf{P}_2=(-3,9)$ and the vectors $\mathbf{v}_1=(1,4)$, $\mathbf{v}_2=(-7,-2)$. Thus,

$$t_1=4-\sqrt{17},\ t_2=\frac{1}{7}\left(2+\sqrt{53}\right),\qquad d_1=\frac{40}{\sqrt{17}},\ d_2=\frac{69}{\sqrt{53}},\qquad e_1=\frac{10}{\sqrt{17}},\ e_2=\frac{3}{\sqrt{53}}.$$

The admissible polynomials we have considered are $q(t)=(1+t^2)^3$ for the first curve (in blue) and for the second curve (in red) q is built considering $r_0=1$ and the free root $z_1=2-\mathbf{i}$. The difference between both curves makes us notice the impact on the choice of the admissible polynomial.

In Figure 10, we have considered an admissible polynomial of degree $n=8$ with the roots $z_1=1+\mathbf{i}$ and $z_2=2-\mathbf{i}$ for both curves. In the left curve we interpolate again the same data we considered in Figure 9. For the right curve we have taken the width $m=30$, the points $\mathbf{P}_1=(10,0)$, $\mathbf{P}_2=(-3,9)$ again, but now the vectors are $\mathbf{v}_1=(-\frac{1}{2},4)$, $\mathbf{v}_2=(-5,0)$. Thus,

$$t_1=-8+\sqrt{65},\ t_2=1,\qquad d_1=16\sqrt{\frac{5}{13}},\ d_2=9,\qquad e_1=-2\sqrt{\frac{5}{13}},\ e_2=3.$$

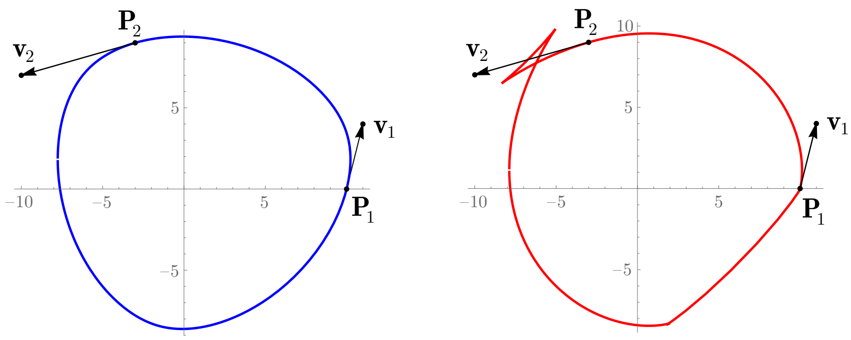

Figure 9: Two rational hedgehogs of the same constant width interpolating two given points and tangents with an admissible polynomial of degree $n = 6$. It is $q(t) = (1 + t^2)^3$ in the first one (in blue) and in the second (in red) q is built considering $r_0 = 1$ and the free root $z_1 = 2 - \mathbf{i}$.

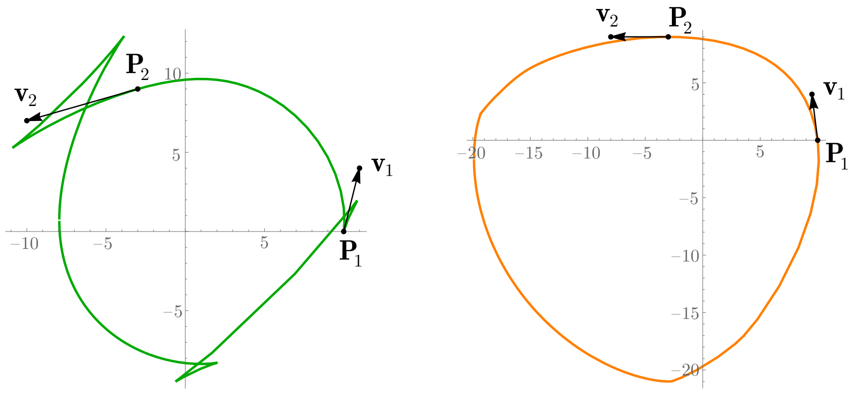

Figure 10: Two rational hedgehogs of constant width interpolating two given points and tangents. Both figures correspond to the admissible polynomial of degree $n = 8$ with the roots $z_1 = 1 + \mathbf{i}$ and $z_2 = 2 - \mathbf{i}$. The first curve interpolates the same data of Figure 9 and also has the same width. A different set of data and a width $m = 30$ is considered for the second figure.

Finally, we graph three more examples in Figure 11, for degree $n = 8$, with the following admissible polynomials:

$$q(t) = \left(4 + t^2\right)^2 \left(\frac{1}{4} + t^2\right)^2, \quad q(t) = \left(1 + t^4\right)^2 \quad \text{and} \quad q(t) = t^8 + 2\,t^6 + 3\,t^4 + 2\,t^2 + 1.$$

We have considered the same initial data for the three figures, specifically, we take again the data we interpolate in the second curve of Figure 10, namely the points $\mathbf{P}_1 = (10, 0)$, $\mathbf{P}_2 = (-3, 9)$, the vectors $\mathbf{v}_1 = (-\frac{1}{2}, 4)$, $\mathbf{v}_2 = (-5, 0)$, but now a width $m = 18$.

5.2. A higher number of points and vectors

Of course, the same methodology we have followed can be applied for any set of points and tangents. From the examples we observe that given an initial set of data (points and tangents) and a width value, the convexity of the interpolated hedgehog of constant width depends heavily on the choice of the admissible polynomial. See some examples in Figure 12, where we interpolate three and five points and tangents using different admissible polynomials.

5.3. A piecewise rational support function

Finally, in this section we want to show that it is possible to join G^1-continuously pieces of rational curves of constant width of different degrees having the same constant width. This increases the construction

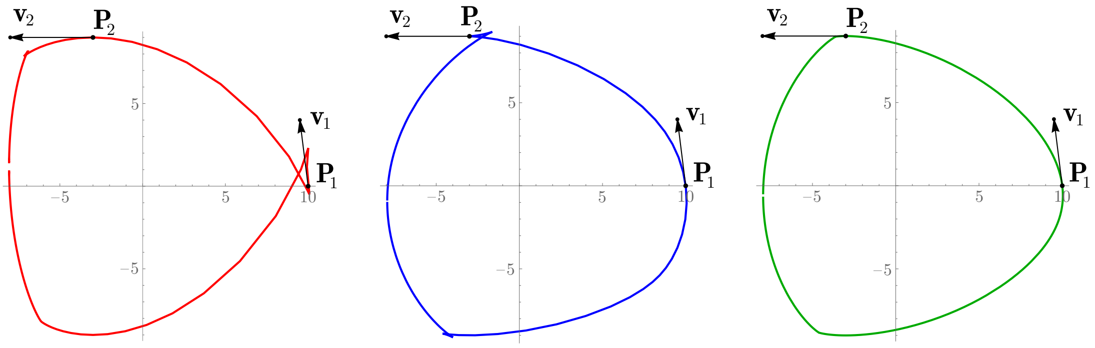

Figure 11: Three rational curves of constant width $m = 18$ obtained by interpolation of the same two points and tangents. The chosen admissible polynomials are $q(t) = (4+t^2)^2\left(\frac{1}{4}+t^2\right)^2$, $q(t) = \left(1+t^4\right)^2$ and $q(t) = t^8 + 2\,t^6 + 3\,t^4 + 2\,t^2 + 1$, respectively.

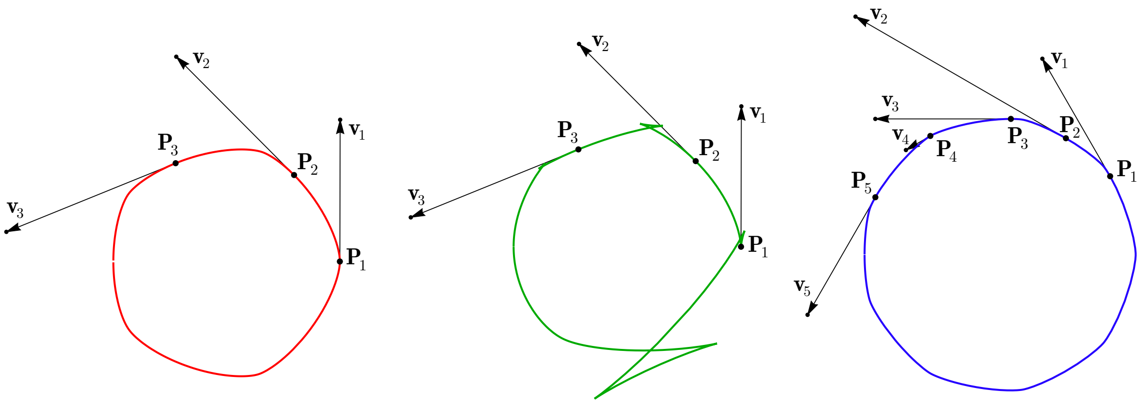

Figure 12: On the left/middle, two rational hedgehogs of the same constant width interpolating the same three points and tangents. The first one (in red) is constructed with $q(t) = (1+t^2)^5$ ($n = 10$). For the second curve (in green), an admissible polynomial q of degree $n = 12$ is chosen taking $r_0 = 0$ and the root $z_1 = 1 - \mathbf{i}$ with multiplicity 3. On the right, a rational hedgehog of constant width interpolating five points and tangents taking the admissible polynomial $q(t) = (1+t^2)^9$ ($n = 18$).

possibilities of rational curves of constant width and allows a better control on the final shape of the curve. This is particularly useful if we want to avoid singularities in the constructed curves.

With this aim we can construct a piecewise support function so that the associated rationally supported constant width hedgehog is a spline curve. Let us illustrate this with an example. If we consider

$$H(t) = \begin{cases} \frac{p_1(t)}{q_1(t)} & \text{if } t \in [t_0, t_1], \\ \frac{p_2(t)}{q_2(t)} & \text{if } t \in [t_1, t_2], \end{cases} \tag{22}$$

and proceed as in Algorithm 2 for each piece we can get a spline curve as the one in Figure 13.

Now, let us explain in detail the way we have generated this example. Given a width m and two interpolating points and tangents, consider an admissible denominator of degree 8 with $r_0 = 0$ and two roots $z_1 = 1 + \mathbf{i}$ and $z_2 = 2 - \mathbf{i}$:

$$\begin{aligned} q_1(t) &= (t - z_1)\,(t - \bar{z}_1)\left(t + \frac{1}{z_1}\right)\left(t + \frac{1}{\bar{z}_1}\right)(t - z_2)\,(t - \bar{z}_2)\left(t + \frac{1}{z_2}\right)\left(t + \frac{1}{\bar{z}_2}\right) \\ &= \frac{1}{10}\left(10\,t^8 - 42\,t^7 + 57\,t^6 + 6\,t^5 - 34\,t^4 - 6\,t^3 + 57\,t^2 + 42\,t + 10\right). \end{aligned}$$

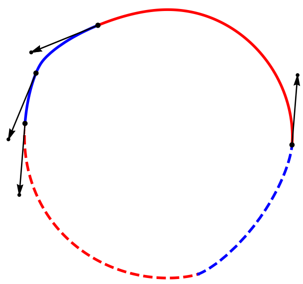

Figure 13: A spline curve of constant width with a piecewise-defined support function as in (22). The admissible denominator q_1 of the first arc (in red) is of degree 8 with $r_0 = 0$ and two roots $z_1 = 1 + \mathbf{i}$ and $z_2 = 2 - \mathbf{i}$. The one for the second arc (in blue) is $q_2(t) = (1 + t^2)^5$, which is of degree 10.

By Theorem 2 we can compute the rational support function $H_1 = \frac{p_1}{q_1}$ of the curve of constant width that interpolates these points and tangents. The result is the curve in Figure 14 (left), which is not regular.

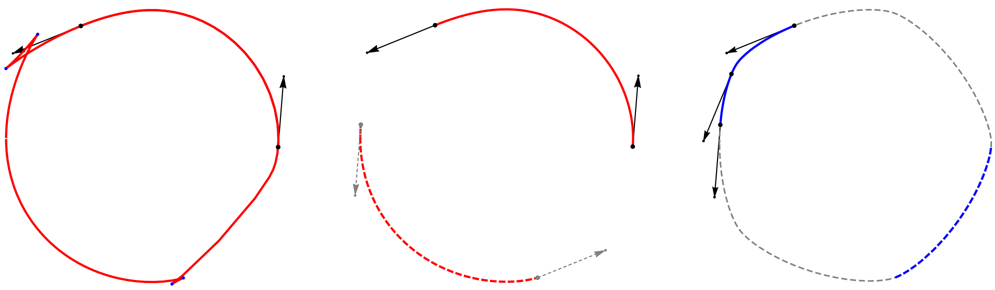

Figure 14: A singular hedgehog of constant width interpolating two points and tangents (left), from which a regular arc together with its antipodal arc is chosen (middle) and another curve of constant width that interpolates three points and vectors is constructed so as to fit the chosen arcs G^1-continuously.

By Remark 2, a rationally parameterized curve by a support function h is convex and regular if and only if the function of (4), namely,

$$4\,h(t) + 2\,t\,(1 + t^2)\,h'(t) + (1 + t^2)^2\,h''(t)$$

has no zero. In our case, this function for H_1 has 4 zeros that correspond to 4 singular points in the constructed hedgehog of constant width; blue points of Figure 14 (left). A plot of this function is in Figure 15.

In this case, we can easily remove these singularities as follows. First, notice that once an arc of a curve of constant width is fixed, the corresponding arc of antipodal points is fixed as well. Therefore, it is enough to choose a regular arc of our curve of constant width such that its corresponding antipodal arc is regular as well. The starting and final points and tangents of these arcs are known. In our example, it is enough if we consider the arc provided by the initial data, see Figure 14 (middle).

Now, we can close the curve of constant width G^1-continuously by means of a new regular arc that interpolates the corresponding start and end points with their tangents and possibly more points and tangents in between. In Figure 14 (right) a third point and tangent has been taken using an admissible polynomial $q_2(t) = (1 + t^2)^5$ of degree $n = 10$. Again, by Theorem 2 we can compute the rational support function $\frac{p_2}{q_2}$ that interpolates these points and tangents.

Joining both regular arcs together with their antipodal arcs results in the convex curve of constant width plotted in Figure 13.

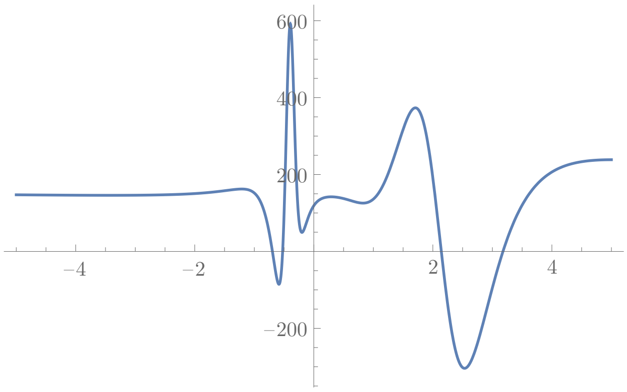

Figure 15: A plot of the function $4\,H_1(t) + 2\,t\,(1+t^2)\,H_1'(t) + (1+t^2)^2\,H_1''(t)$. It has 4 zeros that correspond to 4 singular points in the rationally parameterized hedgehog of constant width.

6. Conclusions and perspectives

In this paper we have studied the class of rationally parameterized curves of constant width by a support function. We obtained an explicit expression of the rational support function such that the constant width condition is satisfied. We focused in the case of characterizing rational projective hedgehogs as any rational hedgehog of constant width can be obtained as an offset to these curves.

First, we characterized all the possible denominators q for this support function, which we called admissible, and then we showed that given an admissible denominator, we could compute the coefficients of the numerator having several degrees of freedom. This produced a constructive method to generate these curves by the user choice of some free parameters.

In the second part of the paper, we translated the degrees of freedom on parameters with geometrical meaning and solved a geometric Hermite problem: given an admissible denominator and a width, there is a unique hedgehog of such a constant width that interpolates a set of chosen points and tangent lines. Our solution is constructive and provide a method to design these curves from user-controlled points and tangents. We showed several examples constructed with this method for different degrees of admissible denominators.

In the examples we have seen that there is a strong dependence on the initial data, both on the points and tangents but also the chosen admissible denominator. The resulting curve is of course not necessarily convex. In such a case, we have shown that a piecewise construction of a curve of constant width can be helpful to avoid singularities, however, this procedure is not entirely automatic. Automatic detection and generation of singularity-free curves of constant width while maintaining the overall shape of a given singular hedgehog of constant width is an interesting problem that can be studied in a future work.

Acknowledgements

The authors wish to thank the anonymous referees for their careful reading of the paper and their comments and suggestions. The first and third authors have been partially supported by Grant PID2021-124577NBI00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”. The first author has also been funded by Ministerio de Ciencia e Innovación (Spain) through project PID2019-104927GB-C21, MCIN/AEI/10.13039/501100011033, by Universitat Jaume I through project UJI-B2022-19, and by Generalitat Valenciana through project CIAICO/2021/180. The fourth author has been partially supported by the Basque Government through the BERC 2022–2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa accreditation of excellence CEX2021-001142-S / MCIN / AEI / 10.13039/501100011033.

References

[1] R. Ait-Haddou, W. Herzog, and L. Biard. Pythagorean-hodograph ovals of constant width. *Computer Aided Geometric Design*, 25(4):258–273, 2008. ISSN 0167-8396. doi: https://doi.org/10.1016/j.cagd.2007.10.008. URL https://www.sciencedirect.com/science/article/pii/S0167839607001252.

[2] M. Bardet and T. Bayen. On the degree of the polynomial defining a planar algebraic curves of constant width, 2013. https://arxiv.org/abs/1312.4358.

[3] L. M. Cruz-Orive. A new stereological principle for test lines in three-dimensional space. *J. Microsc.*, 219(1):18–28, 2005. ISSN 0022-2720. doi: 10.1111/j.1365-2818.2005.01489.x. URL https://doi.org/10.1111/j.1365-2818.2005.01489.x.

[4] L. M. Cruz-Orive. Stereology: A historical survey. *Image Analysis and Stereology*, 36(3):153–177, 2017. ISSN 1854-5165. doi: 10.5566/ias.1767. URL https://www.ias-iss.org/ojs/IAS/article/view/1767.

[5] P. J. Davis. *Interpolation and approximation.* Dover Publications, Inc., New York, 1975. Republication, with minor corrections, of the 1963 original, with a new preface and bibliography.

[6] R. T. Farouki. *Pythagorean-hodograph curves: algebra and geometry inseparable*, volume 1 of *Geometry and Computing.* Springer, Berlin, 2008. ISBN 978-3-540-73397-3. doi: 10.1007/978-3-540-73398-0. URL https://doi.org/10.1007/978-3-540-73398-0.

[7] R. T. Farouki and V. T. Rajan. Algorithms for polynomials in Bernstein form. *Comput. Aided Geom. Design*, 5(1):1–26, 1988. ISSN 0167-8396. doi: 10.1016/0167-8396(88)90016-7. URL https://doi.org/10.1016/0167-8396(88)90016-7.

[8] R. J. Gardner. *Geometric Tomography.* Cambridge University Press, Cambridge, 2006. ISBN 9781107341029. doi: 10.1017/CBO9781107341029. URL http://ebooks.cambridge.org/ref/id/CBO9781107341029.

[9] J. Gravesen, B. Jüttler, and Z. Šír. On rationally supported surfaces. *Comput. Aided Geom. Design*, 25(4-5):320–331, 2008. ISSN 0167-8396. doi: 10.1016/j.cagd.2007.10.005. URL https://doi.org/10.1016/j.cagd.2007.10.005.

[10] B. Kawohl and G. Sweers. On a formula for sets of constant width in 2D. *Commun. Pure Appl. Anal.*, 18(4):2117–2131, 2019. ISSN 1534-0392. doi: 10.3934/cpaa.2019095. URL https://doi.org/10.3934/cpaa.2019095.

[11] Y. Martinez-Maure. Hedgehogs of constant width and equichordal points. *Ann. Polon. Math.*, 67(3):285–288, 1997. ISSN 0066-2216. doi: 10.4064/ap-67-3-285-288. URL https://doi.org/10.4064/ap-67-3-285-288.

[12] Y. Martinez-Maure. Geometric inequalities for plane hedgehogs. *Demonstratio Math.*, 32(1):177–183, 1999. ISSN 0420-1213.

[13] Y. Martinez-Maure. Noncircular algebraic curves of constant width: an answer to Rabinowitz. *Canad. Math. Bull.*, 65(3): 552–556, 2022. ISSN 0008-4395. doi: 10.4153/S0008439521000473. URL https://doi.org/10.4153/S0008439521000473.

[14] H. Martini and Z. Mustafaev. A New Construction of Curves of Constant Width. *Comput. Aided Geom. Des.*, 25(9): 751–755, 2008. ISSN 0167-8396. doi: 10.1016/j.cagd.2008.06.010. URL https://doi.org/10.1016/j.cagd.2008.06.010.

[15] H. Martini, L. Montejano, and D. Oliveros. *Bodies of constant width. An introduction to convex geometry with applications.* Birkhäuser/Springer, Cham, 2019. ISBN 978-3-030-03866-3; 978-3-030-03868-7. doi: 10.1007/978-3-030-03868-7. URL https://doi.org/10.1007/978-3-030-03868-7.

[16] D. S. Meek and D. J. Walton. Geometric Hermite interpolation with Tschirnhausen cubics. *J. Comput. Appl. Math.*, 81(2): 299–309, 1997. ISSN 0377-0427. doi: 10.1016/S0377-0427(97)00066-6. URL https://doi.org/10.1016/S0377-0427(97)00066-6.

[17] C. Panraksa and L. C. Washington. Real algebraic curves of constant width. *Periodica Mathematica Hungarica*, 74(2): 235–244, 2017. ISSN 1588-2829. doi: 10.1007/s10998-016-0149-9. URL https://doi.org/10.1007/s10998-016-0149-9.

[18] V. Pless. *Introduction to the theory of error-correcting codes.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, third edition, 1998. ISBN 0-471-19047-0. doi: 10.1002/9781118032749. URL https://doi.org/10.1002/9781118032749. A Wiley-Interscience Publication.

[19] S. Rabinowitz. A polynomial curve of constant width. *Missouri J. Math. Sci.*, 9(1):23–27, 1997. doi: 10.35834/1997/0901023. URL https://doi.org/10.35834/1997/0901023.

[20] D. Rochera. Algebraic equations for constant width curves and Zindler curves. *J. Symbolic Comput.*, 113:139–147, 2022. ISSN 0747-7171. doi: 10.1016/j.jsc.2022.03.001. URL https://doi.org/10.1016/j.jsc.2022.03.001.

[21] D. Rochera. Offsets and front tire tracks to projective hedgehogs. *Comput. Aided Geom. Design*, 97:Paper No. 102135, 2022. ISSN 0167-8396. doi: 10.1016/j.cagd.2022.102135. URL https://doi.org/10.1016/j.cagd.2022.102135.

[22] D. Rochera. Generalized plane offsets and rational parameterizations. *Comput. Aided Geom. Design*, 103:Paper No. 102196, 2023. ISSN 0167-8396. doi: 10.1016/j.cagd.2023.102196. URL https://doi.org/10.1016/j.cagd.2023.102196.

[23] A. Spitzbart. A generalization of Hermite's interpolation formula. *Amer. Math. Monthly*, 67:42–46, 1960. ISSN 0002-9890. doi: 10.2307/2308924. URL https://doi.org/10.2307/2308924.

[24] Z. Šír, J. Gravesen, and B. Jüttler. Curves and surfaces represented by polynomial support functions. *Theoret. Comput. Sci.*, 392(1-3):141–157, 2008. ISSN 0304-3975. doi: 10.1016/j.tcs.2007.10.009. URL https://doi.org/10.1016/j.tcs.2007.10.009.