Lifespan estimates for the compressible Euler equations with damping via Orlicz spaces techniques†

Ning-An Lai[a], Nico Michele Schiavone[b]

[a] *School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China*
[b] *BCAM – Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Basque Country – Spain*

Abstract

In this paper we are interested in the upper bound of the lifespan estimate for the compressible Euler system with time dependent damping and small initial perturbations. We employ some techniques from the blow-up study of nonlinear wave equations. The novelty consists in the introduction of tools from the Orlicz spaces theory to handle the nonlinear term emerging from the pressure $p \equiv p(\rho)$, which admits different asymptotic behavior for large and small values of $\rho - 1$, being ρ the density. Hence we can establish, in dimensions $n \in \{2, 3\}$, unified upper bounds of the lifespan estimate depending only on the dimension n and on the damping strength, and independent of the adiabatic index $\gamma > 1$. We conjecture our results to be optimal. The method employed here not only improves the known upper bounds of the lifespan for $n \in \{2, 3\}$, but has potential application in the study of related problems.

Keywords: Compressible Euler equations, time dependent damping, blow-up, lifespan, Orlicz spaces

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1. Introduction

In this work, we will consider the compressible isentropic Euler equations with time dependent damping

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, & (1.1a) \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p + \dfrac{\mu\, \rho u}{(1+t)^\lambda} = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, & (1.1b) \\
u(0, x) = \varepsilon u_0(x), \quad \rho(0, x) = \overline{\rho} + \varepsilon \rho_0(x), & (1.1c)
\end{cases}
$$

Email addresses: `ninganlai@zjnu.edu.cn` (Ning-An Lai), `nschiavone@bcamath.org` (Nico Michele Schiavone)

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where $\rho\colon [0,T)\times\mathbb{R}^n\to\mathbb{R}$, $u\colon [0,T)\times\mathbb{R}^n\to\mathbb{R}^n$ and $p\colon [0,T)\times\mathbb{R}^n\to\mathbb{R}$ stand for the density, velocity and pressure of the fluid respectively, $n\in\{1,2,3\}$ is the spatial dimension, while $\frac{\mu}{(1+t)^\lambda}$ is a time dependent frictional coefficient, with $\mu\geq 0$ and $\lambda\geq 1$. The initial values are a small perturbation of constant states (with $\overline{\rho}>0$), where the "smallness" is quantified by the parameter $\varepsilon>0$, and the perturbations $\rho_0, u_0\in C_0^\infty(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\rho_0,\ \operatorname{supp}u_0\subseteq\{x\in\mathbb{R}^n\colon |x|\leq R\} \tag{1.2}$$

for some positive constant R. We assume that the pressure satisfies the γ-law

$$p\equiv p(\rho)=A\rho^\gamma,$$

where $A>0$ is a constant and $\gamma>1$ is the adiabatic index. However, it is not restrictive in the following to assume $R=\overline{\rho}=1$ and $A=\frac{1}{\gamma}$. Our study concerns the estimates for the *lifespan* T_ε of the solution, defined as the largest time such that the solution exists and is C^1 on $[0,T_\varepsilon)\times\mathbb{R}^n$.

1.1. The undamped case

Setting $\mu=0$, the system (1.1) reduces to the classic Euler equations, a set of quasilinear hyperbolic equations fundamental in fluid dynamics to describe the adiabatic and inviscid flow of an ideal fluid. Generally speaking, the compressible Euler equations develop shock waves in finite time for general initial data (see [37] and the references therein).

The investigation of singularity formation in compressible Euler equations began with Sideris in [34], where several blow-up results were established for a $3D$ model of a polytropic, ideal fluid with both large data and small initial perturbations. In [35, 36], the same author established in $3D$ the lifespan estimate

$$\exp\left(C_1\varepsilon^{-1}\right)\leq T_\varepsilon\leq\exp\left(C_2\varepsilon^{-2}\right), \tag{1.3}$$

where the lower bound was obtained assuming the initial velocity irrotational, and the upper bound holds for $\gamma=2$. In particular, Sideris extended the generic lower bound

$$T_\varepsilon\geq C\varepsilon^{-1} \tag{1.4}$$

which typically holds for symmetric hyperbolic systems in any number of space dimensions (see [17, 28] for the general theory). Here and in the following, we will use C, C_1, C_2 to denote generic positive constants independent of ε, the value of which may change in different places.

Rammaha proved in [32] the formation of singularity in finite time for the $2D$ case, and later Alinhac showed in [2] that the lifespan for rotationally invariant data in $2D$ satisfies

$$\lim_{\varepsilon\to 0}\varepsilon^2 T_\varepsilon=C. \tag{1.5}$$

Recently, Jin and Zhou in [14] derived, for $\gamma=2$, the upper bound for the lifespan estimate

$$T_\varepsilon\leq\begin{cases}C\varepsilon^{-1} & \text{if } n=1,\\ C\varepsilon^{-2} & \text{if } n=2,\\ \exp(C\varepsilon^{-1}) & \text{if } n=3,\end{cases} \tag{1.6}$$

which in particular improves the one in (1.3), and shows the optimality of the lifespan estimate in $3D$ when $\gamma = 2$. To be precise, in their work only the $3D$ case is explicated, but their method and Lemma 2.1 therein work also in lower dimensions.

On the other side, putting together (1.4), (1.5) and (1.3), we find that the lower bounds corresponding to (1.6) hold for any $\gamma > 1$, namely

$$T_\varepsilon \geq \begin{cases} C\varepsilon^{-1} & \text{if } n = 1, \\ C\varepsilon^{-2} & \text{if } n = 2, \\ \exp(C\varepsilon^{-1}) & \text{if } n = 3. \end{cases}$$

In particular, we can see that the lifespan estimates in (1.6) are optimal at least for $\gamma = 2$ in $1D$ and $3D$, and for any $\gamma > 1$ if $n = 2$ (due to [2]). Actually, we anticipate that they are optimal for any $\gamma > 1$ also in the $1D$ case thanks to the results in [38], but we will properly discuss them in the next subsection.

1.2. The damped case

Let us turn our attention to the problem in the presence of a time dependent damping term. Wang and Yang in [42] proved global existence in $\mathbb{R}^n$ for the system (1.1) with $\mu > 0$, $\lambda = 0$ and small initial perturbation of some constant state — see also the contribution by Sideris, Thomases and Wang [37] in $\mathbb{R}^3$. Hou and Yin in [11], and Hou, Witt and Yin in [10] studied the system (1.1) in $2D$ and $3D$, proving: global existence for $0 \leq \lambda < 1$, $\mu > 0$ and for $\lambda = 1$, $\mu > 3 - n$; finite time blow-up for $\lambda > 1$, $\mu > 0$ and for $\lambda = 1$, $0 < \mu \leq 1$, $n = 2$. Also, they established for $\gamma = 2$ the following upper bound of the lifespan estimate:

$$T_\varepsilon \leq \exp\left(C\varepsilon^{-2}\right). \tag{1.7}$$

In [30, 31], Pan studied the corresponding problem in 1-dimension, showing that: if $0 \leq \lambda < 1$, $\mu > 0$ or if $\lambda = 1$, $\mu > 2$, there exists a global solution; if $\lambda = 1$, $0 \leq \mu \leq 2$ or if $\lambda > 1$, $\mu \geq 0$, the C^1-solutions will blow up in finite time. Moreover, in the latter case, the same lifespan estimate as (1.7) was established in [30] for $\gamma = 2$. From these results, one infers that the point $(\lambda, \mu) = (1, 3 - n)$ is critical for the problem.

In [38], Sugiyama studied the system (1.1) in $1D$, in its equivalent form obtained by changing the Eulerian coordinates into Lagrangian ones, viz. the so-called p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x + a(t,x)v = 0, \\ u(0,x) = \varepsilon u_0(x), \quad v(0,x) = 1 + \varepsilon v_0(x), \end{cases} \tag{1.8}$$

where $p \equiv p(v) = \frac{v^{-\gamma}}{\gamma}$ and $v = 1/\rho$ is the specific volume. For the space-independent damping $a(t,x) = \frac{\mu}{(1+t)^\lambda}$, making use of Riemann invariants, the author establishes the estimates

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-1} & \text{for } \lambda > 1 \text{ and } \mu \geq 0, \\ C\varepsilon^{-\frac{2}{2-\mu}} & \text{for } \lambda = 1 \text{ and } 0 \leq \mu < 2, \\ \exp(C\varepsilon^{-1}) & \text{for } \lambda = 1 \text{ and } \mu = 2, \end{cases} \tag{1.9}$$

not only from above, but also from below, namely obtaining the sharp lifespan estimates for small ε, provided that $\lim_{x\to-\infty}(u_0(x), v_0(x)) = (u_-, v_-) \in \mathbb{R}^2$.

At the best of our knowledge, except for the above nice results in $1D$ by Sugiyama, the lifespan estimate in $2D$ and $3D$ for (1.1) seems to be still unclear, especially when $\gamma \neq 2$, even in the undamped case for $n = 3$. The method in [30, 31, 11, 10] successfully proves the blow-up of the solutions, but provides, as upper bound for the lifespan, the estimate (1.7), which unfortunately seems to be not the optimal one.

Furthermore, in the literature cited until now often only the case $\gamma = 2$ is explicitly considered. The modifications necessary to generalize the proof for any $\gamma > 1$ are then indicated, pointing to the fact that the lifespan estimates may be modified, but it is not clear how. This happens in the works [34, 14] for $n = 3$ and [32] for $n = 2$ in the undamped case, and similarly in the works [11] and [30] for the case with the time dependent damping term.

Nevertheless, the lower bounds of the lifespan for (1.1) are independent of γ in the undamped case, and they are optimal for $n \in \{1, 2\}$ and any $\gamma > 1$, as we observed at the end of the previous subsection. Thus, it is natural to believe that this phenomena should be true also for the bounds of the lifespan in the damped case, as actually verified in [38] for $n = 1$. Our results go exactly in this direction, see Remark 2.4.

2. Aims and results

In this paper, we are going to employ an argument based on the manipulation of suitable multipliers to absorb the damping term, and a variation of a lemma on differential inequality established in Li and Zhou [24]. Similar techniques were applied in the context of the blow-up study for semilinear damped wave equations with small initial data — as we will show in Section 3, the density satisfies this kind of equation. About the damped wave equation, there exists an extensive literature. Since the case treated in this work can be correspondent to the scattering case (when $\lambda > 1$) and to the scale-invariant case (when $\lambda = 1$) of the damped wave equation according to the Wirth's classification (see [43, 44, 45]), we only cite the works [26, 21, 40] and [5, 7, 6, 29, 15, 41, 16, 13, 23, 12, 39, 25] for the two cases respectively, referring the reader to the Introduction of [20] and references therein for a comprehensive presentation.

The novelty in our work consists in mixing these techniques, which manage the linear part of the equation, with tools from the Orlicz spaces theory, which seems to be a suitable setting to deal with the non-power-like nonlinearity appearing in (3.8) below in the case of $\gamma \neq 2$. In this way, we are able to derive upper bounds of the lifespan not only merely in the case $\gamma = 2$, improving the existing ones, but also in the general case $\gamma \neq 2$, obtaining the *papabili* optimal results (see Remark 2.3). To the best of our knowledge, the use of Orlicz space theory in the context of blow-up and lifespan estimates for nonlinear wave equations seems to be new, and we are confident they can be potentially applied to other problems (see Remark 2.6).

Let us present now our main results.

Theorem 1. *Let $n \in \{1, 2, 3\}$ and $\gamma > 1$. Suppose $\lambda > 1$ and $\mu \geq 0$. Assume that the initial data satisfy* (1.2) *and*

$$\int_{\mathbb{R}^n} \rho_0 \phi \, dx > 0, \qquad \int_{\mathbb{R}^n} u_0 \cdot \nabla \phi \, dx > 0, \tag{2.1}$$

where ϕ is defined in (4.2) *below. Then the system* (1.1) *has no global solutions, and there exists $C \equiv C(n,\lambda,\mu,\gamma,u_0,\rho_0)$ independent of ε such that the lifespan estimate satisfies*

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-\frac{2}{3-n}} & \text{if } n \in \{1,2\}, \\ \exp\left(C\varepsilon^{-1}\right) & \text{if } n = 3, \end{cases}$$

provided $\varepsilon \le \varepsilon_0$ for some positive constant $\varepsilon_0 \equiv \varepsilon_0(n,\lambda,\mu,\gamma,u_0,\rho_0)$.

Theorem 2. *Let $n \in \{1,2\}$ and $\gamma > 1$. Suppose $\lambda = 1$ and $0 \le \mu \le 3-n$. Assume that the initial data satisfy* (1.2) *and* (2.1)*. Then the system* (1.1) *has no global solutions, and there exists $C \equiv C(n,\mu,\gamma,u_0,\rho_0)$ independent of ε such that the lifespan estimate satisfies*

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-\frac{2}{3-n-\mu}} & \text{if } \mu < 3-n, \\ \exp\left(C\varepsilon^{-1}\right) & \text{if } \mu = 3-n, \end{cases} \tag{2.2}$$

provided $\varepsilon \le \varepsilon_0$ for some positive constant $\varepsilon_0 \equiv \varepsilon_0(n,\mu,\gamma,u_0,\rho_0)$.

Remark 2.1 (*On the initial data conditions*)**.** The assumptions in (2.1) involve the Yordanov–Zhang function ϕ, which was firstly introduced in [46] to prove the blow-up of wave equations. We provide its definition later in (4.2). These conditions arise naturally due to the method we employ, see for example [14]. Notice that, since ϕ is positive and by the divergence theorem, assumption (2.1) can be implied by the stronger hypotheses $\rho_0 > 0$ and $\nabla \cdot u_0 < 0$. By rescaling the density and choosing ε sufficiently small, it is easy to see that the condition $\rho_0 > 0$ is not restrictive at all. On the other hand, assuming that the divergence of the initial velocity is negative indicates that the fluid at the initial time is contracting within the unit ball.

Remark 2.2 (*About optimality in the undamped case*)**.** The results in Theorems 1 and 2 hold also for the compressible Euler system without damping, setting $\mu = 0$. Together with the lower bound in (1.3) by Sideris, and recalling the discussion at the end of Subsection 1.1, we completely close the problem of the optimality of the lifespan estimates in the undamped case for any $\gamma > 1$.

Remark 2.3 (*About optimality in the damped case*)**.** Compared to the result (1.7) obtained in [10, 11, 30], we improve the lifespan estimates for $n = 2$ and $n = 3$, whereas for $n = 1$ we re-demonstrate the upper bound in (1.9) with a completely different approach. Since when $\lambda = 1$ it is the size of the positive constant μ that determines whether there is global solution or blow-up in finite time, it is reasonable to believe that the upper bound of the lifespan depends also on μ, as per the results (1.9) in $1D$ obtained by Sugiyama in [38]. And precisely in view of these results, it is natural to conjecture that the lifespan estimates in Theorem 1 and 2 should be indeed optimal, because we already know they are for $n = 1$.

Remark 2.4 (*Independence of γ*)**.** In light of Remark 2.2, we can provide a negative answer to the statements at the end of [34] and [32]: in the undamped case, the lifespan estimates are actually independent of γ, or better, for any $\gamma > 1$ they coincide with the ones in the case $\gamma = 2$. This should be true also in the damped case, on the basis of Remark 2.3.

Remark 2.5 (*Relations with the Glassey's conjecture*)**.** As we anticipated before, we will reduce ourselves to the study of the damped wave equation (3.8), and hence, roughly speaking, to the inequality

$$\widetilde{\rho}_{tt} - \Delta\widetilde{\rho} + \frac{\mu}{(1+t)^\lambda}\widetilde{\rho}_t \geq \Delta R(\widetilde{\rho}) \tag{2.3}$$

where $\lim_{p\to 0}\frac{R(p)}{|p|^2} = \frac{\gamma-1}{2}$ and $\lim_{p\to+\infty}\frac{R(p)}{p^\gamma} = \frac{1}{\gamma}$. But we can be more precise. Indeed, it is interesting to observe that our problem seems to be related to the Glassey's conjecture for the wave equation with derivative-type non-linearity, which affirms that the critical exponent for the equation

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t = |u_t|^p$$

with small initial data is given by the Glassey exponent

$$p_G(n) := 1 + \frac{2}{n-1}.$$

Let us consider the corresponding problem with damping term, namely

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\lambda}u_t = |u_t|^p, \\ u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \end{cases} \tag{2.4}$$

for which the possibly optimal blow-up results in the case $\lambda > 1$ and $\lambda = 1$ are given in [22] and [9] respectively (in the latter reference a more general combined nonlinearity is considered, see also [27, 19, 4, 8] for related problems). For the model with $\lambda = 1$ the critical exponent seems to be $p_G(n+\mu)$, in the sense that we have blow-up for any $1 < p \leq p_G(n+\mu)$ and global existence for $p > p_G(n+\mu)$. But if we set $p = 2$, this requirement is equivalent to $\mu \leq 3-n$, which is exactly the condition on μ in Theorem 2. Similarly, if $\lambda > 1$ the critical exponent should be $p_G(n)$, experiencing blow-up for $1 < p \leq p_G(n)$. Setting $p = 2$ this condition reduce to $n \in \{1,2,3\}$. This analogy holds true also for the lifespan estimates: the corresponding ones proved in [22] and [9], setting $p = 2$, match with the ones provided in Theorem 1 and 2.

In other words, our problem (1.1) seems to share the same blow-up dynamics of the problem (2.4) with the exponent fixed to be $p = 2$. The rising of this particular number is not surprising, in view of Remark 2.4 and our particular nonlinearity. Indeed, even if the function R defined in (3.7) and appearing in (2.3) behaves like $|\cdot|^2$ for small argument and like $|\cdot|^\gamma$ for large argument, it will emerge from the proof of the theorems that it is actually the power $|\cdot|^2$ what essentially influence the behavior of the problem.

Remark 2.6 (*Other problems*)**.** Let us list here some other related problems:

- The method exploited in this work, based on the combination of techniques from the wave equations blow-up theory with Orlicz space tools, could be employed also in the study of blow-up phenomena for wave equation with general non-homogeneous convex non-linearities. For example, the computations we perform can be adapted for an equation like
$$u_{tt} - \Delta u + d(t)u_t + m(t)u = \min\{|u|^p, |u|^q\}$$
with some constants $p, q > 1$ and damping and mass terms $d(t)$, $m(t)$.

- Another potential application of our methods is to the problem corresponding to (1.1) in exterior domain with appropriate boundary conditions.
- For $\gamma = 2$ the problem (1.1) is closely related to the inviscid shallow-water equations: they indeed coincide if the pressure satisfies the law $p(\rho) = \frac{\rho^2}{2F^2}$, being F the Froude number (see e.g. [3, 28]).

The rest of the paper is organized as follows. In Section 3 we will infer from (1.1) the energy formulation of the damped wave equation satisfied by $\rho - 1$. Choosing a suitable test function, in Section 4 we will write a ordinary differential inequality satisfied by a key functional F. In Section 5 we deal with its nonlinear part, employing some techniques inspired by the Orlicz spaces theory — in the strategy here lies our main novelty contribution. Finally we conclude the proofs of Theorems 1 and 2 in Sections 6 and 7 respectively, where we take care of the linear part in the inequality for F with a clever usage of some multipliers. The argument is closed by the employment of a variation of a lemma by Li and Zhou, the proof of which is provided in Appendix A.

Notation. Hereinafter, we will use $A \lesssim B$ (resp. $A \gtrsim B$) in place of $A \leq CB$ (resp. $A \geq CB$), where C is a positive constant independent of ε. We will write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. Finally, $f(x) \sim g(x)$ for $|x| \to r \in \mathbb{R} \cup \{+\infty\}$ means that $\lim_{|x|\to r} \frac{f(x)}{g(x)} = 1$.

3. The damped wave equation

First of all, we recall that the system (1.1) may be reformulated as a symmetric hyperbolic system admitting, in accordance to the general theory in [17, 28], a local C^1-solution on the time interval $[0, T)$ for some $T > 0$, provided the initial data are sufficiently regular. Moreover, the density $\rho(t, x)$ is positive on $[0, T) \times \mathbb{R}^n$, if the initial density $\rho(0, x)$ is positive on $\mathbb{R}^n$ (and in our case this is surely true for ε small enough). Finally, it can be proved (exploiting a parallel method as in the proof of [37, Lemma 3.2]) that for any local C^1-solution (ρ, u) of (1.1), with initial data satisfying (1.2), the following finite speed of propagation result holds:

$$\operatorname{supp}(\rho - 1),\ \operatorname{supp} u \subseteq \{(t, x) \in (0, T) \times \mathbb{R}^n \colon |x| \leq 1 + t\}. \tag{3.1}$$

We omit the details here, and we refer to [34, 35, 37] for a more exhaustive discussion.

With the local existence and the finite speed of propagation at hand, we can proceed deducing the damped wave equation satisfied by the density, the energy formulation of which will be the starting point of our argument. Suppose (ρ, u) is a C^1-solution to (1.1), satisfying (3.1). Let $\Phi(t, x)$ be a real smooth function with compact support on $[0, T) \times \mathbb{R}^n$. Multiplying equation (1.1b) by $\nabla\Phi$, integrating on the strip $[0, t) \times \mathbb{R}^n$ of the space-time, with $t \in [0, T)$, and then by parts with respect to the time, we reach

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho u \cdot \nabla\Phi \,\mathrm{d}x - \int_0^t \int_{\mathbb{R}^n} \rho u \cdot \nabla\Phi_s - d(s)\rho u \cdot \nabla\Phi \,\mathrm{d}x \,\mathrm{d}s \\
={}& \varepsilon \int_{\mathbb{R}^n} (1 + \varepsilon\rho_0) u_0 \cdot \nabla\Phi(0, x) \,\mathrm{d}x \\
&- \int_0^t \int_{\mathbb{R}^n} \left[\nabla \cdot (\rho u \otimes u) + \nabla \frac{\rho^\gamma - 1}{\gamma} \right] \cdot \nabla\Phi \,\mathrm{d}x \,\mathrm{d}s.
\end{aligned} \tag{3.2}$$

where for short we set

$$d(s) := \frac{\mu}{(1+s)^\lambda}.$$

Consider now equation (1.1a). After a multiplication by $\Phi_s(s,x)$, integrating on $[0,t) \times \mathbb{R}^n$ and then by parts with respect to the space, we have

$$\int_0^t \int_{\mathbb{R}^n} \rho u \cdot \nabla \Phi_s \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\mathbb{R}^n} (\rho - 1)_s \Phi_s \,\mathrm{d}x\,\mathrm{d}s. \tag{3.3}$$

Multiplying (1.1a) by Φ instead, integrating only on the space and then by parts, we obtain that

$$\int_{\mathbb{R}^n} \rho u \cdot \nabla \Phi \,\mathrm{d}x = \int_{\mathbb{R}^n} (\rho - 1)_t \Phi \,\mathrm{d}x. \tag{3.4}$$

Finally, multiplying (1.1a) by $d(s)\Phi(s,x)$, integrating on the space-time and then by parts, we get

$$\int_0^t \int_{\mathbb{R}^n} d(s) \rho u \cdot \nabla \Phi \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\mathbb{R}^n} d(s)(\rho - 1)_s \Phi \,\mathrm{d}x\,\mathrm{d}s. \tag{3.5}$$

Now, inserting (3.3), (3.4) and (3.5) into (3.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\rho-1)_t \Phi \,\mathrm{d}x - \int_0^t \int_{\mathbb{R}^n} (\rho-1)_s \Phi_s \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \nabla(\rho-1) \cdot \nabla \Phi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} d(s)(\rho-1)_s \Phi \,\mathrm{d}x\,\mathrm{d}s \\
= &-\varepsilon \int_{\mathbb{R}^n} \nabla \cdot ((1+\varepsilon\rho_0)u_0) \Phi(0,x) \,\mathrm{d}x \\
&\quad - \int_0^t \int_{\mathbb{R}^n} [\nabla \cdot (\rho u \otimes u)] \cdot \nabla \Phi \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}^n} \nabla R(\rho-1) \cdot \nabla \Phi \,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{3.6}$$

where $R \colon [-1,+\infty) \to [0,+\infty)$ is defined by

$$R(p) := \frac{(p+1)^\gamma - 1}{\gamma} - p. \tag{3.7}$$

Notice that equation (3.6) can be regarded as the energy solution formulation of the nonlinear damped wave equation

$$(\rho-1)_{tt} - \Delta(\rho-1) + d(t)(\rho-1)_t = \nabla \cdot [\nabla \cdot (\rho u \otimes u)] + \Delta R(\rho-1) \tag{3.8}$$

with initial data

$$[\rho - 1]_{t=0} = \varepsilon\rho_0, \qquad [(\rho-1)_t]_{t=0} = -\varepsilon \nabla \cdot ((1+\varepsilon\rho_0)u_0).$$

4. The functional

With the energy formulation (3.6) for the density at hand, we need now to choose a suitable test function Φ to insert in it, which will imply an identity for the functional

$$F(t) := \int_{\mathbb{R}^n} (\rho - 1)\Phi \,\mathrm{d}x. \tag{4.1}$$

4.1. The test function

Let us define the positive Yordanov–Zhang (see [46]) function

$$\phi(x) := \begin{cases} \dfrac{1}{n\omega_n}\displaystyle\int_{\mathbb{S}^{n-1}} e^{x\cdot\sigma}\mathrm{d}\sigma & \text{if } n \geq 2, \\ \dfrac{e^x + e^{-x}}{2} & \text{if } n = 1, \end{cases} \tag{4.2}$$

where $\omega_n := \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the volume of the n-dimensional unit ball, so that $n\omega_n$ is the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$. The function ϕ is radial, strictly increasing with respect to $|x|$, solves $\Delta\phi = \phi$, and moreover

$$\phi(0) = 1, \qquad \phi(x) \sim \frac{(2\pi)^{\frac{n-1}{2}}}{n\omega_n}|x|^{-\frac{n-1}{2}}e^{|x|} \quad \text{when } |x| \to +\infty,$$

therefore ϕ satisfies, for any $x \in \mathbb{R}^n$, the relation

$$\phi(x) \approx \langle x\rangle^{-\frac{n-1}{2}}\, e^{|x|}. \tag{4.3}$$

Here and in the following $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$ stands as customary for the Japanese bracket notation.

All the above properties of ϕ can be easily deduced rewriting this function in a closed form. Indeed, using n-dimensional spherical coordinates and choosing the polar axis parallel to x, one can prove that actually

$$\phi(x) = \frac{(2\pi)^{\frac{n}{2}}}{n\omega_n}|x|^{1-\frac{n}{2}} I_{\frac{n}{2}-1}(|x|),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind[2]. From the properties of the latter (see e.g. [1, Sections 9.6 and 9.7]) follow those of ϕ.

Let us consider then the positive function

$$\psi(t, x) := e^{-t}\phi(x),$$

which satisfies $\psi_t = -\psi$, $\Delta\psi = \psi$, and the following bounds we will use later:

$$\psi(t, x) \lesssim \langle t\rangle^{-\frac{n-1}{2}} \tag{4.4}$$

for $|x| \leq 1 + t$, and

$$\int_{|x|\leq 1+t} \psi^b(x)\,\mathrm{d}x \approx \langle t\rangle^{\frac{n-1}{2}(2-b)} \tag{4.5}$$

for any $b \in [0, 2]$ and $t \geq 0$. The estimate (4.4) comes directly from (4.3). For (4.5), observe that, from the one side, we have

$$\begin{aligned} \int_{|x|\leq 1+t} \psi^b(x)\,\mathrm{d}x &\gtrsim \int_{|x|\leq 1+t} \langle x\rangle^{-\frac{n-1}{2}b}\, e^{b|x|-bt}\,\mathrm{d}x \\ &\gtrsim \int_{\frac{1+t}{2}}^{1+t} \langle r\rangle^{\frac{n-1}{2}(2-b)}\, e^{br-bt}\,\mathrm{d}r \\ &\gtrsim \langle t\rangle^{\frac{n-1}{2}(2-b)}, \end{aligned}$$

[2] Curiously, as far as we know, this closed expression seems to be never reported in the related literature.

while, on the other side,

$$
\begin{aligned}
\int_{|x|\le 1+t} \psi^b(x)\,\mathrm{d}x &\lesssim \int_{|x|\le 1+t} \langle x\rangle^{-\frac{n-1}{2}b}\, e^{b|x|-bt}\,\mathrm{d}x \\
&\lesssim \int_0^{1+t} \langle r\rangle^{\frac{n-1}{2}(2-b)}\, e^{br-bt}\,\mathrm{d}r \\
&\lesssim \langle t\rangle^{\frac{n-1}{2}(2-b)}\,.
\end{aligned}
$$

Finally, let χ a smooth compactly supported cut-off function such that $\chi(t,x)\equiv 1$ on the support of $\rho-1$. Our test function is given by

$$
\Phi(t,x) := \psi(t,x)\chi(t,x) = e^{-t}\phi(x)\chi(t,x).
$$

Inserting it in (3.6), using (3.1), integrating by parts with respect to the space variables and deriving with respect to the time, we arrive at

$$
\begin{aligned}
&F'' + 2F' + \frac{\mu}{(1+t)^\lambda}(F'+F) \\
&\quad= \int_{\mathbb{R}^n} \mathrm{tr}\left[(\rho u\otimes u)\nabla^2\psi\right]\,\mathrm{d}x + \int_{\mathbb{R}^n} R(\rho-1)\psi\,\mathrm{d}x,
\end{aligned}
\tag{4.6}
$$

with the functional F defined in (4.1). Due to the support property of χ, notice that

$$
F(t) = \int_{\mathbb{R}^n} (\rho-1)\psi\,\mathrm{d}x.
$$

Moreover, from the conditions (2.1) on the initial data we have

$$
\begin{aligned}
F(0) &= \varepsilon\int_{\mathbb{R}^n} \rho_0\phi\,\mathrm{d}x > 0, \\
F'(0)+F(0) &= -\varepsilon\int_{\mathbb{R}^n} \nabla\cdot((1+\varepsilon\rho_0)u_0)\phi\,\mathrm{d}x \\
&= \varepsilon\int_{\mathbb{R}^n} (1+\varepsilon\rho_0)u_0\cdot\nabla\phi\,\mathrm{d}x > 0.
\end{aligned}
\tag{4.7}
$$

The last inequality holds true for ε small enough, since

$$
\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\left[F'(0)+F(0)\right] = \int_{\mathbb{R}^n} u_0\cdot\nabla\phi\,\mathrm{d}x > 0.
$$

4.2. The key inequality

Our next step is to bound from below the nonlinear term in (4.6), aiming to prove the key inequality

$$
F'' + 2F' + \frac{\mu}{(1+t)^\lambda}(F'+F) \gtrsim \langle t\rangle^{-\frac{n-1}{2}}\,\Upsilon(F)
\tag{4.8}
$$

where the function $\Upsilon\colon \mathbb{R}\to[0,+\infty)$, the role of which will be clear later, is defined by

$$
\Upsilon(p) := R(|p|) = \frac{(|p|+1)^\gamma - 1}{\gamma} - |p|.
\tag{4.9}
$$

First of all, we handle the first term on the right-hand side of (4.6), which can be easily gotten rid of due to its positiveness. Indeed, for $n \geq 2$,

$$\begin{aligned}\operatorname{tr}\left[(\rho u \otimes u)\nabla^2\psi\right] &= \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} \rho \operatorname{tr}\left[(u\otimes u)(\sigma\otimes\sigma)\right] e^{x\cdot\sigma - t}\,\mathrm{d}\sigma \\ &= \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} \rho (u\cdot\sigma)^2 e^{x\cdot\sigma-t}\,\mathrm{d}\sigma \\ &\geq 0\end{aligned}$$

whereas of course

$$\operatorname{tr}\left[(\rho u \otimes u)\nabla^2\psi\right] = \rho u^2\,\frac{e^{x-t}+e^{-x-t}}{2} \geq 0$$

in the 1-dimensional case. Moreover, the functions Υ and R, defined respectively in (4.9) and (3.7), coincide on $[0,+\infty)$, but are different on $[-1,0)$. However, since $R(p), \Upsilon(p) \sim \frac{\gamma-1}{2}|p|^2$ for $p \to 0$ (by Taylor's theorem, see (5.4) below) and R, Υ are bounded on $[-1,0)$, it is easily checked that

$$R(\rho-1) \approx \Upsilon(\rho-1)$$

for $\rho \geq 0$. Therefore, from (4.6) we get

$$F'' + 2F' + \frac{\mu}{(1+t)^\lambda}(F'+F) \gtrsim \int_{\mathbb{R}^n} \Upsilon(\rho-1)\,\psi\,\mathrm{d}x. \tag{4.10}$$

The precise estimation of the right hand-side of the above inequality in order to infer (4.8) in much more delicate, and we will address it in the next section. This is where the Orlicz spaces come to play.

5. Proof of the key inequality via Orlicz spaces tools

We will prove that

$$\int_{\mathbb{R}^n} \Upsilon(\rho-1)\psi\,\mathrm{d}x \gtrsim \langle t\rangle^{-\frac{n-1}{2}}\,\Upsilon(F), \tag{5.1}$$

which inserted in (4.10) would imply the desired (4.8). At this aim, we need to introduce some tools from the Orlicz spaces theory, together with some properties of Υ. For the reader interested in the general theory of Orlicz spaces, we refer to the classic books [18, 33].

5.1. Orlicz tools and properties of Υ

Observe that the function $\Upsilon\colon \mathbb{R} \to [0,+\infty)$ introduced in (4.9) is continuous, even, convex and satisfies the conditions

$$\lim_{p\to 0}\frac{\Upsilon(p)}{p} = 0, \qquad \lim_{|p|\to+\infty}\frac{\Upsilon(p)}{|p|} = +\infty.$$

By definition (see [18, §I.1.5]), Υ is a N-function. Its complementary N-function Υ^* is obtained by taking the Legendre transform of Υ ([18, Equation (2.9)]), viz.

$$\Upsilon^*(q) := \sup_{p\geq 0}\left\{p|q| - \Upsilon(p)\right\} = \frac{(|q|+1)^{\gamma'}-1}{\gamma'} - |q|, \tag{5.2}$$

where γ' is the conjugate exponent of γ, i.e. $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$. For any N-function Υ and $p \geq 0$, the following useful inequalities ([18, Equation (2.10)]) hold:

$$p \leq \Upsilon^{-1}(p) \cdot (\Upsilon^*)^{-1}(p) \leq 2p. \tag{5.3}$$

We are in the position now to define the Orlicz space associated to Υ. Let us denote with $\widetilde{L^\Upsilon}(\mathbb{R}^n)$ the Orlicz class of functions $u\colon \mathbb{R}^n \to \mathbb{R}$ for which

$$\rho(u;\Upsilon) := \int_{\mathbb{R}^n} \Upsilon(u(x))\,\mathrm{d}x < \infty.$$

Then, we denote by $L^\Upsilon(\mathbb{R}^n)$ the set of all functions u satisfying the condition

$$\int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x < \infty$$

for all $v \in \widetilde{L^{\Upsilon^*}}(\mathbb{R}^n)$. The set $L^\Upsilon(\mathbb{R}^n)$ is a Banach space when equipped with the Orlicz norm

$$\|u\|_{(L^\Upsilon)} := \sup_{\rho(v;\Upsilon^*)\leq 1} \left| \int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x \right|.$$

However, for our purposes, it will be more useful the equivalent ([18, Equation (9.24)]) Luxemburg norm, defined as [3]

$$\|u\|_{L^\Upsilon} := \inf\left\{ k>0\colon \int_{\mathbb{R}^n} \Upsilon\left(\frac{u(x)}{k}\right)\,\mathrm{d}x \leq 1 \right\}.$$

A fundamental tool we need from the Orlicz spaces theory is the following Hölder inequality.

Lemma 1 ([18, Theorem 9.3 and Equation (9.24)])**.** *The inequality*

$$\left| \int_{\mathbb{R}^n} u(x)v(x)\,dx \right| \leq 4\,\|u\|_{L^\Upsilon}\,\|v\|_{L^{\Upsilon^*}}$$

holds for any pair of functions $u \in L^\Upsilon(\mathbb{R}^n)$, $v \in L^{\Upsilon^*}(\mathbb{R}^n)$.

The above is generally true for any N-function. Let us come now to the properties of our specific function Υ given in (4.9). Note that, in the simplest case $\gamma = 2$, we have $\Upsilon(p) = \frac{|p|^2}{2}$. In the general case $\gamma > 1$, it is clearly seen that $\Upsilon(p) \sim \frac{|p|^\gamma}{\gamma}$ when $|p| \to +\infty$. On the other side, when $p \to 0$, by Taylor's theorem it holds

$$\Upsilon(p) = \frac{\gamma-1}{2}|p|^2 + o(|p|^2). \tag{5.4}$$

Therefore we can deduce

$$\Upsilon(p) \approx \begin{cases} |p|^2 & \text{if } |p| \leq 1, \\ |p|^\gamma & \text{if } |p| > 1, \end{cases} \tag{5.5}$$

[3] A small remark on the notation: compared with the monograph [18] by Krasnosel'skii and Rutickii, for cosmetic reasons we reverse the symbols used for the Orlicz and Luxemburg norms, since we will employ only the latter norm in our computations.

where the implicit constants in the symbol "$\approx$" depend only on γ. This relation is helpful in order to highlight the asymptotic behavior of Υ, and we will repeatedly use it to approximate the value of Υ.

We need to introduce also the following function $\Xi\colon \mathbb{R} \to [0,+\infty)$, which roughly speaking inverts the asymptotic behaviors of $\Upsilon(p)$ near $p = 0$ and near $p = \infty$:

$$\Xi(p) := \begin{cases} \dfrac{1}{\Upsilon(1/p)} & \text{if } p \neq 0, \\ 0 & \text{if } p = 0, \end{cases} \approx \begin{cases} |p|^\gamma & \text{if } |p| \leq 1, \\ |p|^2 & \text{if } |p| > 1. \end{cases} \tag{5.6}$$

It is also useful to observe that, for $1 < \gamma \leq 2$, Υ is super-multiplicative (apart from a multiplicative constant), while on the contrary it is sub-multiplicative (again apart from a multiplicative constant) when $\gamma \geq 2$.

Lemma 2. *Consider Υ defined in* (4.9)*. If $1 < \gamma \leq 2$, then*

$$\Upsilon(pq) \gtrsim \Upsilon(p)\,\Upsilon(q) \tag{5.7}$$

for any $p, q \in \mathbb{R}$, from which in particular

$$\Upsilon(pq) \lesssim \Upsilon(p)\,\Xi(q), \qquad \Xi(pq) \lesssim \Xi(p)\,\Xi(q), \tag{5.8}$$

for any $p, q \in \mathbb{R}$.

If $\gamma \geq 2$, then

$$\Upsilon(pq) \lesssim \Upsilon(p)\,\Upsilon(q) \tag{5.9}$$

for any $p, q \in \mathbb{R}$, from which in particular

$$\Upsilon(pq) \gtrsim \Upsilon(p)\,\Xi(q), \qquad \Xi(pq) \gtrsim \Xi(p)\,\Xi(q),$$

for any $p, q \in \mathbb{R}$.

All the implicit constants depend only on γ.

Proof. The proof is naïf: we just need to use (5.5). Without loss of generality we can assume $|p| \leq |q|$. Firstly suppose $1 < \gamma \leq 2$. There are four cases:

(i) if $|p| \leq |q| \leq 1$, then in particular $|pq| \leq 1$, thus

$$\Upsilon(p)\Upsilon(q) \approx |p|^2|q|^2 \approx \Upsilon(pq);$$

(ii) if $|p| \leq 1 < |q|$ and $|pq| \leq 1$, then

$$\Upsilon(p)\Upsilon(q) \approx |p|^2|q|^\gamma \approx \Upsilon(pq)|q|^{-(2-\gamma)} \leq \Upsilon(pq);$$

(iii) if $|p| \leq 1 < |q|$ and $|pq| > 1$, then

$$\Upsilon(p)\Upsilon(q) \approx |p|^2|q|^\gamma \approx \Upsilon(pq)|p|^{2-\gamma} \leq \Upsilon(pq);$$

(iv) if $1 < |p| \leq |q|$, then in particular $|pq| > 1$, thus

$$\Upsilon(p)\Upsilon(q) \approx |p|^\gamma|q|^\gamma \approx \Upsilon(pq).$$

The relation on the right of (5.8) follows directly from (5.7) and the definition (5.6) of Ξ, whereas the relation on the left follows by substituting p and q with pq and $1/q$ respectively in (5.7). The case $\gamma > 2$ is completely analogous. $\square$

Remark 5.1. Note that the super-/sub-multiplicative relations above in general can not be inverted if $\gamma \neq 2$, in the sense that (5.9) does not hold if $1 < \gamma < 2$, whereas (5.7) does not hold if $\gamma > 2$. In the special case $\gamma = 2$ instead, we have $\Upsilon(pq) = 2\Upsilon(p)\Upsilon(q)$ for any $p, q \in \mathbb{R}$.

An immediate implication of Lemma 2 is that, if $p \lesssim q$, then $\Upsilon(p) \lesssim \Upsilon(q)$ and $\Xi(p) \lesssim \Xi(q)$. Another consequence is the following observation (which we upgrade at the status of lemma for convenience).

Lemma 3. *Consider Υ defined in* (4.9) *and let $u \in L^{\Upsilon}(\mathbb{R}^n)$. If*

$$\int_{\mathbb{R}^n} \Upsilon\left(\frac{u(x)}{k}\right)\, dx \leq \kappa_0$$

for some $k > 0$ and $\kappa_0 > 0$, then

$$\|u\|_{L^{\Upsilon}} \leq c_{\gamma,\kappa_0} k,$$

where c_{γ,κ_0} is a positive constant depending only on γ and κ_0.

In the following, when we write Υ^{-1}, $(\Upsilon^*)^{-1}$ or Ξ^{-1}, we mean the inverse function of Υ, Υ^* or Ξ respectively, when restricted on the non-negative interval $[0, +\infty)$.

Proof of Lemma 3. From Lemma 2 there exists some positive constant d_γ, depending only on γ, such that $\Upsilon(pq) \leq d_\gamma \Upsilon(p)\, \Xi(q)$ if $1 < \gamma \leq 2$. Set

$$c_{\gamma,\kappa_0} := \left[\Xi^{-1}\left(\frac{1}{d_\gamma \kappa_0}\right)\right]^{-1}.$$

Then

$$1 \geq \int_{\mathbb{R}^n} \Upsilon\left(\frac{u}{k}\right) \frac{1}{\kappa_0}\, \mathrm{d}x = d_\gamma \int_{\mathbb{R}^n} \Upsilon\left(\frac{u}{k}\right) \Xi\left(\frac{1}{c_{\gamma,\kappa_0}}\right) \mathrm{d}x \geq \int_{\mathbb{R}^n} \Upsilon\left(\frac{u}{c_{\gamma,\kappa_0} k}\right) \mathrm{d}x$$

and hence, by definition of Luxemburg norm, we have that $c_{\gamma,\kappa_0} k$ is an upper bound for $\|u\|_{L^{\Upsilon}}$. Employing instead $\Upsilon(pq) \leq \widetilde{d_\gamma} \Upsilon(p)\, \Upsilon(q)$, one can prove the case $\gamma \geq 2$ in the same way. $\square$

5.2. Proof of estimate (5.1)

Finally we are ready to prove (5.1), and consequently obtain the key inequality (4.8). From now on, we will assume $1 < \gamma \leq 2$. Indeed, from (5.5) it is easily seen that $\Upsilon(p) \gtrsim |p|^2$ if $\gamma > 2$, and thus this case can be reduced to the case $\gamma = 2$. From the inequality (5.3), the support property (3.1) and the Hölder inequality in Lemma 1, we obtain, for a fixed positive $\alpha \in (0, 2)$ to be chosen later, that

$$\begin{aligned}
|F(t)| &\leq \int_{\mathbb{R}^n} |\rho - 1| \psi \,\mathrm{d}x \\
&\approx \int_{\mathbb{R}^n} \left[\psi^{1-\alpha} (\Upsilon^*)^{-1}(\psi^\alpha)\right] \left[|\rho - 1| \Upsilon^{-1}(\psi^\alpha)\right] \mathrm{d}x \\
&\approx \int_{\mathbb{R}^n} \psi^{1-\frac{\alpha}{2}} \left[|\rho - 1| \Upsilon^{-1}(\psi^\alpha)\right] \mathrm{d}x \\
&\lesssim \left\|\psi^{1-\frac{\alpha}{2}}\right\|_{L^{\Upsilon^*}(|x| \leq 1+t)} \left\||\rho - 1| \Upsilon^{-1}(\psi^\alpha)\right\|_{L^{\Upsilon}(\mathbb{R}^n)}.
\end{aligned} \tag{5.10}$$

In the penultimate relation we used

$$\psi^{1-\alpha}(\Upsilon^*)^{-1}\left(\psi^\alpha\right) \approx \psi^{1-\frac{\alpha}{2}}$$

which follows from $\psi \lesssim 1$ (due to (4.4)) and the fact that $(\Upsilon^*)^{-1}(p) \approx |p|^{1/2}$ if p is small.

Let us consider $\left\|\psi^{1-\frac{\alpha}{2}}\right\|_{L^{\Upsilon^*}(|x|\le 1+t)}$. Note that, thanks to (4.4) and (4.5), we have

$$\frac{\psi^{1-\alpha/2}}{\left(\int_{|x|\le 1+t}\psi^{2-\alpha}\,\mathrm{d}x\right)^{1/2}} \lesssim \langle t\rangle^{-\frac{n-1}{2}} \lesssim 1$$

for $|x| \le 1+t$, and therefore, since $\Upsilon^*(p) \approx |p|^2$ for small $|p|$,

$$\int_{|x|\le 1+t} \Upsilon^*\left(\frac{\psi^{1-\alpha/2}}{\left(\int_{|x|\le 1+t}\psi^{2-\alpha}dx\right)^{1/2}}\right)\mathrm{d}x \lesssim 1.$$

Hence, by Lemma 3 and using again (4.5), we gain

$$\left\|\psi^{1-\frac{\alpha}{2}}\right\|_{L^{\Upsilon^*}(|x|\le 1+t)} \lesssim \left(\int_{|x|\le 1+t}\psi^{2-\alpha}\,\mathrm{d}x\right)^{1/2} \lesssim \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\alpha}{2}}.$$

Let us insert this information back into (5.10) to obtain

$$|F(t)| \lesssim \left\||\rho-1|\Psi\right\|_{L^{\Upsilon}(\mathbb{R}^n)}, \qquad \Psi \equiv \Psi(t,x) := \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\alpha}{2}}\,\Upsilon^{-1}(\psi^\alpha), \tag{5.11}$$

where we used the homogeneity of the norm. From (4.4) we easily have

$$\Psi \approx \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\alpha}{2}}\,\psi^{\alpha/2} \lesssim \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\alpha}{2}}\,\langle t\rangle^{-\frac{n-1}{2}\cdot\frac{\alpha}{2}} = 1$$

for $|x| \le 1+t$. From (5.6), it follows that

$$\Xi(\Psi) \approx \Psi^\gamma \approx \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\gamma}{2}\alpha}\,\psi^{\frac{\gamma}{2}\alpha}.$$

Employing the above estimate, the relations in (5.8) and the compactness of the support of $\rho-1$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n}\Upsilon\left(\frac{|\rho-1|\Psi}{k}\right)\mathrm{d}x &\lesssim \int_{|x|\le 1+t}\Upsilon(\rho-1)\,\Xi\left(\frac{\Psi}{k}\right)\mathrm{d}x \\
&\lesssim \int_{|x|\le 1+t}\frac{\Upsilon(\rho-1)}{\Upsilon(k)}\,\Xi(\Psi)\,\mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^n}\frac{\Upsilon(\rho-1)\psi^{\frac{\gamma}{2}\alpha}}{\langle t\rangle^{-\frac{n-1}{2}\cdot\frac{\gamma}{2}\alpha}\,\Upsilon(k)}\,\mathrm{d}x
\end{aligned}$$

for any $k>0$. Plugging into the above inequality the choices

$$\alpha = \frac{2}{\gamma}, \qquad k = \Upsilon^{-1}\left(\langle t\rangle^{\frac{n-1}{2}}\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\,\mathrm{d}x\right),$$

we deduce

$$\int_{\mathbb{R}^n} \Upsilon\left(\frac{|\rho-1|\Psi}{\Upsilon^{-1}\left(\langle t\rangle^{\frac{n-1}{2}}\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\,\mathrm{d}x\right)}\right)\mathrm{d}x \lesssim 1$$

and so, by Lemma 3, we have

$$\||\rho-1|\Psi\|_{L^\Upsilon(\mathbb{R}^n)} \lesssim \Upsilon^{-1}\left(\langle t\rangle^{\frac{n-1}{2}}\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\right),$$

inserting which into (5.11) eventually gives us the inequality (5.1).

Before going on, we list some observations.

Remark 5.2. The choice of $\alpha = 2/\gamma$ is of course forced from the fact that we need to recover the expression $\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\,\mathrm{d}x$, and this explain the maybe strange factorization $\psi \approx \psi^{1-\alpha}(\Upsilon^*)^{-1}(\psi^\alpha)\Upsilon^{-1}(\psi^\alpha)$ employed in (5.10). Note also that $\alpha = 2/\gamma$ lies in $(0,2)$ for any $\gamma > 1$.

Remark 5.3. In (5.11), taking advantage of the homogeneity of the norm to squeeze the estimate for $\left\|\psi^{1-\frac{\alpha}{2}}\right\|_{L^{\Upsilon^*}(|x|\le 1+t)}$ inside $\||\rho-1|\Psi\|_{L^\Upsilon(\mathbb{R}^n)}$ is a necessary step in order to not lose information. Indeed, if we do not do that, we would have the estimate

$$|F(t)| \lesssim \langle t\rangle^{\frac{n-1}{2}\cdot\frac{\alpha}{2}}\left\||\rho-1|\Upsilon^{-1}(\psi^\alpha)\right\|_{L^\Upsilon(\mathbb{R}^n)} \tag{5.12}$$

and, proceeding as above, with Ψ replaced by $\Upsilon^{-1}(\psi^\alpha) \approx \psi^{\alpha/2}$, we would obtain

$$\left\||\rho-1|\Upsilon^{-1}(\psi^\alpha)\right\|_{L^\Upsilon(\mathbb{R}^n)} \lesssim \Upsilon^{-1}\left(\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\,\mathrm{d}x\right)$$

with again the choice $\alpha = 2/\gamma$. At this point, since Υ is not sub-multiplicative for $1<\gamma<2$, when we return to (5.12) we would be forced to employ again (5.8), getting

$$\Upsilon(F) \lesssim \langle t\rangle^{\frac{n-1}{\gamma}}\int_{\mathbb{R}^n}\Upsilon(\rho-1)\psi\,\mathrm{d}x,$$

which is a worst estimate respect to (5.1) for $1<\gamma<2$.

6. Proof of Theorem 1

Now that we have (4.8) at our disposal, let us consider the multiplier

$$m \equiv m(t) := \exp\left(\mu\frac{(1+t)^{1-\lambda}}{1-\lambda}\right) \tag{6.1}$$

introduced in [21], which solves the ordinary differential equation

$$\frac{m'(t)}{m(t)} = \frac{\mu}{(1+t)^\lambda} \tag{6.2}$$

and satisfies the bounds

$$0 < \boldsymbol{m}(0) \leq \boldsymbol{m}(t) \leq 1 \tag{6.3}$$

for $\mu \geq 0$ and $\lambda > 1$. Its role is to "absorb" the damping term. Indeed, adding $\frac{\mu}{(1+t)^\lambda} F$ on both side of (4.8) and multiplying by $\boldsymbol{m}(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{m}(t)\left[F'(t)+2F(t)\right]\right\} \gtrsim \frac{\mu}{(1+t)^\lambda}\boldsymbol{m}(t)F(t) + \langle t\rangle^{-\frac{n-1}{2}}\boldsymbol{m}\Upsilon(F(t)). \tag{6.4}$$

We will soon prove that actually F is non-negative, so we can get rid also of the additional first term on the right-hand side. After an integration with respect to the time we have

$$\begin{aligned}
\boldsymbol{m}(t)[F'(t)+2F(t)] \gtrsim\, & \boldsymbol{m}(0)[F'(0)+2F(0)] \\
& + \int_0^t \frac{\mu}{(1+s)^\lambda}\boldsymbol{m}(s)F(s)\,\mathrm{d}s \\
& + \int_0^t \langle s\rangle^{-\frac{n-1}{2}}\boldsymbol{m}(s)\Upsilon(F(s))\,\mathrm{d}s.
\end{aligned}$$

Multiplying now by $\frac{e^{2t}}{\boldsymbol{m}(t)}$, integrating again with respect to the time, and then multiplying by e^{-2t}, we aim to

$$\begin{aligned}
F(t) \gtrsim\, & F(0)e^{-2t} + \boldsymbol{m}(0)[F'(0)+2F(0)]e^{-2t}\int_0^t \frac{e^{2s}}{\boldsymbol{m}(s)}\,\mathrm{d}s \\
& + e^{-2t}\int_0^t \frac{e^{2s}}{\boldsymbol{m}(s)}\int_0^s \frac{\mu}{(1+r)^\lambda}\boldsymbol{m}(r)F(r)\,\mathrm{d}r\,\mathrm{d}s \\
& + e^{-2t}\int_0^t \frac{e^{2s}}{\boldsymbol{m}(s)}\int_0^s \langle r\rangle^{-\frac{n-1}{2}}\boldsymbol{m}(r)\Upsilon(F(r))\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned} \tag{6.5}$$

From this expression we can prove that $F(t) > 0$ for $t \geq 0$, employing a standard comparison argument. Indeed, due to the fact that $F(0) > 0$ by our assumption on the initial data and that F is continuous, we know that $F(t) > 0$ at least for small $t \geq 0$. Assume by contradiction that $t_0 > 0$ is the smallest zero point of F; therefore, setting $t = t_0$ in (6.5) and using also that $F'(0) + 2F(0) \geq 0$ due to (4.7), we get $0 = F(t_0) \gtrsim F(0)e^{-2t_0}$ — a contradiction. Thus $F(t) > 0$ for any $t \geq 0$.

Thanks to this information we can suppress the third term in the right-hand side of (6.5), and using also $\boldsymbol{m}(t) \approx 1$ (due to (6.3)) we have now

$$\begin{aligned}
F(t) \gtrsim\, & F(0)e^{-2t} + [F'(0)+2F(0)]\frac{1-e^{-2t}}{2} \\
& + e^{-2t}\int_0^t e^{2s}\int_0^s \langle r\rangle^{-\frac{n-1}{2}}\Upsilon(F(r))\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned} \tag{6.6}$$

Morally, we would like now to "differentiate" the above estimate to obtain a differential inequality like (4.8) but without the damping term. Let us introduce the auxiliary function $\overline{F} \equiv \overline{F}(t)$ defined by

$$\begin{aligned}
\overline{F}(t) :=\, & \frac{F(0)}{2}e^{-2t} + [F'(0)+2F(0)]\frac{1-e^{-2t}}{2} \\
& + e^{-2t}\int_0^t e^{2s}\int_0^s \langle r\rangle^{-\frac{n-1}{2}}\Upsilon(F(r))\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

From its definition and (6.6), it holds

$$F(t) \gtrsim \frac{F(0)}{2} e^{-2t} + \overline{F}(t) \geq \overline{F}(t) > 0.$$

It is easy to check, multiplying firstly by e^{2t} and deriving, and then multiplying by e^{-2t} and deriving again, that

$$\overline{F}''(t) + 2\overline{F}'(t) = \langle t \rangle^{-\frac{n-1}{2}} \Upsilon(F(t)) \gtrsim \langle t \rangle^{-\frac{n-1}{2}} \Upsilon\left(\overline{F}(t)\right).$$

Moreover

$$\begin{aligned} \overline{F}(0) &= \frac{F(0)}{2} > 0, \\ \overline{F}'(0) &= F'(0) + F(0) > 0. \end{aligned}$$

At this point, recalling (5.5), the conclusion of the proof follows from a straightforward application of the next lemma, which is a variation of Theorem 3.1 in [24], with a non-linear term which is allowed to behave like two different powers when its argument is respectively small or large. Since the proof of the lemma follows step by step the one in [24] with minor changes, we include the demonstration for the sake of completeness but we postpone it in Appendix A.

Lemma 4. *Let $0 \leq \lambda \leq 1$. Assume that $I \in C^2([0, +\infty); \mathbb{R})$ satisfies*

$$I''(t) + I'(t) \gtrsim (1+t)^{-\lambda} N(I(t)) \tag{6.7}$$

where $N(p), N'(p) > 0$ for $p > 0$ and

$$N(p) \approx \begin{cases} p^{1+\alpha} & \text{if } 0 \leq p \leq 1, \\ p^{1+\beta} & \text{if } p > 1, \end{cases}$$

for some $\alpha, \beta > 0$. Suppose also

$$I(0) = \varepsilon > 0, \qquad I'(0) \geq 0.$$

Then, $I(t)$ blows up in a finite time. Moreover, if $\varepsilon > 0$ is small enough, the lifespan T_ε of $I(t)$ satisfies the upper bound

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\dfrac{\alpha}{1-\lambda}} & \text{if } 0 \leq \lambda < 1, \\ \exp(C\varepsilon^{-\alpha}) & \text{if } \lambda = 1, \end{cases}$$

where C is a positive constant dependent on α, β, λ, but independent of ε.

7. Proof of Theorem 2

First of all, notice that, in the case $\lambda = 1$, the solution to the ODE (6.2) is given by

$$m \equiv m(t) = (1+t)^{\mu},$$

which is unbounded for $\mu > 0$. Anyway, the inequality (6.4) still holds but with $\lambda = 1$, and so also (6.5). Hence, with the same comparison argument as in the previous section, we can deduce that

$$F(t) := \int_{\mathbb{R}^n} (\rho - 1)\psi \, \mathrm{d}x > 0$$

for $t \geq 0$. The role of m as multiplier in the case $\lambda = 1$ was only to prove the positivity of F. With this information at hand, let us go back to (4.8) and this time we use as multiplier $\sqrt{m(t)} = (1+t)^{\mu/2}$. Define the functional

$$G(t) := \sqrt{m(t)} F(t) = (1+t)^{\mu/2} F(t),$$

which of course inherits from F its positiveness and the same blow-up dynamic. Multiplying both side of (4.8) by $\sqrt{m}$, we obtain

$$G'' + 2G' + \frac{\mu(2-\mu)/4}{(1+t)^2} G \gtrsim \langle t \rangle^{-\frac{n-1}{2}+\frac{\mu}{2}} \Upsilon(F) \gtrsim \langle t \rangle^{-\frac{n+\mu-1}{2}} \Upsilon(G), \tag{7.1}$$

where we used also (5.7) and that $\Upsilon((1+t)^{-\mu/2}) \approx (1+t)^{-\mu}$. The use of $\sqrt{m}$ is connected to a Liouville-type transform, often employed in the study of the scaling-invariant damped wave equation. For example, D'Abbicco, Lucente and Reissig in [7] inaugurated the beginning of a series of works by various authors where the case $\mu = 2$ is considered. This is due to the fact that this choice simplifies the analysis of the problem, making it related to the undamped wave equation. In our case, setting $\mu = 2$ would eliminate the third term in the left-hand side of (7.1). However, since we are dealing with $\mu \leq 3 - n$, we need another way to suppress the massive term in (7.1).

Let us introduce the new multiplier defined by

$$\ell \equiv \ell(t) := \exp\left(-\frac{\mu(2-\mu)/8}{1+t}\right).$$

Observe that ℓ solves the ODE

$$\frac{\ell'(t)}{\ell(t)} = \frac{\mu(2-\mu)/8}{(1+t)^2}$$

and satisfies the bounds

$$0 < \ell(0) \leq \ell(t) \leq 1, \tag{7.2}$$

since $0 \leq \mu \leq 3 - n \leq 2$ for $n \in \{1, 2\}$. Multiplying (7.1) by $\ell(t)$ we obtain

$$\ell G'' + 2(\ell G)' \gtrsim \ell \, \langle t \rangle^{-\frac{n+\mu-1}{2}} \Upsilon(G)$$

and hence

$$(\ell G)'' + 2(\ell G)' \gtrsim \ell'' G + 2\ell' G' + \ell \, \langle t \rangle^{-\frac{n+\mu-1}{2}} \Upsilon(G). \tag{7.3}$$

We would like to get rid now of the first two terms in the right-hand side of the above inequality. Let us consider another multiplier ϖ, defined by

$$\varpi(t) := (1+t) \exp\left(\frac{\mu(2-\mu)/16}{1+t}\right),$$

which satisfies the ODE

$$\frac{\varpi'(t)}{\varpi(t)} = -\frac{1}{2}\frac{\ell''(t)}{\ell'(t)} = \frac{1}{1+t} - \frac{\mu(2-\mu)/16}{(1+t)^2}.$$

It is straightforward to check that

$$\ell'' G + 2\ell' G' = \frac{2}{\varpi}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varpi \ell' G\right\},$$

and so, integrating by parts the above identity, it holds

$$\int_0^t \left[\ell'' G + 2\ell' G'\right] \mathrm{d}s = 2\ell'(t)G(t) - 2\ell'(0)G(0) + 2\int_0^t \frac{\varpi'}{\varpi}\ell' G \,\mathrm{d}s.$$

Noting that G, ℓ' and $\frac{\varpi'}{\varpi}$ are positive functions, we have

$$\int_0^t \left[\ell'' G + 2\ell' G'\right] \mathrm{d}s \geq -2\ell'(0)G(0) = -\frac{\mu(2-\mu)}{4}\ell(0)F(0). \tag{7.4}$$

Integrating (7.3) with respect to the time and taking into account (7.4), it follows

$$(\ell G)'(t) + 2(\ell G)(t) \gtrsim g_0 + \int_0^t \ell(s)\,\langle s\rangle^{-\frac{n+\mu-1}{2}}\,\Upsilon(G(s))\,\mathrm{d}s, \tag{7.5}$$

where

$$\begin{aligned} g_0 &:= (\ell G)'(0) + 2(\ell G)(0) - \frac{\mu(2-\mu)}{4}\ell(0)F(0) \\ &= \ell(0)\left[F'(0) + F(0) + \frac{\mu^2+2\mu+8}{8}F(0)\right] > 0. \end{aligned}$$

Multiplying (7.5) by e^{2t}, integrating and multiplying by e^{-2t}, we get

$$\begin{aligned} \ell(t)G(t) \gtrsim\, & \ell(0)G(0)e^{-2t} + g_0\frac{1-e^{-2t}}{2} \\ & + e^{-2t}\int_0^t e^{2s}\int_0^s \ell(r)\,\langle r\rangle^{-\frac{n+\mu-1}{2}}\,\Upsilon(G(r))\,\mathrm{d}r\,\mathrm{d}s, \end{aligned}$$

and equivalently, using (7.2),

$$\begin{aligned} G(t) \gtrsim\, & F(0)e^{-2t} + \left[F'(0) + \frac{\mu^2+2\mu+16}{8}F(0)\right]\frac{1-e^{-2t}}{2} \\ & + e^{-2t}\int_0^t e^{2s}\int_0^s \langle r\rangle^{-\frac{n+\mu-1}{2}}\,\Upsilon(G(r))\,\mathrm{d}r\,\mathrm{d}s. \end{aligned}$$

At this stage, the conclusion of the proof follows exactly as that in Section 6. Namely, introduce the auxiliary function $\overline{G} \equiv \overline{G}(t)$ defined by

$$\begin{aligned} \overline{G}(t) :=\, & \frac{F(0)}{2}e^{-2t} + \left[F'(0) + \frac{\mu^2+2\mu+16}{8}F(0)\right]\frac{1-e^{-2t}}{2} \\ & + e^{-2t}\int_0^t e^{2s}\int_0^s \langle r\rangle^{-\frac{n+\mu-1}{2}}\,\Upsilon(G(r))\,\mathrm{d}r\,\mathrm{d}s \end{aligned}$$

and note that

$$G(t) \gtrsim \frac{F(0)}{2}e^{-2t} + \overline{G}(t) \geq \overline{G}(t) > 0.$$

Moreover $\overline{G}$ satisfies

$$\overline{G}''(t) + 2\overline{G}'(t) = \langle t\rangle^{-\frac{n+\mu-1}{2}}\Upsilon(G(t)) \gtrsim \langle t\rangle^{-\frac{n+\mu-1}{2}}\Upsilon(\overline{G}(t)),$$

together with

$$\begin{aligned}\overline{G}(0) &= \frac{F(0)}{2} > 0\\ \overline{G}'(0) &= F'(0) + F(0) + \frac{\mu(\mu+2)}{8}F(0) > 0.\end{aligned}$$

Another application of Lemma 4 concludes our proof.

A. Proof of Lemma 4

Before starting the proof, we need the following two lemmata, which are a slight generalization of Lemmata 3.1, 3.1' and 3.2 in [24]. The proof is exactly the same as the one by Li and Zhou, based on a simple comparison argument, so we will not repeat it. In the referred paper, $N(p) = p^{1+\alpha}$ for some $\alpha > 0$, but in the end the only property exploited is the fact that N is positive and increasing on the positive interval.

Lemma 5. *Consider two functions $h, k \in C^2([0,+\infty);\mathbb{R})$ satisfying*

$$\begin{aligned}a(t)h''(t) + h'(t) &\leq b(t)N(h(t))\\ a(t)k''(t) + k'(t) &\geq b(t)N(k(t))\end{aligned}$$

for any $t \geq 0$, where

$$a(t) > 0, \qquad b(t) > 0,$$

for $t \geq 0$ and

$$N(p) > 0, \qquad N'(p) > 0,$$

for $p > 0$. Suppose also that one of the following couples of assumptions on the initial data holds true:

$$k(0) > h(0), \qquad k'(0) \geq h'(0),$$

or

$$k(0) \geq h(0), \qquad k'(0) > h'(0).$$

Then it holds

$$k'(t) > h'(t)$$

for any $t > 0$.

Lemma 6. *Consider a function $h \in C^2([0,+\infty);\mathbb{R})$ satisfying*

$$a(t)h''(t) + h'(t) = cN(h(t))$$

for any $t \geq 0$, where c is a positive constant and a, N satisfy the same assumptions as in Lemma 5. Suppose also that

$$h(0) > 0, \qquad h'(0) = 0.$$

Then it holds

$$h''(t) > 0$$

for any $t \geq 0$.

Proof of Lemma 4. First of all, note that if $\alpha < \beta$ then $N(p) \gtrsim p^{1+\alpha}$, so we can reduce to the case $\alpha = \beta$. We will consider then only the case $\alpha \geq \beta$. Moreover, it is not restrictive to assume $I'(0) > 0$. If indeed $I(0) = 0$, then we have $I''(0) \gtrsim N(I(0)) > 0$, from which it follows that $I(t), I'(t)$ and $I''(t)$ are positive for small t. In particular, there exists a small $t_0 > 0$ such that $I(t_0), I'(t_0) > 0$, so we can choose this t_0 as initial time.

Case $\lambda = 0$. Let us firstly consider the auxiliary problem

$$\begin{cases} J'(t) = \eta M(J(t)) \\ J(0) = J_0, \end{cases} \tag{A.1}$$

where $\eta, J_0 > 0$ and

$$M(p) := \begin{cases} p^{1+\alpha/2} & \text{if } 0 \leq p \leq 1, \\ p^{1+\beta/2} & \text{if } p > 1. \end{cases}$$

Define the function $\mathscr{M} \colon [0, +\infty) \to \mathbb{R}$ as

$$\mathscr{M}(p) := \int_p^{+\infty} \frac{\mathrm{d}q}{M(q)},$$

and note that this is a strictly positive and strictly decreasing function such that $\lim_{p\to+\infty} \mathscr{M}(p) = 0^+$ (and so, vice versa, $\lim_{p\to 0^+} \mathscr{M}^{-1}(p) = +\infty$). Then, we immediately have that

$$J(t) = \mathscr{M}^{-1}\left(\mathscr{M}(J_0) - \eta t\right)$$

blows up in finite time.

Let us go back to our original problem for $\lambda = 0$. We want to show the blow-up in finite time of I satisfying

$$I''(t) + I'(t) \gtrsim N(I(t)). \tag{A.2}$$

To this aim, it will be sufficient to show that $I(t) > J(t)$ in the existence domain, where J solves (A.1). Additionally, assume now

$$J(0) < I(0) \tag{A.3}$$

and $\eta < I'(0)/M(J_0)$ so that

$$J'(0) = \eta M(J(0)) < I'(0). \tag{A.4}$$

We have that

$$J''(t) + J'(t) = \eta[\eta M'(J(t)) + 1] M(J(t)) = O(J(t))\, N(J(t)), \tag{A.5}$$

where, using the definition of M and the assumptions on N,

$$
\begin{aligned}
O(p) &:= \eta[\eta M'(p)+1]\frac{M(p)}{N(p)} \\
&\sim \begin{cases} \eta^2(1+\alpha/2)+\eta p^{-\alpha/2} & \text{if } p\to 0^+ \\ \eta^2(1+\beta/2)+\eta p^{-\beta/2} & \text{if } p\to +\infty \end{cases} \\
&\lesssim 1+p^{-\alpha/2}+p^{-\beta/2}.
\end{aligned}
$$

Since J is positive and increasing in its existence domain, from (A.5) we have

$$
J''(t)+J'(t)\lesssim (1+J_0^{-\alpha/2}+J_0^{-\beta/2})N(J(t))\lesssim N(J(t)). \tag{A.6}
$$

With (A.2), (A.3), (A.4), (A.6) at hand, an application of Lemma 5 gives that $I'(t)>J'(t)$ for $t\geq 0$, and hence by (A.3) we get $I(t)>J(t)$ for $t\geq 0$, proving the blow-up in finite time of I.

To get the estimate for the lifespan, a scaling argument is required. Firstly, note that, following the proof of Lemma 2, one can easily see that N is super-multiplicative, apart for a multiplicative constant, for $\alpha\geq\beta$, namely:

$$
N(pq)\gtrsim N(p)N(q) \tag{A.7}
$$

for $p,q\geq 0$. Thus, defining

$$
K(t)=\varepsilon^{-1}I(\varepsilon^{-\alpha}t),
$$

we get, from (A.2) and (A.7),

$$
\varepsilon^{\alpha}K''(t)+K'(t)\gtrsim \varepsilon^{-1-\alpha}N(\varepsilon K(t))\gtrsim \varepsilon^{-1-\alpha}N(\varepsilon)N(K(t))\gtrsim N(K(t)),
$$

where we used also that $\varepsilon>0$ is small together with the assumptions on N. Thus, for some $c_0>0$, we have that K satisfy

$$
\begin{cases} \varepsilon^{\alpha}K''(t)+K'(t)\geq c_0N(K(t)) \\ K(0)=1, \qquad K'(0)=\varepsilon^{-1-\alpha}I'(0)>0. \end{cases}
$$

Consider the auxiliary function $\overline{K}$ satisfying

$$
\begin{cases} \varepsilon^{\alpha}\overline{K}''(t)+\overline{K}'(t)=c_0N(K(t)) \\ \overline{K}(0)=1, \qquad \overline{K}'(0)=0. \end{cases}
$$

By Lemma 6 it holds $\overline{K}''(t)>0$ for $t\geq 0$, and so $\overline{K}$ satisfies

$$
\begin{cases} \overline{K}''(t)+\overline{K}'(t)\gtrsim N(K(t)) \\ \overline{K}(0)=1, \qquad \overline{K}'(0)=0. \end{cases}
$$

Therefore, according to the previous discussion, $\overline{K}$ must blow up in a finite time which does not depend on ε. By Lemma 5 we have $K'(t)>\overline{K}'(t)$ for $t\geq 0$, and so $K(t)>\overline{K}(t)$ for $t>0$.

Then also $I(\varepsilon^{-\alpha}t)$ blows up in a finite time independent of ε, from which it follows that the lifespan estimate for I satisfies $T_\varepsilon \lesssim \varepsilon^{-\alpha}$ as desired.

Case $0 < \lambda \leq 1$. Let us define

$$J(t) := \begin{cases} I((1+t)^{1/(1-\lambda)} - 1) & \text{if } 0 < \lambda < 1, \\ I(e^t - 1) & \text{if } \lambda = 1, \end{cases}$$

for which it holds, thanks to (6.7),

$$\begin{cases} a(t)J''(t) + b(t)J'(t) \gtrsim N(J(t)) \\ J(0) = \varepsilon > 0, \qquad J'(0) > 0. \end{cases}$$

where

$$a(t) := \begin{cases} (1-\lambda)^2(1+t)^{-\frac{2\lambda}{1-\lambda}} & \text{if } 0 < \lambda < 1, \\ e^{-t} & \text{if } \lambda = 1, \end{cases}$$

$$b(t) := \begin{cases} (1-\lambda)(1+t)^{-\frac{\lambda}{1-\lambda}} \left[1 - \lambda(1+t)^{-\frac{1}{1-\lambda}}\right] & \text{if } 0 < \lambda < 1, \\ 1 - e^{-t} & \text{if } \lambda = 1, \end{cases}$$

and notice that

$$0 < a(t),\, b(t) \leq 1.$$

Choosing $k = I$ and $h \equiv 0$ in Lemma 5, we get $I'(t) > 0$ for $t > 0$, and so also $J'(t) > 0$ for $t > 0$, from which

$$a(t)J''(t) + J'(t) \geq c_0 N(J(t))$$

for some constant $c_0 > 0$. If we consider the auxiliary function $\overline{J}$ satisfying

$$\begin{cases} a(t)\overline{J}''(t) + \overline{J}'(t) = c_0 N(\overline{J}(t)) \\ \overline{J}(0) = \varepsilon/2 > 0, \qquad \overline{J}'(0) = 0, \end{cases}$$

by Lemma 5 we have $J'(t) > \overline{J}'(t)$ for $t > 0$, and so $J(t) > \overline{J}(t)$ for $t \geq 0$. By Lemma 6 instead we have $\overline{J}''(t) > 0$ for $t \geq 0$, and so

$$\overline{J}''(t) + \overline{J}'(t) \gtrsim N(\overline{J}(t)).$$

We can apply the case $\lambda = 0$ to $\overline{J}$, obtaining that $\overline{J}$ blows up, with lifespan satisfying $T_\varepsilon \lesssim \varepsilon^{-\alpha}$ if $\varepsilon > 0$ is small. The same is then true for J, and so I blows up with the lifespan estimate given in the statement of the lemma. $\square$

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series.* 1964.

[2] S. Alinhac. Temps de vie des solutions réguliéres des équations d'Euler compressibles axisymétriques en dimension deux. *Invent. Math.*, 111(3):627–670, 1993.

[3] D. Bresch. Shallow-water equations and related topics. In *Handbook of differential equations: evolutionary equations. Vol. V*, Handb. Differ. Equ., pages 1–104. Elsevier/North-Holland, Amsterdam, 2009.

[4] W. Chen, S. Lucente, and A. Palmieri. Nonexistence of global solutions for generalized Tricomi equations with combined nonlinearity. *Nonlinear Anal. Real World Appl.*, 61:103354, 2021.

[5] M. D'Abbicco. The threshold of effective damping for semilinear wave equations. *Math. Methods Appl. Sci.*, 38(6):1032–1045, 2015.

[6] M. D'Abbicco and S. Lucente. NLWE with a special scale invariant damping in odd space dimension. In *Dynamical Systems and Differential Equations, AIMS Proceedings 2015 Proceedings of the 10th AIMS International Conference (Madrid, Spain)*, volume 2015, pages 312–319. American Institute of Mathematical Sciences, 2015.

[7] M. D'Abbicco, S. Lucente, and M. Reissig. A shift in the Strauss exponent for semilinear wave equations with a not effective damping. *J. Differ. Equ.*, 259(10):5040–5073, 2015.

[8] M. Hamouda and M. A. Hamza. Blow-up and lifespan estimate for the generalized Tricomi equation with mixed nonlinearities. *Adv. Pure Appl. Math.*, 12(Special):54–70, 2021.

[9] M. Hamouda and M. A. Hamza. Improvement on the blow-up of the wave equation with the scale-invariant damping and combined nonlinearities. *Nonlinear Anal. Real World Appl.*, 59:103275, 2021.

[10] F. Hou, I. Witt, and H. Yin. Global existence and blowup of smooth solutions of 3-D potential equations with time dependent damping. *Pacific J. Math.*, 292(2):389–426, 2018.

[11] F. Hou and H. Yin. On the global existence and blowup of smooth solutions to the multi-dimensional compressible Euler equations with time-depending damping. *Nonlinearity*, 30(6):2485–2517, 2017.

[12] M. Ikeda and M. Sobajima. Life-span of solutions to semilinear wave equation with time dependent critical damping for specially localized initial data. *Math. Ann.*, 372(3):1017–1040, 2018.

[13] T. Imai, M. Kato, H. Takamura, and K. Wakasa. The lifespan of solutions of semilinear wave equations with the scale-invariant damping in two space dimensions. *J. Differ. Equ.*, 269(10):8387–8424, 2020.

[14] Z. Jin and Y. Zhou. Formation of finite-time singularities for nonlinear hyperbolic systems with small initial disturbances. *J. Math. Phys.*, 61(7):071510, 2020.

[15] M. Kato and M. Sakuraba. Global existence and blow-up for semilinear damped wave equations in three space dimensions. *Nonlinear Analysis*, 182:209–225, 2019.

[16] M. Kato, H. Takamura, and K. Wakasa. The lifespan of solutions of semilinear wave equations with the scale-invariant damping in one space dimension. *Differential Integral Equations*, 32(11/12):659–678, 2019.

[17] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58(3):181–205, 1975.

[18] M. A. Krasnosel'skii and Ya. B. Rutickii. *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen, 1961. Translated from the first Russian edition by Leo F. Boron.

[19] N.-A. Lai and N. M. Schiavone. Blow-up and lifespan estimate for generalized Tricomi equations related to Glassey conjecture. *Math. Z.*, 301(4):3369–3393, 2022.

[20] N.-A. Lai, N. M. Schiavone, and H. Takamura. Heat-like and wave-like lifespan estimates for solutions of semilinear damped wave equations via a Kato's type lemma. *J. Differ. Equ.*, 269(12):11575–11620, 2020.

[21] N.-A. Lai and H. Takamura. Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case. *Nonlinear Analysis*, 168:222–237, 2018.

[22] N.-A. Lai and H. Takamura. Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey's conjecture. *Differential Integral Equations*, 32(1/2):37–48, 2019.

[23] N.-A. Lai, H. Takamura, and K. Wakasa. Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent. *J. Differ. Equ.*, 263(9):5377–5394, 2017.

[24] T. Li and Y. Zhou. Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$. *Discrete Contin. Dynam. Systems*, 1(4):503–520, 1995.

[25] J. Lin and Z. Tu. Life-span of semilinear wave equations with scale-invariant damping: Critical Strauss exponent case. *Differential Integral Equations*, 32(5/6):249–264, 2019.

[26] M. Liu and C. Wang. Global existence for semilinear damped wave equations in relation with the Strauss conjecture. *Discrete Contin. Dyn. Syst. Ser. A*, 40(2):709–724, 2020.

[27] S. Lucente and A. Palmieri. A blow-up result for a generalized Tricomi equation with nonlinearity of derivative type. *Milan J. Math.*, 89(1):45–57, 2021.

[28] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[29] A. Palmieri. A global existence result for a semilinear scale-invariant wave equation in even dimension. *Math. Methods Appl. Sci.*, 42(8):2680–2706, 2019.

[30] X. Pan. Blow up of solutions to 1-d Euler equations with time dependent damping. *J. Math. Anal. Appl.*, 442(2):435–445, 2016.

[31] X. Pan. Global existence of solutions to 1-d Euler equations with time dependent damping. *Nonlinear Anal.*, 132:327–336, 2016.

[32] M. A. Rammaha. Formation of singularities in compressible fluids in two-space dimensions. *Proc. Am. Math. Soc.*, 107(3):705–714, 1989.

[33] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.

[34] T. C. Sideris. Formation of singularities in three-dimensional compressible fluids. *Commun. Math. Phys.*, 101(4):475–485, 1985.

[35] T. C. Sideris. The Lifespan of Smooth Solutions to the Three-Dimensional Compressible Euler Equations and the Incompressible Limit. *Indiana Univ. Math. J.*, 40(2):535–550, 1991.

[36] T. C. Sideris. The lifespan of 3D compressible flow. In *Séminaire sur les Équations aux Dérivées Partielles, 1991–1992*, pages Exp. No. V, 12. École Polytech., Palaiseau, 1992.

[37] T. C. Sideris, B. Thomases, and D. Wang. Long Time Behavior of Solutions to the 3D Compressible Euler Equations with Damping. *Comm. Partial Differential Equations*, 28(3-4):795–816, 2003.

[38] Y. Sugiyama. Singularity formation for the 1D compressible Euler equations with variable damping coefficient. *Nonlinear Analysis*, 170:70–87, 2018.

[39] Z. Tu and J. Lin. A note on the blowup of scale invariant damping wave equation with sub-Strauss exponent. 2017.

[40] K. Wakasa and B. Yordanov. On the nonexistence of global solutions for critical semilinear wave equations with damping in the scattering case. *Nonlinear Analysis*, 180:67–74, 2019.

[41] Y. Wakasugi. Critical exponent for the semilinear wave equation with scale invariant damping. In *Fourier Analysis*, volume 63, pages 375–390. Springer International Publishing, 2014.

[42] W. Wang and T. Yang. The pointwise estimates of solutions for Euler equations with damping in multi-dimensions. *J. Differ. Equ.*, 173(2):410–450, 2001.

[43] J. Wirth. Solution representations for a wave equation with weak dissipation. *Math. Methods Appl. Sci.*, 27(1):101–124, 2004.

[44] J. Wirth. Wave equations with time dependent dissipation I. Non-effective dissipation. *J. Differ. Equ.*, 222(2):487–514, 2006.

[45] J. Wirth. Wave equations with time dependent dissipation II. Effective dissipation. *J. Differ. Equ.*, 232(1):74–103, 2007.

[46] B. T. Yordanov and Q. S. Zhang. Finite time blow up for critical wave equations in high dimensions. *J. Funct. Anal.*, 231(2):361–374, 2006.