ON INVERSE CONSTRUCTION OF ISOPTICS AND ISOCHORDAL-VIEWED CURVES

DAVID ROCHERA AND MICHAEL BARTOŇ

Abstract. Given a regular closed curve α in the plane, a ϕ-isoptic of α is a locus of points from which pairs of tangent lines to α span a fixed angle ϕ. If, in addition, the chord that connects the two points delimiting the visibility angle is of constant length ℓ, then α is said to be (ϕ, ℓ)-isochordal viewed. Some properties of these curves have been studied, yet their full classification is not known. We approach the problem in an inverse manner, namely that we consider a ϕ-isoptic curve c as an input and construct a curve whose ϕ-isoptic is c. We provide thus a sufficient condition that constitutes a partial solution to the inverse isoptic problem. In the process, we also study a relation of isoptics to multihedgehogs. Moreover, we formulate conditions on the behavior of the visibility lines so as their envelope is a (ϕ, ℓ)-isochordal-viewed curve with a prescribed ϕ-isoptic c. Our results are constructive and offer a tool to easily generate this type of curves. In particular, we show examples of (ϕ, ℓ)-isochordal-viewed curves whose ϕ-isoptic is not circular. Finally, we prove that these curves allow the motion of a regular polygon whose vertices lie along the (ϕ, ℓ)-isochordal-viewed curve.

1. Introduction

Free-form planar curves possessing a tangential contact with a straight line appear in many applied disciplines, probably the most apposite example being in cam mechanisms [20]. Cam mechanisms can be found in various machines, for example as part of dwells, aircraft drift meters, or engine components, where the curve-line and/or curve-curve entity serves as an actuator-follower pair that, for example, converts a rotational motion to a translation.

There are various cam mechanisms, the one most relevant to our research is probably the double-disk cam with two rigidly connected followers, see Figure 1. There are two radial disks, non-circular planar curves, fixed rigidly together, being in a tangential contact with a flat-faced fork-like follower. Such a mechanism is used e.g. to convert a uniform speed rotation into a back-and-forth swing motion with a non-uniform speed. The geometric ground for such a mechanism is an isoptic curve that spans a fixed angle with a pair of 2D curves.

We start first with the mathematical background for an isoptic of a single curve. Given an angle $\phi \in \,]0, \pi[$, a ϕ-isoptic of a planar curve α is defined as a locus of intersection points of two distinct

Key words and phrases. Isoptic curve, Isochordal-viewed curve, Multihedgehog, Inverse isoptic problem, Regular polygon, Cam mechanism.

This is a preprint of an article published in Journal of Computational and Applied Mathematics. The final authenticated version is available online at: `https://doi.org/10.1016/j.cam.2023.115432`.

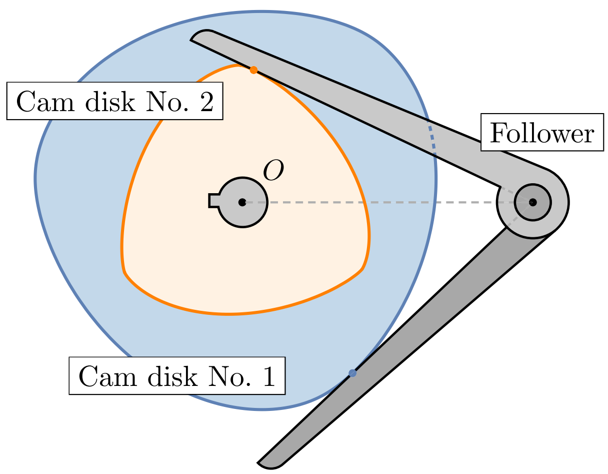


FIGURE 1. Double-disk cam mechanism with an oscillating flat-faced follower. The two curved cam disks are fixed together and rotate around a common rotation center O. The follower touches tangentially the cam disks, converting the rotational motion to a back-and-forth swing. Purely geometrically, this problem leads to a motion of a fixed angle structure along two planar curves.

tangent lines to α that span the given constant angle ϕ, see Figure 2. We refer the interested reader to [3, 4, 7] and references therein for an introduction and relevant research on isoptics theory.

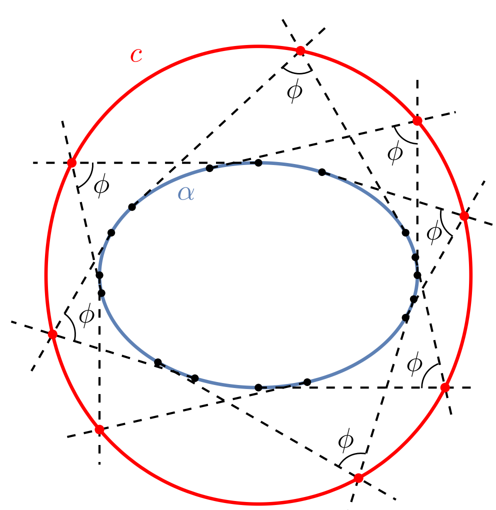


FIGURE 2. A ϕ-isoptic c to a curve α.

The parametric construction of a ϕ-isoptic of α implicitly assumes the existence of a homeomorphism f such that $\alpha(t)$ and $\alpha\big(f(t)\big)$ are the contact points of α where two tangent lines to α meet at an angle ϕ. This homeomorphism is called the Holditch function for the parameterization α and the angle ϕ and its consideration assumes that no retrograde movements are done by the moving chord that joins both contact points, see [17] for more details.

Given a closed curve α, the homeomorphism f is not unique in general and its choice would lead to different curves which can be seen all as ϕ-isoptics of α. If α is regular and convex, there are only two of such homeomorphisms, which correspond to the angles ϕ and $\pi - \phi$ and they also correspond to the angle between the tangent vectors to α at the contact points.

For general non-convex closed curves (not necessarily simple, with or without cusps), each choice of these homeomorphisms will correspond to a value ϕ or $\pi - \phi$ up to a multiple of π and, in general, it would lead to different curves that serve as a ϕ-isoptic.

Notice that from the definition of a ϕ-isoptic, we are only imposing that the pair of tangent lines to α make a constant angle ϕ, not that the tangent vectors to α that define these lines span an angle ϕ. In fact, once a homeomorphism f has been chosen, the angle between tangents could change and be either ϕ or $\pi - \phi$ for different parameter values for curves with cusps (see [17]).

If the curve α is parameterized by a support function, the allowed homeomorphisms f are just translations by the angles ϕ or $\pi - \phi$ up to a multiple of π and, thus, isoptics can be easily computed. In the case of other parameterizations, one can formulate the problem algebraically and look for the intersection point of the two tangents, see e.g. Prop. 2.1 of [17]. This typically leads to an underconstrained problem with two variables (parameters t_1 and t_2) and one constraint (the two tangent lines at $\alpha(t_1)$ and $\alpha(t_2)$ spanning the given angle ϕ). For polynomial or piecewise polynomial curves, one can compute numerically the intersection points using solvers, see e.g. [2], which generically leads to a one-parameter family of solutions (a curve).

If, in addition, the chord that connects the two contact points delimiting the visibility angle is of constant length ℓ, then α is said to be (ϕ, ℓ)*-isochordal viewed*. We refer the reader for more details and further properties of these curves to [6, 17, 18, 19]. Requiring the distance between the contact points to be of constant length, i.e., $\big\|\alpha(t_1) - \alpha(t_2)\big\| = \ell$, however, makes the problem difficult. Looking at the problem from the perspective of systems of non-linear equations, it leads to a well-constrained 2×2 problem with two variables (curve parameters t_1 and t_2) and two constraints (fixed angle and fixed length). Well-constrained problems have, generically, only finitely many solutions which would correspond only to finite positions of a pair lines meeting the (ϕ, ℓ)-isochordal viewed property, not a motion (infinitely many solutions of the 2×2 system). Therefore the property of being (ϕ, ℓ)-isochordal viewed requires some special behavior of the curve α that cannot be arbitrary. Interestingly enough, there exist cases where the solution set is one-parametric, and our work offers a constructive tool to generate these curves from a given isoptic c.

The inverse isoptic problem has been studied by several authors before, see e.g. [12] and [9, 11, 10, 13, 8] for related research. In [12], it is proved that if two convex bodies B_1 and B_2 have a common isoptic for the same angle $\phi \in\ ]0, \pi[$ such that $1 - \frac{\phi}{\pi}$ is irrational or rational with even numerator in its lowest terms, then $B_1 = B_2$. The reader can find a brief introduction to some inverse problems on isoptics in pages 15–17 of [5].

Wunderlich, who made great contributions to the isoptics and their applications in machine theory (see e.g. [22, 25]), refers to the inverse construction as a "difficult problem" in [24] and he himself found partial results for the particular case in which the isoptic is a circle [21, 23] or an ellipse [24].

In this paper, we tackle the inverse isoptic problem from a kinematic perspective. The main contributions are:

- A method to construct curves and, in particular, isochordal-viewed curves with a given regular closed isoptic.
- The answer to an open question whether there exist isochordal-viewed curves whose isoptic is a curve different from a circle.
- A proof that, in case of bounded and differentiable curves, the constructed curves with a prescribed regular closed isoptic are multihedgehogs.
- A proof that a regular polygon is allowed to move with its vertices lying on the constructed isochordal-viewed curves.

The rest of the paper is organized as follows. Section 2 generalizes the isoptics to a pair of curves and explains the direct construction. In Section 3 an inverse isoptic construction is discussed. First, the construction of a pair of curves with a given isoptic is provided (Theorem 1) and it is shown that these curves are multihedgehogs (Corollary 1). Then, a sufficient condition for the construction of a single curve with a given isoptic is obtained (Theorem 2). In Section 4 the isochordal condition is set and Theorem 3 provides a constructive tool to construct isochordal-viewed mates. Section 5 yields a method to construct isochordal-viewed curves (Theorem 4). Section 6 shows that the constructed isochordal-viewed curves admit motions of regular polygons (Theorem 5) and, finally, Section 7 draws some conclusions and discusses a few directions for future research.

2. Isoptics to a pair of curves

We start our analysis by considering a pair of curves (α, β) that are viewed from a given curve c under a constant angle. In [16], a definition of an isoptic to a pair of nested strictly convex curves is given. We now extend this definition to any pair of curves, not necessarily nested nor convex.

Definition 1. Let $\phi \in \,]0, \pi[$ and let α and β be two planar piecewise-regular curves. A *ϕ-isoptic* of the pair (α, β) is defined as a curve consisting of points through which a supporting line of α and a supporting line of β pass making an angle of ϕ.

The terminology of *supporting line* comes from the fact that both α and β are envelopes of a family of these lines. At regular points of α or β such supporting lines are, of course, tangent lines. Notice that even for nested convex curves the definition of isoptic above is not unique [16].

Let α and β be two piecewise-regular curves. If $\mathbf{t}_\alpha$ and $\mathbf{t}_\beta$ are the tangent vectors to α and β (where they can be defined), respectively, then a ϕ-isoptic c of the pair (α, β) can be parameterized in two ways as follows:

$$\begin{aligned} c(t) &= \alpha(t) + \lambda_1(t)\, \mathbf{t}_\alpha(t), \\ &= \beta(t) + \lambda_2(t)\, \mathbf{t}_\beta(t), \end{aligned}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are signed distances from α or β, respectively, to the ϕ-isoptic c (see Figure 3).

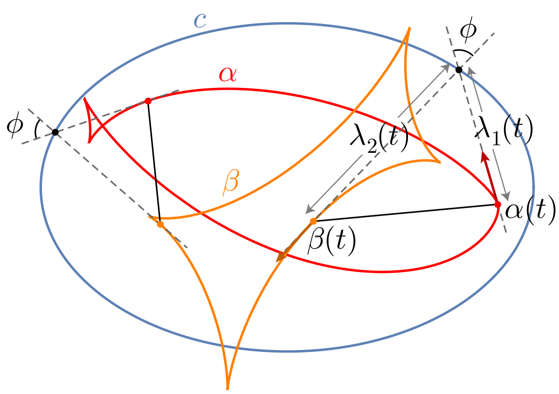


FIGURE 3. A ϕ-isoptic c of a pair of curves (α, β) and the definition of the signed distance functions λ_1 and λ_2.

Immediately from the extension of the definition of an isoptic to a pair of curves, the corresponding generalization for the concept of an isochordal-viewed curve is also possible.

Definition 2. Let $\phi \in \,]0, \pi[$ and let α and β be two planar piecewise-regular curves. The pair (α, β) is said to be a (ϕ, ℓ)-*isochordal-viewed mate* if a ϕ-isoptic c of (α, β) is such that the chord joining the contact points with α and β of their supporting lines meeting at c has constant length ℓ.

If β is another parameterization of the curve α, i.e. $\beta = \alpha \circ f$, for some homeomorphism f, then notice that the definition of a (ϕ, ℓ)-isochordal-viewed mate (α, β) is reduced to the known definition of a (ϕ, ℓ)-isochordal-viewed curve.

Observe that Definition 1 corresponds exactly to a double-disk cam mechanism with a flat-faced follower, such as the one shown in Figure 1. However, the definition does not require convexity. Definition 2 is a bit more restrictive because the distance between the contact points on the flat-faced follower is required to be constant for the entire motion.

3. Inverse construction of isoptics

In this section we address the inverse isoptic problem and describe a method to compute a curve whose isoptic is a given regular curve.

Let $\phi \in \,]0, \pi[$ and let $c : I \to \mathbb{R}^2$ be a closed regular planar curve, where I is some interval. We aim to construct a pair of curves α and β such that c is a ϕ-isoptic of (α, β). Let $\{\mathbf{t}_c, \mathbf{n}_c\}$ be a moving orthonormal frame associated with c, where $\mathbf{t}_c$ and $\mathbf{n}_c$ are the tangent and normal vectors, respectively, of c. Denote further by κ_c the curvature function of c. From the kinematic point of view, the construction of isoptics is determined by the smooth motion of a pair of secant lines (crossing at the isoptic curve) that form an angle ϕ, in such a way that the curves α and β are the envelopes of these two families of supporting lines, see Figure 4.

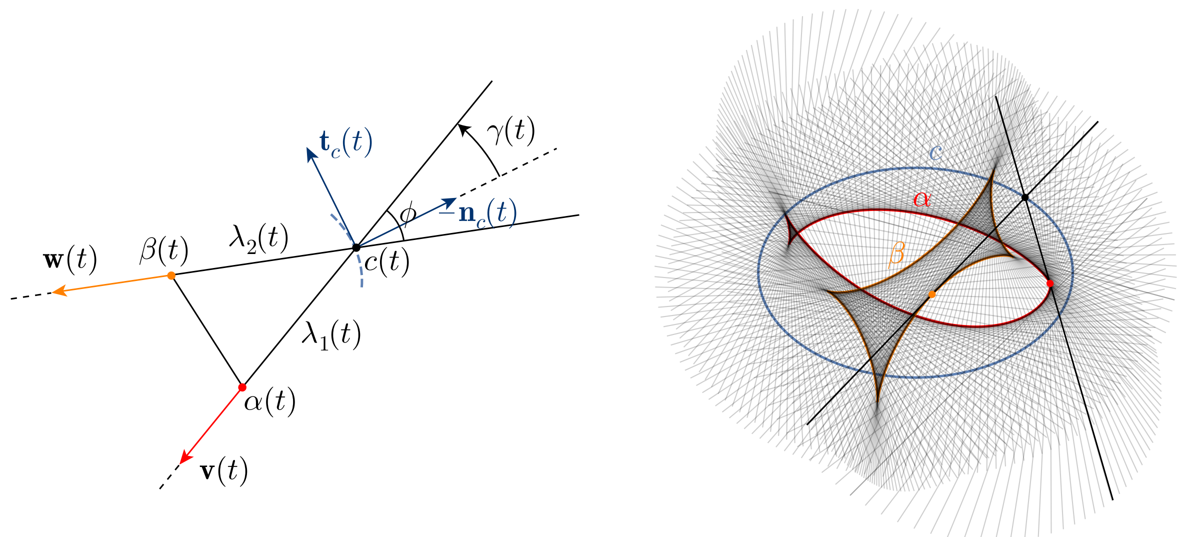


FIGURE 4. Inverse construction of ϕ-isoptics. Left: one time instant of a pair of lines spanning a fixed angle ϕ. Right: a 2D motion of a pair of lines (the intersection point moves along c) creates a pair of envelopes α and β.

Each secant line can be described by a corresponding directional vector field. Let $\mathbf{v}(t)$ be a unit vector in the direction of the supporting line to $\alpha(t)$, and analogously $\mathbf{w}(t)$ to $\beta(t)$, Figure 4-left. Let $\gamma(t)$ be an oriented angle function from $\mathbf{n}_c(t)$ to $\mathbf{v}(t)$. The signed distance from $c(t)$ to $\alpha(t)$ is defined by a function $\lambda_1(t)$ and the signed distance from $c(t)$ to $\beta(t)$ by a function $\lambda_2(t)$.

We have that

$$\alpha(t) = c(t) + \lambda_1(t)\,\mathbf{v}(t), \tag{1}$$

where

$$\mathbf{v}(t) = \cos\gamma(t)\,\mathbf{n}_c(t) - \sin\gamma(t)\,\mathbf{t}_c(t).$$

That is, α depends on the given curve c and two functions λ_1 and γ. Similarly,

$$\beta(t) = c(t) + \lambda_2(t)\,\mathbf{w}(t), \tag{2}$$

where

$$\mathbf{w}(t) = \mathrm{Rot}(-\phi)\mathbf{v}(t) = \cos\big(\phi - \gamma(t)\big)\,\mathbf{n}_c(t) + \sin\big(\phi - \gamma(t)\big)\,\mathbf{t}_c(t).$$

The curve β depends on the given curve c, the angle ϕ and on two functions λ_2 and γ. However, note that the functions λ_1, λ_2 and γ are not independent; they are bind by the fact that $\gamma(t)$

controls the instantaneous rotation of the pair of lines and, consequently the distance between the contact point, $\alpha(t)$ or $\beta(t)$, and $c(t)$. This is stated in the next result.

Theorem 1. *Let $\phi \in \,]0,\pi[$ and let $c : I \to \mathbb{R}^2$ be a single-traced closed regular $\mathcal{C}^3$-curve, where I is some interval. Any pair of $\mathcal{C}^1$-curves (α,β) given by (1) and (2) with c as its ϕ-isoptic is defined through the signed distance functions λ_1 and λ_2 given by*

$$\lambda_1(t) = \frac{\|c'(t)\|}{\|c'(t)\|\,\kappa_c(t) + \gamma'(t)} \cos\gamma(t) \tag{3}$$

and

$$\lambda_2(t) = \frac{\|c'(t)\|}{\|c'(t)\|\,\kappa_c(t) + \gamma'(t)} \cos\big(\phi - \gamma(t)\big), \tag{4}$$

where they are defined, and where γ is an $m|I|$-periodic (up to a multiple of 2π) $\mathcal{C}^2$-function, for some $m \in \mathbb{Z} \setminus \{0\}$.

Proof. First of all, we must impose that $\mathbf{v}(t)$ is, indeed, parallel to $\alpha'(t)$, i.e., parallel to

$$\alpha'(t) = c'(t) + \lambda_1'(t)\,\mathbf{v}(t) + \lambda_1(t)\,\mathbf{v}'(t).$$

Since $\|\mathbf{v}(t)\| = 1$, this happens if and only if $\langle \alpha'(t), \mathbf{v}'(t)\rangle = 0$, that is to say,

$$\big\langle c'(t) + \lambda_1(t)\,\mathbf{v}'(t),\, \mathbf{v}'(t)\big\rangle = 0.$$

This is,

$$\big\langle c'(t),\, \mathbf{v}'(t)\big\rangle + \|\mathbf{v}'(t)\|^2\,\lambda_1(t) = 0. \tag{5}$$

It is straightforward to compute

$$\mathbf{v}'(t) = V(t)\,\big(-\cos\gamma(t)\,\mathbf{t}_c(t) - \sin\gamma(t)\,\mathbf{n}_c(t)\big), \tag{6}$$

where

$$V(t) = \|c'(t)\|\,\kappa_c(t) + \gamma'(t),$$

which is such that $V(t)^2 = \|\mathbf{v}'(t)\|^2$. Thus, from (6) we have

$$\big\langle c'(t),\, \mathbf{v}'(t)\big\rangle = -\|c'(t)\|\,V(t)\cos\gamma(t)$$

and, hence, (5) turns into

$$V(t)^2\,\lambda_1(t) = \|c'(t)\|\,V(t)\cos\gamma(t).$$

Therefore, if $V(t) \neq 0$, we conclude that

$$\lambda_1(t) = \frac{\|c'(t)\|}{V(t)}\cos\gamma(t),$$

which is the expression (3) of the statement.

A similar discussion can be done for $\mathbf{w}(t)$, which must be parallel to

$$\beta'(t) = c'(t) + \lambda_2'(t)\,\mathbf{w}(t) + \lambda_2(t)\,\mathbf{w}'(t).$$

This happens if and only if

$$\big\langle c'(t) + \lambda_2(t)\,\mathbf{w}'(t),\, \mathbf{w}'(t)\big\rangle = 0,$$

that is,

$$\big\langle c'(t),\, \mathbf{w}'(t) \big\rangle + \big\|\mathbf{w}'(t)\big\|^2 \lambda_2(t) = 0. \tag{7}$$

We have that

$$\begin{aligned}\mathbf{w}'(t) &= \mathrm{Rot}(-\phi)\, \mathbf{v}'(t) = \mathrm{Rot}(-\phi)\, V(t) \big(-\cos\gamma(t)\, \mathbf{t}_c(t) - \sin\gamma(t)\, \mathbf{n}_c(t)\big) \\ &= V(t) \Big(-\cos\big(\phi - \gamma(t)\big)\, \mathbf{t}_c(t) + \sin\big(\phi - \gamma(t)\big)\, \mathbf{n}_c(t)\Big).\end{aligned}$$

Thus,

$$\big\langle c'(t),\, \mathbf{w}'(t) \big\rangle = -\big\|c'(t)\big\|\, V(t) \cos\big(\phi - \gamma(t)\big).$$

Therefore, if $V(t) \neq 0$, from (7) we deduce

$$\lambda_2(t) = \frac{\big\|c'(t)\big\|}{V(t)} \cos\big(\phi - \gamma(t)\big),$$

which leads to the expression (4).

Finally, notice that since c is $|I|$-periodic (c is closed) and γ is $m|I|$-periodic (up to a multiple of 2π), then the distance functions λ_1 and λ_2 are $m|I|$-periodic and so the curves α and β are also $m|I|$-periodic. In addition, note that any differentiation we have performed makes sense because α and β are $\mathcal{C}^1$. □

Remark 1. Notice that in Theorem 1 the angle function γ is assumed to be $m|I|$-periodic (up to a multiple of 2π) for some $m \in \mathbb{Z} \setminus \{0\}$. This means that

$$\gamma(t + m|I|) - \gamma(t) = 2\,k\,\pi,$$

for some $k \in \mathbb{Z}$. The geometric interpretation of the two integers k and m is as follows. In general, the isoptic curve c could be traced twice, thrice, etc. when the moving chord travels along the curves α and β. This means that if c is $|I|$-periodic and single-traced, then the curves α and β may be, in general, $m|I|$-periodic, for some $m \in \mathbb{Z} \setminus \{0\}$. Further, notice that the integer m is not necessarily the minimum integer that makes α and β closed (see Example 1 below). Regarding the integer k, it simply indicates how many total revolutions (counted with sign) the pair of crossing lines have performed during the motion.

As an example, consider

$$\gamma(t) = \frac{\pi}{4} + \frac{1}{2}\,t + \sin(t),$$

which satisfies $\gamma(t + 4\pi) - \gamma(t) = 2\pi$, so that γ is 4π-periodic up to 2π. See a plot in Figure 5.

Remark 2. If c is arc-length parameterized, i.e. $\big\|c'(t)\big\| = 1$, then the expressions (3) and (4) from Theorem 1 turn into

$$\lambda_1(t) = \frac{1}{\kappa_c(t) + \gamma'(t)} \cos\gamma(t)$$

and

$$\lambda_2(t) = \frac{1}{\kappa_c(t) + \gamma'(t)} \cos\big(\phi - \gamma(t)\big),$$

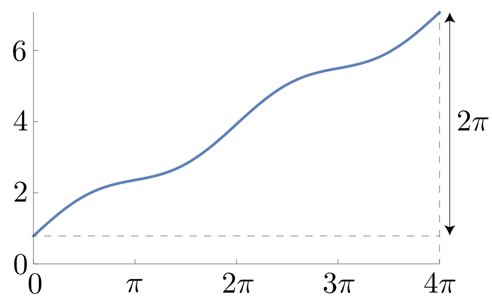


FIGURE 5. A plot of $\gamma(t) = \frac{\pi}{4} + \frac{1}{2}t + \sin(t)$, which is 4π-periodic up to 2π.

respectively. Moreover, a choice $\gamma(t) = 0$ for all $t \in I$ yields

$$\alpha(t) = c(t) + \frac{1}{\kappa_c(t)}\,\mathbf{n}_c(t),$$

which is the well-known expression of the evolute of c.

Example 1. Let $c : [0, 2\pi] \to \mathbb{R}^2$ be the ellipse

$$c(t) = (3\cos t,\ 2\sin t).$$

With Theorem 1 we can construct infinitely many pairs of curves (α, β) such that their ϕ-isoptic is c, for some $\phi \in\ ]0, \pi[$. This is achieved by choosing a $2\pi m$-periodic (up to a multiple of 2π) function γ, for some $m \in \mathbb{Z} \setminus \{0\}$, so that the distance functions λ_1 and λ_2 are determined by (3) and (4).

See in Figure 6 some examples of (α, β) for different choices for a 2π-periodic function γ and the angle ϕ. Notice that for each choice of an angle function γ, the curve α is fully determined, and so is β, linked with α via the constant angle ϕ.

Let us illustrate another example taking the angle function

$$\gamma(t) = \frac{\pi}{4} + \frac{1}{2}t,$$

which satisfies

$$\gamma(t + 4\pi) - \gamma(t) = 2\pi.$$

This function is not periodic, but it is $2\pi m$-periodic up to 2π, for $m = 2$. See in Figure 7 the resulting pair of curves (α, β) given by Theorem 1 in this case.

In this example, the curves (α, β) are 2π-periodic, which shows that $m = 2$ is not the minimum integer that makes (α, β) closed curves. The fact that (α, β) are closed is related to returning the isoptic chord to its initial position after the motion of the pair of secant lines. However, it can happen (as in this case) that $\gamma(t + 2\pi) - \gamma(t) = \pi$ (i.e. the pair of secant lines returns to its initial position but with reverse orientation) and the signed distance functions satisfy $\lambda_1(t)+\lambda_1(t+2\pi) = 0$ and $\lambda_2(t) + \lambda_2(t + 2\pi) = 0$, which implies that their sign "compensates" the reverse orientation.

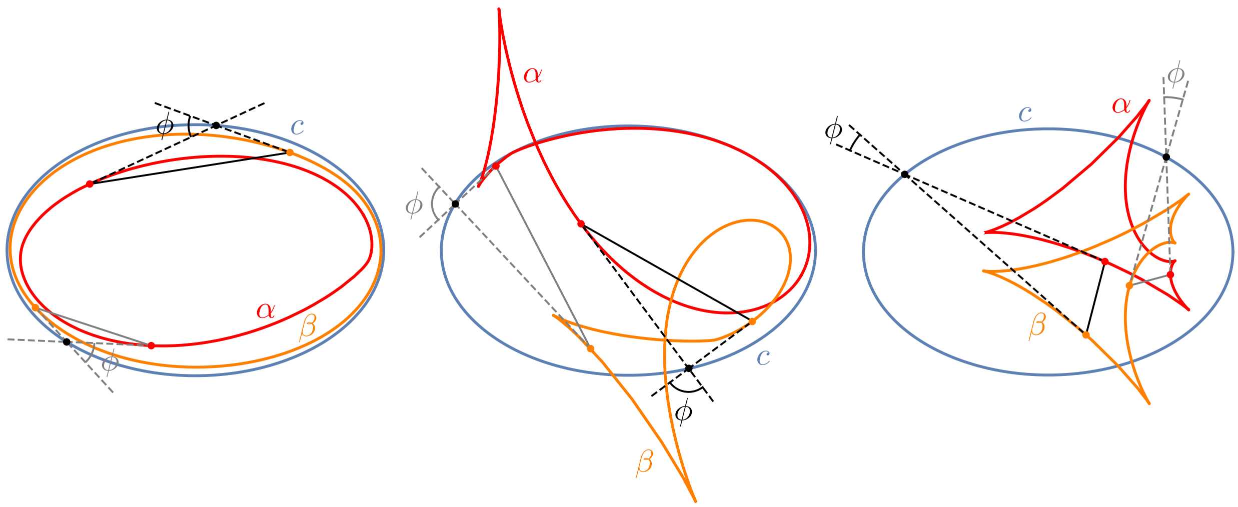


FIGURE 6. Three pairs of curves (α, β) constructed from a single elliptical ϕ-isoptic c. The design parameters are the constant angle ϕ and the $\gamma(t)$ function. From left to right: $\gamma(t) = \frac{2\pi}{3} + \frac{1}{60}\sin(3t)$, $\phi = \frac{\pi}{4}$; $\gamma(t) = \frac{\pi}{3} + \frac{2}{3}\sin(t)$, $\phi = \frac{\pi}{2}$; and $\gamma(t) = \frac{\pi}{20} + \frac{1}{2}\sin(t)$, $\phi = \frac{\pi}{10}$.

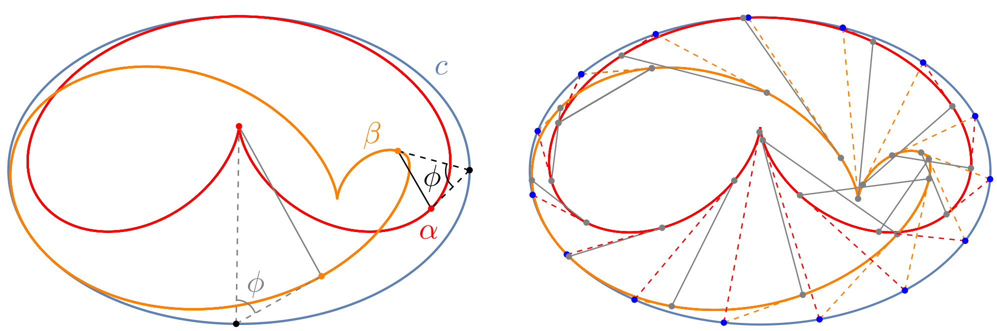


FIGURE 7. Left: a pair of curves (α, β) constructed from its ϕ-isoptic c, where $\phi = \frac{\pi}{3}$. Right: 15 discrete positions of the pair of lines (dashed).

Remark 3. If the denominators of (3) and (4) vanish for some value of t, then although the same construction of Theorem 1 is possible at any other point, it would lead to pairs of curves (α, β) that go to infinity. For example, consider a 2π-periodic function $\gamma(t) = 1 - \frac{5}{2}\sin(t)$, which corresponds to the fact that the term

$$\left\|c'(t)\right\| \kappa_c(t) + \gamma'(t)$$

has two real roots in $[0, 2\pi]$. The pair of curves (α, β) results with asymptotes, see Figure 8. However, this behavior cannot happen if the isochordal condition is also set because in such a case $\left\|c'(t)\right\| \kappa_c(t) + \gamma'(t) \neq 0$ for all $t \in I$ (in order to have a finite chord length, as a consequence of Theorem 3 below).

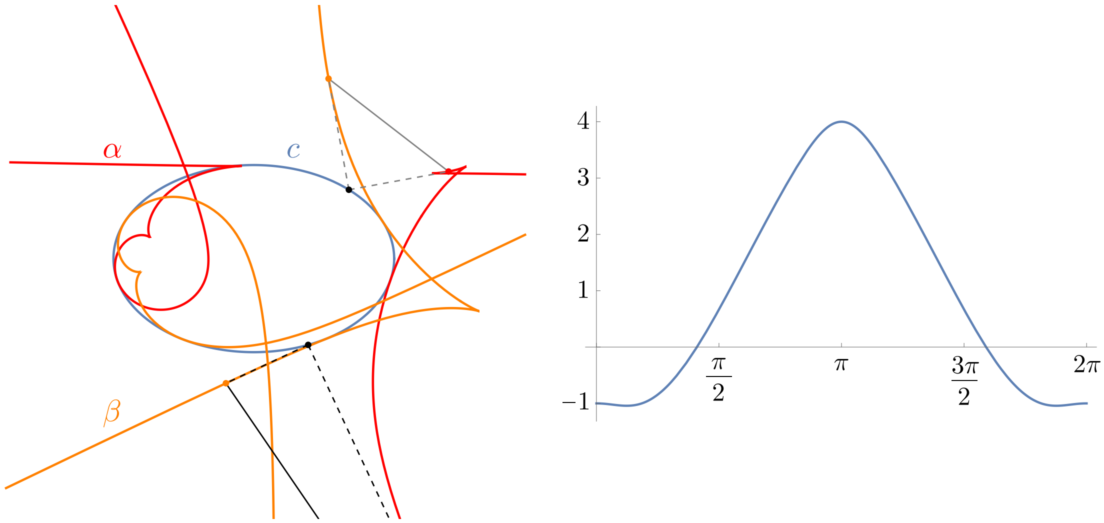


FIGURE 8. Left: a pair of curves (α, β) that go to infinity and such that c is its ϕ-isoptic, for $\phi = \frac{\pi}{2}$. Right: plot of the function $\|c'(t)\|\,\kappa_c(t) + \gamma'(t)$, which has two zeros.

Given $m \in \mathbb{N}$, an m-hedgehog (or, generically, a multihedgehog) is the envelope of a family of supporting lines

$$x \cos t + y \sin t = h(t),$$

where $h : \mathbb{R} \to \mathbb{R}$ is a $2\pi m$-periodic function called *support function* [14, 15].

An m-hedgehog has exactly m cooriented supporting lines with a given normal vector (counted with their multiplicity). Convex curves are singularity-free hedgehogs (that is, 1-hedgehogs). The next result states that the curves that we construct are actually multihedgehogs.

Corollary 1. *Let $\phi \in \,]0, \pi[$, $\ell > 0$ and let $c : I \to \mathbb{R}^2$ be a single-traced closed regular $\mathcal{C}^3$-curve, where I is some interval. If (α, β) is a pair of $\mathcal{C}^1$-curves given by (1) and (2) with c as its ϕ-isoptic, then α and β are multihedgehogs.*

Proof. The supporting lines to α are described by the vectors

$$\mathbf{v}(t) = \cos\gamma(t)\,\mathbf{n}_c(t) - \sin\gamma(t)\,\mathbf{t}_c(t).$$

Notice that we can write

$$\mathbf{v}(t) = \big(-\sin u(t),\, \cos u(t)\big),$$

where

$$u(t) = \gamma(t) + \theta_c(t),$$

with θ_c being the counterclockwise oriented angle from a fixed direction to the tangent vector of c. Similarly, the supporting lines to β are described by

$$\mathbf{w}(t) = \big(-\sin \bar{u}(t),\, \cos \bar{u}(t)\big),$$

with

$$\bar{u}(t) = \gamma(t) - \phi + \theta_c(t).$$

To show that α and β are multihedgehogs, it is enough to prove that $u(t)$ and $\bar{u}(t)$ are strictly monotone. Now,

$$u'(t) = \bar{u}'(t) = \gamma'(t) + \big\|c'(t)\big\|\,\kappa_c(t).$$

Since α and β are $\mathcal{C}^1$ and closed (they do not go to infinity), by Theorem 1 we have that $\gamma'(t) + \big\|c'(t)\big\|\,\kappa_c(t)$ has no zeros (see also Remark 3). Therefore, $u(t)$ and $\bar{u}(t)$ are strictly monotone and thus α and β are multihedgehogs. $\square$

The following theorem provides a sufficient condition on the function γ in order to ensure that the curves α and β describe the same curve.

Theorem 2. *Let $\phi \in \,]0,\pi[$ and let $c : I \to \mathbb{R}^2$ be a single-traced closed regular $\mathcal{C}^3$-curve, where I is some interval. Let (α,β) be $\mathcal{C}^1$-curves defined by (1) and (2) with well-defined distance functions λ_1 and λ_2 given by (3) and (4) for some $m|I|$-periodic (up to a multiple of 2π) $\mathcal{C}^2$-function γ, where $m \in \mathbb{Z}\setminus\{0\}$. If*

$$\phi - \gamma(t) = -\gamma\big(t + \tilde{k}\,|I|\big) + \tilde{m}\,\pi, \tag{8}$$

for some $\tilde{k} \in \mathbb{Z}\setminus\{0\}$ and $\tilde{m} \in \mathbb{Z}$, then the parameterizations α and β describe the same curve. In such a case $m \neq -1, 1$.

Proof. Let $a = |I|$. We have that

$$\alpha(t + \tilde{k}\,a) = c(t + \tilde{k}\,a) + \lambda_1(t + \tilde{k}\,a)\,\mathbf{v}(t + \tilde{k}\,a).$$

Since c is a-periodic, we know that $c(t + \tilde{k}\,a) = c(t)$. Moreover, $c'(t + \tilde{k}\,a) = c'(t)$ and $\kappa_c(t + \tilde{k}\,a) = \kappa_c(t)$. Also, the assumption (8) implies $\gamma'(t + \tilde{k}\,a) = \gamma'(t)$. Thus,

$$\begin{aligned}\lambda_1(t + \tilde{k}\,a) &= \frac{\big\|c'(t)\big\|}{\big\|c'(t)\big\|\,\kappa_c(t) + \gamma'(t)}\cos\gamma(t + \tilde{k}\,a)\\ &= \frac{\big\|c'(t)\big\|}{\big\|c'(t)\big\|\,\kappa_c(t) + \gamma'(t)}\,(-1)^{\tilde{m}}\cos\big(\gamma(t) - \phi\big) = (-1)^{\tilde{m}}\,\lambda_2(t).\end{aligned}$$

In addition, since $\mathbf{t}_c(t + \tilde{k}\,a) = \mathbf{t}_c(t)$ and $\mathbf{n}_c(t + \tilde{k}\,a) = \mathbf{n}_c(t)$, we have

$$\begin{aligned}\mathbf{v}(t + \tilde{k}\,a) &= \cos\gamma(t + \tilde{k}\,a)\,\mathbf{n}_c(t) - \sin\gamma(t + \tilde{k}\,a)\,\mathbf{t}_c(t)\\ &= (-1)^{\tilde{m}}\Big(\cos\big(\phi - \gamma(t)\big)\,\mathbf{n}_c(t) + \sin\big(\phi - \gamma(t)\big)\,\mathbf{t}_c(t)\Big)\\ &= (-1)^{\tilde{m}}\,\mathbf{w}(t).\end{aligned}$$

Therefore, we can conclude

$$\alpha(t + \tilde{k}\,a) = c(t) + \lambda_2(t)\,\mathbf{w}(t) = \beta(t).$$

It is only left to prove that $m \neq \pm 1$. If $m = \pm 1$, we have that the angle function satisfies

$$\gamma(t \pm a) - \gamma(t) = 2\,k\,\pi,$$

for some $k \in \mathbb{Z}$. Therefore, that

$$-\phi + \tilde{m}\,\pi = \gamma(t + \tilde{k}\,a) - \gamma(t)$$

implies that ϕ is a multiple of π, which cannot happen as $\phi \in \,]0, \pi[$. $\square$

Remark 4. Notice that there are single-traced curves whose isoptics are single-traced as well. In these cases, $m = \pm 1$, which shows that the condition of Theorem 2 is sufficient but not necessary.

Example 2. Theorem 2 provides a way to construct a curve α whose ϕ-isoptic is a given curve c. Let $c : [0, 2\pi] \to \mathbb{R}^2$ be the ellipse from Example 1, which is 2π-periodic. Consider

$$\gamma(t) = \frac{\pi}{3} + \frac{t}{4}.$$

This function γ is 8π-periodic up to 2π, that is,

$$\gamma(t + 8\pi) - \gamma(t) = 2\pi.$$

Thus, in this example $m = 4$. Moreover,

$$\gamma(t + 2\pi) - \gamma(t) = \frac{\pi}{2} \in \,]0, \pi[\,.$$

Thus, choosing $\tilde{k} = 1$, $\tilde{m} = 1$ and $\phi = \frac{\pi}{2}$, we have that the curve α given by (1) and the expression of Theorem 1 has c as its $\frac{\pi}{2}$-isoptic. The endpoints of the isoptic chord are given by $\alpha(t)$ and $\alpha(t + 2\pi)$. See the resulting curve in Figure 9. In this case, the ellipse is double-traced as the isoptic chord goes back to its initial position.

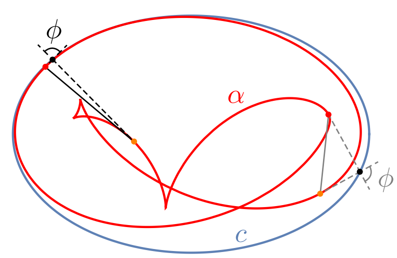


FIGURE 9. A curve α with the ellipse c as its $\frac{\pi}{2}$-isoptic.

Following similar ideas, many other examples of curves having the ellipse c as its ϕ-isoptic can be given. See a few examples in Figures 10 and 11. In particular, observe the impact of the initial value of γ at $t = 0$ on the final shape of α in Figure 11.

A remarkable example is provided if

$$\gamma(t) = -\frac{t}{4} - 3\tan(t)$$

as the curve α presents an infinite number of singularities while having the ellipse c as its $\frac{\pi}{2}$-isoptic (see Figure 12). Notice that here γ is piecewise-$\mathcal{C}^2$ and that α is indeterminate at two points (for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$).

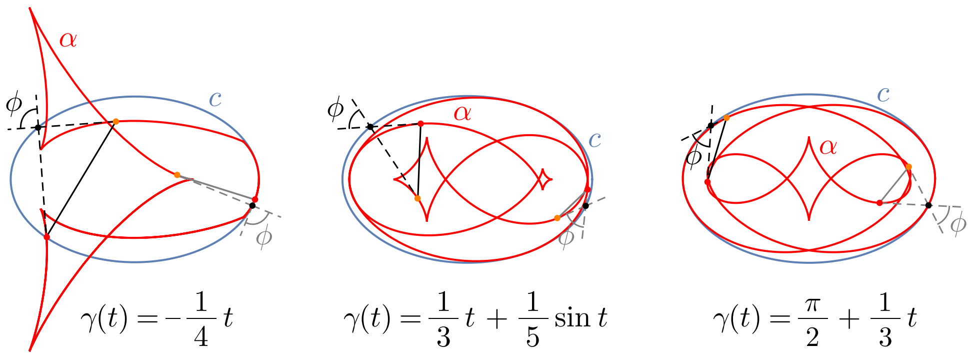


FIGURE 10. Some curves α with the ellipse c as their ϕ-isoptic, where ϕ is $\frac{\pi}{2}$ or $\frac{\pi}{3}$.

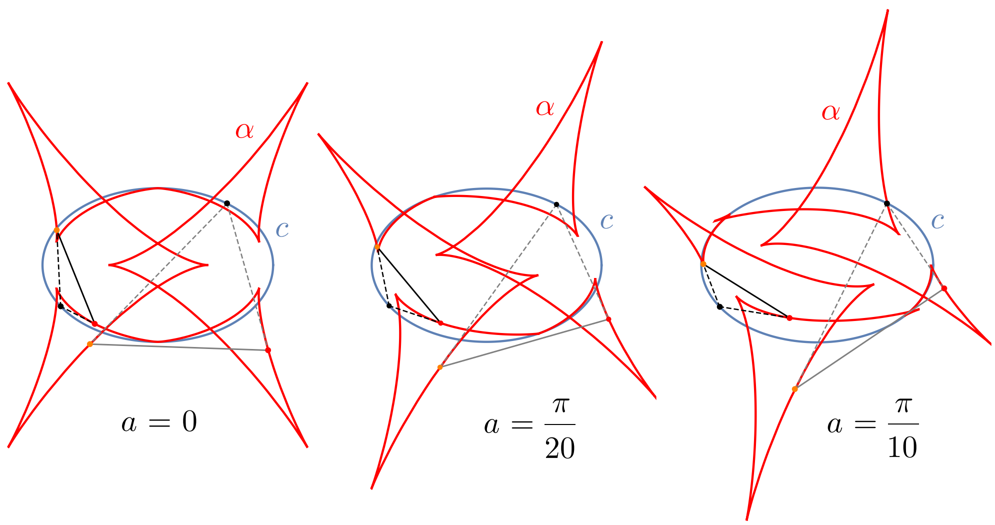


FIGURE 11. Some curves α with the ellipse c as their $\frac{\pi}{3}$-isoptic, where $\gamma(t) = a - \frac{1}{3}t$.

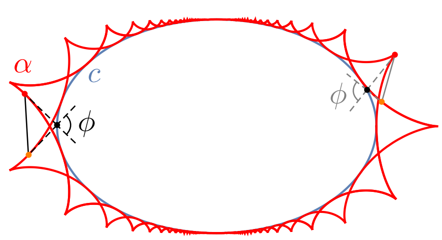


FIGURE 12. A curve α with an infinite number of singularities with the ellipse c as its $\frac{\pi}{2}$-isoptic.

4. Inverse construction of isochordal-viewed mates

Now we want (α, β) to be, in addition, a (ϕ, ℓ)-isochordal-viewed mate. In this case, the function γ is also determined (except for some free parameters) as we show in the following result.

Theorem 3. *Let $\phi \in \,]0, \pi[$, $\ell > 0$ and let $c : I \to \mathbb{R}^2$ be a single-traced closed regular $\mathcal{C}^3$-curve, where I is some interval. Any (ϕ, ℓ)-isochordal-viewed mate (α, β) given by (1) and (2) with c as its ϕ-isoptic is determined by well-defined distance functions λ_1 and λ_2 as in (3) and (4) and an $m|I|$-periodic (up to $2k\pi$) $\mathcal{C}^2$-function*

$$\gamma(t) = k_1 + \frac{2\,\pi\,(k + m\,n)}{\mathcal{L}(c)\,m}\, s_c(t) - \theta_c(t), \tag{9}$$

where $k_1 \in \mathbb{R}$, $m \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}$, with $k + m\,n \neq 0$, $\mathcal{L}(c)$ is the length of c, n is the rotation index of c, s_c is the arc-length parameter of c and θ_c is the counterclockwise oriented angle from a fixed direction to the tangent vector of c, and with

$$\ell = \frac{\mathcal{L}(c)}{2\pi} \left| \frac{m}{k + m\,n} \right| \sin\phi.$$

Proof. By Theorem 1 we have that any pair (α, β) with the conditions on λ_1 and λ_2 have c as its ϕ-isoptic. We must impose now the isochordal condition, which is that

$$\big\|\beta(t) - \alpha(t)\big\| = \ell$$

is constant. By the cosine rule (recall Figure 4) we have that

$$\ell^2 = \lambda_1^2(t) + \lambda_2^2(t) - 2\,\lambda_1(t)\,\lambda_2(t)\cos\phi.$$

If we put in this equation the expressions of $\lambda_1(t)$ and $\lambda_2(t)$ given by (3) and (4) and simplify it, we get

$$\ell^2 = B^2(t)\sin^2\phi, \tag{10}$$

where

$$B(t) = \frac{\big\|c'(t)\big\|}{V(t)} = \frac{\big\|c'(t)\big\|}{\big\|c'(t)\big\|\,\kappa_c(t) + \gamma'(t)}, \tag{11}$$

which is an equation that relates ℓ, ϕ, c and the function γ. From it, since ℓ and ϕ are constant, then $B(t)$ must be constant too (and not zero). Write this constant as $1/k_0$, where $k_0 \in \mathbb{R} \setminus \{0\}$. From (11), we have

$$\gamma'(t) = k_0\,\big\|c'(t)\big\| - \kappa_c(t)\,\big\|c'(t)\big\|.$$

Integrating this expression:

$$\gamma(t) = k_1 + k_0 \int_{\inf I}^{t} \big\|c'(s)\big\|\,\mathrm{d}s - \int_{\inf I}^{t} \kappa_c(s)\,\big\|c'(s)\big\|\,\mathrm{d}s,$$

for any $k_1 \in \mathbb{R}$. Now, recall that $\theta_c(t)$ satisfies

$$\theta_c'(t) = \big\|c'(t)\big\|\,\kappa_c(t).$$

Hence, we can write

$$\gamma(t) = k_1 + k_0\, s_c(t) - \theta_c(t),$$

where s_c is the arc-length parameter of c.

Let $a = |I| \in \mathbb{R}$. Notice that γ must be $m\,a$-periodic (up to a multiple of 2π) for some $m \in \mathbb{Z}\backslash\{0\}$, that is,

$$\gamma(t + m\,a) - \gamma(t) = 2\,k\,\pi,$$

where $k \in \mathbb{Z}$. This will determine the value of k_0. We have

$$\begin{aligned}\gamma(t + m\,a) &= k_1 + k_0\, s_c(t + m\,a) - \theta_c(t + m\,a),\\ \gamma(t) &= k_1 + k_0\, s_c(t) - \theta_c(t).\end{aligned}$$

Therefore,

$$k_0\left(s_c(t + m\,a) - s_c(t)\right) - \left(\theta_c(t + m\,a) - \theta_c(t)\right) = 2\,k\,\pi. \tag{12}$$

Since $\left\|c'(t)\right\|$ is a-periodic, we see that

$$s_c(t + m\,a) - s_c(t) = \int_{\inf I}^{t+m\,a} \left\|c'(s)\right\|\,\mathrm{d}s - \int_{\inf I}^{t} \left\|c'(s)\right\|\,\mathrm{d}s = \int_0^{m\,a} \left\|c'(s)\right\|\,\mathrm{d}s = m\,\mathcal{L}(c),$$

where $\mathcal{L}(c)$ is the length of c. Now, if n is the rotation index of c, then

$$\theta_c(t + m\,a) - \theta_c(t) = 2\,\pi\,m\,n.$$

Thus, from Equation (12),

$$k_0 = \frac{2\pi}{\mathcal{L}(c)}\,\frac{k + m\,n}{m},$$

where $k + m\,n \neq 0$ (this is a consequence of being λ_i well defined or, equivalently, being $k_0 \neq 0$). The expression for the length ℓ comes from Equation (10), which implies

$$\ell = \frac{1}{|k_0|}\sin\phi.$$

Notice that $k_1 = \gamma(\inf I)$, so that the free parameter k_1 represents the initial angle of the angle function $\gamma(t)$. □

Example 3. Consider the same ellipse c as in Example 1. In this case we have

$$\kappa_c(t) = \frac{6}{\left(9\sin^2(t) + 4\cos^2(t)\right)^{3/2}}.$$

Define

$$\theta_c(t) = \int_0^t \kappa_c(s)\left\|c'(s)\right\|\,\mathrm{d}s = \arctan\!\big(2\cos(t), 3\sin(t)\big) + 2\pi\left\lfloor\frac{t}{\pi}\right\rfloor - 2\pi\left\lfloor\frac{t}{2\pi}\right\rfloor,$$

which is defined continuously on the whole $\mathbb{R}$.

The arc-length parameter of c is

$$s_c(t) = \int_0^t \left\|c'(s)\right\|\,\mathrm{d}s = \int_0^t \sqrt{9\sin^2(s) + 4\cos^2(s)}\,\mathrm{d}s,$$

which is an elliptic integral. The length of c is given by

$$\mathcal{L}(c) = s_c(2\pi) \approx 15.8654.$$

By Theorem 3, the angle function γ must be of the form

$$\gamma(t) = k_1 + \frac{2\,\pi\,(k + m\,n)}{\mathcal{L}(c)\,m}\,s_c(t) - \theta_c(t),$$

where $n = 1$, for any $k_1 \in \mathbb{R}$, $k \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$.

Thus, fixed an angle ϕ and two integers k and m, we have a 1-parameter family of pairs of curves (α, β) that have c as its ϕ-isoptic. See in Figure 13 some examples of isochordal-viewed mates with c as their ϕ-isoptic.

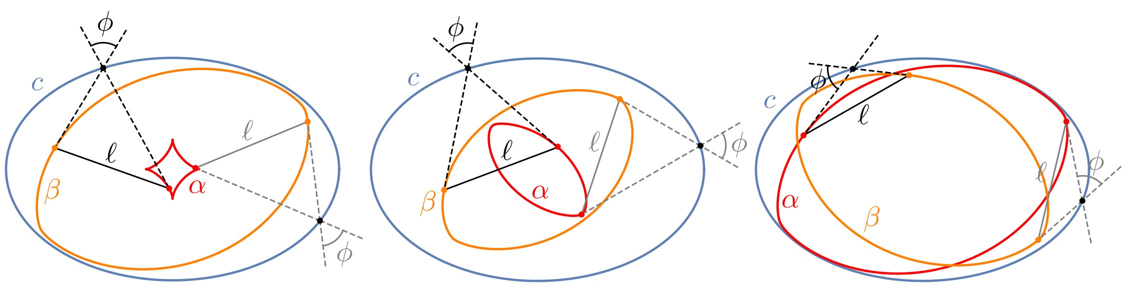
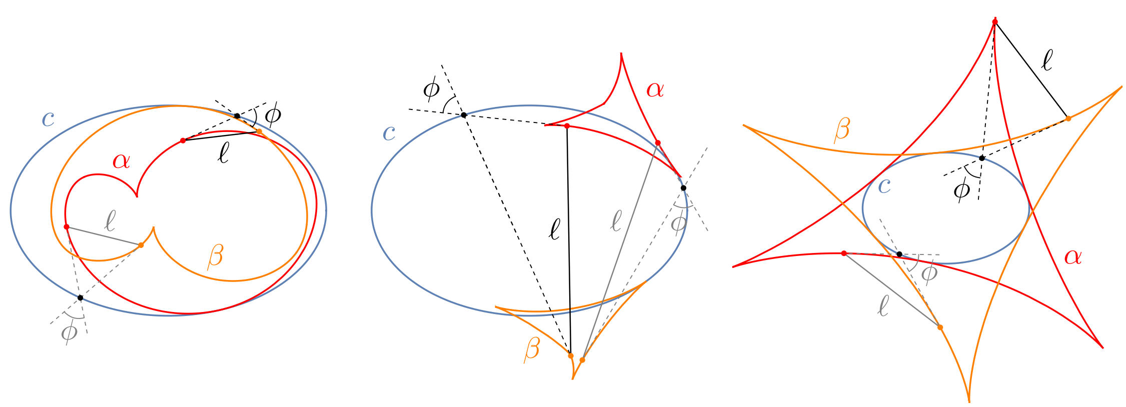


FIGURE 13. Some pairs of (ϕ, ℓ)-isochordal-viewed mates (α, β) with c as their ϕ-isoptic. For a fixed angle $\phi = \frac{\pi}{3}$, $k = 0$ and $m = 1$, we have chosen $k_1 = 0$, $k_1 = \frac{7}{20}$ and $k_1 = 2$, respectively.

The impact of the integer parameters k and m on the shape of the isochordal-viewed mates is shown in Figure 14.



$k = 1, m = 2$ $\quad$ $k = -1, m = 2$ $\quad$ $k = -3, m = 2$

FIGURE 14. Some pairs of (ϕ, ℓ)-isochordal-viewed mates (α, β) with c as their ϕ-isoptic for different choices of k and m. Here, $k_1 = 2$.

5. Inverse construction of isochordal-viewed curves

We are now reaching one of our main results, namely, how to construct isochordal-viewed curves from a given isoptic. In the following theorem we write the expression of these curves in a constructive manner.

Theorem 4. *Let $\ell > 0$ and let $c : I \to \mathbb{R}^2$ be a single-traced closed regular $\mathcal{C}^3$-curve with rotation index n, where I is some interval. For any $k_1 \in \mathbb{R}$ and $k, m \in \mathbb{Z} \setminus \{0\}$ such that $k + m\,n \neq 0$, let $\alpha_{k,m}$ be the curve parameterized by*

$$\alpha_{k,m}(t) = c(t) + \frac{\mathcal{L}(c)\,m}{2\,\pi\,(k+m\,n)} \cos\gamma(t)\,\big(\cos\gamma(t)\,\mathbf{n}_c(t) - \sin\gamma(t)\,\mathbf{t}_c(t)\big),$$

where $\mathcal{L}(c)$ is the length of c and

$$\gamma(t) = k_1 + \frac{2\,\pi\,(k+m\,n)}{\mathcal{L}(c)\,m}\,s_c(t) - \theta_c(t),$$

with s_c being the arc-length parameter of c and θ_c the counterclockwise oriented angle from a fixed direction to the tangent vector of c.

If $\tilde{m} \in \mathbb{Z}$ and $k, \tilde{k}, m \in \mathbb{Z} \setminus \{0\}$ are such that

$$\phi = \tilde{m}\,\pi - 2\,\pi\,\tilde{k}\,\frac{k}{m} \in \,]0, \pi[, \tag{13}$$

with $k + m\,n \neq 0$, then the curve $\alpha_{k,m}$ is (ϕ, ℓ)-isochordal viewed with c as its ϕ-isoptic and with a chord length

$$\ell = \frac{\mathcal{L}(c)}{2\,\pi}\left|\frac{m}{k+m\,n}\right|\sin\phi.$$

Proof. We have that c is a-periodic, where $a = |I|$. Therefore, for any $\tilde{k} \in \mathbb{Z}$, we have

$$c(t + \tilde{k}\,a) = c(t).$$

From the proof of Theorem 3 and with the same notation, recall that

$$\alpha(t) = c(t) + \lambda_1(t)\,\mathbf{v}(t),$$
$$\beta(t) = c(t) + \lambda_2(t)\,\mathbf{w}(t),$$

where

$$\lambda_1(t) = \frac{\mathcal{L}(c)\,m}{2\,\pi\,(k+m\,n)}\cos\gamma(t),$$
$$\lambda_2(t) = \frac{\mathcal{L}(c)\,m}{2\,\pi\,(k+m\,n)}\cos\big(\phi - \gamma(t)\big),$$

and

$$\mathbf{v}(t) = \cos\gamma(t)\,\mathbf{n}_c(t) - \sin\gamma(t)\,\mathbf{t}_c(t),$$
$$\mathbf{w}(t) = \cos\big(\phi - \gamma(t)\big)\,\mathbf{n}_c(t) + \sin\big(\phi - \gamma(t)\big)\,\mathbf{t}_c(t).$$

If we have $\tilde{k} \in \mathbb{Z} \setminus \{0\}$ and $\tilde{m} \in \mathbb{Z}$ such that

$$\phi - \gamma(t) = -\gamma(t + \tilde{k}\, a) + \tilde{m}\, \pi,$$

then we will have $\alpha(t + \tilde{k}\, a) = \beta(t)$, as seen in Theorem 2, and we will finish.

Now, by Theorem 3 notice that

$$\gamma(t + \tilde{k}\, a) - \gamma(t) = \frac{2\,\pi\,(k + m\,n)}{\mathcal{L}(c)\, m} \big(s_c(t + \tilde{k}\, a) - s_c(t)\big) - \big(\theta_c(t + \tilde{k}\, a) - \theta_c(t)\big).$$

Since $\left\|c'(t)\right\|$ is a-periodic, we have

$$s_c(t + \tilde{k}\, a) - s_c(t) = \tilde{k}\, \mathcal{L}(c).$$

Also,

$$\theta_c(t + \tilde{k}\, a) - \theta_c(t) = 2\,\pi\, \tilde{k}\, n,$$

where n is the rotation index of c. Therefore,

$$\gamma(t + \tilde{k}\, a) - \gamma(t) = 2\,\pi\, \tilde{k}\, \frac{k}{m},$$

so that we must demand that

$$\phi = \tilde{m}\, \pi - 2\,\pi\, \tilde{k}\, \frac{k}{m} \in\,]0, \pi[,$$

which is the expression of ϕ of the statement. □

Example 4. Let us use Theorem 4 to construct some examples of isochordal-viewed curves whose isoptic is prescribed. The results for the ellipse are shown in Figure 15. Observe that there are no convexity constraints in Theorem 4 for the isoptic curve c. In Figure 16 we construct some other examples taking as an isoptic curve c a curve of constant width, a symmetric non-convex curve and a trifolium (non-simple) curve.

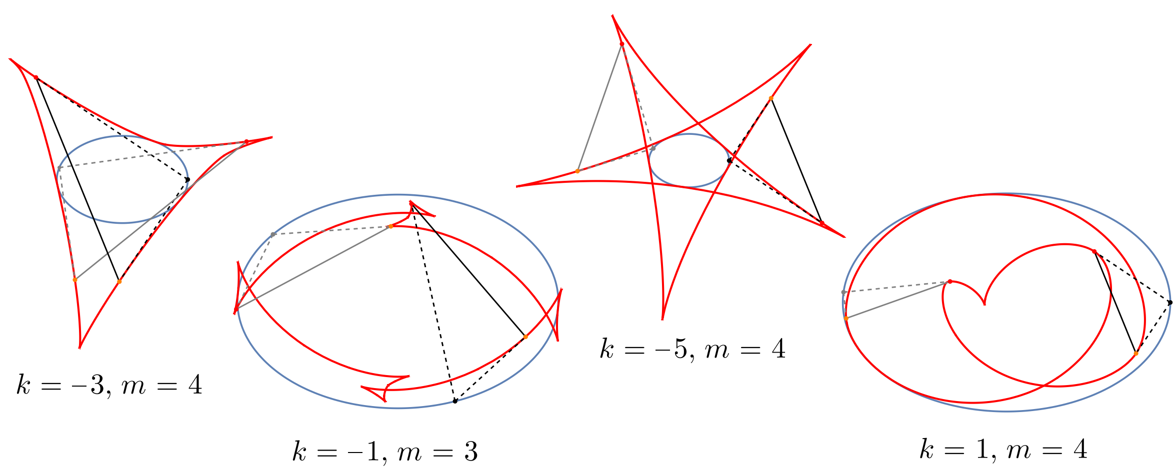


FIGURE 15. Some examples of (ϕ, ℓ)-isochordal-viewed curves with an elliptic ϕ-isoptic, for the angles $\phi = \frac{\pi}{2}$ or $\phi = \frac{\pi}{3}$ and $k_1 = \frac{51}{20}$.

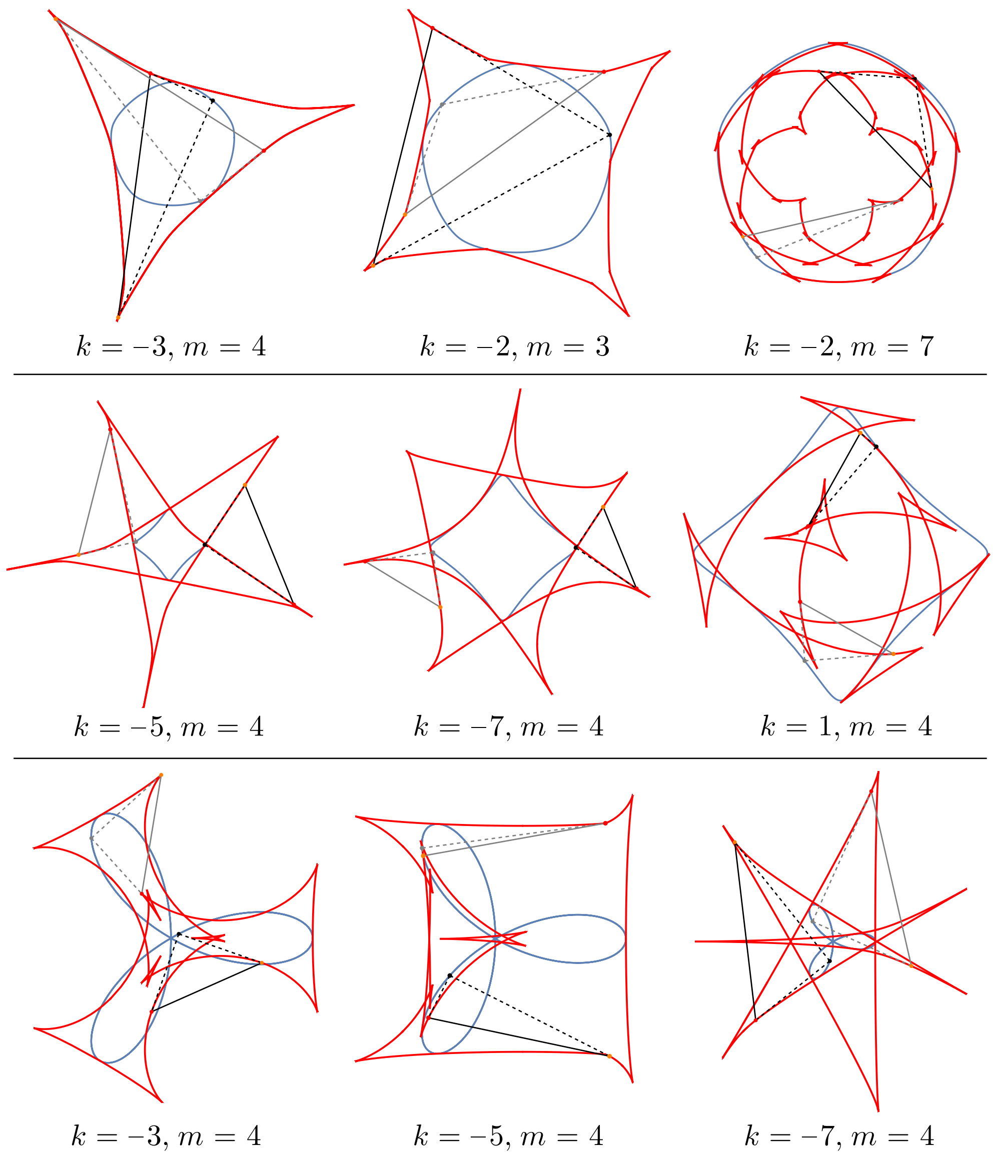


FIGURE 16. Some examples of (ϕ, ℓ)-isochordal-viewed curves for three different isoptic curves: a curve of constant width, a symmetric non-convex curve and a trifolium (non-simple) curve.

Remark 5. By construction, if $\alpha_{k,m}$ is the curve defined in Theorem 4, we can check that

$$\alpha_{k,-m}(t) = \alpha_{-k,m}(t).$$

This means that one of the two integers, k or m, without loss of generality can be supposed to be a natural number. For example, we can consider $m \in \mathbb{N}$ and $k \in \mathbb{Z} \setminus \{0\}$. Notice that $k \neq 0$ implies that we cannot generate isochordal-viewed curves with γ being just $2\pi m$-periodic (it must be up to a multiple of 2π).

In practice, when we use Theorem 4 to construct curves we have just two free parameters, which are the integers k and m. Once these parameters are chosen, the curve $\alpha_{k,m}$ will lead to a (ϕ,ℓ)-isochordal-viewed curve if there exist $\tilde{k}$ and $\tilde{m}$ such that the condition (13) on ϕ is fulfilled (independently of the choice of the prescribed isoptic c). Further, notice that this condition can be fulfilled for different $\tilde{k}$ and $\tilde{m}$. As an example, if $k=-1$ and $m=8$, we get that $\alpha_{k,m}$ is (ϕ,ℓ)-isochordal viewed for $\phi=\frac{\pi}{4}$ (with $\tilde{k}=1$ and $\tilde{m}=0$) and also for $\phi=\frac{\pi}{2}$ (with $\tilde{k}=2$ and $\tilde{m}=0$), see Figure 17.

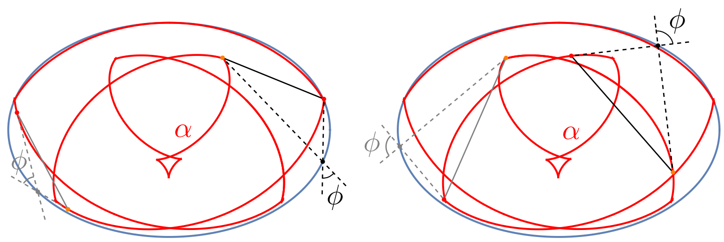


FIGURE 17. A (ϕ,ℓ)-isochordal-viewed curve α whose ϕ-isoptic is an ellipse, both for the angles $\phi=\frac{\pi}{4}$ and $\phi=\frac{\pi}{2}$.

Notice also that once admissible free parameters k and m are chosen, we actually get a 1-parametric family of isochordal-viewed curves $\alpha_{k,m}$ dependent on a parameter $k_1\in\mathbb{R}$.

Example 5. Let us consider now the case of a circular isoptic

$$c(t)=(\cos t,\sin t).$$

The (ϕ,ℓ)-isochordal-viewed curves that we will construct are multihedgehogs (Corollary 1) and the complete classification of (ϕ,ℓ)-isochordal-viewed multihedgehogs with circular ϕ-isoptics is already known [19]. Therefore, the examples that we can construct with our method shall be contained in the known explicit classification. See in Figure 18 some examples of these curves using Theorem 4.

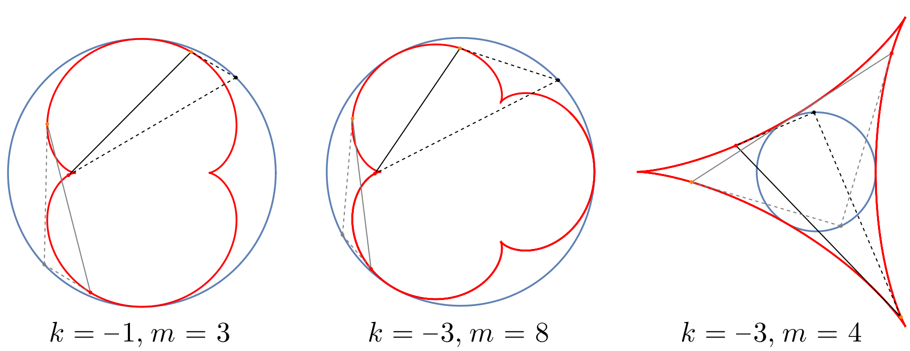


FIGURE 18. Examples of (ϕ,ℓ)-isochordal-viewed curves with a circular isoptic.

6. Regular polygons on isochordal-viewed curves

For (ϕ,ℓ)-isochordal-viewed hedgehogs of constant ϕ-width (i.e., with a circular ϕ-isoptic), a smooth motion of some regular polygons whose vertices moved along the curves was shown in [18]. In the given proof of this fact, the hypothesis of having a circular ϕ-isoptic was used to prove that the polygon was equiangular. Later, in [19] a generalized version to multihedgehogs of constant ϕ-width was given. Now, we address a natural question on what happens for general isochordal-viewed curves, not necessarily having circular isoptics. We prove below that the isochordal-viewed curves constructed by our method (Theorem 4) also satisfy this property.

Theorem 5. *Under the hypothesis of Theorem 4, if* $\gcd(k,m)=1$*, then there exists* $N\in\mathbb{N}$*,* $N\geq 2$*, such that a regular polygon of* N *sides of length* ℓ *is allowed to move with its vertices always lying on the isochordal-viewed curve* $\alpha_{k,m}$*.*

Proof. Denote $\alpha=\alpha_{k,m}$ and $a=|I|$. Since α is (ϕ,ℓ)-isochordal viewed, the polyline $\Gamma(t)$ formed by joining the points $\alpha(t)$, $\alpha(t+\tilde{k}\,a)$, $\alpha(t+2\,\tilde{k}\,a)$, etc. has side length ℓ.

We want to find a natural number N such that $\alpha(t+N\,\tilde{k}\,a)=\alpha(t)$. We know that

$$\gamma(t+\tilde{k}\,a)=\gamma(t)-\phi+\tilde{m}\,\pi.$$

Therefore, for any $N\in\mathbb{N}$, we get

$$\gamma(t+N\,\tilde{k}\,a)=\gamma(t)-N\,\phi+N\,\tilde{m}\,\pi, \tag{14}$$

by applying N times the identity above.

For any $N\in\mathbb{N}$, since $c(t+N\,\tilde{k}\,a)=c(t)$ and using (14), from the expression of α we deduce

$$\begin{aligned}&\alpha(t+N\,\tilde{k}\,a)-\alpha(t)\\ &\qquad=\frac{\mathcal{L}(c)\,m}{2\pi\,(k+m\,n)}\sin(N\phi)\Big(-\sin\big(N\phi-2\,\gamma(t)\big)\,\mathbf{n}_c(t)+\cos\big(N\phi-2\,\gamma(t)\big)\,\mathbf{t}_c(t)\Big).\end{aligned}$$

Notice that $\alpha(t+N\,\tilde{k}\,a)=\alpha(t)$ if and only if $\sin(N\phi)=0$, which happens if and only if there exists $r\in\mathbb{Z}$ such that $\phi=\pi\,\frac{r}{N}$. This trivially holds from the expression

$$\phi=\tilde{m}\,\pi-2\,\pi\,\tilde{k}\,\frac{k}{m}=\pi\,\frac{m\,\tilde{m}-2\,k\,\tilde{k}}{m}$$

by taking $N:=m$ and $r:=m\,\tilde{m}-2\,k\,\tilde{k}$. Notice that this N is not necessarily the minimum natural number that closes the polyline $\Gamma(t)$, but its existence ensures that $\Gamma(t)$ is closed. Notice that $N\neq 1$ because otherwise the chord length ℓ would be zero.

It is only left to prove that $\Gamma(t)$ is equiangular. Without loss of generality, we can check that the angle between the vectors $\alpha(t+\tilde{k}\,a)-\alpha(t)$ and $\alpha(t-\tilde{k}\,a)-\alpha(t)$ is constant.

With the same computations performed above, we have that

$$\alpha(t+\tilde{k}\,a)-\alpha(t)=\pm\ell\,\Big(-\sin\big(\phi-2\,\gamma(t)\big)\,\mathbf{n}_c(t)+\cos\big(\phi-2\,\gamma(t)\big)\,\mathbf{t}_c(t)\Big)$$

and
$$\alpha(t-\tilde{k}\,a)-\alpha(t)=\mp\ell\,\Big(\sin\big(\phi+2\,\gamma(t)\big)\,\mathbf{n}_c(t)+\cos\big(\phi+2\,\gamma(t)\big)\,\mathbf{t}_c(t)\Big).$$
The two possible signs of the expressions above is because of the absolute value in
$$\ell=\frac{\mathcal{L}(c)}{2\pi}\left|\frac{m}{k+m\,n}\right|\sin\phi.$$
Hence,
$$\frac{1}{\ell^2}\left\langle\alpha(t+\tilde{k}\,a)-\alpha(t),\alpha(t-\tilde{k}\,a)-\alpha(t)\right\rangle=\mp\cos(2\,\phi),$$
which means that the angle is either $2\,\phi$ or $\pi-2\,\phi$ up to a multiple of π. In any case, constant in all the vertices of $\Gamma(t)$, which proves that $\Gamma(t)$ is a regular polygon. $\square$

Example 6. Although the regular polygon of Theorem 5 can be degenerated if $N=2$, this is not always the case. See in Figure 19 some examples of (ϕ,ℓ)-isochordal-viewed curves constructed with Theorem 4 together with a regular polygon of N sides that can slide along the curve.

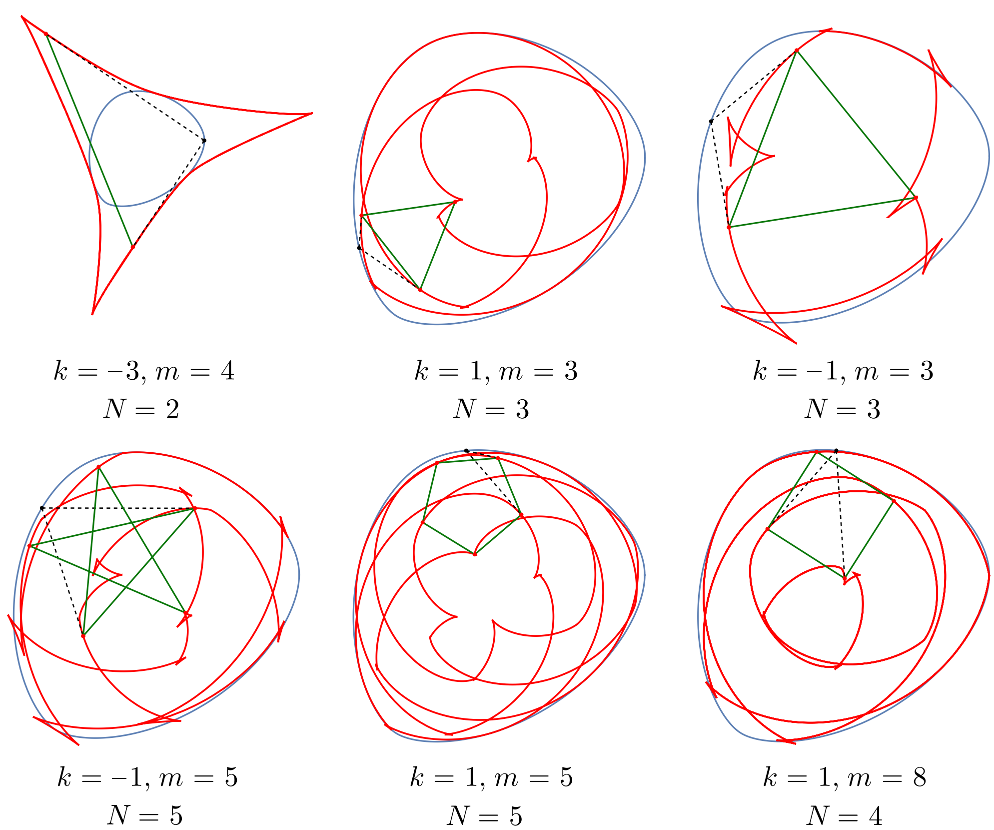


FIGURE 19. Some (ϕ,ℓ)-isochordal-viewed curves with a non-symmetric curve as their ϕ-isoptic, for some values of the free parameters k and m. The motion of a regular polygon of $N\geq 2$ sides along the curve is possible.

A frame-by-frame visualization of the motion of one of these regular polygons over an isochordal-viewed curve can be find in Figure 20.

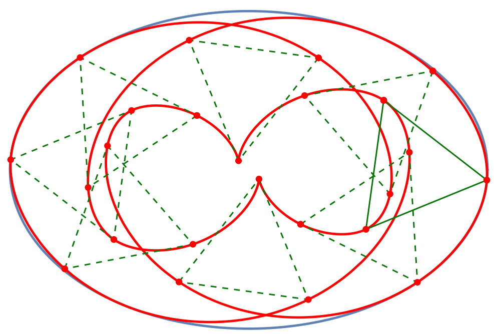


FIGURE 20. Some positions of the motion of a triangle over an isochordal-viewed curve with an elliptic isoptic.

7. Conclusions

We have studied the inverse isoptic problem and have provided a constructive method to generate curves whose ϕ-isoptic is a given curve c. We have incorporated the constant chord length ℓ constraint to our construction to generate also (ϕ, ℓ)-isochordal-viewed curves with a given ϕ-isoptic c. This constructive approach positively answers the open question if there exist isochordal-viewed curves that are not of constant ϕ-width (that is, whose ϕ-isoptic is not circular). We have also studied kinematic motions of a regular polygon along the isochordal-viewed curves generated by our method and have proved that a circular isoptic is not a necessary condition to have this property.

There are a few questions that remained unanswered, namely if our construction generates all (ϕ, ℓ)-isochordal-viewed curves, i.e., if it is a necessary condition, not just sufficient. Notice that Theorem 2 provides a sufficient condition that is not necessary (Remark 4), however this could be different if the isochordal condition is set. In fact, we must point out that we have not found any example of a (ϕ, ℓ)-isochordal-viewed multihedgehog with a circular isoptic that cannot be generated with our construction as well.

Another related open problem is to answer if there exists any convex isochordal-viewed curve other than the circle (see partial results towards this conjecture in [6] and [1]). The results of this paper may be interesting to address this conjecture as well.

Acknowledgements

This work has been partially supported by the Basque Government through the BERC 2022–2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa accreditation of excellence CEX2021-001142-S / MCIN / AEI / 10.13039/501100011033.

References

1. Víctor A. Aguilar-Arteaga, Iván Gónzalez-García, Jesús Jerónimo-Castro, and Francisco G. Jimenez-Lopez, *Some characterizations of the disc by properties of isoptic triangles*, (2022), preprint.
2. Michael Bartoň, Gershon Elber, and Iddo Hanniel, *Topologically guaranteed univariate solutions of underconstrained polynomial systems via no-loop and single-component tests*, Comput. Aided Des. **43** (2011), no. 8, 1035–1044.
3. Waldemar Cieślak, Andrzej Miernowski, and Witold Mozgawa, *Isoptics of a closed strictly convex curve*, Global differential geometry and global analysis (Berlin, 1990), Lecture Notes in Math., vol. 1481, Springer, Berlin, 1991, pp. 28–35.
4. Waldemar Cieślak and Witold Mozgawa, *On curves with circles as their isoptics*, Aequat. Math. **96** (2022), no. 3, 653–667.
5. Csima Géza, *Isoptic curves and surfaces*, Ph.D. thesis, Institute of Mathematics, Budapest University of Technology and Economics, 2017.
6. Jesús Jerónimo-Castro, Francisco G. Jimenez-Lopez, and A. Rosario Jiménez-Sánchez, *On convex bodies with isoptic chords of constant length*, Aequat. Math. **94** (2020), no. 6, 1189–1199.
7. Jesús Jerónimo-Castro, Marco Antonio Rojas-Tapia, Ulises Velasco-García, and Carlos Yee-Romero, *An isoperimetric inequality for isoptic curves of convex bodies*, Results Math. **75** (2020), no. 4, Paper No. 134, 12 pp.
8. János Kincses, *The determination of a convex set from its angle function*, vol. 30, 2003, U.S.-Hungarian Workshops on Discrete Geometry and Convexity (Budapest, 1999/Auburn, AL, 2000), pp. 287–297.
9. János Kincses and Árpád Kurusa, *Can you recognize the shape of a figure from its shadows?*, Beitr. Algebra Geom. **36** (1995), no. 1, 25–35.
10. Árpád Kurusa, *The shadow picture problem for nonintersecting curves*, Geom. Dedicata **59** (1996), no. 1, 103–112.
11. ——, *You can recognize the shape of a figure from its shadows!*, Geom. Dedicata **59** (1996), no. 2, 113–125.
12. ——, *Is a convex plane body determined by an isoptic?*, Beitr. Algebra Geom. **53** (2012), no. 1, 281–294.
13. ——, *The shadow picture problem for parallel straight lines*, J. Geom. **103** (2012), no. 3, 515–518.
14. Yves Martinez-Maure, *Les multihérissons et le théorème de Sturm-Hurwitz*, Arch. Math. (Basel) **80** (2003), no. 1, 79–86.
15. ——, *A Sturm-type comparison theorem by a geometric study of plane multihedgehogs*, Illinois J. Math. **52** (2008), no. 3, 981–993.
16. Andrzej Miernowski and Witold Mozgawa, *Isoptics of pairs of nested closed strictly convex curves and Crofton-type formulas*, Beitr. Algebra Geom. **42** (2001), no. 1, 281–288.
17. David Rochera, *On isoptics and isochordal-viewed curves*, Aequat. Math. **96** (2022), no. 3, 599–620.
18. ——, *Regular polygons on isochordal-viewed hedgehogs*, Aequat. Math. **96** (2022), no. 6, 1285–1301.
19. ——, *An explicit characterization of isochordal-viewed multihedgehogs with circular isoptics*, J. Math. Anal. Appl. **524** (2023), no. 2, Paper No. 127107, 15 pp.
20. Harold A Rothbart, *Cam design handbook*, McGraw-Hill Education, 2004.
21. Walter Wunderlich, *Short Communications: Kurven mit isoptischem Kreis*, Aequat. Math. **5** (1970), no. 2-3, 337–338.
22. ——, *Contributions to the geometry of cam mechanisms with oscillating followers*, J. Mech. **6** (1971), 1–20.
23. ——, *Kurven mit isoptischem Kreis*, Aequat. Math. **6** (1971), no. 1, 71–81.
24. ——, *Kurven mit isoptischer Ellipse*, Monatsh. Math. **75** (1971), 346–362.
25. ——, *Single-disk cam mechanisms with oscillating double roller follower*, Mech. Mach. Theory **19** (1984), 409–415.

Email address, D. Rochera: `drochera@bcamath.org`

Email address, M. Bartoň: `mbarton@bcamath.org`

Alameda de Mazarredo, 14, E-48009 Bilbao (Basque Country), Spain