

# Solving boundary value problems via the Nyström method using spline Gauss rules

Ali Hashemian<sup>a,\*</sup>, Hanna Sliusarenko<sup>a</sup>, Sara Remogna<sup>b</sup>, Domingo Barrera<sup>c,d</sup>, Michael Bartoň<sup>a,e</sup>

<sup>a</sup>BCAM – Basque Center for Applied Mathematics, Alameda Mazarredo 14, 48009 Bilbao, Basque Country, Spain

<sup>b</sup>Department of Mathematics “Giuseppe Peano”, University of Torino, Via Carlo Alberto 10, 10123 Torino, Italy

<sup>c</sup>Department of Applied Mathematics, University of Granada, Campus de Fuentenueva, 18071 Granada, Spain

<sup>d</sup>IMAG – Institute of Mathematics, Ventanilla 11, 18001 Granada, Spain

<sup>e</sup>Iberbasque – Basque Foundation for Sciences, María Díaz de Haro 3, 48013 Bilbao, Basque Country, Spain

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## Abstract

We propose to use spline Gauss quadrature rules for solving boundary value problems (BVPs) using the Nyström method. When solving BVPs, one converts the corresponding partial differential equation inside a domain into the Fredholm integral equation of the second kind on the boundary in the sense of boundary integral equation (BIE). The Fredholm integral equation is then solved using the Nyström method, which involves the use of a particular quadrature rule, thus, converting the BIE problem to a linear system. We demonstrate this concept on the 2D Laplace problem over domains with smooth boundary as well as domains containing corners. We validate our approach on benchmark examples and the results indicate that, for a fixed number of quadrature points (i.e., the same computational effort), the spline Gauss quadratures return an approximation that is by one to two orders of magnitude more accurate compared to the solution obtained by traditional polynomial Gauss counterparts.

*Keywords:* Boundary value problems; Fredholm integral equation; Nyström method; spline Gauss quadratures.

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## 1. Introduction

Boundary value problems (BVPs) are well known in engineering and science, where the given partial differential equation (PDE) is solved with respect to, e.g., Dirichlet or Neumann boundary conditions. When solving BVPs, among other approaches, one converts the corresponding PDE inside the domain into the Fredholm integral equation of the second kind on the boundary in the sense of boundary integral equation (BIE) [1–3]. The main idea is to use the fundamental solution of the governing PDE, so that approximations only occur on the domain’s boundary (the solution inside the domain is then obtained via post-processing). Thus, the order of the problem is reduced by one dimension compared to, e.g., finite element analysis (FEA), where the solution approximation is performed throughout the whole domain. Another advantage is that one avoids meshing the entire domain, thus, reducing the effects of the mesh quality on the computed results.

There exist different methods addressing the numerical solution of the Fredholm integral equation of the second kind. The well-studied approaches are Galerkin (see, e.g., [3–5]) and collocation (see, e.g. [6–9]) methods. The idea of discretizing boundary integrals in the sense of finite elements, referred to as boundary element method (BEM) [10, 11], and its extension to isogeometric analysis (IGA) [12], commonly known as IGA-BEM [13–17], are also among the well-established research in the literature. Another thematically relevant class of research are degenerate kernel methods [18, 19] allowing to approximate the integral equation with a degenerate kernel, whose solution is determined by solving a linear system of equations. Spline quasi-interpolations are known to be very useful for this purpose (c.f. [20–26] and references therein).

We consider the Nyström method [27, 28] as a classical approach for the numerical solution of Fredholm integral equations of the second kind. This method approximates the integral equation using a particular

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\*Corresponding author

Email address: [ahashemian@bcamath.org](mailto:ahashemian@bcamath.org) (Ali Hashemian)

quadrature rule, thus, converting the BIE problem to a linear system of equations. The solution, obtained at quadrature points on the boundary, is then used to approximate the solution over the entire domain. In combination with other numerical methods such as FEA [29, 30] and IGA [31, 32], the Nyström method has been applied to solve problems of great practical interests with applications in, e.g., electromagnetics [33–35], fluid mechanics [36, 37] and structural analysis [38, 39].

Our research aims to point out that the way the quadrature points distribute over the boundary plays an important role in the accuracy and efficiency of the Nyström method. The classical polynomial Gaussian quadrature rules are among the well-known schemes for this purpose, yet not the only option. A survey of research on the application of different quadrature rules in the Nyström method can be found in, e.g., [40]. In this work, we propose to use spline Gauss quadrature rules for solving boundary value problems via the Nyström method. Spline Gauss rules are proven as successful alternatives to classical polynomial Gauss rules when integrating high continuous functions (see, e.g., [41, 42]). When solving the Fredholm integral equation of the second kind arising from a PDE inside a 2D domain, we use a NURBS representation of the boundary curve and place the quadrature points within the knot spans of the NURBS boundary. In this context, we compare the accuracy of the solution obtained by the numerical integration using spline Gauss rules versus the solution obtained by polynomial Gauss rules. We solve the Laplace problem with Dirichlet boundary conditions on different geometries with smooth continuous boundary as well as domains containing corners. We show that when fixing the total number of quadrature points (i.e., with the same computational effort), the spline rules return an approximation that is by one to two orders of magnitude more accurate compared to the solution obtained by traditional polynomial Gauss counterparts.

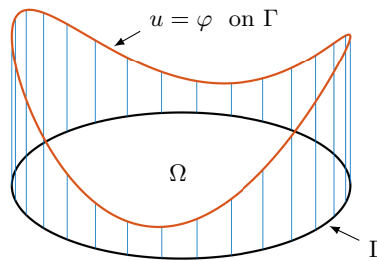
The structure of the remainder of this paper is as follows. Section 2 introduces the boundary value problem and its solution via the Nyström method. Section 3 briefly recalls the notion of B-splines and Gauss quadrature rules for spline spaces. Section 4 discusses the NURBS-based boundary representation and the application of the spline Gauss rules to the solution of the BVPs via the Nyström method. Section 5 shows numerical results on three test cases, and finally, Section 6 draws some conclusions and indicates possible directions for future research.

## 2. Solving boundary value problems using Nyström method

Let us consider a computational domain  $\Omega \subset \mathbb{R}^2$  to be the interior of a closed smooth boundary  $\Gamma := \partial\Omega$ . We consider the following homogeneous Dirichlet boundary value problem (see Fig. 1):

$$\begin{aligned} &\text{Find } u : \Omega \rightarrow \mathbb{R} \text{ such that} \\ &\begin{cases} \mathcal{L}u = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \Gamma, \end{cases} \end{aligned} \quad (2.1)$$

where  $\mathcal{L}$  is the partial differential operator (e.g.,  $\mathcal{L} = -\Delta$  for the Laplace equation).



**Fig. 1.** Dirichlet boundary value problem: an example domain  $\Omega$ , its boundary  $\Gamma$ , and the boundary condition  $\varphi$ .

### 2.1. Fredholm integral equation

When solving (2.1), one converts the corresponding partial differential equation inside  $\Omega$  into the Fredholm integral equation of the second kind on  $\Gamma$  in the sense of boundary integral equation (see, e.g., [1, 43]). Let us consider  $\mathbf{x}$  and  $\mathbf{y}$  as points in  $\mathbb{R}^2$ . We assume that  $\mathbf{x}$  is a point that lies either inside the domain or

on the boundary, and we further use  $\hat{\mathbf{x}}$  to emphasize the case when  $\mathbf{x} \in \Gamma$ . The second point  $\mathbf{y}$  is assumed to lie strictly on the boundary, i.e.,  $\mathbf{y} \in \Gamma$ . Given the solution on the boundary  $\varphi(\hat{\mathbf{x}})$ , the Fredholm integral equation reads

$$\sigma(\hat{\mathbf{x}}) - \int_{\Gamma} K(\hat{\mathbf{x}}, \mathbf{y}) \sigma(\mathbf{y}) d\Gamma_{\mathbf{y}} = -2\varphi(\hat{\mathbf{x}}), \quad \hat{\mathbf{x}}, \mathbf{y} \in \Gamma, \quad (2.2)$$

where  $\sigma$  and  $K$  are the *double-layer density* and *kernel*, respectively. Solving (2.2), one obtains the density  $\sigma$  on  $\Gamma$  and, then, the solution  $u$  of the original problem (2.1) over  $\Omega$ , also called *the double-layer potential*, becomes

$$u(\mathbf{x}) = \frac{1}{2} \int_{\Gamma} K(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) d\Gamma_{\mathbf{y}}, \quad \mathbf{x} \in \Omega \setminus \Gamma, \quad \mathbf{y} \in \Gamma. \quad (2.3)$$

One obtains the kernel in (2.2) and (2.3) from the normal derivative of the *fundamental solution*  $s(\mathbf{x}, \mathbf{y})$  with respect to  $\mathbf{y}$ , that is

$$K(\mathbf{x}, \mathbf{y}) := 2 \frac{\partial}{\partial n_{\mathbf{y}}} s(\mathbf{x}, \mathbf{y}) = 2 \langle \mathbf{n}(\mathbf{y}), \nabla_{\mathbf{y}} s(\mathbf{x}, \mathbf{y}) \rangle, \quad (2.4)$$

where  $\mathbf{n}$  is the outward unit normal vector on the boundary and  $\langle \cdot, \cdot \rangle$  is the Euclidean scalar product. The fundamental solution of the differential operator  $\mathcal{L}$  is a function of the Euclidean norm  $\|\mathbf{y} - \mathbf{x}\|$  satisfying  $\mathcal{L}s = 0$  for  $\mathbf{x} \neq \mathbf{y}$  (see, e.g., [1]). Herein and in the following, we restrict ourselves to the 2D Laplace equation with the following fundamental solution (for other differential equations, c.f. [1, 2, 17]):

$$s(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log \|\mathbf{y} - \mathbf{x}\|. \quad (2.5)$$

Thus, the double-layer kernel, which is a function of the geometry of the domain, is

$$K(\mathbf{x}, \mathbf{y}) = -\frac{1}{\pi} \frac{\langle \mathbf{n}(\mathbf{y}), \mathbf{y} - \mathbf{x} \rangle}{\|\mathbf{y} - \mathbf{x}\|^2}, \quad \mathbf{x} \in \Omega, \quad \mathbf{y} \in \Gamma. \quad (2.6)$$

## 2.2. Nyström method

We start from the general form of the Fredholm integral equation over a univariate domain  $D \subset \mathbb{R}$ , that is

$$\lambda \sigma(x) - \int_D K(x, y) \sigma(y) dy = \psi(x), \quad x, y \in D. \quad (2.7)$$

Considering an  $m_r$ -point quadrature rule, the approximation of the integral reads

$$\int_D K(x, y) \sigma(y) dy \approx \sum_{j=1}^{m_r} \omega_j K(x, \tau_j) \sigma_r(\tau_j), \quad (2.8)$$

where  $\tau_j$  and  $\omega_j$  are quadrature points and weights, respectively. In here, we use subscript  $r \geq 1$  to refer to the level of refinement we consider to make the quadrature rule finer and finer. We assume that for every continuous function, the numerical integrals converge to the true integral as  $r \rightarrow \infty$ . This implies,

$$\sup_{r \geq 1} \sum_{j=1}^{m_r} |\omega_j| < \infty. \quad (2.9)$$

Thus, one obtains

$$\lambda \sigma_r(x) - \sum_{j=1}^{m_r} \omega_j K(x, \tau_j) \sigma_r(\tau_j) = \psi(x). \quad (2.10)$$

The Nyström method approximates the density  $\sigma_r$  at quadrature points, that is

$$\lambda \sigma_r(\tau_i) - \sum_{j=1}^{m_r} \omega_j K(\tau_i, \tau_j) \sigma_r(\tau_j) = \psi(\tau_i), \quad i = 1, 2, \dots, m_r, \quad (2.11)$$

thus, converting the integral equation (2.7) to a  $m_r \times m_r$  linear system

$$(\lambda \mathbf{I} - \mathbf{K}_r) \boldsymbol{\sigma}_r = \boldsymbol{\psi}_r, \quad (2.12)$$

where  $\mathbf{I}$  is the unit matrix and the components of  $\mathbf{K}_r$ ,  $\boldsymbol{\sigma}_r$ , and  $\boldsymbol{\psi}_r$  are obtained as:

$$K_{r,ij} := \omega_j K(\tau_i, \tau_j), \quad i, j = 1, 2, \dots, m_r, \quad (2.13)$$

$$\sigma_{r,i} := \sigma_r(\tau_i), \quad i = 1, 2, \dots, m_r, \quad (2.14)$$

$$\psi_{r,i} := \psi(\tau_i), \quad i = 1, 2, \dots, m_r. \quad (2.15)$$

Let us consider the Banach space  $\mathcal{X} = C(D)$  and the operators  $\mathcal{K}, \mathcal{K}_r : \mathcal{X} \rightarrow \mathcal{X}$  defined as

$$\mathcal{K}\sigma(x) := \int_D K(x, y) \sigma(y) dy, \quad (2.16)$$

$$\mathcal{K}_r \sigma_r(x) := \sum_{j=1}^{m_r} \omega_j K(x, \tau_j) \sigma_r(\tau_j), \quad (2.17)$$

associated respectively with the integral equation (2.7) and a sequence of quadrature rules of the form (2.8) such that

$$\|\mathcal{K}_r\|_\infty = \max_{x \in D} \sum_{j=1}^{m_r} |\omega_j K(x, \tau_j)|. \quad (2.18)$$

Regarding the convergence of the sequence  $\sigma_r$  provided by the Nyström method approximation, we state the following Theorem [44].

**Theorem 1.** Let  $K(x, y)$  be a continuous kernel defined on  $D \times D$ , and suppose that the sequence (2.8) of quadrature rules converges for all continuous functions defined on  $D$ . Moreover, let us suppose that the integral equation (2.7) admits a unique solution for all function  $\psi \in C(D)$  with  $\lambda \neq 0$ . Then, for  $r$  enough large, for instance  $r \geq \tilde{r}$ , the operator  $(\lambda - \mathcal{K}_r)^{-1}$  exists and is uniformly bounded. More precisely, there exists a constant  $c$  such that

$$\|(\lambda - \mathcal{K}_r)^{-1}\|_\infty \leq \frac{1 + \|(\lambda - \mathcal{K})^{-1}\|_\infty \|\mathcal{K}_r\|_\infty}{|\lambda| - \|(\lambda - \mathcal{K})^{-1}\|_\infty \|(\mathcal{K} - \mathcal{K}_r)\mathcal{K}_r\|_\infty} \leq c, \quad r \geq \tilde{r}. \quad (2.19)$$

Furthermore, for the solutions of equations  $(\lambda - \mathcal{K})\sigma = \psi$  and  $(\lambda - \mathcal{K}_r)\sigma_r = \psi$ , it holds

$$\|\sigma - \sigma_r\|_\infty \leq \|(\lambda - \mathcal{K}_r)^{-1}\|_\infty \|(\mathcal{K} - \mathcal{K}_r)\sigma\|_\infty \leq c \|(\mathcal{K} - \mathcal{K}_r)\sigma\|_\infty, \quad r \geq \tilde{r}. \quad (2.20)$$

We recall that the sequences of Gaussian quadrature rules (considered in this work) are convergent for all continuous functions because they have positive weights (see, e.g., [45, Theorem 3] and [46, p. 130]).

**Remark 1.** If there is no danger of confusion, we omit the subscript  $r$  for the number of quadrature points  $m_r$  and will write simply  $m$ . Our limiting process, for  $r \rightarrow \infty$ , will be realized in terms of refinement of the domain  $D$  into  $N$  subdomains, each corresponding to an  $m$ -point quadrature rule. The weighted kernel matrix in (2.13) is then evaluated *globally* for the entire  $D$ , i.e., for  $i, j = 1, 2, \dots, M$ , where  $M := mN$ .

The application of the Nyström method to multidimensional PDEs is analogous. Considering (2.2), we set  $\lambda = 1$  and  $\psi = -2\varphi$ . In the 2D case, particularly, we consider the quadrature points along the boundary curve  $\Gamma$ . Thus, when integrating (2.2), we need to perform a curve *rectification* to transfer from  $\Gamma \subset \mathbb{R}^2$  to the univariate domain  $D \subset \mathbb{R}$ . For this purpose, we consider a NURBS representation of the boundary that maps  $D$  onto  $\Gamma$  (see Section 4.2). Once we compute densities on the boundary from (2.12), we obtain the point-wise solution  $u$  at each arbitrary point  $\mathbf{x}$  inside the domain from (2.3).

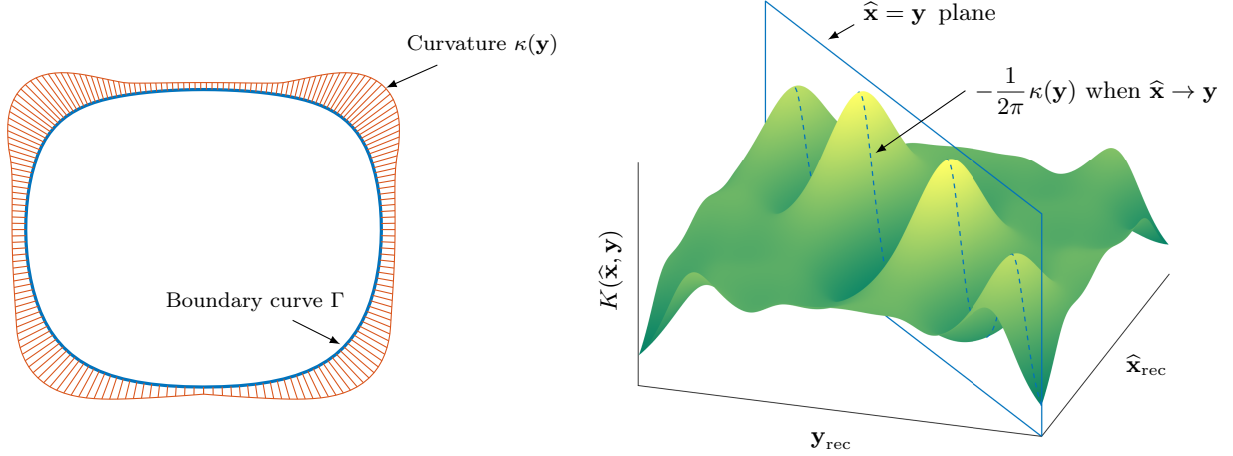
**Remark 2.** For every  $\mathbf{x} \in \Omega \setminus \Gamma$ , the kernel is non-singular. For boundary points  $\hat{\mathbf{x}} \in \Gamma$ , however, singularity may occur in the limit when  $\hat{\mathbf{x}} \rightarrow \mathbf{y}$ . At such points, when  $\Gamma$  is at least  $C^2$  continuous, we have no singularities



as the kernel expresses the (scaled) curvature of the boundary curve (see Fig. 2). More precisely, one obtains the kernel as follows [1]:

$$\lim_{\hat{\mathbf{x}} \rightarrow \mathbf{y}} K(\hat{\mathbf{x}}, \mathbf{y}) = -\frac{1}{2\pi} \kappa(\mathbf{y}), \quad (2.21)$$

where  $\kappa(\mathbf{y})$  denotes the curvature of  $\Gamma$  at  $\mathbf{y}$ . For  $C^0$  boundaries, the singularity occurs at the vicinity of corners, where one observes a discontinuity of the normal vector. For such occasions, the Strain's locally correction algorithm can be used. We refer the readers to [31, 47, 48] for more details.



**Fig. 2.** Left: A general domain represented by its  $C^2$  boundary curve  $\Gamma$ . We show the curvature distribution over the boundary. Right: 2D representation of the kernel  $K(\hat{\mathbf{x}}, \mathbf{y})$  on the boundary. When  $\hat{\mathbf{x}} \rightarrow \mathbf{y}$ , the kernel is equal to the scaled curvature of the boundary curve (dashed line). For better representation, we plot the kernel against  $\hat{\mathbf{x}}_{\text{rec}}$  and  $\mathbf{y}_{\text{rec}}$  as the rectified values of  $\hat{\mathbf{x}}$  and  $\mathbf{y}$ , respectively (see Section 4.2).

**Remark 3.** When dealing with interfaces in the problem domain, the kernel function varies across the interface, leading to discontinuities in the kernel matrix. In such occasions, one can use the high-order Nyström method that handles discontinuities in the kernel function across the interface (see, e.g., [49, 50]). However, since the main objective of this paper is to study the efficiency of spline Gauss rules versus classical polynomial Gauss quadratures, we do not study these cases here.

### 3. Spline Gauss quadrature rules

#### 3.1. B-spline spaces: preliminaries

We use B-spline spaces both for spline Gauss quadrature rules and for the NURBS representation of the domain's boundary (see Section 4). We define the spline space  $\mathcal{S}_{p,c}^N$  as the set of  $p$ -th degree B-spline basis functions spanned over the knot sequence

$$\Xi := \{\underbrace{\xi_0, \dots, \xi_0}_{\mu_0}, \underbrace{\xi_1, \dots, \xi_1}_{\mu_1}, \dots, \underbrace{\xi_N, \dots, \xi_N}_{\mu_N}\} = \{\Xi_0, \Xi_1, \dots, \Xi_{n+p+1}\}, \quad (3.1)$$

where  $N$  knot spans are characterized by non-repeating knots  $\xi_k$ ,  $k = 0, 1, \dots, N$ . We denote knot *multiplicities* by  $\mu_k$  such that  $1 \leq \mu_k \leq p+1$  and  $\sum_{k=0}^N \mu_k = n+p+2$  where  $n+1$  is the dimension of the spline space. We also introduce  $\mathbf{c}$  as the continuity vector whose components are  $c_k := p - \mu_k$ . For any arbitrary parameter  $\xi_0 \leq \xi \leq \xi_N$ , we obtain the  $i$ -th basis function  $B_{i,p}(\xi)$ ,  $i = 0, 1, \dots, n$ , by the Cox–De Boor recursion as follows [51]:

$$B_{i,0}(\xi) = \begin{cases} 1, & \Xi_i \leq \xi < \Xi_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (3.2)$$

$$B_{i,p}(\xi) = \frac{\xi - \Xi_i}{\Xi_{i+p} - \Xi_i} B_{i,p-1}(\xi) + \frac{\Xi_{i+p+1} - \xi}{\Xi_{i+p+1} - \Xi_{i+1}} B_{i+1,p-1}(\xi), \quad (3.3)$$

where its  $l$ -th order derivative ( $l = 1, 2, \dots, p$ ) is given by:

$$B_{i,p}^{(l)}(\xi) = p \left( \frac{B_{i,p-1}^{(l-1)}(\xi)}{\Xi_{i+p} - \Xi_i} - \frac{B_{i+1,p-1}^{(l-1)}(\xi)}{\Xi_{i+p+1} - \Xi_{i+1}} \right). \quad (3.4)$$

When evaluating basis functions and their derivatives at a desired  $\xi$ , we find the corresponding nonzero knot span in (3.2) and efficiently evaluate (3.3) and (3.4) avoiding any division by zero in dealing with repetitive knots. More details are given in [51, Algorithms A2.1–A2.3] and [52, Algorithms A1 and A2].

### 3.2. Spline vs. polynomial Gauss quadrature rules

Let  $D$  be our univariate integration domain and  $T \subset D$  be a set of  $m$  quadrature points  $\tau_j, j = 1, 2, \dots, m$ , inside  $D$ . We consider the  $m$ -point quadrature rule

$$\int_D f(x) dx \approx \sum_{j=1}^m \omega_j f(\tau_j), \quad \tau_j \in T \subset D, \quad (3.5)$$

that approximates the integral of  $f(x)$  over  $D$  and converges to the exact integral as  $m \rightarrow \infty$ . Typically, equation (3.5) is not just an approximation, but it is exact for a certain linear space of functions. A rule is said to be Gaussian if  $m$  is the minimum required number of quadrature points at which  $f$  is evaluated, while guaranteeing the exactness of the integration.

**Remark 4.** When integrating polynomial functions, the optimal rule in terms of the number of quadrature points is known to be the classical Gauss quadrature rule [53] with the order of exactness  $2m - 1$ , that is, a set of  $m$  evaluations are needed to exactly integrate any polynomial of degree at most  $2m - 1$  over  $D$ .

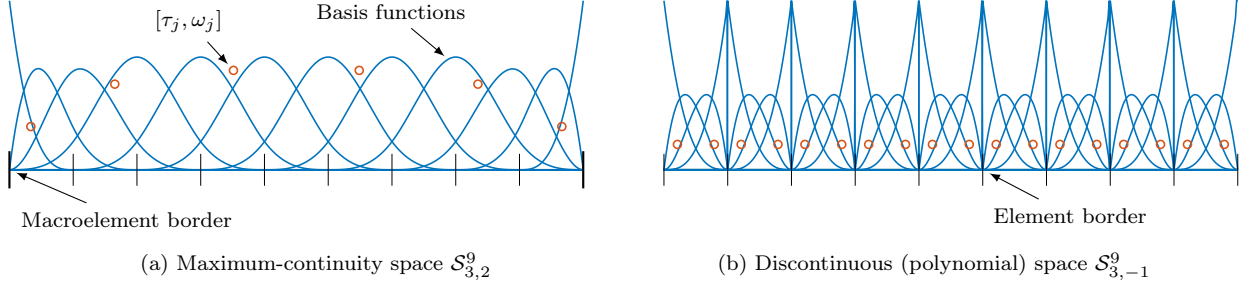
Let the integration domain consist of  $N$  elements  $D_k \subset D, k = 1, 2, \dots, N$ , the element-wise integration over  $D$  using the classical polynomial Gauss quadrature rule entails an  $m$ -point rule within *each* element. This is equal to the discontinuous spline Gauss rule when using a spline space with  $C^{-1}$  continuity at all elements interfaces. However, when integrating continuous functions, it is computationally inefficient to use polynomial Gauss rules on every element. Instead, we use high-continuity spline Gauss rules, thus performing the integration over *macroelements*. In this manner, every macroelement consists of a set of elements corresponding to knot spans of the respective spline space and the quadrature points are distributed over the macroelement. The existence and uniqueness of Gaussian quadrature rules for spline spaces of *uniform* continuity has been studied in [54] and [55, Theorem 3.4], respectively (herein by uniform we refer to spaces with equal multiplicities at all interior knots). Let us consider the spline space  $\mathcal{S}_{\tilde{p}, \tilde{c}}^{\tilde{N}}$  of  $\tilde{p}$ -th degree basis functions spanned over a macroelement of  $\tilde{N}$  uniform elements with a uniform continuity  $\tilde{c} := \tilde{p} - \tilde{\mu}$ , where  $\tilde{\mu}_1 = \tilde{\mu}_2 = \tilde{\mu}_{\tilde{N}-1} = \tilde{\mu}$  (we use  $\tilde{\cdot}$  to distinguish this spline space from the space we employ for the NURBS representation of the boundary in Section 4). For spline spaces, there always exists a Gaussian quadrature rule [55] where the number of necessary evaluations  $m$  is given by:

$$\tilde{p} + \iota + 1 = 2m, \quad (3.6)$$

noting that  $\iota := (\tilde{N} - 1)\tilde{\mu}$  is the total number of interior knots, including their multiplicities. This fact is in accordance with the dimension of the spline space, which is the maximum number of basis functions to be integrated exactly by only half the number of quadrature points. If  $\tilde{p} \ll \iota$ , then  $m \approx \iota/2$ , denoting that in the limit, when  $\tilde{N} \rightarrow \infty$ , the spline rules converge to the half-point rules of Hughes et al. [56], which are exact and Gaussian over the domain.

We use the polynomial homotopy continuation (PHC), a numerical scheme for solving polynomial systems of equations [57], to generate Gaussian quadrature rules for spline spaces of higher continuities (as it has been used in, e.g., [41, 42, 58]). The main advantage of using such spaces is that we can integrate using Gaussian rules with lower number of quadratures (i.e., evaluations), while preserving the same order of exactness (see [59]). In particular, to generate a Gaussian rule in a *target* spline space, we built an associated *source* space with known Gaussian quadratures (e.g., a union of polynomial Gauss rules) and transform the rule from the source space to the target space, while preserving the optimality. The exactness of the quadrature

rule is formulated as a polynomial system where the quadrature points and weights are zeros of this system. Using the homotopy continuation concept, the source space is continuously deformed by changing the source knot sequence towards the target configuration and the quadrature rule gets updated numerically by tracing the unique root of the continuously modified piecewise polynomial system. We omit details for the sake of brevity (see, e.g., [41] for more details). Fig. 3 shows two different cubic spaces with the same order of exactness spanned over the same number of elements. In particular, Fig. 3a represents the maximum-continuity spline space  $\mathcal{S}_{3,2}^9$  (spanned over a macroelement of size  $\tilde{N} = 9$ ) that needs only 6 quadrature points for all elements according to (3.6), while Fig. 3b shows the discontinuous space  $\mathcal{S}_{3,-1}^9$  that needs 18 quadrature points. Using the latter space is equivalent to the element-wise integration by the classical polynomial Gauss rule with two quadrature points within every element. From the computational point of view, with the same order of exactness, the spline rule is more efficient than the polynomial rule because it has a lower number of quadrature points (i.e., lower evaluations).



**Fig. 3.** Two different cubic spaces: (a)  $\mathcal{S}_{3,2}^9$  and (b)  $\mathcal{S}_{3,-1}^9$ , both spanned over 9 elements. The quadrature points  $\tau_j$  and respective weights  $\omega_j$  are shown with red circles. The spline space in (a) represents a 6-point rule for the entire domain and obtains the same order of exactness when compared to the discontinuous space in (b) that needs 18 quadrature points (equivalent to the 2-point element-wise polynomial Gauss rule).

Note that the computation of spline Gaussian rules via homotopy continuation is computationally expensive as it is an iterative procedure and, in each step, one solves a large system of nonlinear equations. For certain spline spaces, e.g., quintic  $C^1$  over uniform knots, there exists a recursive formula to compute the quadratures [60], which is cheap and could be embedded into our algorithm. However, one expects higher computational gain when using quadrature rules for spaces with maximum continuities. This entails computing the spline Gauss quadrature rules in advance, and reading them from look-up tables, which is a common practice. We use spline Gauss quadratures computed in [42, 58] in our algorithm (see later Table 1).

## 4. NURBS-based boundary representation

### 4.1. NURBS curve: a short review

Given the parametric domain  $D \subset \mathbb{R}$ , we represent the boundary curve  $\Gamma \subset \mathbb{R}^2$  as a  $p$ -th degree piecewise continuous NURBS curve  $\mathbf{C} : D \rightarrow \mathbb{R}^2$  with  $n + 1$  control points  $\mathbf{P}_i \in \mathbb{R}^2$  and corresponding non-negative weights  $w_i \in \mathbb{R}^+$ ,  $i = 0, 1, \dots, n$ . Considering the control point  $\mathbf{P}_i = (P_{i,x}, P_{i,y})$ , we use the *homogeneous coordinates* and introduce  $\mathbf{P}_i^w := (w_i P_{i,x}, w_i P_{i,y}, w_i)$  to represent the boundary by a (non-rational) B-spline curve in the homogeneous coordinates as follows [51]:

$$\mathbf{C}^w(\xi) = \sum_{i=0}^n B_{i,p}(\xi) \mathbf{P}_i^w. \quad (4.1)$$

Defining

$$\mathbf{A}(\xi) := (C_x^w(\xi), C_y^w(\xi)), \quad (4.2)$$

$$W(\xi) := \sum_{i=0}^n B_{i,p}(\xi) w_i, \quad (4.3)$$

one writes  $\mathbf{C}^w(\xi) = (\mathbf{A}(\xi), W(\xi))$ . Thus, the NURBS representation of the boundary curve  $\Gamma$  is

$$\mathbf{C}(\xi) = \frac{\mathbf{A}(\xi)}{W(\xi)}, \quad (4.4)$$

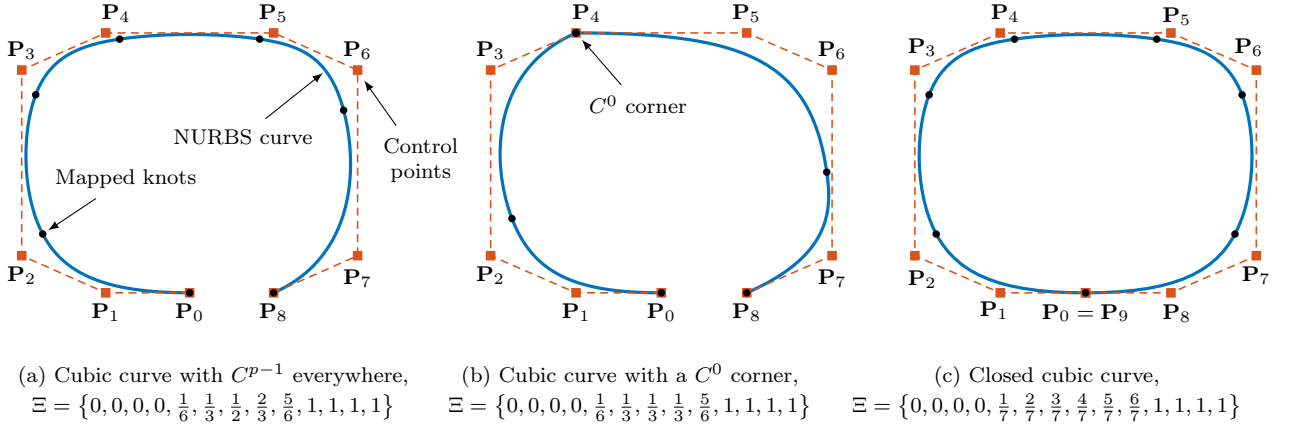
and its  $l$ -th order derivative reads

$$\mathbf{C}^{(l)}(\xi) = \frac{\mathbf{A}^{(l)}(\xi) - \sum_{i=1}^l \binom{l}{i} W^{(i)}(\xi) \mathbf{C}^{(l-i)}(\xi)}{W(\xi)}. \quad (4.5)$$

where we obtain  $\mathbf{A}^{(l)}(\xi)$  and  $W^{(l)}(\xi)$  from

$$\mathbf{C}^{w(l)}(\xi) = \sum_{i=0}^n B_{i,p}^{(l)}(\xi) \mathbf{P}_i^w = (\mathbf{A}^{(l)}(\xi), W^{(l)}(\xi)). \quad (4.6)$$

A piecewise NURBS curve of degree  $p$  with a knot sequence in the form of (3.1) is  $C^{p-\mu_k}$  continuous at  $\xi_k$ . If  $\mu_0 = \mu_N = p + 1$ , the curve is *clamped* at both ends (Figs. 4a–4c). If all *interior* knots are *single*, the curve has the maximum continuity everywhere (see, e.g., Fig. 4a). In the same manner, if the multiplicity of an interior knot equals to  $p$ , the curve is  $C^0$  at that knot, allowing us to model corners. The mapped position of such multiple knot on the curve coincides with the respective control point (Fig. 4b). The boundary  $\Gamma$  in the Fredholm integral equation (2.2) is a closed curve, simply obtained by adding a new control point wrapped on the first one (see Fig. 4c).



**Fig. 4.** Some possible representations of a clamped cubic NURBS curve with 9 (fixed) control points and respective knot sequences over  $D : [0, 1]$ . In (b), the multiplicity of an interior knot reaches  $p$ , thus, creating a corner on the curve. In (c), a new control point is wrapped on the first one to close the curve, thus, updating the knot sequence.

#### 4.2. Application to BVPs and Nyström method

We consider boundary curve  $\Gamma$  as a bijective and differentiable mapping of the parameter space  $D \subset \mathbb{R}$  onto  $\mathbb{R}^2$ , i.e.,  $\Gamma := \{\mathbf{y} = \mathbf{C}(\xi) : \xi \in D \rightarrow \mathbb{R}^2\}$ . We then compute the boundary integrals in (2.2) and (2.3) over the parametric domain  $D$ . Thus, we write

$$d\Gamma_{\mathbf{y}} = \|\mathbf{C}'(\xi)\| d\xi, \quad (4.7)$$

$$\mathbf{n}(\xi) = \frac{[C'_y(\xi), -C'_x(\xi)]}{\|\mathbf{C}'(\xi)\|}, \quad (4.8)$$

$$\kappa(\xi) = \frac{\|\mathbf{C}'(\xi) \times \mathbf{C}''(\xi)\|}{\|\mathbf{C}'(\xi)\|^3}. \quad (4.9)$$

Further, we consider the boundary  $\Gamma$  as the union of  $N$  segments  $\Gamma_k$ ,  $k = 1, 2, \dots, N$ , being the mappings of nonzero knot spans  $D_k := [\xi_{k-1}, \xi_k] \subset D$ . Let  $\xi \in D_k$  and  $\zeta \in D$  be arbitrary parameters, we write  $\mathbf{y}(\xi) = \mathbf{C}(\xi)$  and  $\hat{\mathbf{x}}(\zeta) = \mathbf{C}(\zeta)$ . Then, we obtain the kernel integral in the sense of Remark 1 as follows:

$$\int_{\Gamma} K(\widehat{\mathbf{x}}, \mathbf{y}) \sigma(\mathbf{y}) d\Gamma_{\mathbf{y}} = \sum_{k=1}^N \int_{D_k} K(\widehat{\mathbf{x}}(\zeta), \mathbf{y}(\xi)) \sigma(\mathbf{y}(\xi)) \|\mathbf{y}'(\xi)\| d\xi, \quad \zeta \in D, \xi \in D_k. \quad (4.10)$$

The second kind Fredholm integral equation (2.2) is then

$$\sigma(\widehat{\mathbf{x}}(\zeta)) - \sum_{k=1}^N \int_{D_k} K(\widehat{\mathbf{x}}(\zeta), \mathbf{y}(\xi)) \sigma(\mathbf{y}(\xi)) \|\mathbf{y}'(\xi)\| d\xi = -2\varphi(\widehat{\mathbf{x}}(\zeta)), \quad \zeta \in D, \xi \in D_k. \quad (4.11)$$

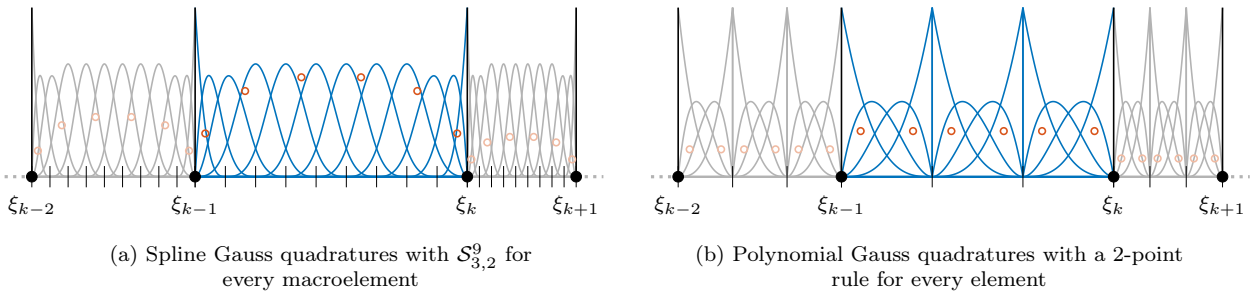
We apply the Nyström method to (4.11), thus selecting the evaluation parameters  $\xi$  and  $\zeta$  at quadrature points. Let us consider  $T_k \subset D_k$  as the set of quadrature points  $\tau_{j,k}$ ,  $j = 1, 2, \dots, m$ , corresponding to an  $m$ -point quadrature rule inside  $D_k$ , and  $T := T_1 \cup T_2 \cup \dots \cup T_N$ . We write the Nyström approximation of the integral equation (4.11) as follows:

$$\sigma(\widehat{\mathbf{x}}(\tau_i)) - \sum_{k=1}^N \sum_{j=1}^m \omega_{j,k} K(\widehat{\mathbf{x}}(\tau_i), \mathbf{y}(\tau_{j,k})) \sigma(\mathbf{y}(\tau_{j,k})) \|\mathbf{y}'(\tau_{j,k})\| = -2\varphi(\widehat{\mathbf{x}}(\tau_i)), \quad \tau_i \in T, \tau_{j,k} \in T_k. \quad (4.12)$$

In order to qualitatively compare the results produced by the polynomial and spline Gauss rules, we fix the *total* number of quadrature points  $M$  for both rules (since it corresponds to the computational effort of the numerical integration) and find the required  $N$  and  $m$  in (4.12) for each method accordingly. We recall that for the spline rule, we perform the integration macroelement-wise. Starting from a NURBS boundary with  $N$  segments, we assume every nonzero knot span  $[\xi_{k-1}, \xi_k]$ ,  $k = 1, 2, \dots, N$ , of  $\Gamma$  as one macroelement associated with the spline space  $\mathcal{S}_{\tilde{p}, \tilde{c}}^{\tilde{N}}$  with the same  $\tilde{p}$ ,  $\tilde{c}$  and  $\tilde{N}$  for all macroelements. Thus, without loss of generality, for the spline Gauss rule, we write  $N = N_{\text{me}}$  where  $N_{\text{me}}$  is the number of macroelements. The number of quadrature points of the spline rule  $m_s$  is governed by (3.6). For the polynomial Gauss rule with an  $m_p$ -point element-wise integration, we use the following formula to obtain the appropriate number of elements  $N_p$  such that we have the same total quadrature points ( $M$ ) as the spline rule:

$$M := N_{\text{me}} m_s = N_p m_p. \quad (4.13)$$

We note that based on the exactness criterion of polynomial Gauss rules (Remark 4), we compare an  $m_p$ -point polynomial rule with a spline rule of degree  $\tilde{p} = 2m_p - 1$ . When fixing  $M$ , we observe a higher flexibility of the continuous spline space than the (discontinuous) polynomial alternative. This is because the spline space spans over  $N_{\text{me}} \tilde{N}$  elements, while its polynomial counterpart spans over  $N_p < N_{\text{me}} \tilde{N}$  elements (see Fig. 5). This reflects lower approximation errors when using spline Gauss rules while preserving the same computational effort (see [59]).



**Fig. 5.** Spline vs. polynomial Gauss quadratures over the non-uniform knot spans of an arbitrary NURBS boundary (we highlight the  $k$ -th span for better visualization). When fixing the total number of quadrature points (in here, six points at every knot span), the spline Gauss rule in (a) corresponds to a higher number elements compared to its polynomial counterpart in (b), thus resulting in a more accurate integration with the same computational efficiency.

## 5. Numerical results

We verify the presented methodology by three case studies. Before reporting the results, some important notes need to be considered:

- To investigate the numerical accuracy of the spline and polynomial Gauss quadrature rules, we set  $m_p = 2, 3$  and compare the corresponding 2- and 3-point element-wise polynomial rules with the spline rules of the cubic  $\mathcal{S}_{3,2}^{39}$  and quintic  $\mathcal{S}_{5,4}^{37}$  spaces, respectively. These spline spaces both contain 21 quadrature points at each macroelement. Table 1 represents the quadrature points  $\tau_j$  and weights  $\omega_j$  of mentioned spaces in  $[0, 1]$ . The corresponding polynomial Gauss counterparts are also tabulated. We note that it is possible to experiment with different number of macroelements, or even higher degrees, (see [41, 42, 58, 59] for more spaces). One could even compute the Gaussian rules recursively, deriving a rule for the whole domain, not just a single block (c.f [60]). However, such a rule is of reduced continuity (e.g., quintic  $C^1$  splines) and is not expected to perform much better in terms of the number of quadrature vs. accuracy.
- In all examples, we construct either spline or polynomial quadrature rules within the nonzero knot spans of the parametric domain of the NURBS boundary curve. To improve the numerical accuracy of the Nyström method, we follow the  $h$ -refinement idea in FEA and IGA (see [31]). Thus, we enrich the knot sequence of the original NURBS geometry and increase the total number of quadrature points  $M$ . Depending the domain's geometry, we may employ the *grading* algorithm to insert knots in appropriate places [31, 61, 62], thus, reaching a better approximation of the solution near corners.
- We study the maximum point-wise error of the Nyström approximation when doubling the number of quadrature points at every refinement level. Let  $e_i$  be the maximum point-wise error at the  $i$ -th level, we obtain the numerical convergence order as

$$\mathcal{NCO}_i := \log_2 \left( \frac{e_{i-1}}{e_i} \right), \quad (5.1)$$

where the theoretical rate is  $\tilde{p} + 1$  (see, e.g., [3]).

**Table 1.** Quadrature points and weights of two spline Gauss rules in  $[0, 1]$ , characterized by cubic  $\mathcal{S}_{3,2}^{39}$  and quintic  $\mathcal{S}_{5,4}^{37}$  spline spaces (see [41, 42, 58, 59] for more spaces). Thanks to the symmetry property of the spline spaces, only the first 11 points and weights are reported. The corresponding points and weights of the 2- and 3-point polynomial Gauss rules are also tabulated.

$j$	$\tau_j$	$\omega_j$	$\tau_j$	$\omega_j$
	$\mathcal{S}_{3,2}^{39}$ ( $C^2$ cubic spline Gauss rule with $\tilde{N} = 39$ )		$\mathcal{S}_{5,4}^{37}$ ( $C^4$ quintic spline Gauss rule with $\tilde{N} = 37$ )	
1	0.0086022074347388	0.0218455595269063	0.0057434073557755	0.0148734239383008
2	0.0423693959303822	0.0433045545577068	0.0306123827542799	0.0343754550330581
3	0.0901289847662636	0.0503213631747089	0.0722191092726851	0.0473488367225914
4	0.1410569521267253	0.0512021143533085	0.1226181175788537	0.0524452767387078
5	0.1923101843694322	0.0512756766459810	0.1758677573121231	0.0537340667590496
6	0.2435899416018961	0.0512815446928528	0.2297657904899482	0.0539935956159231
7	0.2948718106031808	0.0512820110347811	0.2837905183402830	0.0540427493680052
8	0.3461538474036372	0.0512820480845737	0.3378390942637572	0.0540519443566724
9	0.3974358975351839	0.0512820510280155	0.3918921259881264	0.0540536600406546
10	0.4487179487257872	0.0512820512617426	0.4459459881438810	0.0540539780926602
11	0.5000000000000000	0.0512820512788446	0.5000000000000000	0.0540540266687536
	2-point element-wise polynomial Gauss rule		3-point element-wise polynomial Gauss rule	
1	0.2113248654051871	0.5000000000000000	0.1127016653792583	0.2777777777777778
2	0.7886751345948129	0.5000000000000000	0.5000000000000000	0.4444444444444444
3	–	–	0.8872983346207417	0.2777777777777778

### 5.1. Laplace problem on a square domain

We consider the benchmark 2D Laplace problem on a unit square with a sinusoidal boundary condition along one edge while the other three edges have homogeneous boundary conditions (see, e.g., [63, 64]):

Find  $u : \Omega \rightarrow \mathbb{R}$  with  $\Omega := [0, 1]^2$ , such that

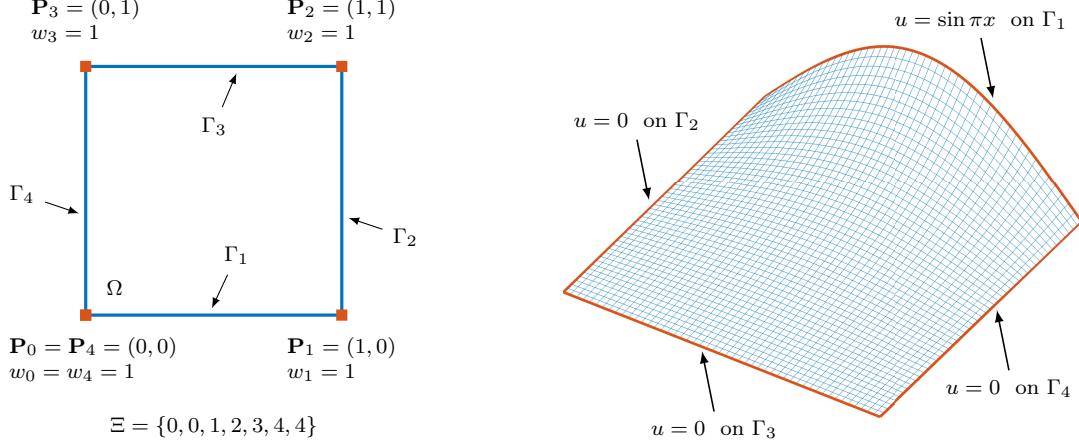
$$\begin{cases} \Delta u(x, y) = 0, & \text{in } \Omega, \\ u(x, 0) = \sin \pi x, \\ u(x, 1) = u(0, y) = u(1, y) = 0. \end{cases} \quad (5.2)$$



The analytical solution of (5.2), obtained by separation of variables, is:

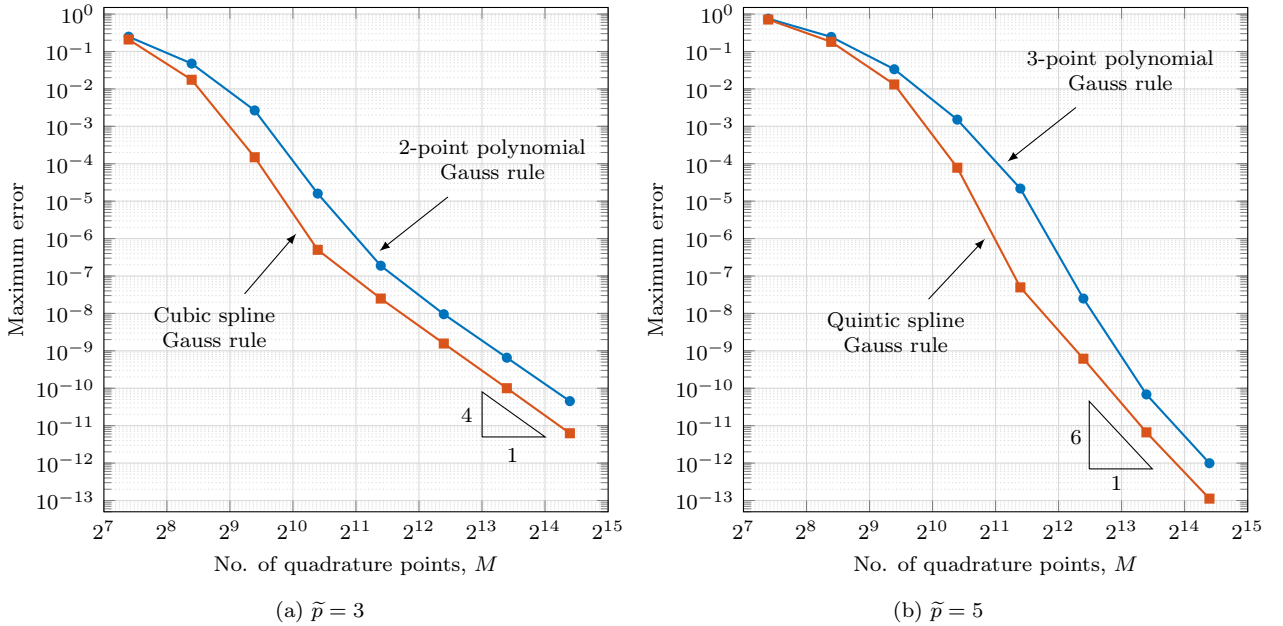
$$u(x, y) = (\cosh \pi y - \coth \pi \sinh \pi y) \sin \pi x. \quad (5.3)$$

Fig. 6 shows a NURBS representation of the boundary characterized by five control points  $\mathbf{P}_0$  to  $\mathbf{P}_4$  and linear B-spline basis functions (i.e.,  $p = 1$ ). The analytical solution over the domain is also depicted in the figure.



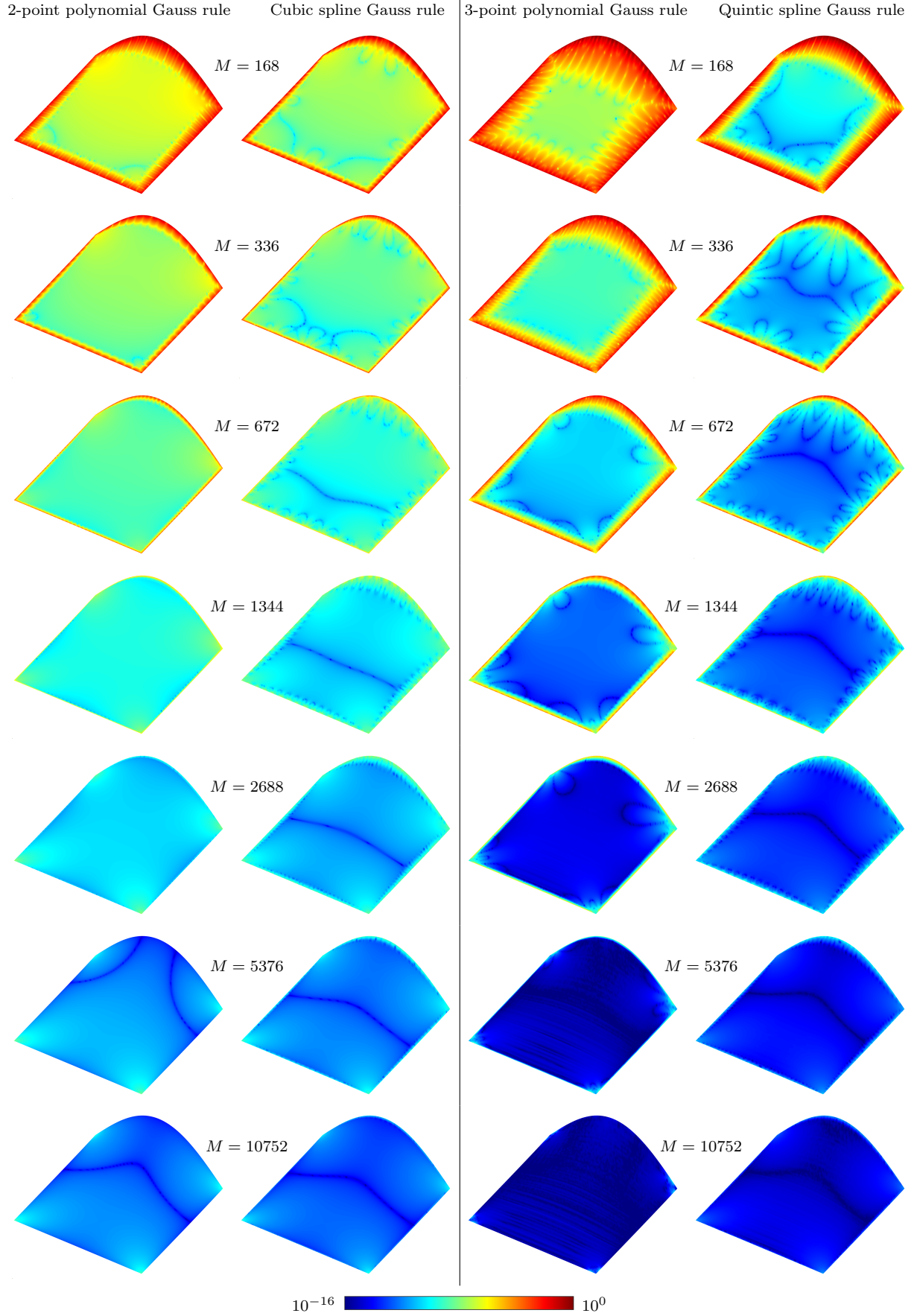
**Fig. 6.** Left: NURBS representation of the square boundary of problem (5.2): control points, weights and knot sequence (for better visualization, we name different parts of the boundary as  $\Gamma_1, \dots, \Gamma_4$ ). Right: analytical solution over the domain.

Fig. 7 shows the maximum point-wise error of the Nyström approximation of the solution of the Dirichlet boundary value problem (5.2) when doubling the number of quadrature points at every refinement step. To do this, we compute the approximation error at different points distributed through the square domain by comparing the approximate solution obtained by (2.3) with the analytical expression (5.3). We compare the solutions when using the polynomial and spline Gauss rules and integrating with the same total number of quadrature points  $M$ . The results indicate that the spline rule is at least one order of magnitude more accurate than the polynomial rule. Additionally, the numerical convergence order is very close to the theoretical one, that is,  $\mathcal{NCO} \approx \tilde{p} + 1$ . It is clear that for a higher  $\tilde{p}$ , we observe a better approximation.



**Fig. 7.** Maximum point-wise error of the Nyström approximation of the solution of (5.2) over the square domain. (a) Gauss cubic spline vs. Gauss 2-point polynomial rule. (b) Gauss quintic spline vs. Gauss 3-point polynomial rule. The respective numerical convergence orders are shown by the triangles.





**Fig. 8.** Error distributions over the square domain at different refinement levels of the Nyström approximation using  $M$  quadrature points. We compare 2- and 3-point polynomial Gauss rules with their cubic and quintic spline counterparts, respectively. We place the error plots over the analytical solution surface and color-code it by the approximation error.

It could be thought of as equal to the  $p$ -refinement idea in FEA and IGA. In order to visualize how the approximation error distributes throughout the domain, we represent the error plots over the square domain in Fig. 8. The results indicate that the spline Gauss quadrature rule has a better approximation quality at all refinement steps.

### 5.2. Laplace problem on a unit disk

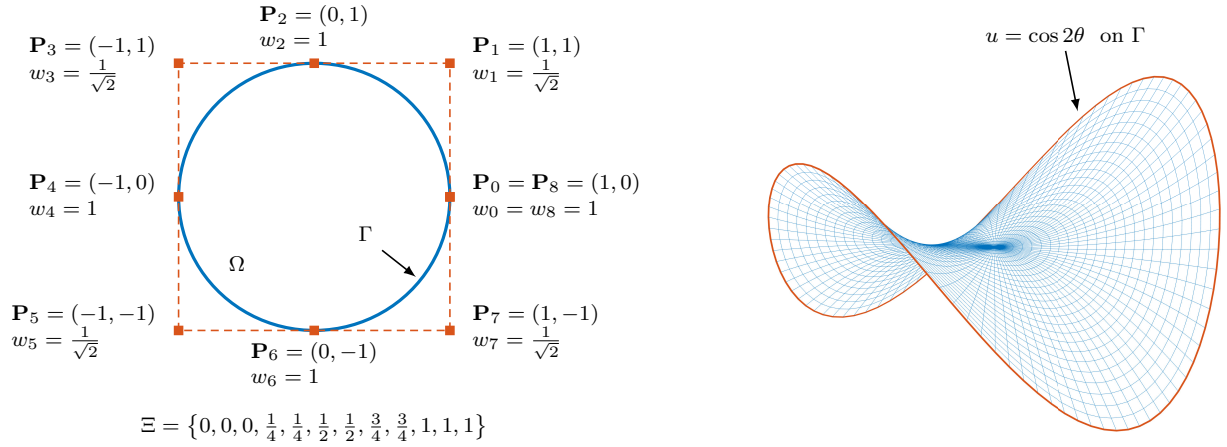
We seek the solution of the Dirichlet boundary value problem of the Laplace equation on a unit disk [65]:

$$\text{Find } u : \Omega \rightarrow \mathbb{R} \text{ with } \Omega := \{(r, \theta) : r \in [0, 1], \theta \in [0, 2\pi]\}, \text{ such that}$$

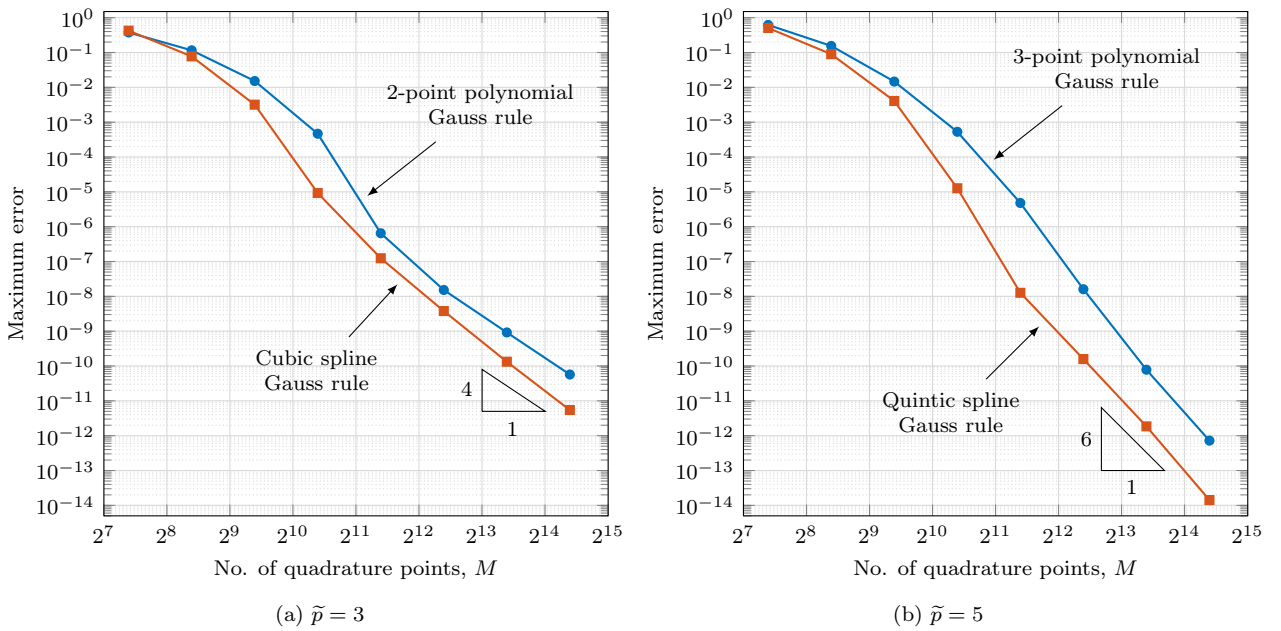
$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = \cos 2\theta, & \text{on } \Gamma. \end{cases} \quad (5.4)$$

The analytical solution of (5.4) in the polar coordinate system is:

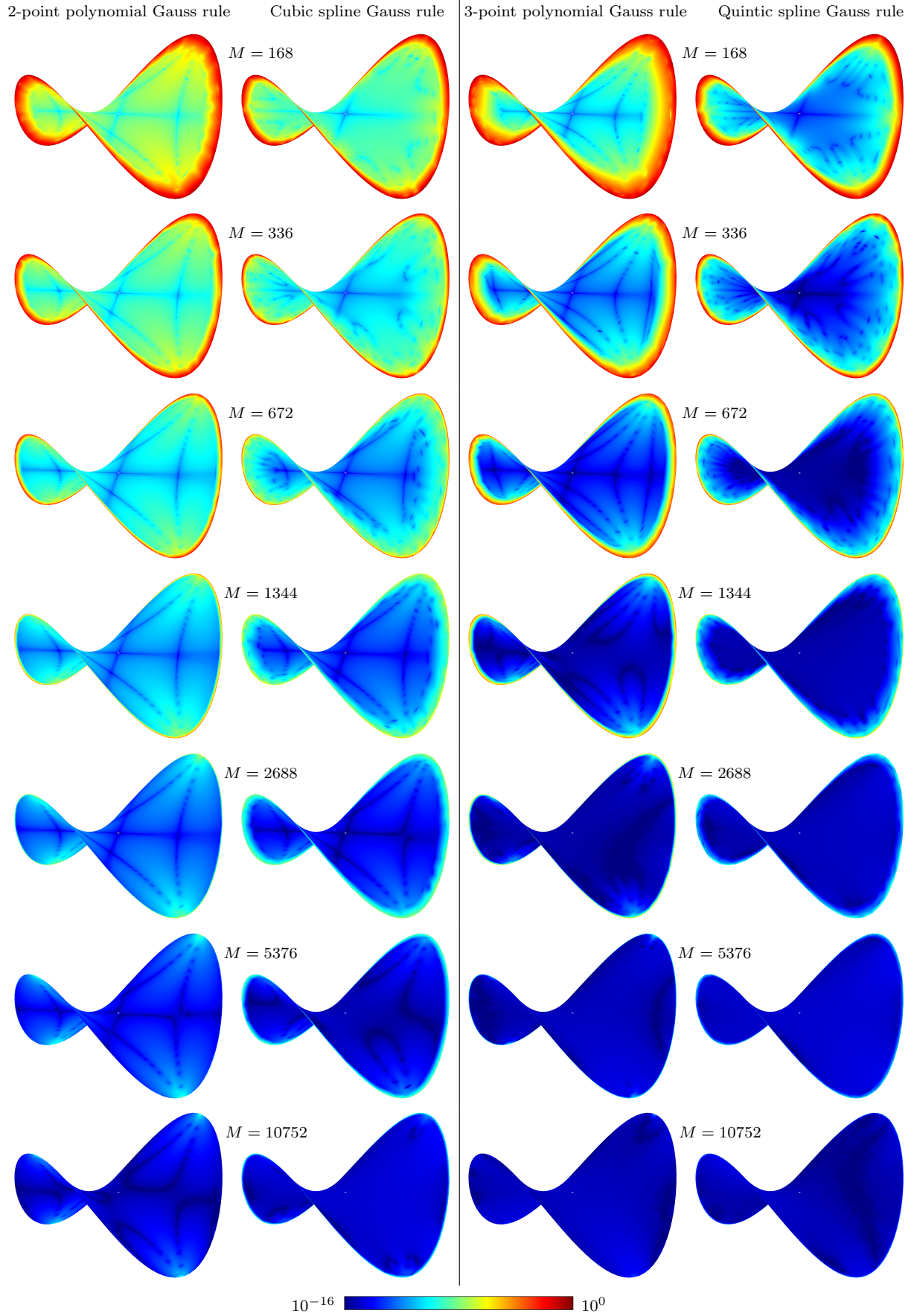
$$u(r, \theta) = r^2 \cos 2\theta. \quad (5.5)$$



**Fig. 9.** Left: NURBS representation of the circular boundary of the problem (5.4): control points, weights and knot sequence. Right: analytical solution over the domain.



**Fig. 10.** Maximum point-wise error of the Nyström approximation of the solution of (5.4) over the unit disk. (a) Cubic spline vs. 2-point polynomial rule. (b) Quintic spline vs. 3-point polynomial rule. The respective numerical convergence orders are shown by the triangles.



**Fig. 11.** Error distributions over the unit disk at different refinement levels when comparing 2- and 3-point polynomial Gauss rules with the cubic and quintic spline rules, respectively. We place the error plots over the analytical solution surface.

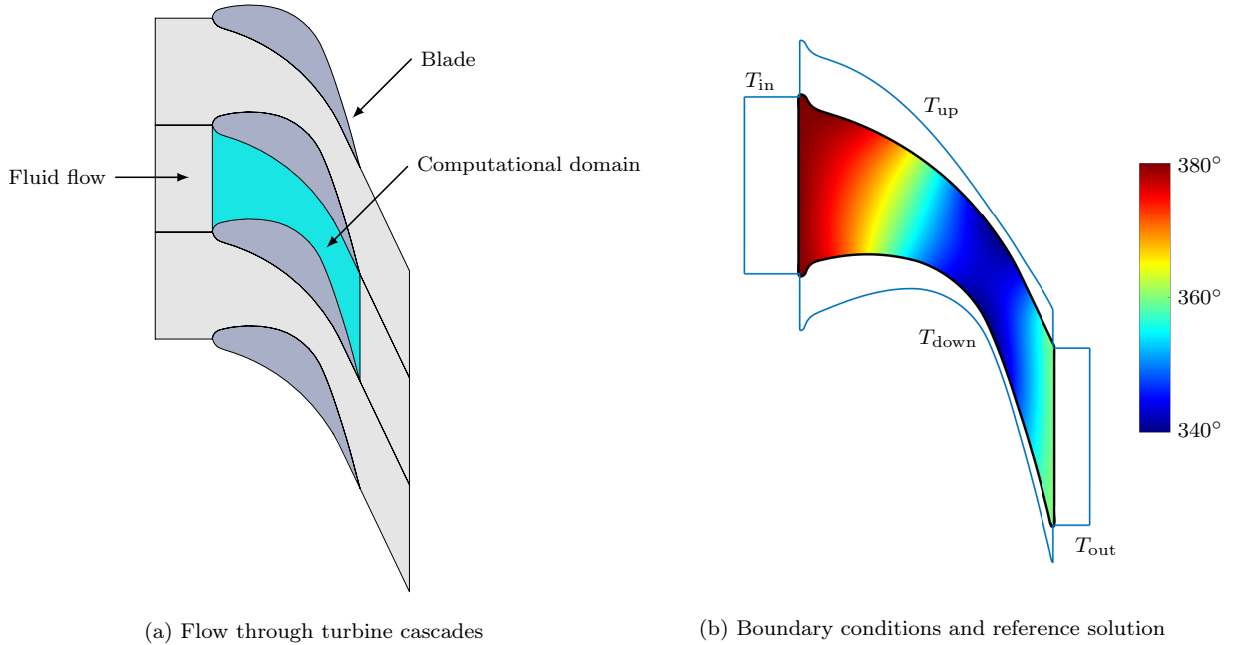
Fig. 9 shows a NURBS representation of the circular boundary as well as the analytical solution over the domain. We use quadratic basis functions spanned over a knot sequence with double multiplicities at interior knots. Fig. 10 shows the maximum point-wise error of the Nyström approximation of the solution of the Laplace problem (5.4) over a unit disk with a Dirichlet boundary condition. Again, the solutions obtained by the spline Gauss quadrature rules are almost one order of magnitude more accurate compared to their polynomial Gauss counterparts when integrating using the same total number of quadrature points  $M$ . The numerical convergence order is also close to the theoretical rate  $\tilde{p} + 1$ . Fig. 11 illustrates the error distributions over the circular domain when doubling the number of quadrature points at every refinement step. The results well confirm the convergence plots of Fig. 10.

### 5.3. Steady-state heat transfer analysis inside a blade cascade

The third example is adopted from the fluid flow analysis through turbine blades, which is a well-known practical case study in the computational fluid dynamics (see, e.g., [66–68]). The main idea is to assess the application of the presented methodology in a real-life model problem for which no analytical solution is available. Fig. 12a schematically shows the flow passage through blade cascades of a steam turbine. We simplify the original problem and only consider the steady-state heat transfer inside our computational domain of interest. The heat transfer is governed by the Poisson’s equation [69]:

$$-k\nabla^2 T = q, \quad (5.6)$$

where  $T$  is the fluid temperature,  $k$  is the thermal conductivity of the fluid, and  $q$  is the rate of heat generation inside the domain. In here, we further assume that no heat source exists, thus, reducing (5.6) to the Laplace equation, i.e.,  $\nabla^2 T = 0$ . We consider Dirichlet boundary conditions as indicated in Fig. 12b and seek to compute the temperature distribution through the blade cascade using the Nyström approximation<sup>1</sup>. In particular, the inlet and outlet temperatures are assumed to be fixed as  $T_{\text{in}} = 380^\circ$  and  $T_{\text{out}} = 360^\circ$ , respectively. While the temperature distribution on the upper and lower parts of the cascade is given by quintic functions with coefficients described in Table 2. We employ a NURBS parameterization of



**Fig. 12.** (a) Schematic view of the fluid flow through blade cascades of a steam turbine. We highlight our computational domain of interest. (b) Dirichlet boundary conditions are overlaid on the four sides of the domain’s boundary. We slightly round the corners to reach a better convergence rate in the numerical solution. The reference temperature distribution is color-coded over the domain.

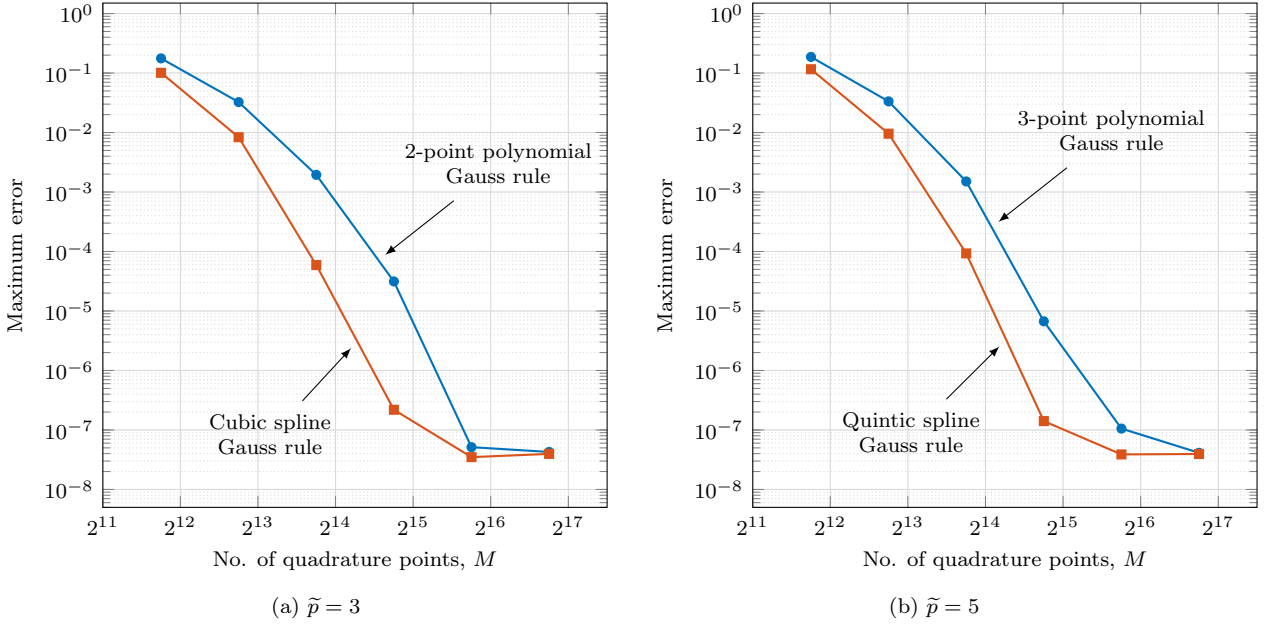
<sup>1</sup>In the original problem, only the inlet temperature is known and the temperature on the other three sides and inside the domain is unknown. It is governed by more parameters such as the pressure, wetness, and turbulence of the fluid.

**Table 2.** Boundary conditions on different sides of the computational domain of the third example.

Side	Temperature	$\xi$ -range	Coefficients ( $\times 10^5$ )					
			$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$
Up	$T = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5$	[0.15,0.47]	-0.0026	0.1227	-0.9063	3.0780	-4.8648	2.9088
Down		[0.62,1.00]	-0.2985	1.8197	-4.3041	5.0020	-2.8601	0.6446
Inlet	380°							
Outlet	360°							

the boundary and slightly round the sharp corners of the domain to improve the convergence rate of our approximation. Thus, we use the same parameter  $\xi$  when imposing boundary conditions  $T_{\text{up}}$  and  $T_{\text{down}}$ . Since the analytical solution for this freeform geometry is not available, we obtain the reference temperature distribution using an *overkill* FEA solution with millions of degrees of freedom.

Fig. 13 shows the maximum point-wise error of the Nyström approximation for the third case study. Taking into account that the reference solution is also obtained by a numerical method, we observe a plateau in the convergence plots. This is mainly because the accuracy of our solution reaches the accuracy of the reference solution after a few refinement steps. For both cases of  $\tilde{p} = 3$  and 5, the solutions obtained by the spline Gauss quadrature rules are orders of magnitude more accurate than their polynomial Gauss counterparts. Fig. 14 illustrates the error distributions over the computational domain when doubling the number of quadrature points at every refinement step. We observe improvements close to the boundary and, particularly, near the corners when using spline Gauss rules. Nevertheless, for the major interior part of the domain, we can hardly observe significant improvements in error contours, again, because of the numerical nature of our reference solution.

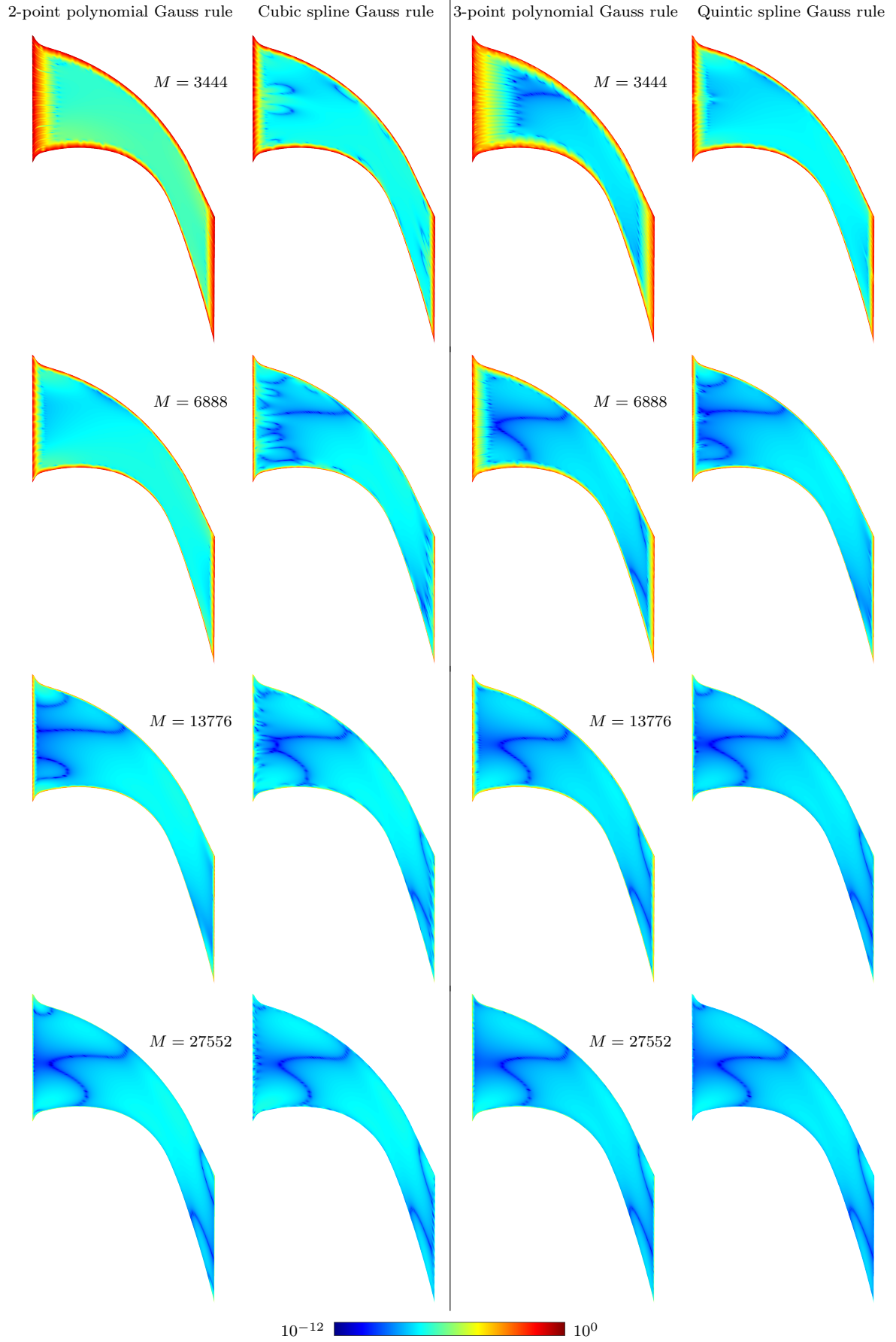


**Fig. 13.** Maximum point-wise error of the Nyström approximation of the solution of the third example. (a) Cubic spline vs. 2-point polynomial rule. (b) Quintic spline vs. 3-point polynomial rule. The numerical convergence graphs reach a plateau after a few refinement steps because the reference solution is obtained by FEA, which is a numerical method as well.

## 6. Conclusions

We propose to use spline Gauss quadrature rules for solving boundary value problems using the Nyström method. The corresponding PDE inside a domain is converted into the Fredholm integral equation of the second kind on the boundary. Then, we use spline Gauss quadratures and convert the integral equation to a linear system via the Nyström method. We consider the solution of the Laplace equation in 2D domains when Dirichlet boundary conditions are applied. We validate our method on three different geometries,





**Fig. 14.** Error distributions through the blade cascade of the third example. We compare 2- and 3-point polynomial Gauss rules with the cubic and quintic spline rules, respectively, when doubling the total number of quadrature points.

namely: square, circular, and freeform domains. When using a NURBS representation of the boundary, we place the quadrature points within the knot spans of the NURBS geometry. In this context, we compare the macroelement-wise integration of the spline Gauss rules with the element-wise integration of the polynomial Gauss rules. The results indicate that, when fixing the total number of quadrature points (i.e., with the same computational effort), the spline rules return an approximation that is by one to two orders of magnitude more accurate compared to the solution obtained by traditional polynomial Gauss counterparts.

As future work, we aim to consider a non-uniform refinement in terms of non-uniform Gaussian spline rules, which are expected to capture better the features of more complicated boundary curves, and possibly further reduce the approximation error.

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