ASYMPTOTIC BEHAVIOR OF THE INTERFACE FOR ENTIRE VECTOR MINIMIZERS IN PHASE TRANSITIONS

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Abstract. We study globally bounded entire minimizers $u : \mathbb{R}^n \to \mathbb{R}^m$ of Allen-Cahn systems for potentials $W \geq 0$ with $\{W = 0\} = \{a_1, ..., a_N\}$ and $W(u) \sim |u - a_i|^\alpha$ near $u = a_i$, $0 < \alpha < 2$. Such solutions are, over large regions, identically equal to some zeroes of the potential a_i's. We establish the estimates

$$\mathcal{L}^n(I_0 \cap B_r(x_0)) \leq c_1 r^{n-1}, \quad \mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(x_0)) \geq c_2 r^{n-1}, \quad r \geq r_0(x_0)$$

for the *diffuse interface* $I_0 := \{x \in \mathbb{R}^n : \min_{1\leq i\leq N} |u(x) - a_i| > 0\}$ and the *free boundary* ∂I_0. Furthermore, if $\alpha = 1$ we establish the upper bound

$$\mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(x_0)) \leq c_3 r^{n-1}, \quad r \geq r_0(x_0).$$

1. Introduction and description of the main results

The object of study in the present paper is a class of entire solutions of the system

$$\Delta u - W_u(u) = 0, \quad u : \mathbb{R}^n \to \mathbb{R}^m, \tag{1.1}$$

$n, m \in \mathbb{N}^+$, where $W : \mathbb{R}^m \to \mathbb{R}$ is a phase transition potential that is nonnegative and vanishes only on a finite set $\{W = 0\} =: A = \{a_1, ...a_N\}$ for some distinct points $a_1, ..., a_N \in \mathbb{R}^m$ that represent the phases of a substance which can exist in $N \geq 2$ different equally preferred phases.

The system (1.1) is the Euler-Lagrange equation corresponding to the Allen-Cahn free energy functional

$$J_D(v) = \int_D \left(\frac{1}{2}|\nabla v|^2 + W(v)\right) dx. \tag{1.2}$$

We restrict ourselves to maps $u \in W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$, which minimize J subject to their Dirichlet data,

$$J_D(u + v) \geq J_D(u), \quad \forall v \in W^{1,2}_0(D, \mathbb{R}^m) \cap L^\infty(D, \mathbb{R}^m) \tag{1.3}$$

for any open, bounded Lipschitz set $D \subset \mathbb{R}^n$. We call such maps *entire minimizers* of J, and note that they clearly satisfy (1.1), under appropriate regularity hypothesis on W.

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In relation to the phase transition interpretation, an interesting and difficult problem is the existence of multi-phase solutions, that are solutions with the following geometrical properties:

There is an $\hat{N} \in \mathbb{N}$ *with* $2 \leq \hat{N} \leq N$, $\hat{a}_1, ..., \hat{a}_{\hat{N}} \in A$, *a small* $\gamma > 0$ *and open sets* $\Omega_1, ..., \Omega_{\hat{N}}$ *such that*

$$\mathbb{R}^n = I \cup \left(\bigcup_{j=1}^{\hat{N}} \Omega_j \right) \tag{1.4}$$

with I *being a set of thickness* $O(1)$ *and*

$$|u(x) - \hat{a}_j| \leq \gamma, \quad \forall x \in \Omega_j, \text{ for some small } \gamma \tag{1.5}$$

The set I plays the role of a diffuse interface that separates the coexisting phases. Understanding the geometrical structure of this diffuse interface is a major point in the study of such solutions [21].

In the scalar case $m = 1$, for $W \in C^2(\mathbb{R}^m, \mathbb{R})$, a_i nondegenerate (i.e. $\frac{\partial^2 W}{\partial u^2} > 0$ at the minima), and for $N = 2$ which is the natural choice, there is a rich literature and many important results. We list some of these works and organize them in two groups: papers that address various general aspects (see [18, 26, 25, 40, 8, 9, 5, 32, 33, 16]); and papers that are motivated by a celebrated conjecture of De Giorgi (see [12, 31, 23, 3, 37, 43, 7, 13]) where a relationship of I with minimal surfaces, and in particular with hyperplanes in low dimensions, is established. The reader could also consult the expository papers [17, 36, 38, 14].

In the vector case $m \geq 2$ and $N \geq 3$, the structure of I is not expected to be planar in any dimension $n \geq 2$ but rather linked to the minimal cones[1] in $\mathbb{R}^n$. This kind of solutions are called *junctions* and have been shown to exist for $n = m = 2$ and $n = m = 3$ with $\hat{N} = N = 3$ and $\hat{N} = N = 4$ respectively, but so far only under symmetry hypotheses. Specifically, the existence is established for:

- $n = 2$ with respect to the symmetries of the equilateral triangle [6],
- $n = 3$ with respect to the symmetries of the tetrahedron [24],

and with (1.3) verified only in their respective equivariance classes. We refer to [1] where the symmetric case is covered in detail for general reflection point groups and also for lattices. For example the solution $u(x)$ for $n = m = 2$ and $N = 3$ (triple junction on the plane) referred above can be described precisely as follows: The reflection point group G in this case has six elements (three reflections, two rotations and the identity). As a fundamental region F we can take the $\frac{\pi}{3}$ sector, one side of which coincides with the x-axis which is associated to one of the reflections. The minimum a_1 is placed on the x-axis. The stabilizer G_{a_1} is the subgroup of G leaving a_1 fixed, and it consists of the reflection with respect to the x-axis and the identity. And $D = \text{int} \bigcup_{g \in G_{a_1}} g \circ F$ is the $\frac{2\pi}{3}$ sector containing a_1. The equivariant u has the following properties:

(1) $|u(x) - a_1| \leq K e^{-k \text{dist}(x, \partial D)}$, $x \in D$, k, K are positive constants.
(2) $u(\bar{F}) \subset \bar{F}$ and $u(D) \subset D$ (positivity).

The singular minimal cone in $\mathbb{R}^2$ is the boundary of the partition of $\mathbb{R}^2$ by the three $\frac{2\pi}{3}$ sectors.

For subquadratic potentials that behave like $|u - a|^\alpha$ near $a \in A$, $\alpha \in (0, 2)$, the interface is less diffuse as we explain below. More precisely we consider potentials that satisfy

H1. $W \in C(\mathbb{R}^m, [0, \infty)) \cap C^2_{loc}(\mathbb{R}^m \backslash A)$ with $\{W = 0\} = A = \{a_1, \ldots, a_N\}$, $N \geq 2$.

[1] The complete classification of the minimal cones in $\mathbb{R}^n$ is known only for n=3 [41]. We refer to Section 7 in the expository paper of David [11] and references therein.

H2. Set

$$r_0 := \frac{1}{2} \min_{1 \le i \ne j \le N} |a_i - a_j|. \tag{1.6}$$

For any $a_i \in A$, $W(u)$ can be written as

$$W(u) = |u - a_i|^{\alpha} g_i(u) \quad \text{in } u \in B_{r_0}(a_i) \tag{1.7}$$

for some function $g_i(u) \in C^2(B_{r_0}(a_i), [c_i, \infty))$ where c_i is a positive constant.

For such subquadratic potentials, (1.5) is replaced by

$$u(x) = \hat{a}_j, \quad \forall x \in \Omega_j. \tag{1.8}$$

Indeed, in this case the entire minimizer possesses a free boundary and the phases $\hat{a}_j$ are attained [2], while for $\alpha = 2$, the solution converges exponentially at infinity to the phases. The subquadratic assumption can be thought as a reduction which simplifies without changing the essential features of this type of solution. In a suitable limit $\alpha \to 0$, (1.2) becomes the Alt-Caffarelli functional.

We now define the appropriate analog of the set I in (1.4) for vector analog of the subquadratic potentials.

Definition 1.1. *Let $0 < \gamma_0 < \frac{1}{2} \min_{i \ne j} |a_i - a_j|$ fixed. Set*

$$\delta(x) = \text{dist}(u(x), \{W = 0\}) \tag{1.9}$$

where dist *stands for the Euclidean distance. Let $0 < \gamma < \gamma_0$ and assume $\gamma_0 < \sup_{\mathbb{R}^n} \delta(x)$. We define the set*

$$I_\gamma = \{x \in \mathbb{R}^n : \delta(x) \ge \gamma\}. \tag{1.10}$$

For entire minimizers satisfying $\|u\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^m)} < \infty$, $\|\nabla u\|_{L^\infty}(\mathbb{R}^n, \mathbb{R}^m) < \infty$, and $0 < \alpha \le 2$, the following estimate is known (see [1, Lemma 5.5])

$$c_1(\gamma) r^{n-1} \le \mathcal{L}^n(I_\gamma \cap B_r(x_0)) \le c_2(\gamma) r^{n-1}, \quad r \ge r(x_0), \tag{1.11}$$

where $x_0 \in \mathbb{R}^n$ is arbitrarily chosen, $r(x_0)$ is a positive constant depending on x_0, and the constants $c_i(\gamma) > 0\,(i = 1, 2)$ is independent of x_0, r. Throughout this paper, $\mathcal{L}^n$ denotes the n dimensional Lebesgue measure and $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure.

Clearly $I_{\gamma_1} \subset I_{\gamma_2}$ if $\gamma_1 > \gamma_2$ and we define

$$\lim_{\gamma \to 0} I_\gamma = I_0 = \{x \in \mathbb{R}^n : \delta(x) > 0\} =: \text{Diffuse Interface}. \tag{1.12}$$

The constants $c_1(\gamma)$, $c_2(\gamma)$ in (1.11) degenerate as $\gamma \to 0$ (see [1]) and so no useful information can be obtained for I_0 out of (1.11). The proof of the upper bound for $\mathcal{L}^n(I_\gamma \cap B_r)$ in (1.11) is an immediate consequence of a well-known energy estimate (see (3.34) in Proposition 3.1). On the other hand the proof of the analogous upper bound for $\mathcal{L}^n(I_0 \cap B_r)$ is highly nontrivial and requires a novel approach that, to a large measure, is the main contribution of this paper.

Our main result in the present paper concerns certain global facts on I_0 that can be summarized in the following theorem.

Theorem 1.2. *Let $\alpha \in (0, 2)$, $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded entire minimizer with $I_{\gamma_0} \ne \varnothing$ (i.e. $u \not\equiv a_i$), and assume H1, H2 above. Then there exists a radius $r_0 > 0$ and positive constants c, c_1, c_2, which only depend on u but not on r, such that for $r \ge r_0$ the following estimates hold:*

$$\mathcal{L}^n(I_0 \cap B_r(0^n)) \le c r^{n-1}, \tag{1.13}$$

$$\mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(0^n)) \ge c_1 r^{n-1}, \tag{1.14}$$

where ∂^* denotes the De Giorgi reduced boundary.

If furthermore $\alpha = 1$ we have the upper bound estimate

$$\mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(0^n)) \leq c_2 r^{n-1}. \tag{1.15}$$

The analogs of the "minimal cone" solutions in $\mathbb{R}^2$ and $\mathbb{R}^3$ mentioned above as well as the cylindrical triple junction cone in $\mathbb{R}^3$ have been shown to exist also for $0 \leq \alpha < 2$ and possess free boundary (see [2, Theorem 1 and Proposition 4])[2]. Combining Theorem 1.2 above with that result we obtain the following

Corollary 1.3. *Under H1 and H2, $0 \leq \alpha < 2$, for the equivariant triple junction in $\mathbb{R}^2$ and the equivariant quadruple junction in $\mathbb{R}^3$ the estimates* (1.13), (1.14) *and* (1.15) *hold for any $r \geq r_0 > 0$.*

Remark 1.1. The solutions in Corollary 1.3 satisfy, instead of the exponential estimate (1) in the definition of junction, the following property

(1)' $u(x) = a_1$ for $x \in D,\ \operatorname{dist}(x, \partial D) \geq d_0$, for some positive constant d_0.

Remark 1.2. We note that the assumption on the existence of minimizers satisfying (1.3) is not restrictive as such nonconstant entire minimizers $U : \mathbb{R}^1 \to \mathbb{R}^m$ have been shown to exist in [34] under the hypothesis of continuity on W together with the mild condition

$$\sqrt{W(u)} \geq f(|u|),\ \text{for some nonnegative } f : (0, +\infty) \to \mathbb{R} \text{ s.t. } \int_0^{+\infty} f(r)\, dr = +\infty, \tag{1.16}$$

that allows even decay to zero of the potential at infinity.

A more convenient sufficient condition for the existence of such nonconstant entire minimizers (called *connections*) is (see [22])

$$\liminf_{|u| \to \infty} W(u) > 0. \tag{1.17}$$

In [8], for the scalar two phase problem, the entire range of potentials $F_0 = \chi_{\{|u|<1\}}$, $F(u) = (1 - u^2)^{\alpha/2}$ $(0 < \alpha \leq 1)$ was already introduced. As $\alpha \to 0$, the minimizers get increasingly localized. In particular for $\alpha = 0$ the connections are affine functions. For these reasons we expect the construction of entire minimizers mentioned in Corollary 1.3 above, under no symmetry hypotheses, to be more accessible for singular potentials.

From the point of view of regularity, our problem can be regarded as a generalization of the more simplified model which studies the minimization problem of the functional

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + |u|^\alpha \right) dx, \quad u : \mathbb{R}^n \to \mathbb{R}^m,\ \alpha \in (0, 2). \tag{1.18}$$

In the scalar case, i.e. $m = 1$, one recovers the two phase free boundary problem

$$\Delta u = \alpha (u^+)^{\alpha - 1} - \alpha (u^-)^{\alpha - 1}.$$

which has been extensively studied under various conditions and settings, see for example [19] for the case $\alpha \in (1, 2)$ and [28] for the 2D problem with $\alpha \in (0, 1)$. One can also refer to [27, 29] for the optimal regularity theory for a functional with a more general form of the potential W. When $\alpha = 1$, the problem becomes the so-called two-phase obstacle-type problem. The regularity of the solution as well as the free boundary regularity has been summarised in detail in [35].

For the vector-valued case, i.e. $m \geq 2$, the minimization problem (1.18) was investigated in [4] for $\alpha = 1$ and [20] for $\alpha \in (1, 2)$, where the authors studied the regularity of the minimizers and the asymptotic behavior for the minimizer near the "regular" point of the free boundary. Also we would like to mention the works [10, 30] studying the vector-valued Bernoulli free boundary

[2]For $\alpha = 0$, $J(u) = \int \frac{1}{2}|\nabla u|^2 + \chi_{\{u \in S_A\}}$, where S_A denotes the interior of the convex hull of $A = \{a_1, ..., a_N\}$. It is the vectorial analog of the Alt-Caffarelli functional.

problem, where the potential function $W(u)$ takes the form $Q^2(x)\chi_{\{|u|>0\}}$. Such a problem is quite close to the $\alpha = 0$ case of the functional (1.18). All these works mainly focus on the behavior and regularity of the free boundary.

The biggest difference between our functional (1.2) and the simplified one (1.18) is that the potential $W(u)$ possesses more than one global minimum. Furthermore, our emphasis is not so much on the local behavior of the free boundary, but on its asymptotic behavior on the large domain $B_R(0^n)$ as $R \to \infty$.

To prove the main Theorem 1.2, we first show in Section 2 that the Euler-Lagrange equation is satisfied by the minimizer u via the regularity of the solution. Then using the regularity property and a non-degeneracy lemma, we prove the first part of the theorem (upper bound estimate for the $\mathcal{L}^n$ measure of I_0) by introducing a method of dividing B_R into identical smaller sub-cubes and classifying all the cubes according to how much measure of the "contact set" they contain. This is done in Theorem 3.3 from Section 3. We also demonstrate in Theorem 3.6 the coexistence of at least two phases for the minimizer at each large scale. In Section 4 we derive a Weiss-type formula and then give a growth estimate for the minimizer near the free boundary. These results from Section 4 are used in Section 5, where we estimate the $\mathcal{H}^{n-1}$ measure of the free boundary when $\alpha = 1$ and prove the second part of Theorem 1.2.

2. Regularity of the minimizer

We first prove the optimal regularity of the entire minimizer u for $\alpha \in (0, 1)$.

Proposition 2.1. *Suppose* $0 < \alpha < 1$ *and let* u *be an entire minimizer of the energy functional* J *satisfying* $|u(x)| \leq M$. *Assuming H1, H2. Then we have*

$$u \in C^{1,\beta}_{loc}(\mathbb{R}^n, \mathbb{R}^m),$$

where β *is a constant defined by* $\beta = \frac{\alpha}{2-\alpha}$. *In particular, this* $C^{1,\beta}$ *regularity is sharp.*

Remark 2.1. We would like to note that in the following proof, unless being specifically stated, all the constants denoted by C only depend on the upper bound M, the potential function W and the dimensions m, n.

Proof. For any ball $B_R(x) \subset \mathbb{R}^n$, let v be the harmonic function (which means each v_i is harmonic) in $B_R(x)$ such that $u = v$ on ∂B_R. Since v is harmonic, then ∇v satisfies a Campanato type growth condition

$$\int_{B_\rho} |\nabla v - (\nabla v)_\rho|^2 \, dx \leq C(\frac{\rho}{R})^{n+2} \int_{B_R} |\nabla v - (\nabla v)_R|^2 \, dx \tag{2.19}$$

For $\rho < R$, we deduce

$$\begin{aligned}\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^2 \, dx &\leq C \int_{B_\rho} |\nabla v - (\nabla v)_\rho|^2 \, dx + C \int_{B_R} |\nabla u - \nabla v|^2 \, dx \\ &\leq C(\frac{\rho}{R})^{n+2} \int_{B_R} |\nabla v - (\nabla v)_R|^2 \, dx + C \int_{B_R} |\nabla u - \nabla v|^2 \, dx.\end{aligned} \tag{2.20}$$

For the second term, we use the minimization property to obtain

$$\int_{B_R} |\nabla u - \nabla v|^2 \, dx = \int_{B_R} \left(|\nabla u|^2 - |\nabla v|^2\right) \, dx \leq \int_{B_R} 2 \left(W(v) - W(u)\right) \, dx. \tag{2.21}$$

By hypothesises H1, H2 and $\|u\|_{L^\infty} \leq M$, we can easily verify that W can be written as

$$W(u) = \left(\prod_{i=1}^N |u - a_i|^\alpha \right) g(u), \tag{2.22}$$

where g is a function such that

$$g \in C^2(\overline{B}_M), \quad g(u) \geq C \text{ for some constant } C > 0. \tag{2.23}$$

When $\alpha \in (0,1)$, since both v and u are uniformly bounded, we compute

$$\begin{aligned}
&W(v) - W(u)\\
=&\left(\prod_{i=1}^N |v-a_i|^\alpha\right) g(v) - \left(\prod_{i=1}^N |u-a_i|^\alpha\right) g(u)\\
\leq&\left(\prod_{i=1}^N (|u-a_i|^\alpha + |u-v|^\alpha)\right)(g(u) + C|u-v|) - \left(\prod_{i=1}^N |u-a_i|^\alpha\right) g(u)\\
\leq& C|u-v|^\alpha,
\end{aligned} \tag{2.24}$$

where C is a constant only depending on W, M, n, m. By Hölder, Poincaré and Young inequalities, we have

$$\begin{aligned}
&\int_{B_R} |u-v|^\alpha \, dx\\
\leq& C\Big(\int_{B_R} |u-v|^2 \, dx\Big)^{\frac{\alpha}{2}} R^{n(1-\frac{\alpha}{2})}\\
\leq& C\Big(\int_{B_R} |\nabla(u-v)|^2 \, dx\Big)^{\frac{\alpha}{2}} R^{n(1-\frac{\alpha}{2})+\alpha}\\
\leq& \delta \int_{B_R} |\nabla(u-v)|^2 \, dx + C(\delta) R^{n+\frac{2\alpha}{2-\alpha}}
\end{aligned} \tag{2.25}$$

where δ is a suitably chosen small number. Combining (2.21), (2.24) and (2.25), we obtain

$$\int_{B_R} |\nabla u - \nabla v|^2 \, dx \leq C R^{n+2\beta}, \quad \beta := \frac{\alpha}{2-\alpha} \tag{2.26}$$

Therefore from (2.19), (2.20) and (2.26) we get the following Campanato type decay estimate for the minimizer u

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^2 \, dx \leq c\Big(\frac{\rho}{R}\Big)^{n+2} \int_{B_R} |\nabla u - (\nabla u)_R|^2 \, dx + C R^{n+2\beta} \tag{2.27}$$

By a standard iteration argument we conclude that there exists a constant C such that

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^2 \, dx \leq C \rho^{n+2\beta}, \tag{2.28}$$

which by the Morrey-Campanato theory implies $u \in C^{1,\beta}$.

To show that this $C^{1,\beta}$ estimate is sharp, we need the following statement that will be proved later in Theorem 3.3 and Remark 3.1: there exists $a_i \in A$ such that $\mathcal{L}^n(\{u(x) = a_i\}) > 0$ and $\{u(x) = a_i\}$ contains interior points. For now we first assume this and proceed with the proof. Let $x_1 \in \{u(x) = a_i\}$ be an interior point, and define

$$r_1 := \sup\{r > 0 : B_r(x_1) \subset \{u(x) = a_i\}\}.$$

When $u \not\equiv a_i$, we know that $r_1 \in (0,\infty)$ and there is a point $x_2 \in \partial B_{r_1}(x_1)$ such that $u(x_2) = a_i$ and $x_2 \in \overline{\{|u-a_i| > 0\}}$. For r sufficiently small, by Lemma 3.2 we have

$$\sup_{B_r(x_2)} |u - a_i| \geq c r^{1+\beta}. \tag{2.29}$$

Also we claim that $\nabla u(x_2) = 0$. Firstly, by $u(x) \equiv a_i$ for any $x \in B_r(x_2) \cap B_{r_1}(x_1)$, we can easily check that $\partial_\nu u(x_2) = 0$ where ν denotes the normal vector on $\partial B_{r_1}(x_1)$. On the other hand, take any unit vector μ which points to one of the tangential directions at x_2 on $\partial B_{r_1}(x_1)$. For any $|t| \leq r_1$,

$$x_2 + t\mu + (\sqrt{r_1^2 - t^2} - r_1)\nu \in \partial B_{r_1}(x_1), \quad u(x_2 + t\mu + (\sqrt{r_1^2 - t^2} - r_1)\nu) = a_i.$$

Thus for any $i \in \{1, 2, ..., m\}$ we have

$$0 = \frac{d}{dt}\bigg|_{t=0} u_i(x_2 + t\mu + (\sqrt{r_1^2 - t^2} - r_1)\nu) = \nabla u_i(x_2) \cdot \mu.$$

Since μ is an arbitrary tangential direction, the claim $\nabla u(x_2) = 0$ is proved.

Finally, the vanishing of $\nabla u(x_2)$ together with (2.29) imply the sharpness of the $C^{1,\beta}$ regularity.

□

Remark 2.2. For $1 \leq \alpha < 2$, by exactly the same argument we can also prove $u \in C^{1,\gamma}_{loc}$ for any $\gamma \in (0, 1)$. Just notice that we need to replace (2.24) with $W(u) - W(v) \leq C|u - v|$. When $\alpha = 1$, in contrast to the scalar case problem (see [39]), it is still open whether $u \in C^{1,1}_{loc}$.

Now with the regularity result we can identify the Euler-Lagrange equation for the entire minimizer u in the following lemma.

Lemma 2.2. *Let u be an entire minimizer of the functional J satisfying $|u(x)| \leq M$. Assume H1, H2. Then we have*

(1) $1 < \alpha < 2$, *u is a strong solution of*

$$\Delta u = W_u(u), \qquad \forall x \in \mathbb{R}^n. \tag{2.30}$$

(2) $\alpha = 1$, *u is a strong solution of*

$$\Delta u = W_u(u)\chi_{\{d(u,A)>0\}}. \tag{2.31}$$

(3) $0 < \alpha < 1$, *in the open set $\{x : d(u(x), A) > 0\}$, u solves the equation* (2.30).

Here $W_u(u)$ denotes the derivative of W with respect to u, χ is the characteristic function and $d(x, A) := \operatorname{dist}(x, A)$.

Proof. Take $D \in \mathbb{R}^n$ to be an arbitrary bounded Lipschitz domain, and for $\phi \in C_0^\infty(D, \mathbb{R}^m)$, we compute the first variation of the energy $J_D(u)$

$$\begin{aligned} 0 &\leq \int_D \left(\frac{1}{2}|\nabla(u + t\phi)|^2 - \frac{1}{2}|\nabla u|^2 + W(u + t\phi) - W(u) \right) dx \\ &= t\int_D \nabla u \cdot \nabla \phi \, dx + \frac{t^2}{2}\int_D |\nabla \phi|^2 \, dx + \int_D (W(u + t\phi) - W(u)) \, dx. \end{aligned} \tag{2.32}$$

When $1 < \alpha < 2$, W can be written as (2.22) with the condition (2.23). One can directly compute

$$W_u(u) = \left(\alpha(u - a_i)|u - a_i|^{\alpha - 2} \prod_{k \neq i} |u - a_k|^\alpha \right) g(u) + \left(\prod_{i=1}^N |u - a_i|^\alpha \right) D_u g(u). \tag{2.33}$$

Since we already proved $u \in C^{1,\gamma}(D)$, it is obvious that $W_u(u) \in C(D)$. Therefore when we divide (2.32) by t and take $t \to 0+$ or $0-$, it follows that u solves (2.30) in D.

For $\alpha = 1$, dividing (2.32) by t and letting $t \to 0$, we can prove that

$$\left| \int_D \nabla u \cdot \nabla \phi \, dx \right| \leq C \int_D |\phi| \, dx,$$

which implies $\Delta u \in L^\infty(D, \mathbb{R}^m)$. Moreover, on any sub-domain $K \subset D \cap \{d(u, A) > 0\}$, the equation (2.30) holds. Combining the fact that $\Delta u = 0$ a.e. on $\{x : u(x) \in A\}$ (which is due to the fact that weak derivatives for Sobolev functions vanish on level sets a.e. [15]), we have (since u is continuous) that u is a strong solution of (2.31) in D.

For $0 < \alpha < 1$, one can easily check that u solves (2.30) in the open set $\{x : d(u, A) > 0\}$. However, since $W_u(u)$ blows up as $d(u, A) \to 0$, the local integrability of $W_u(u)$ is not a priori known, so u may not solve (2.31) on the whole space in distributional sense. □

Remark 2.3. In the case $0 < \alpha < 1$, even though we cannot say u is a distributional solution to (2.31), we can deduce another equation for u from the first domain variation. For any $\varphi \in C_0^1(D, \mathbb{R}^n)$, it holds that

$$\begin{aligned} 0 &= \frac{d}{dt} J_D(u(x + t\varphi(x))) \\ &= \int_D \left((\nabla u \nabla \varphi) \cdot \nabla u - (\operatorname{div} \varphi)(\frac{1}{2}|\nabla u|^2 + W(u)) \right) dx \end{aligned}$$

This formulation has also been utilized in [28, 46]. Here we just present this form of equation for completeness and it will not be utilized in the rest of the paper.

With the Euler-Lagrange equation (2.30) and the formula for $W_u(u)$ (2.33), we can easily improve the regularity of u when $1 < \alpha < 2$.

Proposition 2.3. *When $1 < \alpha < 2$, the entire minimizer $u \in C^{2,\alpha-1}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. According to Lemma 2.2, u satisfies the equation (2.30) when $1 < \alpha < 2$. Using the formula (2.33) for $W_u(u)$ and the rough estimate $u \in C^{1,\gamma}_{loc}$, we see that $W_u(u) \in C^{\alpha-1}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$. Then the $C^{2,\alpha-1}_{loc}$ regularity immediately follows from the classical Schauder estimate. □

3. Estimate of $\mathcal{L}^n(I_0)$ and the existence of the interface

In this section we will prove the estimate (1.13) for any nontrivial entire minimizer u of the functional J. Furthermore, we also show that at every sufficiently large scale, the minimizer u must contain at least two different phases.

Take W satisfying the hypothesises H1 & H2 and assume u is an entire minimizer for the functional J. Before stating our new results, we first recall two estimates from [1] and [2] without proof, which will play an important role in our arguments. Readers can refer to [1] and [2] for detailed proofs. These estimates for the scalar case $m = 1$ are obtained in [8].

Proposition 3.1. *When $0 < \alpha < 2$, for any entire minimizer u satisfying $\|u\|_{L^\infty(\mathbb{R}^n)} < \infty$, the following two estimates hold true:*

(1) *<u>The basic estimate</u> (see [2, Lemma 2.2]) For any $x_0 \in \mathbb{R}^n$, there exists an r_0 such that for $r > r_0$,*

$$J_{B_r(x_0)}(u) \leq C r^{n-1}, \tag{3.34}$$

where the constant $C = C(M)$ is independent of u. We notice that this r_0 can be 0 when $\alpha \in [1, 2)$.

(2) *The density estimate (see* [1, Theorem 5.2]*) Take $a \in A$ to be a minimal point for $W(u)$. If for some r_0, λ, $\mu_0 > 0$,*

$$\mathcal{L}^n(B_{r_0}(x) \cap \{|u-a| > \lambda\}) \geq \mu_0,$$

then there exists a constant $C(\mu_0, \lambda) > 0$ such that

$$\mathcal{L}^n(B_r(x) \cap \{|u-a| > \lambda\}) \geq C(\mu_0, \lambda) r^n, \quad \forall r \geq r_0. \tag{3.35}$$

Another important component of our arguments is the following non-degeneracy lemma.

Lemma 3.2. *Assume $0 < \alpha < 2$. We take the point $a_1 \in A$ and an entire minimizer u for the functional J such that $\|u\|_{L^\infty(\mathbb{R}^n)} < \infty$. There exists a suitably small number $\theta(W)$ and a constant $c = c(n, W)$, such that if $x_0 \in \overline{\{0 < |u - a_1| < \theta\}}$ and $B_r(x_0) \subset \overline{\{|u-a_1| < \theta\}}$, then*

$$\sup_{B_r(x_0)} |u - a_1| \geq c(n, W) r^{\frac{2}{2-\alpha}}, \tag{3.36}$$

where the constant $c(n, W)$ only depends on the dimension n and the potential function W. Moreover $c(n, W) \sim O(\alpha)$ for $\alpha << 1$.

Proof. Without loss of generality, suppose $a_1 = 0^m$. First we require that

$$\theta < \frac{1}{2} \min_{i \neq j} |a_i - a_j| = r_0.$$

By (H2), when $|u| < r_0$, $W(u)$ can be written as $W(u) = |u|^\alpha \cdot g(u)$ for some $g(u) \in C^2(B_{r_0}(0^m))$ satisfying

$$g(u) \geq C_g$$

for some constant $C_g > 0$.

Assume $|u(x_0)| > 0$ (if $|u(x_0)| = 0$, then we simply take a sequence of points $\{x_i\}$ converging to x_0 and satisfy $|u(x_i)| > 0$). Taking $h(x) = |u|^{2-\alpha} - c|x - x_0|^2$ for some constant c which will be determined later, by direct calculation we have that if $|u(x)| > 0$, then

$$\Delta h = (2-\alpha)(-\alpha)\frac{|\nabla |u||^2}{|u|^\alpha} + (2-\alpha)\frac{|\nabla u|^2}{|u|^\alpha} + (2-\alpha) u \cdot D_u(g) + \alpha(2-\alpha) g(u) - 2nc \tag{3.37}$$

If we take

$$\theta < \min\{\frac{\alpha C_g}{4\|D_u g(u)\|_{L^\infty(B_{r_0}(0^m))}}, \frac{1}{2}\min_{i \neq j}|a_i - a_j|\}, \quad c < \frac{\alpha(2-\alpha)C_g}{8n}, \tag{3.38}$$

then (3.37) implies that

$$\Delta h \geq -(2-\alpha)\alpha \frac{|\nabla|u||^2}{|u|^\alpha} + (2-\alpha)\frac{|\nabla u|^2}{|u|^\alpha}$$

When $\alpha \leq 1$, it follows that $\Delta h \geq 0$ in $\{|u(x)| > 0\} \cap B_r(x_0)$. Since $h(x_0) > 0$ and $h(x) < 0$ on $\partial\{|u(x) > 0|\} \cap B_r(x_0)$, we must have

$$\max_{x \in \partial B_r(x_0)} h(x) > 0,$$

which implies the lemma.

For $\alpha \in (1,2)$, combining (3.37) and (3.38) we deduce that in $\{|u|>0\} \cap B_r(x_0)$

$$\begin{aligned}
&\Delta h + (2-\alpha)\alpha \frac{|\nabla |u||^2}{|u|^\alpha} - (2-\alpha)\frac{|\nabla u|^2}{|u|^\alpha} \geq \frac{\alpha(2-\alpha)C_g}{4} \\
\Rightarrow &\Delta h + \frac{\alpha}{2-\alpha}\frac{\nabla h \cdot \nabla(|u|^{2-\alpha} + c|x-x_0|^2) + 4c^2|x-x_0|^2}{|u|^{2-\alpha}} \geq (2-\alpha)\frac{|\nabla u|^2}{|u|^\alpha} + \frac{\alpha(2-\alpha)C_g}{4} \\
\Rightarrow &\Delta h + \left(\frac{\alpha \nabla(h+2c|x-x_0|^2)}{(2-\alpha)|u|^{2-\alpha}}\right)\cdot \nabla h + \frac{4c\alpha(-h+|u|^{2-\alpha})}{(2-\alpha)|u|^{2-\alpha}} \geq \frac{\alpha(2-\alpha)C_g}{4} \\
\Rightarrow &\Delta h + \left(\frac{\alpha \nabla(h+2c|x-x_0|^2)}{(2-\alpha)|u|^{2-\alpha}}\right)\cdot \nabla h - \frac{4c\alpha}{(2-\alpha)|u|^{2-\alpha}}\cdot h \geq 0.
\end{aligned}$$

Here to derive the last inequality we further require that c satisfies

$$c \leq \frac{(2-\alpha)^2 C_g}{16}. \tag{3.39}$$

Then the maximum principle argument can be applied again to get

$$\max_{x\in\partial B_r(x_0)} |u(x)| > cr^{\frac{2}{2-\alpha}}.$$

This completes the proof. □

Now we are ready to prove the first part (1.13) of the Theorem 1.2. For the sake of convenience, we rewrite the statement in the following theorem.

Theorem 3.3 (First part of Theorem 1.2)**.** *Let $x_0 \in \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded nonconstant entire minimizer of the energy J. Then there are positive constants R_0 and c such that*

$$\mathcal{L}^n(B_R(x_0)\cap I_0) \leq cR^{n-1}, \quad R > R_0, \tag{3.40}$$

where I_0 is defined in (1.12)*, which is the region where $W(u) > 0$. The constant c only depends on the dimension n, the potential function W and $\|u\|_{L^\infty(\mathbb{R}^n)}$.*

Proof. Without loss of generality, suppose $x_0 = 0^n$ and write $B_R = B_R(0^n)$. According to the basic estimate (3.34) in Proposition 3.1, we know that there exist positive constants C_0, r_0 such that for any $R > r_0$

$$\int_{B_R} \frac{1}{2}|\nabla u|^2 + W(u)\,dx \leq C_0 R^{n-1}. \tag{3.41}$$

For the sake of convenience, we use the cubes which are centered at 0^n to replace B_R. Define

$$\tilde{S}_R := \{x \in \mathbb{R}^n : x_i \in (-R,R), \text{ for } i = 1,2,...,n\}.$$

Let L be a constant whose value will be specified later. For any positive integer k, we can divide the cube $\tilde{S}_{kL}$ into $(2k)^n$ identical cubes with the side length L. We number all these sub-cubes by $S_1, S_2, ..., S_K$ where $K := (2k)^n$. And we take θ to be the constant $\theta(W)$ in the Lemma 3.2. Then we define

$$\sigma_i^j := \mathcal{L}^n(\{|u-a_j| < \frac{\theta}{2}\} \cap S_i), \quad \text{for } i = 1,...K,\ j = 1,...,N. \tag{3.42}$$

Take $\varepsilon := \varepsilon(\theta)$ to be a small constant to be specified later, depending only on θ and $\|u\|_{L^\infty}$. Also we introduce the notion of adjacent sub-cubes: S_{i_1} and S_{i_2} are called adjacent if and only if

$$\overline{S_{i_1}} \cap \overline{S_{i_2}} \neq \varnothing, \quad 1 \leq i_1,\, i_2 \leq K$$

We divide $\{S_i\}_1^K$ into the following five non-overlapping classes

1 Boundary sub-cubes of $\tilde{S}_{kL}$:

$$T_1 := \{S_i : \text{ the number of adjacent cubes of } S_i \text{ is less than } 3^n - 1\}.$$

2 Sub-cubes that contain two phases:

$$T_2 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} = \max_{1\le j\le N} \sigma_i^j \le (1-2\varepsilon)L^n,\ \max_{j\ne j_0} \sigma_i^j \ge \frac{\varepsilon}{N-1} L^n\}\backslash T_1.$$

3 Sub-cubes that contain regions where u stays away from any a_j.

$$T_3 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} = \max_{1\le j\le N} \sigma_i^j \le (1-2\varepsilon)L^n,\ \max_{j\ne j_0} \sigma_i^j < \frac{\varepsilon}{N-1} L^n\}\backslash T_1.$$

4 "Interior" sub-cubes of the contact set $\{x : u(x) \in A\}$:

$$T_4 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} > (1-2\varepsilon)L^n \text{ and } \sigma_p^{j_0} > (1-2\varepsilon)L^n, \forall S_p \text{ adjacent to } S_i\}\backslash T_1.$$

5 Sub-cubes close to the boundary of the contact set:

$$T_5 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} > (1-2\varepsilon)L^n \text{ and } \sigma_p^{j_0} \le (1-2\varepsilon)L^n,\ \text{for some } S_p \text{ adjacent to } S_i\}\backslash T_1.$$

Now we estimate the number of cubes in each class. First note that $S_i \in T_1$ means S_i is one of the boundary cubes of $\tilde{S}_{kL}$, therefore

$$|T_1| \le c_0(n)k^{n-1} \quad \text{for some dimensional constant } c_0(n). \tag{3.43}$$

For a cube $S_i \in T_2$, assume $\sigma_i^{j_0} = \max_{1\le j\le N} \sigma_i^j \le (1-2\varepsilon)L^n$ and $\sigma_i^{j_1} \ge \frac{\varepsilon}{N-1}L^n$ for some $j_1 \ne j_0$. By the definition of σ_i^j and $\theta < \frac{1}{2}|a_{j_0} - a_{j_1}|$, we can infer that for any $r \in [\frac{\theta}{2}, |a_{j_0} - a_{j_1}| - \frac{\theta}{2}]$, it holds that

$$\mathcal{L}^n(\{|u-a_{j_0}| < r\} \cap S_i) \ge \frac{\varepsilon}{N-1}L^n, \quad \mathcal{L}^n(S_i\backslash\{|u-a_{j_0}| < r\}) \ge \frac{\varepsilon}{N-1}L^n.$$

Applying the co-area formula and the relative isoperimetric inequality (see for example [42]), we have

$$\begin{aligned}
&\int_{S_i} |\nabla u|^2\,dx \\
\ge&\frac{1}{L^n}\left(\int_{S_i} |\nabla(u-a_{j_0})|\,dx\right)^2 \\
\ge&\frac{1}{L^n}\left(\int_{\theta/2}^{|a_{j_0}-a_{j_1}|-\theta/2} \mathcal{H}^{n-1}(\{|u-a_{j_0}| = r\}\cap S_i)\,dr\right)^2 \\
\ge&\frac{1}{L^n}\left(\int_{\theta/2}^{|a_{j_0}-a_{j_1}|-\theta/2} C\left(\min\{\mathcal{L}^n(\{|u-a_{j_0}|<r\}\cap S_i), \mathcal{L}^n(S_i\backslash\{|u-a_{j_0}|<r\})\}\right)^{\frac{n-1}{n}}\,dr\right)^2 \\
\ge&c_1(L,\theta,\varepsilon) > 0
\end{aligned} \tag{3.44}$$

From the basic estimate (3.34) we get

$$|T_2| \le \frac{C(kL)^{n-1}}{c_1} = c_2(L,\theta,\varepsilon)k^{n-1}, \quad k \ge k_0, \tag{3.45}$$

where k_0 is a constant.

For a cube $S_i \in T_3$,

$$\mathcal{L}^n\left(\left\{|u-a_j| > \frac{\theta}{2},\ \forall 1\le j\le N\right\}\cap S_i\right) > \varepsilon L^n$$

By the Hypothesis H1 and H2 on W and the assumption $\|u\|_{L^\infty} < \infty$, there is a constant c_3, which depends on $\|u\|_{L^\infty}$ and θ, such that

$$W(u) \geq c_3, \quad \text{when } |u - a_j| > \frac{\theta}{2}, \ \forall 1 \leq j \leq N$$

Thus by (3.41) the number of sub-cubes T_3 is bounded by

$$|T_3| \leq c_4(L, \theta, \varepsilon, \|u\|_{L^\infty}) k^{n-1}, \quad k \geq k_0. \tag{3.46}$$

From now on we focus on the analysis of cubes in T_4 and T_5. Take S_i in T_4 or T_5, then there is a j_0 such that $\sigma_i^{j_0} > (1 - 2\varepsilon)L^n$. In this case, we claim that when ε is suitably chosen, we can assure that

$$\max_{x \in S_i} |u(x) - a_{j_0}| < \theta$$

If there exists $x_0 \in S_i$ such that $|u(x_0) - a_{j_0}| \geq \theta$, then we have that there exists a constant $c_5(\theta, \|\nabla u\|_{L^\infty})$ such that

$$\mathcal{L}^n(\{|u - a_{j_0}| > \theta/2\} \cap S_i) \geq c_5.$$

We note that the uniform boundedness of $|\nabla u|$ follows from the $C^{1,\beta}$ regularity (Proposition 2.1 and Proposition 2.3) and the assumption that $|u|$ is uniformly bounded. And c_5 doesn't depend on j_0. Then the claim follows if we simply take

$$\varepsilon < \frac{c_5}{2L^n}. \tag{3.47}$$

Lemma 3.4. *When L is suitably chosen depending on θ, in any cube $S_i \in T_4 \cup T_5$, it holds*

$$\mathcal{L}^n(\{u(x) = a_{j_0}\} \cap S_i) \geq \omega_n \left(\frac{L}{4}\right)^n \tag{3.48}$$

where ω_n is the volume of the n-dimensional unit ball.

Proof. We proceed by contradiction and denote the central point of S_i by z_i. So,

$$|\{u(x) = a_{j_0}\} \cap S_i| < \omega_n \left(\frac{L}{4}\right)^n$$

Then there must be a point $x_1 \in B_R(z_i, \frac{L}{4})$ such that $x_1 \in \overline{\{0 < |u - a_{j_0}| < \theta\}}$. Moreover, we have

$$B_{\frac{L}{4}}(x_1) \subset S_i \subset \overline{\{|u - a_{j_0}| < \theta\}}$$

Therefore we are in the position to apply Lemma 3.2 to deduce that

$$\sup_{B_{\frac{L}{4}}(x_1)} |u - a_{j_0}| \geq c(n, W) \left(\frac{L}{4}\right)^{\frac{2}{2-\alpha}}$$

which contradicts with $\max_{x \in S_i} |u - a_{j_0}| < \theta$ if we choose the constant L at the beginning satisfying $c(n, W) \left(\frac{L}{4}\right)^{\frac{2}{2-\alpha}} > 2\theta$. This completes the proof of Lemma 3.4. □

If the cube $S_i \in T_4$, then by definition we have

$$|u(x) - a_{j_0}| < \theta, \quad \forall x \in S_i \cup (\bigcup_{S_p \text{ adjacent to } S_i} S_p)$$

By the same argument as in the proof of the lemma above, we obtain that

$$u(x) \equiv a_{j_0}, \quad x \in S_i.$$

If $S_i \in T_5$, then there must be at least one adjacent cube of S_i, denoted by S_p, such that

$$|\{|u - a_{j_0}| > \frac{\theta}{2}\} \cap S_p| > \varepsilon L^n. \tag{3.49}$$

We set

$$Q_{S_i} := S_i \cup (\bigcup_{S_p \text{ adjacent to } S_i} S_p)$$

Then by (3.48), (3.49) and the co-area formula, we can compute similarly as in (3.44) to get

$$\int_{Q_{S_i}} |\nabla u|^2 \, dx \geq c_6(L, \theta, \varepsilon).$$

Since each point can belong to at most 3^n different Q_{S_i}, utilizing (3.34) we conclude

$$C(n)(kL)^{n-1} \geq \sum_{S_i \in T_5} \int_{Q_{S_i}} |\nabla u|^2 \, dx \geq c_6 |T_5|,$$

which implies

$$|T_5| \leq c_7(n, L, \theta, \varepsilon) k^{n-1}. \tag{3.50}$$

Finally, combining (3.43), (3.45), (3.46) and (3.50) we get

$$\mathcal{L}^n(\tilde{S}_{kL} \cap I_0) \leq (|T_1| + |T_2| + |T_3| + |T_5|) L^n \leq c_8(n, L, \theta, \varepsilon)(kL)^{n-1}. \tag{3.51}$$

Since $B_{kL} \subset \tilde{S}_{kL}$, we can get (3.40) after taking k to be the smallest integer larger than $\frac{R}{L}$ for sufficiently large R. Also if we carefully check the definitions of all the constants in the proof we conclude that c_8 only depends on the dimension n, the potential W and the uniform bound of $|u|$, but not on the specific solution u. This completes the proof of Theorem 3.3. □

Remark 3.1. The proof of Theorem 3.3 implies the existence of $a_i \in A$, such that $\mathcal{L}^n(\{u(x) = a_i\}) > 0$ and $\{u(x) = a_i\}$ contains interior points.

Theorem 3.3 implies that a bounded entire minimizer $u(x)$ should satisfy $W(u) = 0$ in "most of the space". Next we further show that at sufficiently large scales, u must possess at least two different phases, each of which contains some definite measure of order R^n.

Lemma 3.5. *Let $x_0 \in \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded entire minimizer of J. Assume that $u \not\equiv a_i$ for any $i \in \{1, 2, ..., N\}$. We take an arbitrary constant $\theta < r_0 := \frac{1}{2} \min_{1 \leq i \neq j \leq N} |a_i - a_j|$, then there exist positive constants $R_0, c(u, \theta)$ such that for any $R \geq R_0$, there are $a_i, a_j \in A$, which depend on R, satisfying*

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_k| < \theta\}) \geq cR^n, \quad k = i, j. \tag{3.52}$$

Proof. Since u is nonconstant, by $C^{1,\beta}$ regularity of u there is some $R_1 > 0$, $0 < \lambda < r_0$, $\mu_0 > 0$ such that

$$\mathcal{L}^n(B_{R_1}(x_0) \cap \{|u - a_1| > \lambda\}) \geq \mu_0.$$

Then by the density estimate (3.35) in Proposition 3.1, there exists μ_1 such that

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_1| > \lambda\}) \geq \mu_1 R^n, \quad \forall R \geq R_1 \tag{3.53}$$

Take $\theta < r_0$ to be an arbitrary constant. By our hypothesis on W, there is a positive constant $C = C(\lambda, \theta, \|u\|_{L^\infty})$ such that

$$W(u) > C, \ \text{ when } |u - a_1| > \lambda, \ |u - a_j| \geq \theta \text{ for any } j \neq 1.$$

Applying the basic estimate (3.34) in Proposition 3.1, for enough large R,

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_1| > \lambda, \ |u - a_j| \geq \theta \text{ for any } j \neq 1\}) \leq C_2 R^{n-1}, \tag{3.54}$$

for some constant C_2. Combining (3.53) and (3.54), we obtain that

$$\begin{aligned} &\mathcal{L}^n(B_R(x_0) \cap (\bigcup_{j \neq 1} \{|u - a_j| < \theta\})) \\ \geq &\mathcal{L}^n(B_R(x_0) \cap \{|u - a_1| > \lambda, |u - a_j| < \theta \text{ for some } j \neq 1\}) \geq c_1(u, \theta) R^n, \quad \forall R > \tilde{R}_1 \end{aligned}$$

for some constants $\tilde{R}_1$ and c_1. The same argument also works for the set $B_R(x_0) \cap (\bigcup_{j \neq k} \{|u - a_j| < \theta\})$ for any $k \in \{1, 2, ..., N\}$, i.e. there exists $\tilde{R}_k, \ c_k > 0$ such that

$$\mathcal{L}^n(B_R(x_0) \cap (\bigcup_{j \neq k} \{|u - a_j| < \theta\})) \geq c_k(u, \theta) R^n, \forall R \geq \tilde{R}_k$$

Finally, we take $R_0 = \min_k \tilde{R}_k$ and $c = \frac{1}{N-1} \min_k c_k$ and the conclusion of the lemma easily follows. $\square$

In the following theorem, we show that in any ball $B_R(x_0)$ with radius R large enough, the sets $\{u = a_i\}$ and $\{u = a_j\}$ (a_i, a_j from Lemma 3.5) must contain a set of measure of the order R^n.

Theorem 3.6. *Let $x_0 \in \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded entire minimizer of the energy J, and $u \not\equiv a_j$ for any $j \in \{1, 2, ..., N\}$. Then there are positive constants R_0 and c (both depend on u) such that for any $R \geq R_0$, there are a_i, a_j depending on R such that*

$$\min\{\mathcal{L}^n(B_R(x_0) \cap \{u = a_i\}), \mathcal{L}^n(B_R(x_0) \cap \{u = a_j\})\} \geq c R^n, \quad \forall R > R_0 \tag{3.55}$$

Proof. Without loss of generality, suppose $x_0 = 0^n$ and write $B_R = B_R(0^n)$. According to the Proposition 3.1 and Lemma 3.5, we know that for any sufficiently large R, there are $a_i, a_j \in A$ such that (3.52) holds.

The proof relies on the same technique as the proof of Theorem 3.3. So we will only present the main ingredients and omit some technical details. Take L as the same constant in Theorem 3.3 and $k \in \mathbb{N}$. We consider the domain

$$\tilde{S}_{kL} := \{x \in \mathbb{R}^n : x_i \in (-kL, kL)\},$$

and then divide $\tilde{S}_{kL}$ into $K = (2k)^n$ identical sub-cubes $S_1, ..., S_K$, each of which has side of length L. We also recall the definition of σ_i^j in (3.42). By Lemma 3.5, there are two phases a_i, a_j (for simplicity we assume they are a_1, a_2) such that

$$\mathcal{L}^n(\tilde{S}_{kL} \cap \{|u - a_j| < \frac{\theta}{2}\}) \geq c(kL)^n, \quad j = 1, 2. \tag{3.56}$$

Take $\varepsilon := \varepsilon(u, \theta)$ be a small constant such that

a. (3.47) holds. As a result, if $\sigma_i^j > (1 - 2\varepsilon) L^n$, then $|u(x) - a_j| < \theta$ for any $x \in S_i$.

b. $\varepsilon \leq \frac{c}{2^{n+3}}$ where c is the constant in (3.56).

Then we divide $\{S_i\}_1^K$ into the following two classes

1 $U_1 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} = \max_{1\le j\le N} S_i^j \le (1-2\varepsilon)L^n\}$.

2 $U_2 := \{S_i : \exists j_0 \text{ s.t. } \sigma_i^{j_0} = \max_{1\le j\le N} S_i^j > (1-2\varepsilon)L^n.\}$

From the proof of Theorem 3.3, we have

$$|U_1| \le c_0(L,\theta,\varepsilon)k^{n-1}.$$

Let K_1 denote the number of sub-cubes S_i satisfying $\sigma_i^1 > (1-2\varepsilon)L^n$. We obtain from (3.56)

$$\begin{aligned} c(kL)^n &\le \sum_{1\le i\le (2k)^n} \sigma_i^1 \\ &\le |U_1|L^n + K_1L^n + ((2k)^n - |U_1| - K_1)(2\varepsilon L^n) \\ &\le c_0k^{n-1}L^n + K_1L^n + \frac{c}{4}(kL)^n \quad \text{(Property b of } \varepsilon), \end{aligned} \tag{3.57}$$

which immediately implies that $K_1 \ge \frac{c}{2}k^n$ whenever k is large enough. Together with Lemma 3.4 we have

$$\mathcal{L}^n(\tilde{S}_{kL} \cap \{u = a_1\}) \ge \frac{c}{2}k^n\omega_n(\frac{L}{4})^n \ge c_1(kL)^n, \tag{3.58}$$

for some constant $c_1 = c_1(W,u)$. For $\{u = a_2\}$ the estimate (3.58) still holds. One can easily check that (3.58) implies the statement of Theorem 3.6.

□

4. Weiss' Monotonicity formula and a growth estimate in the case $\alpha = 1$

Thanks to Theorem 3.6, we know for a uniformly bounded entire minimizer u, that the free boundary $\partial\{|u-a_i| > 0\}$ $(i = 1,2)$ must exist. In this section we will derive a growth rate estimate for $|u - a_i|$ away from the free boundary in the case $\underline{\alpha = 1}$.

From now on we fix $\alpha = 1$ and assume $a_1 = 0^m$. By the hypothesis H2, $W(u)$ has the form $W(u) = g(u)|u|$ for some $g \in C^2(B_\theta)$. Here θ is the constant in Lemma 3.2. Since u is a local minimizer, it satisfies the Euler-Lagrange equation near the free boundary point,

$$\Delta u = g(u)\frac{u}{|u|} + |u|D_u g(u). \tag{4.59}$$

Also there exists a positive constant $C > 0$ such that $g(u) > C$ when $|u| \le \theta$. We use the notation

$$\Omega(u) := \{|u(x)| > 0\}, \quad \Gamma(u) := \partial^*\Omega(u). \tag{4.60}$$

Here ∂^* denotes De Giorgi's reduced boundary. An easy observation is that for any point $x \in \Gamma(u)$, we must have $|u(x)| = |\nabla u(x)| = 0$. The proof is straightforward: if at some point $x_0 \in \Gamma(u)$, $|\nabla u| > 0$, then by continuity of ∇u we have that in a small neighborhood $B_r(x_0)$, $|\nabla u_i| \ge c$ for some $1 \le i \le m$ and $c > 0$. The inverse function theorem implies that in $B_r(x_0)$, $\{u_i = 0\}$ is a $(n-1)$-dimensional hypersurface, which further gives $x_0 \notin \partial^e\Omega(u)$, where ∂^e denotes the measure theoretic boundary. Finally we arrive at a contradiction thanks to the well-known result $\partial^*E \subset \partial^eE$ for any set E of locally finite perimeter. For the definitions of the reduced boundary and the measure theoretic boundary, as well as their relationship, we refer to [15, Chapter 5.7&5.8] for details.

We first establish an almost monotonicity formula for $|u| < \theta$. The proof closely follows the classical arguments of Weiss (see [44, 45])

Lemma 4.1. *Let u be a solution of (4.59) in $B_r(x_0)$ such that $|u| < \theta$ in $B_r(x_0)$, and set*

$$W(u, x_0, r) = \frac{1}{r^{n+2}} \int_{B_r(x_0)} \left(\frac{1}{2} |\nabla u|^2 + g(u)|u| \right) dx - \frac{1}{r^{n+3}} \int_{\partial B_r(x_0)} |u|^2 \, d\mathcal{H}^{n-1}. \tag{4.61}$$

Then $W(u, x_0, r)$ satisfies

$$\frac{d}{dr} W(u, x_0, r) = r \int_{\partial B_1} |\frac{du_r}{dr}|^2 \, d\mathcal{H}^{n-1} + 2r \int_{B_1} D_u g \cdot u_r |u_r| \, dx \tag{4.62}$$

where

$$u_r(x) := \frac{u(x_0 + rx)}{r^2}.$$

Proof. First we write $W(u, x_0, r)$ as

$$W(u, x_0, r) = \int_{B_1} \left(\frac{1}{2} |\nabla u_r|^2 + g(r^2 u_r)|u_r| \right) dx - \int_{\partial B_1} |u_r|^2 \, d\mathcal{H}^{n-1}.$$

Then by a direct calculation we have

$$\begin{aligned}
&\frac{d}{dr} W(u, x_0, r) \\
=& \int_{B_1} \left(\nabla u_r \cdot \frac{d}{dr}(\nabla u_r) + D_u g(r^2 u_r) \cdot \frac{d}{dr}(r^2 u_r)|u_r| + g(r^2 u_r)|u_r|^{-1} u_r \cdot \frac{d}{dr} u_r \right) dx \\
&- 2 \int_{\partial B_1} u_r \cdot \frac{d}{dr} u_r \, d\mathcal{H}^{n-1} \\
=& \int_{B_1} \left(-\Delta u_r \cdot \frac{d}{dr} u_r + D_u g(r^2 u_r) \cdot \frac{d}{dr}(r^2 u_r)|u_r| + g(r^2 u_r)|u_r|^{-1} u_r \cdot \frac{d}{dr} u_r \right) dx \\
&- 2 \int_{\partial B_1} u_r \cdot \frac{d}{dr} u_r \, d\mathcal{H}^{n-1} + \int_{\partial B_1} (x \cdot \nabla u_r) \cdot \frac{d}{dr} u_r \, d\mathcal{H}^{n-1}. \\
=& \int_{B_1} \left(-\Delta u_r \cdot \frac{d}{dr} u_r + D_u g(r^2 u_r) \cdot \frac{d}{dr}(r^2 u_r)|u_r| + g(r^2 u_r)|u_r|^{-1} u_r \cdot \frac{d}{dr} u_r \right) dx \\
&+ \int_{\partial B_1} r |\frac{d}{dr} u_r|^2 \, \mathcal{H}^{n-1}.
\end{aligned}$$

Here we have used integration by parts in the second step and the formula $\frac{d}{dr} u_r = \frac{1}{r}(x \cdot \nabla u_r - 2u_r)$ in the last step. Since u satisfies the equation (4.59), direct computation implies that

$$\Delta u_r = \left(g(r^2 u_r) \frac{u_r}{|u_r|} + D_u g(r^2 u_r) r^2 |u_r| \right). \tag{4.63}$$

Substituting (4.63) into the above identity, we obtain

$$\begin{aligned}
&\frac{d}{dr} W(u, x_0, r) - \int_{\partial B_1} r |\frac{d}{dr} u_r|^2 \, \mathcal{H}^{n-1} \\
=& \int_{B_1} \Bigg\{ - \left(g(r^2 u_r) \frac{u_r}{|u_r|} + D_u g(r^2 u_r) r^2 |u_r| \right) \frac{d}{dr} u_r \\
&\qquad + D_u g(r^2 u_r) \cdot \frac{d}{dr}(r^2 u_r)|u_r| + g(r^2 u_r) \frac{u_r}{|u_r|} \cdot \frac{d}{dr} u_r \Bigg\} dx \\
=& 2r \int_{B_1} D_u g(r^2 u_r) \cdot u_r |u_r| \, dx
\end{aligned}$$

Hence we have proved (4.62). ∎

Remark 4.1. For other $\alpha \in [0,2)$, the analogue result still holds for

$$W(u,x_0,r) = \frac{1}{r^{n+2\kappa-2}} \int_{B_r(x_0)} \frac{1}{2}|\nabla u|^2 + W(u)\, dx - \frac{\kappa}{2r^{n+2\kappa-1}} \int_{\partial B_r(x_0)} |u|^2 \, d\mathcal{H}^{n-1}.$$

where $\kappa := \frac{2}{2-\alpha}$ and $W(u) = g(u)|u|^\alpha$. The derivative of $W(u,x_0,r)$ is given by

$$\frac{d}{dr} W(u,x_0,r) = r \int_{\partial B_1} |\frac{du_r}{dr}|^2 \, d\mathcal{H}^{n-1} + \kappa r^{\kappa-1} \int_{B_1} D_u g \cdot u_r |u_r|^\alpha$$

For our purpose we only need the statement for $\alpha = 1$. The proof for general $\alpha \in [0,2)$ is identical and we omit it here.

Proposition 4.2. *Let $\alpha = 1$ and u be a bounded entire minimizer and let $\Gamma(u)$ be as defined in* (4.60)*. There exist constants r_0 and C, which only depends on $\|u\|_{L^\infty}$ and the potential function $W(u)$, such that*

$$|u(x)| \leq C \mathrm{dist}(x,\Gamma(u))^2, \quad |\nabla u(x)| \leq C \mathrm{dist}(x,\Gamma(u)) \tag{4.64}$$

whenever $\mathrm{dist}(x,\Gamma(u)) \leq r_0$.

Proof. This proposition and the proof are almost identical to [4, Theorem 2] (except now for a more general potential function). We present the whole argument here for completeness.

The statement of the proposition is equivalent to

$$\sup_{x\in B_r(x_0)} |u(x)| \leq Cr^2, \qquad \sup_{x\in B_r(x_0)} |\nabla u(x)| \leq Cr,$$

whenever $x_0 \in \Gamma(u)$, $r \leq r_0$. By (4.59) and the standard theory of elliptic regularity, it suffices to show

$$\frac{1}{r^n} \int_{B_r(x_0)} |u| \, dx \leq Cr^2, \quad \forall x_0 \in \Gamma(u),\ r \leq r_0 \tag{4.65}$$

where C and r_0 only depend on $\|u\|_{L^\infty}$ and the potential function W.

Note that since $|u|$ is uniformly bounded, we have $u \in C^{1,\gamma}$, which further implies $|\nabla u|$ is uniformly bounded. As a result, there is a constant r_0 such that $\mathrm{dist}(x,\Gamma(u)) \leq 2r_0$ implies $|u(x)| \leq \theta$, where θ is the constant in Lemma 3.2. Also, $W(u(x))$ has the form $g(u(x))|u(x)|$ for some smooth function $g(u) \geq C > 0$ when $\mathrm{dist}(x,\Gamma(u)) \leq 2r_0$.

Since $|\nabla u|$ is bounded and r_0 is a constant, we have that $W(u,x_0,r_0)$ is uniformly bounded by some constant C_1 independent of u and x_0. Here $W(u,x_0,r_0)$ is the quantity defined in (4.61). Using Lemma 4.1, we compute for $r < r_0$

$$\begin{aligned} \frac{1}{r^{n+2}} \int_{B_r(x_0)} g(u)|u| \, dx &= W(u,x_0,r) - \frac{1}{r^{n+2}} \int_{B_r(x_0)} \frac{1}{2}|\nabla u|^2 \, dx \\ &\quad + \frac{1}{r^{n+3}} \int_{\partial B_r(x_0)} |u|^2 \, d\mathcal{H}^{n-1} \\ &= W(u,x_0,r) - \frac{1}{r^{n+2}} \int_{B_r(x_0)} \frac{1}{2}|\nabla(u - p(x-x_0))|^2 \, dx \\ &\quad + \frac{1}{r^{n+3}} \int_{\partial B_r(x_0)} |u - p(x-x_0)|^2 \, d\mathcal{H}^{n-1} \\ &\leq W(u,x_0,r_0) + \int_r^{r_0} 2s \int_{B_1} |D_u g| |\frac{u(x_0+sx)}{s^2}|^2 \, dx \, ds \\ &\quad + \frac{1}{r^{n+3}} \int_{\partial B_r(x_0)} |u - p(x-x_0)|^2 \, d\mathcal{H}^{n-1}, \end{aligned} \tag{4.66}$$

for every $p(x) \in \mathcal{H}$, where $\mathcal{H}$ is defined by

$$\begin{aligned}\mathcal{H} :=\{p(x):\ & p(x) = (p_1(x), ...p_m(x)),\ \text{each } p_i(x) \text{ is a}\\ &\text{homogeneous harmonic polynomial of second order.}\}\end{aligned}$$

We would like to point out that the homogeneity and harmonicity of $p(x)$ is used in the second equality of (4.66).

We already know that the first term in the last step of (4.66) is bounded by a constant C_1 independent of u and x_0. For the second term, since $u(x_0 + x) \le C|x|^{\frac{5}{3}}$ when $|x| \le r_0$ by the $C^{1,\frac{2}{3}}$ regularity (c.f. Remark 2.2 and observation below (4.60)), we have

$$\int_r^{r_0} 2s \int_{B_1} |D_u g| |\frac{u(x_0 + sx)}{s^2}|^2 \, dx \, ds \le C \int_r^{r_0} s^{-3} \int_{B_1} |sx|^{\frac{10}{3}} \, dx \, ds \le C_2$$

for some constant C_2. Because $g(u) \ge C > 0$ in $B_r(x_0)$, in order to prove (4.65), it suffices to show that there is a constant C_3, independent of u and x_0, such that for any $x_0 \in \Gamma(u)$ and $r \le r_0$,

$$\min_{p\in\mathcal{H}} \frac{1}{r^{n+3}} \int_{\partial B_r(x_0)} |u - p(x - x_0)|^2 \, d\mathcal{H}^{n-1} \le C_3. \tag{4.67}$$

Let $p_{x_0,r}$ be the minimizer of the integral $\int_{\partial B_r(x_0)} |u - p(x - x_0)|^2 \, d\mathcal{H}^{n-1}$ among $p \in \mathcal{H}$. Then $p_{x_0,r}$ satisfies

$$\int_{\partial_{B_r}(x_0)} (u(x) - p_{x_0,r}(x - x_0)) \cdot q(x - x_0) \, d\mathcal{H} = 0 \quad \forall q \in \mathcal{H}. \tag{4.68}$$

Suppose by contradiction that (4.67) is not true, then there is a sequence of entire minimizers $\{u_k\}$ (uniformly bounded), a sequence of points $x_k \in \Gamma(u_k)$ as well as a sequence of radii $r_k \to 0$ such that

$$M_k := \frac{1}{r_k^{n+3}} \int_{\partial B_{r_k}(x_k)} |u_k - p_{x_k,r_k}(x - x_k)|^2 \, d\mathcal{H}^{n-1} \to \infty.$$

Define

$$v_k(x) := \frac{u_k(x_k + r_k x)}{r_k^2}, \qquad w_k := \frac{v_k - p_{x_k,r_k}}{\sqrt{M_k}}.$$

Then we immediately get

$$\int_{\partial B_1(0^n)} |w_k|^2 \, d\mathcal{H}^{n-1} = 1$$

and we have

$$\begin{aligned}
&\int_{B_1(0^n)} \frac{1}{2}|\nabla w_k|^2 \, dx - \int_{\partial B_1(0^n)} |w_k|^2 \, d\mathcal{H}^{n-1}\\
=&M_k^{-1} \left(\int_{B_1(0^n)} \frac{1}{2}|\nabla(v_k - p_{x_k,r_k})|^2 \, dx - \int_{\partial B_1(0^n)} |v_k - p_{x_k,r_k}|^2 \, d\mathcal{H}^{n-1} \right)\\
=&M_k^{-1} \left(\int_{B_1(0^n)} \frac{1}{2}|\nabla v_k|^2 \, dx - \int_{\partial B_1(0^n)} |v_k|^2 \, d\mathcal{H}^{n-1} \right)\\
\le&M_k^{-1} W(u_k, x_k, r_k)\\
\le&M_k^{-1} \left(W(u_k, x_k, r_0) + \int_{r_k}^{r_0} 2s \int_{B_1} |D_u g| |\frac{u_k(x_k + sx)}{s^2}|^2 \, dx \, ds \right)\\
\to&0 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.69}$$

So w_k is uniformly bounded in $W^{1,2}(B_1)$. Also we note that by (2.31) each w_k satisfies the equation

$$\Delta w_k = \frac{1}{\sqrt{M_k}}\left(\frac{v_k}{|v_k|}g(u_k) + D_u g(u_k)\right)\chi_{\{|u_k|>0\}},$$

which implies

$$|\Delta w_k| \leq \frac{C}{\sqrt{M_k}} \to 0, \quad \text{as } k \to \infty.$$

By Schauder estimates, w_k is uniformly bounded in $C^{1,\gamma}_{loc}(B_1)$ for any $\gamma < 1$. Therefore we can extract a subsequence, still denoted by w_k, that converges to w_0 with the following properties

(1) $w_k \to w_0$ weakly in $H^1(B_1)$, strongly in $L^2(\partial B_1)$, $\int_{\partial B_1} |w_0|^2 \, d\mathcal{H}^{n-1} = 1$.

(2) $w_k \to w_0$ in $C^{1,\gamma}_{loc}(B_1)$ for any $\gamma < 1$;

(3) $\Delta w_0 = 0$;

(4) $|w_0(0^n)| = |\nabla w_0(0^n)| = 0$;

(5) $\int_{\partial B_1} w_0 \cdot q \, d\mathcal{H}^{n-1} = 0$ for any $q \in \mathcal{H}$. This property follows from (4.68).

By [46, Lemma 4.1], we know that for any w_0 satisfying (3) and (4),

$$\int_{B_1} |\nabla w_0|^2 \, dx \geq 2 \int_{\partial B_1} |\nabla w_0|^2 \, d\mathcal{H}^{n-1}.$$

On the other hand, from (4.69) we know

$$\int_{B_1} |\nabla w_0|^2 \, dx \leq 2 \int_{\partial B_1} |\nabla w_0|^2 \, d\mathcal{H}^{n-1}.$$

Therefore we have $\int_{B_1} |\nabla w_0|^2 \, dx = 2 \int_{\partial B_1} |\nabla w_0|^2 \, d\mathcal{H}^{n-1}$ which implies (again by [46, Lemma 4.1]) that each component of w_0 is a homogeneous harmonic polynomial of second order, i.e. $w_0 \in \mathcal{H}$. This is in contradiction with properties (1) and (5). The proof is complete. □

5. $(n-1)$-Hausdorff measure of the free boundary for $\alpha = 1$

In this section, we continue working with the potential function $W(u)$ satisfying H1 and H2 with $\underline{\alpha = 1}$. Assume u is a bounded entire minimizer of the energy J. We would like to study the $(n-1)$-Hausdorff measure of $\partial^* I_0$ and prove the second part of Theorem 1.2, i.e. the inequality (1.14), (1.15).

Firstly we focus on the local estimate of $\partial^*\{u = a_i\}$ and we use the same notations and assumptions as in Section 4. Take $a_1 = 0^m$ and $W(u) = g(u)|u|$ for some $g \in C^2(B_\theta)$, θ as in Lemma 3.2. u satisfies the Euler-Lagrange equation (4.59).

Thanks to the growth estimate (4) and the non-degeneracy Lemma 3.2, we have for every $x_0 \in \Gamma(u)$ (recall that $\Gamma(u)$ is defined in (4.60)) and small r,

$$c_1 r^2 \leq \sup_{x \in B_r(x_0)} |u(x)| \leq c_2 r^2, \tag{5.70}$$

$$c_1 r \leq \sup_{x \in B_r(x_0)} |\nabla u(x)| \leq c_2 r. \tag{5.71}$$

Theorem 5.1. *(Local estimate of Γ) There are constants r_0 and C_0 such that*

$$\mathcal{H}^{n-1}(\Gamma(u) \cap B_{r_0}(z)) \leq C_0 \quad \textit{for every } \ z \in \Gamma(u). \tag{5.72}$$

Proof. Take the constant r_0 such that for any x that satisfies $\operatorname{dist}(x,\Gamma(u)) \le 2r_0$, $|u(x)| \le \theta$. We will fix a ball $B_{2r_0}(z)$ for some $z \in \Gamma(u)$ in the rest of the proof.

We define

$$v_i := \partial_{x_i} u, \ \ i = 1,2,...,n, \qquad \Sigma_\varepsilon(u) := \{x \in B_{r_0}(z) \cap \{|u|>0\} : |\nabla u| < \varepsilon\}.$$

By differentiating the Euler-Lagrange equation (4.59), formally we have

$$\begin{aligned} \Delta v_i =& |u|^{-1} g(u) v_i + |u|^{-1}(D_u g \cdot v_i)u) - |u|^{-3} g(u)(v_i \cdot u)u \\ &+ |u|^{-1}(v_i \cdot u) D_u g + |u|(D_u^2 g \cdot v_i). \end{aligned} \tag{5.73}$$

Take the function $\psi_\varepsilon(x) : \mathbb{R}^+ \to [0,1]$ defined by

$$\psi_\varepsilon(x) = \begin{cases} 1, & x \ge \varepsilon, \\ \frac{x}{\varepsilon}, & x \in [0,\varepsilon). \end{cases}$$

We also choose a smooth cut-off function $\phi \in C_c^\infty(B_{2r_0}(z), \mathbb{R})$ such that

$$\phi \equiv 1 \text{ in } B_{r_0}(z), \quad |\nabla \phi| \le \frac{C}{r_0}$$

Let

$$\tilde{A} := B_{2r_0}(z) \cap \{|u|>0\}.$$

The key of the proof is to estimate the following integral

$$I := \int_{\tilde{A}} \nabla v_i \cdot \nabla \left[\psi_\varepsilon(|v_i|) \frac{v_i}{|v_i|} \phi \right] dx, \tag{5.74}$$

from which estimate (5.83) below follows.

Claim. There exists a constant $C(g, r_0)$, which is independent of ε, z, such that

$$I \le C(g, r_0). \tag{5.75}$$

Proof of the Claim. Define

$$\eta := \psi_\varepsilon(|v_i|) \frac{v_i}{|v_i|} \phi$$

We first show that $\eta \in W_0^{1,2}(B_{2r_0}(z), \mathbb{R}^m)$. Indeed, by a direct computation we have

$$\begin{aligned} \partial_j \eta = & \psi_\varepsilon'(|v_i|) \partial_j |v_i| \frac{v_i}{|v_i|} \phi + \psi_\varepsilon(|v_i|) \frac{\partial_j v_i}{|v_i|} \phi \\ & - \psi_\varepsilon(|v_i|) v_i \frac{\partial_j |v_i|}{|v_i|^2} \phi + \psi_\varepsilon(|v_i|) \frac{v_i}{|v_i|} \partial_j \phi \end{aligned}$$

By definitions of ψ_ε, ϕ and the $W^{2,2}$ estimate of u, the right-hand side is L^2-integrable. Combining with the fact that $\phi \in C_0^\infty(B_{2r_0}(z))$, we get $\eta \in W_0^{1,2}(B_{2r_0}(z), \mathbb{R}^m)$.

We notice that since Δv_i is very singular when $|u| \to 0$, so we can not directly perform an integration by parts by moving all the derivatives on v_i in the domain $\tilde{A}$. Instead, we will switch ∂_i and ∇.

For any $f \in C_0^\infty(B_{2r_0}(z), \mathbb{R}^m)$, by integration by parts we have

$$\int_{B_{2r_0}(z)} \nabla v_i \cdot \nabla f \, dx = \int_{B_{2r_0}(z)} \Delta u \cdot \partial_i f \, dx$$

This can be generalized to the vector-valued function η in $W_0^{1,2}(B_{2r_0}(z), \mathbb{R}^m)$, so we get

$$\int_{\tilde{A}} \nabla v_i \cdot \nabla \eta \, dx = \int_{B_{2r_0}(z)} \nabla v_i \cdot \nabla \eta \, dx = \int_{B_{2r_0}(z)} \Delta u \cdot \partial_i \eta \, dx = \int_{\tilde{A}} \Delta u \cdot \partial_i \eta \, dx \tag{5.76}$$

Above we have exploited the fact that D^2u and $\nabla\eta$ vanish almost everywhere on $\{|u|=0\}$. So it suffices to prove

$$\int_{\tilde{A}} \Delta u \cdot \partial_i \eta \, dx \le C(g, r_0). \tag{5.77}$$

We define the set $\tilde{A}_\delta := B_{2r_0}(z) \cap \{|u| > \delta\}$, it is obvious that $\tilde{A}_\delta \subset \tilde{A}$ for any $\delta > 0$ and $\tilde{A} = \lim_{\delta\to 0} \tilde{A}_\delta$. Then we have

$$\begin{aligned}
&\int_{\tilde{A}} \Delta u \cdot \partial_i \eta \, dx \\
=& \lim_{\delta\to 0} \int_{\tilde{A}_\delta} \Delta u \cdot \partial_i \eta \, dx \\
=& \lim_{\delta\to 0} \int_{\tilde{A}_\delta} -\Delta v_i \cdot \eta \, dx + \lim_{\delta\to 0} \int_{\partial\tilde{A}_\delta} \Delta u \cdot \eta\gamma_i d\sigma
\end{aligned} \tag{5.78}$$

For the first term in (5.78), we further compute

$$\begin{aligned}
&-\int_{\tilde{A}_\delta} \Delta v_i \cdot [\psi_\varepsilon(|v_i|)\frac{v_i}{|v_i|}\phi] \, dx \\
=& -\int_{\tilde{A}_\delta} \psi_\varepsilon(|v_i|)|v_i|^{-1}\phi\Big(|u|^{-1}g(u)|v_i|^2 - |u|^{-3}g(u)(v_i\cdot u)^2 \\
&\qquad + 2|u|^{-1}(D_u g\cdot v_i)(u\cdot v_i) + |u|(v_i \cdot D_u^2 g \cdot v_i)\Big)\, dx
\end{aligned} \tag{5.79}$$

By the Cauchy-Schwartz inequality,

$$|u|^{-1}g(u)|v_i|^2 - |u|^{-3}g(u)(v_i\cdot u)^2 \ge 0.$$

Substituting this into (5.79) gives

$$\begin{aligned}
&\int_{\tilde{A}_\delta} -\Delta v_i \cdot \eta \, dx \\
\le& \left| \int_{\tilde{A}_\delta} \psi_\varepsilon(|v_i|)|v_i|^{-1}\phi\Big(2|u|^{-1}(D_u g\cdot v_i)(u\cdot v_i) + |u|(v_i \cdot D_u^2 g \cdot v_i)\Big)\, dx \right| \\
\le& C(g, r_0).
\end{aligned} \tag{5.80}$$

The integral is bounded by a constant $C(g, r_0)$ (doesn't depend on the choice of z, δ, ε) because $|v_i|$, $D_u(g)$, $D_u^2 g$, u are all uniformly bounded by a constant in $B_{2r_0}(z)$.

For the second part in (5.78), we apply (4.59) to obtain

$$\begin{aligned}
&\int_{\partial\tilde{A}_\delta} \Delta u \cdot \eta\gamma_i d\sigma \\
=& \int_{\partial\{|u|>\delta\}\cap B_{2r_0}(z)} \Delta u \cdot \eta\gamma_i d\sigma \\
=& \int_{\partial\{|u|>\delta\}\cap B_{2r_0}(z)} \left(g(u)\frac{u}{|u|} + |u|D_u g(u)\right)\left(\psi_\varepsilon(|v_i|)\frac{v_i}{|v_i|}\phi)\right)\gamma_i d\sigma \\
=& \int_{\partial\{|u|>\delta\}\cap B_{2r_0}(z)} g(u)\partial_i|u|\frac{\psi_\varepsilon(|v_i|)}{|v_i|}\phi\gamma_i d\sigma + \int_{\partial\{|u|>\delta\}\cap B_{2r_0}(z)} |u|\partial_i g(u)\frac{\psi_\varepsilon(|v_i|)}{|v_i|}\phi\gamma_i d\sigma \\
=:& \mathrm{I} + \mathrm{II}
\end{aligned} \tag{5.81}$$

We notice that on $\partial\{|u| > \delta\}$, if $|\nabla|u|| \neq 0$, then the outer normal vector can be written as $\gamma = \frac{-\nabla|u|}{|\nabla|u||}$, so we obtain that $\mathrm{I} \leq 0$.

For the term II, we perform integration by parts again to get

$$\begin{aligned}\lim_{\delta\to 0} \mathrm{II} &\leq \lim_{\delta\to 0} \delta \left| \int_{\partial\{|u|>\delta\}\cap B_{2r_0}(z)} \partial_i g(u) \frac{\psi_\varepsilon(|v_i|)}{|v_i|} \phi \gamma_i \, d\sigma \right| \\ &\leq \lim_{\delta\to 0} \delta \int_{\{|u|>\delta|\cap B_{2r_0}(z)\}} \left| \partial_i (\partial_i g(u) \frac{\psi_\varepsilon(|v_i|)}{|v_i|} \phi) \right| \, dx = 0\end{aligned} \tag{5.82}$$

We note that in the last step of (5.82), the limit is zero since it is the multiplication of δ (goes to zero) and a bounded integral (the bound depends on ε, but doesn't depend on δ).

Combining (5.79), (5.80), (5.81) and (5.82) will conclude the proof of the Claim.

□

On the other hand, we compute

$$\begin{aligned} I = &\int_{\tilde{A}} \left(\nabla v_i \nabla \psi_\varepsilon(|v_i|) \frac{v_i}{|v_i|} \phi\right) + \left(\nabla v_i \cdot \nabla (\frac{v_i}{|v_i|}) \psi_\varepsilon(|v_i|)\phi\right) \\ &\qquad + \left(\nabla v_i \frac{v_i}{|v_i|} \nabla\phi \psi_\varepsilon(|v_i|)\right) dx \\ = &\frac{C}{\varepsilon} \int_{\tilde{A}\cap\{0<|v_i|<\varepsilon\}} |\nabla|v_i||^2 \phi \, dx \\ &+ \int_{\tilde{A}\cap\{|v_i|>0\}} \left(|v_i|^{-1}|\nabla v_i|^2 - |v_i|^{-1}|\nabla|v_i||^2\right) \psi_\varepsilon(|v_i|)\phi \, dx \\ &+ \int_{\tilde{A}} \left(\nabla|v_i|\nabla\phi\psi_\varepsilon(|v_i|)\right) \, dx \end{aligned}$$

Note that we have

$$|v_i|^{-1}|\nabla v_i|^2 - |v_i|^{-1}|\nabla|v_i||^2 \geq 0,$$

$$\int_{\tilde{A}} \left(\nabla|v_i|\nabla\phi\psi_\varepsilon(|v_i|)\right) \, dx \leq (\int_{\tilde{A}} |\nabla|v_i||^2)^{\frac{1}{2}} (\int_{\tilde{A}} |\nabla\phi\psi_\varepsilon(|v_i|)|^2)^{\frac{1}{2}} \leq C(r_0).$$

Combining with (5.75), we conclude that

$$\int_{\Sigma_\varepsilon} |\nabla|v_i||^2 \, dx \leq C(g, r_0)\varepsilon, \quad \forall \varepsilon << 1. \tag{5.83}$$

We need the following lemma.

Lemma 5.2. *There are constants ε_0 and C such that for every $\varepsilon \leq \varepsilon_0$,*

$$\sum_{i=1}^n \int_{B_\varepsilon(z)\cap\Omega(u)} |\nabla|v_i||^2 \, dx \geq C\mathcal{L}^n(B_\varepsilon(z)), \quad \forall z \in \Gamma(u).$$

Proof. Recall that $\Omega(u) = \{|u| > 0\}$. If the statement is false, there exists a sequence of uniformly bounded entire minimizers $\{u_j\}_{j=1}^\infty$, $\{z_j \in \Gamma(u_j)\}$, $\{\varepsilon_j\}$ as well as $\{C_j\}$ such that

$$\lim_{j\to\infty} \varepsilon_j = 0, \quad \lim_{j\to\infty} C_j = 0,$$

$$\sum_{i=1}^n \int_{B_{\varepsilon_j}(z_j)\cap\Omega(u)} |\nabla|\partial_i u_j||^2 \, dx < C_j \mathcal{L}^n(B_{\varepsilon_j}(z_j)). \tag{5.84}$$

Define $f_j(x) : B_1(0^n) \to \mathbb{R}^m$ as

$$f_j(x) := \frac{u_j(z_j + \varepsilon_j x)}{\varepsilon_j^2}.$$

Then by Proposition 4.2, (5.70) and (5.84), we have

(1) $|f_j(0^n)| = |\nabla f_j(0^n)| = 0 \quad$ for every j,

(2) $\|f_j\|_{C^{1,\gamma}(B_1(0^n))}(\gamma < 1)$ is uniformly bounded.

(3) $\sum_i^n \int_{B_1} |\nabla|\partial_i f_j||^2 \, dx \le C_j \omega_n$,

(4) $\sup_{B_{\frac{1}{2}}(0^n)} |f_j(x)| \ge C > 0$ for some constant C.

Using all these properties, we can get the following convergence results up to some subsequence,

$$f_j \to f \text{ in } C^1(B_1(0^n)),$$

$$|f(0^n)| = |\nabla f(0^n)| = 0, \quad \sum_i^n \int_{B_1} |\nabla|\partial_i f||^2 \, dx = 0. \tag{5.85}$$

$$\sup_{B_{\frac{1}{2}}(0^n)} |f(x)| \ge C > 0. \tag{5.86}$$

Note that (5.85) implies that $f \equiv 0$ in $B_1(0^n)$, which yields a contradiction with (5.86). The proof of Lemma 5.2 is complete. □

According to Besicovitch covering lemma, we can find a covering of $\Gamma(u) \cap B_{r_0}(z)$ by a finite family of balls $\{B_j\}_{j\in J}$, such that each ball is of radius ε and centered on $\Gamma(u) \cap B_{r_0}$, and no more than N_n balls from this family overlap, where N_n is independent of ε and of the set $\Gamma(u) \cap B_{r_0}$. By the estimate (5.71), we have $B_\varepsilon(z) \cap \Omega(u) \subset \Sigma_{C\varepsilon}$ for some constant C. Consequently, we obtain

$$\begin{aligned}
\sum_{j\in J} \mathcal{L}^n(B_j) &\le C \sum_{j\in J} \sum_{i=1}^n \int_{B_j \cap \{|u|>0\}} |\nabla(|v_i|)|^2 \, dx \quad \text{(by Lemma 5.2)} \\
&\le C \sum_{i=1}^n \int_{\Sigma_{C\varepsilon}} |\nabla(|v_i|)|^2 \, dx \le C(g, r_0)\varepsilon \quad \text{(by (5.83))}.
\end{aligned}$$

This implies

$$\sum_{j\in J} \varepsilon^{n-1} \le C(g, r_0),$$

for some constant $C(n, r_0)$ independent of the choice of z. Finally letting $\varepsilon \to 0$, we get

$$\mathcal{H}^{n-1}(\Gamma(u) \cap B_{r_0}(z)) \le C(g, r_0).$$

The proof of Theorem 5.1 is complete. □

Remark 5.1. Using Theorem 5.1, we can prove that for any R, there exists $C(R)$ such that

$$\mathcal{H}^{n-1}(B_R(x) \cap \Gamma(u)) \le C(R)$$

To prove this, one simply covers the set $B_R(x) \cap \Gamma(u)$ by identical small balls $\{B_{r_0}(z_i)\}$ such that $z_i \in \Gamma(u)$ for every i. We omit the details.

Now we use the local estimate Remark 5.1 and Theorem 3.3 to prove the global estimate of the $\mathcal{H}^{n-1}$ measure of the free boundary $\partial^* I_0$.

Theorem 5.3 (Second part of Theorem 1.2). *Let $\alpha \in (0,2)$, $x_0 \in \mathbb{R}^n$. There are constants c_1, r_0 such that for any $r > r_0$,*

$$\mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(x_0)) \geq c_1 r^{n-1}.$$

And when $\alpha = 1$, there are constants c_2, r_0 such that for $r \geq r_0$,

$$\mathcal{H}^{n-1}(\partial^* I_0 \cap B_r(x_0)) \leq c_2 r^{n-1}.$$

Remark 5.2. Unlike the local estimate, here all the constants depend on u.

Proof. Without loss of generality we take $x_0 = 0^n$. According to Theorem 3.6, for sufficiently large r, there are two phases $a_1, a_2 \in A$, which depend on r, such that

$$\mathcal{L}^n(B_r \cap \{u = a_j\}) \geq c r^n, \quad j = 1, 2.$$

Using the relative isoperimetric inequality we obtain that

$$\begin{aligned}&\mathcal{H}^{n-1}(\partial^*\{|u - a_1| > 0\} \cap B_r)\\ \geq& C\left(\min\{\mathcal{L}^n(B_r \cap \{u = a_1\}), \mathcal{L}^n(B_r \backslash \{u = a_1\})\}\right)^{\frac{n-1}{n}} \geq c_1 r^{n-1},\end{aligned}$$

which gives the lower bound. Note that this estimate is valid for any $0 < \alpha < 2$.

For the upper bound, we fix $\alpha = 1$ and examine more closely the proof of Theorem 3.3. Again we consider the domain $\tilde{S}_{kL}$ and classify all the sub-cubes $\{S_i\}_1^{(2k)^n}$ into five classes T_1–T_5. If $S_i \in T_4$, then $u(x) \equiv a_{j_0}$ for all $x \in \overline{S_i}$, which implies $\mathcal{H}^{n-1}(S_i \cap \partial^* I_0) = 0$. Moreover, for any $x_0 \in \partial S_i$, by the definition of T_4 it holds that

$$\max_{x \in B_L(x_0)} |u(x) - a_{j_0}| \leq \theta.$$

By the proof of Lemma 3.4, we obtain that $B_{L/4}(x_0) \subset \{u = a_{j_0}\}$ and consequently $x_0 \notin \partial^* I_0$. As a result, we have

$$\mathcal{H}^{n-1}(\overline{S_i} \cap \partial^* I_0) = 0.$$

Using the estimates in Theorem 3.3 and Remark 5.1, we have for large enough k

$$\begin{aligned}&\mathcal{H}^{n-1}(\partial^* I_0 \cap \tilde{S}_{kL})\\ \leq& \sum_{S_i \in T_1 \cup T_2 \cup T_3 \cup T_5} \mathcal{H}^{n-1}(\partial^* I_0 \cap \overline{S_i})\\ \leq& (|T_1| + |T_2| + |T_3| + |T_5|) C(L)\\ \leq& c_2(W, u) k^{n-1}.\end{aligned} \tag{5.87}$$

The upper bound follows immediately from (5.87) and the proof is complete. □

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