

# UNIFORM PROFILE NEAR THE POINT DEFECT OF LANDAU-DE GENNES MODEL

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ABSTRACT. For the Landau-de Gennes functional on 3D domains,

$$I_\varepsilon(Q, \Omega) := \int_\Omega \left\{ \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} \left( -\frac{a^2}{2} \operatorname{tr}(Q^2) - \frac{b^2}{3} \operatorname{tr}(Q^3) + \frac{c^2}{4} [\operatorname{tr}(Q^2)]^2 \right) \right\} dx,$$

it is well-known that under suitable boundary conditions, the global minimizer  $Q_\varepsilon$  converges strongly in  $H^1(\Omega)$  to a uniaxial minimizer  $Q_* = s_+(n_* \otimes n_* - \frac{1}{3}\operatorname{Id})$  up to some subsequence  $\varepsilon_n \rightarrow \infty$ , where  $n_* \in H^1(\Omega, \mathbb{S}^2)$  is a minimizing harmonic map. In this paper we further investigate the structure of  $Q_\varepsilon$  near the core of a point defect  $x_0$  which is a singular point of the map  $n_*$ . The main strategy is to study the blow-up profile of  $Q_{\varepsilon_n}(x_n + \varepsilon_n y)$  where  $\{x_n\}$  are carefully chosen and converge to  $x_0$ . We prove that  $Q_{\varepsilon_n}(x_n + \varepsilon_n y)$  converges in  $C_{loc}^2(\mathbb{R}^n)$  to a tangent map  $Q(x)$  which at infinity behaves like a ‘‘hedgehog’’ solution that coincides with the asymptotic profile of  $n_*$  near  $x_0$ . Moreover, such convergence result implies that the minimizer  $Q_{\varepsilon_n}$  can be well approximated by the Oseen-Frank minimizer  $n_*$  outside the  $O(\varepsilon_n)$  neighborhood of the point defect.

## 1. INTRODUCTION

Nematic liquid crystals (NLC) are composed of rigid rod-like molecules which exhibit a locally preferred direction. Sharp variations in the alignment direction of NLC are known as defects, which are generally observed, in experiments, to exist as isolated points or disclination lines in experiments. There are several continuum theories used to describe the local orientation of NLC molecules at equilibrium. In these theories, NLC materials are assumed to occupy a region  $\Omega \in \mathbb{R}^d$  ( $d = 2, 3$ ) and their locally preferred directions are described by functions taking values in some order parameter spaces. The study of variational problems for energy-minimizing configuration of NLC (especially the configuration near defects) within these theories provides many fascinating mathematical problems. Readers are referred to survey articles [3, 33, 51] and references therein for more details.

Among these theories, the simplest one is the Oseen-Frank theory [18]. In the Oseen-Frank theory, the local orientation of NLC is represented by a unit-vector field  $n : \Omega \rightarrow \mathbb{S}^2$ , which minimizes an elastic energy. In the simplest setting, the free energy reduces to

$$\int_\Omega \frac{1}{2} |\nabla n|^2 dx,$$

which is the energy functional for harmonic maps. The singular set of a minimizing harmonic map is very well understood. In particular, in three dimensional space, the singular set contains at most finitely many points [45]. Near each singularity  $x_0$ , the field  $n$  behaves like the rotated ‘‘hedgehog’’ map  $\pm R \frac{x-x_0}{|x-x_0|}$  with some rotation  $R$  [7]. We recall that the major limitations of the Oseen-Frank

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model are that it only accounts for uniaxial nematic states and does not allow for line defects of finite energy (see [24]).

In the physically more realistic Landau-de Gennes theory[14], the order parameter is a  $3 \times 3$  symmetric traceless matrix  $Q$  (the so-called Q-tensors), which can be interpreted as the renormalized second moment of the (formal) probability distribution of the local molecular orientation. The total free energy contains two parts, namely the elastic energy and the bulk potential, whose simplified form reads

$$\begin{aligned} I_\varepsilon(Q, \Omega) &:= \int_{\Omega} \{f_\varepsilon(Q, \nabla Q) + f_b(Q)\} dx \\ &= \int_{\Omega} \left\{ \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} \left( -\frac{a^2}{2} \text{tr}(Q^2) - \frac{b^2}{3} \text{tr}(Q^3) + \frac{c^2}{4} [\text{tr}(Q^2)]^2 + C \right) \right\} dx, \end{aligned}$$

where  $\varepsilon, a, b, c$  are material dependent constants,  $C$  is a constant that ensures  $f_b(Q) \geq 0$ . The Landau-de Gennes theory can predict richer and more complicated local behaviors of the NLC medium because it accounts for both uniaxial and biaxial phases ( $Q$  is called biaxial when it has three distinct eigenvalues, uniaxial when it has only two equal eigenvalues and isotropic when all the three equal eigenvalues are zero). In particular it allows biaxiality in the cores of point defects and disclination lines. Interested readers can refer to [4, 27, 28, 22, 8, 1, 15, 29, 30, 9, 12, 25, 17, 16, 2, 49] for various studies on solutions and defect patterns of the Landau-de Gennes model.

When  $\varepsilon \rightarrow 0$ , the Landau-de Gennes energy will enforce the uniaxial constraint  $Q = s_+(n \otimes n - \frac{1}{3}\text{Id})$  (so that the potential function takes its minimal value, see (2.6)) and one can recover the Oseen-Frank model. Such convergence, which is usually referred to as the *vanishing elasticity limit* (see [19]), was first analysed in [40] and refined later on in [43]. Their results can be briefly summarized as follows: under suitable assumptions on the domain  $\Omega$  and the boundary condition  $Q|_{\partial\Omega}$ , the global minimizers  $Q_\varepsilon$  converges strongly in  $H^1$  to a limiting uniaxial minimizer  $Q_* = s_+(n_* \otimes n_* - \frac{1}{3}\text{Id})$  up to a subsequence, where  $n_* \in H^1(\Omega, \mathbb{S}^2)$  is a minimizing harmonic map. Moreover, the convergence is strong in  $C_{loc}^k(\Omega \setminus \mathcal{S}(n_*))$  for any non-negative integer  $k$ , where  $\mathcal{S}(n_*)$  denotes the singular set of  $n_*$ . Similar limiting problems for Landau-de Gennes model have also been explored in [4, 22, 23, 8, 9, 13] under various settings. The study of vanishing elasticity limits is influenced by similar analyses of the Ginzburg-Landau model for superconductors [5, 6], while the higher dimension of the target space generates greater complexity in analysis for Q-tensors.

The main purpose of this paper is to further investigate the structure of minimizers  $Q_\varepsilon$  in the core of a point defect  $x_0 \in \mathcal{S}(n_*)$  by studying the blow-up profile of  $Q_{\varepsilon_n}(x_n + \varepsilon_n y)$  where  $x_n$  will be carefully chosen and converge to  $x_0$ . We summarise our main results in the following theorem:

**Theorem 1.1.** *Suppose  $Q_{\varepsilon_n}$  is a sequence of global minimizers of  $I_{\varepsilon_n}(\cdot, \Omega)$  subjected to the Dirichlet boundary condition (2.2) and  $Q_{\varepsilon_n}$  converges to the vanishing elasticity limit  $Q_*$  in the sense of [40, 43]. Let  $x_0 \in \mathcal{S}(n_*)$ . There exists a subsequence of  $Q_{\varepsilon_n}$ , denoted as itself, and a sequence  $x_n \rightarrow x_0$  such that the following holds*

- (Proposition 3.5)  $Q_{\varepsilon_n}(x_n + \varepsilon_n x) \rightarrow Q(x)$  in  $C_{loc}^2(\mathbb{R}^3)$  and  $Q(x)$  is a local minimizer of the functional  $I(Q) = \int \{ \frac{1}{2} |\nabla Q|^2 + f_b(Q) \} dx$ .
- (Theorem 4.2, Theorem 4.4)  $Q(x) \rightarrow s_+(n(x) \otimes n(x) - \frac{1}{3}\text{Id})$  as  $|x| \rightarrow \infty$ , where  $n(x) = T(\frac{x}{|x|})$  with  $T \in O(3)$  is determined by the asymptotic profile of  $n_*$  near  $x_0$ .
- (Theorem 5.1) Let  $B_r(x_0)$  be a small neighborhood of  $x_0$  that doesn't contain other singularities of  $n_*$ . Then for any sequence  $R_n \uparrow \infty$  and satisfying  $R_n \varepsilon_n < r$ , there holds

$$\lim_{n \rightarrow \infty} \left( \sup_{R_n \varepsilon_n \leq |x| \leq r} |Q_{\varepsilon_n}(x_n + x) - Q_*(x_0 + x)| \right) = 0,$$

*which implies the uniform convergence of  $Q_{\varepsilon_n}$  to  $Q_*$  outside shrinking domains.*

Our results further improve the convergence results in [40, 43] by showing that the minimizer  $Q_{\varepsilon_n}$  of the Landau-de Gennes model can be well approximated by the Oseen-Frank minimizer  $Q_*$  outside the  $O(\varepsilon_n)$  neighborhood of the point defect (such neighborhood can be regarded as the defect core). The blow-up limit  $Q$  contains the information of the uniform structure of the defect core and its asymptotic behavior at infinity is inherited from the profile near the singularity of  $Q_*$ . The arguments essentially follow [41] by Millot-Pisante, which focuses on the similar problem concerning local minimizers for 3-D Ginzburg-Landau functional. However, there are several major differences from our arguments and those of [41]. On the one hand, the tensor structure gives rise to significant difficulty in our analysis. On the other hand, in [41] the quantification results of the defect measure from [35, 36] play a crucial role in the proof of strong  $H^1$  convergence (see [41, Proposition 3.1, Proposition 4.1]), while in this paper we rule out the possible defect measure and obtain strong  $H^1$  convergence of blow-up/blow-down sequences in a more direct way by simply using minimality and the Luckhaus' Lemma (see the proofs of Lemma 3.3, Theorem 4.1 and Lemma 4.5).

Our study was motivated by [31] where numerical investigations indicated that the behaviour near the singularity of the limiting harmonic maps has a universal profile, that is independent of the boundary conditions or the geometry of the domain. This universal profile has an outer part, resembling a so-called hedgehog pattern, and an inner part that has axial symmetry. Our investigation is capable of providing a rigorous interpretation of the studies in [31] in what concerns the outer part. Studying analytically the universal features in the inner part seems to be a significant analytical challenge.

A complete characterization for the behavior of a global minimizer  $Q_\varepsilon$  inside the defect core is still open. Many research works focus on several typical configurations of the defect core and their stability. Among them, the radial hedgehog solution with the form  $Q(x) = r(x)(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}I_d)$  is most extensively studied (see for example [46, 20, 39, 32, 27, 28]). This configuration is uniaxial everywhere and vanishes at the origin. However, in certain parameter regime the radial hedgehog becomes unstable and biaxiality has to appear near the defect core. Such phenomenon is called "biaxial escape" and one can refer to [28, 12, 25] for rigorous interpretations of this phenomenon within Landau-de Gennes theory at low temperature regime. There are mainly two types of biaxial core structure: the half-degree ring disclination and the split-core solution. These two biaxial configurations have been discovered and studied numerically [42, 26] and recently rigorously constructed in [50, 16, 49] in the axially symmetric setting.

The article is organized as follows. In Section 2, we introduce the basic mathematical setting of our problem and recall some previous results and estimates that will be used in the rest of the paper. In Section 3, we study the properties of  $Q_{\varepsilon_n}$  near a small neighborhood  $B_{r_n}(x_0)$  of the singular point  $x_0$  and establish the existence of the blow-up limit  $Q$  in Proposition 3.5. In Section 4 we study the behavior of the blow-up limit  $Q(x)$  when  $|x| \rightarrow \infty$  by proving its tangent map at infinity is just the asymptotic profile of  $Q_*$  at  $x_0$ . The proof is separated into several steps. We first show that there exists a homogeneous degree-1 tangent map of  $Q$  at infinity; then we prove the uniqueness of the tangent map; for the last step we show this unique tangent map has to coincide with the hedgehog configuration of  $Q_*$  near  $x_0$ . Finally in Section 5 we establish the uniform convergence of  $Q_{\varepsilon_n}(x_n + x)$  to  $Q_*(x_0 + x)$  in varying domains  $B_r \setminus B_{R_n \varepsilon_n}$  for any  $R_n \rightarrow \infty$ .

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## 2. MATHEMATICAL FORMULATION AND PRELIMINARY ESTIMATES

Let  $\Omega$  be an open bounded simply-connected domain in  $\mathbb{R}^3$ . We denote by  $\mathcal{Q}_0$  the set of traceless symmetric  $3 \times 3$  matrices, i.e.

$$\mathcal{Q}_0 := \{Q \in \mathcal{M}^{3 \times 3}, Q = Q^T\}.$$

Consider a Landau-de Gennes functional of the form

$$(2.1) \quad I_\varepsilon(Q, \Omega) = \int_\Omega \left[ \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f_b(Q) \right] dx, \quad Q \in H^1(\Omega, \mathcal{Q}_0),$$

with the Dirichlet boundary condition

$$(2.2) \quad Q|_{\partial\Omega} = Q_b = s_+(n_b \otimes n_b - \frac{1}{3}\text{Id}), \quad n_b \in C^\infty(\partial\Omega, \mathbb{S}^2).$$

That is to say,  $Q_b$  is a smooth function taking values in  $\mathcal{N}$  which is defined later in (2.6).

When  $\varepsilon = 1$ , we write

$$(2.3) \quad I(Q, \Omega) := \int_\Omega \left[ \frac{1}{2} |\nabla Q|^2 + f_b(Q) \right] dx$$

The bulk potential is of the form

$$(2.4) \quad f_b(Q) = -\frac{a^2}{2} \text{tr}(Q^2) - \frac{b^2}{3} \text{tr}(Q^3) + \frac{c^2}{4} [\text{tr}(Q^2)]^2 + C.$$

where  $C$  is the constant that ensures  $\inf_{Q \in \mathcal{Q}_0} f_b(Q) = 0$ .

We introduce the notion of local minimizers of the energy in the following sense.

**Definition 2.1.** Let  $Q \in H_{loc}^1(D_0, \mathcal{Q}_0)$  for some domain  $D_0 \subseteq \mathbb{R}^3$  ( $D_0$  could be  $\mathbb{R}^3$ ).  $Q$  is said to be a local minimizer of  $I(\cdot, D_0)$  if

$$I(Q, D) \leq I(V, D)$$

for any bounded open set  $D \Subset D_0$  and  $V \in H_{loc}^1(\Omega, \mathcal{Q}_0)$  such that  $Q - V \in H_0^1(D, \mathcal{Q}_0)$ .

The Euler-Lagrange equation for the functional  $I_\varepsilon$  is given by

$$(2.5) \quad \Delta Q_\varepsilon = \frac{1}{\varepsilon^2} (-a^2 Q_\varepsilon - b^2 [Q_\varepsilon^2 - \frac{1}{3} \text{tr}(Q_\varepsilon)^2 \text{Id}] + c^2 \text{tr}(Q_\varepsilon)^2 Q_\varepsilon),$$

where the term  $\frac{1}{3\varepsilon^2} b^2 \text{tr}(Q_\varepsilon)^2 \text{Id}$  is a Lagrange multiplier that accounts for the tracelessness constraint.

It is well-known that the bulk potential  $f_b$  takes its minimum value on a sub-manifold of  $\mathcal{Q}_0$  defined by

$$(2.6) \quad \mathcal{N} = \{Q = s_+(n \otimes n - \frac{1}{3}\text{Id}), n \in \mathbb{S}^2\}, \quad s_+ = \frac{b^2 + \sqrt{b^4 + 24a^2c^2}}{4c^2}.$$

In [40, 43], it was shown that subjected to the Dirichlet boundary condition (2.2), up to some subsequence, the minimizers  $Q_\varepsilon$  of  $I_\varepsilon$  converge to the minimizer  $Q_*$  of the functional

$$(2.7) \quad I_*[Q] = \int_{\Omega} |\nabla Q|^2 dx, \quad Q \in H^1(\Omega, \mathcal{N}), \quad Q = Q_b \text{ on } \partial\Omega$$

By direct calculation, we have  $|\nabla Q|^2 = 2s_+^2 |\nabla n|^2$  for  $Q = s_+(n \otimes n - \frac{1}{3}\text{Id})$ . It follows that on the simply-connected domain  $\Omega$  this  $Q_*$  can be written as  $Q_*(x) = s_+(n_*(x) \otimes n_*(x) - \frac{1}{3}\text{Id})$  where  $n_*(x) \in H^1(\Omega, \mathbb{S}^2)$  is a minimizing harmonic map. More precisely, the following results was proved in [40, 43].

**Theorem 2.2.** *Let  $\Omega$  be an open bounded simply-connected subset of  $\mathbb{R}^3$  and  $Q_\varepsilon$  be a minimizer of the minimization problem (2.1)–(2.2). For any sequence  $\varepsilon_k \rightarrow 0$ , there exists a subsequence, still denoted by  $\varepsilon_k$ , such that  $Q_{\varepsilon_k}$  converges strongly in  $H^1$ -norm to a minimizer  $Q_*$  of the (2.7). Let  $\text{Sing}(Q_*)$  denote the singular set of  $Q_*$ , then*

$$Q_{\varepsilon_k} \rightarrow Q_* \text{ in } C_{loc}^j(\Omega \setminus \text{Sing}(Q_*), \mathcal{Q}_0), \quad \forall j \geq 1.$$

The above theorem gives a nice convergence result of  $Q_{\varepsilon_k}$  to  $Q_*$  away from the singular set  $\text{Sing}(Q_*)$ . In this note we would like to investigate the behavior of  $Q_{\varepsilon_k}$  near  $\text{Sing}(Q_*)$ .

For the limiting harmonic map  $Q_* = s_+(n_* \otimes n_* - \frac{1}{3}\text{Id})$ , we recall the classical result of Schoen-Uhlenbeck [45] and Brezis-Coron-Lieb [7, Theorem 1.2] that the singular set  $\text{Sing}(Q_*) = \text{Sing}(n_*)$  is a set of finitely many isolated points, and near each singular point  $x_0$ , one has

$$\lim_{r \rightarrow 0} n_*(r(x - x_0)) = T \frac{x - x_0}{|x - x_0|},$$

for some  $T \in O(3)$ . The convergence is strongly in  $H^1(B_1)$  and uniformly in any compact subset of  $B_1 \setminus \{0\}$ . Moreover, using the technique of integrability of a Jacobi field ( see for instance [48, Theorem 6.3]), the convergence rate can be controlled by a positive power of  $r$ ,

$$(2.8) \quad \left| n_*(x_0 + x) - T \frac{x}{|x|} \right| \leq C|x|^\alpha, \quad \forall |x| < r_0.$$

Here  $C > 0, r_0 > 0, \alpha \in (0, 1)$  are all positive constants depending just on  $n_*$  and  $x_0$ .

Also there are two basic ingredients in our analysis, which are the monotonicity formula and the small energy regularity estimate, which are both established in [40]. We list them below.

**Lemma 2.3.** *(Monotonicity lemma, [40, Section 4, Lemma 2]) Let  $Q_\varepsilon$  be a global minimizer of  $I_\varepsilon$ , then*

$$(2.9) \quad \frac{\partial}{\partial R} \left( \frac{1}{R} \int_{B_R} \frac{1}{2} |\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_b(Q_\varepsilon) dx \right) = \frac{1}{R} \int_{\partial B_R} \left| \frac{\partial Q_\varepsilon}{\partial r} \right|^2 d\sigma + \frac{2}{R^2} \int_{B_R} \frac{f_b(Q_\varepsilon)}{\varepsilon^2} dx$$

Now we define

$$(2.10) \quad e_\varepsilon(Q) := \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f_b(Q),$$

which denotes the energy density of the Landau-de Gennes functional (2.1) for  $Q \in H^1(\Omega, \mathcal{Q}_0)$ . The following small energy argument holds.

**Lemma 2.4.** *(Small energy regularity, [40, Section 4, Lemma 7]) Let  $Q_{\varepsilon_k}$  be global minimizers of (2.1)–(2.2) with coefficient  $\varepsilon_k$  and suppose  $Q_{\varepsilon_k} \rightarrow Q_*$  in  $H^1(\Omega)$ . Let  $K \subset \Omega$  be a compact set which contains no singularity of  $Q_*$ . There exists  $C_1 > 0, C_2 > 0, \varepsilon_0 > 0$  such that for  $a \in K, 0 < r < \text{dist}(a, \partial K), \varepsilon_k < \varepsilon_0$  we have*

$$\frac{1}{r} \int_{B_r(a)} e_{\varepsilon_k}(Q_{\varepsilon_k}) dx \leq C_1,$$

then

$$r^2 \sup_{B_{r/2}(a)} e_{\varepsilon_k}(Q_{\varepsilon_k}) \leq C_2.$$

### 3. CONVERGENCE OF BLOW-UP MAPS

Take  $\varepsilon_k \rightarrow 0$  to be the sequence in Theorem 2.2, satisfying  $Q_{\varepsilon_k} \rightarrow Q_*$  in  $C_{loc}^j(\Omega \setminus \text{Sing}(Q_*))$ . Assume  $\text{Sing}(Q_*) = \{0\}$ , and  $n_*(x) \sim \frac{x}{|x|}$  near the singular point 0. We write the hedgehog map as

$$(3.11) \quad \Phi(x) := s_+ \left( \frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} \text{Id} \right)$$

Now we first fix a sequence of radiuses  $r_n$  such that

$$(3.12) \quad r_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

According to [7, Corollary 7.12] and [47, Section 8], it holds that

$$\|Q_*(r_n x) - \Phi(x)\|_{C^2(B_{3/2} \setminus B_{1/2})} + \|Q_*(r_n x) - \Phi(x)\|_{H^1(B_1)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Once we fix  $\{r_n\}$ , we can then choose a subsequence of  $\{\varepsilon_n\}$ , still denoted by  $\{\varepsilon_n\}$ , such that

$$(3.13) \quad \frac{r_n}{\varepsilon_n} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

and

$$(3.14) \quad \|Q_{\varepsilon_n}(x) - Q_*(x)\|_{C^2(\Omega \setminus B_{r_n})} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

$$(3.15) \quad \|Q_{\varepsilon_n}(r_n x) - \Phi(x)\|_{C^2(B_{3/2} \setminus B_{1/2})} + \|Q_{\varepsilon_n}(r_n x) - \Phi(x)\|_{H^1(B_1)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Here the existence of such  $\{\varepsilon_n\}$  is guaranteed by Theorem 2.2. In the rest of this paper, for convenience we will always work with  $\{(r_n, \varepsilon_n)\}$  that satisfying (3.12), (3.13), (3.14) and (3.15).

We would like to study the convergence property of the sequence of blow-up maps  $Q_{\varepsilon_n}(\varepsilon_n x)$ . Set

$$R_n := \frac{r_n}{\varepsilon_n}.$$

We first look at the rescaled functions

$$(3.16) \quad U_n(x) := Q_{\varepsilon_n}(r_n x) \quad \text{on } B_1(0).$$

Obviously  $U_n$  is a local minimizer, in the sense of Definition 2.1, of the following functional:

$$\int_{B_1} \left\{ \frac{1}{2} |\nabla Q|^2 + R_n^2 f_b(Q) \right\} dx$$

and satisfies  $\|U_n(x) - \Phi(x)\|_{L^\infty(\partial B_1)} \rightarrow 0$  as  $n \rightarrow \infty$  due to (3.15). Then the following lemma holds.

**Lemma 3.1.**  $\lim_{n \rightarrow \infty} \int_{B_1} R_n^2 f_b(U_n) dx = 0.$

*Proof.* Since  $Q_*$  is an admissible map for  $I_{\varepsilon_n}$ , we have

$$\int_{\Omega} \frac{1}{2} |\nabla Q_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f_b(Q_{\varepsilon_n}) dx \leq \int_{\Omega} \frac{1}{2} |\nabla Q_*|^2 dx.$$

On the other hand,  $Q_{\varepsilon_n}$  converges to  $Q_*$  strongly in  $H^1$ , therefore we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{1}{\varepsilon_n^2} f_b(Q_{\varepsilon_n}) dx = 0$$

A straightforward calculation shows that  $\Phi(x)$  satisfies

$$\frac{1}{r} \int_{B_r} \frac{1}{2} |\nabla \Phi|^2 dx = 8s_+^2 \pi, \quad \forall r > 0.$$

Then using the strong convergence of  $Q_{\varepsilon_n}(x)$  to  $Q_*(x)$ , we infer that for any  $\delta > 0$ , there exists a  $r_\delta > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{r_\delta} \int_{B_{r_\delta}} \frac{1}{2} |\nabla Q_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f_b(Q_{\varepsilon_n}) dx \leq 8s_+^2 \pi + \delta.$$

Combining the definition of  $U_n$  and the monotonicity formula (2.9), we deduce that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{B_1} \frac{1}{2} |\nabla U_n|^2 + R_n^2 f_b(U_n) dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{r_n} \int_{B_{r_n}} \frac{1}{2} |\nabla Q_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f_b(Q_{\varepsilon_n}) dx \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{r_\delta} \int_{B_{r_\delta}} \frac{1}{2} |\nabla Q_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f_b(Q_{\varepsilon_n}) dx \\ &\leq 8s_+^2 \pi + \delta \end{aligned}$$

Letting  $\delta \rightarrow 0$  yields

$$(3.17) \quad \lim_{n \rightarrow \infty} \int_{B_1} \frac{1}{2} |\nabla U_n|^2 + R_n^2 f_b(U_n) dx \leq 8s_+^2 \pi.$$

On the other hand, by the  $H^1$  convergence of  $U_n$  to  $\Phi$  (3.15) we have

$$\lim_{n \rightarrow \infty} \int_{B_1} \frac{1}{2} |\nabla U_n|^2 dx = \int_{B_1} \frac{1}{2} |\nabla \Phi|^2 dx = 8s_+^2 \pi,$$

which together with (3.17) implies that

$$\lim_{n \rightarrow \infty} \int_{B_1} R_n^2 f_b(U_n) dx = 0.$$

□

Now we have the following proposition which says that when  $n$  is sufficiently large, the set of points where  $U_n(x)$  leaves  $\mathcal{N}$  has small measure and concentrates near the origin.

**Proposition 3.2.** *For any  $\delta > 0$ , there exist constants  $C_\delta$  and  $n_\delta$  such that*

$$(3.18) \quad \sup\{|x|, x \in \{\text{dist}(U_n, \mathcal{N}) \geq \delta\}\} = o(1) \quad \text{as } n \rightarrow \infty,$$

$$(3.19) \quad \text{diam}(\{\text{dist}(U_n, \mathcal{N}) \geq \delta\}) \leq C_\delta R_n^{-1} \quad \forall n \geq n_\delta.$$

*Proof.* We follow closely the proof of [41, Proposition 3.2]. First recall that (3.15) gives the strong  $H^1$  convergence of  $U_n$  to  $\Phi(x)$  in  $B_1$ . Now we fix  $\delta \in (0, 1)$  and first prove (3.18). Define

$$(3.20) \quad D_n^\delta := \{x \in \bar{B}_1 : \text{dist}(U_n(x), \mathcal{N}) \geq \delta\}$$

It suffices to show for any given  $r < 1$ , the set  $D_n^\delta \subset B_r$  for every  $n$  sufficiently large. Note that according to (3.15),  $(B_1 \setminus B_{1/2}) \cap D_n^\delta = \emptyset$  as  $n \rightarrow \infty$ , so we only need to focus on  $(\bar{B}_{1/2} \setminus B_r) \cap D_n^\delta$ . Since  $\Phi(x)$  is smooth outside the origin, we can find a  $r_0 < \frac{r}{8}$  such that

$$\frac{1}{r_0} \int_{B_{r_0}(x)} |\nabla \Phi(x)|^2 dx \leq \frac{1}{2} C_3 \quad \forall x \in \bar{B}_{1/2} \setminus B_r$$

Here  $C_3$  is a small number to be determined. The strong convergence of  $U_n$  to  $\Phi$  in  $H_1$  implies that

$$\frac{1}{r_0} \int_{B_{r_0}(x)} |\nabla U_n|^2 dx \leq C_3 \quad \forall x \in \bar{B}_{1/2} \setminus B_r, \quad \forall n \geq N_1$$

where  $N_1$  is a large constant. Then we infer from the Lemma 2.4 and Lemma 3.1 that, if  $C_3$  is chosen to be suitably small,

$$r_0^2 \sup_{B_{r_0/2}(x)} e_{R_n^{-1}}(U_n) \leq C_4, \quad \forall x \in \bar{B}_{1/2} \setminus B_r$$

where  $C_4$  is another constant independent of  $n$ . By Arzela-Ascoli lemma the sequence  $\{U_n\}$  is compact in  $L^\infty(\bar{B}_{1/2} \setminus B_r)$ , and therefore  $\text{dist}(U_n(x), \mathcal{N}) \rightarrow 0$  uniformly in  $B_1 \setminus B_r$ . In particular,  $D_n^\delta \subset B_r$  for sufficiently large  $n$ .

Next we show that there exists a  $C_\delta$  such that  $\text{diam}(D_n^\delta) \leq C_\delta R_n^{-1}$  for large enough  $n$ . First we choose a small constant  $\delta_0 > 0$  such that, there exists a smooth orthogonal projection of  $\mathcal{N}_{\delta_0}$  onto  $\mathcal{N}$ . Here  $\mathcal{N}_{\delta_0} \subset \mathcal{Q}_0$  denotes the  $\delta$ -neighborhood of  $\mathcal{N}$  in  $\mathcal{Q}_0$ . We denote  $\mathbb{P}$  as the orthogonal projection from  $\mathcal{N}_{\delta_0}$  onto  $\mathcal{N}$ . It suffices to show (3.19) for all  $\delta < \delta_0$ .

We fix  $\delta < \delta_0$  and argue by contradiction. Let  $d_n := \text{diam}(D_n^\delta)$  and suppose  $\mu_n := d_n R_n \uparrow \infty$ . We take  $a_n, b_n \in D_n^\delta$  such that  $|a_n - b_n| = d_n$ . By (3.18) we have  $\max\{|a_n|, |b_n|\} \rightarrow 0$ . Define

$$c_n := \frac{a_n + b_n}{2}, \quad s_n := \sup\{|x - c_n| : x \in D_n^\delta\}.$$

One can easily verify that  $s_n \in [\frac{d_n}{2}, d_n)$  from definitions.

We perform the following rescaling

$$V_n(x) := U_n(d_n x + c_n), \quad x \in B_2.$$

Note that  $V_n$  is well-defined for all large enough  $n$  due to the fact that  $B_{2d_n}(c_n) \subset B_1$  when  $n$  is sufficiently large. By the relationship of  $s_n$  and  $d_n$ , we have

$$V_n(x) \in \mathcal{N}_\delta, \quad \forall x \in B_2 \setminus \bar{B}_1, \quad n \text{ sufficiently large.}$$

Moreover,  $V_n(x)$  minimizes the energy

$$\int_{B_2} \frac{1}{2} |\nabla Q|^2 + \mu_n^2 f_b(Q) dx.$$

By the definition of  $V_n, U_n$ , (3.17) and the monotonicity formula, for every  $x_0 \in B_2$ ,  $R \in (0, 2 - |x_0|)$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{R} \int_{B_{R(x_0)}} e_{\mu_n^{-1}}(V_n) dx \\ (3.21) \quad & \leq \lim_{n \rightarrow \infty} \frac{1}{1 - |d_n x_0 + c_n|} \int_{B_{1-|d_n x_0 + c_n|}(d_n x_0 + c_n)} e_{R_n^{-1}}(U_n) dx \\ & \leq \lim_{n \rightarrow \infty} \frac{8s_+^2 \pi}{1 - |d_n x_0 + c_n|} = 8s_+^2 \pi. \end{aligned}$$

Denote  $P_1^n := \frac{a_n - c_n}{d_n}$  and  $P_2^n := \frac{b_n - c_n}{d_n}$ . Up to a rotation we assume

$$P_1^n = P_1 = \left(\frac{1}{2}, 0, 0\right), \quad P_2^n = P_2 = \left(-\frac{1}{2}, 0, 0\right).$$

It is well known, via Chen-Struwe[11] and Chen-Lin[10], that up to a subsequence,  $V_n$  converges weakly in  $H^1(B_2, \mathcal{Q}_0)$  and strongly in  $L^2(B_2, \mathcal{Q}_0)$  to a weakly harmonic map  $V \in H^1(B_2, \mathcal{N})$ . Moreover, there exists a nonnegative Radon measure  $\nu$  on  $\Omega$  such that

$$e_{\mu_n^{-1}}(V_n) dx \rightarrow \frac{1}{2} |\nabla V|^2 dx + \nu \text{ in } B_2.$$

Here we note that even though  $V_n$  is a locally rescaled function of  $Q_{\varepsilon_n}$ , the strong  $H^1$  convergence and the minimality of the limiting weak harmonic map  $V$  cannot be derived directly from Theorem



2.2 because Theorem 2.2 requires that all  $Q_\varepsilon$  satisfy the same boundary data (2.2), which is not the case for  $V_n$ . We can apply the Luckhaus-type construction to overcome such deviation of the boundary data and prove the following improvement of the convergence.

**Lemma 3.3.**  $\nu(B_2) = 0$ ,  $V_n \rightarrow V$  strongly in  $H_{loc}^1(B_2, \mathcal{Q}_0)$  and  $V$  is a minimizing harmonic map.

We first admit Lemma 3.3 and proceed with the proof of Proposition (3.2). A direct consequence of Lemma 3.3 is

$$(3.22) \quad \lim_{n \rightarrow \infty} \int_{B_2} \mu_n^2 f_b(V_n) dx = 0$$

For the limiting map  $V$ , first we claim that

$$\lim_{R \rightarrow 0} \frac{1}{R} \int_{B_R(P_i)} |\nabla V|^2 > 0, \quad \text{for } i = 1, 2.$$

Otherwise if  $\lim_{R \rightarrow 0} \frac{1}{R} \int_{B_R(P_i)} |\nabla V|^2 = 0$  for  $i = 1$  or  $2$  (we assume  $i = 1$  without loss of generality), then by strong  $H^1$  convergence and (3.22) we infer that there exist  $R_0 > 0$  and  $N_0$  such that for any  $n \geq N_0$ ,

$$\int_{B_{R_0}(P_1)} e_{\mu_n^{-1}}(V_n) dx \leq C$$

for some suitably small constant  $C$ . Invoking Lemma 2.4, we conclude that there exists a constant, still denoted by  $C$ , such that

$$R_0^2 \sup_{B_{R_0/2}(P_1)} e_{\mu_n^{-1}}(V_n) \leq C, \quad \forall n \geq N_0$$

Again by Arzela-Ascoli lemma we have  $V_n$  converges uniformly to  $V$  in  $B_{R_0/2}(P_1)$ , which contradicts with the assumption  $\text{dist}(V_n(P_1), \mathcal{N}) = \delta > 0$ . So we get the claim.

On the other hand, since  $V$  is a stationary harmonic map, by the quantization results in [34, Corollary 1],

$$\lim_{R \rightarrow 0} \frac{1}{R} \int_{B_R(P_i)} |\nabla V|^2 dx = 16s_+^2 k_i \pi, \quad i = 1, 2; \quad k_i \text{ is a positive integer.}$$

Recall that from (3.21) we have

$$\frac{1}{R} \int_{B_R(P_i)} \frac{1}{2} |\nabla V|^2 dx \leq 8s_+^2 \pi.$$

It follows that  $k_1 = k_2 = 1$ . And by monotonicity formula we have

$$(3.23) \quad \frac{1}{R} \int_{B_R(P_i)} |\nabla V|^2 dx \geq 16s_+^2 \pi, \quad i = 1, 2; \quad R \in (0, 1).$$

For every  $R \in (0, 1)$ , denote  $Q_R := (R - \frac{1}{2}, 0, 0)$ . By (3.21) and (3.22) we have

$$16s_+^2 \pi \geq \int_{B_1(Q_R)} |\nabla V|^2 dx \geq \left( \int_{B_R(P_1)} + \int_{B_{1-R}(P_2)} \right) |\nabla V|^2 dx \geq (R + (1 - R))16s_+^2 \pi = 16s_+^2 \pi.$$

It follows that  $|\nabla V| \equiv 0$  on  $B_1(Q_R) \setminus (B_R(P_1) \cup B_{1-R}(P_2))$  for every  $R \in (0, 1)$ . Note that

$$B_1 \setminus \{(x, 0, 0) : -1 < x < 1\} = \bigcup_{R \in (0, 1)} B_1(Q_R) \setminus (B_R(P_1) \cup B_{1-R}(P_2)),$$

we therefore deduce that

$$\int_{B_1} |\nabla V|^2 dx = 0$$

which clearly contradicts with (3.23). The proof of (3.19) is thus complete.  $\square$

*Proof of Lemma 3.3.* Fix any radius  $\rho_0 \in (1, 2)$ . By Fatou's lemma and Fubini's theorem, there is a radius  $\rho \in (\rho_0, 2)$  and a subsequence of  $\{V_n\}$ , which still denoted by  $\{V_n\}$ , such that

$$(3.24) \quad \lim_{n \rightarrow \infty} \int_{\partial B_\rho} |V_n - V|^2 d\mathcal{H}^2 =: \lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \int_{\partial B_\rho} (e_{\mu_n^{-1}}(V_n) + |\nabla V|^2) d\mathcal{H}^2 \leq C(\rho_0) < \infty, \quad \forall n \in \mathbb{N}.$$

In order to construct an energy competitor, we need the following extension Lemma which was first proved by [45] and later by Luckhaus [38]. Here we present the version in [37, Lemma 2.2.9].

**Lemma 3.4.** *For  $n \geq 2$ , suppose  $u, v \in H^1(\mathbb{S}^{n-1}, \mathcal{N})$ . Then for  $\varepsilon \in (0, 1)$  there is  $w \in H^1(\mathbb{S}^{n-1} \times [1 - \varepsilon, 1], \mathbb{R}^L)$  such that  $w|_{\mathbb{S}^{n-1} \times \{1\}} = u$ ,  $w|_{\mathbb{S}^{n-1} \times \{1 - \varepsilon\}} = v$ ,*

$$\int_{\mathbb{S}^{n-1} \times [1 - \varepsilon, 1]} |\nabla w|^2 \leq C\varepsilon \int_{\mathbb{S}^{n-1}} (|\nabla_T u|^2 + |\nabla_T v|^2) + C\varepsilon^{-1} \int_{\mathbb{S}^{n-1}} |u - v|^2,$$

and

$$\text{dist}^2(w(x), \mathcal{N}) \leq C\varepsilon^{1-n} \left( \int_{\mathbb{S}^{n-1}} (|\nabla_T u|^2 + |\nabla_T v|^2) \right)^{\frac{1}{2}} \left( \int_{\mathbb{S}^{n-1}} |u - v|^2 \right)^{\frac{1}{2}} + C\varepsilon^{-n} \int_{\mathbb{S}^{n-1}} |u - v|^2$$

for a.e.  $x \in \mathbb{S}^{n-1} \times [1 - \varepsilon, 1]$ . Here  $\nabla_T$  is the gradient on  $\mathbb{S}^{n-1}$  and  $\mathbb{R}^L$  is the space in which the manifold  $\mathcal{N}$  is embedded.

let  $W \in H^1(B_2, \mathcal{N})$  satisfying  $W = V$  on  $B_2 \setminus B_{2-\rho_0}$ . We define the energy competitor  $W_n \in H^1(B_2, \mathcal{Q}_0)$  as

$$(3.25) \quad W_n(x) := \begin{cases} V_n(x), & x \in L_1 := B_2 \setminus B_\rho, \\ \frac{|x| - \rho + \sqrt{\varepsilon_n}}{\sqrt{\varepsilon_n}} V_n\left(\frac{\rho x}{|x|}\right) + \frac{\rho - |x|}{\sqrt{\varepsilon_n}} \mathbb{P}(V_n\left(\frac{\rho x}{|x|}\right)), & x \in L_2 := B_\rho \setminus B_{\rho - \sqrt{\varepsilon_n}}, \\ \mathbb{P}(K_n(x)), & x \in L_3 := B_{\rho - \sqrt{\varepsilon_n}} \setminus B_{\rho - \sqrt{\varepsilon_n} - \varepsilon_n^{1/6}}, \\ W\left(\frac{\rho}{\rho - \sqrt{\varepsilon_n} - \varepsilon_n^{1/6}} x\right), & x \in L_4 := B_{\rho - \sqrt{\varepsilon_n} - \varepsilon_n^{1/6}}. \end{cases}$$

Here  $K_n(x)$  is the connecting function obtained by applying Lemma 3.4 to  $\mathbb{P}(V_n|_{\partial B_\rho})$  and  $V|_{\partial B_\rho}$ . To be more precise,  $K_n(x)$  satisfies

$$(3.26) \quad \begin{aligned} K_n(x) &= \mathbb{P}\left(V_n\left(\frac{\rho x}{\rho - \sqrt{\varepsilon_n}}\right)\right) \text{ on } \partial B_{\rho - \sqrt{\varepsilon_n}}, \\ K_n(x) &= V\left(\frac{\rho x}{\rho - \sqrt{\varepsilon_n} - \varepsilon_n^{1/6}}\right) \text{ on } \partial B_{\rho - \sqrt{\varepsilon_n} - \varepsilon_n^{1/6}}, \\ \text{dist}(K_n(x), \mathcal{N}) &\leq C\varepsilon_n^{1/6}, \quad \int_{L_3} |\nabla K_n(x)|^2 \leq C\varepsilon_n^{1/6}. \end{aligned}$$

Here we used

$$\int_{\partial B_\rho} |\mathbb{P}(V_n) - V|^2 \leq 2 \int_{\partial B_\rho} (|\mathbb{P}(V_n) - V_n|^2 + |V_n - V|^2) \leq 4 \int_{\partial B_\rho} |V_n - V|^2 = 4\varepsilon_n$$

and applied Lemma 3.4 with  $n = 3$  and  $\varepsilon = \varepsilon_n^{1/6}$  in order to get (3.26).

Since  $\varepsilon_n \downarrow 0$  we know that  $K_n(x) \in \mathcal{N}_{\delta_0}$  when  $n$  is large enough. Therefore  $\mathbb{P}(K_n)$  is well-defined on  $L_3$ . Direct calculation implies that

$$\begin{aligned} & \left| \int_{B_\rho} e_{\mu_n^{-1}}(W_n) dx - \int_{B_\rho} \frac{1}{2} |\nabla W|^2 dx \right| \\ &= \left| \frac{\sqrt{\varepsilon_n} + \varepsilon_n^{1/6}}{\rho} \int_{B_\rho} \frac{1}{2} |\nabla W|^2 dx \right| + \left| \int_{L_2} e_{\mu_n^{-1}}(W_n) dx \right| + \left| \int_{L_3} \frac{1}{2} |\nabla W_n|^2 dx \right| \\ &=: I_n + II_n + III_n. \end{aligned}$$

For the first term  $I_n$  since  $\varepsilon_n \rightarrow 0$  we have  $\lim_{n \rightarrow \infty} I_n = 0$ . For the second term we calculate using polar coordinates:

$$\begin{aligned} (3.27) \quad & \int_{L_2} e_{\mu_n^{-1}}(W_n) dx \\ &= \int_{\rho - \sqrt{\varepsilon_n}}^\rho r^2 \int_{\mathbb{S}^2} \left\{ \frac{1}{2} |\nabla W_n(r\theta)|^2 + \mu_n^2 f_b \left( \frac{|x| - \rho + \sqrt{\varepsilon_n}}{\sqrt{\varepsilon_n}} V_n \left( \frac{\rho x}{|x|} \right) + \frac{\rho - |x|}{\sqrt{\varepsilon_n}} \mathbb{P} \left( V_n \left( \frac{\rho x}{|x|} \right) \right) \right) \right\} d\theta dr \\ &\leq \int_{\rho - \sqrt{\varepsilon_n}}^\rho r^2 \int_{\mathbb{S}^2} \left\{ \frac{1}{2} \left| \nabla \left( V_n(\rho\theta) + \frac{r - \rho}{\sqrt{\varepsilon_n}} (V_n(\rho\theta) - \mathbb{P}(V_n(\rho\theta))) \right) \right|^2 + C \mu_n^2 f_b(V_n(\rho\theta)) \right\} d\theta dr \\ &\leq \int_{\rho - \sqrt{\varepsilon_n}}^\rho r^2 \int_{\mathbb{S}^2} \left\{ \left(1 + C \frac{(r - \rho)^2}{\varepsilon_n}\right) |\nabla_T V_n(\rho\theta)|^2 + \frac{|V_n(\rho\theta) - \mathbb{P}(V_n(\rho\theta))|^2}{\varepsilon_n} + C \mu_n^2 f_b(V_n(\rho\theta)) \right\} d\theta dr \\ &\leq \int_{\rho - \sqrt{\varepsilon_n}}^\rho r^2 \int_{\mathbb{S}^2} \left\{ C |\nabla_T V_n(\rho\theta)|^2 + \frac{|V_n(\rho\theta) - V(\rho\theta)|^2}{\varepsilon_n} + C \mu_n^2 f_b(V_n(\rho\theta)) \right\} d\theta dr \\ &\leq C \int_{\rho - \sqrt{\varepsilon_n}}^\rho \frac{r^2}{\rho^2} \int_{\partial B_\rho} |\nabla_T V_n(x)|^2 + \frac{|V_n(x) - V(x)|^2}{\varepsilon_n} + \mu_n^2 f_b(V_n(x)) dx dr \\ &\leq C \sqrt{\varepsilon_n}. \end{aligned}$$

From the second line to the third line we used the fact that  $f_b(Q)$  is comparable to  $\text{dist}(Q, \mathcal{N})^2$  when  $Q \in \mathcal{N}_\delta$  with small enough  $\delta$ . From the third line to the fourth line we utilized  $\frac{d}{dr} V_n \left( \frac{\rho x}{|x|} \right) = 0$  and  $|\nabla_T \mathbb{P}(V_n(\rho\theta))|^2 \leq \text{Lip}(\mathbb{P})^2 |\nabla_T V_n(\rho\theta)|^2$ . The final estimate comes from (3.24). Taking  $n \rightarrow \infty$  in (3.27) we get  $\lim_{n \rightarrow \infty} II_n = 0$ .

Finally, by (3.25) and (3.26) we have

$$III_n = \int_{L_3} e_{\mu_n^{-1}}(W_n) dx = \int_{L_3} \frac{1}{2} |\nabla \mathbb{P}(K_n(x))|^2 dx \leq \text{Lip}(\mathbb{P})^2 \int_{L_3} |\nabla K_n|^2 dx \leq C \varepsilon_n^{1/6}.$$

When  $n \rightarrow \infty$ , we obtain  $\lim_{n \rightarrow \infty} III_n = 0$ . Then by minimality of  $V_n$  we conclude that

$$\int_{B_\rho} \frac{1}{2} |\nabla W|^2 dx = \lim_{n \rightarrow \infty} \int_{B_\rho} e_{\mu_n^{-1}}(W_n) dx \geq \lim_{n \rightarrow \infty} \int_{B_\rho} e_{\mu_n^{-1}}(V_n) dx \geq \int_{B_\rho} \frac{1}{2} |\nabla V|^2 dx.$$

Hence  $V \in H^1(B_2, \mathcal{N})$  is an energy minimizing harmonic map. Moreover, if we take  $W \equiv V$  in  $B_2$ , then the calculation above implies that

$$\int_{B_\rho} |\nabla V|^2 dx \geq \lim_{n \rightarrow \infty} \int_{B_\rho} e_{\mu_n^{-1}}(V_n) dx,$$

which further implies the strong  $H_{loc}^1$  convergence of  $V_n$  to  $V$  and  $\nu = 0$ .

□

Now we claim that there is a  $N \in \mathbb{N}$  large enough such that for any  $n \geq N$ , the set  $D_n^{\delta_0}$  defined in (3.20) is not empty, where  $\delta_0$  is defined in the proof of Proposition 3.2. Indeed, suppose the claim is wrong and we can find  $n_j \rightarrow \infty$  such that for every  $j$ ,

$$U_{n_j}(x) \in \mathcal{N}_{\delta_0}, \quad \forall x \in \bar{B}_1.$$

Then  $\mathbb{P}(U_{n_j})$  are smooth maps from  $\bar{B}_1$  to  $\mathcal{N}$ , and therefore  $\deg(\mathbb{P}(U_{n_j}), \partial B_r) = 0$  for any  $r \in (0, 1)$ . On the other hand, using (3.15) we know that when  $j$  is large enough,

$$\deg(\mathbb{P}(U_{n_j}), \partial B_r) = 1, \quad \forall r \in \left(\frac{1}{2}, 1\right),$$

which yields a contradiction. The claim is proved.

Now we can define  $a_n \in B_1$  such that

$$(3.28) \quad a_n \in D_n^{\delta_0} \text{ and } \text{dist}(U_n(a_n), \mathcal{N}) = \sup_{x \in B_1} \text{dist}(U_n(x), \mathcal{N}).$$

According to Proposition 3.2, we have  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $D_n^{\delta_0} \subset B_{r_n}(a_n)$  for some  $r_n \leq C_{\delta_0} R_n^{-1}$ . When  $U_n(\partial B_r) \subset \mathcal{N}_{\delta_0}$ , we also define the topological degree of  $U_n|_{\partial B_r}$  as

$$\deg(U_n, \partial B_r(x)) = \deg(\mathbb{P}(U_n), \partial B_r(x)),$$

By the condition (3.15), the smoothness of the map  $U_n$  and the smallness of  $a_n$ , we infer that

$$(3.29) \quad \deg(U_n, \partial B_r(a_n)) = 1 \quad \text{for every } r \in [C_{\delta_0} R_n^{-1}, 1 - |a_n|].$$

Then we define

$$(3.30) \quad Q_n(x) = U_n\left(\frac{x}{R_n} + a_n\right) = Q_{\varepsilon_n}(\varepsilon_n x + r_n a_n) \text{ for } |x| \leq \bar{R}_n := R_n(1 - |a_n|).$$

**Proposition 3.5.** *Let  $Q_\varepsilon$  be a global minimizer of the minimization problem (2.1)–(2.2). For  $\{Q_n\}$  defined as in (3.30) with  $\{(r_n, \varepsilon_n)\}$  as in (3.12), (3.13) (3.14) and (3.15), there exists a subsequence, still denoted by  $\{Q_n\}$ , such that  $Q_n(x) \rightarrow Q$  in  $C_{loc}^2(\mathbb{R}^3, \mathcal{Q}_0)$ , where the limiting map  $Q$  satisfies*

- (1)  $Q$  locally minimizes the functional  $I(\cdot, \mathbb{R}^3)$  (see (2.3)) in the sense of Definition 2.1.
- (2)  $\text{dist}(Q(x), \mathcal{N}) \rightarrow 0$  as  $|x| \rightarrow \infty$ ,  $\deg_\infty(Q) = 1$  and  $\frac{1}{R}I(Q, B_R) \rightarrow 8s_+^2\pi$  as  $R \rightarrow \infty$ .

*Proof.* Note that  $Q_n$  satisfies the Euler-Lagrange equation

$$\Delta Q_n = -a^2 Q_n - b^2 \left[ Q_n^2 - \frac{1}{3} \text{tr}(Q_n)^2 \text{Id} \right] + c^2 \text{tr}(Q_n)^2 Q_n \text{ in } B_{\bar{R}_n}.$$

Also (3.17) implies that

$$(3.31) \quad \lim_{n \rightarrow \infty} \frac{1}{\bar{R}_n} I(Q_n, B_{\bar{R}_n}) \leq 8s_+^2\pi.$$

Using standard elliptic regularity theory, we can extract a subsequence, still denoted by  $Q_n$ , that converges to  $Q$  in  $C_{loc}^2(\mathbb{R}^3, \mathcal{Q}_0)$ . Here  $Q$  solves the same equation as  $Q_n$  and also inherits the local minimality from  $Q_n$ .

By the definition of  $Q_n$  and (3.29), we have

$$\deg(Q, \partial B_{r_\delta}) = \lim_{n \rightarrow \infty} \deg(Q_n, \partial B_{r_\delta}) = 1, \quad \forall r_\delta > C_\delta.$$

Hence

$$\deg_\infty(Q) = \lim_{r \rightarrow \infty} \deg(Q, \partial B_r) = 1.$$

Similarly, by (3.18) and (3.19) we deduce that  $\text{dist}(Q(x), \mathcal{N}) \rightarrow 0$  as  $|x| \rightarrow \infty$ . It only remains to prove the energy estimate  $\frac{1}{R}I(Q, B_R) \rightarrow 8s_+^2\pi$  as  $R \rightarrow \infty$ . On the one hand, by the monotonicity formula and (3.31) we have

$$\frac{1}{R}I(Q, B_R) = \lim_{n \rightarrow \infty} \frac{1}{R}I(Q_n, B_R) \leq 8s_+^2\pi.$$

On the other hand,  $\text{dist}(Q(x), \mathcal{N}) \rightarrow 0$  as  $|x| \rightarrow \infty$  implies that for any  $\varepsilon > 0$  there exists a  $R_\varepsilon$  such that for all  $R > R_\varepsilon$ ,

$$\frac{1}{R}I(Q, B_R) \geq (1 - \varepsilon)\frac{1}{R}I(\mathbb{P}(Q), B_R).$$

We recall the well-known fact (see e.g. [7, Section VII]) that for  $g : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  with  $\deg(g) = 1$ , it holds that  $\int_{\mathbb{S}^2} \frac{1}{2}|\nabla g|^2 \geq 4\pi$ . Due to  $\deg_\infty(Q) = 1$ , we get

$$\lim_{R \rightarrow \infty} \frac{1}{R}I(Q, B_R) \geq (1 - \varepsilon)8s_+^2\pi.$$

Take  $\varepsilon \rightarrow 0$  and we complete the proof of Proposition 3.5.  $\square$

#### 4. BEHAVIOR OF THE LIMITING MAP $Q$ AT INFINITY

In order to understand the limiting map  $Q$  in Proposition 3.5, we study its tangent map at infinity. A tangent map for  $Q$  is a map  $\Psi : \mathbb{R}^3 \rightarrow \mathcal{Q}_0$  obtained as a weak  $H_{loc}^1(\mathbb{R}^3, \mathcal{Q}_0)$  limit of  $Q_{R_n}(x) := Q(R_n x)$  for some sequence  $R_n \rightarrow \infty$ . Let  $T_\infty$  denote the set of all possible tangent maps of  $Q$  at infinity.  $T_\infty$  can be characterized by the following theorem.

**Theorem 4.1.** *Let  $Q$  be the map defined in Proposition 3.5, then  $T_\infty(Q)$  is not empty. Let  $\Psi \in T_\infty(Q)$  and assume  $Q_{R_n}(x) \rightarrow \Psi$  weakly in  $H_{loc}^1(\mathbb{R}^3, \mathcal{Q}_0)$ . Then  $Q_{R_n}(x) \rightarrow \Psi$  strongly in  $H_{loc}^1(\mathbb{R}^3)$  and*

$$e_{R_n^{-1}}(Q_{R_n}) dx \rightarrow \frac{1}{2}|\nabla \Psi|^2 dx$$

as convergence of Radon measures. Moreover, there exists  $T \in O(3)$  such that

$$(4.32) \quad \Psi(x) = s_+(n(x) \otimes n(x) - \frac{1}{3}\text{Id}), \quad n(x) = T\left(\frac{x}{|x|}\right).$$

*Proof.* Fix a sequence  $R_n \uparrow \infty$ . For any  $R > 0$ , by Proposition 3.5 we have

$$\lim_{R_n \rightarrow \infty} \frac{1}{R} \int_{B_R} e_{R_n^{-1}}(Q_{R_n}) dx = \lim_{R_n \rightarrow \infty} \frac{1}{RR_n} I(Q, B_{RR_n}) = 8s_+^2\pi.$$

Thus  $Q_{R_n}$  is bounded in  $H_{loc}^1(\mathbb{R}^3)$  and up to a subsequence,  $Q_{R_n} \rightarrow \Psi$  weakly in  $H_{loc}^1(\mathbb{R}^3)$  and strongly in  $L_{loc}^2(\mathbb{R}^3)$ . Since for any  $R$ ,  $\lim_{n \rightarrow \infty} \int_{B_R} f_b(Q_{R_n}) dx = 0$ , using Fatou's lemma we obtain  $\Psi(x)$  take values in  $\mathcal{N}$ . Also,  $Q$  satisfies the monotonicity formula (2.9) because  $Q$  locally minimizes the functional  $I(Q, \mathbb{R}^3)$ . Then (2.9) and  $\frac{1}{R}I(Q, B_R) \rightarrow 8s_+^2\pi$  imply that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B_R} \frac{1}{|x|} \left| \frac{\partial Q(x)}{\partial r} \right|^2 dx = 0.$$

It follows that for any  $0 < \rho < R < \infty$ ,

$$\begin{aligned} \int_{B_R \setminus B_\rho} \frac{1}{|x|} \left| \frac{\partial \Psi(x)}{\partial r} \right|^2 dx &\leq \lim_{n \rightarrow \infty} \int_{B_R \setminus B_\rho} \frac{1}{|x|} \left| \frac{\partial Q_{R_n}(x)}{\partial r} \right|^2 dx \\ &= \lim_{n \rightarrow \infty} \int_{B_{RR_n} \setminus B_{\rho R_n}} \frac{1}{|x|} \left| \frac{\partial Q(x)}{\partial r} \right|^2 dx = 0. \end{aligned}$$

This implies that  $\Psi(x) = \Psi(\frac{x}{|x|})$  for  $x \neq 0$ . Also the topological degree of  $\Psi$  on  $\mathbb{S}^2$  is 1 since  $\deg_\infty(Q) = 1$ . Moreover,  $\text{dist}(Q(x), \mathcal{N}) \rightarrow 0$  as  $|x| \rightarrow \infty$  implies that

$$\lim_{n \rightarrow \infty} \sup_{x \in B_R \setminus B_{R/2}} \text{dist}(Q_{R_n}(x), \mathcal{N}) = 0$$

All these properties above enable us to exploit the same argument, as in the proof of Lemma 3.3, to obtain that  $Q_{R_n} \rightarrow \Psi$  strongly in  $H_{loc}^1(\mathbb{R}^3)$  where  $\Psi \in H_{loc}^1(\mathbb{R}^3, \mathcal{N})$  is a homogeneous energy minimizing harmonic map with degree 1.

Now by the classical result of Brezis-Coron-Lieb [7, Theorem 7.3]<sup>1</sup>,  $\Psi = s_+(n(x) \otimes n(x) - \frac{1}{3}\text{Id})$  for  $n(x) = T(\frac{x}{|x|})$ , for some  $T \in O(3)$ .

□

For the limiting map  $Q$  in Proposition 3.5, let's define  $Q^\sharp := \mathbb{P}(Q)$  as the orthogonal projection of  $Q$  onto  $\mathcal{N}$ , i.e.

$$|Q(x) - Q^\sharp(x)| = \text{dist}(Q(x), \mathcal{N})$$

By Proposition 3.5 we know that  $Q(x)$  will stay in a small neighborhood of  $\mathcal{N}$  when  $|x|$  is sufficiently large. Consequently,  $Q^\sharp(x)$  is well-defined for large  $|x|$ . We also denote

$$D(x) := Q(x) - Q^\sharp(x).$$

Then we have the following two lemmas, which mostly rely on the estimates in [43].

**Lemma 4.1.** *For any positive integer  $k$ , there exists a positive constant  $C_k$  such that*

$$|\nabla^k Q| \leq \frac{C_k}{|x|^k}, \quad \text{for all } x \in \mathbb{R}^3.$$

*Proof.* We argue by contradiction. Assume the statement is false, then there would be an integer  $k$  and a sequence of points  $x_n$  such that

$$\begin{aligned} R_n &:= |x_n| \rightarrow \infty \quad \text{as } n \rightarrow \infty \\ R_n^k |\nabla^k Q| &\rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

For each  $n$ , we consider  $Q_n = Q(R_n x)$  as a local minimizer in the sense of Definition 2.1 of the following functional

$$\int_{B_2} \frac{1}{2} |\nabla Q|^2 + R_n^2 f_b(Q) \, dx$$

in the ball  $B_2(0)$ . Thanks to Proposition 3.5 and Theorem 4.1, we have that up to a subsequence,

$$\frac{x_n}{R_n} \rightarrow \bar{x} \text{ for some } \bar{x} \in \partial B_1,$$

$$Q_n \rightarrow \Psi \text{ strongly in } H^1(B_2) \text{ for } \Psi = s_+(\left(T \frac{x}{|x|}\right) \otimes \left(T \frac{x}{|x|}\right) - \frac{1}{3}\text{Id}), \quad T \in O(3).$$

The strong  $H^1$  convergence implies, as in [40, Proposition 4] the uniform convergence of  $f_b(Q_n)$  to 0, which allows to use [40, Lemma 6.7] to get a uniform gradient bound on  $Q_n$  which is updated to convergence in the interior in arbitrarily high norms in [43, Theorem 1]. Thus we have

$$Q_n \rightarrow \Psi \text{ in } C^{k+1}(B_{3/2} \setminus B_{1/2}).$$

Then we can derive

$$\infty = \lim_{n \rightarrow \infty} R_n^k |\nabla^k Q(x_n)| = \lim_{n \rightarrow \infty} |\nabla^k Q_n(\frac{x_n}{R_n})| = |\nabla^k \Psi(\bar{x})| < \infty,$$

<sup>1</sup>[7, Theorem 7.3] is proved for  $\mathbb{S}^2$ -valued maps, however it also holds for the case of  $\mathbb{R}\mathbb{P}^2$ -valued maps, see the discussion in [7, Section VIII-B-c].

which yields a contradiction. The proof is complete.  $\square$

*Remark 4.1.* As a consequence of Lemma 4.1, we can improve the strong  $H^1$  convergence in Theorem 4.1 to  $C_{loc}^k$  convergence, i.e. assume  $\Psi \in T_\infty(Q)$  and  $Q_{R_n} \rightarrow \Psi$  weakly in  $H_{loc}^1(\mathbb{R}^3, \mathcal{Q}_0)$ , then  $Q_{R_n} \rightarrow \Psi$  strongly in  $C^k(K, \mathcal{Q}_0)$  for any integer  $k$  and compact set  $K \subset \mathbb{R}^3 \setminus \{0\}$ .

**Lemma 4.2.** *There exist positive constants  $R_0$  and  $C$  such that for any  $|x| > R_0$ ,*

$$(4.33) \quad |\nabla Q^\sharp(x)| \leq \frac{C}{|x|}, \quad |\nabla(Q^\sharp)^{-1}(x)| \leq \frac{C}{|x|},$$

$$(4.34) \quad |D(x)| \leq \frac{C}{|x|^2}, \quad |\nabla D(x)| \leq \frac{C}{|x|^3}.$$

*Proof.* The proof follows the same strategy of the proof of Lemma 4.1. The key is to show for any sequence  $R_n \rightarrow \infty$  such that  $Q_{R_n} \rightarrow \Psi$  in  $H^1(B_2)$ , the following estimates holds in  $B_{3/2} \setminus B_{1/2}$ :

$$(4.35) \quad |\nabla Q_{R_n}^\sharp(x)| \leq C, \quad |\nabla(Q_{R_n}^\sharp)^{-1}(x)| \leq C,$$

$$(4.36) \quad |D_{R_n}(x)| \leq \frac{C}{|R_n|^2}, \quad |\nabla D_{R_n}(x)| \leq \frac{C}{|R_n|^2},$$

for some constant  $C$ . Here  $D_{R_n} = Q_{R_n} - Q_{R_n}^\sharp$ .

For the estimates in (4.35), we note that it suffices to prove the first estimate, while the second follows from differentiating the identity  $Q_{R_n}^\sharp(Q_{R_n}^\sharp)^{-1} = \text{Id}$ . For the first one, note that since  $Q_{R_n}^\sharp = \mathbb{P} \circ Q_{R_n}$  for some smooth projection map  $\mathbb{P}$ ,

$$(4.37) \quad |\nabla Q_{R_n}^\sharp(x)| \leq \tilde{C} |\nabla Q_{R_n}(x)|, \quad \left| \frac{\partial Q_{R_n}^\sharp}{\partial r} \right| \leq \tilde{C} \left| \frac{\partial Q_{R_n}}{\partial r} \right|, \quad \text{in } B_{3/2} \setminus B_{1/2}.$$

with the constant  $\tilde{C}$  independent of  $n$ , and depending only on the distance between  $Q_{R_n}(x)$  and the manifold  $\mathcal{N}$ . Also, thanks to the proof of Lemma 4.1, we have

$$(4.38) \quad |\nabla Q_{R_n}| \leq C \text{ in } B_{3/2} \setminus B_{1/2}.$$

(4.38) and (4.37) together imply the first estimate in (4.35).

For the estimate (4.36), we recall the definition for the matrix  $X$  in [43]:

$$X_n := R_n^2(Q_{R_n}^2 - \frac{1}{3}s_+Q_{R_n} - \frac{2}{9}s_+^2\text{Id}).$$

According to [43, Proposition 4], we know that there exists a constant  $C$  such that

$$(4.39) \quad \|X_n\|_{C^1(B_{3/2} \setminus B_{1/2})} \leq C,$$

$$(4.40) \quad \frac{1}{CR_n^2}|X_n| \leq |D_{R_n}| \leq \frac{C}{R_n^2}|X_n|,$$

which yields the first estimates in (4.36). For the estimate of  $|\nabla D_{R_n}|$ , we will utilize the estimate for  $|\nabla X_n|$  to derive an upper bound.

For the sake of convenience, in the rest of the proof we simply write  $Q_{R_n}, D_{R_n}$  as  $Q_n, D_n$ . We also denote

$$Y_n := \frac{X_n}{R_n^2}.$$

Note that  $Q_n$  has the decomposition

$$Q_n = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3,$$

where  $\lambda_1 \geq \lambda_2 \geq \lambda_3$  are three eigenvalues and  $e_1, e_2, e_3$  are the corresponding unit eigenvectors where are orthogonal with each other. Note that  $\lambda_i, e_i, i = 1, 2, 3$  also depend on  $n$  and on the location  $x$  where  $Q_n$  is evaluated. For the sake of convenience, we do not indicate this dependence explicitly here. Since  $Q_n$  is very close to  $\mathcal{N}$  when  $n$  is large (and we only care about the case for large  $n$ ), we can assume roughly  $\lambda_1 \approx \frac{2}{3}s_+$  and  $\lambda_i \approx -\frac{1}{3}s_+$  for  $i = 2, 3$ . In particular, one can identify the orthogonal projection of  $Q_n$  onto  $\mathcal{N}$ , which is denoted by  $Q_n^\sharp$ , as

$$Q_n^\sharp = s_+(e_1 \otimes e_1 - \frac{1}{3}\text{Id}).$$

For the validity of this expression, one can refer to [21, Lemma C.1]. Note that since  $\lambda_1$  is an isolated eigenvalue of  $Q_n$ , it is well-known, see for instance [44], that  $e_1$  is as smooth as  $Q_n$  and we have

$$(4.41) \quad |\nabla e_1| \leq C|\nabla Q_n| \leq C \quad \text{in } B_{3/2} \setminus B_{1/2}.$$

By the definitions of  $Y_n$  and  $X_n$ , we compute

$$\begin{aligned} Y_n &= Q_n^2 - \frac{1}{3}s_+Q_n - \frac{2}{9}s_+^2\text{Id} \\ &= (Q_n^\sharp + D_n)(Q_n^\sharp + D_n) - \frac{1}{3}s_+(Q_n^\sharp + D_n) - \frac{2}{9}s_+^2\text{Id} \\ &= Q_n^\sharp D_n + D_n Q_n^\sharp + D_n^2 - \frac{1}{3}s_+D_n \\ &= s_+(2e_1 \otimes e_1 - \text{Id})D_n + D_n^2 \end{aligned}$$

where for the third equality we used that  $Q_n^\sharp$  is a root of the polynomial equation  $Q^2 - \frac{s_+}{3}Q - \frac{2s_+^2}{9}\text{Id} = 0$ , see for instance [43, Lemma 1]. For the fourth equality we used that  $D_n$  and  $Q_n^\sharp$  commute (as they have a common eigenbases), together with the definition of  $Q_n^\sharp$ .

$$\begin{aligned} |\nabla Y_n|^2 &= \sum_{\alpha=1}^3 |\partial_\alpha (s_+(2e_1 \otimes e_1 - \text{Id})D_n + D_n^2)|^2 \\ &= \sum_{\alpha=1}^3 |s_+(2e_1 \otimes e_1 - \text{Id})\partial_\alpha D_n + 2s_+\partial_\alpha(e_1 \otimes e_1)D_n + \partial_\alpha(D_n^2)|^2 \\ (4.42) \quad &= \sum_{\alpha=1}^3 \left\{ s_+^2 |\partial_\alpha D_n|^2 + 4s_+^2 |\partial_\alpha(e_1 \otimes e_1)D_n|^2 + |\partial_\alpha(D_n^2)|^2 \right. \\ &\quad \left. + 4s_+^2 (2e_1 \otimes e_1 - \text{Id})\partial_\alpha D_n : \partial_\alpha(e_1 \otimes e_1)D_n + 4s_+\partial_\alpha(e_1 \otimes e_1)D_n : \partial_\alpha(D_n^2) \right. \\ &\quad \left. + 2s_+(2e_1 \otimes e_1 - \text{Id})\partial_\alpha D_n : \partial_\alpha(D_n^2) \right\} \\ &= (s_+^2 |\nabla D_n|^2 + 4s_+^2 |\nabla(e_1 \otimes e_1)D_n|^2 + |\nabla(D_n^2)|^2) + S_n. \end{aligned}$$

Here  $S_n$  is defined to be the sum of all the cross terms. Also in passing from the second line to the third line we have used the fact that

$$\begin{aligned} &|(2e_1 \otimes e_1 - \text{Id})\partial D_n|^2 \\ &= (2e_1 \otimes e_1 - \text{Id})\partial D_n : (2e_1 \otimes e_1 - \text{Id})\partial D_n \\ &= (2e_1 \otimes e_1 - \text{Id})^2 : (\partial D_n)^2 = |\partial_\alpha D|^2. \end{aligned}$$



By (4.39), (4.40) and (4.41), there is a constant  $C$  such that for  $n$  large enough and  $x \in B_{3/2} \setminus B_{1/2}$ , we have

$$(4.43) \quad |S_n| \leq \frac{C}{R_n^2} |\nabla D_n| + \frac{C}{R_n^2} |\nabla D_n|^2.$$

We obtain

$$(4.44) \quad \frac{C}{R_n^4} \geq |\nabla Y_n|^2 \geq (s_+^2 - \frac{C}{R_n^2}) |\nabla D_n|^2 - \frac{C}{R_n^2} |\nabla D_n|,$$

where the first inequality is obtained out of (4.39) and the definition of  $Y_n$  in terms of  $X_n$ . The second inequality is obtained combining (4.42) and (4.43) for  $n$  large enough. We point out that the constants  $C$  in (4.44) may represent different values, but they are all independent of  $x$  and  $n$ . Finally, it is straightforward to deduce from (4.44) that, there exist  $N$  and a constant  $C$  such that,

$$|\nabla D_n| \leq \frac{C}{R_n^2}, \quad \forall n \geq N, \forall x \in B_{3/2} \setminus B_{1/2},$$

and we conclude the proof.  $\square$

Now we are in the position to establish a decay estimate for the radial derivative of  $Q$ . The argument and the proof are presented in the same spirit as [41, Proposition 5.3], and will also frequently utilize the results from [43].

**Proposition 4.3.** *There exist positive constants  $R_0$  and  $C$  such that for any  $R \geq R_0$ ,*

$$(4.45) \quad \int_{|x| > R} \frac{1}{|x|} \left| \frac{\partial Q}{\partial r} \right|^2 dx \leq \frac{C}{R^2}.$$

*Remark 4.2.* We note that a stronger decay estimate of the radial derivative (see (4.69)) can be derived using the blow-up technique. However, we still want to present the following more explicit proof utilizing the asymptotic properties of  $Q$  at infinity, which we believe to be of independent interest.

*Proof.* Firstly we point out that it suffices to prove (4.45) for  $Q^\sharp$  since  $|\nabla D(x)| \sim |x|^{-3}$  as  $|x| \rightarrow \infty$  thanks to Proposition 4.2. By Proposition 3.5, there exists  $R_0$  such that  $Q^\sharp(x)$  is well defined and  $\deg(Q^\sharp, \partial B_R) = 1$  whenever  $R \geq R_0$ . Recall that  $Q$  satisfies the Euler-Lagrange equation in  $\mathbb{R}^3$ ,

$$\Delta Q = -a^2 Q - b^2 [Q^2 - \frac{1}{3} \text{tr}(Q)^2 \text{Id}] + c^2 \text{tr}(Q)^2 Q$$

We also have the equation for  $Q^\sharp$  by [43, Proposition 2],

$$(4.46) \quad \begin{aligned} \Delta Q^\sharp = & -\frac{2}{s_+^2} |\nabla Q^\sharp|^2 Q^\sharp + \frac{2}{s_+} \left[ \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 - \frac{1}{3} |\nabla Q^\sharp|^2 \text{Id} \right]^2 \\ & - \left[ T^{-1} \left( \frac{1}{s_+} Q^\sharp - \frac{2}{3} \text{Id} \right) W - W \left( \frac{1}{s_+} Q^\sharp - \frac{2}{3} \text{Id} \right) T^{-1} \right], \end{aligned}$$

where

$$(4.47) \quad \begin{aligned} W = & 2 \nabla Q^\sharp \nabla [(Q^\sharp)^{-1} Q] Q^\sharp - 2 Q^\sharp \nabla [(Q^\sharp)^{-1} Q] \nabla Q^\sharp \\ & - \frac{1}{s_+} Q \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 + \frac{1}{s_+} \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 Q, \end{aligned}$$

<sup>2</sup>In reference [43] a different form is used for the first line, namely the expression in (iv) Corollary 1, which is just the equation for the limit harmonic map. The form we use is an equivalent one, which is the form (ii) in Corollary 1.

$$(4.48) \quad T = Q - \frac{2}{9}s_+ \operatorname{tr}[(Q^\sharp)^{-1} Q] \operatorname{Id} + \beta \left[ \frac{1}{s_+} Q^\sharp + \frac{1}{3} \operatorname{Id} \right],$$

and  $\beta$  is an arbitrary nonzero real number.

We note that because  $Q$  is bounded in  $L^\infty$  and close to  $\mathcal{N}$  when  $|x| > R_0$ , we have that there exists a constant  $\tilde{C}$  depending only on  $a^2, b^2, c^2$  and how close the  $Q$  is to  $\mathcal{N}$  such that

$$(4.49) \quad |T^{-1}(\frac{1}{s_+} Q^\sharp - \frac{2}{3} \operatorname{Id})|, |(\frac{1}{s_+} Q^\sharp - \frac{2}{3} \operatorname{Id})T^{-1}| \leq \tilde{C}$$

Recall the following decomposition

$$(4.50) \quad Q = Q^\sharp + D$$

and then we have:

$$\begin{aligned} W &= 2\nabla Q^\sharp \nabla[(Q^\sharp)^{-1} D] Q^\sharp - 2Q^\sharp \nabla[(Q^\sharp)^{-1} D] \nabla Q^\sharp \\ &\quad - \frac{1}{s_+} D \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 + \frac{1}{s_+} \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 D \\ &\quad - \frac{1}{s_+} Q^\sharp \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 + \frac{1}{s_+} \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 Q^\sharp. \end{aligned}$$

We claim that the last two terms vanish. Indeed, for any  $\alpha = 1, 2, 3$ , we have  $\nabla_\alpha Q^\sharp \in T_{Q^\sharp} \mathcal{N}$ , see for instance [43, Lemma 2] for a characterization of the tangent space and the normal space to  $\mathcal{N}$  at a point  $Q$ , which are denoted by  $T_Q \mathcal{N}$  and  $(T_Q \mathcal{N})^\perp$  respectively. And then by [43, Lemma 3] we get that  $\sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 \in ((T_{Q^\sharp} \mathcal{N})^\perp)^\perp$ . On the other hand the characterization of the space  $((T_{Q^\sharp} \mathcal{N})^\perp)^\perp$  in [43, Lemma 2] shows that the elements in this space commute with matrices  $Q^\sharp$ , hence the last two terms vanish as claimed.

We rewrite the equation (4.46) as

$$(4.51) \quad \Delta Q^\sharp = -\frac{2}{s_+^2} |\nabla Q^\sharp|^2 Q^\sharp + \frac{2}{s_+} \left[ \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 - \frac{1}{3} |\nabla Q^\sharp|^2 \operatorname{Id} \right] + H(x)$$

where

$$(4.52) \quad H(x) := -\left[ T^{-1} \left( \frac{1}{s_+} Q^\sharp - \frac{2}{3} \operatorname{Id} \right) W - W \left( \frac{1}{s_+} Q^\sharp - \frac{2}{3} \operatorname{Id} \right) T^{-1} \right] = O(|x|^{-4}), \quad \text{as } |x| \rightarrow \infty$$

where the last estimate results from Lemma 4.2 and relation (4.51) (without the last two terms that vanish). We multiply (4.51) by  $\frac{\partial Q^\sharp}{\partial r}$ . It is straightforward to verify that

$$\left\{ -\frac{2}{s_+^2} |\nabla Q^\sharp|^2 Q^\sharp + \frac{2}{s_+} \left[ \sum_{\alpha=1}^3 (\nabla_\alpha Q^\sharp)^2 - \frac{1}{3} |\nabla Q^\sharp|^2 \operatorname{Id} \right] \right\} \cdot \frac{\partial Q^\sharp}{\partial r} = 0.$$

Thus we have

$$(4.53) \quad 0 = (\Delta Q^\sharp - H(x)) \cdot \frac{\partial Q^\sharp}{\partial r} = \frac{1}{|x|} \left| \frac{\partial Q^\sharp}{\partial r} \right|^2 + \operatorname{div} \left( \nabla Q^\sharp \cdot \frac{\partial Q^\sharp}{\partial r} - \frac{1}{2} |\nabla Q^\sharp|^2 \frac{x}{|x|} \right) - H(x) \cdot \frac{\partial Q^\sharp}{\partial r}.$$

Integrating (4.53) on an annulus  $B_{R_2} \setminus B_{R_1}$  for some  $R_2 > R_1 > R_0$  and then performing integration by parts leads to

$$\begin{aligned}
 (4.54) \quad & \int_{B_{R_2} \setminus B_{R_1}} \frac{1}{|x|} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx - \frac{1}{2} \int_{\partial B_{R_1}} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \\
 &= \frac{1}{2} \int_{\partial B_{R_2}} |\nabla_T Q^\#|^2 d\sigma - \frac{1}{2} \int_{\partial B_{R_1}} |\nabla_T Q^\#|^2 d\sigma - \frac{1}{2} \int_{\partial B_{R_2}} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \\
 & \quad + \int_{B_{R_2} \setminus B_{R_1}} H(x) \cdot \frac{\partial Q^\#}{\partial r} dx,
 \end{aligned}$$

where  $\nabla_T$  means the tangential gradient on the sphere. Note that by the second inequality of (4.37) and the monotonicity formula (2.9) we have

$$\int_0^\infty \left( \frac{1}{R} \int_{\partial B_R} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \right) dR \leq \tilde{C} \int_0^\infty \left( \frac{1}{R} \int_{\partial B_R} \left| \frac{\partial Q}{\partial r} \right|^2 d\sigma \right) dR < \infty.$$

So we can find a sequence  $\{r_k\}_{k=1}^\infty$  such that

$$(4.55) \quad r_k \rightarrow \infty, \quad \int_{\partial B_{r_k}} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

Now we use the compactness property of  $Q_{r_k}$  (see Theorem 4.1 and Remark 4.1) and the closeness between  $Q_{r_k}$  and  $Q_{r_k}^\#$  (see (4.34)) to derive that, up to a subsequence,

$$(4.56) \quad Q_{r_k}^\#|_{\mathbb{S}^2} \rightarrow \Psi(x)|_{\mathbb{S}^2} \quad \text{in } C^1(\mathbb{S}^2, \mathcal{N}).$$

Combining (4.55), (4.56) and the fact that  $\Psi$  is energy-minimizing among all degree-1 map from  $\mathbb{S}^2$  to  $\mathcal{N}$ , we get

$$\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\partial B_{r_k}} |\nabla_T Q^\#|^2 d\sigma - \frac{1}{2} \int_{\partial B_{R_1}} |\nabla_T Q^\#|^2 d\sigma - \frac{1}{2} \int_{\partial B_{r_k}} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \leq 0$$

Substituting the above inequality into (4.54) gives

$$(4.57) \quad \int_{|x| > R_1} \frac{1}{|x|} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx - \frac{1}{2} \int_{\partial B_{R_1}} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \leq \int_{|x| > R_1} H(x) \cdot \frac{\partial Q^\#}{\partial r} dx \quad \forall R_1 \geq R_0$$

Note that by Young's inequality and (4.52) we have

$$\begin{aligned}
 (4.58) \quad \int_{|x| > R_1} H(x) \cdot \frac{\partial Q^\#}{\partial r} dx &\leq \frac{1}{4} \int_{|x| > R_1} |x|^2 H(x) dx + \int_{|x| > R_1} \frac{1}{|x|^2} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx \\
 &\leq CR_1^{-3} + \frac{1}{R_1} \int_{|x| > R_1} \frac{1}{|x|} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx, \quad \text{for some constant } C > 0.
 \end{aligned}$$

Combine (4.58) and (4.57) and write  $R_1 = r$  to obtain

$$(4.59) \quad \left(1 - \frac{1}{r}\right) \int_{|x| > r} \frac{1}{|x|} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx - \frac{1}{2} \int_{\partial B_r} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma \leq Cr^{-3}.$$

Multiplying the above inequality by  $2r$  implies

$$(4.60) \quad \frac{d}{dr} \left( (r^2 - 2r) \int_{|x| > r} \frac{1}{|x|} \left| \frac{\partial Q^\#}{\partial r} \right|^2 dx \right) \leq 2 \int_{\partial B_r} \left| \frac{\partial Q^\#}{\partial r} \right|^2 d\sigma + Cr^{-2}.$$

Denote

$$f(r) := (r^2 - 2r) \int_{|x|>r} \frac{1}{|x|} \left| \frac{\partial Q^\sharp}{\partial r} \right|^2 dx.$$

Then the above inequality implies

$$\begin{aligned} f'(r) &\leq 2 \frac{d}{dr} \left( -\frac{f}{r^2 - 2r} \right) \cdot r + Cr^{-2} \\ \Rightarrow (1 + \frac{2}{r-2}) f'(r) &\leq \frac{4(r-1)}{(r-2)^2 r} f(r) + Cr^{-2} \\ (4.61) \quad \Rightarrow \frac{f'(r)}{f(r)+1} &\leq Cr^{-2} \\ \Rightarrow \ln \frac{f(r)+1}{f(R_0)+1} &\leq \frac{C}{R_0} \Rightarrow f(r) \leq C, \end{aligned}$$

for any  $r > R_0$  with  $R_0$  sufficiently large. The estimate (4.45) follows immediately from (4.61). □

Now we are ready to prove the uniqueness of the tangent map for  $Q$ .

**Theorem 4.2.** *Let  $Q$  be the limiting map defined in Proposition 3.5. Then the tangent map at infinity is unique, i.e. there exists a  $\Psi$  which is of the form (4.32), such that*

$$(4.62) \quad \lim_{R \rightarrow \infty} \|Q_R - \Psi\|_{H_{loc}^1(\mathbb{R}^3)} = 0$$

$$(4.63) \quad \lim_{R \rightarrow \infty} \|Q_R|_{\mathbb{S}^2} - \Psi|_{\mathbb{S}^2}\|_{C^k(\mathbb{S}^2)} = 0, \quad \forall k \in \mathbb{N}^+$$

*Proof.* It suffices to prove (4.62), since (4.63) is a direct consequence of (4.62) and Remark 4.1. We prove by contradiction. Assume the statement is false, then there would be two harmonic maps  $\Psi_1$  and  $\Psi_2$  and two sequences of radiuses  $\{r_i^1\}_{i=1}^\infty$  and  $\{r_j^2\}_{j=1}^\infty$  satisfying

$$\lim_{i \rightarrow \infty} r_i^1 = \lim_{j \rightarrow \infty} r_j^2 = \infty,$$

$$Q_{r_i^1} \rightarrow \Psi_1, \quad Q_{r_j^2} \rightarrow \Psi_2 \quad \text{in the sense of } C_{loc}^2.$$

We take  $i_0$  and  $j_0$  be the integers such that for any  $i \geq i_0$  and  $j \geq j_0$ ,

$$\begin{aligned} \|Q_{r_i^1} - \Psi_1\|_{L^2(\mathbb{S}^2)} &\leq \frac{1}{8} \|\Psi_1 - \Psi_2\|_{L^2(\mathbb{S}^2)}, \\ \|Q_{r_j^2} - \Psi_2\|_{L^2(\mathbb{S}^2)} &\leq \frac{1}{8} \|\Psi_1 - \Psi_2\|_{L^2(\mathbb{S}^2)}, \\ (4.64) \quad \|Q_{r_i^1} - Q_{r_j^2}\|_{L^2(\mathbb{S}^2)} &> \frac{1}{2} \|\Psi_1 - \Psi_2\|_{L^2(\mathbb{S}^2)}. \end{aligned}$$

We fix the  $R_0$  as in Proposition 4.3. For any  $R_0 \leq R_1 < R_2 \leq 2R_1$ , we compute

$$\begin{aligned} \int_{\mathbb{S}^2} |Q_{R_1}(\sigma) - Q_{R_2}(\sigma)|^2 d\sigma &\leq \int_{\mathbb{S}^2} \left( (R_2 - R_1) \int_{R_1}^{R_2} \left| \frac{\partial Q(r\sigma)}{\partial r} \right|^2 dr \right) d\sigma \\ &\leq \int_{\mathbb{S}^2} \left( \int_{R_1}^{R_2} r \left| \frac{\partial Q}{\partial r} \right|^2 dr \right) d\sigma \\ &= \int_{R_1 < |x| < R_2} \frac{1}{|x|} \left| \frac{\partial Q}{\partial r} \right|^2 dx \leq \frac{C}{R_1^2} \end{aligned}$$

Now we fix  $R_1$ , and assume  $R_2$  be an arbitrary large radius such that  $2^k R_1 < R_2 \leq 2^{k+1} R_1$  for some non-negative integer  $k$ . Then we have

$$\begin{aligned} \|Q_{R_1} - Q_{R_2}\|_{L^2(\mathbb{S}^2)} &\leq \sum_{i=0}^{k-1} \|Q_{2^i R_1} - Q_{2^{i+1} R_1}\|_{L^2(\mathbb{S}^2)} + \|Q_{R_2} - Q_{2^k R_1}\|_{L^2(\mathbb{S}^2)} \\ &\leq \sum_{i=0}^k \frac{C}{2^i R_1} < \frac{C}{R_1} \end{aligned}$$

As a consequence, we have that

$$\lim_{i,j \rightarrow \infty} \|Q_{r_i^1} - Q_{r_j^2}\|_{L^2(\mathbb{S}^2)} = 0$$

which yields a contradiction with (4.64). The proof is complete.  $\square$

Recall our assumption (3.15) at the very beginning, which says

$$\lim_{n \rightarrow \infty} \|Q_{\varepsilon_n}(x) - s_+\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\text{Id}\right)\|_{L^\infty(B_{\frac{2}{3}r_n} \setminus B_{\frac{1}{2}r_n})} = 0$$

After a change of variable (making  $r_n a_n$  as the ‘‘central point’’), we have

$$\lim_{n \rightarrow \infty} \|Q_{\varepsilon_n}(x + r_n a_n) - s_+\left(\frac{x + r_n a_n}{|x + r_n a_n|} \otimes \frac{x + r_n a_n}{|x + r_n a_n|} - \frac{1}{3}\text{Id}\right)\|_{L^\infty(B_{\frac{4}{3}r_n} \setminus B_{\frac{2}{3}r_n})} = 0,$$

where  $a_n$  is defined in (3.28). Note that when  $\frac{2}{3}r_n \leq |x| \leq \frac{4}{3}r_n$ ,  $\frac{x+r_n a_n}{|x+r_n a_n|}$  is very close to  $\frac{x}{|x|}$  given  $|a_n|$  sufficiently small (see the remark of  $a_n \rightarrow 0$  after (3.28)). As  $n \rightarrow \infty$ , we obtain

$$(4.65) \quad \lim_{n \rightarrow \infty} \|Q_{\varepsilon_n}(x + r_n a_n) - s_+\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\text{Id}\right)\|_{L^\infty(B_{\frac{4}{3}r_n} \setminus B_{\frac{2}{3}r_n})} = 0$$

On the other hand, by Theorem 4.1 and Theorem 4.2, we have that

$$(4.66) \quad \lim_{R \rightarrow \infty} \|Q(x) - s_+(n \otimes n - \frac{1}{3}\text{Id})\|_{L^\infty(B_{2R} \setminus B_R)} = 0$$

where  $n(x) = T(\frac{x}{|x|})$  for some  $T \in O(3)$ . Since  $Q$  is obtained by taking a  $C_{loc}^2$  limit of  $Q_n(x)$  (see (3.30)), for any fixed  $R$  it holds that

$$(4.67) \quad \lim_{\substack{\frac{r_n}{2\varepsilon_n} \geq R, \\ n \rightarrow \infty}} \|Q_{\varepsilon_n}(x + r_n a_n) - Q\left(\frac{x}{\varepsilon_n}\right)\|_{L^\infty(B_{2R\varepsilon_n} \setminus B_{R\varepsilon_n})} = 0$$

(4.66) and (4.67) together imply that

$$(4.68) \quad \lim_{R \rightarrow \infty} \left( \lim_{\substack{\frac{r_n}{2\varepsilon_n} \geq R, \\ n \rightarrow \infty}} \|Q_{\varepsilon_n}(x + r_n a_n) - s_+(n(x) \otimes n(x) - \frac{1}{3}\text{Id})\|_{L^\infty(B_{2R\varepsilon_n} \setminus B_{R\varepsilon_n})} \right) = 0.$$

Comparing (4.65) and (4.68) we know that  $Q_{\varepsilon_n}(x + r_n a_n)$  is close to  $s_+(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\text{Id})$  at  $|x + r_n a_n| \sim r_n$ , but when  $|x + r_n a_n| \sim R\varepsilon_n$  for large enough  $R$ , it is asymptotically  $s_+(n(x) \otimes n(x) - \frac{1}{3}\text{Id})$  as  $n \rightarrow \infty$ . A natural question would be whether or not  $n(x) = \frac{x}{|x|}$ , so that the behavior of  $Q_{\varepsilon_n}$  on the outer sphere  $\partial B_{r_n}(r_n a_n)$  will match that of the inner sphere  $\partial B_{R\varepsilon_n}(r_n a_n)$ . The answer is positive.

**Theorem 4.4.** *Let  $Q$  be the limiting map in Proposition 3.5 and  $\Psi$  is its unique tangent map at infinity. Then  $\Psi = \Phi$ , i.e.*

$$\Psi = s_+ \left( \frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} \text{Id} \right).$$

To prove the theorem, we need the following lemma, which gives crucial estimates on the decay rate of the radial derivatives.

**Lemma 4.5.** *Let  $Q_{\varepsilon_n}$  be the sequence of minimizers such that  $Q_{\varepsilon_n}(\varepsilon_n x + r_n a_n) \rightarrow Q$  in  $C_{loc}^2(\mathbb{R}^3)$  with  $r_n, \varepsilon_n$  satisfying (3.12), (3.13), (3.14) and (3.15),  $a_n$  as defined in (3.28). Then there is a positive constant  $C$ , such that for any  $\varepsilon_n$  and  $R \leq \frac{r_n}{2\varepsilon_n}$ , it holds that*

$$(4.69) \quad \int_{\{R\varepsilon_n \leq |x| \leq 2R\varepsilon_n\}} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_n}(x + r_n a_n)}{\partial r} \right|^2 dx \leq \frac{C}{R^4}.$$

*Proof.* Without loss of generality, we can assume  $a_n = 0$  for every  $n$ , since we only need the property  $|a_n| \rightarrow 0$  as  $n \rightarrow \infty$  and the exact location of  $a_n$  won't affect our proof.

Assume such constant  $C$  does not exist. Then we can find a sequence  $\varepsilon_k, R_k \leq \frac{r_k}{2\varepsilon_k}$  (in this sequence, one  $\varepsilon_k$  can appear repeatedly) such that

$$(4.70) \quad R_k^4 \int_{\{R_k \varepsilon_k \leq |x| \leq 2R_k \varepsilon_k\}} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_k}}{\partial r} \right|^2 dx \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

In order that (4.70) holds, we must have

$$(4.71) \quad \lim_{k \rightarrow \infty} \varepsilon_k = 0, \quad \lim_{k \rightarrow \infty} R_k = \infty.$$

Here we briefly justify these two limits. Firstly if  $\limsup_{k \rightarrow \infty} \varepsilon_k > 0$ , then there exists an integer  $i_0$  such that  $\varepsilon_k = \varepsilon_{i_0} > 0$  holds for infinitely many  $k$ . For all such  $k$ ,  $R_k$  is uniformly bounded since we require  $R_k \leq \frac{r_k}{2\varepsilon_k} = \frac{r_{i_0}}{2\varepsilon_{i_0}}$ . Then  $R_k^4 \int_{\{R_k \varepsilon_k \leq |x| \leq 2R_k \varepsilon_k\}} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_k}}{\partial r} \right|^2 dx$  is also bounded, which contradicts with (4.70). Secondly, if  $R_k$  doesn't go to infinity, then we assume  $\liminf_{k \rightarrow \infty} R_k = R_0 <$

$\infty$ . By monotonicity formula (2.9) we know  $\int_{R_k \varepsilon_k \leq |x| \leq 2R_k \varepsilon_k} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_k}}{\partial r} \right|^2 dx$  is uniformly bounded. And this will remain bounded after multiplying bounded  $R_k^4$ , which also contradicts with (4.70). Therefore  $R_k$  has to go to infinity.

Now we define

$$P_k := Q_{\varepsilon_k}(R_k \varepsilon_k \cdot x).$$

Then  $P_k$  satisfies the following properties.

- (1)  $P_k$  minimizes the functional  $\int_{B_3} \left\{ \frac{1}{2} |\nabla Q|^2 + R_k^2 f_b(Q) \right\} dx$ .
- (2) For any  $r \in (0, 3)$ , we have

$$\lim_{k \rightarrow \infty} \frac{1}{r} \int_{B_r} \frac{1}{2} |\nabla P_k|^2 + R_k^2 f_b(R_k) = 8s_+^2 \pi.$$

The fact that the limit on the left-hand side is bounded from above by  $8s_+^2 \pi$  comes from (3.17) and the monotonicity formula; the lower bound by  $8s_+^2 \pi$  follows from the  $C_{loc}^2$  convergence of  $Q_{\varepsilon_n}(\varepsilon_n x)$  to  $Q(x)$ , the asymptotic behavior of  $Q(x)$  for large  $|x|$ , and the monotonicity formula as well.

- (3) For any  $0 < r_1 < r_2 \leq 3$ , we have

$$(4.72) \quad \lim_{k \rightarrow \infty} \int_{\{r_1 \leq |x| \leq r_2\}} \frac{1}{|x|} \left| \frac{\partial P_k}{\partial r} \right|^2 dx = 0$$

This property follows from Property 2 and the monotonicity formula (2.9).

(4) For any  $r > 0$ , it holds that

$$\lim_{k \rightarrow \infty} \sup_{x \in B_3 \setminus B_r} \text{dist}(P_k(x), \mathcal{N}) = 0.$$

This follows from (3.19) and  $R_k \rightarrow \infty$ .

All these properties enable us to exploit similar arguments as in the proofs of Lemma 3.3 and Theorem 4.1 to get the strong  $H_{loc}^1$  convergence of  $P_k$  to  $\bar{P}$ , where  $\bar{P} \in H_{loc}^1(B_3, \mathcal{N})$  is a homogeneous minimizing harmonic map of degree 1. In addition  $\bar{P}$  has the form

$$\bar{P} = s_+(m(x) \otimes m(x) - \frac{1}{3}\text{Id}), \quad m = T_m \left( \frac{x}{|x|} \right) \text{ for some } T_m \in O(3).$$

We now apply [43, Proposition 9] to get

$$\|P_k - \bar{P}\|_{C^2(B_2 \setminus B_1)} \leq \frac{C}{R_k^2}, \quad \text{for some constant } C.$$

Consequently, we calculate

$$\begin{aligned} R_k^4 \int_{\{R_k \varepsilon_k \leq |x| \leq 2R_k \varepsilon_k\}} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_k}}{\partial r} \right|^2 dx &= R_k^4 \int_{B_2 \setminus B_1} \frac{1}{|x|} \left| \frac{\partial P_k}{\partial r} \right|^2 dx \\ &= R_k^4 \int_{B_2 \setminus B_1} \frac{1}{|x|} \left| \frac{\partial(P_k - \bar{P})}{\partial r} \right|^2 dx \leq C, \end{aligned}$$

which contradicts with (4.70). The proof is complete. □

*Proof of Theorem 4.4.* We prove by contradiction. Assume the conclusion is false, then

$$(4.73) \quad \|\Psi - s_+ \left( \frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\text{Id} \right)\|_{L^2(\mathbb{S}^2)} = \sigma > 0.$$

Because  $\Psi$  is the tangent map of  $Q$ , we can find a large enough  $r_0$ , such that

$$r_0 > \frac{2C^{1/4}}{\sigma^{1/2}}, \quad C \text{ is the constant in (4.69)}$$

$$\|Q(r_0 x) - \Psi\|_{L^2(\mathbb{S}^2)} < \frac{\sigma}{16}.$$

Also, recall that  $Q$  is obtained by taking limit of  $Q_n$  (see (3.30) and Proposition 3.5), we can find a large integer  $N_0$ , such that for any  $n \geq N_0$ ,

$$(4.74) \quad \|Q_{\varepsilon_n}(r_0 \varepsilon_n x + r_n a_n) - \Psi(x)\|_{L^2(\mathbb{S}^2)} < \frac{\sigma}{8},$$

$$(4.75) \quad \|Q_{\varepsilon_n}(rx + r_n a_n) - s_+ \left( \frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\text{Id} \right)\|_{L^2(\mathbb{S}^2)} < \frac{\sigma}{8} \quad \forall r \in [\frac{1}{2}r_n, \frac{3}{2}r_n],$$

where for (4.75) we used (3.15) and the fact that  $a_n \rightarrow 0$ , see the definition (3.28) of  $a_n$  and the discussion right after (3.28). Using (4.73), (4.74) and (4.75) we have for  $n \geq N_0$ ,  $r \in [\frac{1}{2}r_n, \frac{3}{2}r_n]$ ,

$$(4.76) \quad \|Q_{\varepsilon_n}(rx + r_n a_n) - Q_{\varepsilon_n}(r_0 \varepsilon_n x + r_n a_n)\|_{L^2(\mathbb{S}^2)} \geq \frac{3\delta}{4}.$$

We define  $k_n$  to be the largest integer such that  $k_n \leq |\log_2(\frac{r_n}{r_0\varepsilon_n})|$ . Following the same argument as in the proof of Theorem 4.2, we have

$$\begin{aligned}
& \|Q_{\varepsilon_n}(r_0\varepsilon_n x + r_n a_n) - Q_{\varepsilon_n}(2^{k_n} r_0 x + r_n a_n)\|_{L^2(\mathbb{S}^2)} \\
& \leq \sum_{j=0}^{k_n} \|Q_{\varepsilon_n}(2^j r_0 \varepsilon_n x + r_n a_n) - Q_{\varepsilon_n}(2^{j+1} r_0 \varepsilon_n x + r_n a_n)\|_{L^2(\mathbb{S}^2)} \\
& \leq \sum_{j=0}^{k_n-1} \left( \int_{\{2^j r_0 \varepsilon_n \leq |x| \leq 2^{j+1} r_0 \varepsilon_n\}} \frac{1}{|x|} \left| \frac{\partial Q_{\varepsilon_n}}{\partial r} \right|^2 dx \right)^{1/2} \\
& \leq \sum_{j=0}^{k_n-1} \frac{C^{1/2}}{4^j r_0^2} \leq \frac{2C^{1/2}}{r_0^2} < \frac{\sigma}{2}.
\end{aligned}$$

Here we have used Lemma 4.5 and  $r_0 > \frac{2C^{1/4}}{\sigma^{1/2}}$ . Note that the result of the calculation above already contradicts (4.76), which completes our proof of Theorem 4.4. □

## 5. UNIFORM CONVERGENCE OUTSIDE SHRINKING REGIONS

Using all the characterizations of the limit map  $Q$ , we can further prove the following convergence result.

**Theorem 5.1.** *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^3$  and  $Q_\varepsilon$  be a minimizer of the energy functional  $I_\varepsilon[Q]$  (2.1) with the boundary condition (2.2). For any sequence  $\varepsilon_n \rightarrow 0$ , one can find a subsequence, still denoted by  $\{\varepsilon_n\}$ , and a sequence of points  $\{x_n\}$ , such that*

- (1)  $Q_{\varepsilon_n} \rightarrow Q_*$  in  $H^1(\Omega)$ , where  $Q_*$  is a minimizer of (2.7);
- (2)  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ , where  $x_0 \in \text{Sing}(Q_*)$ ;
- (3) Let  $B_r(x_0)$  be a small neighborhood of  $x_0$  that doesn't contain other singularities of  $Q_*$ . Then for any sequence of radii  $R_n$  such that  $\lim_{n \rightarrow \infty} R_n = \infty$  and  $R_n \varepsilon_n < r$ , there holds

$$\lim_{n \rightarrow \infty} \left( \sup_{R_n \varepsilon_n \leq |x| \leq r} |Q_{\varepsilon_n}(x + x_n) - Q_*(x + x_0)| \right) = 0.$$

*Proof.* The proof will use compactness arguments similar to those that have been applied several times before. Without loss of generality, we assume  $x_0 = 0$  and  $Q_*(x) \sim \Phi(x) = s_+ \left( \frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} \text{Id} \right)$  when  $x$  approaches 0. First the existence of  $H^1$  convergent subsequence  $Q_{\varepsilon_k}$  is given in [40] (see Theorem 2.2). Taking up to a subsequence, we can find a sequence of radii  $\{r_n\}$  such that  $\{r_n, \varepsilon_n\}$  satisfy (3.12), (3.13), (3.14) and (3.15). Let  $a_n$  be as defined in (3.28) and we simply take  $x_n = r_n a_n$ . Then  $x_n \rightarrow 0$  is guaranteed by the definition and Proposition 3.2.

Now we are ready to verify the Property (3) in the theorem. We argue by contradiction. Suppose there exists a subsequence of  $\{\varepsilon_n\}$ , still denoted by  $\{\varepsilon_n\}$ , and a sequence of points  $\{y_n\}$  such that

$$(5.77) \quad \lim_{n \rightarrow \infty} \frac{|y_n|}{\varepsilon_n} = \infty, \quad |y_n| \leq r \\
|Q_{\varepsilon_n}(y_n + x_n) - Q_*(y_n)| \geq \delta > 0, \quad \text{for some constant } \delta$$



First it is obvious that (5.77) implies  $|y_n| \rightarrow 0$ , otherwise it will conflict with the uniform convergence result of  $Q_{\varepsilon_n}$  to  $Q_*$  on any compact set  $K$  that doesn't contain any point in  $\text{Sing}(Q_*)$ . Thus we get

$$(5.78) \quad \lim_{n \rightarrow \infty} |Q_*(y_n) - \Phi(y_n)| = 0.$$

Next by (3.14) we only need to consider the case  $y_n + x_n \in B_{r_n}$ . Now we define

$$Z_n(x) := Q_{\varepsilon_n}(|y_n|x + x_n).$$

Then by exactly the same argument as in the proof of Lemma 4.5 to derive the convergence of  $\{P_k\}$ , we can extract a subsequence, still denoted by  $\{Z_n\}$ , such that

$$\begin{aligned} Z_n(x) &\rightarrow \Psi(x) \text{ in } H^1(B_2) \cap C^2(B_{3/2} \setminus B_{1/2}), \\ \Psi(x) &= s_+(n \otimes n - \frac{1}{3}\text{Id}), \quad n = T \left( \frac{x}{|x|} \right) \text{ for some } T \in O(3). \end{aligned}$$

Using Lemma 4.5 and arguing in the same way as in the proof of Theorem 4.4, one can easily verify that

$$T \left( \frac{x}{|x|} \right) = \frac{x}{|x|}, \text{ i.e. } \Psi(x) = \Phi(x).$$

Therefore we have

$$(5.79) \quad \lim_{n \rightarrow \infty} |Q_{\varepsilon_n}(y_n + x_n) - \Phi(y_n)| = 0.$$

Combining (5.78) and (5.79) yields a contradiction with (5.77), which completes our proof.  $\square$

## 6. DATA AVAILABILITY STATEMENTS

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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