

Algebraic Presentations of Dependent Type Theories

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Abstract

C-systems were defined by Cartmell as models of generalized algebraic theories. B-systems were defined by Voevodsky in his quest to formulate and prove an initiality conjecture for type theories. They play a crucial role in Voevodsky’s construction of a syntactic C-system from a term monad.

In this work, we construct an equivalence between the category of C-systems and the category of B-systems, thus proving a conjecture by Voevodsky. We construct this equivalence as the restriction of an equivalence between more general structures, called CE-systems and E-systems, respectively. To this end, we identify C-systems and B-systems as “stratified” CE-systems and E-systems, respectively; that is, systems whose contexts are built iteratively via context extension, starting from the empty context.

Contents

1	Introduction	2
1.1	Initial Semantics	3
1.2	Voevodsky’s approach towards a theory of type theories	3
1.2.1	Setting the scene	3
1.2.2	B-systems for the construction of C-systems	5
1.3	Models of Type Theory	6
1.3.1	Contextual categories and C-systems	6
1.3.2	B-systems	6
1.3.3	Other Notions of Model	7
1.4	About the present work	8
1.4.1	Foundations	8
1.4.2	About our use of categories	8
1.5	Acknowledgements	9
2	Preliminaries: stratification of categories	9
2.1	Stratification of strict categories	9
2.2	Rooted trees	11
3	The category of C-systems	12
3.1	The category of C-systems	12
3.2	The category of CE-systems	15
3.3	Characterising C-systems as stratified CE-systems	17

4	The category of B-systems	20
4.1	The category of B-systems	21
4.2	The category of E-systems	26
4.2.1	Substitution systems	27
4.2.2	Weakening systems	28
4.2.3	Projection systems	30
4.2.4	The definition of E-systems	31
4.2.5	Pairing and the projections	33
4.3	Characterising B-systems as stratified E-systems	35
4.3.1	From B-systems to stratified E-systems	36
4.3.2	From stratified E-systems to B-systems	41
4.3.3	Equivalence of B-systems and stratified E-systems	44
5	Equivalence between B- and C-systems	46
5.1	From CE-systems to E-systems	46
5.2	From E-systems to CE-systems	48
5.2.1	The strict category of internal morphisms of an E-system	48
5.2.2	The interchange laws	52
5.2.3	The functor from E-systems to CE-systems	55
5.3	Equivalence between E-systems and CE-systems	56
5.4	Equivalence between B-systems and C-systems	60
6	Conclusion	61

1 Introduction

In his unfinished and only partially published [Voe15a, Voe16d, Voe16a, Voe16b, Voe17a] research programme on type theories, Voevodsky aimed to develop a mathematical theory of type theories, similar to the theory of groups or rings. In particular, he aimed to state and prove rigorously an “Initiality Conjecture” for type theories, in line with the initial semantics approach to the syntax of (programming) languages (cf. Section 1.1).

One aspect of this Initiality Conjecture is to construct, from the types and terms of a programming language, a “model”, that is, a mathematical object—typically, a category equipped with some extra structure. To help with this endeavour in the context of initial semantics for type theories, Voevodsky introduced the essentially-algebraic theory of *B-systems*. The models of this theory, he conjectured in [Voe14], are constructively equivalent to the well-known *C-systems* or *contextual categories* first introduced by Cartmell [Car86]. Furthermore, in his Templeton grant application [Voe16c], Voevodsky writes:

The theory of B-systems is conjecturally equivalent to the theory of C-systems that were introduced by John Cartmell under the name “contextual categories” in [2],[3]. Proving this equivalence is among the first goals of the proposed research.

The precise role of B-systems in Voevodsky’s programme is described in [Voe16b]; we give an overview in Section 1.2 below.

In this present work, we construct an equivalence of categories between C-systems and B-systems, each equipped with a suitable notion of homomorphism. Our construction is entirely constructive, in the sense that it does not rely on the law of excluded middle or the axiom of choice.

C- and B-systems are “stratified”, in a sense that will be defined later (in Sections 3.3 and 4.3, respectively). In this work, we also introduce unstratified structures, under the name of E-system and CE-system, respectively. We construct an adjunction between these structures, and obtain the equivalence between B- and E-systems via an equivalence of suitable subcategories. The construction

According to him, constructions and theorems about type theories are currently assumed by **analogy**. Instead, they should be proved by **specialization** of a general theorem.

His goal was thus to build a rigorous connection between, on the one hand, type theories, and, on the other hand, abstract mathematical concepts, via the notion of C-system, introduced by Cartmell [Car86] under the name of *contextual category*. Voevodsky calls a C-system equipped with extra operations corresponding to the inference rules of a type theory a **C-system model**—or just **model**—of type theory. To give semantics of type theory, Voevodsky aims to build two C-system models: (i) one from the formulas and derivations of some type theory, and (ii) one from a category of abstract mathematical objects. Furthermore, one should construct an **interpretation** (a functor) from the first to the second.

Such an interpretation typically needs to be constructed by *recursion* over the derivations of the type theory. To encapsulate the recursive pattern into a result that can then be used as a black box, the methodology of *initial semantics* suggests the following approach:

1. Show that the term model is initial in a suitable category.
2. Then, any model yields automatically a (unique) interpretation from the term model.

Now, for the construction of the two desired models, syntactic and semantic, respectively, Voevodsky developed different methodologies. For the construction of *semantic* models, Voevodsky exhibited several constructions of C-systems from universe categories [Voe15a]. He also sketched a strictification from categories with families to C-systems. For the construction of *syntactic* (or *term*) models, Voevodsky developed a framework outlined across several papers. We summarize the ingredients involved here:

1. Restricted 2-sorted binding signatures (cf. [Voe16d, Section 1]) with sorts for terms and types are used as abstract specification devices for pretypes and preterms.
2. From a restricted 2-sorted binding signature, a “term” monad $R : \mathbf{Set} \rightarrow \mathbf{Set}$ and a “type” module $LM : \mathbf{Set} \rightarrow \mathbf{Set}$ over R are constructed (cf. [Voe16d, Section 1]).
3. Any monad R on \mathbf{Set} gives rise to a C-system $C(R)$, corresponding to the mono-typed (or untyped) syntax of R , cf. [Voe16d, Section 4.2].
4. The *presheaf extension* of $C(R)$ by the module LM over R , called $C(R)[LM]$, constitutes the C-system of pretypes and preterms—but without any typing relation yet, cf. [Voe16d, Section 4.2].
5. Finally, Voevodsky’s theory of *sub-C-systems and regular quotients of C-systems* [Voe16b] allows one to carve out C-systems of types and well-typed terms modulo a convertibility relation.

In the following, we discuss some of these ingredients in slightly more detail, but without any rigorous definitions.

We start with giving an example of a restricted 2-sorted binding signature.

Example 1.1. *An example of a 2-sorted binding signature is the following signature for the raw syntax of the Calculus of Constructions, adapted from Streicher’s Semantics of Type Theory [Str91]:*

A, B	$::=$	$\Pi(A, x.B)$	<i>Product of types</i>
		$Prop$	<i>Type of propositions</i>
		$Proof(t)$	<i>Type of proofs of proposition t</i>
t, u	$::=$	x	<i>Variable</i>
		$\lambda(A, x.t)$	<i>Function abstraction</i>
		$App(A, x.B, t, u)$	<i>Function application</i>
		$\forall(A, x.t)$	<i>Universal quant. over propositions t</i>

This signature specifies a language with two sorts, the sort `type` of “types” and the sort `term` of “terms”. It is restricted because there is no binding of variables of sort `type`, only of variables of sort `term`. Such a signature yields a monad $T : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set} \times \mathbf{Set}$,

$$(X, Y) \mapsto (\text{type}(X, Y), \text{term}(X, Y)).$$

From such a monad on $\mathbf{Set} \times \mathbf{Set}$, Voevodsky [Voe16d] constructs a monad $R = \text{term}$ on \mathbf{Set} , and a module $LM = \text{type}$ over R . Here, the action of the module LM is substitution of term expressions in type expressions. From R and LM , in turn, Voevodsky [Voe16d] constructs two C-systems, called $C(R)$ and $C(R)[LM]$, respectively. The C-system $C(R)$ corresponds to a mono-typed syntax of just terms—in detail:

1. Objects are natural numbers (untyped contexts).
2. Morphisms $m \rightarrow n$ are maps $[n] \rightarrow R([m])$, where $[k]$ is the standard finite set associated to $k \in \mathbb{N}$.
3. The category thus obtained is the opposite of the Kleisli category on R restricted to natural numbers.³
4. The pullback operation extends a substitution map $f : [n] \rightarrow R([m])$ by a variable.

The C-system $C(R)[LM]$, in turn, looks as follows:

1. $C(R)[LM]$ has, as contexts, finite sequences of types (with a suitable number of free variables).
2. Pullback is given by substitution of terms in type expressions.
3. There is no typing relationship yet: $C(R)[LM]$ is a C-system of pretypes and preterms.

In order to build, from $C(R)[LM]$, a C-system of types and well-formed terms, with the intended typing relation, Voevodsky devised (i) sub-C-systems (for eliminating ill-formed pretypes and preterms), (ii) quotients of C-systems (for considering terms and types modulo judgemental equality). To construct such subsystems and quotients, Voevodsky devised the theory of B-systems.

1.2.2 B-systems for the construction of C-systems

Intuitively, the idea is to use the C-system $C(R)[LM]$ to obtain the pretypes and preterms to formulate **judgements**:

- A statement $\Gamma \vdash$ is an element of

$$B(R, LM) := \prod_{n \geq 0} \prod_{i=0}^{n-1} LM([i]) \tag{1}$$

- A statement $\Gamma \vdash t : T$ is an element of

$$\tilde{B}(R, LM) := \prod_{n \geq 0} \left(\prod_{i=0}^{n-1} LM([i]) \times RR([n]) \times LM([n]) \right) \tag{2}$$

Voevodsky [Voe14] defines eight operations on B and \tilde{B} , corresponding to structural rules of type theory. The resulting mathematical structure is captured by the notion of **B-system**, illustrated in more detail in Section 1.3 and studied in detail in Section 4.

Given a C-system C , we call $B(C)$ and $\tilde{B}(C)$ the B-sets associated to C . Voevodsky [Voe16b] constructed a bijection between

³Put differently, it is the Kleisli category of the Jf -relative monad induced by the monad R , as indicated by the title of Voevodsky’s article [Voe16d].

1. Sub-C-systems of a given C-system
2. Subsets of $(B, \tilde{B}(C))$ that are closed under the eight operations

and similar, but more complicated, for quotients. This bijection is used by Voevodsky to construct suitable C-systems; Voevodsky himself [Voe14] positions B-systems as follows:

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

This concludes our overview of the use of B-systems in Voevodsky’s research program. In the remainder of the introduction, we provide more intuition for the notions of B-system and C-system, before giving rigorous definitions and constructions.

1.3 Models of Type Theory

When studying type theories mathematically, one question to answer is: what is the mathematical notion capturing the essential behaviour of type theories? Technically speaking: what are the objects in the category of models of a type theory?

Many different answers have been given to this question. The purpose of this section is to present the two contenders studied and compared in this work, and to relate them to other notions of “model”.

1.3.1 Contextual categories and C-systems

Contextual categories were defined, by Cartmell [Car86, §14], as a mathematical structure for the interpretation of generalized algebraic theories and of the judgements of Martin-Löf type theory. A contextual category comes with a tree structure, in particular, a partial ordering, on its objects; think of the objects of \mathcal{C} as “contexts”, and $\Gamma \leq \Delta$ stating that Γ can be obtained from Δ by truncation. Furthermore, there is a special class of morphisms, closed under pullback along arbitrary morphisms—thought of as substitution by that morphism. In his PhD dissertation [Car78, Section 2.4], Cartmell shows that the category of contextual categories and homomorphisms between them is equivalent to the category of generalized algebraic theories and (equivalence classes of) interpretations between them.

Voevodsky defined C-systems as a mild variant of contextual categories: a C-system is a category coming, in particular, with a length function and a compatible “father” function on objects of the category, signifying truncation of contexts. Again, we have a class of morphisms stable under pullback. Voevodsky rejected the name “contextual category” for these mathematical object, for the reason that the extra structure on top of the underlying category cannot be transported along equivalence of categories and is thus not “categorical” in nature. As an example, consider the terminal category: it can be equipped with exactly one C-system structure. However, there is no C-system structure on any category with more than one, but finitely many, objects.

More recently, Cartmell [Car18] gave two Generalized Algebraic axiomatizations of contextual categories, one of which is using Voevodsky’s s -operator [Voe16b, Definition 2.3] for pullbacks.

1.3.2 B-systems

Voevodsky’s definition of B-systems [Voe14] is inspired by the presentation of type theories in terms of *inference rules*. Specifically, type theories “of Martin-Löf genus” are given by sets of five kinds of judgements:

Well-formed context

$\Gamma \vdash$

Well-formed type in some context

$$\Gamma \vdash A \text{ type}$$

Well-formed term of some type in some context

$$\Gamma \vdash a : A$$

Equality of types

$$\Gamma \vdash A \equiv B$$

Equality of terms

$$\Gamma \vdash a \equiv b : A$$

Interpreting equality of types and terms as actual equality, and expressing $\Gamma \vdash A$ instead as $\Gamma, A \vdash$, this lead Voevodsky to defining a B-system to consist of families of sets $(B_n)_{n \in \mathbb{N}}$ and $(\tilde{B}_n)_{n \in \mathbb{N}}$, intuitively denoting, for any $n \in \mathbb{N}$, contexts of length n and terms in a context of length $n-1$, together with their types, respectively. Furthermore, any B-system has various operations on B and \tilde{B} , such as maps $\partial_n : \tilde{B}_{n+1} \rightarrow B_{n+1}$ specifying, intuitively, for each “term” $t \in \tilde{B}_{n+1}$, the context $\partial_n(t) \in B_{n+1}$ in which t lives.

Voevodsky’s B-systems are very similar to the algebras of the theory *MetaGAT* defined by John Cartmell [Car14], and to the algebras of a monad studied by Richard Garner [Gar15]. Below, we will indicate more precise connections to Garner’s work.

1.3.3 Other Notions of Model

There are many other mathematical structures for the interpretation of type theory. Here, we give some pointers to related literature.

Voevodsky sketched a relation between C-systems and categories with families in his Lectures in the Max Planck Institute in Bonn [Voe, Lecture 5], identifying C-systems as categories with families with a particular property. In the present work, we introduce and study unstratified categorical structures, in the form of E-systems and CE-systems, which we anticipate will be useful in giving a precise construction for Voevodsky’s conjecture.

Categories with families, in turn, are related to categories with attributes (a.k.a. split type categories) in [Bla91] (in a categorical setting) and in [ALV18] (in the univalent setting). Composing these characterizations with the equivalence presented here provides a comparison between B-systems and other mathematical structures for type theory.

Garner [Gar15] studies and compares two structures related to Voevodsky’s B-systems: Generalized Algebraic Theories (GATs) and algebras for a monad on the category of type-and-term structures (see also Examples 4.4 and 4.10).

Remark 1.2. Garner’s and Cartmell’s works, taken together, also point to another possible way to constructing an equivalence between C-systems and B-systems: Cartmell [Car78, Section 2.4] constructs an equivalence of categories between the category of contextual categories and homomorphisms between them, and the category of GATs and (equivalence classes of) interpretations between them. Garner [Gar15] constructs an equivalence of categories between the category of B-frames and the category of \emptyset -GATs (GATs without structural rules) (see also Example 4.4). Garner’s equivalence looks like it could be “upgraded” to an equivalence between the category of B-systems and the category of GATs (see also Example 4.10). Constructing an equivalence between B- and C-systems in this way is, however, both cumbersome (as witnessed by Cartmell’s efforts) and, more importantly, conceptually circular — if not in actuality, then at least in spirit. After all, B- and C-systems were studied by Voevodsky in the context of the Initiality Conjecture; one purpose of the Initiality Conjecture is to give a **specification of dependently-typed syntax**. We believe that such a specification is best given without recourse to such syntax itself.

1.4 About the present work

The main result of this paper is the construction of an equivalence of categories, between the category of C-systems and the category of B-systems. The existence of such an equivalence was conjectured by Voevodsky.

We construct this equivalence as a restriction of an equivalence between more general, unstratified structures introduced in this paper, called CE-systems and E-systems, respectively. While it is not *necessary* to pass via E-systems and CE-systems to construct an equivalence between B-systems and C-systems, it seems *desirable* to us for two reasons:

1. The definitions and constructions are automatically more modular, isolating structure on either side that corresponds to each other.
2. The study of unstratified structures is useful in connecting B-systems and C-systems to other unstratified structures, such as categories with families [Dyb96]. Work on constructing a suitable comparison is already underway.

This paper is organized as follows. In Section 2 we discuss some prerequisites that we build upon in later sections. In Section 3 we review the definition of C-systems given by Voevodsky in [Voe16b, Def. 2.1], itself a mild variant of Cartmell’s definition of contextual categories [Car86, §14]. Here we also introduce CE-systems. In Section 4, we give a variant of Voevodsky’s definition of B-system [Voe14] and introduce E-systems. In Section 5 we construct our equivalence of categories between B-systems and C-systems.

1.4.1 Foundations

The work described in this result can be read to take place in intuitionistic set theory (IZF) or extensional type theory, i.e., a type theory with equality reflection. In particular, we do not make use of classical reasoning principles such as an axiom of choice or excluded middle. We consider in this work categories built from algebraic structures (which sometimes are themselves categories with structure, but see Section 1.4.2). Implicitly, we take these algebraic structures to be built from sets (or types) from a universe \mathcal{U}_1 . The categories of such structures are hence categories built from sets (or types) of objects and morphisms of a universe \mathcal{U}_2 . In the following, we leave the universe levels implicit.

1.4.2 About our use of categories

In this work, categories are used on two different levels.

Firstly, we use categories as algebraic structures, as the basis for C-systems and CE-systems. This use of categories is somewhat “accidental”, and our constructions on these categories are not invariant under equivalence of categories. In particular, we liberally reason about equality of objects in such categories. Consequently, we avoid the unadorned word “category” for these gadgets, and call them *strict categories* instead. We denote by **Cat** the category of strict categories and functors between them.

Secondly, we use categories to compare different mathematical structures to each other, by considering a suitable category of such structures and their homomorphisms. Here, we never consider equality, but only isomorphism, of such mathematical structures; our reasoning on that level is entirely categorical. We reserve the word “category” for such uses of the concept.

We use different fonts for strict categories and categories, respectively: calligraphic font, such as \mathcal{C} , is used for strict categories; boldface, such as **Grph**, is used for categories. We use the same notation for arrows in strict categories and in categories. We write either $g \circ f$ or gf for the composition of $f: a \rightarrow b$ and $g: b \rightarrow c$.

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2 Preliminaries: stratification of categories

In this section we collect definitions and results related to **stratification of strict categories** and morphisms between them. A stratification (see Definition 2.1) associates, to any object of a strict category a natural number, its “length”, and to any length-decreasing morphism a factorization of this morphism into morphisms “of length 1”. Such a stratification can equivalently be described as a rooted tree, see Section 2.2.

2.1 Stratification of strict categories

Definition 2.1 (Stratified strict categories, stratified functors). Let \mathcal{C} be a strict category with terminal object 1. A **stratification** for \mathcal{C} consists of a *stratification functor*

$$L : \mathcal{C} \rightarrow (\mathbb{N}, \geq)$$

such that

1. $L(X) = 0$ if and only if X is the chosen terminal object 1,
2. for any $f : X \rightarrow Y$ we have $L(X) = L(Y)$ if and only if $X = Y$ and $f = \text{id}_X$, and
3. every morphism $f : X \rightarrow Y$ in \mathcal{C} , where $L(X) = n + m + 1$ and $L(Y) = n$, has a unique factorization

$$X = X_{m+1} \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = Y$$

where $L(X_i) = n + i$.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between strict categories with stratifications $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$, respectively, is said to be **stratified** if $L_{\mathcal{C}} = L_{\mathcal{D}} \circ F$.

Remark 2.2. We emphasize that stratifications do not transport along equivalence of categories. For instance, there is a (necessarily unique, see Proposition 2.8) stratification on the terminal strict category, but no stratification on the strict chaotic category with two objects and a choice of terminal object.

Remark 2.3. Those readers familiar with Conduché functors might note that a stratified category (\mathcal{C}, L) is equivalently a Conduché functor $L : \mathcal{C} \rightarrow \mathbb{N}$ with discrete fibers which takes the chosen terminal object of \mathcal{C} to 0.

Definition 2.4. Let \mathcal{C} be a category with a terminal object and $\ell : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}$ a function. An arrow $f : X \rightarrow Y$ in \mathcal{C} is **individual** if $\ell(X) = \ell(Y) + 1$.

Remark 2.5.

1. In a stratified category, there is a unique terminal object 1. More generally, if there is an arrow $1 \rightarrow X$, then $X = 1$.
2. The factorisation of an arrow $f : X \rightarrow Y$ such that $L(X) - L(Y) = m + 1 > 0$ in 2.1.3 consists of $m + 1$ individual arrows.
3. A stratified functor is determined by its action on individual arrows.

Lemma 2.6. *Let \mathcal{C} be a category with a terminal object 1. A function $\ell: \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}$ lifts to a stratification $L: \mathcal{C} \rightarrow (\mathbb{N}, \geq)$ of \mathcal{C} if and only if the following three conditions hold:*

(i) $\ell(1) = 0$,

(ii) *for every object X and $k \leq \ell(X)$, the set*

$$\coprod_{Y \mid \ell(Y)=k} \mathcal{C}(X, Y)$$

is a singleton, i.e. there is a unique arrow $x_k: X \rightarrow X_k$ such that $\ell(X_k) = k$, and

(iii) *for every X and $k > \ell(X)$, the set*

$$\coprod_{Y \mid \ell(Y)=k} \mathcal{C}(X, Y)$$

is empty, i.e. there are no arrows $X \rightarrow Y$ such that $\ell(X) < \ell(Y)$.

Proof. If \mathcal{C} is stratified by L such that $L(X) = \ell(X)$, condition (i) follows from 2.1.1. To show condition (ii), note that every arrow $X \rightarrow 1$ factors uniquely into $l = \ell(X)$ individual arrows

$$X \xrightarrow{x_l} X_{l-1} \xrightarrow{x_{l-1}} \cdots \xrightarrow{x_2} X_1 \xrightarrow{x_1} 1.$$

In particular, for every $n \leq \ell(X)$, the composite $x_{n+1} \cdots x_l: X \rightarrow X_n$ is such that $\ell(X_n) = n$. If $f: X \rightarrow Y$ is also such that $\ell(Y) = n$, then f factors into $l - n$ individual arrows $(f_i)_{i=1}^{l-n}$. Let $y_1 \cdots y_n$ be the factorisation of $Y \rightarrow 1$ into individual arrows. The composite $y_1 \cdots y_n f_0 \cdots f_{l-n-1}$ is a factorisation of $X \rightarrow 1$ into individual arrows. It follows by uniqueness of such factorisations that

$$x_1 = y_1, x_2 = y_2, \dots, x_n = y_n, x_{n+1} = f_1, \dots, x_l = f_{l-n}.$$

In particular, $f = x_{n+1} \cdots x_l$ as required.

Since (\mathbb{N}, \geq) is a poset, condition (iii) is equivalent to the fact that the function ℓ extends uniquely to a functor $L: \mathcal{C} \rightarrow (\mathbb{N}, \geq)$.

Suppose now that conditions (i–iii) above hold. In particular, the function ℓ extends to a functor L .

(1) Let X be such that $L(X) = 0$. Then $X \rightarrow 1$ and $\text{id}_X: X \rightarrow X$ are both such that $\ell(X) = 0 = \ell(1)$. Hence $X = 1$ and the object X is terminal. Conversely, let X be terminal. Then there is $1 \rightarrow X$ and thus $0 \geq L(X)$.

(2) Let $f: X \rightarrow Y$ and suppose $L(X) = L(Y)$, then $Y = X$ and $f = \text{id}_X$ by (ii) with $n = \ell(X)$.

(3) For every X such that $n + 1 = \ell(X) > 0$, let $\overline{X}: X \rightarrow X'$ be the unique arrow such that $\ell(X') = n$ given by (ii). For every $k \leq \ell(X)$, we have a composite x_k of k individual arrows

$$X \xrightarrow{\overline{X}} X' \xrightarrow{\overline{X'}} \cdots \xrightarrow{\overline{X^{(k-2)}}} X^{(k-1)} \xrightarrow{\overline{X^{(k-1)}}} X^{(k)} \quad (3)$$

$\xrightarrow{x_k}$

where $\ell(X^{(k)}) = \ell(X) - k$, which is the unique arrow $X \rightarrow Y$ such that $\ell(Y) = \ell(X) - k$ by (ii). Let us show that (3) is also the unique factorisation of x_k into individual arrows, for every $0 < k \leq \ell(X)$. We proceed by induction on n . If $n = 0$, then factorisations consist of only one individual arrow and uniqueness follow from (ii). For $n > 0$, let $0 < k \leq n + 1$ and consider a factorisation $X \xrightarrow{g_0} Z_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{k-2}} Z_{k-1} \xrightarrow{g_{k-1}} X^{(k)}$ of x_k into individual arrows. Then $\ell(Z_1) = \ell(X) - 1 = \ell(X')$, and so $g_0 = \overline{X}$ by (ii). Again, $g_{k-1} \cdots g_1 = x'_{k-1}: X' \rightarrow (X')^{(k-1)}$ by (ii) and, by inductive hypothesis, $g_i = \overline{X^{(i)}}$ for $0 < i < k$. Therefore (3) is the unique factorisation of x_k into individual arrows.

Given an arrow $f: X \rightarrow Y$ such that $n = \ell(Y) < \ell(X) = m + n + 1$, it must be $Y = X^{(m+1)}$ and $f = x_{m+1}$ by (ii). It follows that f factors uniquely into individuals as $\overline{X^{(m)}} \cdots \overline{X'} \overline{X}$. \square

Remark 2.7. Condition (ii) in Lemma 2.6 is equivalent to requiring that, for every object X :

(ii.a) for every $n \leq \ell(X)$ there is at most one arrow $f: X \rightarrow Y$ such that $\ell(Y) = n$, and

(ii.b) there is an individual arrow $\bar{X}: X \rightarrow X'$.

One direction is clear. For the converse it is enough to show that for every $n < \ell(X)$ there is $f: X \rightarrow Y$ such that $\ell(Y) = n$. Such an arrow is given as the composite of $\ell(X) - n$ individual arrows as in Eq. (3) above.

Proposition 2.8. *Any category can be stratified in at most one way.*

Proof. Consider a category \mathcal{C} with two stratifications $L, M: \mathcal{C} \rightarrow \mathbb{N}$. By Lemma 2.6.ii,

$$L^{-1}(n+1) = \{X \mid \exists Y \in L^{-1}(n) \text{ and an individual } f: X \rightarrow Y\}$$

and similarly for $M^{-1}(n+1)$. Thus, if $L^{-1}(n) = M^{-1}(n)$, we find that $L^{-1}(n+1) = M^{-1}(n+1)$, and the claim follows by induction since $L^{-1}(0) = M^{-1}(0)$ by Definition 2.1.1. \square

Uniqueness of stratification justifies the following definition:

Definition 2.9. We define \mathbf{Cat}_s to be the subcategory of \mathbf{Cat} consisting of stratified strict categories and stratified functors between them.

Lemma 2.10. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between stratified categories. The following are equivalent.*

1. *The functor F is stratified.*
2. *The functor F preserves terminal objects and individual arrows.*

Proof. That 1 implies 2 is clear. The converse is by induction on the length of objects using that, for every $f: X \rightarrow Y$, $L_{\mathcal{C}}(X) = L_{\mathcal{C}}(Y) + 1$ implies $L_{\mathcal{D}}(F(X)) = L_{\mathcal{D}}(F(Y)) + 1$. \square

Lemma 2.11. *Let \mathcal{C} be a stratified category with stratification functor L . Then for every object X and every $f: Y \rightarrow X$,*

$$L_X(f) := L(Y) - L(X)$$

defines a stratification functor L_X for the slice \mathcal{C}/X .

Proof. The above clearly defines a functor $L_X: \mathcal{C}/X \rightarrow (\mathbb{N}, \geq)$ and conditions (1–3) in Definition 2.1 are easily verified. \square

Corollary 2.12. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a stratified functor between stratified categories. For every object X in \mathcal{C} , the functor*

$$\mathcal{C}/X \xrightarrow{F/X} \mathcal{D}/FX$$

is stratified.

2.2 Rooted trees

In this section, we compare stratified categories to rooted trees. Rooted trees were used by Cartmell [Car86] to give his original definition of contextual categories.

Definition 2.13.

1. We define a **rooted tree** T to be a family of sets $(T_n)_{n \in \mathbb{N}}$ indexed by the natural numbers such that T_0 is a singleton, together with functions $(t_n: T_{n+1} \rightarrow T_n)_{n \in \mathbb{N}}$ mapping a node to its parent. A homomorphism of rooted trees $f: T \rightarrow S$ is a family of functions $(f_n: T_n \rightarrow S_n)_{n \in \mathbb{N}}$ such that $f_n \circ t_n = s_n \circ f_{n+1}$ for every $n \in \mathbb{N}$. Let \mathbf{RtTr} be the category of rooted trees and homomorphisms.

2. Let \mathbf{Grph} be the category of directed (multi)graphs and homomorphisms. We define the functor $\mathbf{G} : \mathbf{RtTr} \rightarrow \mathbf{Grph}$ as follows. For a rooted tree T , the directed graph $\mathbf{G}(T)$ has the set of vertices given by the disjoint union $\coprod_{n \in \mathbb{N}} T_n$, and there is an edge $(n+1, X) \rightarrow (n, t_n(X))$ for every $n \in \mathbb{N}$ and $X \in T_{n+1}$. It is straightforward to verify that each homomorphism $T \rightarrow T'$ of rooted trees gives rise to a homomorphism of graphs $\mathbf{G}(T) \rightarrow \mathbf{G}(T')$.
3. Let $\mathbf{F} : \mathbf{Grph} \rightarrow \mathbf{Cat}$ be the well-known functor [Mac98, II.7] that takes a graph to the category freely generated by it.

We now show that the image of the composite

$$\mathbf{RtTr} \xrightarrow{\mathbf{G}} \mathbf{Grph} \xrightarrow{\mathbf{F}} \mathbf{Cat}$$

is the subcategory of *stratified strict categories* defined in Definition 2.9.

Proposition 2.14. *The functor $\mathbf{FG} : \mathbf{RtTr} \rightarrow \mathbf{Cat}$ lifts to an equivalence $\mathbf{RtTr} \xrightarrow{\cong} \mathbf{Cat}_s$.*

Proof. First, observe that, for a rooted tree T , the free category $\mathbf{FG}(T)$ is stratified. We define $L : \mathbf{FG}(T) \rightarrow (\mathbb{N}, \geq)$ by sending an object (n, X) to n and a generating morphism $(n+1, X) \rightarrow (n, t_n(X))$ to $n+1 \geq n$. Given a morphism $f : S \rightarrow T$ of rooted trees, the functor $\mathbf{FG}(f) : \mathbf{FG}(S) \rightarrow \mathbf{FG}(T)$ is stratified by construction. Thus, the functor $\mathbf{FG} : \mathbf{RtTr} \rightarrow \mathbf{Cat}$ lifts to an functor $\mathbf{RtTr} \rightarrow \mathbf{Cat}_s$.

Next we define a functor $\mathbf{I} : \mathbf{Cat}_s \rightarrow \mathbf{RtTr}$. Consider a stratified category (\mathcal{C}, L) and define a rooted tree $\mathbf{I}(\mathcal{C}, L)$ as follows. Let $\mathbf{I}(\mathcal{C}, L)_n := L^{-1}(n)$. By Lemma 2.6, for every $X \in \mathbf{I}(\mathcal{C}, L)_n$ there is exactly one individual arrow, say $X \rightarrow X'$. Then we define $t_n : \mathbf{I}(\mathcal{C}, L)_{n+1} \rightarrow \mathbf{I}(\mathcal{C}, L)_n$ by $t_n(X) := X'$. A stratified functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a homomorphism of rooted trees $\mathbf{I}(\mathcal{C}, L) \rightarrow \mathbf{I}(\mathcal{D}, M)$ since it commutes with the stratification functors and it preserves individual arrows.

It is now straightforward to verify that $\mathbf{FG} \circ \mathbf{I} \cong 1_{\mathbf{Cat}_s}$ and $\mathbf{I} \circ \mathbf{FG} \cong 1_{\mathbf{RtTr}}$. \square

3 The category of C-systems

This section is dedicated to the study of C-systems.

In Section 3.1 we review Voevodsky’s definition of C-system, itself a mild variant of Cartmell’s contextual categories. We then give, in Section 3.2 our definition of CE-system, and identify, in Section 3.3, the category of C-systems as a subcategory of “stratified” objects in the category of CE-systems.

3.1 The category of C-systems

John Cartmell [Car86, Section 14] defined *contextual categories* as mathematical gadgets for the interpretation of type theories. Vladimir Voevodsky [Voe16b, Definition 2.1] gave a slightly modified, but obviously equivalent definition, and coined them *C-systems*.

Definition 3.1 (C-system, [Voe16b, Def. 2.1]). A **C-system** consists of

1. a strict category \mathcal{C} ,
2. a “length” function $\ell : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}$,
3. a chosen object $1 \in \text{Ob}(\mathcal{C})$,
4. a function $\text{ft} : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$,
5. for any object $\Gamma \in \text{Ob}(\mathcal{C})$ such that $\ell(\Gamma) > 0$, a morphism $\text{p}_\Gamma : \Gamma \rightarrow \text{ft}(\Gamma)$,
6. for any $\Gamma \in \text{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$ and any $f : \Delta \rightarrow \text{ft}(\Gamma)$, an object $f^*\Gamma$ and a morphism $\text{q}(f, \Gamma) : f^*\Gamma \rightarrow \Gamma$.

satisfying the following axioms:

- i) $\ell^{-1}(0) = \{1\}$,
- ii) for Γ with $\ell(\Gamma) > 0$, we have $\ell(\text{ft}(\Gamma)) = \ell(\Gamma) - 1$,
- iii) $\text{ft}(1) = 1$,
- iv) 1 is a final object,
- v) for $\Gamma \in \text{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$ and $f : \Delta \rightarrow \text{ft}(\Gamma)$, one has $\ell(f^*\Gamma) > 0$, $\text{ft}(f^*\Gamma) = \Delta$, and the square

$$\begin{array}{ccc}
 f^*\Gamma & \xrightarrow{\mathfrak{q}(f,\Gamma)} & \Gamma \\
 \mathfrak{p}_{f^*(\Gamma)} \downarrow & & \downarrow \mathfrak{p}_\Gamma \\
 \Delta & \xrightarrow{f} & \text{ft}(\Gamma)
 \end{array} \tag{4}$$

commutes and is a pullback square,

- vi) for $\Gamma \in \text{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$ one has $(\text{id}_{\text{ft}(\Gamma)})^* \Gamma = \Gamma$ and $\mathfrak{q}(\text{id}_{\text{ft}(\Gamma)}, \Gamma) = \text{id}_\Gamma$, and
- vii) for $\Gamma \in \text{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$, $g : \Delta \rightarrow \text{ft}(\Gamma)$ and $f : E \rightarrow \Delta$, we have $(g \circ f)^*\Gamma = f^*g^*\Gamma$ and $\mathfrak{q}(g \circ f, \Gamma) = \mathfrak{q}(g, \Gamma) \circ \mathfrak{q}(f, g^*\Gamma)$.

Intuitively, an object Γ of the category underlying a C-system can be thought of as a context of length $\ell(\Gamma)$. Types do not appear explicitly in the definition of C-system; however, intuitively, the types in context Γ are all those contexts Δ with $\text{ft}(\Delta) = \Gamma$ (hence, in particular, $\ell(\Delta) = \ell(\Gamma) + 1$). Similarly, terms are not explicitly given; a term in context $\text{ft}(\Gamma)$ corresponds to a section to a map $\mathfrak{p}_\Gamma : \Gamma \rightarrow \text{ft}(\Gamma)$. This is exactly how terms are defined in the definition of an E-system from a CE-system, see Construction 5.6.

The length function ℓ does not lift in general to a stratification on \mathcal{C} . However, it does on a suitable subcategory, see Corollary 3.9.

Definition 3.2. A **morphism of C-systems** from \mathbb{C} to \mathbb{D} is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between the underlying categories that strictly preserves the rest of the structure, that is:

- i) $F(1_{\mathbb{C}}) = 1_{\mathbb{D}}$,
- ii) $\ell_{\mathbb{D}} \circ \text{Ob}(F) = \ell_{\mathbb{C}} : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}$,
- iii) $\text{Ob}(F) \circ \text{ft}_{\mathbb{C}} = \text{ft}_{\mathbb{D}} \circ \text{Ob}(F) : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$,
- iv) $F\mathfrak{p}_\Gamma = \mathfrak{p}_{F\Gamma}$, for every $\Gamma \in \text{Ob}(\mathcal{C})$,
- v) $F(f^*\Gamma) = (Ff)^*(F\Gamma)$ and $F(\mathfrak{q}(f, \Gamma)) = \mathfrak{q}(Ff, F\Gamma)$, for every $\Gamma \in \text{Ob}(\mathcal{C})$ such that $\ell_{\mathbb{C}}(\Gamma) > 0$ and $f : \Delta \rightarrow \text{ft}(\Gamma)$.

Example 3.3 (C-systems and Lawvere theories). *Fiore and Voevodsky [FV20] construct an isomorphism of categories between the category of Lawvere theories and the category of ℓ -bijective C-systems, that is, of C-systems whose length function is a bijection. Intuitively, such a C-system can be seen as modelling an untyped (or single-sorted) language.*

Example 3.4 (C-systems and contextual categories). *C-systems are equivalent to Cartmell's contextual categories. In his Ph.D. dissertation, Cartmell [Car78, Section 2.4] constructs an equivalence between the category of contextual categories and homomorphisms between them and the category of Generalized Algebraic Theories (GATs) and (equivalence classes of) interpretations between them. Hence C-systems are equivalent to GATs.*

Example 3.5 (C-system from a universe category). *Any universe category gives rise to a C-system, via a construction by Voevodsky [Voe15a, Construction 2.12]. A universe category is a category with a chosen terminal object and a universe, that is, a morphism $p : \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ together with a choice of pullback of p along any morphism $X \rightarrow \mathcal{U}$. Roughly, the C-system constructed from a universe category has, as objects of length n , sequences of n morphisms f_1, \dots, f_n into \mathcal{U} such that the domain of f_{i+1} is the chosen pullback of p along f_i . Such a sequence can be thought of as a sequence of (dependent) types (A_1, A_2, \dots, A_n) such that $A_1, \dots, A_i \vdash A_{i+1}$. Furthermore, any small C-system can be obtained via this construction; given a C-system \mathbb{C} , a universe can be constructed ([Voe15a, Construction 5.2]) on the presheaf category $\hat{\mathbb{C}}$ such that the C-system obtained from that universe is isomorphic to the C-system \mathbb{C} . For a brief overview of these constructions, see [KL21, Section 1.3].*

Voevodsky's simplicial model of univalent foundations [KL21] is built on top of a C-system obtained from a universe in the category of simplicial sets.

Problem 3.6. *To construct a functor $\mathbf{C2RtTr} : \mathbf{Csys} \rightarrow \mathbf{RtTr}$.*

Construction 3.7 (for Problem 3.6). Let $\mathbb{C} = (\mathcal{C}, 1, \ell, \text{ft}, \text{p}, \dots)$ be a C-system. The objects of \mathcal{C} can be arranged into a rooted tree by defining

$$T_n := \{\Gamma \mid \ell(\Gamma) = n\} \quad \text{and} \quad t_n(\Gamma) := \text{ft}(\Gamma) \in T_n, \text{ for } \Gamma \in T_{n+1}.$$

That is, the front square in the diagram of sets and functions

$$\begin{array}{ccccc} T_{n+1} & \xleftarrow{\quad} & \text{Ob}(\mathcal{C}) & & \\ \downarrow & \dashrightarrow^{t_n} & \downarrow & \searrow^{\text{ft}} & \\ & & T_n & \xleftarrow{\quad} & \text{Ob}(\mathcal{C}) \\ & & \downarrow \ell & & \downarrow \ell \\ 1 & \xrightarrow{\quad} & \mathbb{N} & \xrightarrow{\quad} & \mathbb{N} \\ & \searrow & \downarrow \text{pred} & & \\ & & 1 & \xrightarrow{\quad} & \mathbb{N} \end{array}$$

$n+1$ n

is a pullback for every $n \in \mathbb{N}$, and the function t_n is defined by its universal property as the right-hand square commutes by 3.1.ii. The set T_0 is a singleton by (i) in Definition 3.1.

A homomorphism of C-systems $F : \mathbb{C} \rightarrow \mathbb{D}$ restricts, for every $n \in \mathbb{N}$, to a function $F_n : T_n \rightarrow S_n$ between the fibres T_n and S_n of the length function of \mathcal{C} and \mathcal{D} , respectively, by 3.2.ii as in the front part of the diagram below.

$$\begin{array}{ccccccc} T_{n+1} & \xleftarrow{\quad} & \text{Ob}(\mathcal{C}) & & \text{Ob}(\mathcal{C}) & & \\ \downarrow & \dashrightarrow^{t_n} & \downarrow & \searrow^{\text{ft}} & \downarrow & & \\ F_{n+1} & & T_n & \xleftarrow{\quad} & \text{Ob}(\mathcal{C}) & & \\ & & \downarrow \text{Ob}(F) & & \downarrow \ell & & \\ S_{n+1} & \xleftarrow{\quad} & \text{Ob}(\mathcal{D}) & & \text{Ob}(\mathcal{D}) & & \\ \downarrow & \dashrightarrow^{s_n} & \downarrow & \searrow^{\text{ft}} & \downarrow & & \\ & & S_n & \xleftarrow{\quad} & \text{Ob}(\mathcal{D}) & & \\ & & \downarrow \ell & & \downarrow \ell & & \\ 1 & \xrightarrow{\quad} & \mathbb{N} & \xrightarrow{\quad} & \mathbb{N} & & \\ & \searrow & \downarrow \text{pred} & & \downarrow \ell & & \\ & & 1 & \xrightarrow{\quad} & \mathbb{N} & & \end{array} \quad (5)$$

$n+1$ n

The upper-right square commutes by Definition 3.2.iii, thus the upper-left square commutes as well since the rest of the diagram commutes. Functoriality holds since each F_n is defined by a universal property. \square

Lemma 3.8. *Let \mathbb{C} be a C-system with underlying strict category \mathcal{C} and let $\mathfrak{p}(\mathbb{C})$ denote the wide subgraph of \mathcal{C} on the canonical projections \mathfrak{p}_Γ for Γ in \mathcal{C} . Then $\mathfrak{p}(\mathbb{C})$ is isomorphic to the graph $\mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$ naturally in \mathbb{C} , where $\mathbf{G}: \mathbf{RtTr} \rightarrow \mathbf{Grph}$ is from Definition 2.13.*

Proof. The vertices of $\mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$ are pairs $(\ell(\Gamma), \Gamma)$ and edges are of the form $(\ell(\Gamma), \Gamma) \rightarrow (\ell(\mathfrak{ft}(\Gamma)), \mathfrak{ft}(\Gamma))$ for $\ell(\Gamma) > 0$. In particular, every vertex $(n+1, \Gamma)$ has exactly one outgoing edge. The bijection between vertices then extends to an isomorphism between $\mathfrak{p}(\mathbb{C})$ and $\mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$.

Every C-homomorphism $F: \mathbb{C} \rightarrow \mathbb{D}$ induces a morphism of graphs $\mathfrak{p}(F): \mathfrak{p}(\mathbb{C}) \rightarrow \mathfrak{p}(\mathbb{D})$ by 3.2.iv. Naturality then follows from 3.2.ii. \square

Corollary 3.9. *Let \mathcal{F} be the free category on the graph $\mathfrak{p}(\mathbb{C})$ on the canonical projections. Then the terminal object $\mathbf{1}$ of \mathcal{C} is terminal in \mathcal{F} and the function ℓ extends to a stratification functor on \mathcal{F} .*

Proof. By Lemma 3.8 there is an iso $\mathcal{F} = \mathbf{Fp}(\mathbb{C}) \cong \mathbf{F} \circ \mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$. The claim thus follows from Proposition 2.14. \square

3.2 The category of CE-systems

In this section, we define CE-systems and their morphisms.

Definition 3.10. A **CE-system** consists of two strict category structures \mathcal{F} and \mathcal{C} on the same set of objects $\text{Ob}(\mathcal{F}) = \text{Ob}(\mathcal{C})$ and an identity-on-objects functor $I: \mathcal{F} \rightarrow \mathcal{C}$ between them, together with

1. a chosen object $\mathbf{1}$ which is terminal in \mathcal{F} , and
2. for any $f: \Delta \rightarrow \Gamma$ in \mathcal{C} and any $A \in \mathcal{F}/\Gamma$, a functorial choice of a pullback square

$$\begin{array}{ccc} \Delta \cdot f^*A & \xrightarrow{\pi_2(f,A)} & \Gamma \cdot A \\ I(f^*A) \downarrow & & \downarrow I(A) \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

such that $f^*A \in \mathcal{F}/\Delta$. Explicitly, the functoriality requirement is that

- (a) For any $f: \Delta \rightarrow \Gamma$, one has

$$f^*(\text{id}_\Gamma) = \text{id}_\Delta \quad \text{and} \quad \pi_2(f, \text{id}_\Gamma) = f.$$

- (b) For any $A \in \mathcal{F}/\Gamma$, one has

$$(\text{id}_\Gamma)^*A = A \quad \text{and} \quad \pi_2(\text{id}_\Gamma, A) = \text{id}_{\Gamma \cdot A}$$

- (c) For any $f: \Delta \rightarrow \Gamma$, $g: \Xi \rightarrow \Delta$ and $A \in \mathcal{F}/\Gamma$, one has

$$(f \circ g)^*A = g^*(f^*A) \quad \text{and} \quad \pi_2(f \circ g, A) = \pi_2(f, A) \circ \pi_2(g, f^*A)$$

- (d) For any $P \in \mathcal{F}/\Gamma \cdot A$ and $f: \Delta \rightarrow \Gamma$, one has

$$f^*(A \cdot P) = f^*A \circ (\pi_2(f, A))^*P \quad \text{and} \quad \pi_2(f, A \cdot P) = \pi_2(\pi_2(f, A), P)$$

A CE-system is **rooted** if $I(1) = 1$ is terminal in \mathcal{C} .

For any $f : \Delta \rightarrow \Gamma$ we write f^* for the induced functor $\mathcal{F}/\Gamma \rightarrow \mathcal{F}/\Delta$ and refer to the arrows in \mathcal{F} as the **families** of the CE-system. We shall write arrows in \mathcal{F} with a double head as in the above diagram.

We may write $I_{\mathbb{A}} : \mathcal{F}_{\mathbb{A}} \rightarrow \mathcal{C}_{\mathbb{A}}$ for the functor underlying a CE-system \mathbb{A} , whenever we need to make the CE-system explicit.

We show in Section 3.3 that CE-systems generalize C-systems. To provide some intuition, we can think of \mathcal{F} as the category spanned by the projections $\text{pr} : \Gamma \rightarrow \text{ft}(\Gamma)$ of a C-system; in this case, the functor $I : \mathcal{F} \rightarrow \mathcal{C}$ is faithful.

We now give two examples of CE-system.

Example 3.11 (CE-system on finite sets). *Let \mathbb{F} be the category whose objects are natural numbers, and whose morphisms $f : m \rightarrow n$ are functions $f : \text{std}(m) \rightarrow \text{std}(n)$ from the standard finite set of m elements to the standard finite set of n elements. Consider the identity-on-objects functor $[-] : (\mathbb{N}, \geq) \rightarrow \mathbb{F}^{\text{op}}$ given, on $n + k \geq n$, by the opposite of the initial-segment inclusion, which we write $i_n^{n+k} : [n + k] \rightarrow [n]$.*

We equip it with the structure of a CE-system as follows. The chosen pullback of a family $n + k \geq n$ and an arrow $f : [m] \rightarrow [n]$ in \mathbb{F}^{op} is

$$\begin{array}{ccc} [m + k] & \xrightarrow{\pi_2(f, n+k \geq n)} & [n + k] \\ \downarrow i_m^{m+k} & & \downarrow i_n^{n+k} \\ [m] & \xrightarrow{f} & [n] \end{array}$$

where the morphism $\pi_2(f, n + k \geq n)$ is the opposite of the arrow $[f, 1_k] : [n + k] \rightarrow [m + k]$ in \mathbb{F} obtained from the universal property of the coproduct $[n + k]$. Functoriality follows immediately from the definitions.

This CE-system is, of course, rooted — as $[0]$ is terminal in \mathbb{F}^{op} — and stratified in the sense of Definition 3.15 — as initial-segment inclusions factor uniquely into individuals i_n^{n+1} . Note also that the choice of pullback squares is forced by Remark 3.18.

Definition 3.12. Let \mathbb{A} and \mathbb{B} be two CE-systems. A **CE-homomorphism** $F : \mathbb{A} \rightarrow \mathbb{B}$ consists of a commutative square of functors

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{F}}} & \mathcal{F}_{\mathbb{B}} \\ I_{\mathbb{A}} \downarrow & & \downarrow I_{\mathbb{B}} \\ \mathcal{C}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{C}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

such that,

1. $F_{\mathcal{F}}(1_{\mathbb{A}}) = 1_{\mathbb{B}}$, and
2. for every $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$ and $f : \Delta \rightarrow \Gamma$, it is

$$F_{\mathcal{F}}(f^*A) = (F_{\mathcal{C}}f)^*(F_{\mathcal{F}}A) \quad \text{and} \quad F_{\mathcal{C}}(\pi_2(f, A)) = \pi_2(F_{\mathcal{C}}f, F_{\mathcal{F}}A).$$

Remark 3.13. If F is a CE-homomorphism between rooted CE-systems \mathbb{A} and \mathbb{B} , then $F_{\mathcal{C}}(1_{\mathbb{A}}) = 1_{\mathbb{B}}$ and $F_{\mathcal{C}}$ preserves terminal objects in the usual categorical sense.

Definition 3.14. We write **CEsys** for the category of CE-systems and CE-system homomorphisms and **rCEsys** for its full subcategory on rooted CE-systems.

For the comparison of CE-systems with C-systems, the notion of stratification of a CE-system is needed:

Definition 3.15. A CE-system \mathbb{A} is **stratified** if its category of families \mathcal{F} is stratified in the sense of Definition 2.1 and, for every $f: \Delta \rightarrow \Gamma$ in \mathcal{C} , the functor

$$\mathcal{F}/\Gamma \xrightarrow{f^*} \mathcal{F}/\Delta$$

induced by the functorial choice of pullbacks is stratified with respect to the stratification induced on slices in Lemma 2.11.

A CE-homomorphism between stratified CE-systems is **stratified** if its component on families is a stratified functor.

Remark 3.16. It follows from Proposition 2.8 that CE-systems are stratified in at most one way.

Definition 3.17. We denote by $\mathbf{CEsys}_s \hookrightarrow \mathbf{CEsys}$ and $\mathbf{rCEsys}_s \hookrightarrow \mathbf{rCEsys}$ the respective subcategories spanned by *stratified* (rooted) CE-systems and *stratified* CE-homomorphisms between them.

Remark 3.18. In a stratified CE-system, for every $f: \Delta \rightarrow \Gamma$ in \mathcal{C} and $A \in \mathcal{F}/\Gamma$ it is

$$L(\Delta.f^*A) = L(\Delta) + L(\Gamma.A) - L(\Gamma).$$

Lemma 3.19. Let \mathbb{A} and \mathbb{B} be two stratified CE-system. A commuting square of functors

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{F}}} & \mathcal{F}_{\mathbb{B}} \\ I_{\mathbb{A}} \downarrow & & \downarrow I_{\mathbb{B}} \\ \mathcal{C}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{C}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

is a stratified CE-homomorphism $\mathbb{A} \rightarrow \mathbb{B}$ if and only if

1. $F_{\mathcal{F}}$ is a stratified functor, and
2. for every individual arrow $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$ and every $f: \Delta \rightarrow \Gamma$, it is

$$F_{\mathcal{F}}(f^*A) = (F_{\mathcal{C}}f)^*(F_{\mathcal{F}}A) \quad \text{and} \quad F_{\mathcal{C}}(\pi_2(f, A)) = \pi_2(F_{\mathcal{C}}f, F_{\mathcal{F}}A).$$

Proof. One direction is trivial. The other one is proved by induction on the length n of an arrow $A \in \mathcal{F}/\Gamma$. \square

3.3 Characterising C-systems as stratified CE-systems

Recall from Corollary 3.9 that every C-system \mathbb{C} has a stratified wide subcategory \mathcal{F} of its underlying category \mathcal{C} . In this section, we show that the inclusion $\mathcal{F} \rightarrow \mathcal{C}$ has the structure of a stratified CE-system (Construction 3.21). Moreover, we prove that this correspondence is functorial (Construction 3.24) and, in fact, an equivalence between the category of C-systems and the category of stratified CE-systems (Theorem 3.30).

Problem 3.20. To construct a CE-system $\mathbf{CE}(\mathbb{C})$ from a C-system $\mathbb{C} = (\mathcal{C}, 1, \ell, \text{ft}, \dots)$.

Construction 3.21 (for Problem 3.20). Recall from Lemma 3.8 that $\mathfrak{p}(\mathbb{C})$ denotes the wide subgraph of \mathcal{C} on the canonical projections \mathfrak{p}_{Γ} for Γ in \mathcal{C} and let \mathcal{F} be the free category on $\mathfrak{p}(\mathbb{C})$. In particular, \mathcal{F} has the same objects of \mathcal{C} and the object 1 is terminal in \mathcal{F} by Corollary 3.9. It follows that the inclusion $\mathfrak{p}(\mathbb{C}) \hookrightarrow \mathcal{C}$ extends to an identity-on-objects functor $I: \mathcal{F} \rightarrow \mathcal{C}$ that maps a path of length $n > 0$ in $\mathfrak{p}(\mathbb{C})$, *i.e.* a list of composable canonical projections

$$\Gamma \xrightarrow{\mathfrak{p}_{\Gamma}} \text{ft}(\Gamma) \xrightarrow{\mathfrak{p}_{\text{ft}(\Gamma)}} \dots \xrightarrow{\mathfrak{p}_{\text{ft}^2(\Gamma)}} \text{ft}^{n-1}(\Gamma) \xrightarrow{\mathfrak{p}_{\text{ft}^{n-1}(\Gamma)}} \text{ft}^n(\Gamma). \quad (6)$$

to their composite in \mathcal{C} .

It remains to provide I with a suitable choice of pullback squares along an arbitrary arrow $f: \Delta \rightarrow \Gamma$ in \mathcal{C} . As an arrow $p: \Xi \rightarrow \Gamma$ in \mathcal{F} is a path in $\mathbf{p}(\mathbb{C})$, we proceed by induction on the length n of the path p , proving also conditions (2b) and (2c) from Definition 3.10.

If $n = 0$, the path p is the identity on Γ and we take $f^*(\text{id}_\Gamma) := \text{id}_\Delta$ and $\pi_2(f, \text{id}_\Gamma) := f$. This choice is clearly functorial in f and it trivially gives rise to a pullback square. It also ensures condition (2a).

For $n > 0$, it is $I(p) = \mathbf{p}_\Xi \circ I(p')$ where the length of p' is $n - 1$. By inductive hypothesis we have $f^*p' \in \mathcal{F}/\Delta$ and a chosen pullback square of $I(p')$ along f , which is the lower square in the diagram below. The upper square is the canonical pullback square (4) given by the C-system structure.

$$\begin{array}{ccc}
(\pi_2(f, p'))^*\Xi & \xrightarrow{\mathbf{q}(\pi_2(f, p'), \Xi)} & \Xi \\
\mathbf{p}_{(\pi_2(f, p'))^*\Xi} \downarrow & & \downarrow \mathbf{p}_\Xi \\
\Delta.f^*(p') & \xrightarrow{\pi_2(f, p')} & \text{ft}(\Xi) \\
I(f^*p') \downarrow & & \downarrow I(p') \\
\Delta & \xrightarrow{f} & \Gamma
\end{array} \tag{7}$$

Thus we define $\pi_2(f, p) := \mathbf{q}(\pi_2(f, p'), \Xi)$ and f^*p to be the concatenation of f^*p' with $\mathbf{p}_{(\pi_2(f, p'))^*\Xi}$ so that $I(f^*p) = I(f^*p') \circ \mathbf{p}_{(\pi_2(f, p'))^*\Xi}$. Functoriality in f of this choice of pullback squares follows from the fact that both the lower and upper pullback squares are functorial by inductive hypothesis and by assumption, respectively. In more details: given $g: \Theta \rightarrow \Delta$, the inductive hypothesis yields $(f \circ g)^*p' = g^*(f^*p')$ and $\pi_2(f \circ g, p') = \pi_2(f, p') \circ \pi_2(g, f^*p')$. It follows by 3.1.vii that

$$\pi_2(f \circ g, p)^*\Xi = \pi_2(g, f^*p')^*(\pi_2(f, p')^*\Xi)$$

and, in turn, that $(f \circ g)^*p = g^*(f^*p)$. The other component also follows from 3.1.vii:

$$\begin{aligned}
\pi_2(f \circ g, p) &= \mathbf{q}(\pi_2(f \circ g, p'), \Xi) \\
&= \mathbf{q}(\pi_2(f, p'), \Xi) \circ \mathbf{q}(\pi_2(g, f^*p'), \pi_2(f, p')^*\Xi) \\
&= \pi_2(f, p) \circ \pi_2(g, f^*p).
\end{aligned}$$

Finally, condition (2d) for a composite $q \circ p$ in \mathcal{F} is proven by induction on the length of the path p . \square

Lemma 3.22. *Let \mathbb{C} be a C-system and \mathcal{F} the category of families of $\mathbf{CE}(\mathbb{C})$.*

1. *The individual arrows in \mathcal{F} are of the form \mathbf{p}_Γ for some object Γ .*
2. *The CE-system $\mathbf{CE}(\mathbb{C})$ is stratified and $L(\Gamma) = \ell(\Gamma)$, for every object Γ .*
3. *The CE-system $\mathbf{CE}(\mathbb{C})$ is rooted.*

Proof.

1. Immediate from the description of arrows in \mathcal{F} in (6) and 3.1.ii.
2. By Corollary 3.9, the category \mathcal{F} is stratified and $L(\Gamma) = \ell(\Gamma)$. By Lemma 2.10, it is enough to show that the choice of pullback squares in Construction 3.21 preserves individuals. But this follows immediately from the construction of pullbacks in (7) and (1) just shown.
3. The terminal object in \mathcal{F} is terminal in \mathcal{C} by assumption. \square

Problem 3.23. *To construct a functor $\mathbf{CE}: \mathbf{Csys} \rightarrow \mathbf{rCEsys}_s$ into rooted stratified CE-systems and stratified homomorphisms.*

Construction 3.24 (for Problem 3.23). The action of \mathbf{CE} on objects is defined in Construction 3.21. Every morphism $F: \mathbb{C} \rightarrow \mathbb{D}$ of \mathbb{C} -systems restricts to the graphs of canonical projections $\mathfrak{p}(F): \mathfrak{p}(\mathbb{C}) \rightarrow \mathfrak{p}(\mathbb{D})$ by conditions (i,iii,iv) in Definition 3.2 and induces, in turn, a functor between free categories $F_{\mathcal{F}}: \mathcal{F}_{\mathbb{C}} \rightarrow \mathcal{F}_{\mathbb{D}}$ whose action is determined by the action of F on individual arrows. The square

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{C}} & \xrightarrow{F_{\mathcal{F}}} & \mathcal{F}_{\mathbb{D}} \\ I_{\mathbb{C}} \downarrow & & \downarrow I_{\mathbb{D}} \\ \mathcal{C}_{\mathbb{C}} & \xrightarrow{F} & \mathcal{C}_{\mathbb{D}} \end{array}$$

commutes since it does so when precomposed by the unit $\mathfrak{p}(\mathbb{C}) \rightarrow \mathcal{F}_{\mathbb{C}}$. The functor $F_{\mathcal{F}}$ is stratified by 3.2.ii. Lemma 3.19 then ensures that the pair $\mathbf{CE}(F) := (F_{\mathcal{F}}, F)$ lifts to a stratified CE-homomorphism as soon as it preserves pullbacks of individual arrows. But this is precisely condition 3.2.v. Functoriality of \mathbf{CE} follows since $F_{\mathcal{F}}$ is defined by a universal property. \square

Problem 3.25. *To construct a C-system $\mathbf{C}(\mathbb{A})$ from a stratified and rooted CE-system \mathbb{A} .*

Construction 3.26 (for Problem 3.25). Let $I: \mathcal{F} \rightarrow \mathcal{C}$ be the underlying functor of \mathbb{A} . The underlying category of $\mathbf{C}(\mathbb{A})$ is \mathcal{C} and the length function ℓ is given by the action of the stratification functor L on objects. Since \mathbb{A} is rooted, the chosen terminal object 1 in \mathcal{F} is terminal in \mathcal{C} too. Conditions (iv) and (i) are clearly met.

Given an object X with $n = L(X) > 0$, let $X \xrightarrow{x_n} X_{n-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{x_1} 1$ be the factorisation of $X \rightarrow 1$ into n individuals in \mathcal{F} . We define

$$\mathfrak{ft}(1) := 1, \quad \mathfrak{ft}(X) := X_{n-1} \quad \text{and} \quad \mathfrak{p}_X := I(x_n). \quad (8)$$

Conditions (ii) and (iii) hold by construction.

Given also $f: Y \rightarrow \mathfrak{ft}(X)$, let $Y \xrightarrow{y_n} Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \xrightarrow{y_1} 1$ be the factorisation of $Y \rightarrow 1$ into individuals and consider the pullback square below.

$$\begin{array}{ccc} Y.f^*x_n & \xrightarrow{\pi_2(f, x_n)} & X \\ I(f^*x_n) \downarrow & & \downarrow I(x_n) \\ Y & \xrightarrow{f} & \mathfrak{ft}(X) \end{array} \quad (9)$$

It is $L(Y.f^*(x_n)) = L(Y) + 1$ by Remark 3.18, thus $Y.f^*x_n \xrightarrow{f^*x_n} Y \xrightarrow{y_n} Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \xrightarrow{y_1} 1$ is the factorisation of $Y.f^*x_n \rightarrow 1$ into individuals. It follows that $\mathfrak{ft}(Y.f^*x_n) = Y$ and $\mathfrak{p}_{f^*x_n} = I(f^*x_n)$. Condition (v) follows defining $f^*X := Y.f^*(x_n)$ and $\mathfrak{q}(f, X) := \pi_2(f, x_n)$. Condition (vi) holds by 3.10.2b since $\mathfrak{ft}(X).x_n = X$, and (vii) by 3.10.2c as below:

$$\begin{aligned} (f \circ g)^*X &= Z.(f \circ g)^*x_n = Z.(g^*(f^*x_n)) \\ &= g^*(f^*X) \\ \mathfrak{q}(f \circ g, X) &= \pi_2(f \circ g, x_n) = \pi_2(f, x_n) \circ \pi_2(g, f^*x_n) \\ &= \mathfrak{q}(f, X) \circ \mathfrak{q}(g, f^*X). \end{aligned} \quad \square$$

Lemma 3.27. *Let $F: \mathbb{A} \rightarrow \mathbb{B}$ be a stratified homomorphism of rooted stratified CE-systems. Then the underlying functor $F: \mathcal{C}_{\mathbb{A}} \rightarrow \mathcal{C}_{\mathbb{B}}$ is a homomorphism of C-systems $\mathbf{C}(F): \mathbf{C}(\mathbb{A}) \rightarrow \mathbf{C}(\mathbb{B})$.*

Proof. We verify the conditions in Definition 3.2. (i) The functor F maps the chosen terminal object of \mathbb{A} to the one of \mathbb{B} by assumption. (ii) Since F is stratified, its action on objects commutes with

the length functions. (iii–iv) The action on objects also preserves individual arrows by Lemma 2.10, thus it commutes with the father functions and preserves canonical projections. (v) F maps chosen pullback squares in \mathbb{A} to chosen ones in \mathbb{B} by 3.12.2. In particular, it preserves the choice of pullbacks along individual arrows. \square

Definition 3.28. Let $\mathbf{C}: \mathbf{rCEsys}_s \rightarrow \mathbf{Csys}$ be the functor given by Construction 3.26 and Lemma 3.27.

Lemma 3.29. For every \mathcal{C} -system \mathbb{C} , the identity functor on the underlying strict category of \mathbb{C} is an isomorphism $\mathbf{C}(\mathbf{CE}(\mathbb{C})) \cong \mathbb{C}$ of \mathcal{C} -systems, naturally in \mathbb{C} .

Proof. Let \mathcal{C} be the underlying strict category of \mathbb{C} . To see that the identity functor $\text{id}_{\mathcal{C}}$ is a \mathcal{C} -homomorphism note first that the category \mathcal{C} , its terminal object and the length function are the same in $\mathbf{C}(\mathbf{CE}(\mathbb{C}))$ and \mathbb{C} . Since individual arrows in $\mathbf{CE}(\mathbb{C})$ coincide with the canonical projections \mathbf{p}_{Γ} by Lemma 3.22, factorisations in $\mathbf{CE}(\mathbb{C})$ into individual arrows are of the form in (6). It follows that the function ft and the canonical projections as defined in (8) are equal to the ones from \mathbb{C} . Since the choice of pullback squares in $\mathbf{CE}(\mathbb{C})$ is defined inductively by the choice along individual arrows in (7), the choice of pullbacks along canonical projections in (9) coincides with the one in \mathbb{C} .

Naturality follows from the fact that $\mathbf{C}(\mathbf{CE}(F)) = F$ for every \mathcal{C} -homomorphism F . \square

Theorem 3.30. The functor $\mathbf{CE}: \mathbf{Csys} \rightarrow \mathbf{rCEsys}_s$ from Construction 3.24 is an equivalence.

Proof. By Lemma 3.29, it is enough to find, for every stratified rooted CE-system \mathbb{A} , an isomorphism $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \cong \mathbb{A}$ natural in \mathbb{A} . Let $I: \mathcal{F} \rightarrow \mathcal{C}$ be the underlying functor of \mathbb{A} and let $\mathcal{F}_i := \mathbf{p}(\mathbf{C}(\mathbb{A}))$ be the subgraph of \mathcal{F} on the individual arrows. The CE-system $\mathbf{CE}(\mathbf{C}(\mathbb{A}))$ consists, in particular, of a functor $\hat{I}: \hat{\mathcal{F}} \rightarrow \mathcal{C}$, where $\hat{\mathcal{F}}$ is the free category on \mathcal{F}_i , and \hat{I} maps a list of composable individual arrows to the composite of their images in \mathcal{C} under I , by (8), (6) and Lemma 3.22.2.

The inclusion $\mathcal{F}_i \hookrightarrow \mathcal{F}$ induces an identity-on-objects functor $\text{comp}: \hat{\mathcal{F}} \rightarrow \mathcal{F}$. Conversely, the factorisation into individual arrows (3) in \mathbb{A} yields an identity-on-objects functor $\text{fact}: \mathcal{F} \rightarrow \hat{\mathcal{F}}$, which is a (strict) inverse of comp . Since I is a functor, the squares

$$\begin{array}{ccc} \hat{\mathcal{F}} & \xrightarrow{\text{comp}} & \mathcal{F} \\ \hat{I} \downarrow & & \downarrow I \\ \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \end{array} \quad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{fact}} & \hat{\mathcal{F}} \\ I \downarrow & & \downarrow \hat{I} \\ \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \end{array} \quad (10)$$

commute. Since both functors comp and fact are identities on objects and on individual arrows, the squares above are stratified CE-homomorphisms $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \rightarrow \mathbb{A}$ and $\mathbb{A} \rightarrow \mathbf{CE}(\mathbf{C}(\mathbb{A}))$, respectively, by Lemma 3.19. Therefore $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \cong \mathbb{A}$.

To see that this isomorphism is natural in \mathbb{A} , note that $(\text{comp}, \text{id}_{\mathcal{C}})$ is natural in \mathbb{A} since comp is equivalently defined as the composite of the counit of the free-forgetful adjunction at \mathcal{F} with the image under the left adjoint of the graph inclusion $\mathcal{F}_i \hookrightarrow \mathcal{F}$. \square

4 The category of B-systems

In this section, we study Voevodsky’s B-systems.

In Section 4.1 we review the definition of B-systems and their homomorphisms. In Section 4.2 we introduce the notion of E-system and their homomorphisms. Intuitively, E-systems model type theory with strict Σ -types, see Section 4.2.5. Finally, in Section 4.3 we construct an equivalence between the category of B-systems and the subcategory of “stratified” E-systems.

In order to simplify the construction of such equivalence, we structure the definitions in the next sections in three step. In the case of B-systems, for example, we first introduce some piece of structure on sets consisting of functions, which we refer to as *pre*-B-systems, see Definition 4.7. Then we define morphisms between these structures, also called *pre*-homomorphisms, and finally we define

B-systems as those pre-B-systems whose structure functions are themselves pre-homomorphisms. Homomorphisms are then just pre-homomorphisms between B-systems. We shall follow the same pattern when introducing each of the structures that give rise to an E-system, in Sections 4.2.1 to 4.2.3, and when defining E-systems in Section 4.2.4.

4.1 The category of B-systems

In this section, we review the definition of Voevodsky's B-systems [Voe14]. We introduce a few auxiliary intermediate definitions which we will use in later constructions.

Definition 4.1. A **B-frame** \mathbb{B} is a diagram of sets of the following form:

$$\begin{array}{ccccccc} & & & \tilde{B}_1 & & \tilde{B}_2 & \\ & & & \swarrow \partial & & \swarrow \partial & \\ \{*\} \cong B_0 & \xleftarrow{\text{ft}} & B_1 & \xleftarrow{\text{ft}} & B_2 & \xleftarrow{\text{ft}} & \cdots \end{array}$$

In other words, a B-frame consists of:

1. for all $n \in \mathbb{N}$ two sets B_n and \tilde{B}_{n+1} .
2. for all $n \in \mathbb{N}$ functions of the form

$$\begin{aligned} \text{ft}_n &: B_{n+1} \rightarrow B_n \\ \partial_n &: \tilde{B}_{n+1} \rightarrow B_{n+1}. \end{aligned}$$

3. B_0 is a singleton.

For $m, n \in \mathbb{N}$, we denote the composition $\text{ft}_n \circ \cdots \circ \text{ft}_{n+m} : B_{n+m+1} \rightarrow B_n$ by ft_n^m .

A **homomorphism** $H : \mathbb{B} \rightarrow \tilde{\mathbb{A}}$ of **B-frames** is a natural transformation of B-frames, i.e., it consists of maps $H_n : B_n \rightarrow A_n$ and $\tilde{H}_{n+1} : \tilde{B}_{n+1} \rightarrow \tilde{A}_{n+1}$ such that

$$\begin{aligned} \text{ft}(H(X)) &= H(\text{ft}(X)) \\ \partial(\tilde{H}(x)) &= H(\partial(x)) \end{aligned}$$

for any $X \in B_n$ and $x \in \tilde{B}_{n+1}$. The category of B-frames is denoted by **Bfr**.

To provide some intuition for B-frames, we look back at the introduction, where we constructed, implicitly, a B-frame from a module over a monad.

Example 4.2. Recall from Section 1.2.2 the two sets $B(R, LM)$ (see Eq. (1)) and $\tilde{B}(R, LM)$ (see Eq. (2)). From these sets, we obtain a B-frame with the following sets of families (note the shift in the indexing of \tilde{B}),

$$\begin{aligned} B_n &:= B(R, LM)_n := \prod_{i=0}^{n-1} LM([i]) \\ \tilde{B}_{n+1} &:= \tilde{B}(R, LM)_n := \prod_{i=0}^{n-1} LM([i]) \times RR([n]) \times LM([n]) \end{aligned}$$

and the obvious maps for ft and ∂ . We call this B-frame the B-frame generated by a module LM over a monad R . We write elements of B_n as $A_0, \dots, A_{n-1} \vdash A_n$, and elements of \tilde{B}_{n+1} as $A_0, \dots, A_n \vdash t : A_{n+1}$, where $t \in RR([n])$.

More generally, the elements of B_{n+1} of a B-frame can be thought of as a pair of a context of length n , and a type in that context. Hence, the elements of B_1 are the types in the empty context. Just like with C-systems, there is no explicit structure to denote types in a given context. An element $t \in \tilde{B}_{n+1}$ is then a term, and the context and type t lives in is given by $\partial_{n+1}(t)$.

Example 4.3. Recall from Example 3.11 that $\text{std}(n)$ denotes the set $\{0, \dots, n-1\}$. We shall consider the B-frame defined, for each $n \in \mathbb{N}$, by $B_n := \{n\}$ and $\tilde{B}_{n+1} := \text{std}(n)$.

Example 4.4. B-frames are the same as Garner’s “type-and-term structures” [Gar15, Def. 8]. Garner [Gar15, Prop. 13] constructs an equivalence between the category of type-and-term structures and the category of \emptyset -GATs, that is, of Generalized Algebraic Theories [Car86] without weakening, projection, and substitution rules, and interpretations between them.

We now define more structure on B-frames which represents operations on syntax.

The first operation could be called “slicing”; given a B-frame \mathbb{B} and a “context” $X \in B_n$ in that B-frame, we construct the slice of \mathbb{B} over X :

Definition 4.5. For every B-frame \mathbb{B} and any $X \in B_n$, there is a B-frame \mathbb{B}/X given by

$$\begin{aligned} (B/X)_m &:= \{Y \in B_{n+m} \mid \text{ft}^m(Y) = X\} \\ (\tilde{B}/X)_{m+1} &:= \{y \in \tilde{B}_{n+m+1} \mid \text{ft}^{m+1}(\partial(y)) = X\}. \end{aligned}$$

Also, for any homomorphism $H : \mathbb{B} \rightarrow \mathbb{A}$ of B-frames and any $X \in B_n$, there is a homomorphism $H/X : \mathbb{B}/X \rightarrow \mathbb{A}/H(X)$ defined in the obvious way.

Note that for $X \in B_n$ and $Y \in B_{n+m}$ such that $\text{ft}^m(Y) = X$, we have an isomorphism $(\mathbb{B}/X)/Y \cong B/Y$ of B-frames, constructed in the obvious way, which is natural in the sense that for any homomorphism $H : \mathbb{B} \rightarrow \mathbb{A}$ of B-frames, the square

$$\begin{array}{ccc} (\mathbb{B}/X)/Y & \xrightarrow{\cong} & \mathbb{B}/Y \\ (H/X)/Y \downarrow & & \downarrow H/Y \\ (\mathbb{A}/H(X))/H(Y) & \xrightarrow{\cong} & \mathbb{A}/H(Y) \end{array}$$

commutes.

Definition 4.6. Every B-frame \mathbb{B} has an underlying rooted tree given by the sets B_n and the functions $\text{ft}_n : B_{n+1} \rightarrow B_n$, for $n \in \mathbb{N}$. Similarly, a homomorphism of B-frames is in particular a homomorphism of rooted trees. Thus we define

$$\mathbf{Bfr} \xrightarrow{\mathbf{R}} \mathbf{RtTr}$$

to be the forgetful functor from B-frames to rooted trees.

We will now consider different type-theoretic structures on B-frames, specifically substitution, weakening, and projection. Garner considers similar structures in terms of algebras of suitable monads on the category of B-frames a. k. a. type-and-term structures. We have not established a precise relationship (e.g., an equivalence) between our structures and the ones obtained by Garner as the algebras for “his” monads.

Definition 4.7. 1. A **substitution structure** on a B-frame \mathbb{B} is a collection of homomorphisms

$$S_x : \mathbb{B}/\partial(x) \rightarrow \mathbb{B}/\text{ft}(\partial(x))$$

for all $x \in \tilde{B}_{n+1}$ and all $n \in \mathbb{N}$.

2. A **weakening structure** on a B-frame \mathbb{B} is a collection of homomorphisms

$$W_X : \mathbb{B}/\text{ft}(X) \rightarrow \mathbb{B}/X$$

for all $X \in B_{n+1}$ and all $n \in \mathbb{N}$.

3. The **structure of generic elements** on a B-frame \mathbb{B} equipped with weakening structure W is a collection of functions

$$\delta_n : B_{n+1} \rightarrow \tilde{B}_{n+2}$$

such that $\partial(\delta_n(X)) = W_X(X)$ for any $X \in B_{n+1}$.

A **pre-B-system** \mathbb{B} is a B-frame equipped with weakening structure, substitution structure, and generic elements.

Example 4.8. Consider the B-frame generated by a module LM over a monad R of Example 4.2. Given an element $x \in \tilde{B}_{n+1}$, and hence in particular, a term $t \in R([n])$, we obtain a substitution map $S_x : \mathbb{B}/\partial(x) \rightarrow \mathbb{B}/\text{ft}(\partial(x))$ that substitutes the term t for the “last” free variable in any element of \mathbb{B} lying “over” $\partial(x)$. For instance, taking x to be $A_0 \vdash t_1 : A_1$, the substitution S_x maps the element $A_0, A_1 \vdash s : A_2$ to $A_0 \vdash s[t_1] : A_2[t_1]$.

For weakening, consider $X \in B_{1+1}$ to be a context $A_0 \vdash A_1$. The weakening W_X maps any context of the form $A_0, A'_1, \dots, A'_n \vdash A'_{n+1}$ to the weakened context $A_0, A_1, A'_1, \dots, A'_n \vdash A'_{n+1}$, and similar for elements in \tilde{B} .

For the generic element, consider, for instance, a context $X = A_0 \vdash A_1$ in B_2 . This context induces the generic element $A_0, A_1 \vdash \text{var}(1) : A_1$, where $\eta(1) \in RR([2])$ is the “de Bruijn” variable 1 bound by A_1 in the context, and considered as a term by being wrapped in an application of the monadic unit η of the monad RR (the inclusion of variables into terms). We have

$$\partial(A_0, A_1 \vdash \text{var}(1) : A_1) = A_0, A_1 \vdash A_1 = W_{A_0 \vdash A_1}(A_0 \vdash A_1).$$

Example 4.9. Recall the B-frame of finite sets defined in Example 4.3. Here we construct structures of substitution, weakening and generic elements on it.

Note first that its slice on the (unique) element n in B_n is such that

$$(\tilde{B}/n)_{m+1} = \tilde{B}_{n+m+1} = \text{std}(n+m).$$

It follows that a substitution structure must consist of a family of functions $S_{x,j} : \text{std}(n+1+j) \rightarrow \text{std}(n+j)$, for $n, j \in \mathbb{N}$ and $x \in \tilde{B}_{n+1}$. We define $S_{x,j} := s_x + \text{id}_j$, where s_x is the function $[\text{id}_n, x] : \text{std}(n+1) \rightarrow \text{std}(n)$ given by the universal property of the coproduct $\text{std}(n+1)$. In other words, $S_{x,j}$ lists all elements in $\text{std}(n+j)$ repeating the element $x \in \text{std}(n)$ in position $n+1$. In particular, it fixes the first n elements, and decreases the last j by 1.

Similarly, a weakening structure must consist of a family of functions $W_{n,j} : \text{std}(n+j) \rightarrow \text{std}(n+1+j)$. We define $W_{n,j}$ to be the function $i_n + \text{id}_j$, where $i_n : \text{std}(n) \rightarrow \text{std}(n+1)$ is the initial-segment inclusion. In other words, it lists all elements in $\text{std}(n+1+j)$ except for n . Equivalently, it fixes the first n elements, and increases the remaining j by 1.

Finally, the structure of generic elements is given by an element $\delta_n \in \tilde{B}_{n+2} = \text{std}(n+1)$ for every $n \in \mathbb{N}$, which we define to be its maximum, that is, $\delta_n := n$.

Taking advantage of the fact that finite sets are finite coproducts, and slightly abusing notation, we find it convenient to write

$$\begin{array}{ccc} n & \searrow & \\ + & & \\ 1 & \xrightarrow{x} & + \\ + & & \\ j & \searrow & \end{array} \quad \begin{array}{ccc} & & n \\ & \searrow & \\ & & + \\ + & & 1 \\ + & & \\ j & \searrow & \end{array}$$

for the functions $S_{x,j} : \text{std}(n+1+j) \rightarrow \text{std}(n+j)$ and $W_{n,j} : \text{std}(n+j) \rightarrow \text{std}(n+1+j)$, respectively.

Example 4.10 (B-frames with structure and D -GATs). *Garner [Gar15] constructs an equivalence between the category of GATs and a category of algebras for a monad on B-frames. We expect B-systems to be equivalent to Garner's algebras.*

Definition 4.11.

1. Consider two B-frames \mathbb{B} and \mathbb{A} , both equipped with substitution structure. A homomorphism $H : \mathbb{B} \rightarrow \mathbb{A}$ of B-frames is said to **preserve the substitution structure** if the diagram

$$\begin{array}{ccc} \mathbb{B}/\partial(x) & \xrightarrow{H/\partial(x)} & \mathbb{A}/\partial(H(X)) \\ S_x \downarrow & & \downarrow S_{\tilde{H}(x)} \\ \mathbb{B}/\text{ft}(\partial(x)) & \xrightarrow{H/\text{ft}(\partial(x))} & \mathbb{A}/\text{ft}(\partial(H(X))) \end{array}$$

of B-frame homomorphisms commutes for every $x \in \tilde{B}_{n+1}$ and every $n \in \mathbb{N}$.

2. Consider two B-frames \mathbb{B} and \mathbb{A} , both equipped with weakening structure. A homomorphism $H : \mathbb{B} \rightarrow \mathbb{A}$ of B-frames is said to **preserve the weakening structure** if the diagram

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{H/X} & \mathbb{A}/H(X) \\ W_X \uparrow & & \uparrow W_{H(X)} \\ \mathbb{B}/\text{ft}(X) & \xrightarrow{H/\text{ft}(X)} & \mathbb{A}/\text{ft}(H(X)) \end{array}$$

of B-frame homomorphisms commutes for all $X \in B_n$ and all $n \in \mathbb{N}$.

3. Consider two B-frames \mathbb{B} and \mathbb{A} , both equipped with weakening structure, and both equipped with generic elements. A B-frame homomorphism $H : \mathbb{B} \rightarrow \mathbb{A}$ is said to **preserve the generic elements** if

$$\tilde{H}(\delta(X)) = \delta(H(X))$$

for any $X \in B_{n+1}$ and any $n \in \mathbb{N}$.

A **pre-B-homomorphism** $H : \mathbb{B} \rightarrow \mathbb{A}$ is a homomorphism of pre-B-systems preserving the weakening structure, substitution structure and the generic elements.

Definition 4.12. A **B-system** is a pre-B-system for which the following conditions hold:

1. Every S_x is a pre-B-homomorphism.
2. Every W_X is a pre-B-homomorphism.
3. For every $x \in \tilde{B}_{n+1}$ one has $S_x \circ W_{\partial(x)} = \text{id}_{\mathbb{B}/\text{ft}(\partial(x))}$.
4. For every $x \in \tilde{B}_{n+1}$ one has $S_x(\delta(\partial(x))) = x$.
5. For every $X \in B_{n+1}$ one has $S_{\delta(X)} \circ W_X/X = \text{id}_{\mathbb{B}/X}$.

B-homomorphisms are pre-B-homomorphisms between B-systems. We denote the category of B-systems by **Bsys**.

The idea is that first substitution and weakening preserve all the structure of a (pre-)B-system. The third axiom asserts that substitution in weakened type families is constant. Furthermore, the generic elements should behave like internal identity morphisms. Axioms 4 and 5 are akin to two of the well-known monadic laws of substitution.

Lemma 4.13. *The forgetful functor $\mathbf{Bsys} \rightarrow \mathbf{Bfr}$ is faithful.*

Proof. This functor faithful because its action on morphisms only forgets a property. \square

Example 4.14. The structures given in Example 4.9 make the B-frame defined in Example 4.3 into a B-system as follows.

Consider first homomorphism of B-frames $S_y: \mathbb{B}/(k+1) \rightarrow \mathbb{B}/k$, for $k \in \mathbb{N}$ and $y \in \text{std}(k)$. The homomorphism S_y preserves the substitution structure if, for every $n \in \mathbb{N}$, $x \in \text{std}(k+1+n)$ and $j \in \mathbb{N}$, the square

$$\begin{array}{ccc} \text{std}(k+1+n+1+j) & \xrightarrow{S_{y,n+1+j}} & \text{std}(k+n+1+j) \\ S_{x,j} \downarrow & & \downarrow S_{S_y,n(x),j} \\ \text{std}(k+1+n+j) & \xrightarrow{S_{y,n+j}} & \text{std}(k+n+j) \end{array}$$

commutes. This can be readily verified in the three cases $x < k$, $x = k$ or $k < x < n+1+k$. For example, in the last case $S_{y,n}(x) = x-1$ and

$$\begin{array}{ccccc} \begin{array}{c} k \\ + \\ 1 \\ + \\ n \\ + \\ 1 \\ + \\ j \end{array} & \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow j \end{array} & \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow j \end{array} & = & \begin{array}{c} k \\ + \\ 1 \\ + \\ n \\ + \\ 1 \\ + \\ j \end{array} \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow j \end{array} \\ & & & & = & \begin{array}{c} k \\ + \\ 1 \\ + \\ n \\ + \\ 1 \\ + \\ j \end{array} \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow j \end{array} \end{array}$$

The homomorphism S_y preserves the weakening structure if for every $n, j \in \mathbb{N}$, the square

$$\begin{array}{ccc} \text{std}(k+1+n+1+j) & \xrightarrow{S_{y,n+1+j}} & \text{std}(k+n+1+j) \\ W_{k+1+n,j} \uparrow & & \uparrow W_{k+n,j} \\ \text{std}(k+1+n+j) & \xrightarrow{S_{y,n+j}} & \text{std}(k+n+j) \end{array}$$

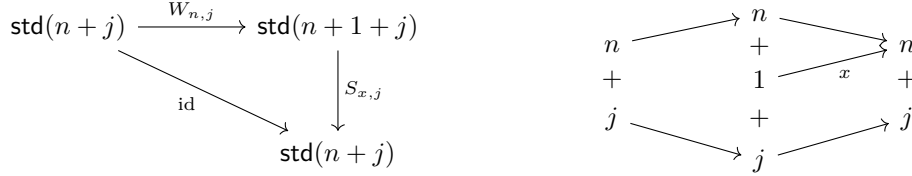
commutes. This is indeed the case:

$$\begin{array}{ccccc} \begin{array}{c} k \\ 1 \\ n \\ j \end{array} & \begin{array}{c} \rightarrow k \\ \rightarrow 1 \\ \rightarrow n \\ \rightarrow j \end{array} & \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow j \end{array} & = & \begin{array}{c} k \\ 1 \\ n \\ j \end{array} \begin{array}{c} \rightarrow k \\ \rightarrow k \\ \rightarrow n \\ \rightarrow 1 \\ \rightarrow j \end{array} \\ & & & & = & \begin{array}{c} k \\ 1 \\ n \\ j \end{array} \begin{array}{c} \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow k \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow n \\ \searrow \\ \rightarrow 1 \\ \searrow \\ \rightarrow j \end{array} \end{array}$$

Finally, for every $n \in \mathbb{N}$, the function $S_{y,n+1}: \text{std}(k+1+n+1) \rightarrow \text{std}(k+n+1)$ preserves the maximum. It follows that S_y preserves the generic elements.

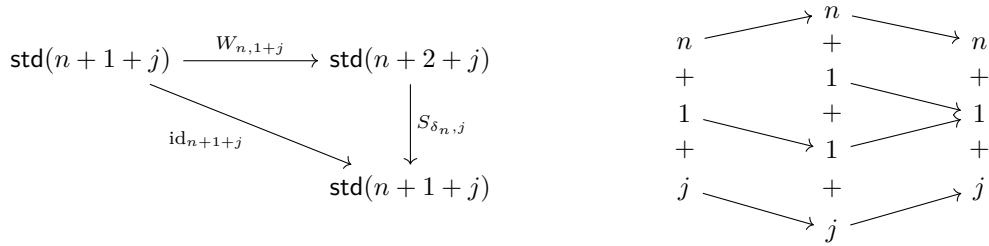
We have shown that S_y is a pre-B-homomorphism. We leave the verification that $W_n: \mathbb{B}/n \rightarrow \mathbb{B}/(n+1)$ is a pre-B-homomorphism to the reader and consider instead the remaining three conditions of Definition 4.12.

Condition 3 amounts to the commutativity of the left-hand diagram below, for every $n, j \in \mathbb{N}$ and $x \in \text{std}(n)$. Its commutativity is shown in the right-hand diagram.



Condition 4 holds since $S_{x,0}(\delta_n) = S_{x,0}(n) = x$, for every $n \in \mathbb{N}$ and $x \in \text{std}(n)$.

Condition 5 amounts to the commutativity of the left-hand diagram below, for every $n, j \in \mathbb{N}$. Its commutativity is shown in the right-hand diagram.



Example 4.15 (B-systems and Generalized Algebraic Theories). *Continuing Example 4.10, any Generalized Algebraic Theory (in Garner’s taxonomy also known as $\{w, p, s\}$ -GATs) gives rise to a B-system. The axioms of Definition 4.12 follow mostly from the definition of substitution and the congruence rules that substitution satisfies.*

Composing Cartmell’s equivalence of categories between contextual categories and GATs with our equivalence between B-systems and C-systems constructed in Section 5.4, we later can establish a more precise relationship between B-systems and GATs, in the form of an equivalence of categories.

4.2 The category of E-systems

In Section 4.3 we will show how for any B-frame we get a category \mathcal{F} with objects (n, X) where $X \in B_n$. As we saw in Definition 4.6, the family of sets B_n induces a tree, with objects (n, X) , and \mathcal{F} is the free category generated by this tree. The sets \tilde{B}_{n+1} then induce a family of sets of terms indexed by the morphisms of \mathcal{F} . In particular for a morphism $(n+1, X) \rightarrow (n, \text{ft}(X))$ we get a set of terms $\partial^{-1}(X)$.

In this section we will define the structure of a type theory directly on \mathcal{F} of the kind that one gets when turning a B-system into a category. Such systems are called E-systems, and in Section 4.3 we will show that the category of B-systems is equivalent to a subcategory of E-systems. Thus, E-systems can be seen as a generalisation of B-systems.

Just like B-systems (and different from C-systems), E-systems have an explicit structure for “terms”. Indeed, the first step towards the definition of E-system is that of a “term structure”:

Definition 4.16. A **category with term structure** is a category \mathcal{F} equipped with a family of sets $(T(A))_{A \in \text{Mor}(\mathcal{F})}$ indexed by the morphisms $\text{Mor}(\mathcal{F})$ of \mathcal{F} . Given two categories \mathcal{F} and \mathcal{D} with term structure, a **functor with term structure** from \mathcal{F} to \mathcal{D} is a functor $F : \mathcal{F} \rightarrow \mathcal{D}$ equipped with a family of functions $T(A) \rightarrow T(F(A))$ for every morphism A in \mathcal{F} .

Any B-frame, and hence any B-system, generates a category with term structure; details will be given in Construction 4.62.

The identity functor with term structure $\text{id}_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}$ is the identity functor on \mathcal{F} equipped with the identity functions $T(A) \rightarrow T(A)$ indexed by the morphisms A in \mathcal{F} . Similarly, the composition

$G \circ F$ of two functors F and G with term structure is defined to be the composition of the underlying functors, equipped with the composites

$$T(A) \longrightarrow T(F(A)) \longrightarrow T(G(F(A))).$$

Definition 4.17. Let \mathcal{F} be a strict category with term structure and Γ an object of \mathcal{F} . The **slice term structure** on the strict slice category \mathcal{F}/Γ is given by $T_{\mathcal{F}/\Gamma}(A) = T_{\mathcal{F}}(A)$.

Remark 4.18. Every functor $F : \mathcal{F} \rightarrow \mathcal{F}'$ with term structure gives rise to a functor with term structure $F/X : \mathcal{F}/X \rightarrow \mathcal{F}'/F(X)$.

In order to illustrate the additional structure that we shall consider on a category with term structure, we introduce the following example.

Example 4.19. Consider the the poset (\mathbb{N}, \geq) . We write $(n, k) : n+k \geq n$ for arrows in (\mathbb{N}, \geq) . Let \mathcal{N} be the category with term structure which consists of the poset (\mathbb{N}, \geq) and the term structure given by $T(n, k) := \mathbf{Set}([k], [n])$, i.e. the set of functions from the standard set with k elements to the standard set with n elements.

4.2.1 Substitution systems

Given a (strict) category \mathcal{F} , an object $\Gamma \in \mathcal{F}$ and an object $A \in \mathcal{F}/\Gamma$, we will write $\Gamma.A$ for the domain of A . In other words, A is a morphism $\Gamma.A \rightarrow \Gamma$.

Definition 4.20. A **pre-substitution structure** on a strict category with term structure \mathcal{F} consists of a functor with term structure $S_x : \mathcal{F}/\Gamma.A \rightarrow \mathcal{F}/\Gamma$ for every $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$, such that $S_x(\text{id}_{\Gamma.A}) = \text{id}_{\Gamma}$.

A **pre-substitution system** is a strict category with term structure together with a pre-substitution structure.

Definition 4.21. A **pre-substitution homomorphism** $F : \mathcal{F} \rightarrow \mathcal{D}$ is a functor with term structure for which the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathcal{D}/F(\Gamma.A) \\ S_x \downarrow & & \downarrow S_{F(x)} \\ \mathcal{F}/\Gamma & \xrightarrow{F/\Gamma} & \mathcal{D}/F(\Gamma) \end{array}$$

commutes for every $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.22. Let \mathcal{F} be a pre-substitution system and Γ an object of \mathcal{F} . The **slice pre-substitution structure** on the strict slice category with term structure \mathcal{F}/Γ from Definition 4.17 is given by $S(\mathcal{F}/\Gamma)_x = S(\mathcal{F})_x$, for every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$ and $x \in T_{\mathcal{F}}(P)$.

Definition 4.23. A **substitution system** is a pre-substitution system for which each S_x is a pre-substitution homomorphism. A **substitution homomorphism** is a pre-substitution homomorphism between substitution systems.

Corollary 4.24. For any object Γ of a substitution system, the strict slice category \mathcal{F}/Γ is a substitution system.

Remark 4.25. The condition that every S_x is a substitution homomorphism, asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A.P.Q & \xrightarrow{S_x/P.Q} & \mathcal{F}/\Gamma.S_x(P).S_x(Q) \\ S_y \downarrow & & \downarrow S_{S_x(y)} \\ \mathcal{F}/\Gamma.A.P & \xrightarrow{S_x/P} & \mathcal{F}/\Gamma.S_x(P) \end{array}$$

commutes for every $y \in T(Q)$.

Example 4.26. We can equip the category with term structure \mathcal{N} from Example 4.19 with a substitution structure as follows. Consider the functor $-k: \mathbb{N}/(n+k) \rightarrow \mathbb{N}/n$ that maps $(n+k+j, l)$ to $(n+j, l)$. It preserves terminal objects since an arrow (m, i) is an identity if and only if $i = 0$. Given $(n, k): n+k \geq n$ and a function $f: [k] \rightarrow [n]$, define $S_f: \mathcal{N}/(n+k) \rightarrow \mathcal{N}/n$ as the functor $-k$ together with functions $T(n+k+j, l) \rightarrow T(n+j, l)$ defined by postcomposition

$$\begin{array}{ccc}
 [l] & & [l] \\
 \downarrow h & \xrightarrow{\quad} & \downarrow S_f(h) \\
 [n+k+j] & & [n+j]
 \end{array}
 \begin{array}{c}
 \nearrow h \\
 [n+k+j] \\
 \nwarrow [\text{id}_n, f] + \text{id}_j
 \end{array}$$

where $[\text{id}_n, f]$ is the function given by the universal property of the coproduct $[n] \leftarrow [n+k] \rightarrow [k]$ in **Set**, and similarly for $[\text{id}_n, f] + \text{id}_j$.

The fact that S_f is a pre-substitution homomorphism follows from the fact that postcomposition distributes on $[-, -]$ as shown below: given $g: [l] \rightarrow [n+k+j]$, then

$$S_f/(n+k, j) \circ S_g = S_{S_f(g)} \circ S_f/(n+k, j+l)$$

since

$$\begin{aligned}
 ([\text{id}_n, f] + \text{id}_j) [\text{id}_{n+k+j}, g] &= [[\text{id}_n, f] + \text{id}_j, S_f(g)] \\
 &= [\text{id}_{n+j}, S_f(g)] ([\text{id}_n, f] + \text{id}_{j+l}).
 \end{aligned}$$

Example 4.27. Consider a group G . A term structure on G consists of a set $T(g)$ for every element g of G .

A pre-substitution structure on G consists of a functor with term structure $S_x: G/\bullet \rightarrow G/\bullet$ (where \bullet denotes the only object in G viewed as a category) for every $g \in G$ and every $x \in T(g)$ such that $S_x(\text{id}_\bullet) = \text{id}_\bullet$. One can show that such functors $S_x: G/\bullet \rightarrow G/\bullet$ correspond to functions $G \rightarrow G$ which preserve the identity, so a pre-substitution structure amounts to functions $S_x: G \rightarrow G$ for every $g \in G$, $x \in T(g)$ preserving the identity together with functions $S_x: T(h) \rightarrow T(S_x(h))$ for every $g, h \in G$, $x \in T(g)$.

A substitution structure T on G is a pre-substitution structure S as described above such that the following diagrams commute for all $g, h, k \in G$, $x \in T(g)$, and $y \in T(h)$.

$$\begin{array}{ccc}
 G & \xrightarrow{S_x} & G \\
 \downarrow S_y & & \downarrow S_{S_x(y)} \\
 G & \xrightarrow{S_x} & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 T(k) & \xrightarrow{S_x} & T(S_x k) \\
 \downarrow S_y & & \downarrow S_{S_x(y)} \\
 T(S_y k) & \xrightarrow{S_x} & T(S_x S_y k)
 \end{array}$$

Now for a particular example, suppose that each $T(g)$ is $\text{Aut}(G)$, that each $S_x: G \rightarrow G$ is just the automorphism x , and that each $S_x: T(h) \rightarrow T(S_x h)$ takes $y \in T(h)$ to xyx^{-1} . Then we find indeed that the first diagram commutes since $(xyx^{-1})x = xy$ for all $x \in S_x = \text{Aut}(G)$ and all $y \in S_y = \text{Aut}(G)$. The second diagram commutes since $(xyx^{-1})xzx^{-1}(xyx^{-1})^{-1} = xyzy^{-1}x^{-1}$ for all $x \in T(g) = \text{Aut}(G)$, $y \in T(h) = \text{Aut}(G)$, and $z \in T(k) = \text{Aut}(G)$.

4.2.2 Weakening systems

Definition 4.28. Consider a category \mathcal{F} with term structure T . A **pre-weakening structure** on \mathcal{F} is a family of functors with term structure $W_A: \mathcal{F}/\Gamma \rightarrow \mathcal{F}/\Gamma$ indexed by the morphisms $A: \Gamma.A \rightarrow \Gamma$ in \mathcal{F} such that

1. $W_{\text{id}_\Gamma} = \text{id}_{\mathcal{F}/\Gamma}$ for every object $\Gamma \in \mathcal{F}$.

2. $W_{A \circ P} = W_P \circ W_A$ for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.
3. W_A strictly preserves the final object, i.e., $W_A(\text{id}_\Gamma) = \text{id}_{\Gamma.A}$.

A **pre-weakening system** is a strict category with term structure equipped with a pre-weakening structure.

Definition 4.29. A **pre-weakening homomorphism** $F : \mathcal{F} \rightarrow \mathcal{D}$ between pre-weakening systems is a functor $F : \mathcal{F} \rightarrow \mathcal{D}$ with term structure such that the square

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathcal{D}/F(\Gamma.A) \\ W_A \uparrow & & \uparrow W_{F(A)} \\ \mathcal{F}/\Gamma & \xrightarrow{F/\Gamma} & \mathcal{D}/F(\Gamma) \end{array}$$

of functors with term structure commutes for any $A \in \mathcal{F}/\Gamma$.

Definition 4.30. Let \mathcal{F} be a pre-weakening system and Γ an object of \mathcal{F} . The **slice pre-weakening system** on the strict slice category with term structure from Definition 4.17 \mathcal{F}/Γ is given by $W(\mathcal{F}/\Gamma)_P = W(\mathcal{F})_P$ for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.31. A **weakening system** is a pre-weakening system \mathcal{F} such that W_A is a pre-weakening homomorphism for every morphism A in \mathcal{F} . A **weakening homomorphism** is a pre-weakening homomorphism between weakening systems.

Remark 4.32. The condition that every W_A is a pre-weakening homomorphism implies that the square

$$\begin{array}{ccc} \mathcal{F}/\Gamma.B.Q & \xrightarrow{W_{A/B.Q}} & \mathcal{F}/\Gamma.A.W_A(B.Q) \\ W_Q \uparrow & & \uparrow W_{W_A(Q)} \\ \mathcal{F}/\Gamma.B & \xrightarrow{W_{A/B}} & \mathcal{F}/\Gamma.A.W_A(B) \end{array}$$

commutes for each $A, B \in \mathcal{F}/\Gamma$ and $Q \in \mathcal{F}/\Gamma.B$. On objects, this property asserts that for any $k \in \mathcal{F}/E$, the dotted arrows in the diagram

$$\begin{array}{ccccc} & \bullet & & \bullet & \\ & \swarrow & & \swarrow & \\ (W_{A/B.Q})(W_Q(R)) & & W_{(W_{A/B}(Q))}((W_{A/B})(R)) & & \\ & \searrow & & \searrow & \\ & \bullet & & \bullet & \\ (W_{A/B})(Q) & & (W_{A/B})(R) & & \\ & \searrow & & \searrow & \\ & \bullet & & \bullet & \\ W_A(B) & & & & \\ \downarrow & & & & \\ \Gamma.A & \xrightarrow{A} & & & \Gamma \\ & & & & \downarrow B \\ & & & & \Gamma.B \\ & & & & \swarrow Q \quad \searrow R \\ & & & & \bullet & \bullet \end{array}$$

are equal.

A useful special case of this property is where $B = \text{id}_\Gamma$. Thus, if W is a weakening system, then the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.C & \xrightarrow{W_{A/C}} & \mathcal{F}/\Gamma.A.W_A(C) \\ W_C \uparrow & & \uparrow W_{W_A(C)} \\ \mathcal{F}/\Gamma & \xrightarrow{W_A} & \mathcal{F}/\Gamma.A \end{array}$$

commutes for every $A, C \in \mathcal{F}/\Gamma$. In particular, we see that $W_A(W_C(D)) = W_{W_A(C)}(W_A(D))$ for any $D \in \mathcal{F}/\Gamma$, i.e. that weakening is a self-distributive operation.

Corollary 4.33. For any object Γ of a weakening system \mathcal{F} , the strict slice category \mathcal{F}/Γ is a weakening system.

Example 4.34. Consider the term category \mathcal{N} from Example 4.19. We can equip \mathcal{N} with a weakening structure as follows. Consider the functor $+k: \mathbb{N}/n \rightarrow \mathbb{N}/(n+k)$ that maps $(n+j, l)$ to $(n+k+j, l)$. It preserves terminal objects as in Example 4.26. Given $(n, k): n+k \geq n$, define $W_{n,k}: \mathcal{N}/n \rightarrow \mathcal{N}/(n+k)$ as the functor $+k$ together with functions $T(n+j, l) \rightarrow T(n+k+j, l)$ defined by postcomposition

$$\begin{array}{ccc} [l] & & [l] \\ \downarrow h & \xrightarrow{\quad} & \downarrow W_{n,k}(h) \\ [n+j] & & [n+k+j] \end{array} \quad \begin{array}{c} \searrow h \\ [n+j] \\ \swarrow i_n^{n+k} + \text{id}_j \\ [n+k+j] \end{array}$$

where $i_n^{n+k}: [n] \rightarrow [n+k]$ is the initial-segment inclusion and $i_n^{n+k} + \text{id}_j$ is the function given by the universal property of the coproduct $[n] \leftarrow [n+j] \rightarrow [j]$ in **Set**.

The fact that $W_{n,k}$ is a pre-weakening homomorphism follows from the fact that initial-segment inclusions factor uniquely into inclusions whose images have codimension 1: given $(n+j, l)$ in \mathcal{N}/n , then

$$W_{n,k}/(n, j+l) \circ W_{n+j,l} = W_{W_{n,k}(n+j,l)} \circ W_{n,k}/(n, j)$$

since

$$(i_n^{n+k} + \text{id}_{j+l})_{i_{n+j}^{n+j+l}} = i_{n+k+j}^{n+k+j+l} (i_n^{n+k} + \text{id}_j).$$

Example 4.35. Consider the situation of Example 4.27 above where the underlying category is a group G with term structure S .

A pre-weakening structure on G is a family of functions $W_g: G \rightarrow G$ for each $g \in G$ which preserves the identity in each coordinate (i.e. $W_e(g) = W_g(e) = g$ for any $g \in G$) and where $W_{hg} = W_g \circ W_h$ together with term structure $W_g: T(h) \rightarrow T(W_g(h))$ for any $g, h \in G$.

If each $W_g: G \rightarrow G$ is a group homomorphism, this structure is a weakening system when the following diagrams commute for every $g, h, k \in G$.

$$\begin{array}{ccc} G & \xrightarrow{W_h} & G \\ W_g \uparrow & & \uparrow W_{W_h(g)} \\ G & \xrightarrow{W_h} & G \end{array} \quad \begin{array}{ccc} T(W_g k) & \xrightarrow{W_h} & T(W_{gh} k) \\ W_g \uparrow & & \uparrow W_{W_h(g)} \\ T(k) & \xrightarrow{W_h} & T(W_h k) \end{array}$$

Now consider the more particular example discussed in Example 4.27, where G is still an arbitrary group, but $T(g) = \text{Aut}(G)$ for all $g \in G$. We can let each $W_g: G \rightarrow G$ be φ_g , the conjugation automorphism sending h to ghg^{-1} , and we can let each $W_g: T(k) \rightarrow T(k)$ be ‘conjugation by conjugation’ taking each automorphism $x \in T(k)$ to $\varphi_g x \varphi_g^{-1}$. Since $\varphi_h \varphi_g = \varphi_{\varphi_h(g)} \varphi_h$, the left-hand diagram above commutes, and using that equation we find that $\varphi_h \varphi_g(-) \varphi_g^{-1} \varphi_h^{-1} = \varphi_{\varphi_h(g)} \varphi_h(-) \varphi_{\varphi_h(g)}^{-1} \varphi_h^{-1}$ so the right-hand diagram commutes.

4.2.3 Projection systems

Definition 4.36. A **pre-projection system** is a pre-weakening system \mathcal{F} equipped with an element $\text{idm}_A \in T(W_A(A))$ for every $A \in \mathcal{F}/\Gamma$ and $\Gamma \in \mathcal{F}$.

Definition 4.37. A **pre-projection homomorphism** $F: \mathcal{F} \rightarrow \mathcal{D}$ is a pre-weakening homomorphism for which

$$F(\text{idm}_A) = \text{idm}_{F(A)}$$

for every $A \in \mathcal{F}/\Gamma$ and $\Gamma \in \mathcal{F}$.

Definition 4.38. Let \mathcal{F} be a pre-projection system and Γ an object of \mathcal{F} . The **slice pre-projection structure** on \mathcal{F}/Γ is given by the slice pre-weakening structure in Definition 4.30 together with $\text{idtm}_P^\Gamma := \text{idtm}_P$, for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.39. A **projection system** is a pre-projection system for which every W_A is a pre-projection homomorphism. A **projection homomorphism** is a pre-projection homomorphism between projection systems.

Corollary 4.40. For any object Γ of a projection system \mathcal{F} , the strict slice category \mathcal{F}/Γ is a projection system.

Example 4.41. Consider the weakening system on \mathcal{N} from Example 4.34. We can equip it with a projection structure defining, for every (n, k) in (\mathbb{N}, \geq) , the element $\text{idtm}_{n,k} \in T(W_{n,k}(n, k)) = \text{Set}([k], [n+k])$ to be the final-segment inclusion

$$[k] \xrightarrow{i_k^{n+k}} [n+k]$$

The fact that each $W_{n,j}: \mathcal{N}/n \rightarrow \mathcal{N}/(n+j)$ is a projection homomorphism is readily verified:

$$\begin{aligned} W_{n,j}(\text{idtm}_{n+m,k}) &= (i_n^{n+j} + \text{id}_{m+k})i_k^{n+m+k} \\ &= i_k^{n+m+j+k} \\ &= \text{idtm}_{W_{n,j}(n+m,k)}. \end{aligned}$$

Example 4.42. Consider the particular example discussed in Example 4.35 where the underlying category is an arbitrary group G , each $T(g)$ is the set of automorphisms of G , and W_g is conjugation by g both on elements of G and terms (automorphisms of G).

A pre-projection system consists of an element $\text{idtm}_g \in \text{Aut}(G)$ for every $g \in G$. We will let idtm_g be the identity automorphism on G .

For a projection system on a group G , we need $W_g(\text{idtm}_h) = \text{idtm}_{W_g(h)}$ for every $g, h \in G$. In our particular example, this means $g1_Gg^{-1} = 1_G$, which holds.

4.2.4 The definition of E-systems

We can now give the definition of E-systems.

Definition 4.43. A **pre-E-system** \mathbb{E} is a strict category \mathcal{F} with term structure equipped with a chosen terminal object \square in \mathcal{F} , the structure of a pre-substitution system S , the structure of a pre-weakening system W , and the structure of a pre-projection system idtm .

Definition 4.44. A **pre-E-homomorphism** from \mathbb{E} to \mathbb{D} is a functor $H: \mathcal{F}_{\mathbb{E}} \rightarrow \mathcal{F}_{\mathbb{D}}$ between the underlying categories with term structure such that $F(\square_{\mathbb{E}}) = \square_{\mathbb{D}}$, which is a pre-substitution homomorphism, a pre-weakening homomorphism, and a pre-projection homomorphism.

Definition 4.45. An **E-system** is a pre-E-system \mathbb{E} such that

1. each S_x is a pre-E-homomorphism,
2. each W_A is a pre-E-homomorphism,
3. $S_x \circ W_A = \text{id}_{\mathbb{E}/\Gamma}$ for any $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$,
4. $S_x(\text{idtm}_A) = x$ for any $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$, and
5. $S_{\text{idtm}_A} \circ W_A/A = \text{id}_{\mathbb{E}/\Gamma.A}$ for any $A \in \mathcal{F}/\Gamma$.

An **E-homomorphism** $H: \mathbb{E} \rightarrow \mathbb{D}$ is a pre-E-homomorphism from an E-system \mathbb{E} to an E-system \mathbb{D} . We write **Esys** for the category of E-systems and E-homomorphisms.

Remark 4.46. The condition that each W_A is a substitution homomorphism asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.B.Q & \xrightarrow{W_{A/B}.Q} & \mathcal{F}/\Gamma.A.W_A(B).W_A(Q) \\ S_y \downarrow & & \downarrow S_{W_A(y)} \\ \mathcal{F}/\Gamma.B & \xrightarrow{W_{A/B}} & \mathcal{F}/\Gamma.A.W_A(B) \end{array}$$

of functors with term structure commutes for every $Q \in \mathcal{F}/\Gamma.B$, $B \in \mathcal{F}/\Gamma$ and each $y \in T(Q)$.

Likewise, the condition that each S_x is a weakening homomorphism asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A.P & \xrightarrow{S_x/P} & \mathcal{F}/\Gamma.S_x(P) \\ W_Q \downarrow & & \downarrow W_{S_x(Q)} \\ \mathcal{F}/\Gamma.A.P.Q & \xrightarrow{S_x/P.Q} & \mathcal{F}/\Gamma.S_x(P).S_x(Q) \end{array}$$

of functors with term structure commutes for every $Q \in \mathcal{F}/\Gamma.A.P$.

Corollary 4.47. *For any object Γ of a E -system \mathcal{F} , the strict slice category \mathcal{F}/Γ is an E -system.*

Example 4.48. *We can finally show that the category with term structure \mathcal{N} from Example 4.19 can be equipped with the structure of an E -system. It can be turned into a pre- E -system because of Examples 4.26, 4.34 and 4.41. Conditions 1 and 2 of Definition 4.45 are left to the reader. The other ones are verified as follows:*

3. Given $f: [k] \rightarrow [n]$, it is $S_f \circ W_{n,k} = \text{id}_{\mathcal{N}/n}$ since $[\text{id}_n, f]i_n^{n+k} = \text{id}_n$.
4. Given $f: [k] \rightarrow [n]$, it is $S_f(\text{idtm}_{n,k}) = [\text{id}_n, f]i_k^{n,k} = f$.
5. Given (n, k) in \mathcal{N} , it is $S_{\text{idtm}_{n,k}} \circ W_{n,k}(n, k) = \text{id}_{\mathcal{N}/(n+k)}$ since $[\text{id}_{n+k}, i_k^{n+k}](i_n^{n+k} + \text{id}_k) = \text{id}_{n+k}$.

Example 4.49. *Consider the situation in Example 4.42 where our underlying category is an arbitrary group G , the terms of each $g \in G$ are Aut , substitution $S_x: G \rightarrow G$ is given by the automorphism x itself and substitution $S_x: T(h) \rightarrow T(S_x h)$, weakening $W_g: G \rightarrow G$, and weakening $W_g: T(h) \rightarrow T(W_g h)$ are given by conjugation.*

Understood as a category, G does not have a terminal object (unless it is trivial), but we can still understand the conditions of Definition 4.45. The condition 2 in that definition means that the following diagrams must commute for $g, h, k \in G$ and $x \in T(h)$

$$\begin{array}{ccc} G & \xrightarrow{W_g} & G \\ \downarrow S_x & & \downarrow S_{W_g(x)} \\ G & \xrightarrow{W_g} & G \end{array} \quad \begin{array}{ccc} T(k) & \xrightarrow{W_g} & T(W_g k) \\ \downarrow S_x & & \downarrow S_{W_g(x)} \\ T(S_x k) & \xrightarrow{W_g} & T(S_{W_g(x)} W_g k) \end{array}$$

Since $\varphi_g x = \varphi_g x \varphi_g^{-1} \varphi_g$, the left-hand diagram above commutes, and since $\varphi_g x - x^{-1} \varphi_g^{-1} = (\varphi_g x \varphi_g^{-1}) \varphi_g(-) \varphi_g^{-1} (\varphi_g x \varphi_g^{-1})^{-1}$, the right-hand square above commutes.

The condition 1 in Definition 4.45 means that the following diagrams must commute for $g, h, k \in G$ and $x \in T(h)$.

$$\begin{array}{ccc} G & \xrightarrow{S_x} & G \\ \downarrow W_g & & \downarrow W_{S_x(g)} \\ G & \xrightarrow{S_x} & G \end{array} \quad \begin{array}{ccc} T(k) & \xrightarrow{S_x} & T(S_x k) \\ \downarrow W_g & & \downarrow W_{S_x(g)} \\ T(W_g k) & \xrightarrow{S_x} & T(S_x W_g k) \end{array}$$

Since $x\varphi_g = \varphi_{x(g)}x$, the left-hand diagram commutes, and since then $x\varphi_g - \varphi_g^{-1}x^{-1} = \varphi_{x(g)}x - x^{-1}\varphi_{x(g)}^{-1}$, the right-hand diagram commutes.

Condition 3 does not hold since (on G) $S_x \circ W_g = x\varphi_g$, and in general this is not the identity.

Condition 4 does not hold since $S_x \text{idtm}_g = x1_G x^{-1}$ which is not in general x .

Condition 5 does not hold since (on G) $S_{\text{idtm}_g} W_g = 1_G \varphi_g = \varphi_g$ which is not the identity in general.

We introduce more convenient notation for weakening and substitution.

Definition 4.50. Let $A \in \mathcal{F}/\Gamma$. Recall that $W_A : \mathcal{F}/\Gamma \rightarrow \mathcal{F}/\Gamma.A$ acts on objects, morphisms and terms. We introduce the infix form of weakening by $A \in \mathcal{F}/\Gamma$ to be $\langle A \rangle -$. Thus, we will write

$$\begin{aligned} \langle A \rangle B &:= W_A(B) && \text{for } B \in \mathcal{F}/\Gamma \\ \langle A \rangle Q &:= W_A(Q) && \text{for } B \in \mathcal{F}/\Gamma \text{ and } Q \in \mathcal{F}/\Gamma.B \\ \langle A \rangle g &:= W_A(g) && \text{for } B \in \mathcal{F}/\Gamma, Q \in \mathcal{F}/\Gamma.B \text{ and } g \in T(Q) \end{aligned}$$

Definition 4.51. Let $x \in T(A)$ for a family $A \in \mathcal{F}/\Gamma$. The infix form of substitution by x is taken to be $-[x]$. Thus, we will write

$$\begin{aligned} P[x] &:= S_x(P) && \text{for } P \in \mathcal{F}/\Gamma.A \\ Q[x] &:= S_x(Q) && \text{for } P \in \mathcal{F}/\Gamma.A \text{ and } Q \in \mathcal{F}/\Gamma.A.P \\ g[x] &:= S_x(g) && \text{for } P \in \mathcal{F}/\Gamma.A, Q \in \mathcal{F}/\Gamma.A.P \text{ and } g \in T(Q) \end{aligned}$$

Definition 4.52. A (pre-)E-system is **stratified** if its underlying category is stratified in the sense of Definition 2.1 and the underlying functor of each W_A and S_x is stratified with respect to the stratification induced on slices.

A morphism of stratified (pre-)E-systems is **stratified** if its underlying functor is stratified.

The category of stratified E-systems and stratified E-homomorphisms between them is denoted by \mathbf{Esys}_s .

Example 4.53. The E-system on \mathcal{N} from Example 4.48 is stratified by the identity functor.

4.2.5 Pairing and the projections

The composition $A.P$ of $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$ behaves like a strict Σ -type. In this section we define the pairing term $\text{pair}^{A,P} := \text{idtm}_{A.P} \in T(W_P(W_A(A.P)))$ and the projections and prove several useful properties about them. The strictness is found, among other things, in the fact that we can prove judgmental η -equality, and that pairing is strictly associative.

In this section we make use of the infix form of the weakening and substitution operations introduced in Definitions 4.50 and 4.51.

Definition 4.54. Let $x \in T(A)$ and $u \in T(S_x(P))$ for $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. We define the **term extension of x and u** to be

$$x.u := \text{idtm}_{A.P}[x][u] \in T(A.P).$$

It is well defined since

$$T(W_{A.P}(A.P)) \xrightarrow{S_x} T(W_{P[x]}(A.P)) \xrightarrow{S_u} T(A.P) \quad (11)$$

where $S_x \circ W_{A.P} = W_{P[x]}$ because S_x is a weakening homomorphism and $S_x \circ W_A = \text{Id}$.

To prove anything about the term $x.u$, we need the following property.

Theorem 4.55. Let $x \in T(A)$ and $u \in T(S_x(P))$ for $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. Then we have

$$S_{x.u} = S_u \circ (S_x/P) : \mathbb{E}/\Gamma.A.P \rightarrow \mathbb{E}/\Gamma$$

Proof.

$$\begin{aligned}
S_{x.u} &= S_{S_u(S_x(\text{idtm}_{A.P}))} && \text{(By 4.54)} \\
&= S_{S_u(S_x(\text{idtm}_{A.P}))} \circ (S_u \circ W_{S_x(P)}) \circ (S_x \circ W_A) && \text{(By 4.45.3)} \\
&= S_u \circ S_{S_x(\text{idtm}_{A.P})} \circ W_{S_x(P)} \circ S_x \circ W_A && \text{(By 4.45.1)} \\
&= S_u \circ (S_x/P) \circ S_{\text{idtm}_{A.P}} \circ W_P \circ W_A && \text{(By 4.45.1)} \\
&= S_u \circ (S_x/P) \circ S_{\text{idtm}_{A.P}} \circ W_{A.P} && \text{(By 4.28.2)} \\
&= S_u \circ (S_x/P). && \text{(By 4.45.5)}
\end{aligned}$$

□

Corollary 4.56. *For every $x \in T(A)$, $u \in T(S_x(P))$ and $v \in T(S_{x.u}(Q))$ we have*

$$(x.u).v = x.(u.v) \in T(A.P.Q).$$

Proof. By Theorem 4.55, we have $S_v \circ (S_{x.u}/Q) = S_v \circ (S_u/Q[x]) \circ (S_x/P.Q) = S_{u.v} \circ (S_x/P.Q)$, so associativity of term extension follows. □

Definition 4.57. Let $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. We define

$$\begin{aligned}
\text{pr}_0^{A,P} &:= \langle P \rangle \text{idtm}_A \\
\text{pr}_1^{A,P} &:= \text{idtm}_P
\end{aligned}$$

Lemma 4.58. *Let $F: \mathbb{E} \rightarrow \mathbb{D}$ be an E-homomorphism. For every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, $x \in T(A)$ and $u \in T(S_x(P))$, it is*

$$F(x.u) = F(x).F(u), \quad F(\text{pr}_0^{A,P}) = \text{pr}_0^{F(A),F(P)}, \quad \text{and} \quad F(\text{pr}_1^{A,P}) = \text{pr}_1^{F(A),F(P)}$$

Proof. We compute:

$$\begin{aligned}
F(x.u) &= F(\text{idtm}_{A.P}[x][u]) = \text{idtm}_{FA.FP}[Fx][Fu] = Fx.Fu, \\
F(\text{pr}_0^{A,P}) &= F(\langle P \rangle \text{idtm}_A) = \langle FP \rangle \text{idtm}_{FA} = \text{pr}_0^{FA,FP}, \\
F(\text{pr}_1^{A,P}) &= F(\text{idtm}_P) = \text{idtm}_{FP} = \text{pr}_1^{FA,FP}
\end{aligned}$$

where the outer equalities hold by definition, and the inner ones since F is an E-homomorphism. □

Lemma 4.59. *For every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, $x \in T(A)$ and $u \in T(S_x(P))$, it is*

$$\begin{aligned}
\text{pr}_0^{A,P}[x.u] &= x, \\
\text{pr}_1^{A,P}[x.u] &= u, \\
\text{pr}_0^{A,P}.\text{pr}_1^{A,P} &= \text{idtm}_{A.P}.
\end{aligned}$$

Proof. To show that $\text{pr}_0^{A,P}[x.u] = x$, we use that $S_{x.u} = S_u \circ S_x/P$ to show that

$$\begin{aligned}
\text{pr}_0^{A,P}[x.u] &= (\langle P \rangle \text{idtm}_A)[x.u] && \text{(By 4.57)} \\
&= (\langle P \rangle \text{idtm}_A)[x][u] && \text{(By Theorem 4.55)} \\
&= (\langle P[x] \rangle \text{idtm}_A[x])[u] && \text{(By 4.45.1)} \\
&= \text{idtm}_A[x] && \text{(By 4.45.3)} \\
&= x && \text{(By 4.45.4)}
\end{aligned}$$

To show that $\text{pr}_1^{A,P}[x.u] = u$, note that

$$\text{pr}_1^{A,P}[x.u] = \text{idtm}_P[x][u] = \text{idtm}_{P[x]}[u] = u$$

Finally note that

$$\langle A.P \rangle A.(W_{A.P}/A)(P) = \langle A.P \rangle A.P$$

and $\text{idtm}_{\langle A.P \rangle A.P} = (W_{A.P}/A.P)(\text{idtm}_{A.P})$. Thus $\text{pr}_0^{A,P}.\text{pr}_1^{A,P} = \text{idtm}_{A.P}$ follows from the commutativity of the outer square in the diagram below.

$$\begin{array}{ccc} T(W_{\langle A.P \rangle A.P}(\langle A.P \rangle A.P)) & \xrightarrow{S_{\text{pr}_0^{A,P}}} & T(W_P/P(\langle A.P \rangle A.P)) \\ \uparrow W_{A.P}/A.P & \nearrow W_P/P & \downarrow S_{\text{pr}_1^{A,P}} \\ T(\langle A.P \rangle A.P) & \xrightarrow{\text{id}} & T(\langle A.P \rangle A.P) \end{array}$$

The bottom-right triangle commutes by 4.45.5. For the top-left one:

$$\begin{aligned} S_{\langle P \rangle \text{idtm}_A} \circ W_{A.P}/A.P &= S_{\langle P \rangle \text{idtm}_A} \circ W_P/(W_A(A.P)) \circ W_A/A.P && \text{(By 4.28.2)} \\ &= (W_P \circ S_{\text{idtm}_A} \circ W_A/A)/P && \text{(By 4.45.2)} \\ &= W_P/P. && \text{(By 4.45.5)} \end{aligned}$$

□

Theorem 4.60. *For every $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$, the map*

$$\begin{aligned} \coprod_{x \in T(A)} T(P[x]) &\longrightarrow T(A.P) \\ (x, u) &\longmapsto x.u \end{aligned}$$

is a bijection.

Proof. The inverse to the given map is defined by $w \mapsto (\text{pr}_0^{A,P}[w], \text{pr}_1^{A,P}[w])$. Thanks to Lemma 4.59 it is enough to show that, for every $w \in T(A.P)$, one has

$$\text{pr}_0^{A,P}[w].\text{pr}_1^{A,P}[w] = w.$$

Lemma 4.58 gives us that y

$$\text{pr}_0^{A,P}[w].\text{pr}_1^{A,P}[w] = (\text{pr}_0^{A,P}.\text{pr}_1^{A,P})[w].$$

Thus the claim follows from $\text{pr}_0^{A,P}.\text{pr}_1^{A,P} = \text{idtm}_{A.P}$, which holds again by Lemma 4.59. □

One consequence of this theorem is that the set $T(\text{id}_\Gamma)$ has exactly one element, see Corollary 5.18.

4.3 Characterising B-systems as stratified E-systems

In this section we construct an equivalence of categories between B-systems and the subcategory of **Esys** on the stratified E-systems and stratified homomorphisms. The functor from B-systems to stratified E-systems is constructed in Section 4.3.1, the one in the other direction in Section 4.3.2. That these form an equivalence is shown in Section 4.3.3.

4.3.1 From B-systems to stratified E-systems

Note first that we obtain a functor $\mathbf{Bfr} \rightarrow \mathbf{Cat}$ as the composition

$$\mathbf{Bfr} \xrightarrow{\mathbf{R}} \mathbf{RtTr} \xrightarrow{\mathbf{G}} \mathbf{Grph} \xrightarrow{\mathbf{F}} \mathbf{Cat}$$

where \mathbf{G} and \mathbf{F} are the functors from Definition 2.13 and \mathbf{R} is the forgetful functor from Definition 4.6. Arrows in $\mathbf{FGR}(\mathbb{B})$ are of the form $(X, k): (n+k, X) \rightarrow (n, \text{ft}^k(X))$, for $X \in B_{n+k}$.

We begin by equipping $\mathbf{FGR}(\mathbb{B})$ with a term structure. The B-frame \mathbb{B} already provides us with sets of terms for the edges of $\mathbf{GR}(\mathbb{B})$, namely $T(X, 1) := \partial^{-1}(X)$. In order to construct sets of terms for (X, k) for each k , which we do in Construction 4.62, we assume that \mathbb{B} comes with a substitution structure in the sense of Definition 4.23. We then show in Construction 4.65 that \mathbf{FGR} gives rise to a functor \mathbf{T} from B-frames with substitution to strict categories with term structure. Next, in Construction 4.68 we provide $\mathbf{T}(\mathbb{B})$ with a pre-E-system structure when \mathbb{B} is a B-system, and prove in Lemma 4.69 that \mathbf{T} preserves and reflects weakening and projection homomorphisms. Finally, we show in Lemma 4.71 that the functor \mathbf{T} lifts to a full and faithful functor from B-systems to stratified E-systems.

Problem 4.61. *For every B-frame \mathbb{B} with substitution structure S , to construct a term structure T on the strict category $\mathcal{F}_{\mathbb{B}} := \mathbf{FGR}(\mathbb{B})$ and to construct, for any $t \in T(X, k)$, a homomorphism of B-frames $S_t^k: \mathbb{B}/X \rightarrow \mathbb{B}/\text{ft}^k(X)$.*

Construction 4.62 (for Problem 4.61). We define the term structure by induction on $n \in \mathbb{N}$. More precisely, for any $X \in B_n$ and $k \leq n$ we will define a set $T(X, k)$ and, for any $t \in T(X, k)$, a homomorphism of B-frames $S_t^k: \mathbb{B}/X \rightarrow \mathbb{B}/\text{ft}^k(X)$.

For every n and $X \in B_n$, let $T(X, 0) := \{*\}$ and $S_*^0 := \text{id}: \mathbb{B}/X \rightarrow \mathbb{B}/X$.

For every n and $X \in B_{n+1}$, let

$$T(X, 1) := \partial^{-1}(X) \subseteq \tilde{B}_{n+1} \quad (12)$$

and $S_x^1 := S_x: \mathbb{B}/X \rightarrow \mathbb{B}/\text{ft}(X)$ which is a homomorphism of B-frames by assumption.

Suppose now that, for every $m \leq n$ and $Y \in B_m$, we have defined sets $T(Y, k)$ for $k \leq m$ and, for every $t \in T(Y, k)$, a homomorphism of B-frames $S_t^k: \mathbb{B}/Y \rightarrow \mathbb{B}/\text{ft}^k(Y)$. Let $X \in B_{n+1}$ and define, for $1 \leq k \leq n$,

$$T(X, k+1) := \coprod_{t \in T(\text{ft}(X), k)} T(S_t(X), 1). \quad (13)$$

and, for $(t, x) \in T(X, k+1)$, a homomorphism of B-frames $S_{(t,x)}^{k+1}$ as the composite below

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{S_{(t,x)}^{k+1}} & \mathbb{B}/\text{ft}^{k+1}(X) \\ & \searrow S_t^k/X & \nearrow S_x \\ & \mathbb{B}/S_t^k(X) & \end{array} \quad (14)$$

where S_x comes from the substitution structure and S_t^k from the inductive hypothesis. \square

Remark 4.63.

1. For every B-frame \mathbb{B} and $X \in B_n$, there is an isomorphism of strict categories $\mathbf{FGR}(\mathbb{B}/X) \cong \mathbf{FGR}(\mathbb{B})/(n, X)$ natural in \mathbb{B} which maps (i, Y) to $(n+i, Y)$ and it is the identity on arrows. It follows that, when \mathbb{B} is a B-system, we can choose the identity as the action on the term structure. Therefore this isomorphism of categories lifts to an isomorphism of categories with term structure $(\mathbf{FGR}(\mathbb{B}/X), T_{\mathbb{B}}) \cong (\mathbf{FGR}(\mathbb{B})/(n, X), T_{\mathbb{B}})$.

Once we establish an E-system structure on $\mathbf{FGR}(\mathbb{B})$, we will see that this isomorphism is in fact an isomorphism of E-systems.

2. Let \mathbb{A} and \mathbb{B} be B-frames with substitution structure and $H: \mathbb{A} \rightarrow \mathbb{B}$ be a homomorphism of B-frames. If H preserves the substitution structure, then for every $X \in B_{n+k}$ and $t \in T(X, k)$ the square

$$\begin{array}{ccc} \mathbb{A}/X & \xrightarrow{H/X} & \mathbb{B}/H(X) \\ S_t^k \downarrow & & \downarrow S_{\tilde{H}(t)}^k \\ \mathbb{A}/\text{ft}^k(X) & \xrightarrow{H/\text{ft}^k(X)} & \mathbb{B}/\text{ft}^k H(X) \end{array} \quad (15)$$

commutes in \mathbf{Bfr} , where $\tilde{H}(t) := (\tilde{H}(t_1), \dots, \tilde{H}(t_k))$. Indeed, by definition of S_t^k in (14), the square (15) factors vertically into k squares of the form in Definition 4.11.1, each of which commutes if H preserves the substitution structure.

3. Let \mathbb{A} and \mathbb{B} be B-frames with substitution structure and $H: \mathbb{A} \rightarrow \mathbb{B}$ be a homomorphism of B-frames. Suppose that \mathbb{A} and \mathbb{B} have weakening structure and define, for every $X \in B_{n+k}$, the homomorphism of B-frames $W_X^k: \mathbb{B}/\text{ft}^k(X) \rightarrow \mathbb{B}/X$ as the composite

$$\mathbb{B}/\text{ft}^k(X) \xrightarrow{W_{\text{ft}^{k-1}(X)}} \cdots \xrightarrow{W_{\text{ft}(X)}} \mathbb{B}/\text{ft}(X) \xrightarrow{W_X} \mathbb{B}/X \quad (16)$$

W_X^k (curved arrow above the sequence)

which we take to be $\text{id}_{\mathbb{B}}$ if $n = k = 0$.

If H preserves weakening structure, then for every $X \in B_{n+k}$ the square

$$\begin{array}{ccc} \mathbb{A}/X & \xrightarrow{H/X} & \mathbb{B}/H(X) \\ W_X^k \uparrow & & \uparrow W_{H(X)}^k \\ \mathbb{A}/\text{ft}^k(X) & \xrightarrow{H/\text{ft}^k(X)} & \mathbb{B}/\text{ft}^k H(X) \end{array} \quad (17)$$

commutes in \mathbf{Bfr} . Indeed, by definition of W_X^k in (16) the square (17) factors vertically into k squares of the form in Definition 4.11.2, each of which commutes if H preserves the weakening structure.

Problem 4.64. *To lift the functor $\mathbf{FGR}: \mathbf{Bfr} \rightarrow \mathbf{Cat}$ to a functor $\mathbf{T}: \mathbf{SubBfr} \rightarrow \mathbf{TCat}$ from the category \mathbf{SubBfr} of B-frames with substitution structure and homomorphisms of B-frames that preserve the substitution structure, to the category of strict categories with term structure.*

Construction 4.65 (for Problem 4.64). Let \mathbb{A} and \mathbb{B} be B-frames with substitution structure. For every homomorphism of B-frames $H: \mathbb{A} \rightarrow \mathbb{B}$, the functor $\mathbf{FGR}(H): \mathcal{F}_{\mathbb{A}} \rightarrow \mathcal{F}_{\mathbb{B}}$, maps an object (n, X) to $(n, H(X))$ and an arrow (X, k) to $(H(X), k)$. Since $H(*) = *$, the functor $\mathbf{FGR}(H)$ strictly preserves the (unique) terminal object.

To make $\mathbf{T}(H) := \mathbf{FGR}(H)$ into a functor with term structure note that, for every $t = (t_1, \dots, t_k) \in T(X, k)$ and $1 \leq j \leq k$, the function \tilde{H} restricts as follows

$$\begin{array}{ccc} T(\text{ft}^{k-j} S_{t_{j-1}} \cdots S_{t_1}(X), 1) & \hookrightarrow & \tilde{A}_{n-k+1} \\ \downarrow \tilde{H} & & \downarrow \tilde{H} \\ T(\text{ft}^{k-j} S_{\tilde{H}(t_{j-1})} \cdots S_{\tilde{H}(t_1)} H(Y), 1) & \hookrightarrow & \tilde{B}_{n-k+1} \end{array} \quad (18)$$

since H commutes with the functions ft and preserves the substitution structure in the sense of Definition 4.11. It follows that

$$\mathbf{T}(H)(t) := (\tilde{H}(t_1), \dots, \tilde{H}(t_k)) \in T(\tilde{H}(X), k). \quad (19)$$

This makes $\mathbf{T}(H): \mathcal{F}/(n, X) \rightarrow \mathcal{F}/(n-k, \text{ft}^k(X))$ into a functor with term structure.

The action of H on the sets $T(X, k)$ is clearly functorial in H . \square

Remark 4.66.

1. The functor $\mathbf{T}: \mathbf{SubBfr} \rightarrow \mathbf{TCat}$ from Construction 4.65 is faithful, since the functors $\mathbf{R}: \mathbf{Bfr} \rightarrow \mathbf{RtTr}$, $\mathbf{G}: \mathbf{RtTr} \rightarrow \mathbf{Grph}$ and $\mathbf{F}: \mathbf{Grph} \rightarrow \mathbf{Cat}$ are faithful and the sets $T(X, 1)$ for $X \in B_n$ form a partition of \tilde{B}_n .
2. For every B-frame with substitution structure \mathbb{B} , it follows by Propositions 2.8 and 2.14 that the underlying category of $\mathbf{T}(B)$ is stratified by the functor that maps $(X, k): (n+k, X) \rightarrow (n, \text{ft}^k(X))$ to $n+k \geq n$.

Problem 4.67. For every B-system \mathbb{B} , to construct a pre-E-system structure on the category with term structure $\mathbf{T}(\mathbb{B})$ from Construction 4.62.

Construction 4.68 (for Problem 4.67). Construction 4.62 provides a homomorphism of B-frames $S_t^k: \mathbb{B}/X \rightarrow \mathbb{B}/\text{ft}^k(X)$ for every $X \in B_{n+k}$ and $t \in T(X, k)$. The homomorphism S_t^k preserves the substitution structure since it factors, as in Remark 4.63.2, into k B-homomorphisms of the form S_{x_j} , where $x_j \in \tilde{B}_{n+k-j}$ for $j < k$. Construction 4.65 and Remark 4.63.1 then yield a functor with term structure

$$(\mathbf{T}(\mathbb{B})/(n+k, X), T_{\mathbb{B}}) \xrightarrow{S_t := \mathbf{T}(S_t^k)} (\mathbf{T}(\mathbb{B})/(n, \text{ft}^k(X)), T_{\mathbb{B}}) \quad (20)$$

as required.

For the pre-weakening structure, consider the homomorphism of B-frames $W_X^k: \mathbb{B}/\text{ft}^k(X) \rightarrow \mathbb{B}/X$ defined in Remark 4.63.3. Since \mathbb{B} is a B-system, each factor of W_X^k in (16) is a homomorphism of B-systems and so is W_X^k . Construction 4.65 and Remark 4.63.1 provide a functor with term structure

$$\mathbf{T}(\mathbb{B})/(n, \text{ft}^k(X)) \xrightarrow{W_{(X,k)} := \mathbf{T}(W_X^k)} \mathbf{T}(\mathbb{B})/(n, X). \quad (21)$$

It remains to construct the pre-projection structure. In fact, we will prove a little bit more. We construct by induction on $n \in \mathbb{N}$, for every $X \in B_n$ and $k \leq n$, an element $\text{idtm}_{(X,k)} \in T(W_{(X,k)}(X, k)) = T(W_X^k(X), k)$ with the property that the triangle of B-homomorphisms

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{\text{id}} & \mathbb{B}/X \\ & \searrow W_X^k/X & \nearrow S_{\text{idtm}_{(X,k)}}^k \\ & \mathbb{B}/W_X^k(X) & \end{array}$$

commutes. This additional condition is needed in the inductive construction. For every n and $X \in B_n$, let

$$\text{idtm}_{(X,0)} := * \in T(W_{(X,0)}(X, 0), 0) = T(X, 0).$$

For every n and $X \in B_{n+1}$, it is $\partial \circ \delta(X) = W_X(X) \in B_{n+2}$. Thus we can define

$$\text{idtm}_{(X,1)} := \delta(X) \in T(W_{(X,1)}(X, 1)) = T(W_X(X), 1) \quad (22)$$

and $S_{\text{idtm}_{(X,1)}}^1 \circ W_X/X = \text{id}_{\mathbb{B}/X}$ by condition 5 in Definition 4.12.

Suppose now that we have defined, for every $m \leq n$, $Y \in B_m$ and $i \leq m$, an element $\text{idtm}_{(Y,i)} \in T(W_{(Y,i)}(Y, i))$ such that $S_{\text{idtm}_{(Y,i)}}^i \circ W_Y^i/Y = \text{id}_{\mathcal{T}/(m,Y)}$. Let $X \in B_{n+1}$. It follows from (13) that, for every $1 \leq k \leq n$

$$T(W_{(X,k+1)}(X, k+1)) = \coprod_{t \in T(W_X^{k+1}(\text{ft}(X)), k)} T(S_t^k \circ W_X^{k+1}(X), 1).$$

But $W_X^{k+1} = W_X \circ W_{\text{ft}(X)}^k$, thus

$$\bar{t} := \tilde{W}_X(\text{idtm}_{(\text{ft}(X), k)}) \in T(W_X^{k+1}(\text{ft}(X)), k)$$

and

$$\begin{aligned} S_{\bar{t}}^k \circ W_X^{k+1}/\text{ft}(X) &= S_{\bar{t}}^k \circ W_X/W_{\text{ft}(X)}^k(\text{ft}(X)) \circ W_{\text{ft}(X)}^k/\text{ft}(X) \\ &= W_X \circ S_{\text{idtm}_{(\text{ft}(X), k)}}^k \circ W_{\text{ft}(X)}^k/\text{ft}(X) \\ &= W_X \end{aligned} \quad (23)$$

by Remark 4.63.2 and the fact that W_X preserves the substitution structure, and assumption (4.68). In particular, $T(S_{\bar{t}}^k \circ W_X^{k+1}(X), 1) = T(W_X(X), 1)$ and we can define

$$\text{idtm}_{(X, k+1)} := (\bar{t}, \delta(X)). \quad (24)$$

It remains to check that $S_{\text{idtm}_{(X, k+1)}}^{k+1} \circ W_X^{k+1}/X = \text{id}_{\mathbb{B}/X}$. This is indeed the case by (14), (23) and condition 5 in Definition 4.12:

$$\begin{aligned} S_{\text{idtm}_{(X, k+1)}}^{k+1} \circ W_X^{k+1}/X &= S_{\delta(X)} \circ (S_{\bar{t}}^k \circ W_X^{k+1}/\text{ft}(X)) / X \\ &= S_{\delta(X)} \circ W_X / X \\ &= \text{id}_{\mathbb{B}/X}. \end{aligned}$$

This completes the construction of the pre-E-system structure. \square

Lemma 4.69. *Let \mathbb{A} and \mathbb{B} be B-systems and $H: \mathbb{A} \rightarrow \mathbb{B}$ a homomorphism of B-frames that preserves the substitution structure.*

1. *The functor with term structure $\mathbf{T}(H): \mathcal{F}_{\mathbb{A}} \rightarrow \mathcal{F}_{\mathbb{B}}$ is a pre-substitution homomorphism.*
2. *H preserves the weakening structure if and only if $\mathbf{T}(H)$ is a pre-weakening homomorphism.*
3. *H preserves the structure of generic elements if and only if $\mathbf{T}(H)$ is a pre-projection homomorphism.*

Proof.

1. By definition of the pre-substitution structure in Construction 4.68 and Remark 4.63.1, $\mathbf{T}(H)$ is a pre-substitution homomorphism if every image under \mathbf{T} of any square in \mathbf{Bfr} of the form (15) commutes. By Remark 4.63.2, such squares commute since H preserves the substitution structure.
2. By definition of the pre-weakening structure (16) and Remark 4.63.1, $\mathbf{T}(H)$ is a pre-weakening homomorphism if and only if the image under $\mathbf{T}: \mathbf{Bsys} \rightarrow \mathbf{TCat}$ of any square in \mathbf{Bfr} of the form (17) commutes. By Remark 4.63.3, such squares commute if H preserves the weakening structure. The converse holds since \mathbf{T} is faithful by Remark 4.66.1.
3. By (18) and (19), $\mathbf{T}(H)$ acts componentwise as \tilde{H} on a term $t \in T(X, t)$. It follows that $\mathbf{T}(H)$ preserves the terms $\text{idtm}_{(X, k+1)} = (\delta(\text{ft}^k(X)), \dots, \delta(X))$ for $X \in B_n, k < n$ if and only if H preserves generic elements. \square

Lemma 4.70. *For every B-system \mathbb{B} , the pre-E-system constructed in 4.68 is a stratified E-system.*

Proof. First, we need to verify conditions 1–4 in Definition 4.45, as condition 5 holds by construction.

1. It follows from Lemma 4.69 and (20) since S_t^k , as defined in (14), is a homomorphism of B-systems when \mathbb{B} is a B-system.
2. As above, it follows by Lemma 4.69 and (21) since W_X^k , as defined in (16), is a homomorphism of B-systems.
3. The case $X \in B_n, * \in T(X, 0)$ holds trivially. The case $X \in B_{n+1}, x \in T(X, 1)$ follows from condition 3 in Definition 4.12 and functoriality of \mathbf{T} . The case $X \in B_{n+k+1}, (t, x) \in T(X, k+1)$, where $t \in T(\text{ft}(X), k)$ and $x \in T(S_t^k(X), 1)$, holds by induction and functoriality of \mathbf{T} as

$$\begin{aligned} S_{(t,x)}^k \circ W_X^{k+1} &= S_x \circ S_t^k / X \circ W_X \circ W_{\text{ft}(X)}^k \\ &= S_x \circ W_{S_t(X)} \circ S_t^k \circ W_{\text{ft}(X)}^k \\ &= \text{id}_{\mathbb{B}/\text{ft}^{k+1}(X)} \end{aligned}$$

by (14) and (16), the fact that S_t is a pre-E-homomorphism, and Definition 4.12.2 and the inductive hypothesis.

4. As above, the case $k = 0$ holds trivially and the case $k = 1$ holds by condition 4 in Definition 4.12. For $X \in B_{n+1}, k \leq n$ and $(t, x) \in T(X, k+1)$,

$$\begin{aligned} \tilde{S}_{(t,x)}^{k+1}(\text{idtm}_{(X,k+1)}) &= \tilde{S}_x \circ \tilde{S}_t^k(\tilde{W}_X(\text{idtm}_{(\text{ft}(X),k)}), \delta(X)) \\ &= \tilde{S}_x(\tilde{W}_{S_t^k(X)} \circ \tilde{S}_t^k(\text{idtm}_{(\text{ft}(X),k)}), \delta(S_t^k(X))) \\ &= (\tilde{S}_x \circ \tilde{W}_{S_t^k(X)}(t), \tilde{S}_x(\delta(S_t^k(X)))) \\ &= (t, x) \end{aligned}$$

by (14) and (24), the fact that S_t is a pre-E-homomorphism, the inductive hypothesis, and conditions 3 and 4 in Definition 4.12.

Finally, the underlying category $\mathcal{F} = \mathbf{FGR}(\mathbb{B})$ is stratified by Remark 4.66.2. By definition, weakening and substitution functors preserve the \mathbb{N} -component of objects and arrows. It follows that $\mathbf{T}(\mathbb{B})$ is a stratified E-system. \square

Lemma 4.71.

1. *The functor $\mathbf{T}: \mathbf{SubBfr} \rightarrow \mathbf{TCat}$ from Construction 4.65 lifts to a functor $\mathbf{B2E}: \mathbf{Bsys} \rightarrow \mathbf{Esys}_{\mathfrak{s}}$.*
2. *The functor $\mathbf{B2E}$ is full and faithful.*

Proof.

1. By Lemma 4.70, it is enough to show that, for every homomorphism of B-systems $H: \mathbb{A} \rightarrow \mathbb{B}$, the functor with term structure $\mathbf{T}(H): \mathbf{T}(\mathbb{A}) \rightarrow \mathbf{T}(\mathbb{B})$ is a stratified homomorphism of E-systems. By Lemma 4.69, $\mathbf{T}(H)$ is a homomorphism of E-systems. It is stratified since it preserves the \mathbb{N} -component of objects and arrows by definition.
2. The functor $\mathbf{B2E}$ is faithful by Remark 4.66.1. Let then $K: \mathbf{B2E}(\mathbb{A}) \rightarrow \mathbf{B2E}(\mathbb{B})$ be a stratified homomorphism of E-systems. Since K is stratified, the function on objects $K: \coprod_m A_m \rightarrow \coprod_n B_n$ is the identity on indices and gives rise to a family of functions $H: \prod_n (A_n \rightarrow B_n)$ such that

$$K(n, X) = (n, H_n(X)). \tag{25}$$

This function commutes with \mathbf{ft} functions since, for every $X \in A_{n+1}$, the arrow $K(X, 1): (n + 1, H_{n+1}(X)) \rightarrow (n, H_n(\mathbf{ft}(X)))$ in $\mathcal{F}_{\mathbb{B}}$ is necessarily of the form $(n + 1, Y) \rightarrow (n, \mathbf{ft}(Y))$.

The sets $T(X, 1)$ for $X \in A_{n+1}$ form a partition of \tilde{A}_{n+1} , thus the functions $K_X: T(X, 1) \rightarrow T(H_{n+1}(X), 1)$ for $X \in A_{n+1}$ glue together to form a function $\tilde{H}_{n+1}: \tilde{A}_{n+1} \rightarrow \tilde{B}_{n+1}$ such that

$$\tilde{H}_{n+1}(x) = K_{\partial(x)}(x). \quad (26)$$

It follows that $\partial \circ \tilde{H} = H \circ \partial$ since $\tilde{H}_{n+1}(x) \in T(H_{n+1}(\partial(x)), 1)$. Thus we have defined a homomorphism of B-frames $H: \mathbb{A} \rightarrow \mathbb{B}$.

Let $x \in \tilde{B}_{n+1}$. Since K is a substitution homomorphism, for every $Y \in B_{n+k+1}$ such that $\mathbf{ft}^k(Y) = \partial(x)$, it is

$$\begin{aligned} (n + k, \tilde{H}_{n+k} \circ S_x(Y)) &= K \circ S_x(n + k + 1, Y) \\ &= S_{K_{\partial(x)}(x)} \circ K(n + k + 1, Y) \\ &= (n + k, S_{\tilde{H}_{n+1}(x)} \circ H_{n+k+1}(Y)) \end{aligned}$$

and, for every $y \in \tilde{B}_{n+k+1}$ such that $\mathbf{ft}^k \circ \partial(y) = \partial(x)$, it is

$$\begin{aligned} \tilde{H}_{n+k} \circ \tilde{S}_x(y) &= K_{\partial \circ S_x(y)} \circ S_x(y) \\ &= S_{K_{\partial(x)}(x)} \circ K_{\partial(y)}(y) \\ &= \tilde{S}_{\tilde{H}_{n+1}(x)} \circ \tilde{H}_{n+k+1}(y). \end{aligned}$$

It follows that the homomorphism of B-frames H preserves the substitution structure. Moreover, it is $\mathbf{T}(H) = K$ by (25), (26) and Construction 4.65. Thus $\mathbf{B2E}(H) = \mathbf{T}(H) = K$ once we show that H is a homomorphism of B-systems.

It remains to verify that H also preserve the weakening structure and the structure of generic elements. Since K is a projection homomorphism, for every $X \in B_{n+1}$ it is

$$\mathbf{T}(H/X \circ W_X) = K/(n + 1, X) \circ W_{(X,1)} = W_{K(X,1)} \circ K/(n, \mathbf{ft}(X)) = \mathbf{T}(W_{H_{n+1}(X)} \circ H/\mathbf{ft}(X)).$$

The first claim then follows from faithfulness of $\mathbf{B2E}$. Finally, H preserves generic elements

$$\tilde{H}_{n+2} \circ \delta(X) = K_{W_X(X)}(\mathbf{idtm}_{(X,1)}) = \mathbf{idtm}_{K(X,1)} = \delta \circ H_{n+1}(X)$$

since K is a projection homomorphism. \square

4.3.2 From stratified E-systems to B-systems

We have constructed a full and faithful functor $\mathbf{Bsys} \rightarrow \mathbf{Esys}_s$. Here we construct a functor in the opposite direction. We begin in Construction 4.75 with a functor $\mathbf{E2B}$ from stratified categories with term structures to B-frames. In Construction 4.77 we consider substitution, weakening and projection structures and prove in Lemma 4.78 that $\mathbf{E2B}$ maps homomorphisms into homomorphisms. This allows us to lift $\mathbf{E2B}$ to a functor $\mathbf{Esys}_s \rightarrow \mathbf{Bsys}$ in Construction 4.80.

Problem 4.72. *Given a stratified category with term structure (\mathcal{F}, T) , to construct a B-frame $\mathbf{B2E}(\mathcal{F}, T)$.*

Construction 4.73 (for Problem 4.72). For every object X in \mathcal{F} , let \overline{X} denote the unique individual arrow with domain X . For every $n \in \mathbb{N}$, define sets

$$B(\mathcal{F}, T)_n := \{X \in \text{Ob}(\mathcal{F}) \mid L(X) = n\} \quad (27)$$

$$\tilde{B}(\mathcal{F}, T)_{n+1} := \coprod_{X \in B(\mathcal{F}, T)_{n+1}} T(\overline{X}) \quad (28)$$

and functions $\text{ft}_n: B(\mathcal{F}, T)_{n+1} \rightarrow B(\mathcal{F}, T)_n$ and $\partial_n: \tilde{B}(\mathcal{F}, T)_{n+1} \rightarrow B(\mathcal{F}, T)_{n+1}$ by

$$\text{ft}(X) := \text{cod}(\overline{X}) \quad (29)$$

$$\partial(X, x) := X. \quad (30)$$

These definitions give rise to a B-frame $\mathbf{E2B}(\mathcal{F}, T)$. \square

Problem 4.74. *To construct a functor $\mathbf{E2B}: \mathbf{TCat}_s \rightarrow \mathbf{Bfr}$ from the category of stratified categories with term structure and stratified functors with term structure to the category of B-frames and homomorphisms.*

Construction 4.75 (for Problem 4.74). The action on objects is given by Construction 4.73. Let then $F: (\mathcal{F}, T) \rightarrow (\mathcal{F}', T')$ be a stratified functor with term structure. We need to construct a homomorphism of B-frames $\mathbf{E2B}(F): \mathbf{E2B}(\mathcal{F}, T) \rightarrow \mathbf{E2B}(\mathcal{F}', T')$. Since F is stratified, it maps $B(\mathcal{F}, T)_n$ into $B(\mathcal{F}', T')_n$. For every $X \in B(\mathcal{F}, T)_{n+1}$, the functor F maps the individual arrow \overline{X} to the individual arrow $\overline{F(X)}$ by Lemma 2.10. It follows first that

$$\begin{aligned} F \circ \text{ft}(X) &= F \circ \text{cod}(\overline{X}) \\ &= \text{cod}(\overline{F(X)}) \\ &= \text{ft} \circ F(X), \end{aligned}$$

and secondly that we can define, for every $n \in \mathbb{N}$, a function $\tilde{F}: \tilde{B}(\mathcal{F}, T)_{n+1} \rightarrow \tilde{B}(\mathcal{F}', T')_{n+1}$ such that $\partial \circ \tilde{F}(X, t) = F \circ \partial(X, t)$ by

$$\tilde{F}(X, t) := (F(X), F(t)). \quad (31)$$

This defines a homomorphism of B-frames $\mathbf{E2B}(F) := (F, \tilde{F})$. \square

Problem 4.76. *Let (\mathcal{F}, T) be a stratified category with term structure and consider the B-frame $\mathbf{E2B}(\mathcal{F}, T)$ from Construction 4.73*

1. *From a stratified pre-substitution structure on (\mathcal{F}, T) , construct a substitution structure on $\mathbf{E2B}(\mathcal{F}, T)$.*
2. *From a stratified pre-weakening structure on (\mathcal{F}, T) , construct a weakening structure on $\mathbf{E2B}(\mathcal{F}, T)$.*
3. *From a pre-projection structure on (\mathcal{F}, T) , construct a structure of generic elements on $\mathbf{E2B}(\mathcal{F}, T)$.*

Construction 4.77 (for Problem 4.76).

1. For every $(X, t) \in \tilde{B}(\mathcal{F}, T)_{n+1}$, the functor with term structure $S_t: (\mathcal{F}, T)/X \rightarrow (\mathcal{F}, T)/\text{ft}(X)$ is stratified. Construction 4.75 then yields a homomorphism of B-frames

$$\mathbf{E2B}(\mathcal{F}, T)/X \xrightarrow{S_{(X,t)} := \mathbf{E2B}(S_t)} \mathbf{E2B}(\mathcal{F}, T)/\text{ft}(X). \quad (32)$$

2. For every $X \in B(\mathcal{F}, T)_n$, the functor with term structure $W_{\overline{X}}: (\mathcal{F}, T)/\text{ft}(X) \rightarrow (\mathcal{F}, T)/X$ is stratified, where \overline{X} denotes the unique individual arrow with domain X . Construction 4.75 then yields a homomorphism of B-frames

$$\mathbf{E2B}(\mathcal{F}, T)/\text{ft}(X) \xrightarrow{W_{\overline{X}} := \mathbf{E2B}(W_{\overline{X}})} \mathbf{E2B}(\mathcal{F}, T)/X. \quad (33)$$

3. For every $X \in B(\mathcal{F}, T)_{n+1}$, we can define

$$\delta(X) := (W_X(X), \text{idtm}_{\overline{X}}) \in \tilde{B}(\mathcal{F}, T)_{n+2} \quad (34)$$

since $\overline{W_X(X)} = W_{\overline{X}}(\overline{X})$. \square

Lemma 4.78. *Let $F: (\mathcal{F}, T) \rightarrow (\mathcal{F}', T')$ be a stratified functor with term structure.*

1. *If (\mathcal{F}, T) and (\mathcal{F}', T') have stratified pre-substitution structure and F is a pre-substitution homomorphism, then $\mathbf{E2B}(F): \mathbf{E2B}(\mathcal{F}, T) \rightarrow \mathbf{E2B}(\mathcal{F}', T')$ preserves the substitution structure.*
2. *If (\mathcal{F}, T) and (\mathcal{F}', T') have stratified pre-weakening structure and F is a pre-weakening homomorphism, then $\mathbf{E2B}(F): \mathbf{E2B}(\mathcal{F}, T) \rightarrow \mathbf{E2B}(\mathcal{F}', T')$ preserves the weakening structure.*
3. *If (\mathcal{F}, T) and (\mathcal{F}', T') have stratified pre-projection structure and F is a pre-projection homomorphism, then $\mathbf{E2B}(F): \mathbf{E2B}(\mathcal{F}, T) \rightarrow \mathbf{E2B}(\mathcal{F}', T')$ preserves the structure of generic elements.*

Proof.

1. We need to show that, for every $(X, t) \in \tilde{B}(\mathcal{F}, T)_{n+1}$, it is $\mathbf{E2B}(F)/\text{ft}(X) \circ S_{(X,t)} = S_{(F(X), F(t))} \circ \mathbf{E2B}(F)/X$. This follows from (32), functoriality of $\mathbf{E2B}$ and $F/\text{ft}(X) \circ S_t = S_{F(t)} \circ F/X$, which holds because F is a pre-substitution homomorphism.
2. We need to show that, for every $X \in B(\mathcal{F}, T)_n$, it is $\mathbf{E2B}(F)/X \circ W_X = W_{F(X)} \circ \mathbf{E2B}(F)/\text{ft}(X)$. This follows from (33), functoriality of $\mathbf{E2B}$ and $F/X \circ W_{\overline{X}} = W_{\overline{F(X)}} \circ F/\text{ft}(X)$, which holds because F is a pre-substitution homomorphism and $\overline{F(X)} = F(\overline{X})$.
3. For every $X \in B(\mathcal{F}, T)_{n+1}$, it is

$$\mathbf{E2B}(F) \circ \delta(X) = (F(W_X(X)), F(\text{idm}_{\overline{X}})) = (W_{F(X)}(F(X)), \text{idm}_{\overline{F(X)}}) = \delta \circ \mathbf{E2B}(F)(X)$$

where the first and last equality hold by (31) and (34), and the middle one because F is a pre-projection homomorphism. \square

Problem 4.79. *To lift the functor $\mathbf{E2B}: \mathbf{TCat}_s \rightarrow \mathbf{Bfr}$ to a functor $\mathbf{E2B}: \mathbf{Esys}_s \rightarrow \mathbf{Bsys}$.*

Construction 4.80 (for Problem 4.79). Let \mathbb{E} be a stratified E-system. Then $\mathbf{E2B}(\mathcal{F}, T)$ can be given the structure of a pre-B-system $\mathbf{E2B}(\mathbb{E})$ by Construction 4.77. To show that $\mathbf{E2B}(\mathbb{E})$ is a B-system, we need to verify conditions 1–5 of Definition 4.12.

- 1,2. Since every stratified functor with term structure of the form S_t, W_A is an E-homomorphism, it follows by Lemma 4.78 that the homomorphisms of B-frames constructed in (32) and (33) are homomorphisms of B-systems.
3. For $(X, t) \in \tilde{B}(\mathbb{E})_{n+1}$, it is

$$S_{(X,t)} \circ W_X = \mathbf{E2B}(S_t \circ W_{\overline{X}}) = \text{id}_{\mathbf{E2B}(\mathbb{E})/\text{ft}(X)}$$

by (32), (33), functoriality of $\mathbf{E2B}$ and 4.45.3.

4. For $(X, t) \in \tilde{B}(\mathbb{E})_{n+1}$, it is

$$S_{(X,t)} \circ \delta(X) = ((S_{(X,t)} \circ W_X)(X), S_t(\text{idm}_{\overline{X}})) = (X, t)$$

by (32), (34), condition 3 just proved and 4.45.4.

5. For every $X \in B(\mathbb{E})_{n+1}$, it is

$$S_{\delta(X)} \circ W_X/X = \mathbf{E2B}(S_{\text{idm}_{\overline{X}}} \circ W_{\overline{X}}/X) = \text{id}_{\mathbf{E2B}(\mathbb{E})/X}$$

by (32–34), functoriality of $\mathbf{E2B}$ and 4.45.5.

Finally, for every stratified E-homomorphism $F: \mathbb{E} \rightarrow \mathbb{D}$, the homomorphism of B-frames $\mathbf{E2B}(F): \mathbf{E2B}(\mathbb{E}) \rightarrow \mathbf{E2B}(\mathbb{D})$ is a homomorphism of B-systems by Lemma 4.78. \square

4.3.3 Equivalence of B-systems and stratified E-systems

Here we show in Theorem 4.83 that the functors **B2E** from Lemma 4.71 and **E2B** from Construction 4.80 form an equivalence of categories. We do so by showing in Construction 4.82 that **E2B**: **Esys_s** → **Bsys** is an essential section of the full and faithful functor **B2E**.

Problem 4.81. *For every stratified E-system \mathbb{E} , to construct an isomorphism $\mathbf{B2E}(\mathbf{E2B}(\mathbb{E})) \cong \mathbb{E}$ of stratified E-systems, natural in \mathbb{E} .*

Construction 4.82 (for Problem 4.81). In this construction we decorate the structures from **B2E**(**E2B**(\mathbb{E})) with a hat, as in $\hat{\mathcal{F}}$. Since \mathcal{F} is stratified, the function mapping $(n, X) \in \coprod_n B(\mathbb{E})_n$ to X extends to an isomorphism φ between the underlying strict category $\hat{\mathcal{F}}$ of **B2E**(**E2B**(\mathbb{E})), constructed in 4.62, and \mathcal{F} . In particular, it maps an arrow (X, k) to the arrow $\overline{X}^k := \overline{\text{ft}^{k-1}(X)} \circ \dots \circ \overline{X}: X \rightarrow \text{ft}^k(X)$ in \mathcal{F} as in (3).

In order to lift φ to an isomorphism of categories with term structure, we need to show that $\hat{T}(X, k) \cong T(\overline{X}^k)$ for every $X \in B(\mathbb{E})_n$ and $k \leq n$, where $\hat{T}(X, k)$ is the set defined in Construction 4.62. For every $X \in B_{n+1}$, by (12) it is

$$\hat{T}(X, 1) = \partial^{-1}(X) = \{(Y, y) \in \tilde{B}(\mathbb{E})_{n+1} \mid Y = X, y \in T(\overline{Y})\} \cong T(\overline{X}).$$

Suppose that $\hat{T}(Y, j) \cong T(\overline{Y}^j)$ for every $m < n, Y \in B_m$ and $j \leq m$. It follows by (13) that

$$\hat{T}(X, k+1) = \coprod_{t \in T(\text{ft}(X), k)} T(S_t(X), 1) \cong \coprod_{t \in T(\overline{\text{ft}(X)}^k)} T(\overline{S_t(X)}) \cong T(\overline{X}^{k+1})$$

where the last bijection follows from Theorem 4.60 since $\overline{X}^{k+1} = \overline{\text{ft}(X)^k} \circ \overline{X}$ and $\overline{S_t(X)} = S_t(\overline{X})$. In other words, elements of $\hat{T}(X, k)$ are lists of length k of pairs $(Y, y) \in \tilde{B}(\mathbb{E})_{n+j}$ for $j = 1, \dots, k$, where $y \in T(\overline{Y})$, and the action on terms of φ first acts componentwise dropping the first component of each pair and then applies the bijection from Theorem 4.60.

Next, we show that this choice of isos is natural in \mathbb{E} . Given a stratified E-homomorphism $F: \mathbb{E} \rightarrow \mathbb{D}$, we need to show that $\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F)) = F \circ \varphi_{\mathbb{E}}$. The functor **B2E**(**E2B**(F)) maps an arrow (X, k) to $(F(X), k)$, thus

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(X, k) = \overline{F(X)}^k = F(\overline{X}^k) = F \circ \varphi_{\mathbb{E}}(X, k).$$

since F preserves individual arrows by Lemma 2.10. The functor with term structure **B2E**(**E2B**(F)) maps $(X, x) \in \hat{T}(X, 1)$ to $(F(X), F(x))$ by (18) and (31), thus

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(X, x) = F(x) = F \circ \varphi_{\mathbb{E}}(X, x).$$

Suppose now that, for every $m \leq n, Y \in B(\mathbb{E})_m, i \leq m$ and $(Y, t) \in \hat{T}(Y, i)$, it is $\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(Y, t) = F \circ \varphi_{\mathbb{E}}(Y, t)$. Let $X \in B(\mathbb{E})_{n+1}$ and $(t, (X, x)) \in \hat{T}(X, k+1)$, then

$$\begin{aligned} \varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(t, (X, x)) &= (\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(t)).F(x) \\ &= (F \circ \varphi_{\mathbb{E}}(t)).F(x) \\ &= F(\varphi_{\mathbb{E}}(t).x) \\ &= F \circ \varphi_{\mathbb{E}}(t, (X, x)) \end{aligned}$$

by definition of φ , inductive hypothesis, Lemma 4.58, and definition of φ again. Therefore we conclude that, for every E-homomorphism $F: \mathbb{E} \rightarrow \mathbb{D}$,

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F)) = F \circ \varphi_{\mathbb{E}}. \quad (35)$$

It remains to show that each component $\varphi_{\mathbb{E}}$ is an E-homomorphism.

To show that φ is a weakening homomorphism, note that for every $X \in B(\mathbb{E})_{n+k}$, it is $\hat{W}_{(X,k)} = \mathbf{B2E}(W_X^k)$ by (21) and

$$\begin{aligned} W_X^k &= W_{\text{ft}^{k-1}(X)} \circ \cdots \circ W_X \\ &= \mathbf{E2B}(W_{\text{ft}^{k-1}(X)} \circ \cdots \circ W_X) \\ &= \mathbf{E2B}(W_{\varphi(X,k)}) \end{aligned}$$

by, in order, (16); (33) and functoriality of $\mathbf{E2B}$; condition 4.28.1 in the case $k = 0$ and condition 4.28.2 for $k > 0$; and definition of φ . Moreover, $W_{\varphi(X,k)}$ is an E-homomorphism, thus φ is a weakening homomorphism by (35).

To show that φ is a substitution homomorphism we reason by induction. The case $X \in B(\mathbb{E})_n$ and $* \in \hat{T}(X, 0)$ is trivial. For every $X \in B(\mathbb{E})_{n+1}$ and $(X, x) \in \hat{T}(X, 1)$, it is $\hat{S}_{(X,x)} = \mathbf{B2E}(S_{(X,x)})$ by (20) and

$$S_{(X,x)} = \mathbf{E2B}(S_x) = \mathbf{E2B}(S_{\varphi(X,x)})$$

by (32) and definition of φ . Suppose now that, for every $m \leq n$, $Y \in B(\mathbb{E})_m$, $i \leq m$ and $t \in \hat{T}(Y, i)$, it is $S_t = \mathbf{E2B}(S_{\varphi(t)})$ as homomorphisms of B-systems. Then for every $X \in B(\mathbb{E})_{n+1}$, $k \leq n$ and $(t, (X, x)) \in \hat{T}(X, k+1)$, it is $\hat{S}_{(t,(X,x))} = \mathbf{B2E}(S_{(t,(X,x))})$ by (20) and

$$\begin{aligned} S_{(t,(X,x))} &= S_{(X,x)} \circ S_t / X \\ &= \mathbf{E2B}(S_x \circ S_{\varphi(t)} / X) \\ &= \mathbf{E2B}(S_{\varphi(t),x}) \\ &= \mathbf{E2B}(S_{\varphi(t,(X,x))}) \end{aligned}$$

by, in order, (14); inductive hypothesis, (32) and functoriality of $\mathbf{E2B}$; Theorem 4.55; and definition of φ . Therefore $S_t = \mathbf{B2E}(\mathbf{E2B}(S_{\varphi(t)}))$ for every $X \in B(\mathbb{E})_{n+k}$ and $t \in \hat{T}(X, k)$. We conclude that φ is a substitution homomorphism by naturality (35).

To show that φ is a projection homomorphism we reason by induction. The case $(X, 0)$ for $X \in B(\mathbb{E})_n$ is again trivial. Let $X \in B(\mathbb{E})_{n+1}$, then

$$\text{idtm}_{(X,1)}^\wedge = \delta(X) = (W_X(X), \text{idtm}_{\overline{X}})$$

by (22) and (34). Therefore $\varphi(\text{idtm}_{(X,1)}^\wedge) = \text{idtm}_{\varphi(X,1)}$ by definition of φ . Suppose that, for every $m \leq n$, $Y \in B(\mathbb{E})_m$, $i \leq m$, it is $\varphi(\text{idtm}_{(Y,i)}^\wedge) = \text{idtm}_{\varphi(Y,i)}$. Let $X \in B(\mathbb{E})_{n+1}$ and $k \leq n$. Then

$$\begin{aligned} \varphi(\text{idtm}_{(X,k+1)}^\wedge) &= \varphi(W_{(X,1)}(\text{idtm}_{(\text{ft}(X),k)}^\wedge), \delta(X)) \\ &= \left(W_{\overline{X}}(\varphi(\text{idtm}_{(\text{ft}(X),k)}^\wedge)) \right) \cdot \text{idtm}_{\overline{X}} \\ &= \left(W_{\overline{X}}(\text{idtm}_{\varphi(\text{ft}(X),k)}) \right) \cdot \text{idtm}_{\overline{X}} \\ &= \text{idtm}_{\text{ft}(X)^k \circ \overline{X}} \\ &= \text{idtm}_{\varphi(X,k+1)} \end{aligned}$$

by (24), definition of φ and the fact that φ is a weakening homomorphism, the inductive hypothesis, Lemma 5.20, and definition of φ again. Therefore $\varphi(\text{idtm}_{(X,k)}^\wedge) = \text{idtm}_{\varphi(X,k)}$ for every $X \in B(\mathbb{E})_{n+k}$. This concludes the proof that φ is an E-homomorphism. \square

Finally we reach the main result of this section.

Theorem 4.83. *The functors $\mathbf{B2E}: \mathbf{Bsys} \rightarrow \mathbf{Esys}_s$ from Lemma 4.71 and $\mathbf{E2B}: \mathbf{Esys}_s \rightarrow \mathbf{Bsys}$ from Construction 4.80 form an equivalence of categories.*

Proof. As the functor $\mathbf{B2E}$ is fully faithful by Lemma 4.71.2, it is enough to show that $\mathbf{E2B}$ is an essential section of $\mathbf{B2E}$. This holds by Construction 4.82. \square

5 Equivalence between B- and C-systems

In this section, we construct an equivalence between B-systems and C-systems, in several steps. We first construct an adjunction between the categories of CE-systems and of E-systems. To this end, we construct, in Section 5.1, a functor from CE-systems to E-systems, and, in Section 5.2, a functor in the other direction, from E-systems to CE-systems. In Section 5.3 we show that these functors form an adjunction that restricts to an equivalence when considering *rooted* CE-systems. Finally, in Section 5.4, we give our equivalence between B-systems and C-systems, obtained by restricting the aforementioned equivalence to *stratified* rooted CE-systems and E-systems, respectively.

5.1 From CE-systems to E-systems

Definition 5.1. Let \mathbb{A} be a CE-system. For any $\Gamma \in \mathcal{C}$, we define the **slice CE-system** \mathbb{A}/Γ as follows. Let $\mathcal{C}(\Gamma)$ be the strict category with the same objects as \mathcal{F}/Γ and with all arrows from $I(A)$ to $I(B)$ in \mathcal{C}/Γ as arrows from A to B . The functor $I/\Gamma: \mathcal{F}/\Gamma \rightarrow \mathcal{C}/\Gamma$ factors as an identity-on-objects I_Γ followed by a full and faithful one as shown in the diagram below.

$$\begin{array}{ccc} \mathcal{F}/\Gamma & \xrightarrow{I/\Gamma} & \mathcal{C}/\Gamma \\ & \searrow I_\Gamma & \nearrow \\ & \mathcal{C}(\Gamma) & \end{array}$$

We take I_Γ to be the underlying functor of \mathbb{A}/Γ . The choice of pullback squares is induced by \mathbb{A} .

Remark 5.2. Let \mathbb{A} be a CE-system.

1. For every object Γ , the identity id_Γ is terminal in $\mathcal{C}_\mathbb{A}(\Gamma)$. It follows that any slice CE-system is rooted.
2. For every $f: \Delta \rightarrow \Gamma$ in \mathcal{C} , the functor $f^*: \mathcal{F}/\Gamma \rightarrow \mathcal{F}/\Delta$ lifts to a functor $f^*: \mathcal{C}(\Gamma) \rightarrow \mathcal{C}(\Delta)$ making the square below commute.

$$\begin{array}{ccc} \mathcal{F}/\Gamma & \xrightarrow{f^*} & \mathcal{F}/\Delta \\ I_\Gamma \downarrow & & \downarrow I_\Delta \\ \mathcal{C}(\Gamma) & \xrightarrow{f^*} & \mathcal{C}(\Delta) \end{array}$$

3. For every $f: \Delta \rightarrow \Gamma$ the commutative square in Remark 5.2 lifts to a CE-homomorphism $f^*: \mathbb{A}/\Gamma \rightarrow \mathbb{A}/\Delta$.

Lemma 5.3. *Let $F: \mathbb{A} \rightarrow \mathbb{B}$ be a CE-homomorphism. Then for every $f: \Delta \rightarrow \Gamma$ in $\mathcal{C}_\mathbb{A}$ the diagram below commutes.*

$$\begin{array}{ccccc} & & \mathcal{F}_\mathbb{A}/\Gamma & \xrightarrow{F_{\mathcal{F}/\Gamma}} & \mathcal{F}_\mathbb{B}/F\Gamma \\ & \swarrow I_\Gamma & \downarrow & & \swarrow I_{F\Gamma} \\ \mathcal{C}_\mathbb{A}(\Gamma) & \xrightarrow{F_{\mathcal{C}/\Gamma}} & \mathcal{C}_\mathbb{B}(F\Gamma) & & \\ & \downarrow f^* & \downarrow & & \downarrow (Ff)^* \\ & & \mathcal{F}_\mathbb{A}/\Delta & \xrightarrow{F_{\mathcal{F}/\Delta}} & \mathcal{F}_\mathbb{B}/F\Delta \\ & \swarrow I_\Delta & \downarrow & & \swarrow I_{F\Delta} \\ \mathcal{C}_\mathbb{A}(\Delta) & \xrightarrow{F/\Delta} & \mathcal{C}_\mathbb{B}(F\Delta) & & \end{array}$$

Proof. Commutativity of the back face follows from the fact that $(Ff)^*(FA) = F(f^*A)$ for every $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$, commutativity of the front face follows from the universal property of pullbacks, and commutativity of the other faces is immediate. \square

Lemma 5.4. *Let \mathbb{A} be a CE-system. For every $f: \Delta \rightarrow \Gamma$ in \mathcal{C} and every $g: A \rightarrow B$ in $\mathcal{C}(\Gamma)$ the diagram below commutes.*

$$\begin{array}{ccccc}
& & \mathcal{F}/\Gamma.B & \xrightarrow{\quad} & \mathcal{F}/\Delta.f^*B \\
& \swarrow I/\Gamma.B & \downarrow & & \swarrow I/\Delta.f^*B \\
\mathcal{C}(\Gamma.B) & \xrightarrow{\quad} & \mathcal{C}(\Delta.f^*B) & & \\
& \downarrow g^* & \downarrow f^*/B & & \downarrow (f^*g)^* \\
& & \mathcal{F}/\Gamma.A & \xrightarrow{\quad} & \mathcal{F}/\Delta.f^*A \\
& \swarrow I/\Gamma.A & \downarrow (f^*g)^* & & \swarrow I/\Delta.f^*A \\
\mathcal{C}(\Gamma.A) & \xrightarrow{\quad} & \mathcal{C}(\Delta.f^*A) & &
\end{array}$$

Proof. This is Lemma 5.3 applied to f^* seen as a homomorphism of CE-systems thanks to Remark 5.2.3. \square

Problem 5.5. *To construct a functor $\mathbf{CE2E}: \mathbf{CEsys} \rightarrow \mathbf{Esys}$.*

Construction 5.6 (for Problem 5.5). Let \mathbb{A} be a CE-system with underlying functor $I: \mathcal{F} \rightarrow \mathcal{C}$. The underlying category of the E-system $\mathbf{CE2E}(\mathbb{A})$ is \mathcal{F} . The chosen terminal object is the one in \mathbb{A} . To equip \mathcal{F} with a term structure we define, for every $A \in \mathcal{F}/\Gamma$, the set

$$T(A) := \{x: \Gamma \rightarrow \Gamma.A \mid I(A) \circ x = \text{id}_{\Gamma}\}. \quad (36)$$

We define for any $A \in \mathcal{F}/\Gamma$, the functor

$$W_A := A^*: \mathcal{F}/\Gamma \rightarrow \mathcal{F}/\Gamma.A. \quad (37)$$

Likewise, we define for any $x \in T(A)$, the functor

$$S_x := x^*: \mathcal{F}/\Gamma.A \rightarrow \mathcal{F}/\Gamma. \quad (38)$$

These clearly extend to functors with term structure. We also define $\text{idtm}_A: T(W_A(A))$ by the universal property of pullbacks as in the diagram below.

$$\begin{array}{ccccc}
\Gamma.A & & & \xrightarrow{\quad \text{id}_{\Gamma.A} \quad} & \Gamma.A \\
& \searrow \text{idtm}_A & & & \downarrow \\
& & \Gamma.A.W_A(A) & \xrightarrow{\quad \pi_2(I(A),A) \quad} & \Gamma.A \\
& \searrow \text{id}_{\Gamma.A} & \downarrow & & \downarrow \\
& & \Gamma.A & \xrightarrow{\quad I(A) \quad} & \Gamma
\end{array}$$

As an immediate consequence of Lemma 5.4, we get that each functor W_A and S_x is both a weakening functor and a substitution functor. It follows by the definitions that weakening and substitution preserve the terms idtm_A .

It remains to verify the remaining conditions of E-systems.

3. To show that substitution in weakened families is constant, note that

$$S_x \circ W_A = x^* \circ A^* = (A \circ x)^* = (\text{id}_{\Gamma})^* = \text{id}_{\mathcal{C}_{\mathcal{F}}/\Gamma}.$$

5. The identity terms are neutral for pre-composition:

$$S_{\text{idm}_A} \circ W_A/A = S_{\text{idm}_A} \circ \pi_2(A, A)^* = (\pi_2(A, A) \circ \text{idm}_A)^* = (\text{id}_{\Gamma.A})^* = \text{id}_{\mathcal{C}_{\mathcal{F}}/\Gamma.A}.$$

4. The identity terms behave like identity functions: by the universal property, $S_x(\text{idm}_A)$ is the unique section of A such that the square

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi_2(x, \text{id}_{\Gamma.A})} & \Gamma.A \\ S_x(\text{idm}_A) \downarrow & & \downarrow \text{idm}_A \\ \Gamma.A & \xrightarrow{\pi_2(x, W_A(A))} & \Gamma.A.W_A(A) \end{array}$$

commutes. Thus, it suffices to show that this square also commutes with x in the place of $S_x(\text{idm}_A)$. Note that $\pi_2(x, \text{id}_{\Gamma.A}) = x$. Since $\Gamma.A.W_A(A)$ is itself a pullback, it suffices and it is straightforward to verify the equalities

$$\begin{aligned} W_A(A) \circ \pi_2(x, W_A(A)) \circ x &= W_A(A) \circ \text{idm}_A \circ x \\ \pi_2(A, A) \circ \pi_2(x, W_A(A)) \circ x &= \pi_2(A, A) \circ \text{idm}_A \circ x. \end{aligned}$$

Let now $F: \mathbb{A} \rightarrow \mathbb{B}$ be a CE-system homomorphism. The underlying functor of $\mathbf{CE2E}(F)$ is $F_{\mathcal{F}}: \mathcal{F}_{\mathbb{A}} \rightarrow \mathcal{F}_{\mathbb{B}}$, which clearly preserves the choice of terminal objects, while the action on terms is given by F_C . This functor with term structure is both a weakening and a substitution homomorphism because of Lemma 5.3. Note that commutativity of the front square in the diagram in Lemma 5.3 is needed for the equations on the action on terms. Finally, it is a projection homomorphism since it preserves identities. \square

Remark 5.7. It follows immediately from the above construction that, for every CE-system \mathbb{A} , the E-system $\mathbf{CE2E}(\mathbb{A})$ has the property that $T(\text{id}_{\Gamma})$ is a singleton set for every $\Gamma \in \mathcal{F}$. As we shall see in Corollary 5.18, this is true for every E-system. In fact, it will follow from Lemma 5.40.1 that $\mathbf{CE2E}$ is essentially surjective on objects.

5.2 From E-systems to CE-systems

In this section we construct a functor from \mathbf{Esys} to \mathbf{CEsys} . We proceed in several steps: In Section 5.2.1 we define the strict category of internal morphisms of an E-system. There are two kinds of morphisms in this category: internal morphisms from A to B in context Γ , and for any internal morphism $f: A \rightarrow B$ in context Γ there are morphisms over f . There are also two kinds of composition, and in Section 5.2.2 we prove an interchange law for them. In Section 5.2.3 we complete the construction of the functor from \mathbf{Esys} to \mathbf{CEsys} .

5.2.1 The strict category of internal morphisms of an E-system

In this section we define for every E-system \mathbb{E} , and every context Γ in \mathbb{E} , a category $\mathcal{C}_{\mathbb{E}}(\Gamma)$. This goal is accomplished in Theorem 5.17. The empty context \square of \mathbb{E} , i.e. a terminal object in \mathcal{F} , allows us to have a non-trivial category structure on the contexts of \mathbb{E} . In this case, the category structure is inherited from the category $\mathcal{C}_{\mathbb{E}} := \mathcal{C}_{\mathbb{E}}(\square)$.

Definition 5.8. For every $A, B \in \mathcal{F}/\Gamma$ we define the set

$$\text{thom}(A, B) := T(\langle A \rangle B).$$

An element $f \in \text{thom}(A, B)$ is called an **internal morphism in context** Γ . We sometimes write $f \in \text{thom}(A, B)$ to indicate that f is an internal morphism over Γ , or we may draw a diagram of the

form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \swarrow \\ & \Gamma & \end{array}$$

but we shall generally omit the arrows down to Γ and say instead that we have a diagram in context Γ .

Remark 5.9. Note that $\text{thom}(\text{id}_\Gamma, A) = T(A)$ for any $A \in \mathcal{F}/\Gamma$, because we have $W_{\text{id}_\Gamma, A} = \text{id}_{\mathcal{F}/\Gamma, A}$.

Note also that $\text{thom}(A.P, B) = \text{thom}(P, \langle A \rangle B)$ for any $P \in \mathcal{F}/\Gamma, A$ and $B \in \mathcal{F}/\Gamma$, because $W_{A.P} = W_P \circ W_A$. Once we have established a strict category of which the morphisms are given by $\text{thom}(-, -)$, we therefore get that

$$A.(-) \dashv W_A.$$

The right adjoint to weakening by A , if it exists, will be the dependent product Π_A .

Definition 5.10. Let $A, B \in \mathcal{F}/\Gamma$. For any $f \in \text{thom}(A, B)$ we define the **pre-composition** E -homomorphism

$$f^* := S_f \circ W_A/B : \mathbb{E}/\Gamma.B \rightarrow \mathbb{E}/\Gamma.A.$$

We shall denote the action of f^* on a family $Q \in \mathcal{F}/\Gamma.B$ as $Q \circ f$. Similarly, for every $C \in \mathcal{F}/\Gamma$, we shall write $g \circ f$ for the action of f^* on $g \in \text{thom}(B, C) = T(W_B(C))$.

Lemma 5.11. Let $F: \mathbb{E} \rightarrow \mathbb{D}$ be an E -homomorphism. Then for every $f \in \text{thom}(A, B)$ in \mathbb{E} , the square of E -homomorphisms below commutes.

$$\begin{array}{ccc} \mathbb{E}/\Gamma.B & \xrightarrow{F/\Gamma.B} & \mathbb{D}/F\Gamma.FB \\ f^* \downarrow & & \downarrow (Ff)^* \\ \mathbb{E}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathbb{D}/F\Gamma.FA \end{array}$$

Proof. As F is both a weakening and a substitution homomorphism, it is

$$\begin{aligned} (F/\Gamma.A) \circ f^* &= (F/\Gamma.A) \circ S_f \circ (W_A/B) = S_{Ff} \circ (F/\Gamma.A.W_A(B)) \circ (W_A/B) \\ &= S_{Ff} \circ (W_{FA}/FB) \circ (F/\Gamma.B) = (Ff)^* \circ (F/\Gamma.B). \end{aligned} \quad \square$$

Definition 5.12. Let $A, B \in \mathcal{F}/\Gamma$, $Q \in \mathcal{F}/\Gamma.A$ and $R \in \mathcal{F}/\Gamma.B$. For every $f \in \text{thom}(A, B)$ we define

$$\text{thom}_f(Q, R) := \text{thom}(Q, R \circ f).$$

Remark 5.13.

1. The terms $\text{pr}_0^{A,P}$ and $\text{pr}_1^{A,P}$ from Definition 4.57 are internal morphisms:

$$\text{pr}_0^{A,P} \in \text{thom}(A.P, A) \quad \text{and} \quad \text{pr}_1^{A,P} \in \text{thom}_{\text{pr}_0^{A,P}}(A.P, P).$$

2. Note that for $g \in \text{thom}(B, C)$, we have $g \circ f \in T(S_f(W_A/B(W_B(C))))$, whereas we would like that $g \circ f \in \text{thom}(A, C)$. More generally, we can show that

$$S_f \circ (W_A/B) \circ W_B = W_A.$$

Since weakening is a weakening homomorphism, we have

$$S_f \circ (W_A/B) \circ W_B = S_f \circ W_{W_A(B)} \circ W_A.$$

By condition 3 in Definition 4.45 we get that

$$S_f \circ W_{W_A(B)} \circ W_A = W_A.$$

Remark 5.14. The term structure of f^* , for $f \in \text{thom}(A, B)$, provides us also with an action on the internal morphisms over B . To see how this extended notion of precomposition works, suppose we have a diagram

$$\begin{array}{ccc} Q & \xrightarrow{g} & R \\ & \searrow & \swarrow \\ & B & \end{array}$$

in context Γ , i.e. $g \in \text{thom}(Q, R)$. Precomposing by f gives us a diagram

$$\begin{array}{ccc} Q \circ f & \xrightarrow{g \circ f} & R \circ f \\ & \searrow & \swarrow \\ & A & \end{array}$$

in context Γ .

Remark 5.15. Note that condition 5 in Definition 4.45 asserts precisely that $(\text{idtm}_A)^* = \text{id}_{\mathcal{F}/\Gamma, A}$ for any $A \in \mathcal{F}/\Gamma$. In particular, it follows that $g \circ \text{idtm}_A = g$ for any $g \in \text{thom}(A, B)$

Lemma 5.16. *For any $f \in \text{thom}(A, B)$ and $g \in \text{thom}(B, C)$ we have $f^* \circ g^* = (g \circ f)^*$.*

Proof.

$$\begin{aligned} f^* \circ g^* &= S_f \circ (W_A/B) \circ S_g \circ (W_B/C) \\ &= S_f \circ S_{W_A(g)} \circ (W_A/B.W_B(C)) \circ (W_B/C) \\ &= S_{S_f(W_A(g))} \circ (S_f/W_A(W_B(C))) \circ (W_A/B.W_B(C)) \circ W_B/C \\ &= S_{S_f(W_A(g))} \circ ((S_f \circ (W_A/B) \circ W_B)/C) \\ &= S_{S_f(W_A(g))} \circ ((S_f \circ W_{W_A(B)} \circ W_A)/C) \\ &= S_{S_f(W_A(g))} \circ W_A/C \\ &= (g \circ f)^*. \end{aligned} \quad \square$$

Theorem 5.17.

1. For every E -system \mathbb{E} and every object Γ in its underlying strict category \mathcal{F} , objects of \mathcal{F}/Γ and internal morphisms of \mathbb{E} over Γ form a strict category $\mathcal{C}_{\mathbb{E}}(\Gamma)$.
2. Every E -homomorphism $F: \mathbb{E} \rightarrow \mathbb{D}$ induces a functor $F_\Gamma: \mathcal{C}_{\mathbb{E}}(\Gamma) \rightarrow \mathcal{C}_{\mathbb{D}}(F(\Gamma))$ for every Γ in $\mathcal{F}_{\mathbb{E}}$.

Proof.

1. For $A, B \in \mathcal{F}/\Gamma$, the set of arrows from A to B is $\text{thom}(A, B)$. The fact that composition is associative is a direct corollary of Lemma 5.16. The axiom $(\text{idtm}_A)^* = \text{id}_{\Gamma, A}$ implies that the identity morphisms satisfy the right identity law. It remains to show that $\text{idtm}_B \circ f = f$, which is a simple calculation:

$$\text{idtm}_B \circ f = S_f \circ W_A(\text{idtm}_B) = S_f(\text{idtm}_{W_A B}) = f.$$

2. The action of F_Γ on arrows is given by the term structure of F . Functoriality of F_Γ follows from Lemma 5.11 and the fact that F is a projection homomorphism. \square

Now that we have a category structure, we can state and prove the following consequence of Theorem 4.60.

Corollary 5.18.

1. Let $A \in \mathcal{F}/\Gamma$ and $Q \in \mathcal{F}/\Gamma.B$, then for every $f \in \text{thom}(A, B)$ there is a bijection

$$\varphi: T(Q \circ f) \xrightarrow{\sim} \left\{ h \in \text{thom}(A, B.Q) \mid \text{pr}_0^{B.Q} \circ h = f \right\}.$$

given by $\varphi(t) = f.t$.

2. For every object Γ , $T(\text{id}_\Gamma) = \{\text{idtm}_{\text{id}_\Gamma}\}$.

Proof.

1. Theorem 4.60 yields the following bijection:

$$\begin{aligned} \text{thom}(A, B.Q) &= T(\langle A \rangle(B.Q)) \\ &= T(\langle A \rangle B. \langle A \rangle Q) \\ &\cong \coprod_{f \in T(\langle A \rangle B)} T(\langle A \rangle Q[f]) \\ &= \coprod_{f \in \text{thom}(A, B)} T(Q \circ f). \end{aligned}$$

Also, we find $\text{pr}_0^{\langle A \rangle B. \langle A \rangle Q}[h] = \langle A \rangle \text{pr}_0^{B.Q}[h] = \text{pr}_0^{B.Q} \circ h$.

2. The above bijection becomes in this case

$$T(\text{id}_{\Gamma.A}) \cong \{h \in \text{thom}(A, A) \mid \text{pr}_0^{A, \text{id}_{\Gamma.A}} \circ h = \text{idtm}_A\} = \{\text{idtm}_A\}$$

where the second equality follows from $\text{pr}_0^{A, \text{id}_{\Gamma.A}} = \text{idtm}_A$. Since $\text{id}_\Gamma = W_{\text{id}_\Gamma}(\text{id}_\Gamma)$, the only element in $T(\text{id}_\Gamma)$ is $\text{idtm}_{\text{id}_\Gamma}$. \square

Theorem 5.19. Let $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. Precomposition with $\text{pr}_0^{A,P}$ is weakening by P , i.e.

$$\mathbb{E}/\Gamma.A \xrightarrow{(\text{pr}_0^{A,P})^* = W_P} \mathbb{E}/\Gamma.A.P$$

Proof.

$$\begin{aligned} (\text{pr}_0^{A,P})^* &= S_{\text{pr}_0^{A,P}} \circ W_{A.P}/A \\ &= S_{\langle P \rangle \text{idtm}_A} \circ W_P/W_A(A) \circ W_A/A \\ &= W_P \circ S_{\text{idtm}_A} \circ W_A/A \\ &= W_P \end{aligned} \quad \square$$

We conclude this section with a description of the projections and the pairing operation of an E-system in the image of the functor **CE2E** from Construction 5.6 in terms of the underlying CE-system structure.

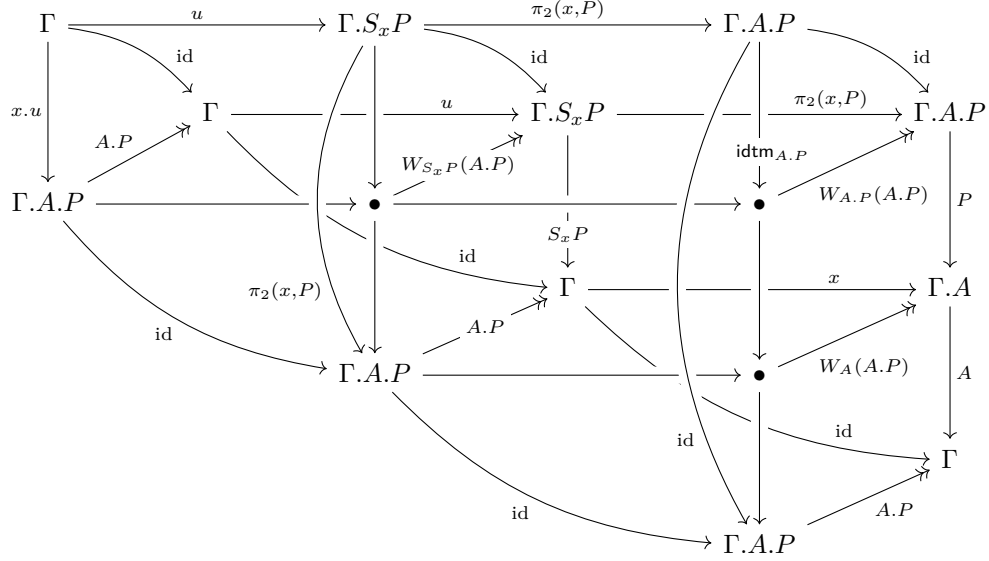
Lemma 5.20. Let \mathbb{A} be a CE-system and consider the E-system $\mathbb{E} := \mathbf{CE2E}(\mathbb{A})$. For every object Γ , every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, it is

$$\begin{aligned} \text{pr}_0^{A,P} &= \langle \text{id}_{\Gamma.A.P}, P \rangle \in \mathcal{C}_{\mathbb{A}}(\Gamma.A.P, \Gamma.A.P. \langle A.P \rangle A) \\ \text{pr}_1^{A,P} &= \langle \text{id}_{\Gamma.A.P}, \text{id}_{\Gamma.A.P} \rangle \in \mathcal{C}_{\mathbb{A}}(\Gamma.A.P, \Gamma.A.P. \langle P \rangle P) \end{aligned}$$

and, for every $x \in T(A)$ and $u \in T(S_x P)$, it is

$$x.u = \pi_2(x, P) \circ u \in \mathcal{C}_{\mathbb{A}}(\Gamma, \Gamma.A.P).$$

Proof. The first two claims follow immediately from Definition 4.57 and the definitions in Construction 5.6. The third claim follows from commutativity of the front-left face in the diagram below.



This diagram commutes by definition, in the sense that every square not involving the top row is a chosen pullback in \mathbb{A} , and the remaining part commutes by definition of $\text{idtm}_{A,P}$ and $x.u$ in Construction 5.6 and Definition 4.54, respectively. In this diagram we drop occurrences of the functor I and freely use notation from the E-system $\mathbf{CE2E}(\mathbb{A})$ to increase readability. \square

5.2.2 The interchange laws

We are now in the position to define vertical and horizontal composition, and prove properties of them. In particular, we conclude the section showing in Theorem 5.29 that every pair $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$ induces a morphism, *i.e.* a commuting square, from $\text{pr}_0^{A,P}$ to $\text{pr}_0^{B,Q}$.

Definition 5.21. Let $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$. Then we define

$$f \times F := \langle P \rangle f.F \in \text{thom}(A.P, B.Q)$$

Whenever we say that we have a diagram of the form

$$\begin{array}{ccc} R & \xrightarrow{f_2} & S \\ \downarrow & & \downarrow \\ P & \xrightarrow{f_1} & Q \\ \downarrow & & \downarrow \\ A & \xrightarrow{f_0} & B \end{array}$$

we mean that we have $f_0 \in \text{thom}(A, B)$, $f_1 \in \text{thom}_{f_0}(P, Q)$ and $f_2 \in \text{thom}_{f_0 \times f_1}(R, S)$.

Lemma 5.22. Let $H : \mathbb{E} \rightarrow \mathbb{D}$ be an E-homomorphism. For every $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$ it is

$$H(f \times F) = H(f) \times H(F).$$

Proof. $H(f \times F) = H(\langle Q \rangle f.F) = \langle HQ \rangle Hf.HF = H(f) \times H(F)$. \square

Lemma 5.23. Vertical composition is associative.

Proof. Consider the diagram

$$\begin{array}{ccc} R & \xrightarrow{f_2} & S \\ \downarrow & & \downarrow \\ P & \xrightarrow{f_1} & Q \\ \downarrow & & \downarrow \\ A & \xrightarrow{f_0} & B \end{array}$$

in context Γ . Because weakening distributes over term extension, and term extension is associative, we have

$$\begin{aligned} (f_0 \times f_1) \times f_2 &= \langle R \rangle \langle \langle P \rangle f_0 \cdot f_1 \rangle \cdot f_2 \\ &= (\langle R \rangle \langle P \rangle f_0 \cdot \langle R \rangle f_1) \cdot f_2 \\ &= \langle P \cdot R \rangle f_0 \cdot (\langle R \rangle f_1 \cdot f_2) && \text{(By Corollary 4.56)} \\ &= f_0 \times (f_1 \times f_2). && \square \end{aligned}$$

Definition 5.24. Let $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$. Then we define the E-homomorphism

$$F^\bullet := F^\bullet \circ (f^*/Q) : \mathbb{E}/\Gamma B.Q \rightarrow \mathbb{E}/\Gamma A.P.$$

The infix notation of F^\bullet is taken to be $- \bullet F$.

Lemma 5.25. Let $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$. Then we have the equality

$$F^\bullet = (f \times F)^*.$$

Proof.

$$\begin{aligned} F^\bullet \circ (f^*/Q) &= S_F \circ W_P \circ S_f / (W_A(Q)) \circ W_A/B.Q \\ &= S_F \circ S_{W_P(f)} / W_P(W_A(Q)) \circ W_P/W_A(B.Q) \circ W_A/B.Q \\ &= S_F \circ S_{W_P(f)} / W_P(W_A(Q)) \circ W_{A.P}/B.Q \\ &= S_{W_P(f).F} \circ W_{A.P}/B.Q && \text{(By Theorem 4.55)} \\ &= (f \times F)^*. && \square \end{aligned}$$

In the next theorem we prove the interchange law of horizontal and vertical composition. Its proof uses the following fact.

Lemma 5.26. Let $f \in \text{thom}(A, B)$ be an internal morphism in context Γ . Then one has

$$f^* \circ W_B = W_A.$$

Proof. The proof is a simple calculation:

$$f^* \circ W_B = S_f \circ W_A/B \circ W_B = S_f \circ W_{W_A(B)} \circ W_A = W_A. \quad \square$$

Theorem 5.27. Consider the diagram

$$\begin{array}{ccccc} P & \xrightarrow{F} & Q & \xrightarrow{G} & R \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in context Γ . Then the equality

$$(g \times G) \circ (f \times F) = (g \circ f) \times (G \bullet F)$$

of morphisms from $A.P$ to $C.R$ in context Γ holds.

Proof. By Lemma 5.25, we have

$$\begin{aligned}
(g \times G) \circ (f \times F) &= (\langle Q \rangle g.G) \circ f \circ F && \text{(By Lemma 5.25)} \\
&= (\langle Q \circ f \rangle g \circ f).(G \circ f) \circ F \\
&= (\langle Q \circ f \rangle g \circ f \circ F).(G \circ f \circ F) \\
&= (\langle Q \circ f \rangle g \circ f \circ F).(G \bullet F) \\
&= (\langle P \rangle g \circ f).(G \bullet F) && \text{(By Lemma 5.26)} \\
&= (g \circ f) \times (G \bullet F). && \square
\end{aligned}$$

Theorem 5.28. *Consider the diagram*

$$\begin{array}{ccccc}
P & \xrightarrow{F} & Q & \xrightarrow{G} & R \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}$$

in context Γ . Then $F^\bullet \circ G^\bullet = (G \bullet F)^\bullet$. In other words the composition $-\bullet-$ is associative.

Proof.

$$\begin{aligned}
F^\bullet \circ G^\bullet &= (f \times F)^\bullet \circ (g \times G)^\bullet && \text{(By Lemma 5.25)} \\
&= (g \times G \circ f \times F)^\bullet && \text{(By Lemma 5.16)} \\
&= (g \circ f \times G \bullet F)^\bullet && \text{(By Theorem 5.27)} \\
&= (G \bullet F)^\bullet. && \text{(By Lemma 5.25)}
\end{aligned}$$

□

Theorem 5.29. *Let $f \in \text{thom}(A, B)$ and $F \in \text{thom}_f(P, Q)$. Then $f \times F$ is the unique morphism from $A.P$ to $B.Q$ with the property that both the diagram*

$$\begin{array}{ccc}
A.P & \xrightarrow{f \times F} & B.Q \\
\text{pr}_0^{A,P} \downarrow & & \downarrow \text{pr}_0^{B,Q} \\
A & \xrightarrow{f} & B
\end{array}$$

commutes and $\text{pr}_1^{B,Q} \circ f \times F = F$.

Proof. We first note that

$$\begin{aligned}
\text{pr}_0^{B,Q} \circ f \times F &= \text{pr}_0^{B,Q} \circ f \circ F && \text{(By Lemma 5.25)} \\
&= \langle Q \circ f \rangle \text{idtm}_B \circ f \circ F \\
&= \langle Q \circ f \rangle f \circ F \\
&= \langle P \rangle f && \text{(By Lemma 5.26)} \\
&= f \circ \left(\text{pr}_0^{A,P} \right). && \text{(By Theorem 5.19)}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\text{pr}_1^{B,Q} \circ f \times F &= \text{idtm}_Q \circ f \circ F && \text{(By Lemma 5.25)} \\
&= \text{idtm}_{Q \circ f} \circ F \\
&= F.
\end{aligned}$$

Thus, we conclude that $f \times F$ has indeed the stated property. For the uniqueness, let $G : A.P \rightarrow B.Q$ be a morphism such that $\text{pr}_0^{B,Q} \circ G = f \circ \left(\text{pr}_0^{A,P} \right)$ and $\text{pr}_1^{B,Q} \circ G = F$. Then it follows that

$$G = f \circ \left(\text{pr}_0^{A,P} \right).F = \langle P \rangle f.F = f \times F. \quad \square$$

5.2.3 The functor from E-systems to CE-systems

Let \mathbf{Esys}_* be the category of pointed E-systems: objects are pairs (\mathbb{E}, Γ) of an E-systems \mathbb{E} and an object Γ in its underlying strict category, and arrows are E-homomorphisms that preserve the distinguished object. There is an evident forgetful functor $\mathbf{Esys}_* \rightarrow \mathbf{Esys}$ together with an embedding $\mathbf{E2E}_* : \mathbf{Esys} \hookrightarrow \mathbf{Esys}_*$ which picks out the terminal object of an E-system.

Problem 5.30. *To construct a functor $\mathbf{E}_* \mathbf{2CE} : \mathbf{Esys}_* \rightarrow \mathbf{CEsys}$.*

Construction 5.31 (for Problem 5.30). Let (\mathbb{E}, Γ) be a pointed E-system and consider the category of terms $\mathcal{C}_{\mathbb{E}}(\Gamma)$ from Theorem 5.17. Define a functor $I_{\mathbb{E}}^{\Gamma} : \mathcal{F}/\Gamma \rightarrow \mathcal{C}_{\mathbb{E}}(\Gamma)$ as follows. It is the identity on objects and maps an arrow $Q : A.Q \rightarrow A$ in \mathcal{F}/Γ to $\text{pr}_0^{A,Q} \in \text{thom}(A.Q, A)$. For functoriality, we compute $\text{pr}_0^{A, \text{id}_{\Gamma.A}} = \langle \text{id}_{\Gamma.A} \rangle \text{idtm}_A = \text{idtm}_A$ and

$$\begin{aligned} \text{pr}_0^{A,Q.R} &= \langle Q.R \rangle \text{idtm}_A = \langle R \rangle (\langle Q \rangle \text{idtm}_A) = \langle R \rangle \text{pr}_0^{A,Q} \\ &= \left(\text{pr}_0^{A,Q} \right)^* (\langle R \rangle \text{idtm}_Q) \\ &= \text{pr}_0^{A,Q} \circ \text{pr}_0^{Q,R}. \end{aligned}$$

Given $f \in \text{thom}(A, B)$ and R in $\mathcal{F}/\Gamma.B$, there is $R \circ f$ in $\mathcal{F}/\Gamma.A$. We define

$$\pi_2(f, R) := f \times \text{idtm}_{R \circ f} : A.R \circ f \rightarrow B.R. \quad (39)$$

Then the following diagram in $\mathcal{C}_{\mathbb{E}}(\Gamma)$

$$\begin{array}{ccc} A.R \circ f & \xrightarrow{\pi_2(f, R)} & B.R \\ \text{pr}_0^{A, R \circ f} \downarrow & & \downarrow \text{pr}_0^{B, R} \\ A & \xrightarrow{f} & B \end{array} \quad (40)$$

commutes. The functoriality conditions follow immediately from the interchange laws proven in Section 5.2.2.

To show that (40) is a pullback square, consider a morphism $g : X \rightarrow A$ in $\mathcal{C}_{\mathbb{E}}(\Gamma)$. Then we have the isomorphisms

$$\{h \in \text{thom}(X, B.Q) \mid \text{pr}_0^{B,Q} \circ h = f \circ g\} \cong T((f \circ g)^*(Q)) \cong \{u \in \text{thom}(X, A.Q \circ f) \mid \text{pr}_0^{A, Q \circ f} \circ u = g\}$$

Thus, we find for every $h : X \rightarrow B.Q$ satisfying $\text{pr}_0^{B,Q} \circ h = f \circ g$, a unique morphism $u : X \rightarrow A.Q \circ f$ satisfying $\text{pr}_0^{A, Q \circ f} \circ u = g$. It is easy to verify that $\pi_2(f, Q) \circ u = h$, so the universal property of pullbacks holds.

Let now (\mathbb{D}, Δ) be a pointed E-system and let $F : \mathbb{E} \rightarrow \mathbb{D}$ be an E-homomorphism such that $F\Gamma = \Delta$. In particular, for every $A, B \in \mathcal{F}_{\mathbb{E}}/\Gamma$ there is a function $F : T(\langle A \rangle B) \rightarrow T(\langle FA \rangle FB)$. These functions give the action on arrows of a functor $F_{\Gamma} : \mathcal{C}_{\mathbb{E}}(\Gamma) \rightarrow \mathcal{C}_{\mathbb{D}}(F\Gamma)$ whose action on objects is given by $F/\Gamma : \mathcal{F}_{\mathbb{E}}/\Gamma \rightarrow \mathcal{F}_{\mathbb{D}}/F\Gamma$. Functoriality of F_{Γ} follows from the fact that F is a projection homomorphism and Lemma 5.11. Using Lemma 4.58, we see that $F_{\Gamma} \circ I_{\mathbb{E}}^{\Gamma} = I_{\mathbb{D}}^{F\Gamma} \circ (F/\Gamma)$. Finally, it follows from Lemma 5.11 and Lemma 5.22 that F_{Γ} preserves the choice of pullback squares.

We have described the action of $\mathbf{E}_* \mathbf{2CE}$ on objects and arrows. Its functoriality is straightforward. \square

We obtain a functor $\mathbf{E2CE} : \mathbf{Esys} \rightarrow \mathbf{CEsys}$ defining $\mathbf{E2CE} := \mathbf{E}_* \mathbf{2CE} \circ \mathbf{E2E}_*$.

Remark 5.32. For every E-system \mathbb{E} and every Γ , the CE-system $\mathbf{E}_* \mathbf{2CE}(\mathbb{E}, \Gamma)$ is rooted. The canonical terminal object id_{Γ} of $\mathcal{F}_{\mathbb{E}}/\Gamma$ is terminal in $\mathcal{C}_{\mathbb{E}}(\Gamma)$ by Corollary 5.18 since for every $A \in \mathcal{F}_{\mathbb{E}}/\Gamma$

$$\text{thom}(A, \text{id}_{\Gamma}) = T(W_A(\text{id}_{\Gamma})) = T(\text{id}_{\Gamma.A}).$$

Next we give the choice of pullbacks in a CE-system in the image of **E2CE** in terms of the underlying E-system structure.

Lemma 5.33. *For \mathbb{E} an E-system and Γ an object in \mathbb{E} , consider the CE-system $\mathbb{A} := \mathbf{E}_* \mathbf{2CE}(\mathbb{E}, \Gamma)$. For every $A \in \mathcal{F}/\Gamma$ and $P, Q \in \mathcal{F}/\Gamma.A$ it is*

$$\left(\mathrm{pr}_0^{A,P}\right)^* Q = \langle P \rangle Q \in \mathcal{F}/\Gamma.A.P$$

and

$$\pi_2(\mathrm{pr}_0^{A,P}, Q) = \mathrm{pr}_1^{P, \langle P \rangle Q} \in \mathrm{thom}(P, \langle P \rangle Q, Q).$$

Proof. The first equality follows from Theorem 5.19. For the second one:

$$\begin{aligned} \pi_2(\mathrm{pr}_0^{\mathrm{id}_\Gamma, A}, B) &= \mathrm{pr}_0^{\mathrm{id}_\Gamma, A} \times \mathrm{idtm}_{\langle A \rangle B} \\ &= (W_{\langle A \rangle B} \langle A \rangle \mathrm{idtm}_{\mathrm{id}_\Gamma}) \cdot \mathrm{idtm}_{\langle A \rangle B} \\ &= (W_A \langle B \rangle \mathrm{idtm}_{\mathrm{id}_\Gamma}) \cdot (W_A \mathrm{idtm}_B) \\ &= \langle A \rangle (\langle B \rangle \mathrm{idtm}_{\mathrm{id}_\Gamma} \cdot \mathrm{idtm}_B) \\ &= \langle A \rangle \left(\mathrm{pr}_0^{\mathrm{id}_\Gamma, B} \cdot \mathrm{pr}_1^{\mathrm{id}_\Gamma, B} \right) \\ &= \langle A \rangle \mathrm{idtm}_B = \mathrm{idtm}_{\langle A \rangle B} \\ &= \mathrm{pr}_1^{A, \langle A \rangle B}. \end{aligned} \quad \square$$

5.3 Equivalence between E-systems and CE-systems

In this section, we show that the functors constructed in Sections 5.1 and 5.2 form an adjunction that, when suitably restricted, yields an equivalence of categories between rooted CE-systems and E-systems.

Specifically, we prove the following results:

Theorem 5.34.

1. The functor **E2CE** is left adjoint to **CE2E**

$$\mathbf{CEsys} \begin{array}{c} \xleftarrow{\mathbf{E2CE}} \\ \perp \\ \xrightarrow{\mathbf{CE2E}} \end{array} \mathbf{Esys}$$

2. The functor **E2CE** is full and faithful.
3. The image of **E2CE** is equivalent to **rCEsys**.

Consider the functors **E2rCE** and **rCE2E** obtained by (co)restricting **E2CE** and **CE2E** to **rCEsys**.

Corollary 5.35. *The pair of adjoint functors **E2CE** and **CE2E** induces an equivalence between the category **Esys** of E-systems and the category **rCEsys** of rooted CE-systems.*

Proof. The equivalence follows from Theorem 5.34.3 and the fact that a coreflective subcategory is equivalent to its image. \square

To prove Theorem 5.34 we construct unit and counit and prove the triangular identities. In this proof we denote as

$$\mathcal{F}/1 \begin{array}{c} \xrightarrow{d} \\ \longleftarrow \\ \downarrow \end{array} \mathcal{F} \quad (41)$$

the canonical isomorphism of strict categories, for any strict category \mathcal{F} with a terminal object 1. We may still leave this isomorphism implicit when doing so creates no confusion.

Problem 5.36. To construct, for each E-system \mathbb{E} , an E-homomorphism $\eta_{\mathbb{E}}: \mathbb{E} \rightarrow \mathbf{CE2E} \circ \mathbf{E2CE}(\mathbb{E})$, naturally in \mathbb{E} .

Construction 5.37 (for Problem 5.36). Let \mathbb{E} be an E-system and denote its terminal object by \square . In this proof we shall decorate with a hat the constituents of the E-system structure of $\hat{\mathbb{E}} := \mathbf{CE2E} \circ \mathbf{E2CE}(\mathbb{E})$. The underlying pre-category of $\hat{\mathbb{E}}$ is $\mathcal{F}_{\mathbb{E}}/\square$ and, for $X \in (\mathcal{F}_{\mathbb{E}}/\square)/!\Gamma$

$$\hat{T}(X) = \left\{ h \in \text{thom}(!\Gamma, !\Gamma.X) \mid \text{pr}_0^{!\Gamma, !\Gamma.X} \circ h = \text{idtm}_{!\Gamma} \right\}.$$

We define $\eta_{\mathbb{E}}$ as the functor $!: \mathcal{F}_{\mathbb{E}} \rightarrow \mathcal{F}_{\mathbb{E}}/\square$ in (41) with term structure given by the bijections

$$T(A) \xrightarrow{\varphi} \hat{T}(!A) \tag{42}$$

from Corollary 5.18, that is, for $A \in \mathcal{F}_{\mathbb{E}}/\Gamma$ and $t \in T(A)$, it is $\eta_{\mathbb{E}}(t) := \text{idtm}_{!\Gamma}.t$.

To conclude that this defines an E-homomorphism we compute for $A \in \mathcal{F}_{\mathbb{E}}/\Gamma$

$$\begin{aligned} \hat{W}_{!A} \circ (!/\Gamma) &= \left(\text{pr}_0^{!\Gamma, A} \right)^* \circ (!/\Gamma) = W_A \circ (!/\Gamma) \\ &= (!/\Gamma.A) \circ W_A, \end{aligned}$$

and for $t \in T(A)$,

$$\begin{aligned} \hat{S}_{!(t)} \circ (!/\Gamma.A) &= (\text{idtm}_{!\Gamma}.t)^* \\ &= S_{\text{idtm}_{!\Gamma}.t} \circ (W_{!\Gamma}/!\Gamma.A) \\ &= S_t \circ (S_{\text{idtm}_{!\Gamma}} \circ (W_{!\Gamma}/!\Gamma))/A \\ &= (!/\Gamma) \circ S_t, \end{aligned}$$

and finally

$$\begin{aligned} \varphi(\text{idtm}_A) &= \text{idtm}_{!\Gamma}.\text{idtm}_A \\ &= \text{idtm}_{!\Gamma.A} \\ &= \hat{\text{idtm}}_{!(A)}. \end{aligned}$$

Naturality in \mathbb{E} requires that any E-homomorphism $F: \mathbb{E} \rightarrow \mathbb{D}$ commutes with η as functors, this is clear, with term structures, and this follows from Lemmas 5.11 and 4.58. \square

Problem 5.38. To construct, for each CE-system \mathbb{A} , a CE-homomorphism $\varepsilon_{\mathbb{A}}: \mathbf{E2CE} \circ \mathbf{CE2E}(\mathbb{A}) \rightarrow \mathbb{A}$, naturally in \mathbb{A} .

Construction 5.39 (for Problem 5.38). Let \mathbb{A} be a CE-system and let $\mathbb{E} := \mathbf{CE2E}(\mathbb{A})$ be the associated E-system. The underlying functor of the CE-system $\hat{\mathbb{A}} := \mathbf{E2CE} \circ \mathbf{CE2E}(\mathbb{A})$ is $I_{\mathbb{E}}: \mathcal{F}/\square \rightarrow \mathcal{C}_{\mathbb{E}}$ defined in Construction 5.31. As before, we decorate with a hat the constituents of the CE-system structure of $\hat{\mathbb{A}}$. For Γ, Δ in \mathcal{F} , recall that $\text{thom}(!\Delta, !\Gamma) = \{ \Delta \xrightarrow{x} \Delta.(!\Delta *!\Gamma) \mid I(!\Delta *!\Gamma) \circ x = \text{id}_{\Delta} \}$ and let

$$\text{thom}(!\Delta, !\Gamma) \xrightarrow{\psi} \mathcal{C}(\Delta, \Gamma) \tag{43}$$

be the function that maps x to the arrow $\pi_2(I(!\Delta), !\Gamma) \circ x$ of \mathcal{C} . The functions ψ give rise to a functor $\Psi: \mathcal{C}_{\mathbb{E}} \rightarrow \mathcal{C}$ as follows. It maps identities to identities since the identity on Γ in $\mathcal{C}_{\mathbb{E}}$ is the only $h \in \text{thom}(!\Gamma, !\Gamma)$ such that $\pi_2(I(!\Gamma, !\Gamma)) \circ h = \text{id}_{\Gamma}$. To see that it preserves composites, consider the commutative diagram below which defines the composite $y \circ x$ of $x \in \text{thom}(!\Delta, !\Gamma)$ and $y \in \text{thom}(!\Gamma, !\Xi)$

in $\mathcal{C}_{\mathbb{E}}$.

$$\begin{array}{ccccccc}
 \Delta & \xrightarrow{x} & \Delta.(!_{\Delta}^{*}!_{\Gamma}) & \xrightarrow{\pi_2(!_{\Delta},!_{\Gamma})} & \Gamma & \xrightarrow{y} & \Gamma.(!_{\Gamma}^{*}!_{\Xi}) \\
 & \searrow \text{id}_{\Delta} & \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\
 & & \Delta.(!_{\Delta}^{*}!_{\Xi}) & \xrightarrow{\pi_2(!_{\Delta},!_{\Xi})} & \Xi & \xrightarrow{\pi_2(!_{\Gamma},!_{\Xi})} & \Gamma \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Delta & \xrightarrow{I(!_{\Delta})} & 1 & & \Gamma \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

Functoriality of Ψ amounts to the commutativity of the upper face.

To conclude that (d, Ψ) is a CE-homomorphism it remains to show that it preserves chosen pull-backs, since the square below commutes by definition of Ψ .

$$\begin{array}{ccc}
 \mathcal{F}/\square & \xrightarrow{d} & \mathcal{F} \\
 I_{\mathbb{E}} \downarrow & & \downarrow I_A \\
 \mathcal{C}_{\mathbb{E}} & \xrightarrow{\Psi} & \mathcal{C}
 \end{array}$$

Let then $x \in \text{thom}(!_{\Delta}, !_{\Gamma})$ and $A \in \mathcal{F}/\Gamma$. It is

$$\begin{aligned}
 x^{\hat{*}}(!_A) &= S_x \circ (W_{!_{\Delta}}/!_{\Gamma}) \circ !_A \\
 &= x^* \circ (I(!_{\Delta})^*/!_{\Gamma}) \circ !_A \\
 &= !((\pi_2(I(!_{\Delta}), !_{\Gamma}) \circ x)^* A) \\
 &= !(\Psi(x)^* A)
 \end{aligned}$$

whereas

$$\Psi(\hat{\pi}_2(x, !_A)) = \pi_2(I(!_{\Delta}, (\Psi(x)^* A)), !_{\Gamma, A}) \circ \hat{\pi}_2(x, !_A) = \pi_2(\Psi(x), A)$$

holds by commutativity of the upper face in

$$\begin{array}{ccccccc}
 \Delta.\Psi(x)^* A & & & & & & \\
 \downarrow \text{idtm}_{\Psi(x)^* A} & \searrow \text{id} & & & & & \\
 \bullet & \xrightarrow{\pi_2(x, !_A)} & \Delta.\Psi(x)^* A & \xrightarrow{\pi_2(\Psi(x), A)} & \Gamma.A & & \\
 \downarrow \text{id} & \searrow & \downarrow \Psi(x)^* A & \searrow & \downarrow A & & \\
 \Delta.\Psi(x)^* A & \xrightarrow{\Psi(x)^* A} & \Delta & \xrightarrow{\Psi(x)} & \Gamma & & \\
 \downarrow \text{id} & \searrow & \downarrow x & \searrow & \downarrow \pi_2(I(!_{\Delta}), !_{\Gamma}) & & \\
 \Delta.\Psi(x)^* A & \xrightarrow{\Psi(x)^* A} & \Delta & \xrightarrow{!_{\Delta}} & 1 & & \\
 & & \downarrow \text{id} & & \downarrow !_{\Gamma} & & \\
 & & \Delta & \xrightarrow{!_{\Delta}} & 1 & &
 \end{array}$$

This diagram commutes because all the squares not involving the top-left object are chosen pullback squares in \mathbb{A} , two of the remaining triangles commute by definition of idtm , and the third one involving $\hat{\pi}_2(x, !(A))$ commutes by (39) and Lemma 5.20.

The component $\varepsilon_{\mathbb{A}}: \mathbf{E2CE} \circ \mathbf{CE2E}(\mathbb{A}) \rightarrow \mathbb{A}$ of the counit at \mathbb{A} is defined to be the pair (d, Ψ) . To see that this choice is natural in \mathbb{A} it is enough to show that the square of functors

$$\begin{array}{ccc} \mathcal{C}_{\mathbf{CE2E}(\mathbb{A})} & \xrightarrow{\Psi_{\mathbb{A}}} & \mathcal{C}_{\mathbb{A}} \\ \downarrow & & \downarrow F_{\mathcal{C}} \\ \mathcal{C}_{\mathbf{CE2E}(\mathbb{B})} & \xrightarrow{\Psi_{\mathbb{B}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

commutes for every CE-homomorphism $F: \mathbb{A} \rightarrow \mathbb{B}$. Note that the action of the left-hand functor coincide with that of F . Commutativity of the square thus follows from

$$F(\pi_2(x, A)) = \pi_2(Fx, FA)$$

which holds by definition of CE-homomorphism. \square

Lemma 5.40.

1. For every E-system \mathbb{E} , the E-homomorphism $\eta_{\mathbb{E}}$ from Problem 5.36 is invertible.
2. For every CE-system \mathbb{A} , the CE-homomorphism $\varepsilon_{\mathbb{A}}$ from Problem 5.38 is invertible if and only if \mathbb{A} is rooted.

Proof. 1. By construction, the E-homomorphism $\eta_{\mathbb{E}}$ is invertible as a functor with term structure. The inverse is an E-homomorphism too:

$$\begin{aligned} (d/!_{\Gamma.A}) \circ \hat{W}_{!(A)} &= \left(\text{pr}_0^{!_{\Gamma.A}} \right)^* \circ (d/!_{\Gamma}) = S_{W_A(\text{idtm}_{!_{\Gamma}})} \circ (W_{!_{\Gamma.A}}/!_{\Gamma}) \circ (d/!_{\Gamma}) \\ &= W_A \circ S_{\text{idtm}_{!_{\Gamma}}} \circ (W_{!_{\Gamma}}/!_{\Gamma}) \circ (d/!_{\Gamma}) \\ &= W_A \circ (d/!_{\Gamma}), \end{aligned}$$

and for $t \in T(A)$,

$$\begin{aligned} (d/!_{\Gamma}) \circ \hat{S}_{!(t)} &= (\text{idtm}_{!_{\Gamma}}.t)^* \circ (d/!_{\Gamma.A}) = S_{\text{idtm}_{!_{\Gamma}}.t} \circ (W_{!_{\Gamma}}/!_{\Gamma.A}) \circ (d/!_{\Gamma.A}) \\ &= S_t \circ (S_{\text{idtm}_{!_{\Gamma}}} \circ (W_{!_{\Gamma}}/!_{\Gamma}))/A \circ (d/!_{\Gamma.A}) \\ &= S_t \circ (d/!_{\Gamma.A}), \end{aligned}$$

and finally

$$\begin{aligned} d(\hat{\text{idtm}}_{!(A)}) &= \text{pr}_1^{!_{\Gamma.A}, W_A(A)}[\text{idtm}_{!_{\Gamma.A}} \cdot \text{pr}_1^{!_{\Gamma.A}}[\text{idtm}_{!_{\Gamma.A}}]] \\ &= \text{pr}_1^{!_{\Gamma.A}}[\text{idtm}_{!_{\Gamma}} \cdot \text{idtm}_A] \\ &= \text{idtm}_A. \end{aligned}$$

2. Note first that each function ψ in (43) induces a bijection

$$\text{thom}(!_{\Delta}, !_{\Gamma}) \xrightarrow{\sim} \{f \in \mathcal{C}(\Delta, \Gamma) \mid I(!_{\Gamma}) \circ f = I(!_{\Delta})\} \quad (44)$$

with inverse given by the universal property of the canonical pullback square below.

$$\begin{array}{ccc} \Delta \cdot !_{\Delta} \cdot !_{\Gamma} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow I(!_{\Gamma}) \\ \Delta & \xrightarrow{I(!_{\Delta})} & \mathbf{1} \end{array}$$

As soon as $\mathbf{1}$ is terminal in \mathcal{C} , the right-hand set in (44) coincide with $\mathcal{C}(\Delta, \Gamma)$. Conversely, if the counit components are invertible it follows from (44) that $\mathcal{C}(\Delta, \mathbf{1}) = \{!_{\Delta}\}$. \square

Proof of Theorem 5.34. 1. To complete the proof we show that, for an E-system \mathbb{E} and a CE-system \mathbb{A}

$$\mathbf{CE2E}(\varepsilon_{\mathbb{A}}) \circ \eta_{\mathbf{CE2E}(\mathbb{A})} = \text{Id}_{\mathbf{CE2E}(\mathbb{A})} \quad \text{and} \quad \varepsilon_{\mathbf{E2CE}(\mathbb{E})} \circ \mathbf{E2CE}(\eta_{\mathbb{E}}) = \text{Id}_{\mathbf{E2CE}(\mathbb{E})}.$$

It is clear that these equations hold between functors on families by the isomorphism in (41). It remains to show that they hold also between the term structures in the left-hand one, and between functors on substitutions in the right-hand one.

For a CE-system \mathbb{A} , a family $A \in \mathcal{F}/\Gamma$ and $y \in T(A) = \{x: \Gamma \rightarrow \Gamma.A \mid I(A) \circ x = \text{id}_{\Gamma}\}$, Lemma 5.33 yields $\eta_{\mathbf{CE2E}(\mathbb{A})}(y) = \pi_2(\text{idm}_{\Gamma}, \pi_2(!_{\Gamma}, !_{\Gamma})^*A) \circ y$. It follows that

$$\begin{aligned} \mathbf{CE2E}(\varepsilon_{\mathbb{A}}) \circ \eta_{\mathbf{CE2E}(\mathbb{A})}(y) &= \pi_2(!_{\Gamma}, !_{\Gamma.A}) \circ \pi_2(\text{idm}_{\Gamma}, \pi_2(!_{\Gamma}, !_{\Gamma})^*A) \circ y \\ &= \pi_2(\pi_2(!_{\Gamma}, !_{\Gamma}), A) \circ \pi_2(\text{idm}_{\Gamma}, \pi_2(!_{\Gamma}, !_{\Gamma})^*A) \circ y \\ &= \pi_2(\pi_2(!_{\Gamma}, !_{\Gamma}) \circ \text{idm}_{\Gamma}, A) \circ y \\ &= y. \end{aligned}$$

For an E-system \mathbb{E} , objects Δ and Γ and $f \in \text{thom}(!_{\Delta}, !_{\Gamma})$, Lemmas 5.33 and 4.59 yield

$$\varepsilon_{\mathbf{E2CE}(\mathbb{E})} \circ \mathbf{E2CE}(\eta_{\mathbb{E}})(f) = \text{pr}_1^{!_{\Delta}, (!_{\Delta})!_{\Gamma}}[\text{idm}_{\Gamma}.f] = f.$$

This concludes the proof of the adjunction.

2. It is a consequence of Lemma 5.40.1.

3. We have already observed in Remark 5.32 that, for every E-system \mathbb{E} , the CE-system $\mathbf{E2CE}(\mathbb{E})$ is rooted. The converse follows from Lemma 5.40.2. \square

5.4 Equivalence between B-systems and C-systems

Here we describe the main contribution of our work: the construction of an equivalence of categories between the category of C-systems of Section 3 and the category of B-systems of Section 4.

Lemma 5.41. *The functor $\mathbf{CE2E}: \mathbf{CEsys} \rightarrow \mathbf{Esys}$ restricts to a functor $\mathbf{CE2E}: \mathbf{rCEsys}_s \rightarrow \mathbf{Esys}_s$ between stratified systems.*

Proof. To see that the E-system $\mathbf{CE2E}(\mathbb{A})$ is stratified whenever the rooted CE-system \mathbb{A} is stratified, note first that the underlying category \mathcal{F} is stratified by assumption. Weakening and substitution homomorphisms are stratified since the pullback functor that defines them in Construction 5.6.(37,38) is stratified.

For a stratified CE-homomorphism F , the underlying functor of the E-homomorphism $\mathbf{CE2E}(F)$ is the component $F_{\mathcal{F}}$ of F on families, which is stratified by assumption. \square

Lemma 5.42. *The functor $\mathbf{E2CE}: \mathbf{Esys} \rightarrow \mathbf{rCEsys}$ restricts to a functor $\mathbf{E2CE}: \mathbf{Esys}_s \rightarrow \mathbf{rCEsys}_s$ between stratified systems.*

Proof. Let \mathbb{E} be a stratified E-system. In particular, the underlying category \mathcal{F} is stratified. Since weakening and substitution homomorphisms are also stratified by assumption, so is the precomposition homomorphisms from Definition 5.10. It follows that the CE-system $\mathbf{E2CE}(\mathbb{E})$ is stratified.

For a stratified E-homomorphism F , the component on families of the CE-homomorphism $\mathbf{E2CE}(F)$ is the underlying functor of F , which is stratified by assumption. \square

Lemma 5.43.

1. *For every stratified E-system \mathbb{E} , the unit component $\eta_{\mathbb{E}}$ of Construction 5.37 is a stratified E-homomorphism.*
2. *For every stratified CE-system \mathbb{A} , the counit component $\varepsilon_{\mathbb{A}}$ of Construction 5.39 is a stratified CE-homomorphism.*

3. The adjunction $\mathbf{E2CE} \dashv \mathbf{CE2E}$ from Theorem 5.34.1 restricts to an adjunction

$$\mathbf{CEsys}_s \begin{array}{c} \xleftarrow{\mathbf{E2CE}} \\ \xrightarrow{\mathbf{CE2E}} \end{array} \mathbf{Esys}_s.$$

between subcategories of stratified structures.

Proof.

1. The underlying functor of the unit component $\eta_{\mathbb{E}}$ is the functor $! : \mathcal{F} \rightarrow \mathcal{F}/[\!]$ from (41). This functor is stratified since $L([\!]) = 0$.
2. The underlying functor of the counit component $\varepsilon_{\mathbb{A}}$ on families is the inverse $d : \mathcal{F}/1 \rightarrow \mathcal{F}$ of $! : \mathcal{F} \rightarrow \mathcal{F}/1$, and it is stratified for the same reason.
3. This is a consequence of Lemmas 5.41 and 5.42 and Items 1 and 2 just proved. □

Define a functor $\mathbf{C2B} : \mathbf{Csys} \rightarrow \mathbf{Bsys}$ as the composite

$$\mathbf{Csys} \xrightarrow{\mathbf{CE}} \mathbf{rCEsys}_s \xrightarrow{\mathbf{CE2E}} \mathbf{Esys}_s \xrightarrow{\mathbf{E2B}} \mathbf{Bsys} \quad (45)$$

where the functors are, in order, \mathbf{CE} from Construction 3.24 $\mathbf{CE2E}$ from Construction 5.6 and $\mathbf{E2B}$ from Construction 4.80. Similarly, we obtain a functor $\mathbf{B2C} : \mathbf{Bsys} \rightarrow \mathbf{Csys}$ in the other direction as the composite

$$\mathbf{Bsys} \xrightarrow{\mathbf{B2E}} \mathbf{Esys}_s \xrightarrow{\mathbf{E2CE}} \mathbf{rCEsys}_s \xrightarrow{\mathbf{C}} \mathbf{Csys} \quad (46)$$

where the functors are, in order, $\mathbf{B2E}$ from Lemma 4.71, $\mathbf{E2CE}$ from Construction 5.31 and \mathbf{C} from Definition 3.28.

Theorem 5.44. *The pair of functors $\mathbf{C2B}$ and $\mathbf{B2C}$ establish an equivalence between the category of C-systems and the category of B-systems.*

Proof. The functors defining $\mathbf{C2B}$ in (45) and $\mathbf{B2C}$ in (46) are essentially inverse to each other by Theorems 3.30 and 4.83 and Corollary 5.35. The claim follows since equivalences compose. □

6 Conclusion

We have constructed an equivalence between the category of C-systems and the category of B-systems, each equipped with a suitable notion of morphism. The equivalence does not rely on classical reasoning principles such as the axiom of choice or excluded middle. This equivalence constitutes a crucial piece in Voevodsky's research program on the formulation and solution of an initiality conjecture.

Some questions that remain open:

- Voevodsky has studied different type constructions on C-systems, in particular, dependent function types [Voe16a, Voe17b] and identity types [Voe15b]. The equivalence constructed in the present paper should be extended to type and term constructors on C-systems and B-systems.
- Via Generalized Algebraic Theories, B-systems and C-systems relate to Garner's algebras for a monad on type-and-term systems [Gar15], in the form of an equivalence of categories. It would be very useful to have an explicit description of the maps back and forth, without passing through GATs.
- E-systems and CE-systems should be related to other unstratified categorical structures for the interpretation of type theory, such as categories with families [Dyb96].

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