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# Identification of Linear State-Space Models Subject to Truncated Gaussian Disturbances



R. A. González, A. L. Cedeño, K. Tiels and T. Oomen

## Identification under bounded disturbances

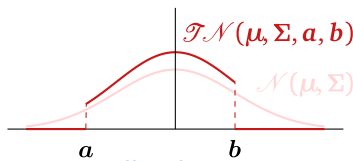
- Constraints are integral to state estimation for process optimization, monitoring, and control
- Yet, Gaussian assumptions are prevalent in identification  $\Rightarrow$  inaccurate for bounded-support noise

## Problem formulation

Discrete-time SISO system in state-space form:

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{B}u_t + \mathbf{w}_t \\ y_t &= \mathbf{C}\mathbf{x}_t + Du_t + v_t \end{aligned} \iff \xi_t = \Theta\zeta_t + \eta_t$$

- Truncated Gaussian noise  $\eta_t \sim \mathcal{T}\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{a}, \mathbf{b})$
- Initial condition  $\mathbf{x}_1$  known or  $\mathbf{x}_1 \sim \mathcal{N}(\mathbf{e}, \mathbf{H}, \mathbf{a}_1, \mathbf{b}_1)$
- Unknowns: system matrices  $\Theta$ , noise parameters  $\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{a}, \mathbf{b}$ , and possibly  $\mathbf{e}, \mathbf{H}, \mathbf{a}_1, \mathbf{b}_1$  as well



How can we estimate all unknown parameters (stacked in  $\boldsymbol{\beta}$ ) using  $\{u_t, y_t\}_{t=1}^N$ ?

## EM for truncated Gaussian identification

Direct maximum likelihood estimation ( $\arg \max_{\boldsymbol{\beta}} \mathcal{L}(\boldsymbol{\beta})$ ) is intractable. Instead, we consider the EM algorithm:

1. For fixed  $\hat{\boldsymbol{\beta}}^{(k)}$ , obtain  $\mathcal{Q}(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}}^{(k)}) = \mathbb{E} \left\{ \log \mathcal{L}(\boldsymbol{\beta}) | y_{1:N}, \hat{\boldsymbol{\beta}}^{(k)} \right\}$
2. Compute  $\hat{\boldsymbol{\beta}}^{(k+1)} = \arg \max_{\boldsymbol{\beta}} \mathcal{Q}(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}}^{(k)})$
3. Set  $k \rightarrow k+1$  and return to Step 1.

For our problem, the  $\mathcal{Q}$  function is given by

$$\mathcal{Q}(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}}^{(k)}) = \underbrace{\mathcal{Q}_1(\boldsymbol{\beta}_1, \hat{\boldsymbol{\beta}}^{(k)})}_{\text{Only depends on } \mathbf{x}_1} + \sum_{t=1}^N \mathbb{E} \left\{ \underbrace{\log p(\mathbf{x}_{t+1}, y_t | \mathbf{x}_t, \boldsymbol{\beta}_2)}_{\text{Depends on } \boldsymbol{\beta}_2 = (\boldsymbol{\Theta}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{a}, \mathbf{b})} | y_{1:N}, \hat{\boldsymbol{\beta}}^{(k)} \right\}$$

where  $\mathbf{x}_{t+1}, y_t | \mathbf{x}_t, \boldsymbol{\beta}_2 \sim \mathcal{T}\mathcal{N}(\boldsymbol{\Theta}\zeta_t + \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Theta}\zeta_t + \mathbf{a}, \boldsymbol{\Theta}\zeta_t + \mathbf{b})$ . Due to space constraints, we assume  $\mathbf{x}_1$  known.

**Theorem:** Define  $\mathcal{A}_t := \text{supp}(p(\mathbf{x}_t, \mathbf{x}_{t+1} | y_{1:N}, \hat{\boldsymbol{\beta}}^{(k)}))$ . The next EM iteration of  $\boldsymbol{\beta}_2$  given  $\hat{\boldsymbol{\beta}}^{(k)}$  is obtained by

$$\hat{\boldsymbol{\beta}}_2^{(k+1)} = \arg \max_{\boldsymbol{\beta}_2} \gamma_2^{(k)}(\boldsymbol{\beta}_2)$$

$$\text{s.t. } \mathbf{a} \leq \xi_t - \boldsymbol{\Theta}\zeta_t \leq \mathbf{b}, \quad (\mathbf{x}_t, \mathbf{x}_{t+1}) \in \mathcal{A}_t, t=1, \dots, N,$$

where

$$\begin{aligned} \gamma_2^{(k)}(\boldsymbol{\beta}_2) &= \frac{-1}{2} \text{tr} \left\{ \boldsymbol{\Sigma}^{-1} \left( \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\Theta}^\top \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\Xi} - \boldsymbol{\Phi}\boldsymbol{\Phi}^\top & -\boldsymbol{\Psi} + \boldsymbol{\Phi}\boldsymbol{\Lambda}^\top \\ -\boldsymbol{\Psi}^\top + \boldsymbol{\Lambda}\boldsymbol{\Phi}^\top & \boldsymbol{\Delta} - \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\Theta}^\top \end{bmatrix} \right. \right. \\ &\quad \left. \left. + (\boldsymbol{\Phi} - \boldsymbol{\Theta}\boldsymbol{\Lambda} - \boldsymbol{\mu})(\%)^\top \right) \right\} - \log \int_a^b \exp \left\{ \frac{(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})}{-2} \right\} d\mathbf{x}, \end{aligned}$$

and where we have defined the following quantities

$$\begin{bmatrix} \boldsymbol{\Xi} & \boldsymbol{\Psi} & \boldsymbol{\Phi} \\ \boldsymbol{\Psi}^\top & \boldsymbol{\Delta} & \boldsymbol{\Lambda} \\ \boldsymbol{\Phi}^\top & \boldsymbol{\Lambda}^\top & \mathbf{1} \end{bmatrix} = \frac{1}{N} \sum_{t=1}^N \int \begin{bmatrix} \xi_t \\ \zeta_t \\ 1 \end{bmatrix} \begin{bmatrix} \xi_t \\ \zeta_t \\ 1 \end{bmatrix}^\top p(\mathbf{x}_{t+1}, \mathbf{x}_t | y_{1:N}, \hat{\boldsymbol{\beta}}^{(k)}) d\mathbf{x}_{t+1} d\mathbf{x}_t.$$

How to compute  $\boldsymbol{\Xi}, \boldsymbol{\Psi}, \boldsymbol{\Phi}, \boldsymbol{\Lambda}, \boldsymbol{\Delta}$ ? Particle smoothing at each EM iteration (also required for computing  $\mathcal{A}_t$ ).

How to maximize  $\gamma_2^{(k)}(\boldsymbol{\beta})$ ? We require two lemmas:

**Lemma 1:** For fixed  $(\hat{\boldsymbol{\Theta}}, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ , the  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  parameters that maximize  $\gamma_2^{(k)}(\boldsymbol{\beta})$  satisfy

$$\hat{\boldsymbol{\mu}}_1 = \boldsymbol{\Phi} - \hat{\boldsymbol{\Theta}}\boldsymbol{\Lambda} - \boldsymbol{\mu},$$

$$\hat{\boldsymbol{\mu}}_2 - \hat{\boldsymbol{\mu}}_1 \hat{\boldsymbol{\mu}}_1^\top = \begin{bmatrix} \mathbf{I} \\ \hat{\boldsymbol{\Theta}}^\top \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{\Xi} - \boldsymbol{\Phi}\boldsymbol{\Phi}^\top & -\boldsymbol{\Psi} + \boldsymbol{\Phi}\boldsymbol{\Lambda}^\top \\ -\boldsymbol{\Psi}^\top + \boldsymbol{\Lambda}\boldsymbol{\Phi}^\top & \boldsymbol{\Delta} - \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \hat{\boldsymbol{\Theta}}^\top \end{bmatrix},$$

where  $\hat{\boldsymbol{\mu}}_i$  is the  $i$ th moment of the truncated Gaussian  $\mathcal{T}\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}, \hat{\mathbf{a}} - \boldsymbol{\mu}, \hat{\mathbf{b}} - \boldsymbol{\mu})$ .

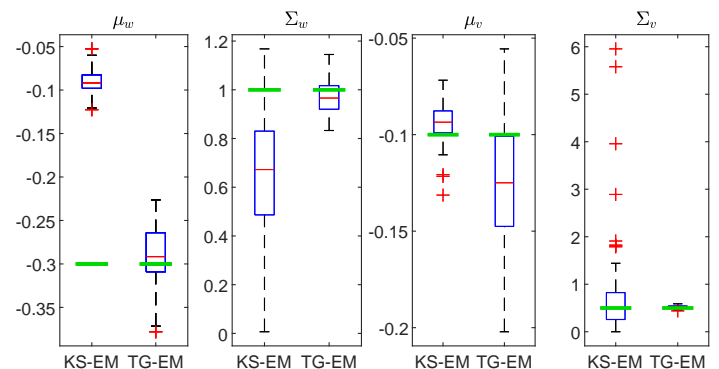
**Lemma 2:** For fixed  $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$ , the cost  $\gamma_2^{(k)}(\boldsymbol{\beta})$  is maximized by  $(\text{vec}\{\hat{\boldsymbol{\Theta}}\}, \hat{\mathbf{a}}, \hat{\mathbf{b}}) = \arg \max_{\mathbf{z}, \mathbf{a}, \mathbf{b}} \mathbf{z}^\top \text{vec}(\hat{\boldsymbol{\Sigma}}^{-1} [\boldsymbol{\Psi} - \hat{\boldsymbol{\mu}}\boldsymbol{\Lambda}^\top]) - \frac{1}{2} \mathbf{z}^\top (\boldsymbol{\Delta} \otimes \hat{\boldsymbol{\Sigma}}^{-1}) \mathbf{z}$

$$\text{s.t. } \begin{bmatrix} \zeta_t \otimes \mathbf{I} \\ -\zeta_t \otimes \mathbf{I} \end{bmatrix} \mathbf{z} \leq \begin{bmatrix} \xi_t - \mathbf{a} \\ \mathbf{b} - \xi_t \end{bmatrix}, \quad (\mathbf{x}_t, \mathbf{x}_{t+1}) \in \mathcal{A}_t, t=1, \dots, N.$$

Implementation aspects? fixed-point iterations + control variates + quadratic programming

## Simulation study

Setup:  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$  fixed,  $\boldsymbol{\mu} = [-0.3, -0.1]$ ,  $\boldsymbol{\Sigma} = \text{diag}\{1, 0.5\}$ ,  $\mathbf{a} = [-1.5, -50]^\top$ ,  $\mathbf{b} = [2.5, 50]^\top$ ,  $N = 5000$ .



## Reference

[1] R. A. González, A. L. Cedeño, K. Tiels and T. Oomen. "Identification of Linear State-Space Models Subject to Truncated Gaussian Disturbances". In preparation.