H-BASIS IN GEOMETRIC STRUCTURES WITH A DENSE/CODENSE INDEPENDENT SUBSET

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Abstract. We discuss H-basis in geometric structures with a dense/codense independent subset, and algebraic n-gons in geometric structures to calculate the ranks of H-structures in trivial/non-trivial independence of base geometric structures.

1. Notations and algebraic dimension

Let T be a complete L-theory and let $\mathcal{M}$ be a sufficiently saturated model of T. $a, b, c, \ldots$ denote elements of $\mathcal{M}$. $\bar{a}, \bar{b}, \bar{c}, \ldots$ denote finite tuples of $\mathcal{M}$. A, B, C denote small subsets of $\mathcal{M}$. We write $\bar{a} \in \mathrm{acl}(A)$ if $|\{\sigma(\bar{a}) : \sigma \in \mathrm{Aut}(\mathcal{M}/A)\}|$ is finite. The algebraic closure of A.
Let $i(\geq 1)$ be a natural number. We put $\bar{a}_{\leq i} := \bar{a}_1, \ldots, \bar{a}_i$. $\bar{a}_{<i} = \bar{a}_1, \ldots, \bar{a}_{i-1}$. $\bar{a}_{<1} := \emptyset$.
We say that $a_{\leq n} = a_1, a_2, \ldots, a_n$ is algebraically independent over B if $a_i \notin \mathrm{acl}(Ba_{<i})$ for each $i \leq n$.
We say that $(\mathcal{M}, \mathrm{acl}(*))$ has Steinitz exchange property if $a \in \mathrm{acl}(Bb) \setminus \mathrm{acl}(B)$ implies $b \in \mathrm{acl}(Ba)$.
We say that $(\mathcal{M}, \mathrm{acl}(*))$ is geometric if it has Steinitz exhchange property and eliminates $\exists^\infty$.

Assume that $(\mathcal{M}, \mathrm{acl}(*))$ has Steinitz exchange property. Then for any $a_{\leq n} = a_1, a_2, \ldots, a_n$ and B, after renumbering indices, there exists unique $m(\leq n)$ such that

$$a_i \notin \mathrm{acl}(Ba_{<i}) \text{ for each } i \leq m$$
$$a_j \in \mathrm{acl}(Ba_{\leq m}) \text{ for each } j > m$$

We write $m = \dim(a_{\leq n}/B)$, the dimension of $a_{\leq n}$ over B.

Basic properties on dimension

(1) If $A \subseteq B$, then $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$.
(2) Transitivity: If $A \subseteq B \subseteq C$, then $\dim(\bar{a}/A) = \dim(\bar{a}/C)$ iff $\dim(\bar{a}/A) = \dim(\bar{a}/B)$ and $\dim(\bar{a}/B) = \dim(\bar{a}/C)$
(3) Sub-additivity: $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A) + \dim(\bar{b}/A\bar{a})$.
(4) Finite coding: There exists a finite tuple $\bar{b} \subseteq B$ such that $\dim(\bar{a}/B) = \dim(\bar{a}/\bar{b})$.

Date: April 6, 2023.
1991 *Mathematics Subject Classification.* 03C45.
Key words and phrases. geometric structures, dense/codense subset, H-structures, H-basis.

The independence relation by dimension
We write $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{c}$ if $\dim(\bar{a}/\bar{b}) = \dim(\bar{a}/\bar{b}\bar{c})$.

Symmetry: $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{c}$ implies $\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{a}$.
Proof. By using sub-additivity, we have $\dim(\bar{c}/\bar{a}\bar{b}) + \dim(\bar{a}/\bar{b}) = \dim(\bar{a}\bar{c}/\bar{b}) = \dim(\bar{a}/\bar{b}\bar{c}) + \dim(\bar{c}/\bar{b})$. So $\dim(a/\bar{b}\bar{c}) = \dim(\bar{a}/b)$ implies $\dim(\bar{c}/\bar{a}\bar{b}) = \dim(\bar{c}/\bar{b})$.

Monotonicity: if $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{c}$ and $\bar{a}_0 \subseteq \bar{a}$, then $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{c}$.
Proof. By symmetry $\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{a}$. By transitivity $\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{a}_0$. By symmetry again, $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}}^{\dim} \bar{c}$.

Now we define $A \mathop{\smile\!\!\!\!\!\!\mid}_{B}^{\dim} C$ if $\dim(\bar{a}/B) = \dim(\bar{a}/BC)$ for any finite tuple $\bar{a} \subseteq A$.
We intruduce the imaginarr element $e \in \mathcal{M}^{\mathrm{eq}}$ if $e = \bar{a}/E$, where $E(\bar{x}, \bar{y})$ is an $\emptyset$-definible equivalence relation with $\mathrm{lh}(\bar{a}) = \mathrm{lh}(\bar{x}) = \mathrm{lh}(\bar{y})$ and some $\bar{a} \subset \mathcal{M}$.
For $e \in \mathcal{M}^{\mathrm{eq}}$ and $A \subset \mathcal{M}^{\mathrm{eq}}$ we write $e \in \mathrm{acl}^{\mathrm{eq}}(A)$ if $|\{\sigma(e) : \sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)\}|$ is finite.
The independence calculus: See [A].
A *symmetric* ternary relation $* \mathop{\smile\!\!\!\!\!\!\mid}_{*} *$ on $\mathcal{M}^{\mathrm{eq}}$ has the independence calculus if the following 8 conditions hold:

1. Normality: $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} BC$.
2. Invariance: $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$ and $ABC \equiv A'B'C'$ imply $A' \mathop{\smile\!\!\!\!\!\!\mid}_{B'} C'$
3. Monotonicity: $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$ and $A_0 \subseteq A$ imply $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$
4. *Transitivity* : If $B \subseteq C \subseteq D$, then
   $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} D$ iff $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} D$
5. Extention: There exists $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$.
6. Finite character: If $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$ for any finite tuple $\bar{a} \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_{B} C$.
7. *Local character*: For any $\bar{a}, A \subset \mathcal{M}$, there exists $A_0 \subseteq A$ such that $|A_0| \leq |T|$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A_0} A$.
8. Anti-reflexivity: $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A} \bar{a}$ implies $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(A)$.

symmetric⇔*transitive*⇔*local character* holds modulo other properties of the independence calculus. We have the following: stable⇒simple⇒rosy(i.e.having the independence calculus)⇐o-minimal
superstable($\mathrm{U} < \infty$)⇒supersimple($\mathrm{SU} < \infty$) ⇒superrosy($\mathrm{U}^{\mathfrak{p}} < \infty$).
strongly minimal($0 < \mathrm{U} \leq \mathrm{RM} = 1, \deg_{\mathrm{RM}} = 1$)⇒ $\mathrm{SU} = 1 \Rightarrow \mathrm{U}^{\mathfrak{p}} = 1$ ⇐o-minimal.

2. Non-triviality and algebraic n-gons

Assume that $(\mathcal{M}, \mathrm{acl}(*))$ has Steinitz exchange property and put $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\dim}$.

We say that $a \in \mathcal{M}$ is *non-trivial* if there exists $a_2, a_3, \bar{c}$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{c}} a_2, a \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{c}} a_3, a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{c}} a_3$ and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{\bar{c}} a_2, a_3$. Then we say that a, a_2, a_3 is an *algebraic triangle* over $\bar{b}$.

We say that $a_{\leq n} = a_1, \cdots, a_n$ is an *algebraic n-gon* over A if $\dim(a_{\leq n}/A) = n-1$ and $\dim(a_{\leq n} \setminus \{a_i\}/A) = n-1$ for each $1 \leq i \leq n$.
Then a_i, a_j, a_k is an algebraic triangle over $Aa_{\leq n} \setminus \{a_i, a_j, a_k\}$, because $a_i \not\mathop{\smile\!\!\!\!\!\!\mid}_{Aa_{\leq n} \setminus \{a_i, a_j, a_k\}} a_j, a_k$. so any point of n-gon is non-trivial.

Fact 2.1. *If $a \in \mathcal{M}$ is non-trivial, then for each $n < \omega$, there exists $b_{\leq n-1}, A$ such that $ab_{\leq n-1}$ is an algebraic n-gon over A.*

Proof. The case $n = 3$ is clear. By induction hypothesis, assume that $ab_{\leq n-1}$ is an algebraic n-gon over A. As b_{n-1} is non-trivial, there exist $b_n, b_{n+1}, \bar{c}$ such that $b_{n-1}b_nb_{n+1}$ is an algebraic triangle over $\bar{c}$. By an automorphism fixing b_{n-1}, we may assume that $b_nb_{n+1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} b_{\leq n-2}A$.

CLAIM: $ab_{\leq n-2}b_nb_{n+1}$ is an algebraic $(n+1)$-gon over $A\bar{c}$.

Subclaim 1: $\dim(ab_{\leq n-2}b_nb_{n+1}/A\bar{c}) = n$.
As $b_{n+1} \in \mathrm{acl}(b_{n-1}b_n\bar{c})$ and $b_{n-1} \in \mathrm{acl}(ab_{\leq n-2}A)$, we have $b_{n+1} \in \mathrm{acl}(ab_{\leq n-2}b_nA\bar{c})$. On the other hand, we have $b_n\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} b_{\leq n-2}A$, $b_n \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} b_{n-1}$ and $a \in \mathrm{acl}(b_{\leq n-1}A)$, we have $b_n \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} ab_{\leq n-2}A$. As $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} ab_{\leq n-2}A$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!|} b_{n-1}$, we have $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_A ab_{\leq n-2}$. Therefore $\dim(ab_{\leq n-2}b_nb_{n+1}/A\bar{c}) = n$.

Subclaim 2: $ab_{\leq n-2}b_n$ is independent over $A\bar{c}$. Similarly for $ab_{\leq n-2}b_{n+1}$.
By $b_n \notin \mathrm{acl}(b_{n-1}\bar{c})$ and $b_nb_{n+1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} b_{\leq n-2}A$, we have $b_n \notin \mathrm{acl}(b_{\leq n-1}A\bar{c}) = \mathrm{acl}(ab_{\leq n-2}A\bar{c})$ as $ab_{\leq n-1}$ is an algebraic n-gons over A.

We use the following : We have $b_nb_{n+1} \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} A$ by $b_nb_{n+1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} A$ and $b_{n-1} \mathop{\smile\!\!\!\!\!\!|} A$. Note that $b_n \mathop{\smile\!\!\!\!\!\!|}_{A\bar{c}} b_{n+1}$.

Subclaim 3: $b_{\leq n-2}b_nb_{n+1}$ is independent over $A\bar{c}$.
Since $b_{n-1}b_nb_{n+1}$ is an algebraic triangle over $\bar{c}$ and $b_n \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}\bar{c}} b_{\leq n-2}A$, we see that $b_n \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} b_{\leq n-2}A$. So $b_{\leq n-2}b_n$ is independent over $A\bar{c}$. By $b_nb_{n+1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} b_{\leq n-2}A$ and $b_{n-1} \mathop{\smile\!\!\!\!\!\!|}_A b_{\leq n-2}$, we have $b_nb_{n+1}b_{n-1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_A b_{\leq n-2}$. If we had $b_{n+1} \in \mathrm{acl}(b_nb_{\leq n-2}A\bar{c})$, we would have $b_{n+1} \in \mathrm{acl}(b_nA\bar{c})$. As we have $b_nb_{n+1} \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} A$, $b_{n+1} \in \mathrm{acl}(b_n\bar{c})$ follows, a contradiction to $b_n \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} b_{n+1}$.

Subclaim 4: $ab_{\leq n-2}b_nb_{n+1} \setminus \{b_j\}$ is independent over $A\bar{c}$ for each $1 \leq j \leq n-2$.
As $b_{n-1} \mathop{\smile\!\!\!\!\!\!|}_A ab_{\leq n-2}\setminus\{b_j\}$ and $b_nb_{n+1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{b_{n-1}} ab_{\leq n-2}A$, we have $b_nb_{n+1}b_{n-1}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_A ab_{\leq n-2}\setminus \{b_j\}$. So $ab_{\leq n-2} \setminus \{b_j\}b_n$ is independent over $A\bar{c}$. If we had $b_{n+1} \in \mathrm{acl}(b_nab_{\leq n-2} \setminus \{b_j\})$, we would have $b_{n+1} \in acl(b_nA\bar{c})$, a contradiction. □

3. H-structure

Let $\mathcal{M}$ be a sufficiently saturated model of a complete L-theory T. Suppose that $(\mathcal{M}, \mathrm{acl}(*))$ has Steinitz exchange property. We add a unary predicate $H(x)$ for a dense/codense independent subset. L_H denotes $L \cup \{H\}$.

Definition 3.1. We say that $(\mathcal{M}, H(\mathcal{M}))$ is an H-structure if

1. $H(\mathcal{M})$ is *independent*: If $a_1, \cdots, a_n \in H(\mathcal{M})$ are distinct, they are independent over $\emptyset$.
2. density: If $A \subset \mathcal{M}$ is finite dimentional and $p(x)$ is a unary non-algebraic L-type over A, then there exists $a \in H(\mathcal{M})$ such that $a \models p(x)$.
3. codensity: If $A \subset \mathcal{M}$ is finite dimentional and $q(x)$ is a unary non-algebraic L-type over A, then there exists $b \notin \mathrm{acl}(AH(\mathcal{M}))$ such that $b \models q(x)$.

For $A \subset \mathcal{M}$ we say A is H-independent in $\mathcal{M}$ if $A \downarrow_{H(A)} H(\mathcal{M})$, where $H(A) = H(\mathcal{M}) \cap A$. Note that if A is H-independent, then $\mathrm{acl}(A)$ is also H-independent. tp denotes L-type and tp_H denotes L_H-type.

Fact 3.2. (1) *For any $(M, H(M))$, there exists a sufficiently saturated H-structure $(M, H(M)) \subseteq (\mathcal{M}, H(\mathcal{M}))$ with M is H-independent in $\mathcal{M}$.*

(2) *Suppose that $(\mathcal{M}, H(\mathcal{M}))$ and $(\mathcal{N}, H(\mathcal{N}))$ are H-structures. If H-independent tuples $\bar{a} \subset \mathcal{M}, \bar{b} \subset \mathcal{N}$ with $\mathrm{tp}(\bar{a}, H(\bar{a})) = \mathrm{tp}(\bar{b}, H(\bar{b}))$, then $\mathrm{tp}_H(\bar{a}) = \mathrm{tp}_H(\bar{b})$. In particular, all H-structures are elementarily equivalent, let T^{ind} be the common theory. If T eliminates $\exists^\infty$, then T^{ind} is axiomatizable such that all $|T|^+$-saturated model of T^{ind} are H-structures.*

The following theories eliminate $\exists^\infty$: strongly minimal theories, SU $= 1$ theories, dense o-minimal theories and the p-adics in a single sort.

Let T be the theory of infinite dimensional countable vector space V over a finite field. Put $H(V) := \{v_i : i < \omega\}$ a basis of V. Then $(V, H(V))$ is a model of T^{ind} but not H-structure because it does not satisfy codense property. Put $H_j(V) := \{v_i : i > j\}$. Then $(V, H_j(V))$ is a model of T^{ind} but not H-structure and $(V, H_j(V))$ is not isomorphic to $(V, H_k(V))$ for $j \neq k < \omega$, so T^{ind} is not ω-categorical. Put $H_{even}(V) = \{v_{2i} : i < \omega\}$. Then $(V, H_{even}(V))$ is an H-structure.

4. H-basis

Let $A \subset \mathcal{M}$ be H-independent. For any $\bar{a} \subset \mathcal{M}$ we can take a finite tuple $\bar{h} \subset H(\mathcal{M})$ such that $\bar{a} \downarrow_{A\bar{h}} H(\mathcal{M})$ and $A \cap \bar{h} = \emptyset$. Suppose that $\bar{h}$ is minimal length. We show the uniqueness of $\bar{h}$ up to permutation.

Take such another $\bar{h}'$ and let $\bar{h}_1 := \bar{h} \cap \bar{h}', \bar{h} = \bar{h}_1\bar{h}_2$ and $\bar{h}' = \bar{h}_1\bar{h}_2'$. As $H(\mathcal{M})$ is an independent subset and $\bar{h} \cap A = \bar{h}' \cap A = \emptyset$, we have $\bar{h}_2 \downarrow_{H(A)\bar{h}_1} \bar{h}_2'$. As A is H-independent, $A\bar{h}_1 \downarrow_{H(A)\bar{h}_1} \bar{h}_2\bar{h}_2'$, we have $\bar{h}_2 \downarrow_{A\bar{h}_1} \bar{h}_2'$.

Let $\bar{a} = \bar{a}_1\bar{a}_2$ be such that $\bar{a}_1 \downarrow AH(\mathcal{M})$ and $\bar{a}_2 \in \mathrm{acl}(\bar{a}_1 AH(\mathcal{M}))$. Note that $\bar{a}_2 \in \mathrm{acl}(\bar{a}_1 A\bar{h}_1\bar{h}_2) \setminus \mathrm{acl}(\bar{a}_1 A\bar{h}_1)$ and $\bar{a}_2 \in \mathrm{acl}(\bar{a}_1 A\bar{h}_1\bar{h}_2') \setminus \mathrm{acl}(\bar{a}_1 A\bar{h}_1)$ by minimality of $\bar{h}$ and $\bar{h}'$. Note that $\bar{h}_2' \not\downarrow_{\bar{a}_1 A\bar{h}_1} \bar{h}_2$ witnessed by $\bar{a}_2$. By $\bar{a}_1 \downarrow CH(\mathcal{M})$, we have $\dim(\bar{h}_2'/A\bar{h}_1\bar{h}_2) = \dim(\bar{h}_2'/\bar{a}_1 A\bar{h}_1\bar{h}_2) < \dim(\bar{h}_2'/\bar{a}_1 A\bar{h}_1) = \dim(\bar{h}_2'/A\bar{h}_1)$, $\bar{h}_2' \not\downarrow_{A\bar{h}_1} \bar{h}_2$, a contradiction to the independency of $H(\mathcal{M})$.

We write $HB(\bar{a}/A) := \bar{h}$, which is called H-basis of $\mathrm{tp}(\bar{a}/A)$, where A is H-independent. Note that $HB(\bar{a}/A) \in \mathrm{acl}_H(\bar{a}, A)$ and $HB(\bar{a}/A) = HB(\bar{a}/\mathrm{acl}(A))$.

Fact 4.1. (1) *$HB(\bar{a}\bar{b}/A) = HB(\bar{a}/A)HB(\bar{b}/A\bar{a}HB(\bar{a}/A))$ for any H-independent set A.*

(2) *If $A \subseteq B$ are H-independent, then $HB(\bar{a}/A) \subseteq HB(\bar{a}/B)H(B)$.*

Proof. (1): Put $\bar{h}_1 := HB(\bar{a}/A), \bar{h}_2 = HB(\bar{b}/\bar{a}A\bar{h}_1)$ and $\bar{h} = HB(\bar{a}\bar{b}/A)$.
As $\bar{a} \downarrow_{A\bar{h}_1} H(\mathcal{M})$, $\bar{a}A\bar{h}_1$ is H-independent, so we can consider $\bar{h}_2 = HB(\bar{b}/\bar{a}A\bar{h}_1)$.
Claim 1: $\bar{h} \subseteq \bar{h}_1\bar{h}_2$.
As $\bar{a} \downarrow_{A\bar{h}_1} H(\mathcal{M})$ and $\bar{b} \downarrow_{\bar{a}A\bar{h}_1\bar{h}_2} H(\mathcal{M})$, we have $\bar{a}\bar{b} \downarrow_{A\bar{h}_1\bar{h}_2} H(\mathcal{M})$ as desired.
Claim 2: $\bar{h}_1\bar{h}_2 \subseteq \bar{h}$.
As $\bar{a}\bar{b} \downarrow_{A\bar{h}} H(\mathcal{M})$, we have $\bar{a} \downarrow_{A\bar{h}} H(\mathcal{M})$, so $\bar{h}_1 \subseteq \bar{h}$ follows. On the other hand,

we have $\bar{b} \downarrow_{\bar{a}A\bar{h}} H(\mathcal{M})$, so $\bar{b} \downarrow_{\bar{a}A\bar{h}_1\bar{h}} H(\mathcal{M})$, $\bar{h}_2 \subseteq \bar{h}$ follows.
(2): Put $\bar{h} = HB(\bar{a}/B)$. As $\bar{a} \downarrow_{B\bar{h}} H(\mathcal{M})$, we have $\bar{a}B \downarrow_{B\bar{h}} H(\mathcal{M})$. As B is H-independent, $B\bar{h} \downarrow_{H(B)\bar{h}} H(\mathcal{M})$. So we get $\bar{a} \downarrow_{H(B)\bar{h}A} H(\mathcal{M})$. So $HB(\bar{a}/A) \subseteq H(B)\bar{h} = H(B)HB(\bar{a}/B)$. □

Question 4.2. *If $A \subseteq B$ are H-independent, then $HB(\bar{a}/B) \subseteq HB(\bar{a}/A)$?*

Fact 4.3. *Let $(\mathcal{M}, H(\mathcal{M}))$ be an H-structure.*

(1) *Suppose that A is H-independent, then* $\mathrm{acl}(A) = \mathrm{acl}_H(A)$.
(2) $\mathrm{acl}(AHB(A)) = \mathrm{acl}_H(A)$ *for any $A \subset \mathcal{M}$.*

Proof. (1): $\mathrm{acl}(A) \subseteq \mathrm{acl}_H(A)$ is clear.
We show that $a \notin \mathrm{acl}(A)$ implies $a \notin \mathrm{acl}_H(A)$.
The case that $a \notin acl(AH(\mathcal{M}))$: As $a \downarrow_A H(\mathcal{M})$ and A is H-independent, we see that Aa is H-independent. By extension property take $(a_i : i < \omega)$ realizations of $\mathrm{tp}(a/A)$ which are *acl*-independent over $AH(\mathcal{M})$. As A is H-independent, we see that Aa_i is H-independent. By Fact 3.2 (2), we see that $\mathrm{tp}_H(a_i/A) = \mathrm{tp}_H(a/A)$ as desired.
The case that $a \in acl(AH(\mathcal{M}))$: Take $\bar{b} \in H(\mathcal{M})$ such that $a \in \mathrm{acl}(A\bar{b})$. By coheir property take $(\bar{b}_i : i < \omega)$ acl-independent realizations in $H(\mathcal{M})$ of $\mathrm{tp}(\bar{b}/A)$. Let a_i be such that $a_i\bar{b}_i \models \mathrm{tp}(a\bar{b}/A)$. As Ab_i is H-independent and $Aa_i\bar{b}_i \subseteq \mathrm{acl}(A\bar{b}_i)$, $Aa_i\bar{b}_i$ is H-independent. By Fact 3.2 (2), $\mathrm{tp}_H(a_i\bar{b}_i/A) = \mathrm{tp}_H(a\bar{b}/A)$ follows, as desierd.
(2): $\mathrm{acl}_H(A) \subseteq \mathrm{acl}_H(AHB(A))$ is clear. By (1) and H-independence of $AHB(A)$, we have $\mathrm{acl}_H(AHB(A)) = \mathrm{acl}(AHB(A))$. As we have $HB(A) \in \mathrm{acl}_H(A)$, we see $\mathrm{acl}(AHB(A)) \subseteq \mathrm{acl}_H(A)$. □

Proposition 4.4. *Let $b \in \mathrm{acl}(h_1, \cdots, h_nA)$ where A is H-independent, $\bar{h} = h_1, \cdots, h_n \subset H(\mathcal{M})$ and n is minimal. Then* $\mathrm{acl}_H(bA) = \mathrm{acl}_H(\bar{h}A)$.

Proof. We have $\mathrm{acl}_H(bA) = \mathrm{acl}(bAHB(bA))$ by 4.3 (2). By 4.1 (1), $HB(Ab) = HB(A)HB(b/AHB(A))$ follows. Since A is H-independent, we see $\mathrm{acl}(HB(A)) = \mathrm{acl}(H(A))$. On the other hand we have $b \downarrow_{A\bar{h}} H(\mathcal{M})$. By minimality of $\bar{h}$, $\bar{h} = HB(b/A)$ follows. Thus $\mathrm{acl}(HB(Ab)) = \mathrm{acl}(H(A)HB(b/\mathrm{acl}(A))) = \mathrm{acl}(H(A)\bar{h})$. So we have $\mathrm{acl}_H(bA) = \mathrm{acl}(bA\bar{h}) = \mathrm{acl}(A\bar{h})$. As $A\bar{h}$ is H-indepedent, $\mathrm{acl}(A\bar{h}) = \mathrm{acl}_H(A\bar{h})$ follows. Therefore we have $\mathrm{acl}_H(bA) = \mathrm{acl}_H(\bar{h}A)$. □

5. Rank in $(\mathcal{M}, H(\mathcal{M}))$, where $\mathcal{M}$ is strongly minimal

Let X_H be an L_H-definable set over A. We say that X_H is small if $X \subset \mathrm{acl}(AH(\mathcal{M}))$. Otherwise, we say that X_H is large.
Let X be an L-definable set over A. By codensity of H, X is infinite iff X is large. Clearly $H(x)$ is small.

Fact 5.1. *In a sufficiently saturated H-structure $(\mathcal{M}, H(\mathcal{M}))$, let X_H be an L_H-definable set in $\mathcal{M}$. Then there exists an L-definable set X in $\mathcal{M}$ such that $X_H \triangle X$ is small.*

Recall that $X \triangle Z \subseteq (X \triangle Y) \cup (Y \triangle Z)$.

The case that the base theory is strongly minimal
(1) large L_H-type is unique: If X_H, Y_H be large L_H-definable, then $X_H \triangle Y_H$ is

small. (Because there exist infinite L-definable sets X, Y in $\mathcal{M}$ such that $X_H \triangle X, Y_H \triangle Y$ are small and $X \triangle Y$ is finite.) This argument holds if there exists the *unique* non-algebraic 1-type in geometric L-structures.
(2) $H(x)$ has the unique non-algebraic type over H-independent set A: Suppose that $b, b' \in H(\mathcal{M})$ such that $\mathrm{tp}_H(b/A), \mathrm{tp}_H(b'/A)$ are non-algebraic. Clearly $b, b' \notin \mathrm{acl}(A)$. As $\mathrm{tp}(AbH(Ab)) = \mathrm{tp}(Ab'H(Ab'))$ (by strong minimality) and Ab, Ab' are H-independent, we have $\mathrm{tp}_H(Ab) = \mathrm{tp}_H(Ab')$.
(3) It is shown that $(\mathcal{M}, H(\mathcal{M}))$ is ω-stable by counting of types over countable sets. By (2) $RM_H(H(x)) = 1$.

The case that the base theory is strongly minimal and trivial
(1) Suppose that $b \in \mathrm{acl}(AH(\mathcal{M}))$. Then by triviality either $b \in \mathrm{acl}(A)$ or $b \in \mathrm{acl}(h) \setminus \mathrm{acl}(A)$ for some $h \in H(\mathcal{M})$. So we see that $RM_H(b/A) = RM_H(h/A) = RM_H(h) = 1$. So RM_H(small type)≤ 1. As any large type has the unique large extension, RM_H(large type)≤ 2.
(2) If $\mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ is finite for all non-algebraic $a \in \mathcal{M}$, $RM_H(x = x) = 1$.
(3) If $\mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$ is infinite for all non-algebraic $a \in \mathcal{M}$, $RM_H(x = x) = 2$.

The case that the base theory is strongly minimal and non-trivial
(1) Suppose that $b \in \mathrm{acl}(AH(\mathcal{M}))$, where A is H-independent. Take $h \subset H(\mathcal{M})$ be minimal length such that $b \in \mathrm{acl}(A\bar{h})$. Then $\mathrm{acl}_H(Ab) = \mathrm{acl}_H(A\bar{h})$. So $RM_H(b/A) = |\bar{h}|$.
(2) Let $a_{\leq n}$ be an algebraic n-gon over H-independent set A. By density, we may assume that $a_{\leq n-1} \subseteq H(\mathcal{M})$. Then $a_n \notin H(\mathcal{M})$ follows. Then we see that $HB(a_n/A) = a_{\leq n-1}$ and $\mathrm{acl}_H(Aa_n) = \mathrm{acl}(Aa_{\leq n-1})$ by Proposition 4.4. So we see that $\mathrm{RM}_H(a_n/A) = n - 1$.
(3) Suppose that $b \notin \mathrm{acl}(AH(\mathcal{M}))$. If $b \in \mathrm{acl}(BH(\mathcal{M}))$, then $RM_H(b/B) < \omega$. Large extension of $\mathrm{tp}(b/A)$ over B is unique, we see that $RM_H(b/A) = \omega$.

6. Ampleness and Triviality

$\bar{a}_{\leq i}$ denotes $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_i$, $\bar{a}_{<i}$ denotes $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{i-1}$ and $\bar{a}_{<0} = \emptyset$.
We say that $(\mathcal{M}, \mathop{\smile\!\!\!\!\!\!|}\,)$ is n-ample over $\bar{c}$ if there exist $\bar{c}, \bar{a}_{\leq n}$ such that for any $0 \leq i < n$

(1) $\bar{a}_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{\bar{c}\bar{a}_i} \bar{a}_{<i}$
(2) $\mathrm{acl}^{\mathrm{eq}}(\bar{c}\bar{a}_{<i}\bar{a}_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{c}\bar{a}_{<i}\bar{a}_i) = \mathrm{acl}^{\mathrm{eq}}(\bar{c}\bar{a}_{<i})$
(3) $\bar{a}_n \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{c}} \bar{a}_0$

If $(\mathcal{M}, \mathop{\smile\!\!\!\!\!\!|}\,)$ has weak canonical bases and $\bar{a}_{\leq n+1} = \bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{n+1}$ is $(n+1)$-ample over $\bar{c}$, then $\bar{a}_1, \cdots, \bar{a}_{n+1}$ is n-ample over $\bar{c}\bar{a}_0$, so $(n+1)$-ampleness implies n-ampleness.

n-dimesional free pseudospace ((n+1)-many sorts and n-many incident relations) is n-ample but not $(n+1)$-ample. we have one-basedness⇔non-1-ampleness and CM-triviality⇔non-2-ampleness.

For any sequence $(\bar{a}_i : i \in I)$, we write $\bar{a}_{\neq i} := (\bar{a}_i : j \in I \setminus \{i\})$.
We say that $(\bar{a}_i : i \in I)$ is independent over A if $\bar{a}_i \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}_{\neq i}$.
We say that $(\bar{a}_i : i \in I)$ is pairwise independent over A if $\bar{a}_i \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}_j$ for any $i \neq j \in I$.

We say that T is *trivial* if any pairwise independent sequence in $\mathcal{M}^{\mathrm{eq}}$ is independent.

We say that $(\mathcal{M}, \mathrm{acl}(*))$ is trivial if $\mathrm{acl}(A) = \bigcup_{a\in A} \mathrm{acl}(\{a\})$ for any $A \subset \mathcal{M}$.

If $(\mathcal{M}, \mathrm{acl}(*))$ is not trivial, there exists $a, b, c, B \subset \mathcal{M}$ such that $a \in \mathrm{acl}(bcB) \setminus (\mathrm{acl}(bB) \cup \mathrm{acl}(cB))$. (Take a minimal size $A =: Bc$ such that $\mathrm{acl}(A) = \bigcup_{a\in A} \mathrm{acl}(\{a\})$ and if $b \notin \mathrm{acl}(A)$, then $\emptyset \neq \mathrm{acl}(Ab) \setminus (\mathrm{acl}(A) \cup \mathrm{acl}(b))$. Take $a \in \mathrm{acl}(Bcb) \setminus (\mathrm{acl}(Bc) \cup \mathrm{acl}(b))$. As $|A| = |Bb|$, we have $\mathrm{acl}(Bb) = \bigcup_{d\in Bb} \mathrm{acl}(\{d\})$. So $a \notin \mathrm{acl}(Bb)$, as desired.)
The triviality of $(\mathcal{M}, \mathrm{acl}(*))$ coincides with the triviality of $(\mathcal{M}, \downarrow)$.

7. A lovely pair and ampleness

Let $L_P = L \cup \{P(x)\}$, where $P(x)$ is a new unary predicate. Let $\kappa \geq |T|^+$
We say that $(M, P(M))$ is a κ-lovely pair if

1. $P(M) \prec M \models T$
2. Coheir property: For any $A \subset M$ with $|A| < \kappa$ and any fininary L-type $p(\bar{x})$ over A with $\bar{a} \downarrow_{P(A)} A$ and $\bar{a} \models p(\bar{x})$, then $p(\bar{x})$ is realized in $P(M)$
3. Extension property: For any $A \subset M$ with $|A| < \kappa$ and any fininary L-type $p(\bar{x})$ over A there exists $\bar{a} \models p(\bar{x})$ such that $\bar{a} \downarrow_A AP(M)$.

If $(\mathcal{M}, \downarrow)$ has independence calculus, then a κ-lovely pair exists.
κ-lovely pairs are elementarily equivalent, T_P denotes the theory of lovely pair, where $T = \mathrm{Th}(\mathcal{M})$. Any $|T|^+$-saturated model of T_P is a lovely pair.
If T is simple, T_P is axiomatizable iff T is low and has weak non-finite cover property.
If T is simple and T_P is axiomatizable, then T_P is also simple. [B-YPiV].

Fact 7.1. (1) *If T is simple, one-besed and T_P is axiomatizable, then T_P is simple and one-based.* [B-YPiV].

(2) *If T is stable, trivial and does not have finite cover property, then T is n-ample iff T_P is n-ample for any $n \geq 1$ assuming a nice characterization of forking in T_P: In general for $A, B, C \subset \mathcal{M}$, we have*

$$A \overset{P}{\underset{C}{\downarrow}} B \Leftrightarrow AP \underset{CP}{\downarrow} BP, \mathrm{Cb}(AC/P) \underset{\mathrm{Cb}(C/P)}{\downarrow} \mathrm{Cb}(BC/P).$$

The nice characterization of forking in T_P is as follows: For any P-independent subsets A, B, C,

$$A \overset{P}{\underset{C}{\downarrow}} B \Leftrightarrow AP \underset{CP}{\downarrow} BP, AC \underset{C}{\downarrow} BC.$$

[CM-PP].

(3) *If T has* SU*-rank one with QE, then T is one-based iff* $\mathrm{acl} = \mathrm{acl}_P$ *in T_P iff T_P has* SU*-rank≤ 2 iff T_P is model-complete iff* $(\mathcal{M}, \mathrm{acl})$ *is modular over $P(\mathcal{M})$.* [V1].

(4) *If T is geometric(=having Steinitz exchange property for* acl *and elimination of $\exists^\infty$), then T is weakly locally modular iff* $(\mathcal{M}, \mathrm{acl})$ *is modular over $P(\mathcal{M})$ iff* $\mathrm{acl} = \mathrm{acl}_P$ *in T_P.* [BV1].

(5) *If T eliminates $\exists^\infty$ and $U^{\mathfrak{p}}(T)=1$, then $U^{\mathfrak{p}}_P(T_P)\le\omega$.* [B].
Moreover if T is weakly local modular, then $U^{\mathfrak{p}}_P(T_P)\le 2$. [BV1].

Question 7.2. *If T eliminates $\exists^\infty$, $U^{\mathfrak{p}}(T)=1$ and T is weakly local modular, then is T_P weakly locally modular? There is a proof containing serious gaps in* [BB-WKAP]. *(We do not know whether $A\downarrow^{\mathfrak{p}}_B C$ and $AP\downarrow^{\mathfrak{p}}_{BP} CP$ imply $A\downarrow^{P,\mathfrak{p}}_B C$ or not. And we can not find the reason for $\bar a\downarrow^{\mathfrak{p}}_{\bar a_2\bar a'} C$ in the proof of Theorem 4.2 in* [BB-WKAP].*)*

8. A dense independent subset in supersimple structures and ampleness

This section is almost due to [BV4]. We assume that $(\mathcal{M},\downarrow)$ is supersimple and Let $T=\mathrm{Th}(\mathcal{M})$. Fix a partial unary type $\Pi(x)$ over $\emptyset$ and let $H(x)$ be a new predicate.

We say that $(M,H(M))$ is H-structure associated to Π if

(1) If $h\in H(M)$, then $\models\Pi(h)$.
(2) $h_1,\cdots,h_n\in H(M)$ are distinct, then $\{h_1,\cdots,h_n\}$ is $\downarrow$-independent.
(3) Density: Let $\bar b\subset M$ and let $\Pi(x)\subset p(x)$ be a complete type over $\bar b$ such that if $a\models p(x)$ then $a\downarrow\bar b$. Then there exists $h\in H(M)$ such that $h\models p(x)$.
(4) Codensity: Let $\bar b\subset M$ and let $p(x)$ be a complete type over $\bar b$. Then there exists $a\in M$ such that $a\models p(x)$ and $a\downarrow_{\bar b}H(M)$.

An H-structure associated to $\Pi(x)$ exists. H-structures associated to $\Pi(x)$ are elementarily equivalent, T^{ind}_Π denotes the common theory of H-structure associated to $\Pi(x)$. T^{ind}_Π is axiomatizable if T has two conditions (1) (2):

(1) For each formula $\varphi(x,\bar y)$ there exists a formula $\psi(\bar y)$such that (there exists a such that $a\models\varphi(x,\bar b)$ and $a\downarrow\bar b$) iff $\bar b\models\psi(\bar y)$.
(2) Let $\varphi(\bar x,\bar y)$ and $\psi(\bar y,\bar z)$ be formulas. The following condition on $\bar c$ is type-definable: For any $\bar b\models\psi(\bar y,\bar c)$ there exists $\bar a\models\varphi(\bar x,\bar b)$ we have $\bar a\downarrow_{\bar c}\bar b$.

Any $|T|^+$-saturated model of T^{ind}_Π is a H-structure associated to $\Pi(x)$.
If T is supersimple and T^{ind}_Π is axiomatizable, then T^{ind}_Π is supersimple.

Fact 8.1. (1) [BV4] *Let $(\mathcal{M},H(\mathcal{M}))$ be an H-structure associated to $\Pi(x)$, where $\mathcal{M}$ is supersimple. Then*
$\bar a\downarrow^H_A B\Leftrightarrow \bar a\downarrow_{AH(\mathcal{M})} BH(\mathcal{M}), HB(\bar a/A)=HB(\bar a/B)$.

(2) [BV3] *Let $(\mathcal{M},H(\mathcal{M}))$ be an H-structure, where* $\mathrm{SU}(\mathcal{M})=1$. *Suppose that $A=\mathrm{acl}_H(A)$.*
(a) *$a\in A$ iff* $\mathrm{SU}_H(a/A)=0$

(b) *$a\in\mathrm{acl}(AH(\mathcal{M}))\setminus A$ iff* $\mathrm{acl}_H(aA)=\mathrm{acl}(AHB(a/A))$ *iff* $\mathrm{SU}_H(a/A)=|HB(a/A)|$.

(c) *If a is trivial over A, then $a\in\mathrm{acl}(AH(\mathcal{M}))\setminus A$ then* $\mathrm{SU}_H(a/A)=1$.

(d) *If a is trivial over A and $a\notin\mathrm{acl}(AH(\mathcal{M}))$ then* $\mathrm{SU}_H(a/A)=1$.

(e) *If a is non-trivial over A, then* $a \notin \mathrm{acl}(AH(\mathcal{M}))$ *iff* $\mathrm{SU}_H(a/A) = \omega$.

Proof. (c) : By triviality there exists $h \in H(\mathcal{M})$ such that $HB(a/A) = h$. The conclusion follows from (b).
(d) : If $a \not\perp^H_A B$ with $B = \mathrm{acl}_H(B)$, by $HB(a/B) \subseteq HB(a/A) = \emptyset$, we have $a \not\perp_{AH(\mathcal{M})} BH(\mathcal{M})$. So $a \in \mathrm{acl}(BH(\mathcal{M}))$. By triviality and $a \notin \mathrm{acl}(AH(\mathcal{M}))$ we have $a \in \mathrm{acl}(B) \setminus \mathrm{acl}(AH(\mathcal{M}))$. So $\mathrm{SU}_H(a/B) = 0$ follows by (a). Therefore we have $\mathrm{SU}_H(a/A) = 1$. □

Fact 8.2. (1) [BV3] *There exists a non-trivial one-based strongly minimal theory T whose T^{ind} is not one-based. $(V, +, 0, H)$ be a vector space over $\mathbb{Q}$, where $H(V) = \{v_i \in V : i < \omega\}$. Then $(V, +, 0, H) \models T^{ind}$. Let $u \in V \setminus H(V)$. Put $t = u + v_1, t' = u + v_2$. Then $HB(tt') = v_1v_2$ because $\mathrm{acl}_H(tt') = \mathrm{acl}(v_1v_2t')$ as $t - t' = v_1 - v_2$ and $tt' \perp_{v_1v_2} H(V)$ and $tt' \not\perp_{v_i} H(V)$. We have $t \perp^H_u ut'$ as $v_1 \perp^H_u v_2$. If T^{ind} was one-based and $\mathrm{acl}_H(t) \cap \mathrm{acl}_H(ut') = \mathrm{acl}_H(\emptyset)$, then $t \perp_\emptyset ut'$. On the other hand $\mathrm{RM}_H(t/ut') = \mathrm{RM}_H(v_1/ut') \le 1$ and $\mathrm{RM}_H(t/t') = \mathrm{RM}_H(v_1v_2/t') = 2$. So $t \not\perp^H_{t'} ut'$, a contradiction.*

(2) *If T is one-based and $T^{ind}_{x=x}$ is axiomatizable, then T is trivial iff $T^{ind}_{x=x}$ is one-based.*

(3) *For any partial type $\Pi(x)$ over $\emptyset$, if $T^{ind}_{\Pi(x)}$ is axiomatizable, T is n-ample iff $T^{ind}_{\Pi(x)}$ is n-ample for any $n \ge 2$.*

Let $\mathrm{SU}(T) = \omega^\alpha$. Let $\mathrm{cl}(A) := \{x \in \mathcal{M} : \mathrm{SU}(x/A) < \omega^\alpha\}$. We say that $\mathrm{cl}(*)$ is trivial if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for any $A \subset \mathcal{M}$.

Fact 8.3. *Suppose that* SU*-rank is continuous. Let $\Pi(x)$ be the union of all the types over $\emptyset$ of $\mathrm{SU} = \omega^\alpha$. Assume that T^{ind}_Π is axiomatizable and T is one-based. Then $\mathrm{cl}(*)$ on $\mathcal{M}$ is trivial iff T^{ind}_Π is one-based.*

9. Triviality and T^{ind}

This section is due to [BV3].

Fact 9.1. (1) *If T is strongly minimal and trivial, then $\mathrm{RM}_H(T^{ind}) \le 2$.*

(2) *If T is strongly minimal and non-trivial, then $\mathrm{RM}_H(T^{ind}) = \omega$.*

(3) *If $\mathrm{SU}(T) = 1$ and trivial, then $\mathrm{SU}_H(T^{ind}) = 1$.*

(4) *If $\mathrm{SU}(T) = 1$ and non-trivial, then $\mathrm{SU}_H(T^{ind}) = \omega$.*

(5) *If $\mathrm{U}^{\mathfrak{p}}(T) = 1$ and trivial, then $\mathrm{U}^{\mathfrak{p}}_H(T^{ind}) = 1$.*

(6) *If $\mathrm{U}^{\mathfrak{p}}(T) = 1$ and non-trivial, then $\mathrm{U}^{\mathfrak{p}}_H(T^{ind}) = \omega$.*

Acknowledgement: This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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