Harmonic transplantation and its applications to Sobolev embeddings, functional inequalities and PDEs

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1 Introduction

Let $\Omega \subset \mathbf{R}^N$ be a domain and $0 \in \Omega$. If the exponent $p > 1$ and the dimension $N \geq 2$ satisfy $p < N$, then the Hardy inequality (1) and the Sobolev inequality (2) hold for any $u \in \dot{W}_0^{1,p}(\Omega)$, where $\dot{W}_0^{1,p}(\Omega)$ is a completion of $C_c^\infty(\Omega)$ with respect to $\|\nabla \cdot \|_{L^p(\Omega)}$.

$$\left(\frac{N-p}{p} \right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \int_\Omega |\nabla u(x)|^p \, dx, \tag{1}$$

$$S_{N,p} \left(\int_\Omega |u(x)|^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq \int_\Omega |\nabla u(x)|^p \, dx, \tag{2}$$

$$\text{where } p^* = \frac{Np}{N-p}, \; S_{N,p} = \pi^{\frac{p}{2}} N \left(\frac{N-p}{p-1} \right)^{p-1} \left(\frac{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}{\Gamma(N)\Gamma(1+\frac{N}{2})} \right)^{\frac{p}{N}}.$$

These two inequalities are fundamental and important. Also, these two inequalities appear in analyzing existence, non-existence and stability of solution to nonlinear partial differential equations and so on. Their best constants and their attainability are well-studied. The Sobolev inequality (2) denotes the embedding : $\dot{W}_0^{1,p} \hookrightarrow L^{p^*}$, and the Hardy inequality (1) denotes the embedding : $\dot{W}_0^{1,p} \hookrightarrow L^{p^*,p} (\subsetneq L^{p^*})$.

What about the critical case where $p = N$? Although $p^* \nearrow \infty$ as $p \nearrow N$, the embedding : $\dot{W}_0^{1,N} \hookrightarrow L^\infty$ does not hold. Furthermore, these two inequalities look degenerate, because their best constants $(\frac{N-p}{p})^p, S_{N,p}$ go to zero as $p \nearrow N$. In the next section, we give some relation between the subcritical ($p < N$) and the critical ($p = N$) Sobolev spaces via harmonic transplantation.

2 Harmonic transplantation and its applications

The harmonic transplantaion is proposed by Hersch [1]. It is a generalization of the conformal transplantation and is a powerful tool for the construction of comparison functions or approximate solutions of variational problems. Here, we introduce a result in [6] as an

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application of harmonic transplantation, which is the equivalence between two norms of the subcritical Sobolev space $\dot{W}_0^{1,p}(\mathbf{R}^m)$ and the critical Sobolev space $\dot{W}_0^{1,N}(B^N)$ for radial functions. For other kinds of applications of the harmonic transplantation, see [4] §3.3.. Let $G_{\Omega,\mathrm{O}}$ be p-Green's functions on Ω with the pole O. The following transformation and the equality are given for radial functions u, v in [6]:

$$u(|x|) = v(|y|), \text{ where } G_{\mathbf{R}^m,\mathrm{O}}(|x|) = G_{B^N,\mathrm{O}}(|y|), \; p = N < m,$$
$$\int_{\mathbf{R}^m} |\nabla u(x)|^p \, dx = \int_B |\nabla v(y)|^N \, dy$$

This equality helps us to investigate embeddings, functional inequalities and PDEs in the critical case where $p = N$. However, the harmonic transplantation is available only for radial functions. Therefore, we need further analysis of critical problems for non-radial functions. Recently, we have obtained some results about the embedding, the functional inequality and the PDE in the critical case where $p = N$. For the details, see [2, 3, 5].

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References

[1] Hersch, J., *Transplantation harmonique, transplantation par modules, et théorèmes isopérimétriques. (French. English summary)*, Comment. Math. Helv. 44 (1969), 354-366.

[2] Machihara, S., Sano, M., *On the compactness of the (non)radial Sobolev spaces*, arXiv:2206.13775.

[3] Sano, M., *Extremal functions of generalized critical Hardy inequalities*, J. Differential Equations 267 (2019), no. 4, 2594-2615.

[4] Sano, M., Takahashi, F., *Critical Hardy inequality on the half-space via the harmonic transplantation*, Calc. Var. Partial Differential Equations 61 (2022), no. 4, Paper No. 158.

[5] Sano, M., Takahashi, F., *On eigenvalue problems involving the critical Hardy potential and Sobolev type inequalities with logarithmic weights in two dimensions*, in preparation.

[6] Sano, M., Takahashi, F., *Scale invariance structures of the critical and the subcritical Hardy inequalities and their improvements*, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Paper No. 69, 14 pp.