European University Institute

DEPARTMENT OF ECONOMICS

Essays on the Economics of Social Interactions

Julie Pinole

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

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European University Institute
Department of Economics

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Examining Board

Prof. David K. Levine, EUI, Supervisor
Prof. Andrea Mattozzi, EUI
Prof. Yann Bramoullé, CNRS, Aix-Marseille School of Economics, External Co-Supervisor
Prof. Dino Gerardi, Collegio Carlo Alberto

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Monday 17th of June, 2019

Abstract

This dissertation consists of three self-contained essays on the economics of social interactions.

The first chapter is coauthored with Lorenzo Verstraeten. Knowing that Individuals interact with their peers, we study how a social planner can intervene, changing these interactions, in order to achieve a particular objective. When the objective is welfare maximization, we describe the interventions for games of strategic complements and strategic substitutes. We show that, for strategic complements, the planner uses resources to target central players; while she divides individuals into separated communities in the case of strategic substitutes. We study which connections she targets in order to achieve these goals.

The second chapter is coauthored with Lorenzo Verstraeten and analyzes a model of contagion on social network. We ask how a social planner should intervene to prevent contagion. We characterize the optimal intervention and the cost associated. We discuss the intuition behind the choice of the planner and we provide comparative static on the cost of intervention for different type of network.

In the third chapter I develop a theoretical study about groups relationship and ask whether intragroup cooperation crowd-out intergroup cooperation. I consider a gift-giving game where cooperation endogenously arises, within and across groups. Cooperation is sustained through peer punishment with the help of a group specific monitoring technology. I specify under which conditions cooperation crowding-out occur. I identify two classes of equilibrium: a Sorting equilibrium where guilty players prefer to be matched outside their group due to a less efficient Out-Group monitoring technology, and a Non Sorting equilibrium where the higher level of In-Group cooperation makes it more attractive for everybody. I then compare their welfare properties and draw conclusions on optimal punishment levels.

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Contents

Chapter 1
Optimal intervention for network games

1 Introduction

We study a classical network game. Players' payoffs are a function of their own characteristics, their actions and the actions of their peers. We ask how a social planner, with limited resources, should intervene on the connections among players when she has in mind a particular objective. The planner might want to implement a given outcome, in that case we show how she can achieve this goal at minimal cost; or she might have some resources to spend, in that case we ask how she should allocate them in order to maximize aggregate welfare.

The type of interventions we have in mind focus on changing both the absolute sum of links weight as well as the relative weight of links. Examples of quantitative interventions could be to increase (or decrease) the number of extracurricular activities provided in a school or neighborhood context: adding book clubs, sport clubs, or any types of groups, both physical and virtual. The social planner can also play on the cost of those activities to incentivize or desincentivize participation. This will likely create or destroy links between individuals. As far as relative weight interventions are concerned, we can think of a regulator influencing the partnership structure of firms, by encouraging local links between firms of different size or different centrality in a national or global scale versus promoting networks of firms similar in terms of characteristics. For instance the regulator may have voice to the chapter when financial cross-participations of big firms are realized. When agents are individuals, the planner can devise policies promoting the integration of newly arrived immigrants. The principal of a school could organize study groups where she decides their composition. For example, Algan et al. (2015) ([3]) ran an experiment in a French university by randomly assigning students into first-year groups. This design allowed to measure both the actual change in the network structure and whether it affected the outcome of interest. [1]

We assume that the intervention of the planner has increasing marginal cost. The more she wants to change a connection between two individuals the larger is the marginal cost she has to pay. We first ask how she could achieve, at minimal cost, a specific outcome for the network game. Modifying the connection

[1] This paper validates our assumption that in some circumstances a social planner can actually affect a social network.

between two individuals changes their incentives. This affects the decisions of all their peers and therefore the equilibrium of the network game. When the planner has in mind a specific equilibrium to implement we describe how she should modify the network structure in order to obtain her goal at minimal cost. The first set of results in Section 4 describe the interventions the planner should take and the cost she will incur. Borrowing from the literature in computer science we manage to be very general in the results we obtain. We provide results for directed and undirected network. We also describe what happens when we do not allow the planner to change some connections.

Using the results in section 4 we ask how a social planner, with limited resources, can modify interactions among players in order to maximize aggregate welfare. Galeotti, Golub, Goyal (2018) ([22]), GGG from now on, address the same question. However, they focus on how a social planner should change players' incentives. More precisely the intervention targets individual private values. We allow the social planner to intervene on the network structure instead. Following GGG, we distinguish two kind of network games. The first category are games of strategic complements; in this case players are incentivized to engage in an activity if their peers do. The second category are games of strategic substitute; in this case players players' incentives to engage in an activity are smaller the higher is the involvement of their neighbors. As in GGG, we draw conclusions on how qualitatively different the intervention is depending on whether we play a game of strategic complements or strategic substitutes.

We first try to understand which players will be more affected by the intervention of the planner. As in GGG, we decompose the effect of the intervention on a particular system of coordinates. The orthonormal basis obtained by diagonalizing the matrix representing the interactions between individuals. We show that the equilibrium of the network game can be measured in terms of the singular vectors of the adjacency matrix of the initial network This is interesting because it allows us to understand how the planner change the players' incentives. We show that our intervention shares common features with the characteristics-intervention problem of GGG: in game of strategic complements central players will be mostly affected by the intervention; in game of strategic substitutes, instead, the incentives of neighbors are moved in opposite directions.

In section 7 we try to analyze how the social planner affects the network structure. In the complement case, we show that the decision of the planners depends on two aspects.It is important whether players are central or not, and whether they have a high private marginal benefit for the action or not. If central players tend to have high private value, then the effort of the planner is unambiguously directed at them. The planner will sponsor links from and towards these players. Otherwise the result would depend on which aspect is the most important. The substitute case is more delicate. The planner will try to eliminate links in order to form bipartite network. [2]She will try to form two group in the population. Individuals inside one group share few link across them, while most of the links are across individuals of the two different groups. She achieves this by destroying links of low intensity.

The paper is organized as follows. Section 2 compares our with the related literature. Section 3 describes the model. In section 4, starting from a given network structure, we depict the closest network enabling to implement a given outcome. Section 5 exposes the planner's problem and give properties of the equilibrium profile played on the new network. Section 6 provides a comparison between those properties and the results of GGG. Section 7 provides an analysis of the changes that the network structure goes through. Section 8 concludes.

2 Literature Review

We study a model of game on networks that covers many important situations. They fall into two main categories, as described by Bramoullé and Kranton (2016) ([14]): peer effects and local public goods [3]. Examples of outcomes where peer effects play a role range from smoking (Robalino and Macy, 2018, [41]), obesity (Trogdon et a., 2008, [44]), school achievement (Boucher et al., 2014, [8]), delinquent behaviors (Glaeser et al., 1996, [24], to retirement savings (Saez and Duflo, 2003, [19]). [4] In all those situations, I am more likely to engage in an activity if my peers do. Those games are called games of strategic complementarities. On the contrary local public goods games exhibit strategic substituability. I am less likely to contribute to a non-excludable good if my peers do and I can benefit from their contributions at zero cost. Bramoullé and Kranton, 2007 ([13]) depicts various interactions of this type. For instance information and innovation are often non-excludable. If my friends engage in information acquisition on a new consumption good, I may take advantage of it. Research and development expenses in enterprise generate innovations that also profit connected partners. Another dimension of technological spillovers is

[2]In graph theory, a bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V.

[3]we abstract from the third topic they address, technology adoption, as adapting our model to discrete decisions is left for future work)

[4]Bramoullé et al., 2009, [11], and Boucher and Fortin, 2016, [9] provide interesting studies of peer effects with a focus on the associated econometrics challenges.

geography, as evidenced by Bloom et al., 2013, ([6]). A last example is what the literature refers to as crime games. Ballester et al. , 2010 ([4]) quote the criminology literature to support their assumption that criminal skills are mostly learnt through peers, and thus there is spillover of crime activities from one individual to his connections. In all those examples, the exact structure of the network matters and the aggregate outcome is relevant to policy makers.

Our work represents a new application of the computer science literature on nearest matrices. Higham, 2000 ([27]) proposes answers to different mathematical problems searching for a matrix with specific properties that is as close as possible to an initial matrix deprived of this property, where different closeness metrics are possible. We provide a specific application of the nearest matrix problem by defining the network structure through its adjacency matrix and interpreting the nearest matrix solving an adequate optimization problem as the adjacency matrix of the desired network structure.

Finally, our work contributes to the literature on optimal strategy in the presence of social interactions. Zenou, 2016 ([12]) provides, among other things, a review of the literature on network intervention in games. Among the economics literature we quote other recent works: Fainmesser and Galeotti, 2016 ([21]), Akbarpour, Malladi, and Saberi, 2017 ([2]), Banerjee, Chandrashekhar, Duflo, and Jackson, 2016 ([5]), Candogan, Bimpikis, and Ozdaglar, 2012 ([16]). Other disciplines investigates the topic. In marketing and computer science, the problem is often whom to target: Borgatti, 2006 ([7]), Kempe, Kleinberg, and Tardos, 2003 ([30]).

3 The Model

3.1 The setup

We study a game where n players are located on a directed network described by the weighted adjacency matrix $G \in \mathcal{M}_{n,n}$. [5] The set of player is called $\mathcal{N} = 1, \ldots, n$. The element g_{ij} of G represents how strong the connection between players i and j is. We impose $g_{ij} \geq 0$ for all i, j.

Each player i chooses an action a_i from $\mathbb{R}_+$. We call $a \in \mathbb{R}_+^n$ the vector containing the action profile of all the players: $\boldsymbol{a} = (a_i)_{i \in \mathcal{N}}$.

The payoffs to individual i are given by:

$$W_i(\boldsymbol{a}) = b_i a_i - \frac{1}{2} a_i^2 + \beta a_i \sum_{j \in \mathcal{N}} g_{ij} a_j$$

where $b_i \in \mathbb{R}_+^*$ is an individual-specific characteristic measuring individual i's marginal return of the direct effect of his action. We call b the vector containing the characteristics of the n players. Each player incurs a quadratic cost. Finally, each player's payoffs are affected by the interaction between his own action and the action of his connections. If β is positive, actions are strategic complement, whereas if β is negative they are strategic substitutes.

For notation purposes we call A the following transformation of the network structure:

$$A \equiv I - \beta G \tag{3.1}$$

3.2 Equilibrium

We impose the following assumption:

Assumption 1. *The spectral radius of $\beta \boldsymbol{G}$ is less than* 1.

Under assumption (1), Bramoullé, Kranton and D'Amours (2014) [15] shows that there exists a unique equilibrium $\boldsymbol{a}^*$ for the network game described above. Furthermore the equilibrium satisfies the following system of linear equations:

$$[I - \beta G]\, \boldsymbol{a}^* = b \tag{3.2}$$

[5] For some of our results we allow g_{ii} to be different from 0, with the interpretation that it is a factor that influence the cost of player i when he chooses action a_i. In terms of the network structure this is equivalent to assume the presence of self-loops.

Assumption (1) ensures that $I - \beta G$ is invertible. Hence we can write

$$\boldsymbol{a}^* = [I - \beta G]^{-1} b$$

4 Closest network structure to implement a chosen vector of actions

We assume that the social planner can intervene to change the structure of the network. She could do this by changing the intensity of any link across players. We first ask how she should do it if there is a specific action profile $\bar{\boldsymbol{a}}$ that she wants to implement at minimal cost. We make the following assumption about the cost she incurs to alter the network:

Assumption 2. *Changing the structure of the network from G to G^* has a cost of $||G^* - G||_F$, where $||M||_F = \sqrt{\sum_{ij} m_{ij}^2}$ is the Frobenius norm of M.*

The previous assumption captures the idea that the social planner can modify the interaction between i and j at a convex cost. The planner faces increasing marginal costs of intervention. The Frobenius norm can be seen as an extension of the euclidean norm to $\mathcal{R}^{n \times n}$.

In order to determine how the social planner can reach her goal we rely on important results from the computer science's literature. [6]. Those results focus on finding the closest matrices, satisfying some properties, to a given one.

Definition 1. *We call $Q(y, x)$ the set of matrix quotients of y by x, with $y, x \in \mathbb{R}^n$:*

$$Q(y, x) = \{M \in \mathcal{A} | Mx = y\} \tag{4.1}$$

where $\mathcal{A} \subseteq \mathcal{M}_{n,n}$ is a set of matrices with some desirable properties.

Definition 2. *We call $\mu^{x,A}(y)$ the minimal cost of altering the matrix A so that the resulting matrix belongs to $Q(y, x)$:*

$$\mu^{x,A}(y) = \min_{E \in \mathcal{M}_{n,n}} \{||E||_F : (A + E) \in Q(x, x) \cap \mathcal{A}\} \tag{4.2}$$

where $\mathcal{A} \subseteq \mathcal{M}_{n,n}$ is a set of matrices with some desirable properties.

[6] see Higham (2000) [27].

Higham, 2000 ([27]) reviews results of the computer science literature that solves the minimization problem (4.2) for different constraints on the type of matrices to work with (constraints defined in the set $\mathcal{A}$). This is of interest for the problem at hand as the equilibrium condition (3.2) makes our problem equivalent to the minimization problem (4.2) with $A = I - \beta G$, $x = b$ and $y = \bar{\boldsymbol{a}}$.

We focus on the three types of constraints that we consider the most relevant: the unconstrained case, the case where the starting network is undirected and we wish to reach an undirected network as well, and the case where we want to preserve some sparseness properties of the network.

4.1 Case 1: Unconstrained intervention

The first type of intervention we consider is one were we don't impose any constraint to the planner. A given initial network is given and she can change any connection at a cost specified in Assumption 2. We will provide a closed form solution to the problem of the planner and specifies the cost that is associated to the intervention

For simplicity, we are now using the notation A defined in equation (3.1). Not imposing any condition on the intervention of the planner translates in the language of the previous definitions in $\mathcal{A} = \mathcal{M}_{n,n}$. The following proposition gives us the result:

Proposition 1. *The least costly intervention such that the action profile $\bar{\boldsymbol{a}}$ is played in equilibrium in the game played on the transformed network is:*

$$A + E_{\min}(\bar{\boldsymbol{a}}) \tag{4.3}$$

with

$$E_{\min}(\bar{\boldsymbol{a}}) = \frac{(\boldsymbol{b} - A\bar{\boldsymbol{a}})\bar{\boldsymbol{a}}^T}{\bar{\boldsymbol{a}}^T\bar{\boldsymbol{a}}} \tag{4.4}$$

and this closest matrix is reached at a cost of:

$$\mu^{b,A}(\bar{\boldsymbol{a}}) = \frac{||b - A\bar{\boldsymbol{a}}||}{||\bar{\boldsymbol{a}}||} \tag{4.5}$$

Note that $||.||$ denotes the euclidean norm of $\mathbb{R}^n$.

Proof. We first quote the following lemma, which is a result coming from the section 8 of Higham, 2000 ([27]):

Lemma 1. *Given $y, x \in \mathbb{R}^n$, $A \in \mathcal{M}_{n,n}$, the following minimization problem:*

$$\min_{E \in \mathcal{M}_{n,n}} \{||E||_F : (A + E) \in Q(x, x)\}$$

admits $E_{\min}$ as a solution with

$$E_{\min} = \frac{(y - Ax)x^T}{x^T x}$$

and reach $\mu^{x,A}(y)$ as minimum with

$$\mu^{x,A}(y) = \frac{||y - Ax||}{||x||}$$

This result tells us how to change a matrix in the sense of minimizing the Frobenius norm of the difference between the initial and the final matrix, subject to the matrix belonging to $Q(y, x)$.

The proof of Proposition (1) is a direct application of this lemma for $y, x = \bar{\boldsymbol{a}}, \boldsymbol{b}$.

The following corollary simply uses (3.1) to go from the transformed matrix of A to the transformed matrix of G:

Corollary 1. *The new network structure in the modified game is:*

$$G(\bar{\boldsymbol{a}}) = G - \frac{1}{\beta} E_{\min}(\bar{\boldsymbol{a}})$$

reached at a cost of $\frac{||b - (I - \beta G)\bar{\boldsymbol{a}}||}{||\bar{\boldsymbol{a}}||.}$

Remark 1. *Mathematically $E_{\min}$ is what the planner adds to A to reach the transformed matrix. That is why we call $E_{\min}$ the optimal intervention. The first thing to note is that the matrix $E_{\min}$ is of rank 1. Therefore we say that the planner's optimal intervention is a rank-1 intervention, that is of low computational complexity. This is true independently of the initial condition. The second thing to note is that the optimal intervention $E_{\min}$ is a priori non symmetric; this is the case even when the initial matrix is symmetric.*

With this type of intervention, we are not constraining the social planner in any way. This is something that might not be desirable in some situations. When we start from an undirected network, represented by a symmetric adjacency matrix, it may be questionable to reach a directed network (with an asymmetric adjacency matrix) as the outcome of the planner's optimal intervention. One desirable property that we might ask is the preservation of symmetry.

4.2 Case 2: Symmetric intervention

In this case $\mathcal{A} = \{M \in \mathcal{M}_{n,n}$ such that $M = M^T\}$. The following results is another direct application of Higham, 2000 ([27]):

Proposition 2. *The closest matrix - in the Frobenius norm - in $\mathcal{A} = \{M \in \mathcal{M}_{n,n} : M = M^T\}$ to A such that the action profile $\bar{\boldsymbol{a}}$ is played in the game played on the transformed network is:*

$$A + E_{\min}^{Sym}(\bar{\boldsymbol{a}}) \tag{4.6}$$

with

$$E_{\min}^{Sym}(\bar{\boldsymbol{a}}) = \frac{(b - A\bar{\boldsymbol{a}})\bar{\boldsymbol{a}}^T + \bar{\boldsymbol{a}}^T(b - A\bar{\boldsymbol{a}})^T}{\bar{\boldsymbol{a}}^T\bar{\boldsymbol{a}}} - (b - A\bar{\boldsymbol{a}})^T\bar{\boldsymbol{a}}\frac{\bar{\boldsymbol{a}}\bar{\boldsymbol{a}}^T}{\bar{\boldsymbol{a}}^T\bar{\boldsymbol{a}}} \tag{4.7}$$

Remark 2. [7] *-The intervention of the planner is a rank-2 matrix -It is possible to show that the cost of intervention μ^{Sym} is close to the cost of the intervention without constraint*

$$\mu <= \mu^{Sym} <= \sqrt{2}\mu$$

4.3 Case 3: Sparse intervention

In some situations the planner might not be able to modify some features of the network. For example she might not be able to create a link between player $\tilde{i}$ and $\tilde{j}$. One important case is when we do not allow for self-loops, that is $g_{ii} = 0,$ for any i.

Let Y be a matrix such that $y_{ij} \in \{0, 1\}$, $\forall i, j$. We define the following set:

Definition 3.

$$S_2(Y) = \{M \in \mathcal{M}_{n,n} \text{ such that } M_{ij} = 0 \text{ if } Y_{ij} = 0\}$$

[7] See

From Higham, 2000 ([27]) we know that, given Y and S_2 the solution to the minimization problem (4.2) with $\mathcal{A} = S_2(Y)$ is:

$$E_{\min}^{Sp}(\bar{\boldsymbol{a}}) = \sum_{i=1..n} \frac{1}{(\bar{a}_i^T \bar{\boldsymbol{a}})} \epsilon_i^T (b - A\bar{\boldsymbol{a}})\epsilon_i s_i^T$$

where $\bar{a}_i \in \mathbb{R}^n$, with $j-$th element $\bar{a}_i(j)$ defined as:

$$\bar{a}_i(j) = \begin{cases} \bar{\boldsymbol{a}}(j) & \text{if } Y_{ij} = 1 \\ 0 & \text{if } Y_{ij} = 0 \end{cases}$$

For example we want to study the case where the initial and post-intervention do not have self-loops (that is $a_{kk} = 0$ for all k), we define the restrictions matrix:

$$B = \begin{bmatrix} 0 & 1 & 1 & ... \\ 1 & 0 & 1 & ... \\ & ... & & \\ ... & 1 & 1 & 0 \end{bmatrix}$$

Consequently we get:

$$E_{\min}^{Sp}(\bar{\boldsymbol{a}}) = \begin{bmatrix} \frac{b_1 - \epsilon_1 A\bar{\boldsymbol{a}}}{\sum_{i\neq 1} a_i^2} & 0 & 0 & ... \\ 0 & \frac{b_2 - \epsilon_2 A\bar{\boldsymbol{a}}}{\sum_{i\neq 2} a_i^2} & 0 & ... \\ & ... & & \\ ... & 0 & 0 & \frac{b_n - \epsilon_n A\bar{\boldsymbol{a}}}{\sum_{i\neq n} a_i^2} \end{bmatrix} \begin{bmatrix} 0 & a_{12} & a_{13}... & \\ a_{21} & 0 & a_{23} & ... \\ & ... & & \\ ... & a_{n,n-2} & a_{n,n-1} & 0 \end{bmatrix}$$

Remark 3. *This intervention is a rank-n intervention. As noted in Dennis and Schnabel, 1979 ([18]) this type of correction is of no computational significance and can be made "one row at the time". Column k of the intervention is proportional to the non-zero entries to the objective equilibrium profile k-th component.*

Dennis and Schnabel, 1979 ([18]) and other work in the computer science study other interesting case of $\mathcal{A}$. In the next section we focus on studying a problem that we consider of particular interest: How should a planner, with a given budget, intervene on the network structure in order to increase total welfare.

5 Closest network structure that maximizes welfare

In the previous section we specified how a social planner can implement an equilibrium profile at minimal cost. While this might be of interest per se, in this section, we will use the results we obtained to address an important policy question: how should the social planner allocates resources to maximize total welfare.

5.1 The planner's problem

The goal of the social planner is to maximize aggregate welfare knowing that agents are utility-maximizers and that they will play the Nash equilibrium $\boldsymbol{a}^* \in \mathbb{R}^n$ of the game described in the model. To reach this goal the social planner can intervene on the network structure provided she respects a cost constraint. Given $A \in \mathcal{M}_{nn}$ (as defined in (3.1) as a function of the initial network structure G), a budget $C > 0$, and a vector of characteristics $b \in \left(\mathbb{R}_+^*\right)^n$, the planner's problem $\mathcal{P}[A, C, b]$ is the following:

$$\begin{aligned} \max_{E \in \mathcal{M}_{nn}} & \sum_{i \in \mathcal{N}} W_i \left[\boldsymbol{a}^*(E)\right] \\ s.t. & \left[A + E\right] \boldsymbol{a}^*(E) = b, \\ & ||E||_F^2 \leq C \end{aligned} \tag{$\mathcal{P}[A,C,b]$}$$

where $\boldsymbol{a}^*(E)$ is the Nash equilibrium of the game played on the transformed matrix $A + E$.

5.2 Equilibrium profile's reaction to planner's intervention

Even if the solution to problem ($\mathcal{P}[A, C, b]$) would be very sensible to the initial conditions, we try to describe what is the general idea behind the planner's intervention. In particular we will show how the equilibrium of the network game moves after intervention. This will give us an idea of which player will be more affected and why. We will first rewrite the problem of the planner in a equivalent form. After we will recall some notions of matrix algebra that will use in Proposition 3 to get our result.

We use the result of proposition (1) to express the problem of the planner in ($\mathcal{P}[A, C, b]$):

Given $b \in \mathbb{R}^n$, $A \in \mathcal{M}_{n,n}$, $C > 0$,

$$\begin{aligned} \max_{\boldsymbol{a}\in\mathbb{R}^n} \quad & \frac{||\boldsymbol{a}||^2}{2} \\ s.t. \quad & C||\boldsymbol{a}||^2 - ||A\boldsymbol{a} - b||^2 \geq 0 \end{aligned} \tag{$\mathcal{P}_2[A,C,b]$}$$

We used a classical result of the quadratic cost network game literature, for which at the Nash equilibrium $\boldsymbol{a}^*$: [8]

$$W = \sum_{i\in\mathcal{N}} W_i(\boldsymbol{a}^*) = \frac{||\boldsymbol{a}^*||^2}{2} \tag{5.1}$$

A solution to ($\mathcal{P}_2[A,C,b]$) exists as the objective function is continuous and the constrained set is compact. Applying the extreme value theorem yields existence of a solution. We call $\tilde{\boldsymbol{a}}^*$ a solution of ($\mathcal{P}_2[A,C,b]$).

The idea is to compare the equilibrium of the initial game, with the equilibrium of the game after the intervention of the planner. The equilibria we want to compare are n-dimensional vector and therefore a metric for comparison is difficult to obtain. What we will do is to choose an appropriate set of coordinates and try to compare the projections of the two equilibria on these. In order to do this we will have to recall some notion of matrix algebra.

Singular value decomposition In order to analyze the changes that the equilibrium action profile is going through when the planner's intervention takes place, we introduce a common tool of linear algebra, the singular value decomposition. Given a matrix $M \in \mathcal{M}_{n,n}$ there exists a factorization, called singular value decomposition (SVD) of M of the form

$$M = U\Sigma V^T,$$

where U and V are unitary matrices of $\mathcal{M}_{n,n}$ and Σ is a diagonal matrix with non-negative real numbers on the diagonal. The diagonal entries $\{s_i\}_i$ of Σ are known as the singular values of M, the column of U (or V) are known as left (or right) singular vectors of M.

[8] see Bramoullé, Kranton and D'Amours, 2014, [15]

Case of symmetric positive definite matrices If M is symmetric and positive definite, its singular value decomposition coincides with its eigendecomposition. In this case $U = V$, the column of the matrix U are the eigenvectors of M and the singular values are its associated eigenvalues. The advantage of the singular value decomposition over the eigendecomposition is that it always exists.

Let us consider the singular value decomposition of A:

$$A = U\Sigma V^T \tag{5.2}$$

with u^i (respectively v^i) the i-th column of U (respectively V), and s_i the $i-$th singular value, when ranking the singular values in decreasing order. u^i_j (respectively v^i_j) is the $j-$th element of the vector u^i (respectively v^i). Let p_i be the projection of the initial equilibrium action profile $\boldsymbol{a}^*$ (before intervention) on v^i and $\tilde{p}_i$ the projection of the new action profile $\tilde{\boldsymbol{a}}^*$ (after intervention).

We add an assumption about the size of the budget. This condition is sufficient for our next result to hold but not necessary.

Assumption 3.

$$C < \frac{\left(\min_i s_i^2\right)^2}{\sum_{i=1}^{n} s_i^2}$$

The following theorem allows us to rank the ratio of any two projections before and after intervention.

Theorem 1. *Under assumption 3, for $i, j = 1, \dots, n$, $i < j$:*

$$\frac{p_i}{p_j} \leq \frac{\tilde{p}_i}{\tilde{p}_j} \tag{5.3}$$

Proof. See section 9.1 of appendix. For the version where symmetry is imposed to the transformed network, see section 9.3 of appendix.

The ratio of the projections of the resulting action profile on different right singular vectors of A tells us how close the action profile is from one right singular vector relative to another one. By comparing this ratio before and after intervention, we understand whether the action profile moves towards one right singular vector relative to another one with the intervention. The next corollary formalizes this idea.

For any $\boldsymbol{a} \in \mathbb{R}^n$, let $\theta_i(\boldsymbol{a})$ be the angle between the vector $\boldsymbol{a}$ and v^i in the 2-dimensional subspace of $\mathbb{R}^n$ spanned by $\boldsymbol{a}$ and v^i .

Corollary 2. *Under assumption 3, for $i, j = 1, \ldots, n$, $i < j$:*

$$\frac{\theta_i(\boldsymbol{a}^*)}{\theta_j(\boldsymbol{a}^*)} \leq \frac{\theta_i(\tilde{\boldsymbol{a}}^*)}{\theta_j(\tilde{\boldsymbol{a}}^*)} \tag{5.4}$$

Proof. See appendix.

Corollary (2) means that the action profile moves towards v^i and away from v^j, for all $i < j$.

This result holds in the particular case when the singular value decomposition is the eigendecomposition as well as in the general case when it is not. But the interpretation of the result is easier when the $\{v^i\}_i$ are the n eigenvectors of A. In this case it exists the eigenvectors of A and of G are the same and because the eigenvectors of an adjacency matrix have a nice interpretation in terms of network structure, we can give an interpretation to our result.

Remember the relationship between A and G from (3.1):

$$A = I - \beta G$$

As I is a diagonal matrix, the correspondence between eigenvectors and eigenvalues of the two matrices is straightforward and depends on the sign of β. Let's call $\{\lambda_i\}_i$ the eigenvalues of A and $\{\lambda_i(G)\}_i$ the eigenvalues of G.

Lemma 2.

$$\lambda_i(G) = \frac{1 - \lambda_i}{\beta} \qquad \text{with } v^i \text{ the associated eigenvector} \tag{5.5}$$

and the n eigenvectors $\{\bar{v}^i\}_{i=1}^n$ of G are the eigenvectors of A ranked:

1. *If $\beta > 0$: in the opposite order as $\{v^i\}_{i=1}^n$*
2. *If $\beta < 0$: in the same order as $\{v^i\}_{i=1}^n$*

Proof. From (3.1) we get the following equivalence:

$$Av^i = \lambda_i v^i \qquad \Leftrightarrow \qquad Gv^i = \frac{1 - \lambda_i}{\beta} v^i \tag{5.6}$$

The results directly follows.

The order of the eigenvectors refers to the orders of their associated eigenvalues ranked in decreasing order, from the largest to smallest. The eigenvectors of the adjacency matrix capture important characteristic of the network. We will use these to interpret the result on the projections of the equilibria. Each component of the first eigenvector of the adjacency matrix represents the eigenvector centrality of the corresponding player. A high eigenvector centrality means that a node is connected to many nodes who themselves have high centrality. The higher the $j-$element of the first eigenvector is, the more central the $j-$th player is. The last eigenvector, instead, in a bipartite network, assign negative values to players in one of the two sets and positive values to the one in the other set.[9] We put together corollary 2 with lemma 2 to give a result that has a clear interpretation in terms of network structure:

Proposition 3. *Under assumption 3, for $i, j = 1, \ldots, n$, $i < j$:*

1. *If $\beta > 0$:*

   *The equilibrium responds to the planner's intervention moving **from the higher-ranked to the lower-ranked eigenvector** in the subspace of $\mathbb{R}^n$ spanned by those two eigenvectors.*

2. *If $\beta < 0$:*

   *The equilibrium responds to the planner's intervention moving **from the lower-ranked to the higher-ranked eigenvector** in the subspace of $\mathbb{R}^n$ spanned by those two eigenvectors.*

Figure (1) is a graphical representation of corollary (2), for $i = 1, j = 2$ in the case of strategic complements. Proposition (3) says that more central agents (of the initial network), in this case player 1, as his component of the first eigenvector is larger, increase more their action relative to another weighting of the agents (the weighting described by $\bar{v}^2$ for instance). This result tells us that the planner is changing the incentives in the game in such a way that central players are the one more affected. They will respond more than others to the intervention. We see the decision of the planner to target central player as a result of the importance of these players typical of network games of strategic complements. When a central player, after intervention, increases is action he incentivizes the players that are connected to him to increase their action as well, bringing benefits to all the populations. The more a player is central, the more his action is

[9] A bipartite network is a network whose nodes can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V.

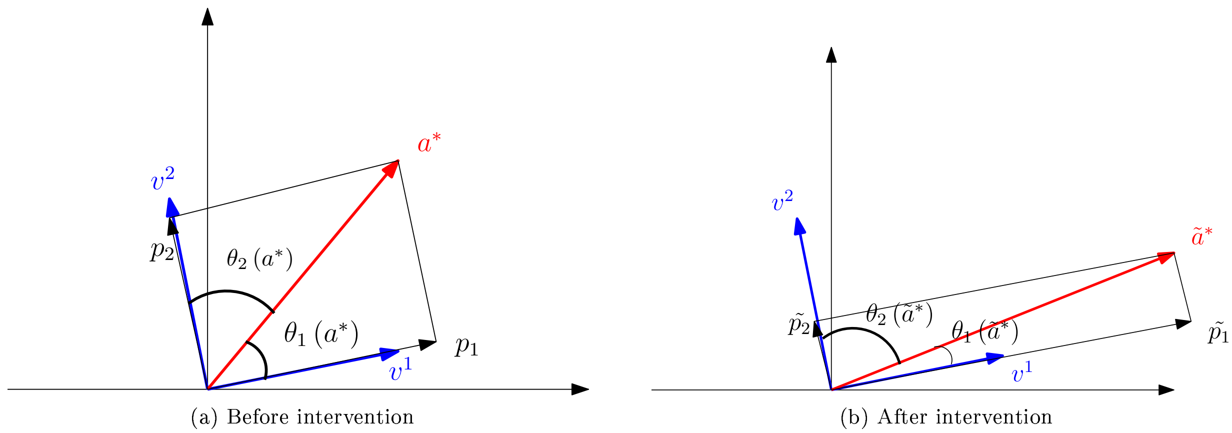

Figure 1: Change in the relative distance between the action profile and the first two right singular vectors of A

important to incentivize other players. In section 7 we will try to investigate how the social planner actually targets central players, but first we want to explain what happens in game of strategic substitutes.

When we consider a game of strategic substitutes the two graphs of Figure 1 follow the opposite order. To get an interpretation we focus on the last eigenvector of the network. Proposition 3 says that the equilibrium will tend to mirror the last eigenvector (with respect to any other eigenvector) after the intervention of the planner. To fix ideas consider Figure (2). Here all the nodes are connected but some links are stronger (dark blue) and form a biparite graph. The last eigenvector of the adjacency matrix will have positive entries for the red nodes and negative for the green. Proposition 3 is telling us that the social planner is changing the incentives of red and green players in opposite directions. Red players, after interventions will increase their actions while green will decrease them. Changing actions of closed neighbors in opposite direction increases total welfare. If the planner would instead move incentives in the same direction an increase in the action of a player would crowed out the incentive of his neighbor. In section 7 we try to understand how the planner reaches her goal. Before, in the next section, we will compare our result to GGG.

6 Comparison with GGG

We now want to compare our result with the proposition 1 of GGG. We focus on the special case where the singular value decomposition is the eigendecomposition. The results follow through in terms of singular vectors. We find that the changes in the action profile have the same **direction** in terms of eigenvectors, though the **amplitude** of the changes is surely different.

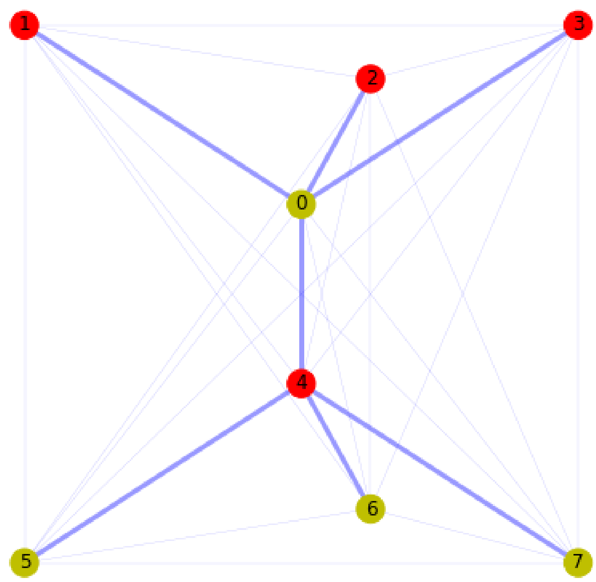

Figure 2: Bipartite Network

Definition 4. *$q^i(G, b)$ is the projection of b on the i-th eigenvector of G.*

The result of proposition 1 of GGG is:

1. If $\beta > 0$,
$$\frac{q^l(G, b^{\text{new}}) - q^l(G, b)}{q^l(G, b)} \text{ is weakly decreasing in } l \tag{6.1}$$

2. If $\beta < 0$,
$$\frac{q^l(G, b^{\text{new}}) - q^l(G, b)}{q^l(G, b)} \text{ is weakly increasing in } l \tag{6.2}$$

where b is the initial vector of individual characteristics, and b^{new} is the new vector, after intervention (remember that they intervene on b when we intervene on G).

In order to compare their result and our result, let us rename our projections:

Definition 5. *$p^i(b, G, C)$ is the projection of the action profile solution of ($\mathcal{P}_2[A, C, b]$), with parameters $b \in \mathbb{R}^n$, $G \in \mathcal{M}_{n,n}$, $C > 0$, on the i-th eigenvector of G*

Simple algebra yields the following theorem:

Theorem 2. *For $i < j$:*

(1) (6.1) *implies*
$$\frac{p^i(b, G, C)}{p^j(b, G, C)} \leq \frac{p^i(b^{new}, G, C)}{p^j(b^{new}, G, C)} \tag{6.3}$$

(2) (6.2) *implies*
$$\frac{p^i(b, G, C)}{p^j(b, G, C)} \geq \frac{p^i(b^{new}, G, C)}{p^j(b^{new}, G, C)} \tag{6.4}$$

In words, it means that the results they find when $\beta > 0$ and $\beta < 0$ leads to the same direction of the change in the action profile $\boldsymbol{a}^*$ with respect to any two eigenvectors of G as in our result of proposition 3 . Note that the inequalities (6.3) and (6.4) are inequalities, telling us nothing about the amplitude of the variation. But the variation in $\boldsymbol{a}^*$ (our object of interest as it determines aggregate welfare) goes in the same direction in both cases of β.

An open questions remains which method yields the highest welfare gain at a given cost C.

7 Network structure analysis

In section 6 we saw how the social planner change the incentives of the players, studying how the equilibrium moves after the interventions. In this section we want to give insights on how she actually achieves her goal, in terms of which connections will be modified and how.

Following the notations of equilibrium profiles, we call $\tilde{A} = A + E_{\min}$ (how the matrix A is transformed after intervention) and $\tilde{G}$ the new network structure. As we will study the object $E_{\min}$, we rename it ΔA as it represents what is added to the matrix A in the optimal intervention. Similarly we define $\Delta G = \tilde{G} - G$. From the definition (3.1) we get:

$$\tilde{G} - G = -\frac{1}{\beta}(\tilde{A} - A) \tag{7.1}$$

Therefore studying $\tilde{A} - A = \Delta A$ sheds light on the change of network structure (ΔG). We starts from an undirected network so that we can use the eigendecomposition and it will be easier to interpret the resutlts. We study the unrestricted case as it is more tractable, meaning that we do not impose that the transformed network is undirected too. As a consequence we study separately whether the links are targets of the planner:

1. links arriving to a set of players

2. links starting from a set ofl players

Following the approach of the previous sections, we work on the projection of ΔA on the basis B of the eigendecomposition of A:

$$B = \{(v_1 v_1^T), (v_1 v_2^T), ..., (v_1 v_n^T), (v_2 v_1^T), ..., (v_n v_1^T), ..., (v_n v_n^T)\} \tag{7.2}$$

The decomposition is a sum of n^2 elements:

$$\Delta A = \sum_{i=1,...,n, j=1,...n} \mu_{i,j} \left[v^i \left(v^j \right)^T \right] \tag{7.3}$$

We define the following two objects of interest:

Definition 6. *For $i = 1, \dots, n$:*

$$S^I(v^i) = \sum_{k=1}^{n} \left(v^i\right)^T \Delta A v^k$$
$$S^O(v^i) = \sum_{k=1}^{n} \left(v^k\right)^T \Delta A v^i \tag{7.4}$$

$S^I(v^i)$ is the sum of the coefficients of decomposition (7.3) on the basis elements of B with vector v^i as left vector of the outer product. With this quantity we want to capture how relevant is the eigenvector v^i to explain the structure of ΔA (the in-links are captured with $S^I(v^i)$ and the out-links with $S^O(v^i)$.

Proposition 4. *When $C \to 0$, for any i, j:*

$$\frac{S^I(v^i)}{S^I(v^j)} = \left(\frac{1 - \beta\lambda_j(G)}{1 - \beta\lambda_i(G)}\right)^2 \frac{v_i^T b}{v_j^T b} \tag{7.5}$$

and

$$\frac{S^O(v^i)}{S^O(v^j)} = \left(\frac{1 - \beta\lambda_j(G)}{1 - \beta\lambda_i(G)}\right) \frac{v_i^T b}{v_j^T b} \tag{7.6}$$

with $\lambda_i(G)$ defined in equation (5.5) as a function of the eigenvalues of A in the following way:

$$\lambda_i(G) = \frac{1 - \lambda_i}{\beta}$$

where $\lambda_i(G)$ is:

1. *the $(n - i + 1)-th$ largest eigenvalue of G if $\beta > 0$*

2. *the $i-th$ largest eigenvalue of G if $\beta < 0$*

Proof. See appendix

Proposition 4 gives us information about ΔA but we are eventually interested in ΔG. Recalling (7.1), we notice that ΔG and ΔA are proportional, and thus :

$$\frac{\sum_{k=1}^{n} \left(v^i\right)^T \Delta G v^k}{\sum_{k=1}^{n} \left(v^j\right)^T \Delta G v^k} = \frac{S^I(v^i)}{S^I(v^j)} \tag{7.7}$$

As a consequence we directly use the ratio $\frac{S^I(v^i)}{S^I(v^j)}$ to inform us about ΔG. To avoid ambiguity, let us call r_i the $i-$th largest eigenvalue of G regardless of the sign of β, and w_i its associated eigenvector. Therefore, when $\beta > 0$, $r_1 = \lambda_n(G)$ and $r_p = \lambda_{n-p+1}(G)$. Those notations are merely relabeling to provide clearer intuition of proposition 4 , as when $\beta > 0$, $\lambda_i(G)$ is the $n - i + i$-th largest eigenvalue of G, but v^i stays its associated eigenvector, that is $v^i = w^{n-i+1}$. We can then rewrite the expressions of proposition 4 in a more intuitive way in the following corollary.

Corollary 3. *When $C \to 0$, for $p > q$:*

$$\frac{S^I(w^q)}{S^I(w^p)} = \left(\frac{1-\beta r_p}{1-\beta r_q}\right)^2 \frac{w_q^T b}{w_p^T b} \tag{7.8}$$

and

$$\frac{S^O(w^q)}{S^O(w^p)} = \left(\frac{1-\beta r_p}{1-\beta r_q}\right) \frac{w_q^T b}{w_p^T b} \tag{7.9}$$

Interpretation for strategic complements ($\beta > 0$) In the case of strategic complements we would like to explain how the social planner incentivize central players, the result we obtained from section 6. We call

$$W(p) = \frac{w_1^T b}{w_p^T b} \tag{7.10}$$

and $R(p)$:

$$R(p) = \left(\frac{1-\beta r_p}{1-\beta r_1}\right) \tag{7.11}$$

We can thus rewrite the results of corollary 7.8, when $q = 1$ as:

$$\frac{S^I(w^1)}{S^I(w^p)} = R(p)^2 W(p) \quad \text{and} \quad \frac{S^O(w^1)}{S^O(w^p)} = R(p)W(p) \tag{7.12}$$

The larger is the first of the ratio in (7.14) the more the planner is increasing the intensity of links toward central players. The larger the second of the two ratio in (7.14) the more the planner is increasing the intensity of links from central players to the other individuals. We can now study the two ratio and make some interesting observations:

- From assumption 1 we know that both the numerator and the denominator of $R(p)$ are between 0 and 1 and as $r_1 \geq r_p$, as $|r_1 - r_p|$ grows the more the social planner is targeting links from and towards central players. Other thing beeing equal we can compare networks that have different eigenvalues and understand how much centrality of players has a key role in the decision of the planner.

- As $R(p) \geq 1$ (strict inequality for $r_1 \neq r_p$), $R(p)^2$ grows faster than $R(p)$ with $|r_1 - r_p|$, and thus the ratio for in-links grows faster than the ratios for out-links.

- The inner product $\langle w^i, b\rangle$ describes how high the private benefits of the action is for the players that matter in the $i-$th direction of the network structure. For instance, w^1 singles out central players and $\langle w^1, b\rangle$ is high when central players have high private marginal benefits from the action. The more central players have high private value for the action relative to players important according to other the $p-$th dimension, the more $W(p)$ is large. The idea is that the planner has an incentive to induce links with players that have high private benefits for the action as a link with one of this player is more valuable than a link with a similar player with lower private benefit. In other words $W(p)$ gives us the relative size of externalities in the direction of the structural aspect described by eigenvector w^1 versus by eigenvector w^p.

- $W(p)$ can either be smaller or bigger than 1.
  - If $W(p) > 1$, then $S^k(w^1)/S^k(w^p)$, $k = I, O$ are both strictly greater than 1, (this) which means that the planner focuses more budget on central players relative to the players singled out by the dimension w^p. The effects contained in the ratios $R(p)$ and $W(p)$ reinforce each other.
  - On the other hand, if $W(p) < 1$, the effects of $R(p)$ and $W(p)$ go compensate each other. The first ratio pushes the planner to focus on central players, but on the other hand those players do not have a high marginal return on the action (indicated by $W(p) < 1$) and the planner may want to split her budget between central players and players with high private returns.

Interpretation for strategic substitutes ($\beta < 0$) We showed in section 6 that the social planner, in the case of strategic substitutes, moves incentive of neighbors' players in opposite directions. We try to understand how she achieves this goal.

We denote by $R^*(q)$:

$$R^*(q) = \left(\frac{1-\beta r_n}{1-\beta r_q}\right) \tag{7.13}$$

We can thus rewrite the results of corollary 7.8, when $p = n$ as:

$$\frac{S^I(w^q)}{S^I(w^n)} = R^*(q)^2 \frac{w_q^T b}{w_n^T b} \quad \text{and} \quad \frac{S^O(w^q)}{S^O(w^n)} = R^*(q) \frac{w_q^T b}{w_n^T b} \tag{7.14}$$

As opposed to the complement case, $R^*(q) \leq 1$ and decreases as the distance between r_q and r_n increases. The smaller is $R^*(q)$ the more planner lower the intensity of links players between green players and links between red playersc in Figure (2). In this way the network will tend to become a bipartite network. Green players will invest a lot in the action and red players will do the opposite.

8 Other Interventions

Started from the results given in Section 3, it is possible to investigate other interesting problem for the planner, as, for example, how to modify the network in order to give a minimum utility to all the players or how to garantee some form of equality among players or again how to increase the utility (or action) of all the players by the same amount. The social planners, in the case where a represents effort might also be interested in maximizing total effort. We leave these and other questions for future studies.

9 Appendix

9.1 Proof of theorem 1

We remind that the mathematical definition of the projection of any vector $x \in \mathbb{R}^n$ on the vector v^i is:

$$\text{projection of } x \text{ on } v^i = \quad \langle x, v^i \rangle \tag{9.1}$$

The steps of the proof are the following: first we derive an expression for p_i by using the singular value decomposition of A. Then, we derive properties on the equilibrium profile after optimal intervention $\tilde{\boldsymbol{a}}^*$ by differentiating the Lagrangian of the optimization problem. From there we get an expression for $\tilde{p_i}$. Finally we use the expressions for p_i and $\tilde{p_i}$ to deliver a ranking of the ratios of projections at different indices.

9.1.1 Projection of the equilibrium profile before intervention

The equilibrium before intervention is a vector $\boldsymbol{a}^*$ such that

$$A\boldsymbol{a}^* = b \tag{9.2}$$

Pre-multiplying both sides of the equation by the vector $\left(v^i\right)^T A^T$ we get:

$$\left(v^i\right)^T A^T A\boldsymbol{a}^* = \left(v^i\right)^T A^T b \tag{9.3}$$

Right hand side of (9.3) We are now using the singular value decomposition of A: $U\Sigma V^T$. As $\left(v^1, \ldots, v^n\right)$ forms an orthonormal basis of $\mathbb{R}^n$, we have the following equality:

$$\left(v^i\right)^T V = \begin{bmatrix} \langle v^i, v^1 \rangle & \langle v^i, v^2 \rangle & \ldots & \langle v^i, v^n \rangle \end{bmatrix} = \left(e_i^n\right)^T \tag{9.4}$$

where e_i^n is the i-th vector of the Euclidean basis of $\mathbb{R}^n$, that is its i-th element is 1 and all its other elements are 0. This helps us to simplify $\left(v^i\right)^T A^T$:

$$\left(v^i\right)^T A^T = \left(v^i\right)^T V\Sigma U^T = s_i \left(u^i\right)^T \tag{9.5}$$

By post-multiplying by the vector of characteristics b we reach:

$$\left(v^i\right)^T A^T b = s_i \left(u^i\right)^T b \tag{9.6}$$

Left hand side of (9.3)

$$\left(v^i\right)^T A^T A = s_i \left(u^i\right)^T A \tag{9.7}$$

(using the equality in (9.5)). Re-using the singular value decomposition, we get:

$$\left(v^i\right)^T A^T A = s_i \left(u^i\right)^T U \Sigma V^T \tag{9.8}$$

Similarly to (9.4) we have

$$\left(u^i\right)^T U = \begin{bmatrix} \langle u^i, u^1 \rangle & \langle u^i, u^2 \rangle & \dots & \langle u^i, u^n \rangle \end{bmatrix} = \left(e_i^n\right)^T \tag{9.9}$$

This helps us to simplify (9.8):

$$\left(v^i\right)^T A^T A = s_i \left(e_i^n\right)^T \Sigma V^T = s_i^2 \left(v^i\right)^T \tag{9.10}$$

By post-multiplying by $\boldsymbol{a}^*$ we reach:

$$\left(v^i\right)^T A^T A \boldsymbol{a}^* = s_i^2 \left(v^i\right)^T \boldsymbol{a}^* \tag{9.11}$$

Expression for the projection of $\boldsymbol{a}^*$ on v^i Putting together left-hand side and right-hand side of (9.3), we reach an expression for $\left(v^i\right)^T \boldsymbol{a}^*$, that is p_i:

$$p_i = \langle v^i, \boldsymbol{a}^* \rangle = \left(v^i\right)^T a^* = \frac{\left(u^i\right)^T b}{s_i} \tag{9.12}$$

Note on the symmetric case (undirected network) When the initial matrix G (and therefore A) is symmetric:

1. $v_i = u_i$ is an eigenvector of G (and therefore of A)

2. $s_i = \lambda_i$ is the eigenvalue i-th of the matrix A

Hence:

$$v_i^T a^* = \frac{v_i^T b}{\lambda_i} \tag{9.13}$$

9.1.2 Projection of the equilibrium profile after intervention

Let us first derive conditions on the solution $\tilde{\boldsymbol{a}}^*$ of ($\mathcal{P}_2[A, C, b]$) by writing and differentiating the lagrangian $\mathcal{L}(.)$ of the problem.

$$\mathcal{L}(a, \mu) = \frac{||a||^2}{2} + \mu\Big[C||a||^2 - ||Aa - b||^2\Big]$$

$$\frac{\partial \mathcal{L}}{\partial a}(a, \mu) = a + \mu 2ca - \mu\left(2A^T Aa - 2A^T b\right) \tag{9.14}$$

Setting this expression equal to zero we get:

$$(1 + 2\mu c)\tilde{\boldsymbol{a}}^* - 2\mu A^T A\tilde{\boldsymbol{a}}^* = -2\mu A^T b \tag{9.15}$$

Dividing both sides by -2μ (as the constraint is binding and thus the multiplier is strictly positive) and calling $K = \dfrac{1}{2\mu} + c$ we get :

$$-K\tilde{\boldsymbol{a}}^* + A^T A\tilde{\boldsymbol{a}}^* = A^T b \tag{9.16}$$

Pre-multiplying both sides by $\left(v^i\right)^T$ and directly using the results of (9.6) and (9.10), we reach

$$-Kv_i^T\tilde{\boldsymbol{a}}^* + s_i^2\left(v^i\right)^T\tilde{\boldsymbol{a}}^* = s_i\left(u^i\right)^T b \tag{9.17}$$

and finally we have an expression for the projection of $\tilde{\boldsymbol{a}}^*$ on the vector v^i:

$$\tilde{p_i} = v_i^T\tilde{\boldsymbol{a}}^* = \frac{s_i}{s_i^2 - K}\left(u^i\right)^T b \tag{9.18}$$

9.1.3 Comparing the ratio of projections on two different right singular vectors

From (9.13) and (9.18) we compute the ratio of interest:

$$\frac{\tilde{p}_i/\tilde{p}_j}{p_i/p_j} = \frac{s_i^2}{s_j^2}\frac{s_j^2 - K}{s_i^2 - K} = \frac{s_i^2}{s_i^2 - K}\Bigg/\frac{s_j^2}{s_j^2 - K} \tag{9.19}$$

Let's observe that assumption (3) together with the Kuhn-Tucker conditions of ($\mathcal{P}_2[A, C, b]$) tells us that $s_i^2 - K > 0 \ \ \forall i$. In order to establish this consider the FOCs of the lagrangian:

$$\tilde{\boldsymbol{a}}^* + \mu 2C\tilde{\boldsymbol{a}}^* - \mu 2A^T A\tilde{\boldsymbol{a}}^* = -\mu 2A^T b$$

Simple algebra gives:

$$\left(\frac{1}{2\mu} + C\right)^2 = \frac{(A\tilde{\boldsymbol{a}}^* - b)^T AA^T (A\tilde{\boldsymbol{a}}^* - b)}{||\tilde{\boldsymbol{a}}^*||^2} = \frac{||A^T(A\tilde{\boldsymbol{a}}^* - b)||^2}{||\tilde{\boldsymbol{a}}^*||^2} \tag{9.20}$$

Cauchy-Schwartz inequality tells us that for a matrix $M \in \mathcal{M}_{n,n}$, a vector $x \in \mathbb{R}^n$, $||Mx||_2 \leq ||M||_F ||x||_2$. We apply this result (note also that the Frobenius norm of a matrix and that of its transposed are the same) to our previous equality:

$$(9.20) \Rightarrow \left(\frac{1}{2\mu} + C\right)^2 \leq \frac{||Aa^* - b||^2 ||A||^2}{||\tilde{\boldsymbol{a}}^*||^2} \tag{9.21}$$

Besides, we know that the constraint of ($\mathcal{P}_2[A, C, b]$) is binding:

$$\frac{||A\tilde{\boldsymbol{a}}^* - b||^2}{||\tilde{\boldsymbol{a}}^*||^2} = C$$

Therefore, plugging this into (9.21), we get:

$$\left(\frac{1}{2\mu} + C\right)^2 \leq C||A||_F^2$$

Which is equivalent to:

$$K^2 \leq C||A||_F^2 = C\sum_{i=1}^n s_i^2 \tag{9.22}$$

As K is > 0 (the lagrangian multiplier is > 0), we get the following:

$$K \leq \sqrt{C}\sqrt{\sum_{i=1}^n s_i^2} \leq \frac{\min_i s_i^2}{\sqrt{\sum_{i=1}^n s_i^2}}\sqrt{\sum_{i=1}^n s_i^2} = \min_i s_i^2 \tag{9.23}$$

where the second inequality comes from assumption (3)

(9.23) implies, for all i:

$$s_i^2 \geq K \tag{9.24}$$

We now use this result to determine whether the ratio in (9.19) is bigger or smaller than 1. The function $f : x \mapsto \frac{x}{x-K}$ is increasing in x iff $x > K$ (which we know from (9.24)). This implies that

$$(9.20) \Rightarrow \frac{\tilde{p}_i/\tilde{p}_j}{p_i/p_j} > 1 \quad \text{iff} \quad s_i > s_j \tag{9.25}$$

In the symmetric case the projection on the left singular vectors corresponds to the projection on the eigenvector of the adjacency matrix.

9.1.4 Proof of Corollary (2)

Corollary (2) comes directly from the fact that the projection of $\boldsymbol{a}^*$ on v^i can also be written as:

$$p_i = \langle \boldsymbol{a}^*, v^i \rangle = ||\boldsymbol{a}^*||\ ||v^i|| \cos\theta_i(\boldsymbol{a}^*) \tag{9.26}$$

Besides, the vectors of the family $\{v^i\}_i$ are orthonormal, therefore the norm of each of them is 1. The ratio of projections thus boils down to:

$$\frac{p_i}{p_j} = \frac{\cos\theta_i(\boldsymbol{a}^*)}{\cos\theta_j(\boldsymbol{a}^*)} \qquad \text{and} \qquad \frac{\tilde{p}_i}{\tilde{p}_j} = \frac{\cos\theta_i(\tilde{\boldsymbol{a}}^*)}{\cos\theta_j(\tilde{\boldsymbol{a}}^*)} \tag{9.27}$$

We pause a moment to consider the sign of the projections. In principle, $\langle \boldsymbol{a}^*, v^i \rangle$ can be either negative or positive (depending whether the angle $\theta_i(\boldsymbol{a}^*)$ is bigger or smaller than 90°. We can abstract from this sign ambiguity by "choosing" the (right) singular vector that makes the projection positive. If a vector x is a singular vector of a matrix with norm 1, then the vector $-x$ will also be a singular vector of this matrix (provided we change the sign of both the left and the right singular vectors associated to the same singular value), of norm 1 too. Therefore we choose all the right singular vectors of A such that $\langle \boldsymbol{a}^*, v^i \rangle > 0$ for all i. Then, by continuity of the inner product, for a budget small enough, $\langle \tilde{\boldsymbol{a}}^*, v^i \rangle$ will also be positive.

Then, combining (9.27) and theorem 1 yields the desired result.

9.2 Proof of proposition 4

Step 1: expression for $\tilde{\boldsymbol{a}}^*$ From (9.16), we get:

$$\tilde{\boldsymbol{a}}^* = \left(A^T A - KI\right)^{-1} A^T b \tag{9.28}$$

(remember that K is a scalar, equal to $\frac{1}{2\mu} - C$ and that the matrix $A^T A - KI$ is invertible as it is symmetric) . Going back to the eigendecomposition of A, and taking into account that A is symmetric and thus that $A = A^T$, we have:

$$\left(A^T A - KI\right) = A^2 - KI = V\Sigma^2 V^T - KI = V \begin{pmatrix} \lambda_1^2 - K & & \\ & \ddots & \\ & & \lambda_n^2 - K \end{pmatrix} V^T \tag{9.29}$$

It is then easy to compute $\left(A^T A - KI\right)^{-1}$:

$$\left(A^T A - KI\right)^{-1} = V \begin{pmatrix} \frac{1}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n^2 - K} \end{pmatrix} V^T \tag{9.30}$$

Finally:

$$\tilde{\boldsymbol{a}}^* = \left(A^T A - KI\right)^{-1} V\Sigma V^T b = V \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & \frac{\lambda_n}{\lambda_n^2 - K} \end{pmatrix} V^T b \tag{9.31}$$

Step 2: expression for $(\boldsymbol{b} - A\tilde{\boldsymbol{a}}^*)$ Another useful factorization is the one of $(\boldsymbol{b} - A\tilde{\boldsymbol{a}}^*)$. Directly from the previous result in (9.31), we get:

$$(\boldsymbol{b} - A\tilde{\boldsymbol{a}}^*) = b - AV \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & \frac{\lambda_n}{\lambda_n^2 - K} \end{pmatrix} V^T b = \left(I - V \begin{pmatrix} \frac{\lambda_1^2}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & \frac{\lambda_n^2}{\lambda_n^2 - K} \end{pmatrix} V^T \right) b$$

or again

$$(\boldsymbol{b} - A\tilde{\boldsymbol{a}}^*) = V \begin{pmatrix} 1 - \frac{\lambda_1^2}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & 1 - \frac{\lambda_n^2}{\lambda_n^2 - K} \end{pmatrix} V^T \ b \tag{9.32}$$

Step 3: expression for $\left(v^i\right)^T \Delta A \ v^j$ We now compute $\left(v^i\right)^T \Delta A \ v^j$ for all i, j by using the two previous steps together with (4.4) of proposition 1 (with $\bar{\boldsymbol{a}} = \boldsymbol{a}^*$):

$$\left(v^i\right)^T \Delta A \ v^j = \left(v^i\right)^T \frac{(\boldsymbol{b} - A\tilde{\boldsymbol{a}}^*)(\tilde{\boldsymbol{a}}^*)^T}{(\tilde{\boldsymbol{a}}^*)^T \tilde{\boldsymbol{a}}^*} \ v^j \tag{9.33}$$

whose right-hand side is equal to:

$$\frac{1}{(\tilde{\boldsymbol{a}}^*)^T \tilde{\boldsymbol{a}}^*} \left(v^i\right)^T V \begin{pmatrix} 1 - \frac{\lambda_1^2}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & 1 - \frac{\lambda_n^2}{\lambda_n^2 - K} \end{pmatrix} V^T \ b \ b^T V \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2 - K} & & \\ & \ddots & \\ & & \frac{\lambda_n}{\lambda_n^2 - K} \end{pmatrix} V^T \ \ v^j$$

or again is equal to

$$\frac{1}{(\tilde{\boldsymbol{a}}^*)^T \tilde{\boldsymbol{a}}^*} \frac{-K}{\lambda_i^2 - K} \left(v^i\right)^T b \ b^T \frac{\lambda_j}{\lambda_j^2 - K} v^j$$

finally we get:

$$\left(v^i\right)^T \Delta A \ v^j = -\frac{1}{(\tilde{\boldsymbol{a}}^*)^T \tilde{\boldsymbol{a}}^*} \frac{K \lambda_j}{(\lambda_i^2 - K)(\lambda_j^2 - K)} \langle v^i, b \rangle \langle v^j, b \rangle \tag{9.34}$$

Step 4: computing $S^I(v^i)$ Now we want to compute the sum $S^I(v^i)$ from previous result:

$$S^I(v^i) = \sum_{j=1}^{n} \left(v^i\right)^T \Delta A \ v^j = -\frac{1}{(\tilde{\boldsymbol{a}}^*)^T \tilde{\boldsymbol{a}}^*} \langle v^i, b \rangle \frac{1}{\lambda_i^2 - K} \sum_{j=1}^{n} \frac{K \ \lambda_j}{(\lambda_j^2 - K)} \langle v^j, b \rangle \tag{9.35}$$

Step 5: computing the ratio $S^I(v^1)/S^I(v^p)$ For $p = 2, \dots, n$:

$$\frac{S^I(v^1)}{S^I(v^p)} = \frac{\langle v^1, b \rangle}{\langle v^p, b \rangle} \frac{\lambda_p^2 - K}{\lambda_1^2 - K} \tag{9.36}$$

Step 6: taking the limit as $C \to 0$ Following the previous approach, we take $C \to 0$ and work for a transformed network close to the initial network. We know from (9.22) that:

$$K^2 \leq C||A||_F^2 = C\sum_{i=1}^{n}\lambda_i^2$$

Therefore when $C \to 0$, $K \to 0$ too and we get:

$$\frac{S^I(v^1)}{S^I(v^p)} = \frac{\langle v^1, b\rangle}{\langle v^p, b\rangle}\frac{\lambda_p^2}{\lambda_1^2} \tag{9.37}$$

Step 7: Ratio as a function of the eigenvalues of G The final step is again to transform expression (9.41) as a function of the eigenvalues of G and not of A. Using the correspondence stated in lemma 2, we get:

$$\frac{S^I(v^1)}{S^I(v^p)} = \frac{\langle v^1, b\rangle}{\langle v^p, b\rangle}\left(\frac{1-\beta\lambda_p(G)}{1-\beta\lambda_1(G)}\right)^2 \tag{9.38}$$

However it is to be noted that when $\beta > 0$, the order of the eigenvalues of A and G is inverted and therefore $\lambda_1(G)$ is the **smallest** eigenvalue of G and $\lambda_n(G)$ is the **largest** eigenvalue of G

Step 8: computation of $S^O(v^i)$ Remember that

$$S^O(v^i) = \sum_{k=1}^{n}\left(v^k\right)^T \Delta A v^i$$

We restart from (9.34) above where we just change the index over which we sum:

$$S^O(v^j) = -\frac{1}{(\tilde{\boldsymbol{a}}^*)^T\tilde{\boldsymbol{a}}^*}\frac{\lambda_j}{\lambda_j^2 - K}\langle v^j, b\rangle\sum_{i=1}^{n}\frac{K}{(\lambda_i^2 - K)}\langle v^i, b\rangle \tag{9.39}$$

Step 9: computing the ratio $S^O(v^1)/S^O(v^p)$ For $p = 2, \ldots, n$:

$$\frac{S^O(v^1)}{S^O(v^p)} = \frac{\langle v^1, b\rangle}{\langle v^p, b\rangle}\frac{\lambda_p^2 - K}{\lambda_1^2 - K}\frac{\lambda_1}{\lambda_p} \tag{9.40}$$

Following the previous approach, we take $C \to 0$ and can thus neglect the constant K and we get:

When $C \to 0$:

$$\frac{S^O(v^1)}{S^O(v^p)} = \frac{\langle v^1, b\rangle}{\langle v^p, b\rangle}\frac{\lambda_p}{\lambda_1} \tag{9.41}$$

Step 10: Ratio as a function of the eigenvalues of G The final step is again to transform expression (9.41) as a function of the eigenvalues of G and not of A. Using the correspondence stated in lemma 2, we get:

$$\frac{S^O(v^1)}{S^O(v^p)} = \frac{\langle v^1, b\rangle}{\langle v^p, b\rangle} \frac{1 - \beta\lambda_p(G)}{1 - \beta\lambda_1(G)} \tag{9.42}$$

However it is to be noted that when $\beta > 0$, the order of the eigenvalues of A and G is inverted and therefore $\lambda_1(G)$ is the **smallest** eigenvalue of G and $\lambda_n(G)$ is the **largest** eigenvalue of G.

This proves the result.

9.3 Optimal intervention under symmetry constraint

From (4.7) we have that

$$E_{\min}(\bar{a}) = \frac{(b - A\bar{a})\bar{a}^T + \bar{a}^T(b - A\bar{a})^T}{\bar{a}^T\bar{a}} - (b - A\bar{a})^T\bar{a}\frac{\bar{a}\bar{a}^T}{\bar{a}^T\bar{a}}$$

Let us first derive conditions on the solution $\tilde{\boldsymbol{a}}^*$ of the planner's problem under symmetry constraint by writing and differentiating the Lagrangian $\mathcal{L}(.)$ of the problem.

$$\mathcal{L}(a, \mu) = (a^T a) + \mu\Big[C(a^T a)^2 - 2(b - Aa)^T(b - Aa)a^T a + ((b - Aa)^T a)^2\Big]$$

$$\frac{\partial \mathcal{L}}{\partial a}(a, \mu) = a + 2\mu c(a^T a)a - 2\mu(a^T a)(2A^2 a - 2Ab) - 2\mu(w^T w)a + 2\mu(w^T a)(b - Ay) \tag{9.43}$$

where we pose

$$w \equiv b - Ay$$

Rearranging we get

$$[(1 + 2\mu c(a^T a) - 2\mu(w^T w))I - 4\mu(a^T a)A^2 - 2\mu(w^T a)A]a = [-4\mu(a^T a)A - -2\mu(w^T a)I]b$$

This give us an expression for the new equilibrium profile action (that we call a^* in this section only). Using the properties of eigenvectors-eigenvalues, as in the case examined before we get:

$$a^* = V \begin{bmatrix} \frac{-4\mu(a^Ta)\lambda_1 - 2\mu(w^Ta)}{(1+2\mu c(a^Ta) - 2\mu(w^Tw)) - 4\mu(a^Ta)\lambda_1^2 - 2\mu(w^Ta)\lambda_1} & & \\ & \ddots & \\ & & \frac{-4\mu(a^Ta)\lambda_n - 2\mu(w^Ta)}{(1+2\mu c(a^Ta) - 2\mu(w^Tw)) - 4\mu(a^Ta)\lambda_n^2 - 2\mu(w^Ta)\lambda_n} \end{bmatrix} V^T b \tag{9.44}$$

where λ_i is the i-th eigenvalue of A

Dividing numerator and denominator by $2\mu(a^Ta)$, calling $\gamma = -\frac{w^Ta}{a^Ta} > 0$ and $K = \frac{1}{2\mu a^Ta} + c - \frac{w^Tw}{a^Ta}$ we get:

$$a^* = V \begin{bmatrix} \frac{2\lambda_1 - \gamma}{2\lambda_1^2 - \gamma\lambda_1 - K} & & \\ & \ddots & \\ & & \frac{2\lambda_n - \gamma}{2\lambda_n^2 - \gamma\lambda_n - K} \end{bmatrix} V^T b \tag{9.45}$$

Projecting on eigenvector v^i:

$$a^* v^i = \frac{2\lambda_i - \gamma}{2\lambda_i^2 - \gamma\lambda_i - K}(v^i)^T b$$

As before I want to compare the ratio of projection on different eigenvectors before and after intervention. This boils down to the study of the function

$$f(\lambda_i) = \frac{2\lambda_i^2 - \gamma\lambda_i}{2\lambda_i^2 - \gamma\lambda_i - K}$$

When $C \to 0$, $\gamma \to 0$ as $w \to 0$ and $K > 0$. This yields the result. For $K > 0$, multiply by a^T the equation of the derivative of the lagrangian and divide by $2\mu a^T a$. This gives us

$$K = a^T(2A^2 - \gamma A)(a - A^{-1}b)$$

which is >0 when γ is small enough

Chapter 2
Stopping contagion: optimal network intervention

1 Introduction

We study contagion processes on social networks. We investigate how a social planner should optimally intervene on the network structure to prevent them. Many welfare-relevant phenomena can be described as contagion processes in networks. The most studied one is epidemics, in this case we investigate which kind of prevention programs the planner should promote. Other interesting applications are the diffusion of bad rumors and fake news, or risky behaviors such as crime and smoking. We ask in all those cases which preventive measures the planner could take. [10]

We use the Susceptible-Infected-Susceptible (SIS) model from the epidemiology literature as a convenient way to address such processes.[11] Individuals are in one of two possible states: Susceptible S or Infected I. The probability that a susceptible individual becomes infected is increasing in the number of individuals he interacts with who are infected. The probability that an infected individual becomes susceptible again is exogenous and given by a parameter. This gives tractability and fits well the epidemics example. [12] The key characteristics of those diffusion processes is that my behavior or my state (sick or sound) may evolve over time, and the transition from one state to the other may depend on the states of the individuals I interact with. For instance, I am more likely to get infected by a disease if I meet individuals that are infected themselves, and the more such individuals I meet, the more likely it becomes. The biggest the number of my friends knowing about a rumor, the more probable it is that I learn it, and afterwards transmit it. In coordination games, my best response is to cooperate if my friends do so and to cheat if they do. In this environment we work on long run outcomes and study the steady-state of the system, with the interpretation that this represents the fraction of time each individual spends in the infected state over a long period of

[10] The empirical literature on peer effects shows that teenagers are more inclined to start smoking if their friends do (Robalino and Macy, 2018, [41]).

[11] see Pastor-Satorrás and Vespignagni, 2001, [39])

[12] It may become a limitation for other contagion processes. In the context of social conventions, the probability to switch from one convention to another seems to depend on the convention my friends adopted in any of the two directions. We leave the study of cases where the transition between one state to the other is symmetric across the states for future work.

time. [13]

In this classical SIS framework we ask how a social planner could intervene to prevent an outbreak. We allow the planner to decrease the probability that an infection is transmitted when a meeting with an infected individual occurs. A politician could promote a campaign to increase the use of protective measures to decrease the spread of sexually transmitted diseases. Notifying social media users about dubious sources of information could prevent the spread of fake news. All these intervention are costly, and the larger is the intervention the higher is the cost the planner has to sustain. Of course multiple interventions constitute possible way to stop the epidemic diffusion, so we ask how to reach our goal at minimal cost. In this way we try to address the question of which connections are the most important for diseases' diffusion.

The first important contribution of the paper is the characterization of the intervention of the planner (Proposition 1). We managed to have an analytical solution to the problem that does not require any limitation on the initial structure of the network. This result gives the social planner a useful guidance when deciding how to allocate resources. It is important to stress that network-nature problem like the one we face, where we have to deal with a large amount of information, are very difficult to solve and require a lot of computational power. Therefore providing a closed-form solution becomes of even larger importance. We show that mathematically, the problem corresponds to the transformation of the matrix representing the interactions across individuals into its "closest" Negative Definite matrix. We use some results from the computer science literature to solve this problem. [14] Even if the interventions would differ a lot depending on the initial structure of the network, we argue that the planner intervenes in a systematic way. She tries to decrease interactions across different communities in the population and she focus on eliminating the disease in each community separately.

If, on one side the result is unique in the literature for its generality, on the other side, there is an important limitation. The optimal intervention might result in making an interaction with an infected individual decrease the probability to spread contagion. This means that the planner's prevention program

[13] In these models the outcomes of interest are divided between short run and long run. We focus on the second one. It is to be noted that the steady-state has a different meaning when working on approximate versus exact networks. In the first case, the system is deterministic and once the steady-state is reached, the system does not move anymore. The steady-state then describes the fraction of individuals of each type (for instance their degree) in the infected state at each period. Of course some individuals change state at each periods but for each type, individuals leaving the infected state are replaced by the same quantity leaving the susceptible state, provided the population is large enough. In the case of exact networks, each individual is unique and thus the system is stochastic and its state (which is a n-dimensional vector collecting the states of each individuals for a population of size n) changes at each period. The steady-state can be seen as a measure of time spent in the Infected state.

[14] See Higham for a survey of the literature.

should not only make the probability of contagion smaller than the initial one but it should be able to promote immunization from interaction across individuals. While this would make sense if the infected state has the flavor of a rival good, the interpretation is not straightforward when we think about diseases. Aware of this limitation, in example 1 and example 2, we show that when we analyze two of the most-studied networks' structures in the literature, we should not be concerned with the problem. Furthermore, we argue that, even when the optimal intervention requires immunization to increase from some interactions, our result is useful. In fact, corollary 2 gives us the exact cost of the planner's optimal program. If we only allow her to decrease the probability of contagion from the initial one, our measure of the cost gives a useful lower bound on the total expenditure the planner needs to prevent the spread of the disease.

In section 5 we study two families of networks. First, we consider a situation where players differ in their degrees. Some players have an higher number of connections (degree) with respect to others. We compare populations with different degree distributions. We want to compare a population where all the players are similar (have the same degree) with an heterogeneous population (some players have an high degree and other a small one). Using Corollary 2 we show that it is easier for the planner to prevent the spread of the disease in the first case. In proposition 2 we extend this result showing that, for comparable networks, the easiest scenario to face for the social planner is one where all the players are connected among them. Finally, we compare populations with two communities that are more or less integrated between them. We show that the social planner's optimal strategy is to limit interactions between individuals of the two communities no matter what is the initial configuration. We argue that these interactions are crucial to the spread of the disease in the population and that is why are the target of the planner. Fighting the disease separately in the two communities is the best way for the planner to defeat the disease.

2 Literature Review

Our paper inscribes itself in the literature on diffusion processes among a population of individuals who interact with one another. [15]. There are two main modeling choices that impact the properties of those types of processes: the structure of connections between individuals and the details of how the transmission takes place from one individual to another. The SIS model has been examined under a wide range of connection structures. In the benchmark version, the pattern of interactions is not fixed and individuals have the same probability of meeting any other individual at each period. This is called the homogeneous mixing

[15] To see a full review on modeling dynamical systems on networks, see Porter and Gleeson, 2016 ([40])

assumption. A refined version of this model is one where individuals are defined by the number of connections they have, their degree. The homogeneous mixing assumption is maintained but some individuals meet more people than other. Consequently the outcome varies by type of individuals. Lopez-Pintado, 2007 ([34]) and Jackson and Rogers, 2007 ([35]) are examples of this approach. The pattern of connections is thus expressed through the degree distribution of the population, that is, the fraction of individuals having exactly d friends for all possible d. This modeling choice yields tractability but fails to grasp various important features of connection patterns, such as geography or the long-term aspect of interactions. This is why we chose to study exact arbitrary networks. In exact networks, every individual is different in principle and has his own set of links.

Regarding the transmission function, Gleeson, 2013, ([25]) compares diffusion processes under different contagion mechanisms. The probability of contagion can be an increasing function of the absolute number of my friends who are infected (SIS model), of the relative number of my friends who are infected (voter model), it can follow a majority rule, a threshold rule, or an Ising Glauber model. [16] Beyond those well-defined transmission function, Lopez-Pintado, 2007 ([34]) brings an interesting contribution to the literature by working with an unspecified function. Lopez-Pintado, 2007 varies the properties of this function and analyzes how her outcomes of interest change. As opposed to those papers, we chose the SIS model for its tractability as it allowed us to deal with more complexity on the network structure side, which is our focus.

While it is really interesting to analyze the dynamic of the SIS model, we decide to focus on long run outcomes (Steady-States). We do this for tractability and to compare our work to the economic literature. Among the papers studying the steady-state of contagion processes, we distinguish two main objects of interest. A strand of literature (like Jackson and Rogers, 2007, [35]) focuses on positive steady-states (where at least some individuals or some types have a strictly positive steady-state value) and derive properties of those steady-states. For instance, Jackson and Rogers 2007 provides comparative statics on the average level of infection as a function of the dispersion of the degree distribution representing the network. Another branch of literature explores when the steady-state is null (nobody is infected, an initial seeding of a disease dies out before spreading) versus strictly positive (an outbreak of the epidemics occurs). Lopez-Pintado, 2007 ([34]) is an example.

We belong to this last group, but as opposed to most of this literature that focuses on which characteristics of the process (ratio of the individual transmission and remission rate) allow to prevent an outbreak for

[16] See Gleeson (2013) for a discussion

different network structures, we take the parameters of the process as given and we ask which network structure reaches the zero-steady-state for those parameters. Galeotti and Rogers (2013) consider an intervention where part of the individuals get vaccination and therefore cannot be infected. Differently from them the planner in our case targets links and individuals. While Galeotti and Rogers (2013) considers a specific type of network we try to give a result that generalize to all networks' structures.

3 The Model

In this section we present the SIS model and We recall some important results from the literature on epidemic diffusion.

We study a population of n individuals located on a network. The set of player is called $\mathcal{N} = 1, \ldots, n$. The network is described by the adjacency matrix $G \in \mathcal{M}_{n,n}(\mathcal{R}^{+})$. Each element g_{ij} of G represents the intensity of the link between individual i and j. [17] [18]

We model the epidemic process in continuous time. At each time t, each individual can either be susceptible, or infected. Let $X_i(t)$ be the Bernoulli random variable that takes value 1 if node i is infected at time t and value 0 if it is not. An infected node may become susceptible at a constant rate $\delta > 0$. The infection rate of a susceptible node i is $\lambda \sum_{j=1}^{n} g_{ij} X_j(t)$. Here λ is a parameter measuring the contagiousness of the disease. The second term of the product $\sum_{j=1}^{n} g_{ij} X_j(t)$ capture the importance of the interaction with the other players. The probability of becoming infected is increasing in the probability of nodes j being infected weighted by the link that i shares with j.

Following Pastor-Satorras et al (2015) [38], we write the equations governing the evolution of the expectation of $X_i(t)$ (which is also the probability that node i is infected at time t as $X_i(t)$ is Bernoulli).

$$\frac{dE\left[X_i(t)\right]}{dt} = E\left[-\delta X_i(t) + (1 - X_i(t))\, \lambda \sum_{j=1}^{n} g_{ij} X_j(t)\right] \tag{3.1}$$

The term inside the expectation on the RHS of equation (3.1) when i is infected is equal to -δ (the recovery rate) while it is equal to $\lambda \sum_{j=1}^{n} g_{ij} X_j(t)$ (the probability of infection) when i is susceptible.

Since $X_i(t)$ is a Bernoulli we can rewrite equation (3.1) as:

$$\frac{dx_i(t)}{dt} = -\delta x_i(t) + \lambda \sum_{j=1}^{n} g_{ij} x_j(t) - \lambda \sum_{j=1}^{n} g_{ij} E\left[X_i(t) X_j(t)\right] \tag{3.2}$$

where $x_i(t)$ the probability that i is infected at time t.

[17] In the literature there are two different approaches to model networks. It is possible to impose that all connections among players have the same intensity. In this case the adjacency matrix has only entrances 0 or 1. g_{ij} is 1 when there is a link between i and j, 0 otherwise. We decide, instead, to model the network using a weighted adjacency matrix. Connections between players can vary in intensity. We believe that this approach fits well when analyzing the diffusion of diseases. This approach is also necessary for the results we obtain

[18] We limit the study to symmetric networks. Links are bidirectional. i can be affected by j with the same probability as j can be infected by i.

3.1 Discussion of the epidemic threshold

It is a well known result that equation (3.2) has always one steady-state where $x_i(t) = 0$ for all i. Sometimes a positive steady-state exists as well. A crucial goal of the epidemics literature is to determine a threshold for the diffusion parameters λ, δ such that the initial seed of the disease does not result in an outbreak, or again such that the zero-steady-state is the only one. We will recall two important results from the literature on epidemics (Lemma 1 and Lemma 2) that explain the role of the largest eigenvalue of the adjacency matrix G on the existence of an outbreak (non-zero steady state).

Epidemic threshold lower bound in the exact SIS model

Lemma 3. *A lower bound for the threshold of the epidemic process* $\frac{\lambda}{\delta}$ *in the exact version of the SIS model is* $\frac{1}{\lambda_1}$*:*

$$\frac{\lambda}{\delta} \leq \frac{1}{\lambda_1} \quad \Rightarrow \quad \textit{there is no outbreak} \tag{3.3}$$

Proof. Following Pastor-Satorras et al (2015) [38], we revisit equation (3.2) and note that for all i, $\sum_{j=1}^{n} g_{ij} X_i(t) X_j(t) \geq 0$. We can thus transform (3.2) into an inequality by removing the last term, and replacing expectations of Bernoulli random variables by their probability of success:

$$\frac{dx_i(t)}{dt} \leq -\delta x_i(t) + \lambda \sum_{j=1}^{n} g_{ij} x_j(t) \tag{3.4}$$

This inequality holds for all i. We create a system of n inequalities, $i = 1, \ldots, n$. We observe that setting inequalities to equations, the system boils down to (3.12) , whose solution has been found in (3.19). We deduce that:

$$\boldsymbol{x}(t) \leq \sum_{r=1}^{n} a_r(0) e^{(\lambda\lambda_r - \delta)t} v_r \tag{3.5}$$

The fastest growing term (as t increases) of (3.5) is the one associated with the highest positive eigenvalue λ_1. This expression shows that in order to get $\boldsymbol{x}(t)$ going to 0, we need all exponential factors to be negative, or again:

$$\frac{\lambda}{\delta} \leq \frac{1}{\lambda_1} \quad \Rightarrow \quad \text{the right hand side of (3.5) decays exponentially} \tag{3.6}$$

Therefore the inverse of the highest eigenvalue of the adjacency matrix G is a lower bound for the threshold of the epidemic process (meaning $\frac{\lambda}{\delta}$) at which the disease does not degenerate in an outbreak in the exact model.

□

This first result tells us that a sufficient condition not to have an outbreak is that the largest eigenvalue of the adjacency matrix has to be small enough with respect to the ratio $\frac{\delta}{\lambda}$. Intuitively this ratio measure how fast an individual recovers from the disease (δ) with respect to how contagious the disease is (λ). We would like the implication in (3.6) to be a double implication. While this is not always true, we show how adding and additional assumption gives us the result.

SIS-epidemic threshold under individual-based mean-field approximation (IBMF) We assume that the two random variables of neighboring nodes are uncorrelated. While it seems a strong assumption, this is the standard in the literature.[19] We will keep this assumption throughout the paper.

Assumption 4.

$$E\left[X_j(t)X_i(t)\right] \quad = \quad E\left[X_j(t)\right]E\left[X_i(t)\right] \qquad \textit{for all } t, i, j \tag{3.7}$$

Under assumption 4 it is possible to prove the following

Lemma 4. *The disease dies out and an epidemic is avoided in the IBMF approximation of the SIS model iff:*

$$\frac{\lambda}{\delta} \leq \frac{1}{\lambda_1} \tag{3.8}$$

Proof. We remind the equations governing the evolution of the expectation of $X_i(t)$:

$$\frac{dE\left[X_i(t)\right]}{dt} = E\left[-\delta X_i(t) + \left(1 - X_i(t)\right)\lambda \sum_{j=1}^{n} g_{ij} X_j(t)\right] \tag{3.9}$$

We use an individual-based mean-field approximation (IBMF), assuming that the status of neighboring nodes are independent, or again:

Using (3.7) in (3.9), and replacing the expectation of the Bernoulli variables by their probability, we get:

[19] Pastor-Satorrar et al. (2015) study the accuracy of the individual mean-field approximation for different type of networks.

$$\frac{dx_i(t)}{dt} = -\delta x_i(t) + (1 - x_i(t))\lambda \sum_{j=1}^{n} g_{ij} x_j(t) \tag{3.10}$$

We choose the initial conditions so that at $t = 0$ we have a small number c of infected individuals and everyone else is susceptible, so that $x_i(0) = c/n$. One way to get insight from (3.10) is to see that as n grows large, $1 - x_i(0)$ goes to 1, and following Pastor-Satorras et al (2015) [38], we approximate (3.10) by replacing $1 - x_i(0)$ by its limit value:

$$\frac{dx_i(t)}{dt} = -\delta x_i(t) + \lambda \sum_{j=1}^{n} g_{ij} x_j(t) \tag{3.11}$$

or in matrix form:

$$\frac{d\boldsymbol{x}(t)}{dt} = \lambda A(G)\boldsymbol{x}(t) \tag{3.12}$$

with $\boldsymbol{x}(t)$ the vector of the $\{x_i(t)\}_i$ and:

$$A(G) = G - \frac{\delta}{\lambda} I \tag{3.13}$$

To find a solution to the system of differential equations in (3.12), we decompose $\boldsymbol{x}(t)$ on the orthonormal basis composed of the eigenvectors of A (which are the same as the eigenvectors of G) that we call $\{v_r\}_{r=1}^{n}$ where v_r is the eigenvector associated with λ_r, the r-th eigenvalue of G, where eigenvalues are ranked in decreasing order. We call $a_r(t)$ the coefficient of the decomposition associated to v_r:

$$\boldsymbol{x}(t) = \sum_{r=1}^{n} a_r(t) v_r \tag{3.14}$$

Differentiating this expression we get:

$$\frac{d\boldsymbol{x}(t)}{dt} = \sum_{r=1}^{n} \frac{a_r(t)}{dt} v_r$$

Combining the previous equation with (3.12):

$$\sum_{r=1}^{n} \frac{da_r(t)}{dt} v_r = \lambda A \boldsymbol{x}(t) = \lambda A \sum_{r=1}^{n} a_r(t) v_r = \lambda \sum_{r=1}^{n} a_r(t) A v_r \tag{3.15}$$

and finally, using the fact that λ_r is the eigenvalue of G associated with the eigenvector v_r and that from

(3.13) we can write the r-th eigenvalue of A as $\lambda_r - \frac{\delta}{\lambda}$:

$$\sum_{r=1}^{n} \frac{da_r(t)}{dt} v_r = \lambda \sum_{r=1}^{n} a_r(t) \left(\lambda_r - \frac{\delta}{\lambda} \right) v_r \tag{3.16}$$

As $\{v_r\}_{r=1}^n$ constitutes a basis of $\mathbb{R}^n$, the decomposition of any vector on it is unique and thus, $\forall r, t$:

$$\frac{da_r(t)}{dt} = a_r(t) \left(\lambda \lambda_r - \delta \right) \tag{3.17}$$

For each $, r, t$, the above differential equation as for solution:

$$a_r(t) = a_r(0) e^{(\lambda \lambda_r - \delta)t} \tag{3.18}$$

Finally, plugging (3.19) into the decomposition (3.14), we get:

$$\boldsymbol{x}(t) = \sum_{r=1}^{n} a_r(0) e^{(\lambda \lambda_r - \delta)t} v_r \tag{3.19}$$

The same argument as in the previous proof tells us that:

$$\lambda_1 \leq \frac{\delta}{\lambda} \tag{3.20}$$

□

4 Optimal immunization

We determined in the previous section the relationship between the largest eigenvalue of G and the epidemic threshold under different aproximations. The inverse of the highest eigenvalue of G, $\frac{1}{\lambda_1}$ is the exact epidemic threshold in the IBMF approximation, and the upper bound of this threshold if we remove assumption 4.

In this section we model the intervention of the planner and we state the main result of the paper.

We assume that the social planner can intervene to change the structure of the network determining the diffusion of the disease. If $g_{i,j}$ is the weight that determine the probability that i is infected from j the planner can invest resources to to lower this probability to $g'_{i,j} < g_{i,j}$. We make the following assumption about the cost she incurs to alter the network:

Assumption 5. *Changing the structure of the network from G to G^* has a cost of $||G^* - G||_F$, where $||M||_F = \sqrt{\sum_{ij} m_{ij}^2}$ is the Frobenius norm of M.*

The previous assumption captures the idea that the social planner can modify the interaction between i and j at a convex cost. The planner faces increasing marginal costs of intervention. [20].

Once we established a metric on the cost of intervention of the planner we can ask what is the least costly intervention that prevent the diffusion of the disease in the population. As we discussed in the previous section this is equivalent to impose conditions so that in steady state all the variable measuring infection are equal to zero. Lemma 2 gives us a good starting point to solve the problem of the planner. Having the highest eigenvalue of G less or equal to $\frac{\delta}{\lambda}$ is equivalent to having the matrix $A(G)$ (defined in (3.13)) be semi-definite negative. As a result we have that:

Corollary 4. *The least costly intervention for the planner is the one that change the network from G to G', where G' is the solution to:*

$$min\{||A(G) - A(G')||, such\ that\ A(G')\ is\ negative\ semi-definite\}$$

This corollary enables us to resort to a famous result of the computer science literature: we use the theorem 2.1 of Higham(1998)[26] to find the nearest symmetric negative semi-definite matrix of $A(G)$, and to see how to perform this manipulation. Higham(1998) [26] also gives us a closed form solution for the cost of intervening on any given network structure, as a function of $\frac{\lambda}{\delta}$.

In order to state the theorem, we remind that $A(G)$ and G have the same eigenvectors $\{v_r\}_{r=1}^n$. We call $\{\mu_r\}_{r=1}^n$ the n eigenvalues of $A(G)$, μ_r being associated with the eigenvector v_r. They can be expressed as a function of the eigenvalues of G in the following way:

$$\mu_r = \lambda_r - \frac{\delta}{\lambda} \tag{4.1}$$

The following theorem specify the optimal intervention of the planner:

Theorem 3. *The planner intervenes on the network changing $A(G)$ to $\tilde{A}(G)$, where:*

[20]The Frobenius norm can be seen as an extension of the euclidean norm to $\mathcal{R}^{n\times n}$

$$\tilde{A}(G) = V\Delta V^T \tag{4.2}$$

with V the matrix whose r-th column is the the r-th eigenvector of G, v_r, and where Δ is the following diagonal matrix:

$$\Delta = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

with

$$d_r = \begin{cases} \mu_r & \textit{if } \lambda_r < \frac{\delta}{\lambda} \\ 0 & \textit{otherwise} \end{cases}$$

The cost to reach it from A is:

$$||A - \tilde{A}||_F^2 = \sum_{\lambda_r > \frac{\delta}{\lambda}} \left(\lambda_r - \frac{\delta}{\lambda}\right)^2 \tag{4.3}$$

Proof. The result directly follows from theorem 2.1 of Higham(1998)[26] and its proof. □

We can restate the theorem in term of the new network G instead of $A(G)$ to have a better interpretation of the result

Corollary 5. *The closest network structure $\tilde{G}(G)$ of G such that no outbreak occurs is:*

$$\tilde{G}(G) = V\Delta V^T + \frac{\delta}{\lambda} I = V\Lambda V^T \tag{4.4}$$

with

$$\Lambda = \begin{bmatrix} \tilde{\lambda}_1 & & \\ & \ddots & \\ & & \tilde{\lambda}_n \end{bmatrix}$$

and

$$\tilde{\lambda}_r = \begin{cases} \lambda_r & if\ \lambda_r < \frac{\delta}{\lambda} \\ \frac{\delta}{\lambda} & otherwise \end{cases}$$

We are changing the all the eigenvalues of the matrix G that exceed the ratio $\frac{\delta}{\lambda}$, while leaving unchanged the ones below this ratio. All the eigenvectors of the original adjacency matrix remained unchanged.The first comment on the property of the new network structure regards what is known in the network literature as the spectral gap:

Definition 7. *The spectral gap is the difference between the largest and the second largest eigenvalues of the adjacency matrix.*

A straightforward application of 5 is:

Corollary 6. *If*

$$\lambda_2 \geq \frac{\delta}{\lambda}$$

then the spectral gap of the transformed network structure is 0.

This is interesting because the spectral gap has an interpretation in terms of network structure. A strictly positive spectral gap corresponds to a network with only one component. A small but positive spectral gap corresponds to a network with at least two communities with many within-group links and few between-group links. After intervention and under the condition stated in proposition 6 the spectral gap will close to zero. The new network will be characterized by communities that do not share links across them. The social planner isolate players into separate communities and then reduces the spread of the disease inside each community until elimination.

It is important to note that Theorem 1 also gives us the cost of the transformation, as a simple function of λ, δ, and the eigenvalues of G. It means that given two network structures and an epidemic threshold, we directly have a number enabling us to rank those two networks in terms of how costly it would be for the social planner to intervene. It can also be interpreted as how close each network is from a structure that prevents an epidemic outbreak. In the next section we will use this information to do comparative statics on different networks.

5 Applications

We discuss limitations and benefits of our method, and illustrate them in three examples hereafter.

When applying the intervention described in the previous section, we see that the post-intervention matrix described in equation (4.4) may have negative entries. Our method requires to make the matrix $A(G) = G - \frac{\delta}{\lambda}$ semi-definite negative. There is not a condition that preclude the new matrix (after intervention) to exhibit negative entries. In the epidemics framework, a link with negative intensity between two individuals is translated into a decreased probability to be infected for one of the individuals when the other becomes infected. This interpretation is difficult to justify in some of the applications we described.

When the minimal-cost intervention we presented make some entries of the modified network negative, our theorem does not give information about other interventions to prevent epidemics diffusion. The theorem 2.1 of Higham (1988) [26] only addresses minimization problems that do not apply constraints on the outcome matrix properties. We therefore regard our result as a lower bound of the cost of intervention. The problem that the planner has to solve is the same one but with the additional constraint of having the post-intervention matrix with non-negative entries only. When this additional constraint is not binding, meaning that the matrix $\tilde{G}$ defined in (4.4) has no negative entries, theorem 3 hits its full potential. It tells us what is the intervention that corresponds to the minimal-cost intervention, and gives us a simple formula of the cost of the intervention, allowing for quick comparative statics. We give hereafter examples where this lower bound for the cost of intervention is reached.

5.1 Example 1

5.1.1 Initial structure

We consider a population of n individuals who differ in their total intensity of interaction (or equivalently their probability of spreading the disease). Each individual i is characterized by a coefficient c_i measuring his propensity of interaction. c_i is drawn from a distribution $\mathcal{C}$ with support $\mathcal{R}^+$. We assume random mixing , that is the probability of having an interaction with an individual j is proportional to j's propensity of interaction. The strength of the link between individuals i and j is therefore:

$$g_{ij} = c_i c_j$$

Two individuals are very likely to have an interaction if they both have an propensity of interaction. A different interpretation is that if c_i measures the contagiousness of individual i it is likely that an interaction with him will result in an high probability of infection. The network framework described in this example is the intensity counterpart of the degree distribution framework where links are non-weighted, 0 or 1, but exist with a probability depending on the degree of two nodes. If nodes i and j have degree d_i and d_j respectively, the probability that they are linked is proportional to $d_i d_j$. In our example, all nodes are linked, and the weight of the link between i and j is $c_i c_j$.

We will try to determine the optimal intervention of the planner and to compare the cost of intervention for different distributions. In order to that, we will compute the eigenvalues and eigenvectors of the adjacency matrix to apply Theorem 1.

Eigenvalues and eigenvectors of G The only positive eigenvalue of G is λ_1. For $i = 2, \ldots, n$, $\lambda_i = 0$. The eigenvector associated to λ_1 is v_1. We have:

$$\lambda_1 = c_1^2 + c_2^2 + ... + c_n^2$$

and

$$v_1 = \begin{bmatrix} \dfrac{c_1}{\sqrt{c_1^2 + c_2^2 + ... + c_n^2}} \\ \dfrac{c_2}{\sqrt{c_1^2 + c_2^2 + ... + c_n^2}} \\ ... \\ \dfrac{c_n}{\sqrt{c_1^2 + c_2^2 + ... + c_n^2}} \end{bmatrix}$$

5.1.2 Post-intervention structure

Applying theorem 1 we can see that order to prevent an epidemics, the planner transforms the structure from G to $\tilde{G}$:

$$\tilde{G} = \frac{\delta}{c_1^2 + c_2^2 + ... + c_n^2} G$$

This means that each link is affected proportionally to its initial intensity.

5.1.3 Comparative statics of the cost of intervention

We wish to derive comparative statics with respect to the distribution $\mathcal{C}$ which determines the intensities $\{c_i\}_i$.

We are interested in comparative statics with respect to the variance of the $\{c_i\}_i$, as it is trivial that increasing the average of $\mathcal{C}$ increases the diffusion of the disease and thus the cost of the intervention. The idea is to compare a network where individuals are similar in their propensity to interaction to one where individuals are instead heterogeneous.

In order to neutralize the effect of the mean, we compare mean-preserving spreads of $\mathcal{C}$. Given the weight of each c_i is one, the mean-preserving spread will express itself through the support of the $\{c_i\}_i$.

Theorem 3 gives us the cost of the intervention:

$$||\tilde{G} - G||_F^2 = \lambda_1 - \frac{\delta}{\lambda} = c_1^2 + c_2^2 + ... + c_n^2$$

We see that by taking a mean-preserving spread of the initial distribution of $\{c_i\}_i$, we increase the cost of intervention.

The less spread the distribution $\mathcal{C}$ is, the easier it is for the planner to intervene on the network. Another interpretation is that as links intensities are more homogeneous, it is unambiguously easier to intervene to prevent epidemics.

Comparison with Jackson, Rogers (2007) [36] We wish to draw a parallel with proposition 2 of Jackson, Rogers (2007) [36] (hereafter JR), which states that when the epidemic characteristics $\frac{\lambda}{\delta}$ is low enough, a mean-preserving spread of the degree distribution of the network yields higher average infection. This result echoes ours that by spreading the intensities, it becomes more difficult to immunize the population (in terms of higher cost of intervention). The comparison is delicate however because of the following two reasons:

- Both JR and we make comparative statics with respect to spreads of the degree distribution of the network. The object of interest is different though: they show results on the average level of infection in the (positive) steady-state, while we provide the cost of reaching a zero steady state level from an initial network that exhibits a positive level of infection. A parallel may thus be drawn between lowering the steady-state level of infection (in JR) and lowering the cost of reaching a zero-steady state (in this paper). Even though it intuitively makes sense, we don't have the proof of it.

- Proposition 2 of Jackson, Rogers (2007) [36] has two elements. It states the existence of two thresholds $\underline{\lambda}$ and $\bar{\lambda}$. When $\frac{\lambda}{\delta}$ is low enough ($\frac{\lambda}{\delta} < \underline{\lambda}$), a spread in the degree distribution yields *higher average infection*. But when $\frac{\lambda}{\delta}$ is high enough ($\frac{\lambda}{\delta} > \bar{\lambda}$), a spread in the degree distribution yields *lower average infection*. In the first case, the behavior of JR's model and ours is comparable (comparable in the sense defined in the previous element). Taking a mean-preserving spread of the degree distribution increases the average level of infection in JR, and increases our cost of intervention. This statement naturally raises the question of why we focus on the first case. Our answer is the following: it makes sense to think that what matters for total eradication is the case where $\frac{\lambda}{\delta}$ is low enough and we have a low average infection, that is the first case of JR. However this claim disregards the fact that we don't know the dynamics of the average infection rate if we were to progressively increase (or decrease) the spread of the degree distribution. This matters if our initial starting point places us in the case where $\frac{\lambda}{\delta} > \bar{\lambda}$.

Hypothetical dynamics when increasing the spread in Jackson, Rogers (2007) [36] We start from $\frac{\lambda}{\delta} > \bar{\lambda}$. Figure 3 is a picture of the $\frac{\lambda}{\delta}$ with respect to the thresholds:

Figure 3: Thresholds for the cases in proposition 2, Jackson, Rogers (2007)

In this case of JR, an increase in the spread of the degree distribution decreases average infection. To understand why a starting point at $\frac{\lambda}{\delta} > \bar{\lambda}$ is delicate, we refer to the proof of proposition 2 of JR. They resort to an intermediary variable, that they call θ, which is the average level of infection taken with respect to a transformation of the degree distribution. The existence of two opposite reactions to a mean-preserving spread comes from the fact that this variable θ is always increasing with a mean-preserving spread, but the relationship between the actual average infection (that they call ρ) and θ is hump-shaped (see figure 4. The road towards the zero steady state from $\theta = \theta_1$ (on the same figure 4) consists of continuously decreasing the mean-preserving spread, and thus θ, even though the average steady-state ρ (for instance, $\rho_2 > \rho_1$) increases first before decreasing. It is especially difficult to understand in which case one stands as when passing from $\theta = \theta_2$ to $\theta = \theta_3$, one switches from the high zone to the low zone, but does not realize by observing the relationship between θ and ρ (note that $\rho_3 > \rho_2$ even though we are now in the zone where ρ and θ co-move). Therefore we do not share the ambiguity of the role of the mean-preserving spread present in JR paper.

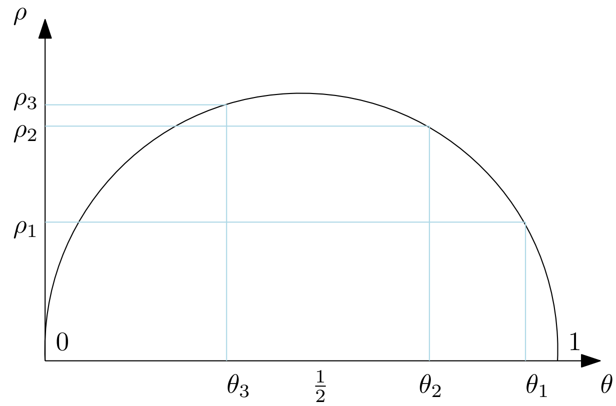

Figure 4: Relationship between average infection ρ and θ while changing the degree distribution, at fixed λ, in Jackson, Rogers (2007)

Insights from the comparison This potentially cyclical relationship between spread and average infection limits the comparison between Jackson, Rogers' result and ours. Yet we refer their result as it shed lights on one possible mechanism for our finding. In their paper, they claim that a spread in degree distribution boosts average infection because when the exogenous contagiousness is not favorable to the disease (low $\frac{\lambda}{\delta}$), it is crucial to have very high degree nodes that serve as conductors of the disease, otherwise the epidemics would die out. If this behavior is the last resort of the disease before vanishing, it makes sense that the planner prevents it by containing the structural inequality in the network structure.

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5.2 Example 2

5.2.1 Initial structure

We consider a population of n agents divided into two groups of equal size: agents $1 ... \frac{n}{2}$ belong to group 1 and agents $\frac{n}{2}+1,...,n$ belong to group 2. Intragroup links have strenght $1-\epsilon$, intergroup links have strength ϵ, with $\epsilon \leq 1/2$. Therefore links are more intense within than between groups. Such a network is represented by the following adjacency matrix:

$$G = \begin{bmatrix} (1-\epsilon) & (1-\epsilon) & ... & (1-\epsilon) & \epsilon & ... & \epsilon & \epsilon \\ (1-\epsilon) & (1-\epsilon) & ... & (1-\epsilon) & \epsilon & ... & \epsilon & \epsilon \\ & & \ddots & & & & & \\ (1-\epsilon) & (1-\epsilon) & ... & (1-\epsilon) & \epsilon & ... & \epsilon & \epsilon \\ \epsilon & \epsilon & ... & \epsilon & (1-\epsilon) & ... & (1-\epsilon) & (1-\epsilon) \\ & & & & & \ddots & & \\ \epsilon & \epsilon & ... & \epsilon & (1-\epsilon) & ... & (1-\epsilon) & (1-\epsilon) \end{bmatrix} \tag{5.1}$$

ϵ represents the strength of inequality between the two types of links: intragroup and intergroup links. As ϵ grows from 0 to $\frac{1}{2}$, the total intensity of links does not move, but the distribution of this intensity does, shifting from maximum heterogeneity (1 versus 0) to total homogeneity ($\frac{1}{2}$ for all links). Note that by restricting ϵ to be less or equal to $\frac{1}{2}$, we focus on positive assortative matching, and do not consider negative assortative matching (where individuals are more linked with members of the other group than with members of their own).

Eigenvalues and eigenvectors of G The matrix G has two positive eigenvalues:

$$\lambda_1 = \frac{n}{2}, \quad \lambda_2 = (1-2\epsilon)\frac{n}{2}, \quad \lambda_3 = \ldots \lambda_n = 0$$

The associate eigenvectors are:

$$v_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ ... \\ 1 \\ 1 \\ ... \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ ... \\ 1 \\ -1 \\ ... \\ -1 \end{bmatrix}$$

The sign of the elements in v_2 depends on the group of each individual. The $\frac{n}{2}$ elements corresponding to individuals of the first group are positive, the $\frac{n}{2}$ elements corresponding to individuals of the second groups are negative.

5.2.2 Post-intervention structure

The post-intervention structure $\tilde{G}$ is:

$$\tilde{G} = \begin{bmatrix} \frac{\delta}{\lambda}\frac{2}{n} & \dots & \frac{\delta}{\lambda}\frac{2}{n} & 0 & \dots & 0 \\ \dots & & \dots & & & \\ \frac{\delta}{\lambda}\frac{2}{n} & \dots & \frac{\delta}{\lambda}\frac{2}{n} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\delta}{\lambda}\frac{2}{n} & \dots & \frac{\delta}{\lambda}\frac{2}{n} \\ & & & \dots & & \dots \\ 0 & \dots & 0 & \frac{\delta}{\lambda}\frac{2}{n} & \dots & \frac{\delta}{\lambda}\frac{2}{n} \end{bmatrix}$$

The first thing to note is that the outcome structure does not depend on ϵ. Regardless of the value of ϵ, the planner removes all the intergroup links, and set all the intragroup links to $\frac{\delta}{\lambda}\frac{2}{n}$.

5.2.3 Comparative statics of the cost of intervention

We want to study the behavior of the cost of intervention as a function of the inequality of relationships between intra and intergroup, ϵ. Theorem 3 gives us the exact value of the cost K of the intervention:

$$K = \sum_{\lambda_r > \frac{\delta}{\lambda}} \left(\lambda_r - \frac{\delta}{\lambda}\right)^2 = \left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2 \mathbf{1}_{\{\frac{n}{2} > \frac{\delta}{\lambda}\}} + \left((1-2\epsilon)\frac{n}{2} - \frac{\delta}{\lambda}\right)^2 \mathbf{1}_{\{(1-2\epsilon)\frac{n}{2} > \frac{\delta}{\lambda}\}}$$

We can rewrite the cost as a function of n:

$$K = \begin{cases} 0 & \text{if} \quad n < 2\frac{\delta}{\lambda} \quad \text{(no eigenvalue is lowered)} \\ \left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2 & \text{if} \quad 2\frac{\delta}{\lambda} \leq n < 2\frac{\delta}{\lambda}\frac{1}{1-2\epsilon} \quad \text{(one eigenvalue is lowered)} \\ \left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2 + \left((1-2\epsilon)\frac{n}{2} - \frac{\delta}{\lambda}\right)^2 & \text{if} \quad 2\frac{\delta}{\lambda}\frac{1}{1-2\epsilon} \leq n \quad \text{(two eigenvalues are lowered)} \end{cases} \tag{5.2}$$

Let us focus on the case where there is intervention (that is $n > 2\frac{\delta}{\lambda}$). We call $\bar{\epsilon}$ the following threshold for ϵ:

$$\bar{\epsilon} = \frac{1}{2} - \frac{\delta}{\lambda n}$$

As a direct application of Theorem 1 we have:[21]

[21] The decomposition of the support of ϵ makes sense as $\bar{\epsilon}$ is positive. For any ϵ bigger than $\bar{\epsilon}$, the cost of intervention remains at $\left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2$ as the second eigenvalue remains under the threshold $\frac{\delta}{\lambda}$ and the intervention concentrates on lowering the highest eigenvalue λ_1, which is independent of ϵ. However on $[0, \bar{\epsilon}]$, the second eigenvalue is beyond the threshold $\frac{\delta}{\lambda}$, decreases with ϵ

- For $\epsilon \in [0, \bar{\epsilon}]$, the cost of intervention strictly decreases with ϵ, from $\left((1-2\epsilon)\frac{n}{2} - \frac{\delta}{\lambda}\right)^2$ to $\left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2$
- For $\epsilon > \bar{\epsilon}$, the cost of intervention is constant as a function of ϵ, at $\left(\frac{n}{2} - \frac{\delta}{\lambda}\right)^2$

As a whole, the cost of intervention for the planner is weakly decreasing in ϵ (strictly decreasing for $\epsilon < \bar{\epsilon}$ and then stable). The more isolated the two communities are, the more difficult it is to eliminate the disease in this setup. One potential explanation would be that decreasing the diffusion in one group has spillovers on the other group, reducing the spread of the disease there too. The more linked the communities are ex-ante, the bigger this effect is. This can seems contradictory to the ex-post structure described in (5.4) where the intergroup links intensity is lowered from ϵ to 0. Our result shows that the extra cost of intervention on intergroup links resulting from an increase in ϵ is more than compensated by the extra saving made on intragroup links. We can see it analytically, by computing the derivative of the cost with respect to ϵ. In order to separate the two effects, we rewrite the costs under another form, directly coming from the formula of the Frobenius norm for $||G - \tilde{G}||_F^2$. There is the same number of links of each type, $\frac{n^2}{2}$, we can thus compare the change of cost per link with respect to a change in ϵ:

- Marginal cost of intragroup link change: 2ϵ
- Marginal cost of intergroup link change: $2\epsilon - 2\left(1 - \frac{\delta}{\lambda}\frac{2}{n}\right)$

Therefore, the total marginal cost (divided by the number of links of each type) is:

$$4\epsilon - 2\left(1 - \frac{\delta}{\lambda}\frac{2}{n}\right) \tag{5.3}$$

which is strictly negative for $\epsilon < \bar{\epsilon}$. The cost savings on the intergroup links more than compensate for the extra cost on intragroups links when we increase ϵ.

We can compare this effect of ϵ on the cost of eradicating the disease with Galeotti, Rogers (2013) [23]. They intervene at the group level, as in this example. They consider an intervention where part of the individuals get vaccination and therefore cannot be infected, as opposed to our intervention that targets links. They find that, under positive assortative matching (our setup), the planner should spread its immunization effort equally across both groups (Proposition 2). Our result is compatible with theirs. The planner acts symmetrically with regard to groups. However their cost of immunization necessary to eradicate the disease does not depend on the relative weight of intragroup and intergroup links. It would be

and consequently lowers the cost of intervention.

interesting to further study the difference between the two methods to understand whether passing from a n dimension intervention to a n^2 dimension intervention grants substantial benefits.

5.3 Example 3: homogeneous versus heterogeneous networks

We want to generalize what we found in the previous two examples. In both cases, in fact, it results that for the planner it is more difficult to intervene when the population is heterogeneous in terms of the link structure.

We define homogeneous networks in the following way:

Definition 8. *Let $G \in \mathcal{M}_{n,n}$ be the adjacency matrix of a network. If there exists a number $x \in \mathbb{R}$, $x \geq 0$ such that:*

$$g_{ij} = x \quad \textit{for all } i, j$$

then the network is said to be homogeneous.

The network defined in Definition 2 is one where all the individuals are equally connected to the others. We want to compare this network with one where the population is instead heterogenous.

The next proposition tell us that it is always cheaper to intervene in an homogeneous network:

Proposition 5. *Consider two networks where the sum of the intensity of all connections is equal to 1. The first network is homogeneous, the second is not. The cost of intervention in the first network is strictly smaller than the cost in the second.*

Proof. We can associate to the adjacency matrix of any homogeneous network a corresponding stochastic matrix and use the theory of Markov chains to derive some insights on the planner's intervention. (see Levin et al., 2006, [32])

When the sum of elements of each column of the matrix is constant and equal to 1, the largest eigenvalue is equal to 1 and the corresponding eigenvector is $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$.

The corresponding adjacency matrix is given by:

$$G = \lambda_1^* \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ & & \ddots & \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} \tag{5.4}$$

where λ_1^* si the eigenvalue of G. It can be rewritten as:

$$\lambda_1^* \frac{1}{\sqrt{n}} (\frac{1}{\sqrt{n}})^T$$

The adjacency matrix of a generic network can be written, using the singular value decomposition, as:

$$\lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + ... + \lambda_n v_n v_n^T$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the adjacency matrix and $v_1, v_2, ..., v_n$ are the corresponding eigenvectors.

We can compute the total number of connections in a network summing all the entries of the adjacency matrix. The number of links for the homogeneous network is:

$$\lambda_1^* (\frac{1}{\sqrt{n}})^T (\frac{1}{\sqrt{n}})$$

Similarly, summing all the entries of the matrices obtained from the eigendecomposition (weighted by the corresponding eigenvalues) we obtained that the number of links for the non-homogeneous network is:

$$\lambda_1 (v_1^T (\frac{1}{\sqrt{n}}))^2 + \lambda_2 (v_2^T (\frac{1}{\sqrt{n}}))^2 + ... + \lambda_n (v_n^T (\frac{1}{\sqrt{n}}))^2$$

Given that we want to compare two network with the same number of links we ask that the two previous expressions are the same:

$$\lambda_1^* = \lambda_1^* (\frac{1}{\sqrt{n}})^T (\frac{1}{\sqrt{n}}) = \lambda_1 (v_1^T (\frac{1}{\sqrt{n}}))^2 + \lambda_2 (v_2^T (\frac{1}{\sqrt{n}}))^2 + ... + \lambda_n (v_n^T (\frac{1}{\sqrt{n}}))^2$$

We note that the dot product $v_i^T (\frac{1}{\sqrt{n}})$ can be written as $||v_i|| * ||\frac{1}{\sqrt{n}}|| cos\theta_i$ where θ_i is the angle between the vector $\sqrt{n}$ and eigenvector v_i. Hence, we can express the RHS of the previous equation as:

$$\lambda_1||v_1||^2 * ||\frac{1}{\sqrt{n}}||^2 cos^2\theta_1 + \lambda_2||v_2||^2 * ||\frac{1}{\sqrt{n}}||^2 cos^2\theta_2 + ... + \lambda_n||v_n||^2 * ||\frac{1}{\sqrt{n}}||^2 cos^2\theta_n =$$

$$= \lambda_1 cos^2\theta_1 + \lambda_2 cos^2\theta_2 + ... + \lambda_n cos^2\theta_n$$

whre the last equality comes from the fact that the eigenvectors have length 1 (from the singular value decomposition). Note also that $cos^2\theta_i \in [0.1]$. Finally observe that $cos^2\theta_1 + cos^2\theta_2 + ... + cos^2\theta_n = 1$, being $v_1, v_2, ..., v_n$ an orthonormal basis for $\mathbb{R}^n$

Therefore we know that λ_1^* can be written as a convex combination of the eigenvalues of the non-homogeneous network. This implies that at least one of the eigenvalue of the non-homogeneous network is larger than λ_1^*. As a direct consequence of Theorem 1, we know that the cost of intervention for the non-homogeneous network must be bigger. □

6 Conclusion

We investigate contagion processes among a networked population. We use results from linear algebra applied to computer science to find how to prevent contagion from an initial seed. For a given diffusion process and an arbitrary given initial network, we give the closest network to the initial one such that no outbreak occurs. We provide intuition on what the intervention on the network structure looks like by analyzing relevant examples of connection patterns. We discuss the limitations of our result together with the potential for future research.

Chapter 3

Does intragroup cooperation crowd-out intergroup cooperation?

1 Introduction

We are interested in the intersection between cooperation and Groups relationships. We define Groups are large anonymous entities where members do not know the identity of all the other members. However, Group membership is observable. We can think of religious or immigrants communities as examples. Groups are the ideal place where cooperation can occur. If Group members join their forces, they can get higher payoffs whenever a game is cooperative and there is a commitment device helping them not to deviate from the Group utility maximizing strategies. However such a commitment device is usually a strong assumption we are not willing to make. Without commitment device, Groups are confronted to the free-riding problem, where agents have a profitable deviation from the Group utility maximizing strategies, leading to a suboptimal outcome. Nevertheless, it is often seen that some Groups manage to reach cooperative outcomes, depending on various characteristics (their size, heterogeneity, the threat of outside forces...). Olson (1965) [37] pioneers the study of Groups confronted to the free-riding problem and discusses the role of the size of the Group on the ability to sustain cooperation within the Group. Levine and Modica (2013) [33] also investigate how within group incentives vary with the size of Groups and find a non-monotonic relationship, in a peer discipline set-up close to ours.

One particular characteristic influencing cooperation within the Group is the behavior of the outside environment and we want to further on it. Groups do not live in autarky and group members have to interact with individuals from both the In-Group and the Out-Group. Many experimental evidence investigates the relationship between Out-Group behaviors towards the In-Group and cooperative outcomes inside the Group. We can distinguish two types of Out-Group behaviors: unilateral Out-Group hostility towards the In-Group, or bilateral relationship between the In-Group and the Out-Group. As our definition of Group is not geographical, we consider that Groups overlap and that actions go both ways. Also, we focus on the part of the Out-Group made up of members of other Groups, as opposed to members of no Groups, as we want to understand the dynamics of intergroup relationships.

Empirical and experimental evidence show increased In-Group cooperation when rising the stakes of intergroup conflict. Shayo, Zussman (2011) [42] find evidence of judicial In-Group bias in Israel, the more the stronger the neighboring conflict. Charness et al. (2007) [17] and Tan, Bolle (2007) [43] find a causal link from intergroup conflict to intragroup cooperation in experimental games. In reality, we often observe correlation between intergroup hostility and intragroup cooperation: when two communities are at war, more powerful solidarity mechanisms within groups mean more soldiers, more contributions, more weapon, thus more conflict. It is thus legitimate to ask whether there is also a reverse causality and whether intragroup cooperation crowds out intergroup cooperation.

However, we do not want to talk about the mechanical crowding-out that would occur when agents face a resource constraint (Bramoulle, Goyal, (2014), [10]) . Of course in this case one can either give a unit of resource inside or outside, and total crowding-out will be observed. We rather want to investigate how the mechanisms of intragroup cooperation impact intergroup cooperation. We thus open the black box of intragroup cooperation and develop a model where cooperation is sustained by peer punishment, based on Kandori (1992) [29]: agents may cooperate today with the agent they are matched with because they hope to benefit from another agent that will cooperate with them tomorrow. The system is equipped with a local information device which helps agents determine whether their match cooperated or not yesterday, and thus whether to cooperate or punish today.

We introduce the Kandori local information device in a simple overlapping generation gift-giving game (inspired from Johnson, Levine, Pesendorfer (2001) [28]), with that particularity that society is divided into two Groups and that matches can be realized within or across Groups. Thus Groups are only a label. They are exogenous and symmetric in any characteristics. But the local information device performs differently according to whether matches are within or across Groups. In other words the monitoring technology is unambiguously more efficient within Groups than across Groups. We can easily imagine that it is the case in reality: agents have usually more information about the In-Group than the Out-Group, or is better at distinguishing guilty In-Group members because one knows correlates of guilt or innocence when regarding In-Group members. Furthermore, this is the interesting case because we could expect that cooperation would only occur where information is better, whereas we sees in our model that it is not always the case.

There is on path punishment first because monitoring technologies are imperfect and may fail to transmit the information that cooperation has previously taken place. But punishment also rightly takes place as a proportion of agents, called the guilty players, are not complying with the social norm of cooperation. Agents are identical except in the dimension of their idiosyncratic cost of cooperation, bringing about high-cost players to deviate and low-cost players to cooperate (called Innocent players). This heterogeneity in cooperation is what brings action in the model through the proportion of In-Group and Out-Group which becomes an endogenous variable.

We make another significant change: we introduce imperfect directed search into the matching process, in order to link cooperation decisions to the Group or your matching partner. This leads to two types of equilibrium. First a Non-Sorting Equilibrium where both innocent and guilty players prefer to be matched in the In-Group (and direct their search accordingly), where more cooperation is sustained thanks to the better monitoring technology. Second, a Sorting Equilibrium sees Innocent players still targeting the In-Group, but Guilty players now preferring to be matched outside, as the weaker monitoring technology makes it less likely to be recognized as guilty. We then compare their welfare properties for Innocent and for Guilty players.

For The question boils down to whether agents prefer a matching pool with guilty players in majority from the In-Group or from the Out-Group.

The paper proceeds as follows: section 2 briefly discusses related literature. Section 3 presents the model. Section 4 lays out players incentives. Section 5 summarizes the typology of equilibria this game possesses. Section 6 discusses results on crowding-out for Non Sorting equilibria, whereas section 7 compares outcomes for Sorting versus Non Sorting equilibria. Section 8 concludes.

2 Literature Review

We locate the paper at the intersection of the literature on non cooperative repeated games sustaining cooperation through social norms and on matching. The difference between our paper and the first type of literature has already been discussed earlier. Regarding the second one, we want to single out the paper by Eeckhout (2005) [20] which has a similar flavor: society is divided into two groups, matched players play a non-cooperative game, matching is endogenous as a consequence of the decisions made during the first stage of the game, with segregation or integration as potential outcomes. However, the separation between the two groups is made along the lines of a payoff-irrelevant characteristic, whereas in this paper being matched inside or outside the group matters int terms of efficiency of the monitoring technology. Furthermore, Eeckhout deprive agents of any information beyond this payoff-irrelevant information, whereas we are interested in situations where local information devices make monitoring possible.

Our paper is also related to the literature investigating substitution between formal versus informal arrangements, formal arrangements being the parallel of our notion of In-Group, and informal ones comparing with the Out-Group. Kranton (1996) [31] presents a set-up where substitution between the market and informal exchanges arise,the two modes competing for providing consumption goods to the agents. Acemoglu and Wolitsky (2015) [1] compare formal and informal monitoring, and establish that it is optimal to have only one form of monitoring. In this sense formal and informal monitoring crowd each other out. The main difference between those two papers and ours is that here In-Group and Out-Group cooperation are identical, be it for the identity of the group, whereas in Kranton (1996) and Acemoglu and Wolitsky (2015), the nature of the formal and informal arrangements are different. We want to see under which conditions crowding-out arise in a set-up where the In-Group and the Out-Group provide an identical service.

3 The Model

We study a community made up of 2 Groups $g \in G = \{1; 2\}$. Time is discrete $t = 0, 1, ...$ and agents live 2 periods. They are of the type Young in the first period of their life and of the type Old in the second, and last, period of their life. We consider an overlapping generation model where, each period t, population is made up of N Young just born at period t and N Old born at period $t-1$ and dying at the end of the period.Therefore population remains constant of size $2N$. Additionally we assume that it is equally split across Groups, each gathering N members $\left(\frac{N}{2}\text{ Young and }\frac{N}{2}\text{ Old }\right)$. We denote by g^i the Group of individual i.

We study an infinitely repeated game made up of one-shot 2-person matching games with *local and Group specific information processing*. Each period, each Young is matched with an Old. Matches can occur within or across Groups.

The match of individual i can therefore either be a Young, in which case we call the matching partner $y(i)$ (thus when i is Old), or an Old, $z(i)$ (in this case i is Young and thus matched with an Old). Whenever it is obvious that we talk about the Young or Old matching partner of individual i, we omit the i and just call them y and z.

Players maximize expected utility, with discount factor $\delta > 0$.

3.1 Stage Games

We describe the one-shot game played between an individual i meeting today his Old matching partner $z(i)$. This game has 2 stages and the following choices faced by the Players:

3.1.1 Pre-matching Stage: Choice of the Targets of the Signalling Technology

Before any matching takes place, each Young makes the following decision,
$(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$, identifying a particular signalling technology, of a binary signal $s \in \mathcal{S} = \{0; 1\}$. In our story, the signal 0 stands for Innocent and the signal 1 for Guilty.

We restrict ourselves to a class of signalling technologies that target a particular action of the choice set of the agent. The technology then assigns to the agent a signal 0 or 1 with a different probability, depending on some arguments and on whether the agent chose the targeted action.

Formally in this environment, we define a signalling technology $\sigma(.)$ associated to the target $(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$ as a particular map of the type:

$$\mathbb{R}_+ \times G^3 \to \Delta(\mathcal{S})$$

To each element $(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$ corresponds a different target signalling technology. The elements of $\mathbb{R}_+^3 \times G^3$ are then classical arguments of a signalling technology (choices and characteristics of the players), but the mechanical device is fixed once chosen the target $(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$.

To summarize, the first action of the Young in this environment is to fix this target $(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$ of a signalling technology. This action is observable by the entire population. Therefore we omit to analyze the incentive compatibility of this target setting, for instance by assuming that a deviation on this choice triggers punishment for sure.

Then come choices that are not observable anymore and therefore that will need to be incentivized in equilibrium.

3.1.2 Matching Process

Matching is not random. We consider a model of directed search where the Old choose whether to direct his search effort towards the In-Group or the Out-Group. Let the variable μ^i be the choice of the Old i:

$$\mu^i = \begin{cases} 0 & \text{if the Old wishes to be matched with an In-Group partner.} \\ 1 & \text{if the Old wishes to be matched with an Out-Group partner.} \end{cases}$$

We introduce an imperfection in the matching technology: even though in a symmetric equilibrium (on which we will focus), and with our Group and generation size assumptions, all demands could be satisfied, the result of the directed search will be stochastic. If in each Group, n_0 and n_1 Old members wish to be matched respectively inside and outside their own Group (with $n_0 + n_1 = \frac{N}{2}$), only a fraction of them will achieve their goal.

$$P\big[y(i) \in g^i | \mu^i = 0\big] = P\big[y(i) \notin g^i | \mu^i = 1\big] = 1 - q,$$

The parameter q is therefore a measure of the imperfection of the matching technology. It represents the probability of failure, for individuals targeting their In-Group and their Out-Group respectively. We assume $q < \frac{1}{2}$ otherwise players would anticipate the failure and target the opposite Group, which is not interesting. Note that failure means failure to be matched in the desired Group. But in any case, each Old gets matched with a Young at each period. No Old remains unmatched, as in traditional directed search models.

The parameter q is common knowledge. The choice μ^i however is not observable to the Young partner of the Old. The Young can only see which group the Old belongs to.

3.1.3 Post-matching Stage: Gift-Giving Stage

Matches are realized. Each Young is paired with an Old, be it from his Group or not. Only Young make a decision. Each Young observes:

1. His random idiosyncratic unit cost of giving a gift: $C^i \sim \mathcal{U}[0; \bar{C}], \bar{C} > 0$;

2. The choice of the target signalling technology: $(x_0, x_1, x_0^P, x_1^P) \in \mathbb{R}_+^4$;

3. The signal sent by the Old he meets: $s^{z(i)} \in \mathcal{S} = \{0; 1\}$;

Then he takes an action $x \in \mathbb{R}_+$, which corresponds to a gift of unit cost $C^i > 0$ for him and of unit benefit $F > 0$ for the Old. As $F > C^i, \forall C^i$ by assumption, there are gains of trade at any level of gift. Stage payoffs are:

$$U : \mathbb{R}_+ \to \mathbb{R}^2$$
$$U(x) = (-C^i x, Fx)$$

to the Young and the Old respectively.

We can observe that in the absence of community enforcement, the optimal period game strategy for the Young is $x = 0$.

3.2 Signal Processing

As commented earlier, the environment is equipped with a target signalling technology that sends a signal on behalf of each Old before his partner decides whether and how much to give to him. The target is made up of 4 elements: a gift when meeting an Innocent agent (sending a signal $s = 0$), a gift when meeting a guilty agent (sending a signal $s = 1$), both gifts varying whether meeting an In-Group or an Out-Group member.

Those gifts can be summarized by a function $\alpha : G^2 \times \mathcal{S} \to \mathbb{R}_+$, which tells agent i, meeting an Old j with signal s^j how much he is supposed to give as a function of his own group, the group of his match and the signal sent by his match:

$$\alpha(g^i, g^j, s^j) = \begin{cases} x_0 & \text{if } (g^j, s^j) = (g^i, 0) \\ x_0^P & \text{if } (g^j, s^j) = (g^i, 1) \\ x_1 & \text{if } (g^j, s^j) = (-g^i, 0) \\ x_1^P & \text{if } (g^j, s^j) = (-g^i, 1) \end{cases}$$

The notation $-g^i$ refers to the group agent i does not belong to.

The target signalling technology can thus be written as a probability distribution $\sigma : G^3 \times \mathcal{S} \times \mathbb{R}_+ \to \Delta(\mathcal{S})$:

$$\sigma\big(g^i, g^j, g^k, s_t^j, x_t^i\big)[1] = \begin{cases} \pi_0 & \text{if } (g^k, x_t^i) = \big(g^i, \alpha(g^i, g^j, s_t^j)\big) \\ \pi_0^P & \text{if } g^k = g^i \text{ and } x_t^i \neq \big(\alpha(g^i, g^j, s_t^j)\big) \\ \pi_1 & \text{if } (g^k, x_t^i) = \big(-g^i, \alpha(g^i, g^j, s_t^j)\big) \\ \pi_1^P & \text{if } g^k = -g^i \text{ and } x_t^i \neq \big(\alpha(g^i, g^j, s_t^j)\big) \end{cases}$$

with $\pi_\gamma^P > \pi_\gamma, \forall \gamma \in \{0, 1\}$. $\gamma = 0$ refers to In-Group and $\gamma = 1$ refers to Out-Group.

We denote by $\lambda_\gamma \equiv \pi_\gamma^P - \pi_\gamma, \forall \gamma \in \{0, 1\}$, that is, how much more likely the agent is to send a bad signal when deviating from the norm with respect to complying with the norm, when the matching takes place respectively In-Group or Out-Group.

Group-Specific Efficiency of the monitoring technology and Sorting

One major assumption is the following:

$$\lambda_0 > \lambda_1 \tag{3.1}$$

It means that the In-Group is technologically better at distinguishing Innocent players from Guilty players than the Out-Group. We assume this as we are interested in the impact of this imbalance on desired matching groups from the Old.

We can expect that the Old who cheated when they were Young will be attracted to matches in the Out-Group and sort accordingly as they will be less likely to be recognized as Guilty.

On the other hand, the Out-Group is less attractive as a lower monitoring efficiency will allow to reach less cooperation, as we will show later. The maximum cooperation level sustainable with the Out-Group will be further depressed as Young matched with the Out-Group expect a lower proportion of cooperative players in this pool, due to the sorting process described above.

We are interested in analyzing this very trade-off.

In order to make the In-Group unambiguously better at monitoring than the Out-group, we also start with the following assumptions:

$$\pi_0 < \pi_1 \text{ and } \pi_0^P > \pi_1^P \tag{3.2}$$

3.3 Strategies and Equilibrium definition

3.3.1 Strategies

Period 2 (Old)

Strategies for the Old are maps $\varsigma^O : G \times G \times \mathcal{S} \times \mathbb{R}_+ \to \{0;1\}$ that state whether the Old targets the In-Group or the Out-Group, depending on whether he acted as Innocent or Guilty yesterday. We measure Innocence and Guilt as conformity or not to the strategy prescribed by the target signalling technology, which will then lead to an Innocent or Guilty signal.

Period 1 (Young)

Strategies for the Young are maps $\varsigma^Y : G \times G \times \mathcal{S} \times [0;F] \to \mathbb{R}_+$.

Depending on the cost the Young draws on $[0;F]$, he decides whether to act Innocent or Guilty (always in terms of mimicking the strategy prescribed by the signalling technology).

3.3.2 Equilibrium Definition

We need the following notation: $\rho_0, \rho_1, \rho_0^P, \rho_1^P$ are the proportions of agents which comply with the prescribed rule (play innocent), when meeting respectively an Innocent In-Group member, an Innocent Out-Group member, a Guilty In-Group member, a Guilty Out-Group member. Innocent and Guilty correspond to the signal sent by the Old.

A *Peer Punishment Stationary Equilibrium* are strategies $\left(\varsigma^{Oi}\right)_{i=1,\ldots,N}$, $\left(\varsigma^{Yi}\right)_{i=1,\ldots,N}$, proportions $\left(\rho_0, \rho_1, \rho_0^P, \rho_1^P\right)$ such that:

1. Given $\rho_{0,t}, \rho_{1,t}, \rho_{0,t}^P, \rho_{1,t}^P$, given everybody else plays the equilibrium strategy profile, ς_t^{Oi} maximizes the utility of the agent i as an Old in each period.

2. Given $\rho_{0,t}, \rho_{1,t}, \rho^P_{0,t}, \rho^P_{1,t}$, given everybody else plays the equilibrium strategy profile, ς^{Yi}_t maximizes the utility of the agent i as a Young in each period and has the following form, for the Young i meeting the Old j sending signal s^j:

$$\varsigma^Y(g^i, g^j, s^j, C^i) = \begin{cases} \alpha(g^i, g^j, s^j) & \text{if } C^i \leq \bar{C}(g^i, g^j, s^j) \\ 0 & \text{if } C^i > \bar{C}(g^i, g^j, s^j) \end{cases}$$

where $\bar{C}(g^i, g^j, s^j)$ is a cut-off cost, depending on whether the match was In-Group or Out-Group, and the signal Innocent or Guilty:

$$\bar{C}(g^i, g^j, s^j) = \begin{cases} \bar{C}(x_0) & \text{if } (g^j, s^j) = (g^i, 0) \\ \bar{C}(x_1) & \text{if } (g^j, s^j) = (-g^i, 0) \\ \bar{C}(x^P_0) & \text{if } (g^j, s^j) = (g^i, 1) \\ \bar{C}(x^P_1) & \text{if } (g^j, s^j) = (-g^i, 1) \end{cases}$$

3. $\rho_{0,t}, \rho_{1,t}, \rho^P_{0,t}, \rho^P_{1,t}$ are the proportions of Young who play innocent respectively in the In-Group and the Out-Group interactions at time t.

4 Players Incentives

4.1 Old Players Incentives

There are two possible strategies for the Old, leading to two different equilibria:

1. Non Sorting Equilibrium: Even though monitoring is weaker with the Out-Group, the Guilty players of period 1 still prefer to target the In-Group during the matching process.

2. Sorting Equilibrium: Due to the weaker monitoring with the Out-Group, the Guilty players of period 1 prefer to target the Out-Group during the matching process.

We restrict the parameters to the case where the Innocent players always want to target the In-Group.

Strategies for the Old are maps: $\varsigma^O : G \times G \times \mathcal{S} \times \mathbb{R}_+ \to \{0; 1\}$ but boil down to two different payoff-relevant cases: $x^i = \alpha(g^i, g^j, s^j)$ or $x^i \neq \alpha(g^i, g^j, s^j)$.

4.1.1 Optimal Choice when $x^i = \alpha(g^i, g^j, s^j)$

$U^O\big(\mu^i = 0 | x^i = \alpha(g^i, g^j, s^j)\big) \geq U^O\big(\mu^i = 1 | x^i = \alpha(g^i, g^j, s^j)\big) \Leftrightarrow$

$$\rho_0(1-\pi_0)x_0 + \rho_0^P \pi_0 x_0^P \geq \rho_1(1-\pi_1)x_1 + \rho_1^P \pi_1 x_1^P \tag{4.1}$$

This condition is almost always satisfied except in some extreme cases that we rule out, as we are interested in equilibria where Innocent players choose to match inside the Group. We therefore assume that (4.1) holds and we will discuss it later.

4.1.2 Optimal Choice when $x^i \neq \alpha(g^i, g^j, s^j)$

$U^O\big(\mu^i = 0 | x^i \neq \alpha(g^i, g^j, s^j)\big) \geq U^O\big(\mu^i = 1 | x^i \neq \alpha(g^i, g^j, s^j)\big) \Leftrightarrow$

$$\rho_0(1-\pi_0^P)x_0 + \rho_0^P \pi_0^P x_0^P \geq \rho_1(1-\pi_1^P)x_1 + \rho_1^P \pi_1^P x_1^P \tag{4.2}$$

When comparing Groups, agents are trading:

- Higher probability of being discovered as guilty: $\pi_0^P > \pi_1^P$
- Against higher cooperation gains: $x_0 > x_1$

4.2 Young Players Incentives

Young players incentives depend on the idiosyncratic cost of giving they draw. By definition of the equilibrium, under a certain cut-off they will cooperate, above they will not. Obviously, those cut-offs are endogenously determined in equilibrium, as the limit cost such that players want to play innocent versus guilty. Given that $C^i \sim \mathcal{U}[0; \bar{C}]$, we have

$$\bar{C}(x) = \rho(x)\bar{C}$$

with the following abuse of notation: $\big(\rho(x_0), \rho(x_1), \rho(x_0^P), \rho(x_1^P)\big) = \big(\rho_0, \rho_1, \rho_0^P, \rho_1^P\big)$.

4.2.1 Non-Sorting Equilibrium

$\forall x = x_0, x_1, x_0^P, x_1^P$, $U^Y(x | x = \alpha(g^i, g^j, s^j)) \geq U^Y(0 | x = \alpha(g^i, g^j, s^j)) \Leftrightarrow$

$$C^i x \leq \delta F\Big(E_{1-q}\big[\lambda\rho\Delta\big] - E_{1-q}\big[(\rho^P - \rho)x^P\lambda\big]\Big) \tag{4.3}$$

where $E_p(Y)$ is the expectation of a random variable Y when the probability to be matched in the In-Group is p.

The left hand side can be considered as the cost of cooperation over defection, whereas the right hand side as its gain.

Using the cut-off costs, we therefore get four incentive constraints necessary and sufficient for a peer punishment stationary equilibrium to exist. $\forall x = x_0, x_1, x_0^P, x_1^P$:

$$\rho(x)\bar{C}x \leq \delta F\Big(E_{1-q}\big[\lambda\rho\Delta\big] - E_{1-q}\big[(\rho^P - \rho)x^P\lambda\big]\Big) \qquad (IC^{NS}_{\rho(x)})$$

it can also be written in the following way:

$$x\rho_x\bar{C} \leq \delta F\Big(a_0x_0\rho_0 + a_1x_1\rho_1 + a_0^Px_0^P\rho_0^P + a_1^Px_1^P\rho_1^P\Big) \qquad (IC^{NS}_{\rho_x})$$

for some function of the parameters $a_0, a_1, a_0^P, a_1^P > 0$

4.2.2 Sorting Equilibrium

$\forall x = x_0, x_1, x_0^P, x_1^P$, $U^Y(x|x = \alpha(g^i, g^j, s^j)) \geq U^Y(0|x = \alpha(g^i, g^j, s^j)) \Leftrightarrow$

$$\rho(x)\bar{C}x \leq \delta F\, \underbrace{E_{1-q}\big[(1-\pi)\rho x\big] - E_q\big[(1-\pi^P)\rho x\big]} + \underbrace{E_{1-q}\big[\pi\rho x^P\big] - E_q\big[\pi^P\rho x^P\big]}$$
$$- \underbrace{\Big(E_q\big[\pi^P(\rho^P - \rho)x^P\big] - E_{1-q}\big[\pi(\rho^P - \rho)x^P\big]\Big)} \qquad (IC^S_{\rho(x)})$$

5 Typology of equilibrium

Both Sorting and Non Sorting equilibria are determined by four incentive constraints. Some can be slack and other binding, but given that the right hand side of each incentive constraint is identical but the left hand side can be ranked, some incentive constraints imply other, depending on the ranking of x_0, x_1, x_0^P, x_1^P. This leads to the following typology of equilibria, $\forall k = NS, S$, for Non Sorting, Sorting:

Case	Conditions on x	Incentive Constraints	Proportions ρ
1	$0 < x_0^p, x_1^P < x_1 \leq x_0$	$IC^k_{\rho_0^P}, IC^k_{\rho_1^P}, IC^k_{\rho_0}, IC^k_{\rho_1}$ binding	$0 < \rho_0 < \rho_1 < \rho_1^p, \rho_0^P < 1$
2	$0 < x_0^p \leq x_1^P < x_1 \leq x_0$	$IC^k_{\rho_1^P}, IC^k_{\rho_0}, IC^k_{\rho_1}$ binding, $IC^k_{\rho_0^P}$ slack	$0 < \rho_0 < \rho_1 < \rho_1^p < \rho_0^P = 1$
3	$0 < x_1^p \leq x_0^P < x_1 \leq x_0$	$IC^k_{\rho_0^P}, IC^k_{\rho_0}, IC^k_{\rho_1}$ binding, $IC^k_{\rho_1^P}$ slack	$0 < \rho_0 < \rho_1 < \rho_0^p < \rho_1^P = 1$
4	$0 < x_0^p, x_1^P < x_1 \leq x_0$	$IC^k_{\rho_0}, IC^k_{\rho_1}$ binding, $IC^k_{\rho_0^P}, IC^k_{\rho_1^P}$ slack	$0 < \rho_0 < \rho_1 < \rho_1^p = \rho_0^P = 1$
5	$0 < x_0^p, x_1^P < x_1 \leq x_0$	$IC^k_{\rho_0}$ binding, $IC^k_{\rho_1}, IC^k_{\rho_0^P}, IC^k_{\rho_1^P}$ slack	$0 < \rho_0 < \rho_1 = \rho_1^p = \rho_0^P = 1$
6	$0 < x_0^p, x_1^P < x_1 \leq x_0$	$IC^k_{\rho_0}, IC^k_{\rho_1}, IC^k_{\rho_0^P}, IC^k_{\rho_1^P}$ slack	$\rho_0 = \rho_1 = \rho_1^p = \rho_0^P = 1$

6 Non Sorting Equilibrium and Crowding-out

During all this section, we assume that (4.2) holds and consequently that we are in a Non Sorting equilibrium. Next section we will study when we are in a Sorting versus Non Sorting Equilibrium. We now focus on the case 4 ,5 and 6 where $IC^{NS}_{\rho_0^P}, IC^{NS}_{\rho_1^P}$ are satisfied with strict inequality and $\rho_0^P, \rho_1^P = 1$, as they already allow us to draw interesting conclusions on In-Group/ Out-Group crowding-out. We are interested in the case where $x_0 \geq x_1$ even though in principle it could be the opposite. Thus:

$$IC^{NS}_{\rho_0} \text{ is satisfied} \Rightarrow IC^{NS}_{\rho_1} \text{ is satisfied}$$

but not the other way around.

6.1 Case 4: $\rho_0, \rho_1 < 1$ and $IC^{NS}_{\rho_0}, IC^{NS}_{\rho_1}$ are binding.

Proposition 6. *Whenever the following condition holds:*

$$\bar{C} < \delta F E_{1-q}(\lambda) \tag{6.1}$$

a Non Sorting equilibrium exists where $(x_0, x_1, x_0^P, x_1^P), (\rho_0, \rho_1, \rho_0^P, \rho_1^P)$ *are such that:*

$$
\begin{array}{rrlcl}
 & & x_0^P, x_1^P & \geq & 0 \\
 & & x_0 & > & \max(x_0^P, \tilde{x}) \text{ where } \tilde{x} \equiv \frac{\delta F E_{1-q}(\lambda x^P)}{\delta F E_{1-q}(\lambda) - \bar{C}} \\
x_0 & \geq & x_1 & > & \max(x_1^P, \tilde{x}) \\
 & & \rho_0^P, \rho_1^P & = & 1 \\
 & & \rho_0 & = & \frac{\delta F E_{1-q}(\lambda x^P)}{x_0 \left(\delta F E_{1-q}(\lambda) - \bar{C}\right)} \\
 & & \rho_1 & = & \frac{\delta F E_{1-q}(\lambda x^P)}{x_1 \left(\delta F E_{1-q}(\lambda) - \bar{C}\right)}
\end{array} \tag{6.2}
$$

and where $IC^{NS}_{\rho_0}, IC^{NS}_{\rho_1}$ are binding.

Notice that there is no upper bound on x_0. In-Group cooperation requirement can be infinitely big, as long as (6.1) holds, an equilibrium will exist. As x_0 increases, rho_0, the proportion of agents which will cooperate when facing a required gift of x_0, will decrease accordingly. As x_0 goes to infinity, ρ_0 goes to 0 but there will always be low-cost agents cooperating. This relies on the assumption that the lower bound of the idiosyncratic cost of giving is 0.

Proposition 7. *In the equilibrium described in Proposition 6, there is no crowding-out between In-Group and Out-Group "actual cooperation". We define "actual cooperation" as the product of the required amount of cooperation and the proportion of agents of a certain type of matches (In-Group, Out-Group) that cooperate. Furthermore, In-Group and Out-Group actual cooperation levels are independent from both x_0 and x_1:*

$$
\rho_0 x_0 = \rho_1 x_1 = \frac{\delta F E_{1-q}(\lambda x^P)}{\delta F E_{1-q}(\lambda) - \bar{C}} \tag{6.3}
$$

The mechanism is the following: each change in x_0, x_1 is compensated by an adjustment in ρ_0, ρ_1. This is due to the fact that $\rho_0, \rho_1 < 1$ and that $IC^{NS}_{\rho_0}, IC^{NS}_{\rho_1}$ hold with equality. If x_0 increases, the marginal agent who waas cooperating drops and gives no gift, decreasing ρ_0.

6.2 Case 5: $\rho_0 < 1, \rho_1 = 1$, $IC^{NS}_{\rho_0}$ is binding and $IC^{NS}_{\rho_1}$ is not binding.

Proposition 8. *Whenever the following condition holds:*

$$\bar{C} < \delta F(1-q)\lambda_0 \tag{6.4}$$

a Non Sorting equilibrium exists where (x_0, x_1, x_0^P, x_1^P), $(\rho_0, \rho_1, \rho_0^P, \rho_1^P)$ *are such that:*

$$\begin{array}{rrlll} & & x_0^P, x_1^P & \geq & 0 \\ \min\left(\tilde{x}, \frac{E_{1-q}(\lambda x^P)}{q\lambda_1}\right) & \geq & x_1 & > & x_1^P \\ & & x_0 & > & \max(x_0^P, \hat{x}(x_1)) \text{ where } \hat{x}(x_1) \equiv \frac{\delta F\left(E_{1-q}(\lambda x^P) - q\lambda_1 x_1\right)}{\delta F(1-q)\lambda_0 - \bar{C}} \\ & & \rho_1, \rho_0^P, \rho_1^P & = & 1 \\ & & \rho_0 & = & \frac{\delta F\left(E_{1-q}(\lambda x^P) - q\lambda_1 x_1\right)}{x_0\left(\delta F(1-q)\lambda_0 - \bar{C}\right)} \end{array} \tag{6.5}$$

and where $IC_{\rho_0}^{NS}$ *is binding.*

Proposition 9. *In the equilibrium described in Proposition 8, there is crowding-out between In-Group and Out-Group actual cooperation. In-Group and Out-Group actual cooperation are constant with respect to* x_0 *but Out-Group actual cooperation is increasing in* x_1 *when In-Group actual cooperation is decreasing in* x_1*:*

$$\rho_0 x_0 = \frac{\delta F\left(E_{1-q}(\lambda x^P) - q\lambda_1 x_1\right)}{\delta F(1-q)\lambda_0 - \bar{C}} \tag{6.6}$$

$$\rho_1 x_1 = x_1 \tag{6.7}$$

In this unbalanced case where $IC_{\rho_0}^{NS}$ is binding and where $IC_{\rho_1}^{NS}$ is slack, an increase in x_1 does not lead to a decrease in ρ_1, and we lack the adjustment mechanism that was preventing crowding-out to occur previously.

The following picture (figure 5) illustrates the results of propositions (7) and (9). We show In-Group and Out-Group actual level of cooperation, that is the level of the gift corrected by the cooperating proportion: $\rho_0 x_0$ and $\rho_1 x_1$. They are drawned as a function of x_1, for $x_0 > \tilde{x}$, which ensures that full In-Group cooperation cannot be reached and thus that $\rho_0 < 1$.

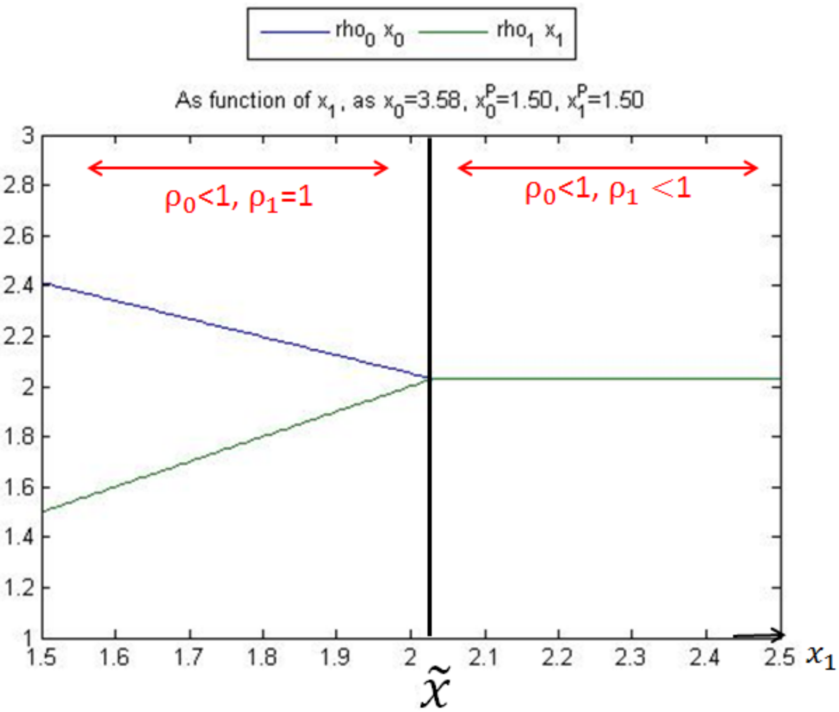

Figure 5: $\rho_g x_g$ as a function of x_1, with $x_0 > \tilde{x}$

For $x_1 > \tilde{x}$, we are in case 4, any increase in x_1 is washed out by a decrease in the related cooperation proportion, ρ_1. It follows that the future gains of cooperation are unchanged, and thus the incentives for In-Group cooperation, when moving x_1. Therefore $x_0\rho_0$ too stays constant as a function of x_1.

Furthemore, those two levels are equal. You find this from the computations, or directly looking at the incentive constraints $(IC^{NS}_{\rho_x})$. This comes from the fact that the unit cost of cooperation is the same in intragroup and intergroups interactions. In this case, there is neither crowding-out nor crowding-in between In-Group and Out-Group cooepration. The key factor here is that we are in a world of partial cooperation, a proportion strictly less than 1 of agents cooperate inside and outside the group, because the required level of cooperation is too expensive for high-cost individuals. It is understood throughout the population that increasing or decreasing the In-Group or Out-Group targeted gift will drive more or less people out of cooperation, independently from one another.

On the other hand, for $x_1 > \tilde{x}$, we are in case 5, and we do see crowding-out between $\rho_0 x_0$ and $\rho_1 x_1$. In this case, x_1 is low enough such that everybody comply with the Out-Group target. ρ_1 is a corner solution, stuck at 1, and therefore cannot adjust downwards as x_1 increases. This leads to a direct increase of $\rho_1 x_1 = x_1$.

What may seem surprising is that ρ_0 adjusts downwards proportionally, leading to a decrease in $\rho_0 x_0$. Looking at the incentive constraint ($IC_{\rho_0}^{NS}$), we see that the gains of future cooperation increase with x_1, which should lead more people into cooepration, thus increasing ρ_0. This turns out to be a possible outcome but not in an interior equilibrium (with respect to ρ_0). Indeed, as ρ_0 increases, the cost of cooperation (Left Hand Side) increases less fast than its gain (Right Hand Side). With a higher x_1 (uncompensated by ρ_1), either we jump to full cooperation equilibrium ($\rho_0 = 1$), and ($IC_{\rho_0}^{NS}$) holds with strict inequality, or we stay in a partial cooperation equilibrium where ($IC_{\rho_0}^{NS}$) holds with equality and rho_0 adjusts downwards. As future gains of cooperation increase, today's cooperation decreases, because we are focusing on the partial equilibrium, the worst equilibrium, and agents are too small to take into account the effect of their actions on the population rate of cooperation. And this proposition, in the spirit of Folk Theorems, tells us that as the best equilibrium becomes better, the worst equilibrium becomes worse. The situation is not totally symmetric as "becoming better" does not refer to a change in equilibrium payoffs due to a change in the exogenous variables. Here the exogenous variables stay fixed and equilibrium quantities only are moving with respect to one another. But we can keep the intuition: as the payoffs of cooperation increase, and thus punishment becomes more costly (for the punished agents), it is easier to have agents bloked in a suboptimal behavior (on another dimension, ρ_0).

6.3 Case 6: $\rho_0 = 1, \rho_1 = 1$, no Incentive Constraint is binding.

Proposition 10. *A Non Sorting equilibrium exists where* (x_0, x_1, x_0^P, x_1^P), $(\rho_0, \rho_1, \rho_0^P, \rho_1^P)$ *are such that:*

$$\begin{array}{rclcl} & & x_0^P, x_1^P & \geq & 0 \\ \min\left(\tilde{x}, \frac{E_{1-q}(\lambda x^P)}{q\lambda_1}\right) & \geq & x_1 & > & x_1^P \\ \frac{\delta F\left(E_{1-q}(\lambda x^P) - q\lambda_1 x_1\right)}{\delta F(1-q)\lambda_0 - C} \equiv \hat{x}(x_1) & \geq & x_0 & > & x_0^P \\ & & \rho_0, \rho_1, \rho_0^P, \rho_1^P & = & 1 \end{array} \tag{6.8}$$

and where no Incentive Constraint is binding.

Proposition 11. *In the equilibrium described in Proposition 10, there is a new form of crowding-out: the* *threshold for full versus partial cooperation $\hat{x}(x_1)$ of x_0 decreases with x_1, or the maximum value of x_0 for full cooperation (with $\rho_0 = 1$).*

It means that when x_1 is small enough such that everybody cooperates when facing a required gift of x_1 (that is when $x_1 < \tilde{x}$), and when this constraint is strict, it gives more room for cooperation inside the group,which you can see considering the x_0-threshold value for full versus partial cooperation,

$$\hat{x}(x_1) \equiv \frac{\delta F\big(E_{1-q}(\lambda x^P) - q\lambda_1 x_1\big)}{\delta F(1-q)\lambda_0 - \bar{C}}$$

$\hat{x}(x_1) > \tilde{x}$ for $x_1 < \tilde{x}$ but decreases towards $\tilde{x}$ as x_1 increases until reaching:

$$\hat{x}(x_1) = \tilde{x} \text{ when } x_1 = \tilde{x} \tag{6.9}$$

$\hat{x}(x_1)$ is stricly above its level when there is partial cooperation on both dimension x_0 and x_1. Furthermore, the extra room for In-Group cooperation, $\hat{x}(x_1) - \tilde{x}$, is increasing as x_1 becomes smaller.

We can illustrate this in figure 6.

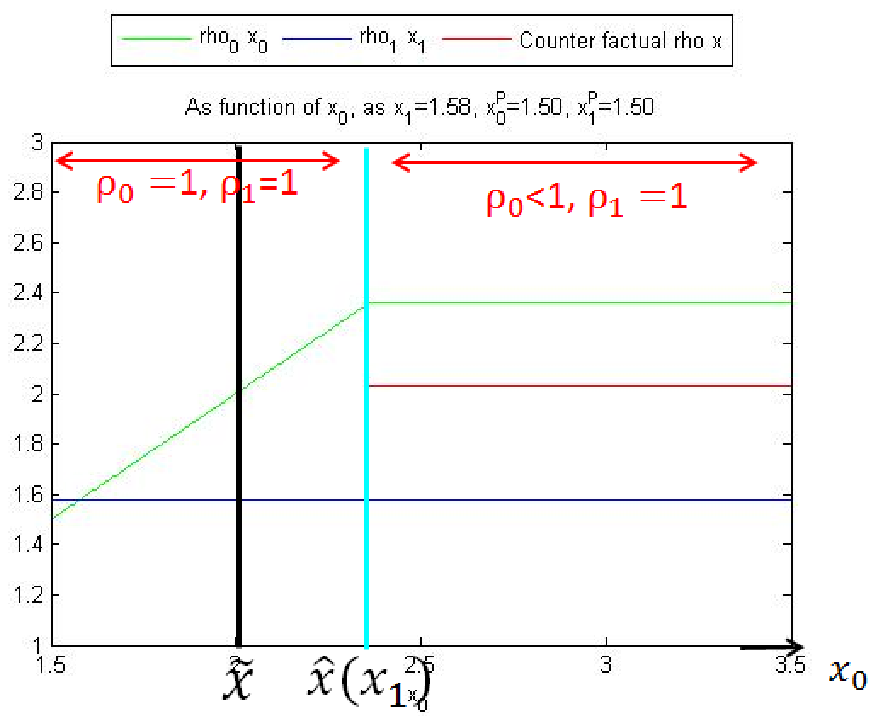

Figure 6: $\rho_g x_g$ as a function of x_0, with $x_1 < \tilde{x}$

We draw actual levels of cooperation $\rho_0 x_0, \rho_1 x_1$, in case 5 and 6. In this picture, $x_1 < \tilde{x}$, to ensure that there is full Out-Group cooperation ($\rho_1 = 1$). We also draw a line showing the value of partial actual cooperation of case 4 (that is when x_0 and x_1 are high enough so that $\rho_0, \rho_1, < 1$, and that any increase in x_0, x_1 is compensated by a decrease in ρ_0, ρ_1). We remind that this value is as given in (6.3) and represents our reference point.

We see that $\rho_0 x_0$ increases with x_0 until the threshold $\hat{x}(x_1)$ from which ρ_0 starts decreasing below 1 and actual cooperation becomes constant. This is the maximum amount of cooperation that can be reached within the Group.

Now, consider the following picture, figure 7, where x_1 is higher, but still under $\tilde{x}$

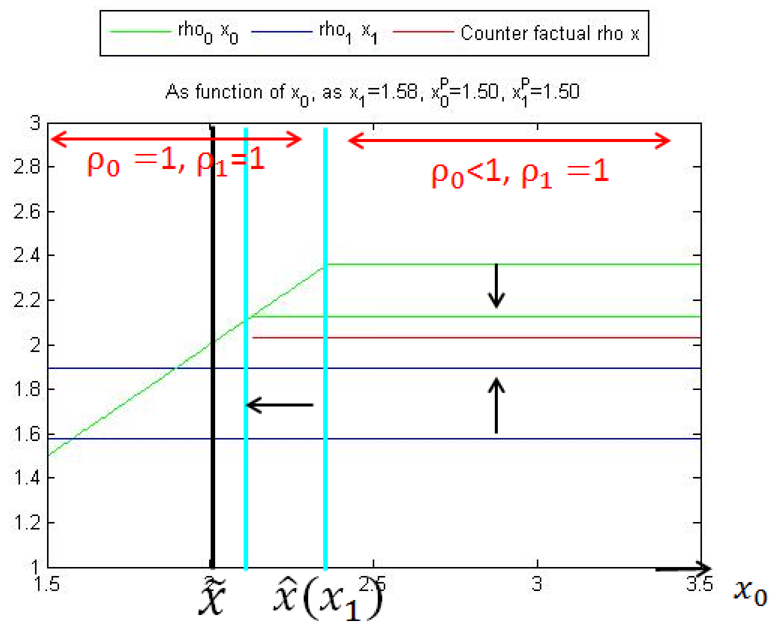

Figure 7: $\rho_g x_g$ as a function of x_0, with $x_1 < \tilde{x}$

The threshold of full In-Group cooperation, $\hat{x}(x_1)$, has decreased, and as a result the maximal level of cooperation that can be sustained within the group, $\rho_0 \hat{x}(x_1) = 1 \times \hat{x}(x_1)$ too. Also, Out-Group cooperation has increased proportionaly. Thus In-Group and Out-Group cooperation are getting closer to one another and to the benchmark level of case 4.

The limiting point of this phenomenon is when x_1 reaches $\tilde{x}$, and then $\hat{x}(x_1)$ decreases to $\tilde{x}$. We see now that the maximal amount of cooperation is the same within and across groups. In other words, the maximum amounts of cooperation that can be sustained within and across groups crowd each other out.

7 Sorting versus Non Sorting Equilibria

In this section we focus on the case where $\rho_0, \rho_1 < 1$ and $IC^{NS}_{\rho_0}, IC^{NS}_{\rho_1}$ are binding, comparing Sorting and Non Sorting equilibria. Equilibria are of the following form, for $k = NS, S$:

Proposition 12. *Whenever the condition (Ex^k) holds:*

a k- equilibrium exists where $(x_0, x_1, x_0^P, x_1^P), (\rho_0^k, \rho_1^k, \rho_0^{P,k}, \rho_1^{P,k})$ are such that:

$$\begin{array}{rrlcl} & & x_0^P, x_1^P & \geq & 0 \\ & & x_0 & > & \max(x_0^P, \tilde{x}^k) \\ x_0 & \geq & x_1 & > & \max(x_1^P, \tilde{x}^k) \\ & & \rho_0^{P,k}, \rho_1^{P,k} & = & 1 \\ & & \rho_1^k & = & \frac{x_0}{x_1}\rho_0^k \end{array} \tag{7.1}$$

and where $IC^k_{\rho_0}, IC^k_{\rho_1}$ are binding.

Existence constraints (Ex^k), threshold levels $\tilde{x}^k$ and cooperation proportions ρ_0^k, ρ_1^k are different across equilibrium types.

Existence Constraints

$$\begin{array}{lccl} Ex^{NS}: & \bar{C} & < & \delta F E_{1-q}(\lambda) \\ Ex^{S}: & \bar{C} & < & \delta F\Big((1-q)(\pi_1^P - \pi_0) + q(\pi_0^P - \pi_1)\Big) \end{array} \tag{7.2}$$

The two equations have the same flavor in the sense that the expression $(1-q)(\pi_1^P - \pi_0) + q(\pi_0^P - \pi_1)$ represents the gains in terms of probability of being punished when cooperating versus when deviating, weighted over the probability of success or failure of the directed search.

Threshold levels

$$\begin{aligned}
\tilde{x}^{NS} &= \frac{\delta F E_{1-q}(\lambda x^P)}{\delta F E_{1-q}(\lambda) - \bar{C}} \\
\tilde{x}^{S} &= \frac{\delta F \left(x_0^P \left((1-q)\pi_0 - q\pi_0^P \right) + x_1^P \left(-(1-q)\pi_1^P + q\pi_1 \right) \right)}{\delta F \left((1-q)(\pi_1^P - \pi_0) + q(\pi_0^P - \pi_1) \right) - \bar{C}}
\end{aligned} \tag{7.3}$$

Cooperation proportion levels We draw in (fig: 8) ρ_0^{NS} versus ρ_0^S and ρ_0^{NS} versus ρ_0^S for the following parametrization:

q	δ	F	$\bar{C}$	π_0	π_1	π_0^P	π_1^P	x_0	x_1	x_0^P
0.4	0.99	10	2	0.01	0.1	0.9	0.7	15	5	5.99

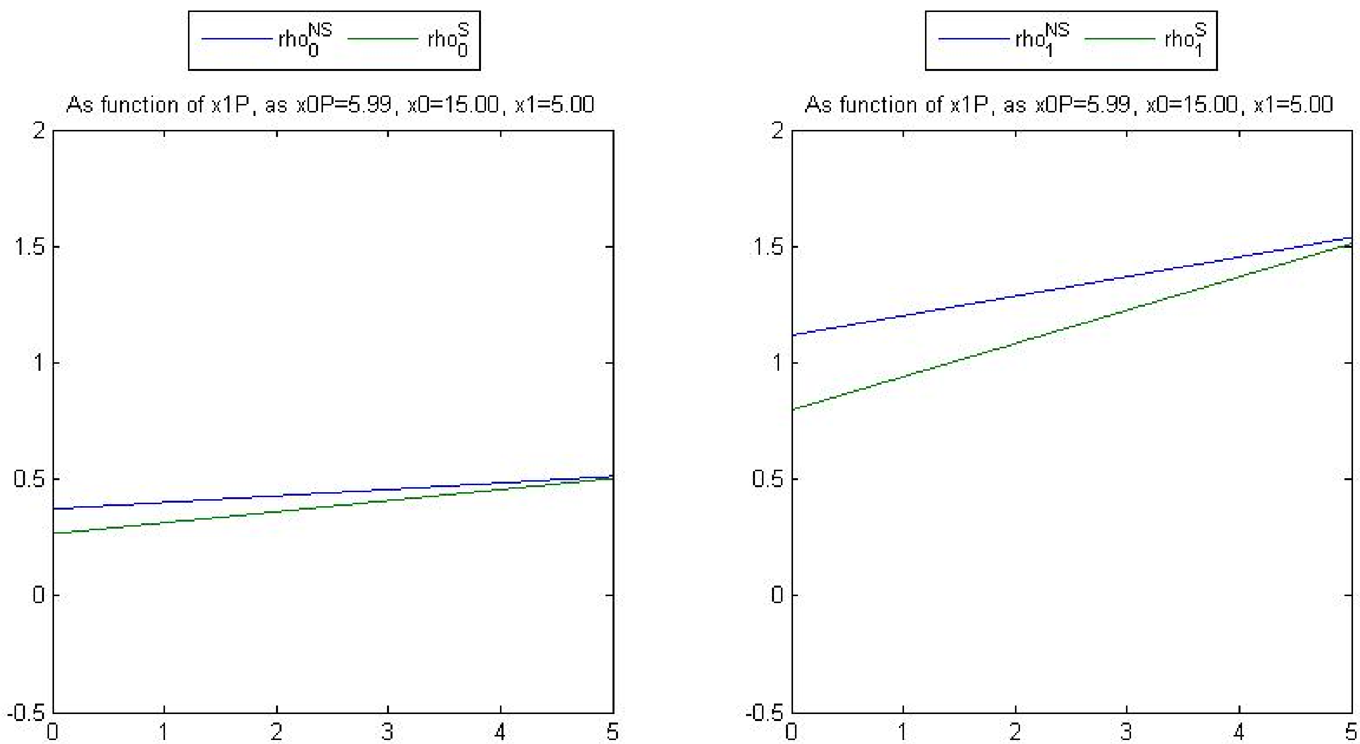

Figure 8: ρ_g^{NS} versus ρ_g^S

In both cases, the proportion increase as a function of x_1^P (as the reward of cooperation minus deviation decreases all other things equal, proportions have to increase to keep the two binding incentive constraints $IC_{\rho_0}^k, IC_{\rho_1}^k$ satisfied). But our numerical analysis yields:

$$\forall g = 0, 1, \rho_g^{NS} > \rho_g^S \tag{7.4}$$

7.1 Drivers of Equilibrium type

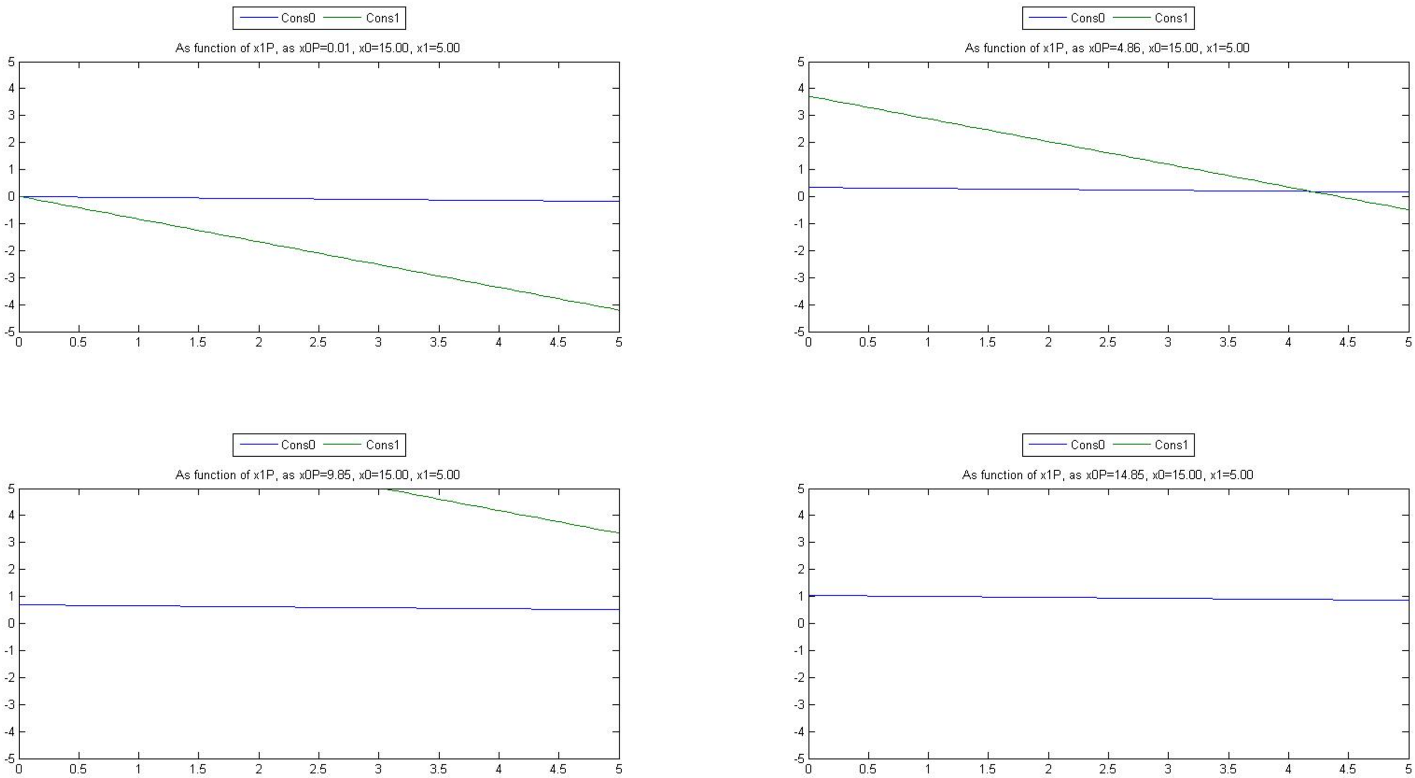

Figure 9: Constraints for Innocent and Guilty players, Sorting

Given in this kind of equilibrium where $IC_{\rho_0}^k, IC_{\rho_1}^k$ are binding and the other constraints are slack, we have $\rho_0 x_0 = \rho_1 x_1$ is a constant with respect to x_0, x_1 (from Proposition (7)). With numerical simulations, we see that x_0^P, x_1^P are determining variables. We draw how the constraints for Innocent Players (4.1) and for Guilty players (4.2) vary as a function of x_1^P, for different x_0^P, with the Sorting (figure 9) and Non Sorting (figure 10) values for ρ_0, ρ_1 as a function of the other variables. We take the same parametrization as before. When the constraint of guilty players (line labeled "Cons1" in the two above mentioned figures) is positive, Guilty players choose to direct their search inside the group, when it is negative, outside the group. The results are presented in figure 9 for the Sorting equilibrium, and in figure 10 for the Non Sorting equilibrium.

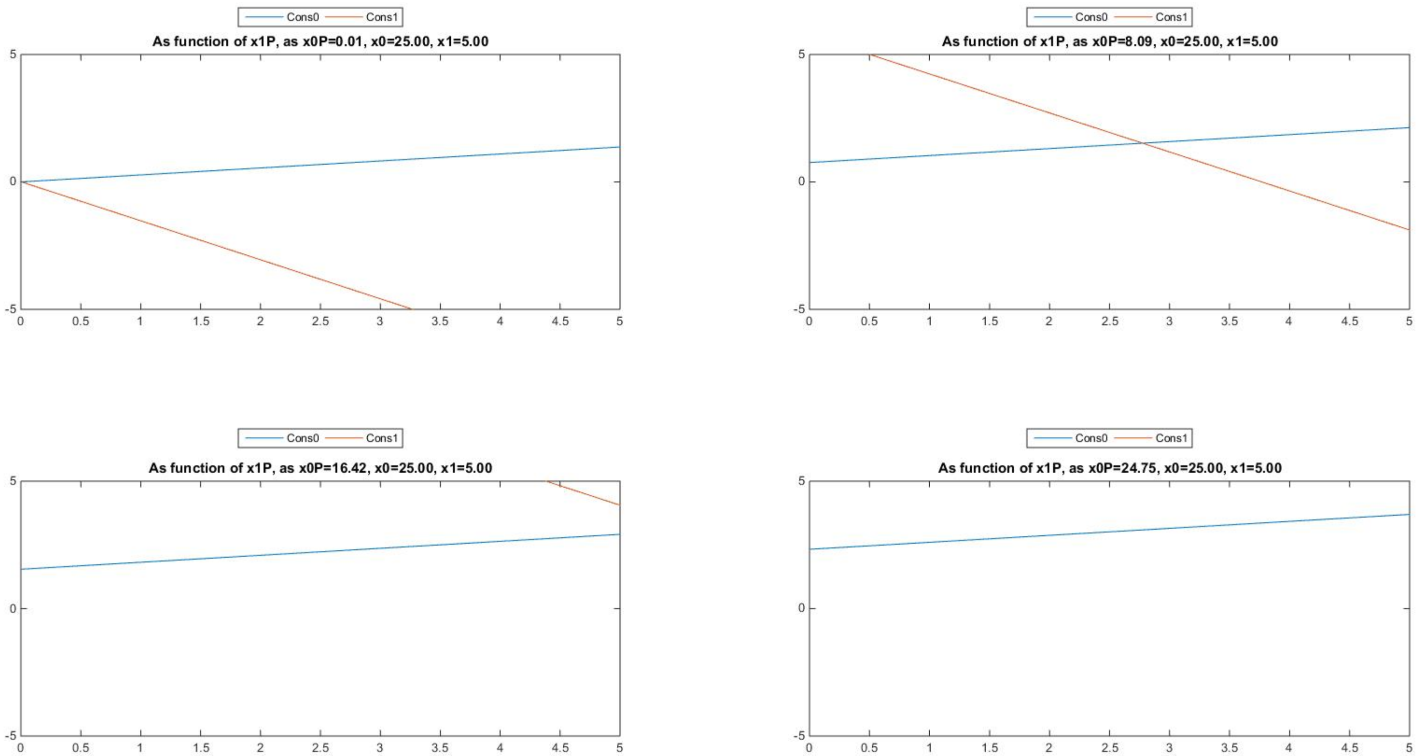

Figure 10: Constraints for Innocent and Guilty players, Non Sorting

We ask whether there can be multiple equilibria, that is, whether for some equilibrium allocations, both a Sorting and a Non Sorting equilibria could exist (as ρ_0, ρ_1 are different in both cases). Our numerical simulations suggest it is not the case.

7.2 Welfare analysis

We now want to compare the utility of Innocent and Guilty agents under both equilibria. Again we resort to numerical analysis. We draw three utilities per graph: the counter-factual utility in case of Non Sorting equilibrium (green), the counter-factual utility in case of Sorting equilibrium (red), and the actual utility (purple), depending on whether the equilibrium played at those values of the variables is Sorting or Non Sorting. Therefore the actual utility can be superposed with the Non Sorting Utility, the Sorting Utility, or be zero if no equilibrium exist for those values (in the range of equilibria where $IC^k_{\rho_0}, IC^k_{\rho_1}$ and only $IC^k_{\rho_0}, IC^k_{\rho_1}$ are binding).

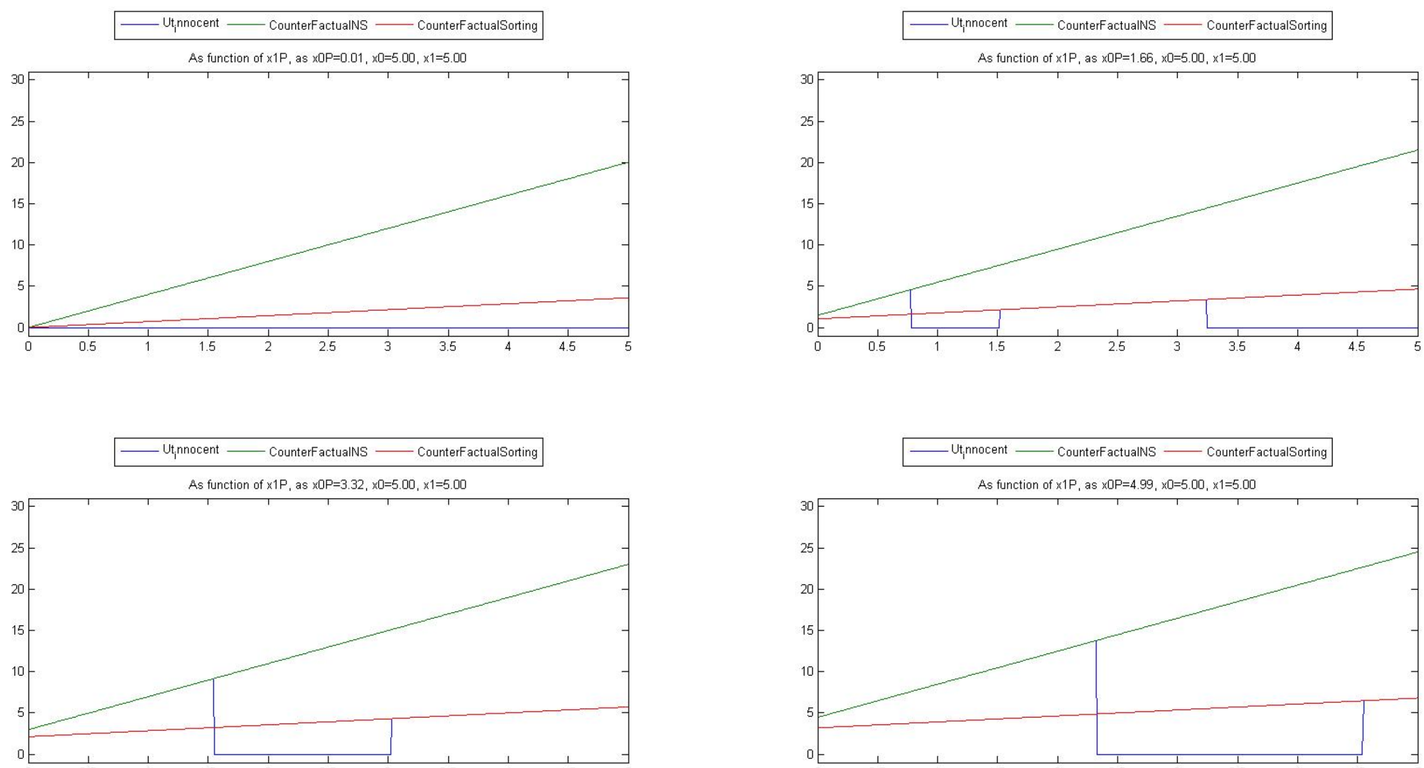

Figure 11: Utility of Innocent players, Balanced x_0, x_1

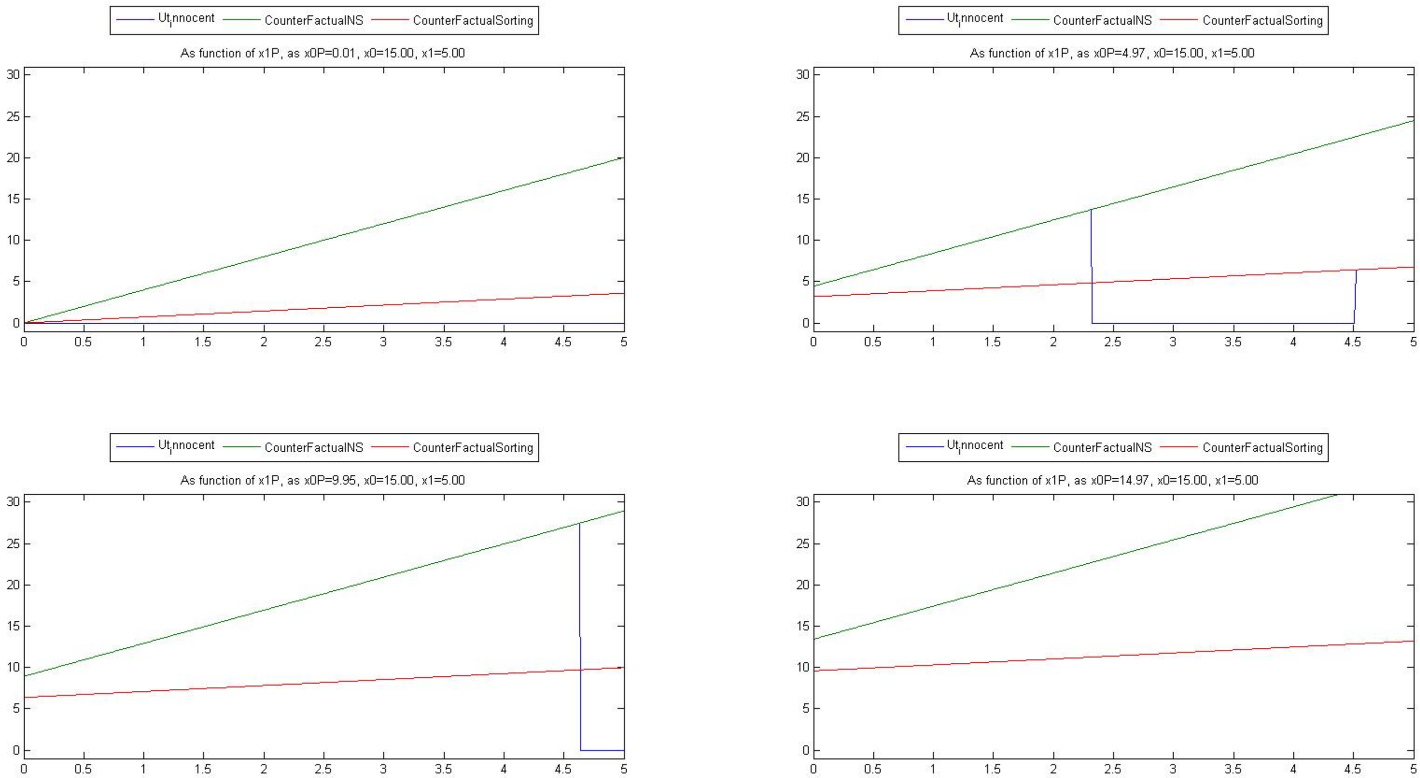

Figure 12: Utility of Innocent players, Unbalanced x_0, x_1

We stress the following observations:

- Utility under Non Sorting Equilibrium is higher than under Sorting Equilibrium, both for Innocent and Guilty players, but it is more striking for Innocent players. This is due to the fact that under Sorting, a Group can get rid of its own Guilty players who go outside, but receive the Guilty players of the other Group, who also chose to go outside. This is relying on the assumption that there are only two groups in the society and that going Out-Group means joining the other Group. The question for agents is thus whether they are better-off living with their own Guilty players or the ones of the other Group. Given the In-Group technology is more efficient, players are better-off when matching with their own Guilty players that they can recognize as guilty more often.

- In a non-sorting equilibrium, crowding-out operates not only in terms of cooperation, but in terms of interactions. More interactions happen within groups, and less across grouops. But it also means that

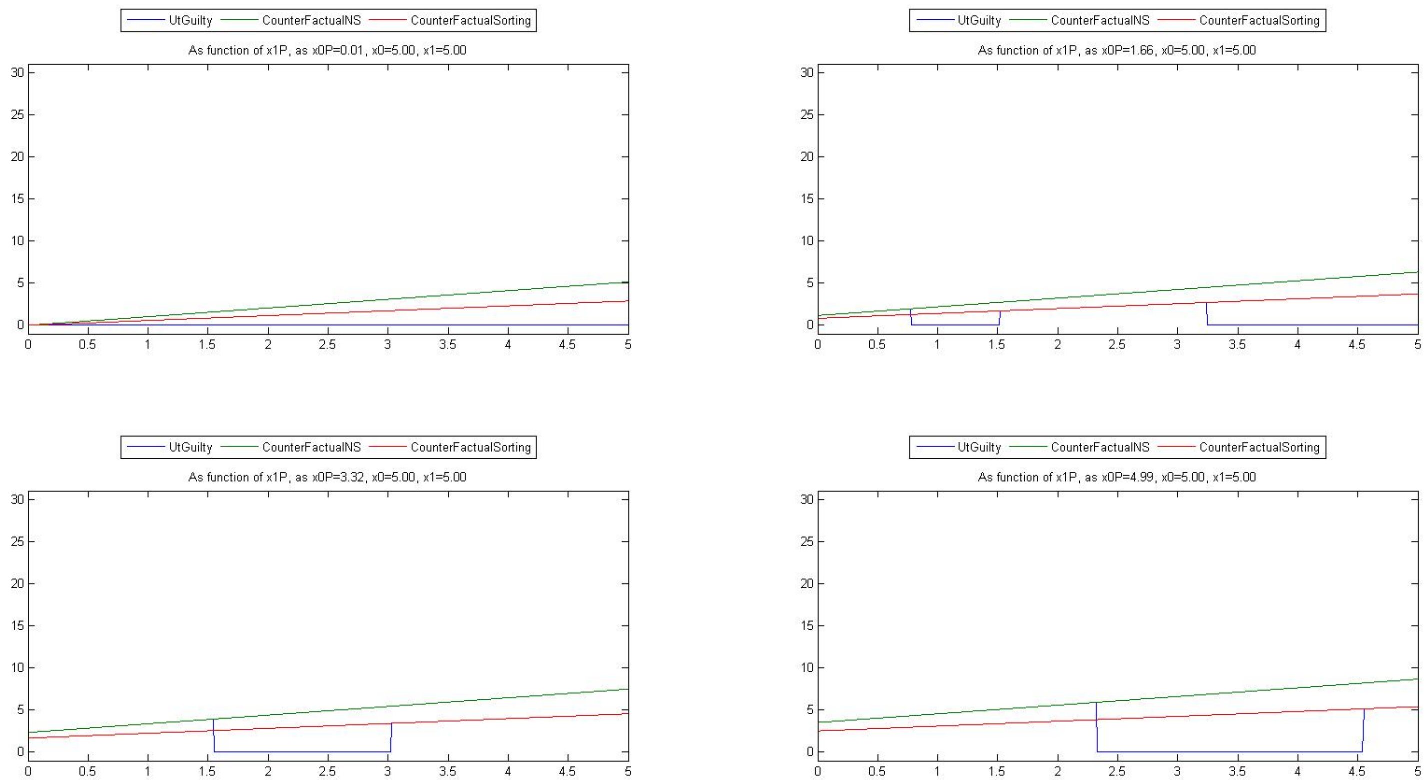

Figure 13: Utility of Guilty players, Balanced x_0, x_1

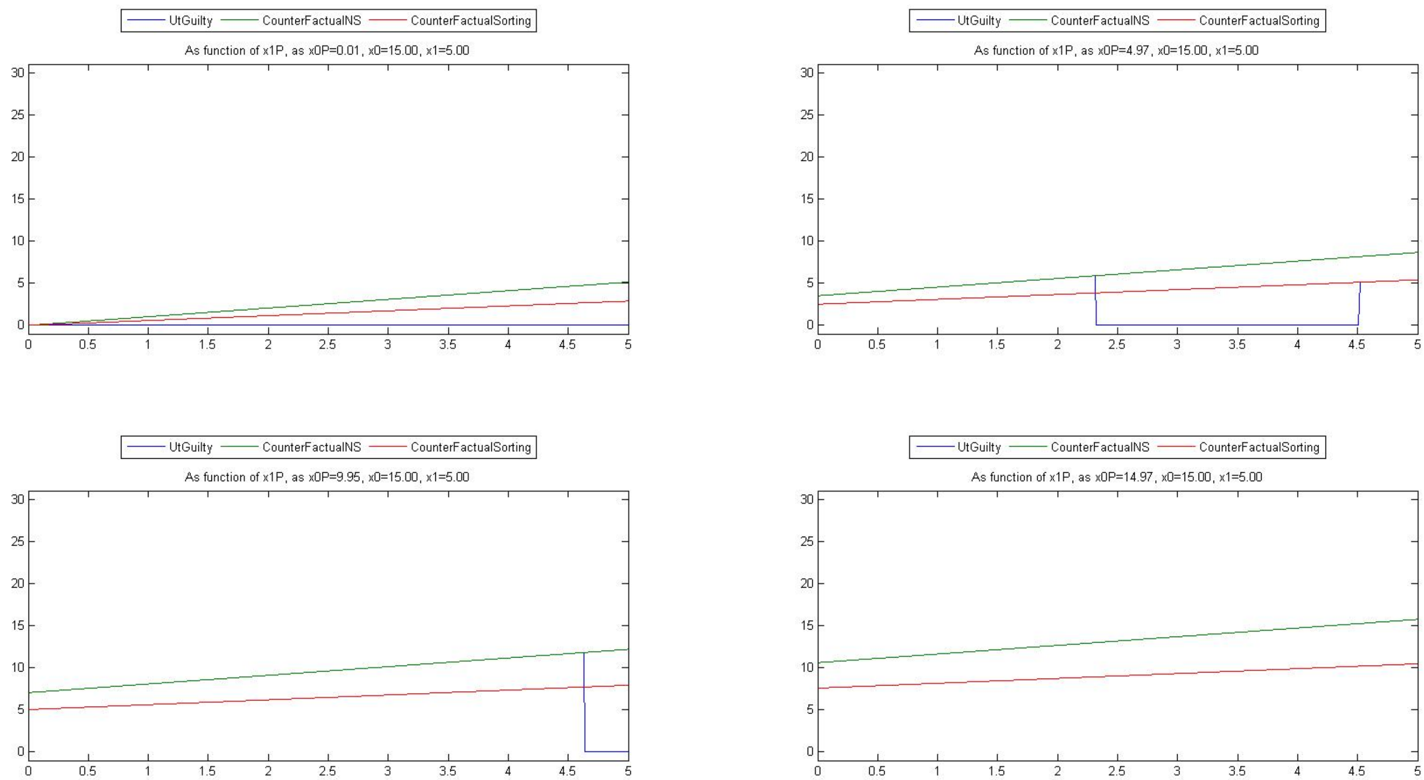

Figure 14: Utility of Guilty players, Unbalanced x_0, x_1

each intergroup interaction is more cooperative (as the individuals who drew a high cost of cooperation, and are therefore likely to deviate, choose to stay inside the group). On the other hand, in a sorting equilibrium, there is no interactions crowding-out, but intergroup interactions are on average less cooperative.

- The fact that Sorting Equilibria arise even though Guilty players would be better-off in a Non Sorting equilibrium present evidence of the externalities at play in the model: Guilty players are better-off going outside of the Group but they do not internalize the effect they have on other Guilty players going out of the Group.

- Non Sorting equilibria seem to arise for lower value of x_1^P than Sorting equilibria. As x_1^P increases, the Out-Group pay-off become too attractive for Guilty players to stay inside. As Non Sorting equilibria yield higher payoffs than Sorting equilibria, we can therefore ask whether agents are always better-off by diminishing punishment (once punishment is enough to sustain cooperation), as is the case int traditional models of peer punishment.

8 Conclusion

We find that crowding-out between intragroup cooperation and intergroup cooperation may or may not occur. Sorting equilibria can arise even though they are Pareto-dominated by Non Sorting equilibria (at least under our numerical simulations), shedding light on the existence of negative externalities non internalized by Old Guilty players. We aim at furthering this work in progress by formal proofs for the last section, and comparative statics with respect to the parameters. We are in particular interested in the role of q: how does the imperfection of the matching process impact our results?

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