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Union-closed sets and Horn Boolean functions

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A B S T R A C T

A family $\mathcal{F}$ of sets is union-closed if the union of any two sets from $\mathcal{F}$ belongs to $\mathcal{F}$. The union-closed sets conjecture states that if $\mathcal{F}$ is a finite union-closed family of finite sets, then there is an element that belongs to at least half of the sets in $\mathcal{F}$. The conjecture has several equivalent formulations in terms of other combinatorial structures such as lattices and graphs. In its whole generality the conjecture remains wide open, but it was verified for various important classes of lattices, such as lower semimodular lattices, and graphs, such as chordal bipartite graphs. In the present paper we develop a Boolean approach to the conjecture and verify it for several classes of Boolean functions, such as submodular functions and double Horn functions.

1. Introduction

Let U be a finite set (the universe) and $\mathcal{F} \subseteq 2^U$ a set system, i.e. a family of subsets of U. $\mathcal{F}$ is *union-closed* if for any two sets $A, B \in \mathcal{F}$ the union $A \cup B$ belongs to $\mathcal{F}$. The

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following conjecture is known as *union-closed sets conjecture* and sometimes is referred to as Frankl's conjecture.

Conjecture 1. *Any finite union-closed family $\mathcal{F} \neq \{\emptyset\}$ of finite sets contains an element that belongs to at least half of the sets in the family.*

For an equivalent formulation, let us say that $\mathcal{F}$ is *intersection-closed* if for any two sets $A, B \in \mathcal{F}$ the intersection $A \cap B$ belongs to $\mathcal{F}$. Also, assume without loss of generality that every element of the universe appears in at least one set of $\mathcal{F}$. Then $\mathcal{F}$ is intersection-closed if and only if the family $\{U - A \ : \ A \in \mathcal{F}\}$ is union-closed. Therefore, Conjecture 1 admits the following equivalent formulation.

Conjecture 2. *Any finite intersection-closed family of at least two finite sets contains an element that belongs to at most half of the sets in the family.*

The conjecture admits various other equivalent formulations, in particular, in the language of lattice theory [8] and in the terminology of graph theory [2].

In spite of its simple formulation, the conjecture is wide open and was verified only for some special classes of sets, lattices or graphs. In the present paper, we develop a Boolean approach to the conjecture and verify it for several classes of Boolean functions, such as submodular functions and double Horn functions.

2. Horn Boolean functions

Let $\mathcal{F}$ be an intersection-closed family over the universe $U = \{x_1, x_2, \ldots, x_n\}$, and let $A \in \mathcal{F}$ be a member-set of $\mathcal{F}$, i.e. a subset of U that belongs to $\mathcal{F}$. We represent A by its characteristic vector c_A, i.e. a 0-1 vector of length n in which the i-th coordinate equals 1 if and only if $x_i \in A$. This allows us to interpret $\mathcal{F}$ as a Boolean function over variables $\{x_1, x_2, \ldots, x_n\}$ whose false points are precisely the member-sets of $\mathcal{F}$. According to the following theorem proved in [7] (also see [3]) this is a Horn function.

Theorem 1. *A Boolean function is Horn if and only if the set of its false points is closed under conjunction.*

We denote the set of false points of a Horn Boolean function f by $F = F(f)$. Adapting set theory terminology, we will say that a variable x_i belongs to a Boolean point $X \in \{0,1\}^n$ if it appears in X with $x_i = 1$, i.e. if the i-th coordinate of X equals 1. Therefore, in the terminology of Boolean functions Frankl's conjecture can be restated as follows: any Horn Boolean function with at least two false points contains a variable that belongs to at most half of the false points of the function.

We denote the set of true points of a Horn Boolean function f by $T = T(f)$ and associate with T one more set system, denoted $\mathcal{T}$, over the same universe U, such that a set $A \subseteq U$ belongs to $\mathcal{T}$ if and only if the characteristic vector c_A belongs to T.

We observe that a variable belongs to at most half of the false points if and only if it belongs to at least half of the true points of the function, which suggests that the relation between $\mathcal{F}$ and $\mathcal{T}$ is similar to the relation between intersection-closed and union-closed families. However, in general $\mathcal{T}$ is neither intersection-closed nor union-closed.

In the terminology of set systems, an element that appears in at least half of the member-sets is known as *abundant* and an element that appears in at most half of the member-sets is known as *rare*. In the terminology of Boolean functions, every variable that is abundant for true points is rare for false points, and vice versa. In our study of Boolean functions we will frequently switch between the two roles of the same variable. To avoid any ambiguities, we will call such a variable *good*, i.e. a variable is good if it belongs to at most half of the false points, or equivalently, if it belongs to at least half of the true points of the function. In this terminology Frankl's conjecture can be restated as follows.

Conjecture 3. *Any Horn Boolean function with at least two false points contains a good variable.*

We will say that a Horn Boolean function satisfies Frankl's conjecture if it satisfies Conjecture 3.

Up to now, we just translated Frankl's conjecture to a different language, but we did not discuss the advantage of this translation. In addition to possibility of playing simultaneously with two set systems in the search for a good element (variable), the importance of Boolean formulation is that Horn functions admit a very specific disjunctive normal form (DNF) representation. By definition, a Boolean function is a *Horn function* if it can be represented by a DNF in which every term contains at most one negated literal. We use this representation in order to verify Conjecture 3 for some special classes of Horn functions in Sections 4 and 5. All preliminary information can be found in Section 3. Section 6 concludes the paper with open problems.

3. Preliminaries

In this section, we fix terminology and notation and prove some preliminary results.

Given a Boolean function f and a variable x, we denote by $f_{|x=0}$ and $f_{|x=1}$ the *restriction* of f to $x = 0$ and to $x = 1$, respectively, i.e. these are the functions obtained from f by restricting it to the sets of Boolean points with $x = 0$ and $x = 1$, respectively. With this notation, good variables can be characterized as follows: a variable x of a function f is *good*

- if the number of true points of $f_{|x=0}$ does not exceed the number of true points of $f_{|x=1}$, or equivalently,
- if the number of false points of $f_{|x=1}$ does not exceed the number of false points of $f_{|x=0}$, or equivalently,

- if there is an injective mapping from the set of true points of $f_{|x=0}$ to the set of true points of $f_{|x=1}$, or equivalently,
- if there is an injective mapping from the set of false points of $f_{|x=1}$ to the set of false points of $f_{|x=0}$.

A *literal* is either a Boolean variable or its negation. We will refer to literals as *positive* or *negative* depending on whether they are unnegated or negated, respectively.

A *term* is a conjunction (product) of literals. Any variable that does not appear as a literal in a term t, neither positively nor negatively, will be called a *free* variable for t.

A term is *linear* if it consists of one literal, and *quadratic* if it consists of two literals. A term is *Horn* if it contains at most one negated literal, and *pure Horn* if it contains exactly one negated literal.

We will say that a term t *covers* a Boolean point X if $t(X) = 1$, i.e. t evaluates to 1 at X. A term t is an *implicant* of a Boolean function f if every point covered by t is a true point of f. Also, t is a *prime implicant* of f if no proper subset of literals of t is an implicant of f.

A *DNF* is a disjunction of terms. A DNF is *Horn* if each of its terms is Horn, and it is *pure Horn* if each of its terms is pure Horn. A *Horn function* is a Boolean function that admits a Horn DNF representation. A *pure Horn function* is a Boolean function that admits a pure Horn DNF representation. It is not difficult to see that a Horn function f is pure Horn if and only if $f(1, 1, \ldots, 1) = 0$. If f is not pure Horn, then by changing its value at the point $(1, 1, \ldots, 1)$ from 1 to 0 we obtain a pure Horn function f' such that if f' has a good variable, then so does f. Therefore, Conjecture 3 is valid if and only if it is valid for pure Horn functions.

We now translate to the language of Boolean functions some facts that are known in non-Boolean terminology (see e.g. [1]).

Lemma 1. *If Conjecture 3 holds for Horn functions without linear prime implicants, then it holds for all Horn functions.*

Proof. If a Horn function f has a linear prime implicant x, then x is a good variable, because $f_{|x=1} \equiv 1$ and hence x does not belong to any false point. If f has a linear prime implicant $\overline{x}$, then $f_{|x=0} \equiv 1$ and x belongs to every false point. In this case, f can be restricted to $f_{|x=1}$, and a good variable for the function $f_{|x=1}$ is also good for f. □

Lemma 2. *Let f be a Horn function represented by a Horn DNF D_f. If a variable x of f does not appear in D_f negatively, then x is a good variable for f.*

Proof. If the function $f_{|x=0}$ does not have true points, then the number of false points of f containing x cannot be larger than the number of false points that do not contain x, and hence x is a good variable for f.

Now let X be a true point of f with $x = 0$, and let t be a term of D_f with $t(X) = 1$. Then t does not contain x, since x does not appear in D_f negatively. Therefore, for the point X' obtained from X by changing x to 1, we have $t(X') = 1$. This establishes an injective mapping $\phi : X \to X'$ from the set of true points of $f_{|x=0}$ to the set of true points of $f_{|x=1}$, and shows that x is good. □

Lemmas 1 and 2 allow us to restrict ourselves to Horn functions that have no linear prime implicants and in which every variable appears negatively in all Horn DNFs representing f. We call such functions *non-trivial*.

The book [3] distinguishes four special classes of Horn functions: submodular functions, bidual Horn functions, double Horn functions and acyclic Horn functions.

For the acyclic Horn functions, the validity of Conjecture 3 follows directly from the definition with the help of Lemma 2. To define this notion, let us associate with a pure Horn DNF ϕ representing a function f of n variables $x_1, \dots, x_n$ a directed graph G_ϕ, *the implicant graph*, with vertex set $\{x_1, \dots, x_n\}$ containing an arc (x_i, x_j) whenever ϕ contains a term involving both $\overline{x}_i$ and x_j. A pure Horn function is called *acyclic* if G_ϕ is acyclic. Since any acyclic graph contains a sink vertex, i.e. a vertex with no out-going arcs, we conclude with the help of Lemma 2 that every acyclic Horn function contains a good variable.

For submodular functions and double Horn (and more general) functions, we verify Conjecture 3 in Sections 4 and 5, respectively. For bidual Horn functions the conjecture remains open and we discuss it in Section 6.

4. Submodular functions

In this section we study a subclass of Horn functions known as submodular.

Definition 1. A function $f(X)$ is called *submodular* if $f(X \vee Y) \vee f(X \wedge Y) \leq f(X) \vee f(Y)$.

To reveal a relationship between submodular functions and Horn functions, let us say that a Boolean function $f(x_1, \dots, x_n)$ is *co-Horn* if the function $f(\overline{x}_1, \dots, \overline{x}_n)$ is Horn. The following characterization of submodular functions was proved in [6].

Theorem 2. *A Boolean function is submodular if and only if it is both Horn and co-Horn. All prime implicants of a submodular function are either linear or quadratic pure Horn.*

Theorem 3. *Submodular Boolean functions satisfy Frankl's conjecture.*

Proof. Let $f(x_1, x_2, \dots, x_n)$ be a submodular function. Without loss of generality we assume that f is non-trivial and that D_f is a DNF representation of f in which every term is quadratic pure Horn.

Let G_f be the implicant graph associated with D_f. Given a Boolean point $X = (\alpha_1, \dots, \alpha_n)$, we denote by $G_f(X)$ a labelled graph obtained from G_f by assigning label α_i to vertex x_i for each i. It is not difficult to see that

(∗) X is a true point of f if and only if there is an arc (or equivalently, a directed path) in $G_f(X)$ from a 0-vertex (i.e. a vertex labelled 0) to a 1-vertex.

Now we contract each strongly connected component of G_f into a single vertex obtaining in this way a directed acyclic graph G^*_f. By Theorem 2, this graph represents a new submodular function f^* with a DNF D_{f^*} whose terms correspond to the arcs of G^*_f.

For a vertex x of G_f we denote by c_x the strongly connected component of the graph containing x. Note that c_x is a subset of $V(G_f)$, as well as a vertex of G^*_f and a variable of f^*.

If X is a false point of f, then, according to (∗), within each strongly connected component of $G_f(X)$ either all vertices are labelled 0 or all vertices are labelled 1. This can be viewed as a 0-1 labelling of the vertices of G^*_f, and we denote a labelled graph G^*_f corresponding to a false point X of f by $G^*_f(X)$. Also, since X is false, there is no directed path from a 0-vertex to a 1-vertex in $G^*_f(X)$, since otherwise such a path could be found in $G_f(X)$. Therefore, the 0-1 labelling of the vertices of G^*_f corresponding to a false point of f defines a false point of the function f^*. Similarly, a false point of f^* corresponds to a false point of f. It is not difficult to see that this is a one-to-one correspondence, showing that $|F(f)| = |F(f^*)|$. Moreover, for any variable x of f, $|F(f_{|x=0})| = |F(f^*_{|c_x=0})|$ and $|F(f_{|x=1})| = |F(f^*_{|c_x=1})|$ in view of the above discussion.

Since G^*_f is acyclic, it contains sink vertices, i.e. vertices with no out-going arcs. Sink variables do not appear in D_{f^*} negatively and hence, by Lemma 2, each of them is a good variable of f^*.

Let x be a vertex of G_f such that c_x is a sink vertex of G^*_f. Since c_x is a good variable of f^*, we have $|F(f^*_{|c_x=1})| \leq |F(f^*_{|c_x=0})|$. Therefore,

$$|F(f_{|x=1})| = |F(f^*_{|c_x=1})| \leq |F(f^*_{|c_x=0})| = |F(f_{|x=0})|$$

and hence x is a good variable. □

5. Double Horn and more general functions

A Boolean function f is *double Horn* if both f and $\overline{f}$ are Horn. According to [4], double Horn functions admit a DNF representation where each variable appears negatively at most once. We will show that any such Horn function satisfies Frankl's conjecture. More generally, we will verify the conjecture for Horn Boolean functions that admit a Horn DNF satisfying the following property, which we call *dependency*: for each variable x there is a variable $d(x)$ such that $d(x)$ appears (positively) in every term containing x negatively. Clearly, any non-trivial Horn DNF containing each variable negatively at most once satisfies the dependency property.

Theorem 4. *Let f be a non-trivial Boolean function that admits a Horn DNF $D = D_f$ satisfying the dependency property. Then Frankl's conjecture is valid for f.*

Proof. Each true point of f is covered by at least one term in D. To prove the theorem, we construct a binary matrix M with n rows corresponding to the variables of f and $|T(f)|$ columns representing the true points of f. A good variable (if exists) corresponds to a row of the matrix containing at least as many 1s as 0s. To prove that such a variable exists, we will show that the entire matrix M contains at least as many 1s as 0s. To this end, we will map the set of 0s of M injectively to the set of 1s of M according to the following procedure. Consider a 0 in M and denote it by z. Also, we denote by i the row containing z and by X the column containing z. In other words, X is a true point with a zero in the i-th coordinate.

(a) If the true point X is covered by a term t of D for which x_i is free, then the point X' obtained by switching the i-th coordinate to a 1 also is a true point covered by t and we map the zero z to the 1 in X' in the same row (coordinate). We call it a *horizontal map.*
(b) If X is not covered by any term of D for which x_i is free, then it is covered by some term t containing x_i negatively. According to the dependency property, t also contains (positively) the variable $x_j = d(x_i)$, i.e. the j-th coordinate of X is a 1, in which case we map the zero z to the 1 in the j-th coordinate of X. We call it a *vertical map* (mapping a 0 to a 1 in the same column).

It remains to show that the mapping described above is injective. It is not difficult to see that no two horizontal maps send two different 0s to the same 1. The same is true for any two vertical maps. Indeed, when we apply a vertical map to a zero in column X, this zero represents a negative literal $\overline{x}_i$ in some term t. Any other zero in the same column corresponds to a variable x_j which is free for t, since otherwise either X is not a true point (if x_j appears in t positively) or D is not a Horn DNF (if x_j appears in t negatively). Therefore, when we apply a vertical map to a zero in column X, all other zeros in the same column are mapped horizontally.

Now assume that a vertical map sends a zero z_1 to a 1, denote it by u, and a horizontal map sends a zero z_2 to the same u. We denote the row containing z_1 by i and the column containing z_1 (and u) by X_1. Also, let j be the row containing z_2 (and u) and let X_2 be the column containing z_2. Finally, let t_1 be a term covering X_1, as in the definition of the vertical map applied to z_1, and let t_2 be a term covering X_2, as in the definition of the horizontal map applied to z_2.

According to the definition of the horizontal map, X_1 and X_2 differ only in the j-th coordinate, and x_j is free for t_2, i.e. t_2 also covers X_1. Also, according to the definition of the vertical map, x_i appears in t_1 negatively, and $x_j = d(x_i)$ appears in t_1 positively. Since t_2 does not contain x_j, and t_1 contains x_j, the two terms t_1 and t_2 are different.

We know that t_2 contains x_i, since otherwise we had to apply a horizontal map to z_1. Moreover, t_2 contains x_i negatively, because the i-th coordinate of both X_1 and X_2 is zero. But then, by the dependency property, t_2 also contains (positively) x_j, which is not possible, since the j-th coordinate of X_2 is zero, and t_2 covers X_2. This contradiction completes the proof. □

6. Concluding remarks and open problems

According to [1], one of the major obstacles for proving the conjecture is that we do not know where to expect a good element. Theorem 3 suggests that such an element should be expected in a strongly connected component C of the implicant graph, which becomes a sink vertex after contracting each strongly connected component to a single vertex. Theorem 4, on the other hand, suggests a possible way of proving the existence of a good element in C by showing that the matrix M of true points restricted to the elements of C contains at least as many 1s as 0s.

Taking into account the special role of duality in this topic, the next natural step towards proving the conjecture via a Boolean approach is to consider the class of bidual Horn functions studied in [5]. These are Horn functions f such that the dual of f is also Horn, where the dual of f is the function f^d such that $f^d(X) = \overline{f(\overline{X})}$ for all Boolean points X. We believe that this is the core step towards proving the conjecture. Even more specifically, we believe that the class of self-dual Horn functions, i.e. Horn functions f such that $f = f^d$, which forms a proper subclass of bidual functions, is a key to cracking the conjecture. These are functions where an instance of the problem comes simultaneously in both forms: intersection-closed (over the false points) and union-closed (over the true points). Proving the conjecture for self-dual Horn functions seems to be a challenging problem.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication.

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