

Transient and Asymptotic Properties of Robust Adaptive Controllers in the Presence of Non-Coercive Lyapunov Functions

Aditya A. Paranjape, Vivek Natarajan and Supratim Ghosh

Abstract—Adaptive control architectures often make use of Lyapunov functions to design adaptive laws. We are specifically interested in adaptive control methods, such as the well-known \mathcal{L}_1 adaptive architecture, which employ a parameter observer for this purpose. In such architectures, the observation error plays a critical role in determining analytical bounds on the tracking error as well as robustness. In this paper, we show how the non-existence of coercive Lyapunov operators can impact the analytical bounds, and with it the performance and the robustness of such adaptive systems.

I. INTRODUCTION

Lyapunov equations with non-coercive solutions are a peculiar feature of infinite dimensional systems [3], [14]. In a finite dimensional setting, the Lyapunov equation corresponding to a Hurwitz matrix yields a positive definite solution. In semilinear systems of the form $\dot{w} = Aw + f(w)$, where A is Hurwitz, one can use this solution to determine permissible bounds on $f(w)$ as well as the associated bounds on the solution $w(t)$.

In infinite dimensional systems, the impact of non-coercivity can be felt on the nature of bounds that can be derived for $w(t)$; see [8], [5] for example. There are ways to get around the non-coercivity, by invoking additional assumptions on the system (e.g., a stronger form for the Lyapunov equation [1]) or delicate fictitious modifications which aid the derivation of a coercive Lyapunov function [16].

In this paper, we will consider robust adaptive control of systems of semilinear partial differential equations (PDEs) of the form $\dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}u(t) + f(w)$, $y(t) = \mathcal{C}w(t)$, where $w(t)$ denotes the system state, $u(t)$ is the control input, and $y(t)$ is the output. The operators \mathcal{A} , \mathcal{B} and \mathcal{C} are the state, control, and output operators, respectively. Coercive Lyapunov functions feature prominently in the derivation of adaptive laws, and help ensure appropriate bounds on the tracking error [9], [10], [11]. Our objective is to determine how the guaranteed bounds change in the absence of a coercive solution to the usual, unmodified Lyapunov equation.

A. Contribution

In this paper, we examine the effects of non-coercive Lyapunov functions on the performance and stability of semilinear infinite dimensional systems controlled by an adaptive controller based on the \mathcal{L}_1 philosophy [4]. In particular, we

AP and SG are with TCS Research, a part of Tata Consultancy Services Limited, India. Email: {aditya.paranjape@tcs.com; supratim.ghosh2@tcs.com}. VN is with the Systems and Control Engineering Group at the Indian Institute of Technology Bombay in Mumbai, India. Email: vivek.natarajan@iitb.ac.in.

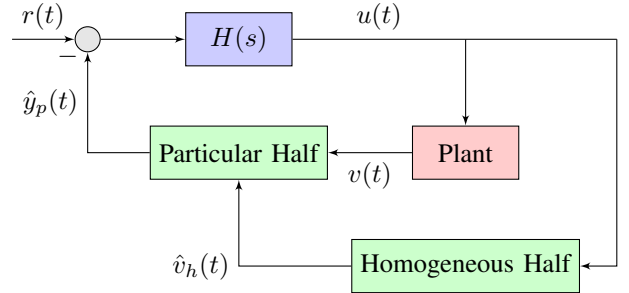


Fig. 1: A block diagram of the DAC framework, with the subscripts p and h denoting signals from the particular and homogeneous components. The symbols $v(t)$, $y(t)$, and $r(t)$ denote the system state, output and reference signal, respectively.

consider a semilinear system with unmatched uncertainties, and a dyadic adaptive architecture based on [11], see Fig. 1.

It has been shown previously [9], [11] how a coercive Lyapunov function helps derive tight bounds on the tracking performance and the control inputs. In this paper, we extend the analysis to derive weaker bounds when a coercive Lyapunov function cannot be found.

Although we consider a specific dyadic adaptive architecture in this paper, our conclusions or results can be extended to other adaptive architectures such as model reference adaptive control (MRAC) where Lyapunov analysis is used to derive the adaptive laws and prove that the error between the reference model and the system is suitably bounded.

The paper is organized as follows. We introduce the mathematical preliminaries in Sec. II, and the problem formulation in Sec. III. In Sec. IV, we present the design of the control law. In Sec. V, we show the boundedness of the observation error. We discuss closed-loop stability and model-following in Sec. VI.

II. PRELIMINARIES

A. Spaces, operators and norms

Definition 1 (\mathcal{L}_∞ and \mathcal{L}_1 norms): Given $q(t) \in \mathbb{R}^n$ with components $q_i(t)$ ($1 \leq i \leq n$), we define

$$\|q(t)\|_\infty = \max_{1 \leq i \leq n} |q_i(t)|, \quad \|q\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t \geq 0} \|q(t)\|_\infty$$

$$\|q\|_{\mathcal{L}_\infty, \tau} = \operatorname{ess\,sup}_{0 \leq t \leq \tau} \|q(t)\|_\infty$$

If $\|q\|_{\mathcal{L}_\infty} < \infty$, then we denote $q \in \mathcal{L}_\infty^n$. The 1-norm of a matrix $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined as $\|F\|_1 =$

$\sup_{\|p\|_\infty=1} \|Fp\|_\infty$, $p \in \mathbb{R}^m$. The \mathcal{L}_1 norm of a linear operator $\mathcal{F} : \mathcal{L}_\infty^m \mapsto \mathcal{L}_\infty^n$ is defined as $\|\mathcal{F}\|_{\mathcal{L}_1} = \sup_{\|q\|_{\mathcal{L}_\infty^m}=1} \|\mathcal{F}q\|_{\mathcal{L}_\infty^n}$, $q \in \mathcal{L}_\infty^m$.

The spatial domain of interest in this paper is the closed interval $[0, L]$ for some finite $L > 0$. Let $\mathbb{Z} = \mathcal{L}_2([0, L], \mathbb{R}^n)$ denote the Hilbert space of square integrable functions with the usual inner product and norm.

Definition 2: We define \mathbb{W} as the space of \mathbb{Z} -valued functions on $\mathbb{R}_{\geq 0}$ satisfying $\text{ess sup}_{t \geq 0} \|w(t)\|_{\mathbb{Z}} < \infty$. The space \mathbb{W} is a Banach space with the norm $\|w\|_{\mathbb{W}} = \text{ess sup}_{t \geq 0} \|w(t)\|_{\mathbb{Z}}$. For $\tau \geq 0$, we define the truncated norm given by $\|w\|_{\mathbb{W}, \tau} = \text{ess sup}_{0 \leq t \leq \tau} \|w(t)\|_{\mathbb{Z}}$ and the associated Banach space by \mathbb{W}_τ .

Definition 3: We denote a linear operator between spaces X and Y by $\mathcal{L}(X, Y)$.

Definition 4 ([15], Definition 1.1, Ch. 6): Consider a system $\dot{w} = \mathcal{A}w + f(t, w)$, $w(t=0) = w_0 \in \mathbb{Z}$, where \mathcal{A} is the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ and $f(t, w)$ is continuous in t and satisfies a Lipschitz condition in w . The mild solution $w(t)$ is given by

$$w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t-\tau)f(\tau, w(\tau)) d\tau, \quad (1)$$

Definition 5 (Convolution): Given a C_0 semigroup $\mathcal{T}(t)$ and $\tau \geq 0$, we define the operator $\mathcal{T} \star (t)(\tau) : \mathbb{W}_\tau \mapsto \mathbb{W}_\tau$ as $\mathcal{T} \star (\tau)f = \int_0^\tau \mathcal{T}(\tau-s)f(s, w(s)) ds \forall f \in \mathbb{W}_\tau$. We define the induced norm $\|\mathcal{T} \star\| \triangleq \text{ess sup}_{(t \geq 0)} \|\mathcal{T} \star (t)\|$.

We recall the following result from [15] for solutions of initial value problems in Definition 4.

Theorem 1 (Theorems 6.1.4, 6.1.5, [15]): Let \mathcal{A} be the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ on the Hilbert space \mathbb{Z} . If $f : [0, T] \times \mathbb{Z} \rightarrow \mathbb{Z}$ is continuously differentiable with respect to both arguments, for $T > 0$, then the mild solution (1) is a classical solution of the initial value problem in Definition 4 for $t \in [0, T]$. If the solution exists only up to $T_{\max} < T$, then $\|w(t)\|_{\mathbb{Z}} \rightarrow \infty$ as $t \rightarrow T_{\max}$.

Next, we define the projection operator [6] which will be used for constructing the adaptive laws. Let $\pi : \mathbb{R}^k \rightarrow \mathbb{R}$ be defined by

$$\pi(\alpha) \equiv \pi(\alpha; \kappa, \epsilon) = \frac{\langle \alpha, \alpha \rangle - \kappa^2}{\epsilon \kappa^2}, \quad \alpha \in \mathbb{R}^k, \quad \kappa \in \mathbb{R}^+$$

The number $\epsilon \in \mathbb{R}^+$ is chosen to be arbitrarily small. The Fréchet derivative of π at $\alpha_1 \in \mathbb{R}^k$ is denoted by $\pi'(\alpha_1) \in \mathbb{R}^k$ and it satisfies

$$\langle \pi'(\alpha_1), \alpha_2 \rangle = \frac{2\langle \alpha_1, \alpha_2 \rangle}{\epsilon \kappa^2} \quad \forall \alpha_2 \in \mathbb{R}^k$$

Definition 6: The projection operator $\text{Proj} : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is defined as

$$\text{Proj}(\alpha_1, \alpha_2) = \begin{cases} \alpha_2, & \text{if } \pi(\alpha_1) \leq 0 \text{ or } \langle \pi'(\alpha_1), \alpha_2 \rangle \leq 0 \\ \alpha_2 - \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2} \times \left\langle \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2}, \alpha_2 \right\rangle \pi(\alpha_1), & \\ \text{otherwise} & \end{cases} \quad (2)$$

Lemma 1 (Lemma 9 in [6]): Let Ω_0 and Ω_1 denote the convex sets satisfying

$$\Omega_0 = \{\alpha \mid \pi(\alpha) \leq 0\}, \quad \Omega_1 = \{\alpha \mid \pi(\alpha) \leq 1\}$$

Suppose that $\alpha_1^* \in \Omega_0$. Then, for all $\alpha_1, \alpha_2 \in \mathbb{R}^k$, $(\alpha_1 - \alpha_1^*)(\text{Proj}(\alpha_1, \alpha_2) - \alpha_2) \leq 0$. Moreover, the solution of the initial value problem $\dot{\alpha}_1 = \text{Proj}(\alpha_1, \alpha_2)$, $\alpha_1(0) = \alpha_{10}$, has the property that if $\alpha_{10} \in \Omega_1$, then $\alpha_1(t) \in \Omega_1$ for all t .

B. Stability

We will need the following weaker notion of asymptotic stability, in addition to the more usual notions of stability.

Definition 7: We say that a function $f : \mathbb{R}_{\geq \nu} \rightarrow \mathbb{R}$ converges to 0 almost asymptotically if

$$\lim_{n \rightarrow \infty} f(nx) = 0 \quad \text{for almost all } x \geq 0$$

Lemma 2 ([7], Theorem 1): If $f(t) \in \mathcal{L}_2(\mathbb{R}_{\geq \nu}, \mathbb{R})$, then $f(t)$ converges to 0 almost asymptotically, in the sense of Definition 7.

Consider the abstract system $\dot{z} = \mathcal{A}_m z + g(z)$, $z(0) = z_0$, where $z_0 \in \mathcal{D}(\mathcal{A}_m)$ and \mathcal{A}_m is the infinitesimal generator of an exponentially stable semi-group $\mathcal{T}(t)$ and g satisfies a Lipschitz condition in z . The following lemma asserts the existence of a Lyapunov function corresponding to \mathcal{A}_m .

Assumption 1 (based on [9]): Let $\mathcal{Q} > 0$ be a self-adjoint, boundedly invertible operator on \mathbb{Z} ; i.e., $\mathcal{Q}^{-1} \in \mathcal{L}(\mathbb{Z})$ and with $\mathcal{D}(\mathcal{A}_m) \subseteq \mathcal{D}(\mathcal{Q})$. We assume that there exists $\mathcal{P} \in \mathcal{L}(\mathbb{Z})$ with $\mathcal{P} > 0$ such that,

$$\begin{aligned} \langle \mathcal{A}_m z, \mathcal{P} z \rangle_{\mathbb{Z}} + \langle \mathcal{P} z, \mathcal{A}_m z \rangle_{\mathbb{Z}} &\leq -\langle z, \mathcal{Q} z \rangle_{\mathbb{Z}}, \\ \forall z \in \mathcal{D}(\mathcal{A}_m) & \end{aligned} \quad (3)$$

We note that a solution $\mathcal{P} \in \mathcal{L}(\mathbb{Z})$ exists if $\mathcal{Q} \in \mathcal{L}(\mathbb{Z})$ [2].

Remark 1: If $\mathcal{A}_m + \mathcal{A}_m^* < 0$ or if \mathcal{A}_m is the infinitesimal generator of C_0 group [14], then it is possible to find \mathcal{Q} satisfying Assumption 1 such that the $\mathcal{P} \in \mathcal{L}(\mathbb{Z})$ is coercive.

III. PROBLEM FORMULATION

This paper is concerned with the control of systems of semilinear infinite dimensional systems described by

$$\dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}u(t) + \alpha f(w), \quad y(t) = \mathcal{C}w(t) \quad (4)$$

where $w(t) \in \mathbb{Z}$ and $u(t) \in \mathbb{R}$, $\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathbb{Z})$ and $\mathcal{C} \in \mathcal{L}(\mathbb{Z}, \mathbb{R})$. The control objective is to design $u(t)$ so that the output $y(t)$ tracks a reference signal $r(t)$, and the resulting closed-loop system is stable and robust in the sense of \mathcal{L}_∞ .

Assumption 2: The nonlinearity $f(w)$ is a known C^1 function of w , while $\alpha \in \mathbb{R}^n$ is unknown but satisfies $|\alpha_i| < \nu_\alpha$ for all $i \in \{1, 2, \dots, n\}$.

The analysis in the paper does not require that α_i be a constant, and neither is it necessarily restricted to a single ‘‘basis’’ function $f(w)$ (see [13]). This assumption does, however, simplify the presentation.

Assumption 3: The permissible initial conditions are restricted by $\|w_0\|_{\mathbb{Z}} < \rho_0$, and $w_0 \in \mathcal{D}(\mathcal{A})$.

Assumption 4 (Stabilizability): There exists $\mathcal{K} \in \mathcal{L}(\mathbb{Z}, \mathbb{R})$ such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is the infinitesimal generator of an exponentially stable semi-group.

Lemma 3: For every $\rho > 0$, there exist constants $\nu_1(\rho)$ and $\nu_2(\rho)$ such that if $\|w\|_{\mathbb{W}, \tau} < \rho$ for some $\tau > 0$, then $\|f(w)\|_{\mathbb{W}, \tau} \leq \nu_1(\rho)\|w\|_{\mathbb{W}, \tau} + \nu_2(\rho)$.

IV. CONTROL DESIGN

A. Control Signal

Consider the system

$$\dot{w}_h(t) = \mathcal{A}w_h(t) + \mathcal{B}u(t), \quad y_h = Cw_h \quad (5)$$

which is found by neglecting the nonlinearity in (4). Using Assumption 4, we deduce that there exists a bounded stabilizing gain $\mathcal{K} : \mathbb{Z} \rightarrow U$ such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ generates an exponentially stable C_0 semigroup. We formalize this as follows.

Definition 8: The operator $\mathcal{A}_m = \mathcal{A} - \mathcal{B}\mathcal{K}$ generates an exponentially stable semigroup $\mathcal{T}(t)$; i.e., there exist constants $M, \beta > 0$ such that $\|\mathcal{T}(t)\|_i \leq Me^{-\beta t}$. Moreover, $\|\mathcal{T} * \cdot\|$ is bounded.

Based on our prior work [12], [10], we use the following control law for the system (4):

$$u(t) = -\mathcal{K}w(t) - H_C p(t) \quad (6)$$

$$\dot{p}(t) = H_A p(t) + H_B \sigma(t), \quad p(0) = p_0 \quad (7)$$

with H_A Hurwitz. The term $\sigma(t)$, on which $p(t)$ depends, will be defined presently. The terms H_C and H_B are chosen to satisfy the DC gain condition $C(-\mathcal{A}_m)^{-1}\mathcal{B}H_C(-H_A)^{-1}H_B = -1$.

The system (4) can now be written as

$$\dot{w}(t) = \mathcal{A}_m w(t) - \mathcal{B}H_C p(t) + \alpha f(w(t)) \quad (8)$$

Using the linear term as a pivot, we decompose the system in (4) into two sub-systems

$$\dot{w}_p = \mathcal{A}_m w_p + \alpha f(w), \quad y_p = Cw_p \quad (9)$$

$$\dot{w}_h = \mathcal{A}_m w_h - \mathcal{B}H_C p(t), \quad y_h = Cw_h \quad (10)$$

The two systems (9) and (10) are referred to as the *particular* and *homogeneous* halves, respectively. In the next section, we will derive an observer for estimating the states; for now, we use (9) and (10) to investigate tracking.

If we could choose $\sigma(t) = r(t) - y_p(t)$, we would get that the tracking error $y(t) - r(t) = y_h(t) - \sigma(t)$; therefore, $\sigma(t)$ can serve as the reference signal for $y_h(t)$. Since $y_p(t)$ is not known, we will choose

$$\sigma(t) = r(t) - \hat{y}_p(t) \quad (11)$$

where $\hat{y}_p(t)$ is the output of an observer which will be designed presently (see (12)).

B. Observer Design

We use the symbol “ \wedge ” to denote observer states, and the subscripts p and h to denote states of the particular and the homogeneous halves, respectively. The dynamics of the two halves are given by

$$\dot{\hat{w}}_p = \mathcal{A}_m \hat{w}_p + \hat{\alpha}(t)f(w), \quad \hat{y}_p = C\hat{w}_p \quad (12)$$

$$\dot{\hat{w}}_h = \mathcal{A}_m \hat{w}_h - \mathcal{B}H_C p(t), \quad \hat{y}_h = C\hat{w}_h \quad (13)$$

with the initial conditions $\hat{w}_h(0) = w(0)$ and $\hat{w}_p(0) = 0$.

The predicted values $\hat{\alpha}(t)$ are found using the projection operator (see [6], [9] for details).

$$\begin{aligned} \hat{\alpha}_i(t) &= \gamma \text{Proj}(\hat{\alpha}_i, -\langle \mathcal{P}\tilde{w}(t), f(w)e_i \rangle_{\mathbb{Z}}), \\ |\hat{\alpha}_i(t)| &< \nu_\alpha(1 + \epsilon) \end{aligned} \quad (14)$$

where $\epsilon \in \mathbb{R}^+$ is arbitrarily small; $\tilde{w} = \hat{w}_p + \hat{w}_h - w$; $\hat{\alpha}_i$ ($1 \leq i \leq n$) is the i^{th} component of $\hat{\alpha}$, e_i denotes the i^{th} column of the $n \times n$ identity matrix, and $\gamma > 0$ is the adaptation gain.

The operator $\mathcal{P} > 0$ in (14) is found by solving the Lyapunov equation (3) with \mathcal{Q} chosen as follows:

$$\mathcal{Q} = \begin{cases} -(\mathcal{A}_m + \mathcal{A}_m^*) & \text{if } \mathcal{A}_m + \mathcal{A}_m^* < 0 \\ \mathcal{I} & \text{otherwise} \end{cases} \quad (15)$$

where \mathcal{I} is the identity operator on \mathbb{Z} . In the first case, it can be seen that $\mathcal{P} = \mathcal{I}$, which is coercive.

In summary, the closed-loop system consists of the original system (4), together with the controller (6), and the dyadic observer (12), (13) and (14).

C. Well-Posedness

To analyze the well-posedness of the closed-loop system, we construct the augmented vector $\mathbf{w} = [w, \hat{w}_p, \hat{w}_h, p(t)]^\top \in \mathbb{V} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^{n_p}$. The dynamics of \mathbf{w} is given by

$$\dot{\mathbf{w}}(t) = \bar{\mathcal{A}}\mathbf{w} + \bar{f}(\hat{\alpha}(t), w(t), r(t)) \quad (16)$$

$$w(0) = \hat{w}_h = w_0, \quad \hat{w}_p(0) = 0, \quad p(0) = p_0$$

$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_m & 0 & 0 & -\mathcal{B}k_r \\ 0 & \mathcal{A}_m & 0 & 0 \\ 0 & 0 & \mathcal{A}_m & -\mathcal{B}k_r \\ 0 & -H_B & 0 & H_A \end{bmatrix}$$

where the exogenous signal $\hat{\alpha}(t)$ is known to be C^1 in time. Therefore, it can be checked readily that $\bar{f}(\cdot)$ is a C^1 function of its arguments. Furthermore, the operator $\bar{\mathcal{A}}$ is the infinitesimal generator of a semigroup. We state the following result without proof, but as a direct application of Thm. 1.

Lemma 4: There exists $T_{\max} > 0$ such that the system (16) has a unique classical solution $\mathbf{w}(t)$ for $t \in [0, T_{\max}]$. Moreover, if $T_{\max} < \infty$, then $\lim_{t \rightarrow T_{\max}} \|\mathbf{w}(t)\|_{\mathbb{V}} \rightarrow \infty$.

D. A Necessary Condition for Tracking

Lemma 5 (Necessary condition for tracking): Suppose that w is suitably bounded and we design $u(t)$ to ensure that $\hat{y}_h(t)$ tracks $\sigma(t) = r(t) - \hat{y}_p(t)$, where the signals have been defined in (11), (12) and (13). Then, $y(t)$ tracks $r(t)$ only if $y(t)$ tracks $\hat{y}(t)$.

The necessary condition stated here is quite obvious, but its role will become clear in the subsequent analysis. Informally speaking, when coercivity is lost, it may no longer be possible to prove asymptotic bounds on the observer states themselves, but one can prove asymptotic bounds on the observer output. In the next section, we will prove output error regulation.

V. OBSERVER ERROR REGULATION

In this section, we derive bounds on the observation error between \hat{w}_h and \hat{w}_p on the one hand and w_p and w_h , respectively, on the other. Let $\hat{w} = \hat{w}_h + \hat{w}_p$, and let $(\hat{\cdot}) = (\cdot) - (\cdot)$ denote the error between predicted and the actual terms. We have two objectives: derive tight bounds on \tilde{w} and \tilde{y} , and show that \tilde{y} converges to zero asymptotically if an arbitrarily tight bound (in a sense that will become clear presently) cannot be derived.

From (8), (12) and (13), the observation error dynamics are given by

$$\dot{\tilde{w}}(t) = \mathcal{A}_m \tilde{w}(t) + \tilde{\alpha}(t) f(w(t)), \quad \tilde{y}(t) = \mathcal{C} \tilde{w}(t), \quad \tilde{w}(0) = 0 \quad (17)$$

We recall that \mathcal{C} is bounded.

We start by proving a bound on $\tilde{w}(t)$ that relies only on the boundedness of $\tilde{\alpha}(t)$. Understandably, this is a weak bound and we will subsequently make it stronger in the following subsections under additional assumptions. A key point is that it does not rely on the coercivity of \mathcal{P} in the projection-based adaptive laws.

Lemma 6: Suppose that $\|w\|_{\mathbb{W},\tau} < \rho_w$ for some constant $\rho_w > 0$. Then, the adaptive laws in (14) ensure that $\|\tilde{w}\|_{\mathbb{W},\tau}$ and $|\tilde{y}(t)|$ are bounded for $t < \tau$.

Proof: From (17), note that

$$\|\tilde{w}\|_{\mathbb{W},\tau} \leq \|\mathcal{T} \star (\tau)\| \|\tilde{\alpha} f(w)\|_{\mathbb{W},\tau}$$

Notice that $\tilde{\alpha}(t)$ is bounded for all t due to the projection-based laws. Furthermore, from Lemma 3, it follows that $\|f(w)\|_{\mathbb{W},\tau}$ is bounded. Since \mathcal{T} is exponentially stable and since all other terms on the RHS are bounded, it follows that $\|\tilde{w}\|_{\mathbb{W},\tau}$ is bounded. Since the output operator \mathcal{C} is bounded, it follows that $|\tilde{y}(t)|$ is bounded for $t < \tau$. ■

In the subsequent sections, we will strengthen the bounds on \tilde{w} and \tilde{y} . In particular, coercivity of \mathcal{P} will play an essential part in strengthening the bounds on \tilde{w} . We will show that it is possible to obtain stronger bounds on \tilde{y} (but not necessarily \tilde{w}) in the absence of coercivity.

A. Case 1: \mathcal{A}_m permits a coercive \mathcal{P}

We start with the case where \mathcal{A}_m permits a coercive solution to the Lyapunov equation. This result is a combination of those in [9] and [11].

Lemma 7: Suppose that $\|w\|_{\mathbb{W},\tau} < \rho_w$ for some constant $\rho_w > 0$. Suppose that a coercive solution \mathcal{P} exists for (3) and is used in the projection operator (14). Then, we have that all of the following terms are uniformly bounded for $t < \tau$: (i) the total observation errors $\|\tilde{w}(t)\|_{\mathbb{Z}}$ and $|\tilde{y}(t)|$; (ii) the observation errors $\|\tilde{w}_p(t)\|_{\mathbb{Z}}$ and $|\tilde{y}_p(t)|$ for the particular half, and (iii) the observation errors $\|\tilde{w}_h(t)\|_{\mathbb{Z}}$ and $|\tilde{y}_h(t)|$ for the homogeneous half. Moreover, the bounds can be made arbitrarily small by increasing γ .

Proof: We start by proving the bounds for the total observer error. We consider the Lyapunov function

$$V(t) = \langle \tilde{w}(t), \mathcal{P} \tilde{w}(t) \rangle + \frac{1}{\gamma} \tilde{\alpha}(t)^\top \tilde{\alpha}(t) \quad (18)$$

where the choice of $\mathcal{P} > 0$ is explained in Sec. IV-B.

Differentiating the Lyapunov function gives

$$\begin{aligned} \dot{V}(t) &= \langle \mathcal{P} \dot{\tilde{w}}, \mathcal{A}_m \tilde{w} \rangle + \langle \mathcal{A}_m \tilde{w}, \mathcal{P} \tilde{w} \rangle \\ &\quad + 2 \langle \mathcal{P} \tilde{w}, \tilde{\alpha}(t) f(w) \rangle + \frac{1}{\gamma} \tilde{\alpha}(t)^\top \dot{\tilde{\alpha}}(t) \end{aligned} \quad (19)$$

Using (14) and the properties of the projection operator in Lemma 1, it follows that

$$\dot{V} \leq -\langle \tilde{w}, \mathcal{Q} \tilde{w} \rangle \quad (20)$$

Since \mathcal{Q} is boundedly invertible, exists a constant $\lambda_p > 0$ satisfying

$$\dot{V} \leq -\lambda_p \langle \tilde{w}, \mathcal{P} \tilde{w} \rangle$$

Substituting into (20), and by adding and subtracting $\tilde{\alpha}^\top \tilde{\alpha}$ with suitable scaling, we get

$$\dot{V} \leq -\lambda_p V + \frac{\lambda_p}{\gamma} \tilde{\alpha}(t)^\top \tilde{\alpha}(t)$$

Since $\|w\|_{\mathbb{Z}} < \rho$, and $\tilde{\alpha}$ is bounded, it follows that there exists constant $\rho_1 > 0$, which is independent of γ , such that

$$V(t) \leq V(0) e^{-\lambda_p t} + \frac{\rho_1}{\lambda_p \gamma} (1 - e^{-\lambda_p t}) \quad (21)$$

Since $\tilde{w}(0) = 0$, we have that

$$V(0) = \frac{1}{\gamma} \tilde{\alpha}(0)^\top \tilde{\alpha}(0) \quad (22)$$

Using the coercivity of \mathcal{P} , we deduce that

$$\|\tilde{w}(t)\|_{\mathbb{Z}} \leq \frac{\mu_{p,1} + \mu_{p,2} e^{-\lambda t}}{\sqrt{\gamma}} \leq \frac{\mu_{p,1} + \mu_{p,2}}{\sqrt{\gamma}},$$

where the constant $\mu_{p,1}$ and $\mu_{p,2}$ depend on \mathcal{P} and ρ_1 . A similar bound for \tilde{y} follows from the fact that \mathcal{C} is bounded. Clearly, the bounds can be made arbitrarily small by increasing γ .

The proof for the boundedness of $\|\tilde{w}_p(t)\|_{\mathbb{Z}}$ and $|\tilde{y}_p(t)|$ for $t < \tau$ is identical to that for $\|\tilde{w}\|_{\mathbb{Z}}$ and $|\tilde{y}(t)|$. This is because the error equation for \tilde{w}_p is identical to (17), except with \tilde{w} therein replaced by \tilde{w}_p . Thereafter, we infer the bounds on $\|\tilde{w}_h(t)\|_{\mathbb{Z}}$ and $|\tilde{y}_h(t)|$ for $t < \tau$ using the triangle inequality. This completes the proof. ■

B. Case 2: \mathcal{A}_m does not permit a coercive \mathcal{P}

In this section, we consider the case where a coercive solution to (3) cannot be found. We prove two results here; informally speaking, these are either weaker results for the same set of assumptions as earlier, or equally strong results under stronger assumptions on the system.

The first result is motivated by [1] (Theorem 2 therein). We use the conditions of the Kalman-Yakubovich-Popov (KYP) lemma to derive a strong bound on $|\tilde{y}(t)|$, similar to Lemma 7. The KYP lemma is used routinely when dealing with output feedback problems, as in [1]. In our paper, it provides a way to deal with non-coercive settings when its conditions are met.

Theorem 2: Consider the observer error dynamics (17) and let $\|w\|_{\mathbb{W},\tau} < \rho_w$ be suitably bounded. Suppose that

there exists a constant $\mathcal{F} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_m), \mathbb{Z})$, $\mathcal{P} \in \mathcal{L}(\mathbb{Z})$ with $\mathcal{P} \geq 0$, $\mathcal{Q} : \mathcal{D}(\mathcal{A}_m) \rightarrow \mathbb{Z}$ with $\langle z, \mathcal{Q}z \rangle \geq \epsilon_Q \|z\|_{\mathbb{Z}}^2$ for all $z \in \mathcal{D}(\mathcal{A}_m)$ and an operator $\mathcal{E} \in \mathcal{L}(\mathbb{Z}, \mathbb{R})$ such that

$$\begin{aligned} \mathcal{A}_m^* \mathcal{P}z + \mathcal{P} \mathcal{A}_m z &= -\mathcal{F}^* \mathcal{F}z - \mathcal{Q}z \\ \mathcal{E} \mathcal{P}z &= \mathcal{C}z \end{aligned} \quad (23)$$

for all $z \in \mathcal{D}(\mathcal{A}_m)$. Then, $\|P^{1/2} \tilde{w}\|_{\mathbb{Z}}$ and $|\tilde{y}|$ are bounded and, moreover, the bound can be made arbitrarily small by increasing γ .

Proof: The proof is a continuation of that for Lemma 7. Since $\langle \mathcal{F}z, \mathcal{F}z \rangle \geq 0$, we recover (21) and (22) to obtain

$$V(t) \leq \frac{1}{\gamma} \left(\left(\tilde{\alpha}(0)^\top \tilde{\alpha}(0) - \frac{\rho_1}{\lambda_p} \right) e^{-\lambda_p t} + \frac{\rho_1}{\lambda_p} \right)$$

which, via $V(t) \geq \langle P^{\frac{1}{2}} \tilde{w}(t), P^{\frac{1}{2}} \tilde{w}(t) \rangle$, implies that

$$\|P^{\frac{1}{2}} \tilde{w}(t)\|_{\mathbb{Z}} \leq \sqrt{\frac{1}{\gamma}} \sqrt{\left(\left(\tilde{\alpha}(0)^\top \tilde{\alpha}(0) - \frac{\rho_1}{\lambda_p} \right) e^{-\lambda_p t} + \frac{\rho_1}{\lambda_p} \right)} \quad (24)$$

Notice that the term on the RHS is bounded, and can be made arbitrarily small by reducing γ .

Since $\tilde{y} = \mathcal{C} \tilde{w}$, we get using (23)

$$|\tilde{y}(t)| = |\mathcal{C} \tilde{w}(t)| = \|\mathcal{E} P^{\frac{1}{2}} \mathcal{P}^{\frac{1}{2}} \tilde{w}(t)\|_{\mathbb{Z}} \leq \|\mathcal{E} \mathcal{P}^{\frac{1}{2}}\| \|P^{\frac{1}{2}} \tilde{w}(t)\|_{\mathbb{Z}}$$

Since $\|\mathcal{E} \mathcal{P}^{\frac{1}{2}}\|$ is bounded, we conclude using (24) that $|\tilde{y}(t)|$ is bounded and the bound can be made arbitrarily small by increasing γ . ■

Remark 2: If the operator \mathcal{E} in Thm. 2 exists, it must satisfy the condition that $(\mathcal{A}_m, \mathcal{E})$ form a controllable pair. For the conditions of the KYP lemma to be satisfied, $(\mathcal{A}_m, \mathcal{E}, \mathcal{C})$ must satisfy a strictly positive real (SPR) condition.

Next, we show the almost asymptotic convergence of \tilde{y} (the output of the observer error dynamics) to 0 for more general cases when the stronger assumptions of Thm. 2 cannot be met.

Theorem 3: Suppose that $\|w\|_{\mathbb{W}, \tau} < \rho_w$ for some $\tau > 0$ and some constant $\rho_w > 0$. Then, the adaptive laws in (14) ensure that $\int_0^t |\tilde{y}(s)|^2 ds$ is bounded for $t < \tau$. Furthermore, if the solution to (16) exists for all t and $\|w\|_{\mathbb{W}} < \rho_w$, then \tilde{y} converges to 0 almost asymptotically in the sense of Definition 7.

Proof: We start by defining a state $v \in \mathbb{R}$ whose dynamics is defined via

$$\frac{d}{dt} \left(\frac{v^2}{2} \right) = \tilde{y}^2, \quad v(0) = 0$$

Restricting v to satisfy $v \geq 0$, this equation has a well-defined solution for all t .

Recall that \mathcal{I} is the identity operator (on \mathbb{Z}). Let ϵ_1 be an arbitrarily small number such that $\mathcal{I} > \epsilon_1 \mathcal{C}^* \mathcal{C}$. We define a Lyapunov function

$$V = V_0 + \epsilon_1 \frac{v^2}{2}$$

where V_0 is the same Lyapunov function as in (18), and \mathcal{P} is chosen to satisfy (3) with $\mathcal{Q} = \mathcal{I}$, the identity operator. Differentiating with respect to time, we get

$$\dot{V} \leq -\langle \tilde{w}, \tilde{w} \rangle + \epsilon_1 \tilde{y}^2 < 0$$

Hence, $V(t)$ is bounded for $t < \tau$ and it follows that $v(t)^2$ is bounded for $t < \tau$.

If the solution to (16) exists for all t , it follows that $\|\tilde{y}\|_{\mathcal{L}_2}$ is bounded. Lemma 2 then implies that $\tilde{y} \rightarrow 0$ in the almost asymptotic sense of Definition 7. ■

VI. PERFORMANCE AND STABILITY

A. Stability

In this section, we assert the boundedness of the control input and the stability of the closed-loop system. These results, and their proofs, are identical to those in our prior work [9], [11]. These results are not altered by the lack of a coercive solution to the Lyapunov function

We start by asserting the boundedness of \hat{y}_p in (12).

Lemma 8: Suppose that $\|w\|_{\mathbb{W}, \tau} \leq \rho_w$ for some $\rho_w > 0$. Then, there exist constants δ_0 and δ_1 such that $\|\hat{y}_p\|_{\mathcal{L}_\infty, \tau} \leq \delta_0 + \delta_1 \|w\|_{\mathbb{W}, \tau}$.

The boundedness of \hat{y}_p allows us to assert that the control input $u(t)$, given by (6) and (7), is bounded. Let $u_r = -H_C p(t)$, the second term on the RHS of (6). Let $H(s) = H_C (sI - H_A)^{-1} H_B$.

Lemma 9: Let $\|w\|_{\mathbb{W}, \tau} < \rho_w$ for some τ and $\rho_w > 0$. Then, the control input $u(t)$ is bounded and a C^1 function of time for $t < \tau$. Moreover, there exist constants $\delta_{0w} \equiv \delta_{0w}(H(s), \rho)$, $\delta_{0r} \equiv \delta_{0r}(H(s), \rho)$ and $\delta_{0u} \equiv \delta_{0u}(H(s), \rho)$ such that $\|u_r\|_{\mathcal{L}_\infty, \tau} \leq \delta_{0w} \|w\|_{\mathbb{W}, \tau} + \delta_{0r} \|r\|_{\mathcal{L}_\infty, \tau} + \delta_{0u}$.

Finally, we assert the stability of the complete closed-loop system, in the sense of \mathbb{W} -boundedness of signals, using the following small gain.

Assumption 5 (Small-gain condition): We assume that there exists a constant ρ_w , an arbitrarily small $\epsilon_s > 0$, and a stable strictly proper $H(s)$ such that the following inequality is satisfied:

$$\frac{M \rho_0 + \|\mathcal{T} * \|(v_2(\rho_w) + \delta_{0r} \|r\|_{\mathcal{L}_\infty} + \delta_{0u})\|}{1 - \|\mathcal{T} * \|(v_1(\rho_w) + \|\mathcal{B}\| \delta_{0w})} \leq \rho_w - \epsilon_s$$

where the constants have been defined in Lemmas 3 and 9.

Theorem 4: The closed-loop system (8), (12), (13), (14), (6) and (7) is bounded-input-bounded-state stable in the sense of \mathbb{W} if Assumption 5 is satisfied. Moreover, the solution exists for all time and $\|w\|_{\mathbb{W}} < \rho_w$.

B. Reference signal tracking

Ideally, we would design $u(t)$ to ensure that $\hat{y}_h(t)$ in (13) tracks $\sigma(t)$ in (11). Guarantees on the tracking error between $y(t)$ and $r(t)$ depend, therefore, on the provable bounds on $\tilde{y}(t)$. Notice that Theorems 2 and 3 provide relatively strong bounds, albeit of different natures, on $|\tilde{y}|$. In particular, Theorem 3 shows that it \hat{y}_h tracks σ asymptotically, then y tracks r almost asymptotically.

However, in neither of these cases (Theorems 2 and 3) is it possible to guarantee a strong bound on $\|\tilde{w}(t)\|_{\mathbb{Z}}$. This

implies that bounds on the *transient characteristics* of the output tracking error cannot be asserted, unlike in the case where \mathcal{P} is coercive.

C. Impact on Model-Following

In \mathcal{L}_1 adaptive control, unlike MRAC, there is no explicitly prescribed reference model. Instead, we define an *auxiliary reference system*

$$\begin{aligned} \dot{w}_{\text{ref}}(t) &= \mathcal{A}_m w_{\text{ref}}(t) + \mathcal{B}u_{\text{ref},r}(t) + f(w_{\text{ref}}), \quad w_{\text{ref}}(0) = w(0) \\ y_{\text{ref}} &= \mathcal{C}w_{\text{ref}} \end{aligned} \quad (25)$$

where $f(\cdot)$ is assumed to be known. For the auxiliary system, we don't need state observers for the homogeneous and particular halves. Instead we write their dynamics as

$$\begin{aligned} \dot{w}_{\text{ref},h} &= \mathcal{A}_m w_{\text{ref},h} + \mathcal{B}u_{\text{ref},r} \\ \dot{w}_{\text{ref},p} &= \mathcal{A}_m w_{\text{ref},p} + f(w_{\text{ref}}) \end{aligned} \quad (26)$$

and calculate $u_{\text{ref},r}$ as

$$\begin{aligned} u_{\text{ref},r} &= -H_c p_{\text{ref}}(t), \quad \dot{p}_{\text{ref}}(t) = H_A p_{\text{ref}}(t) + H_B \sigma_{\text{ref}}(t) \\ \sigma_{\text{ref}}(t) &= r(t) - y_{\text{ref},p}(t) \end{aligned} \quad (27)$$

Note that this equation is similar to (6).

We define the model-tracking error $e_w = w - w_{\text{ref}}$, and the error $e_p = p - p_{\text{ref}}$. It can be checked that the dynamics of these variables are given by

$$\begin{aligned} \dot{e}_w &= \mathcal{A}_m e_w - \mathcal{B}H_c e_p + f(w) - f(w_{\text{ref}}) \\ \dot{e}_p &= H_A e_p - H_B \mathcal{C}e_w - H_B \tilde{y}_p \end{aligned} \quad (28)$$

Subject to the small gain condition in Assumption 5, (28) represents a stable system driven by \tilde{y}_p . This allows us to assert the following result.

Proposition 1: Suppose that the closed-loop system is well-posed and bounded for all time. Then, if \mathcal{A}_m permits a coercive solution \mathcal{P} to (3) or if the conditions of Theorem 2 are satisfied, the error $\|e_w\|_{\mathbb{W}}$ is bounded and the bound can be made arbitrarily small by increasing γ .

The above proposition does not cover the case of Thm 3. It is not possible to provide strong guarantees on the transient model-following error in the absence of stronger bounds on $\|e_w\|_{\mathbb{W}}$.

Remark 3 (Implications for MRAC): In traditional model reference adaptive control (MRAC), adaptive laws are designed by considering the error between the actual system and a reference model, rather than between the actual system and an observer. Equation (17) is sufficiently representative of the dynamics of the model-tracking error. Since the lack of a coercive solution to (3) prevents us from deriving a strong bound on \tilde{w} , it is not possible to guarantee that the transient response of the actual system matches that of the reference model. This is significant for MRAC because the primary role of the reference model is to specify desirable closed-loop transient response characteristics.

VII. CONCLUSION

In this paper, we examined adaptive control problems where the Lyapunov equation used to derive adaptive laws and closed-loop performance guarantees does not permit a coercive solution. We showed how, in the absence of coerciveness, it is still possible to provide limited guarantees on the tracking error. We showed, in particular, that is generally possible to show that the tracking errors decay in a weakly asymptotic manner. Under extra assumptions resembling those in the KYP lemma, we derived tracking performance guarantees closer to the case where the Lyapunov equation permits a coercive solution. We demonstrated the effect of the lack of coercivity on the nature of the provably guarantees for the transient response of the closed-loop system.

REFERENCES

- [1] R. F. Curtain, M. A. Demetriou, and K. Ito, "Adaptive compensators for perturbed positive real Infinite-Dimensional systems," *International Journal of Applied Mathematics and Computer Science*, vol. 13, pp. 441 – 452, 2003.
- [2] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, ser. Texts in Applied Mathematics (Vol. 21). Springer-Verlag, 1995.
- [3] R. Datko, "An extension of a theorem of A. M. Lyapunov to Semi-Groups of operators," *Journal of Mathematical Analysis and Applications*, vol. 24, pp. 290 – 295, 1968.
- [4] N. Hovakimyan and C. Cao, *LI Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation*, ser. Advances in Design and Control. SIAM, 2010.
- [5] B. Jacob, A. Mironchenko, J. R. Partington, and F. Wirth, "Non-coercive lyapunov functions for input-to-state stability of infinite-dimensional systems," *SIAM Journal on Control and Optimization*, vol. 58, no. 5, pp. 2952–2978, 2020.
- [6] E. Lavretsky, T. E. Gibson, and A. M. Annaswamy, "Projection operator in adaptive systems," 2011, arXiv preprint arXiv:1112.4232.
- [7] E. Lesigne, "On the behavior at infinity of an integrable function," *American Mathematical Monthly*, vol. 117, no. 2, pp. 175 – 181, 2010.
- [8] A. Mironchenko and F. Wirth, "Non-coercive lyapunov functions for infinite-dimensional systems," *Journal of Differential Equations*, vol. 266, no. 11, pp. 7038 – 7072, 2019.
- [9] V. Natarajan and J. Bentsman, "Adaptive projection-based observers and l_1 adaptive controllers for infinite dimensional systems with full-state measurement," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 585 – 598, 2014.
- [10] A. A. Paranjape and S. Chung, "Sub-optimal boundary control of semilinear PDEs using a dyadic perturbation observer," in *Proc. 55th IEEE Conference on Decision and Control (CDC), Las Vegas, NV, 2016*, pp. 1382 – 1387.
- [11] —, "Robust adaptive boundary control of semilinear PDE systems using a dyadic controller," *International Journal of Robust and Non-linear Control*, vol. 28, no. 8, pp. 3174 – 3188, 2018.
- [12] A. A. Paranjape and S.-J. Chung, "Dyadic perturbation observer framework for control of a class of nonlinear PDE/ODE systems," in *Proc. 53rd IEEE Conference on Decision and Control (CDC), Los Angeles, LA, 2014*, pp. 2065 – 2070.
- [13] A. A. Paranjape, J. Guan, S.-J. Chung, and M. Krstic, "PDE boundary control for Euler-Bernoulli beam using a two stage perturbation observer," in *Proc. 51st IEEE Conference on Decision and Control (CDC), Maui, HI, 2012*, pp. 4442 – 4448.
- [14] A. Pazy, "On the applicability of Lyapunov's theorem in Hilbert space," *SIAM Journal of Mathematical Analysis*, vol. 3, no. 2, pp. 291 – 294, 1972.
- [15] —, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, ser. Applied Mathematical Sciences; v.44. Springer-Verlag, New York, 1983.
- [16] J. T.-Y. Wen and M. J. Balas, "Robust adaptive control in hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 143, pp. 1 – 26, 1989.