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Propagation and dispersion of waves in composite media with resonant inclusions

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Yi Sheng Lim

Summary

This thesis concerns the effective behaviour of waves in high-contrast composite media. We formulate our study in the language of operators as follows: Fix dimension $d \geq 2$. For $\varepsilon > 0$, consider the operator on $L^2(\mathbb{R}^d)$,

$$A_\varepsilon = -\operatorname{div}\left(a_\varepsilon \nabla \cdot\right), \qquad \text{where } a_\varepsilon \text{ is } \varepsilon\mathbb{Z}^d\text{-periodic.}$$

Here, A_ε has the following features

- **("Stiff-soft-stiff" setup)** If we write $a_\varepsilon = \widetilde{a}_c(\frac{\cdot}{\varepsilon})$ for a $\mathbb{Z}^d$-periodic matrix $\widetilde{a}_c$ depending on $c > 0$, then we define

$$\widetilde{a}_c(y) = \begin{cases} cI, & y \in \cup_{n \in \mathbb{Z}^d}(Q_{\text{soft}} + n), \\ I, & y \in \cup_{n \in \mathbb{Z}^d}((Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}}) + n). \end{cases}$$

  Here, the sets Q_{soft}, $Q_{\text{stiff-int}}$ ("stiff interior"), and $Q_{\text{stiff-ls}}$ ("stiff landscape") partition the reference period cell $Q = [0,1)^d$, and are arranged as follows: We have an annular "soft" region Q_{soft} with remainder filled by the "stiff" regions $Q_{\text{stiff-int}}$ and $Q_{\text{stiff-ls}}$. We impose transmission boundary conditions on the soft-stiff interfaces.

- **(High-contrast/resonant inclusions)** We allow c to depend on ε. That is, A_ε depends on ε in two ways, namely, in the periodicity and in the material coefficients. In particular, we will focus on the case $a_\varepsilon = \widetilde{a}_{\varepsilon^2}(\frac{\cdot}{\varepsilon})$.

These features, together with the following requirement, makes our problem new

- **(Mode of convergence)** Identify the limiting behavior of A_ε, as $\varepsilon \downarrow 0$, in the *norm-resolvent* sense.

After an introductory chapter, Chapter 2 details the process of homogenization for the stiff-soft-stiff composite. We identify an operator $\mathcal{A}_{\varepsilon,\text{hom}}$ that is asymptotically equivalent to A_ε in the norm-resolvent sense, using an operator framework developed by Cherednichenko, Ershova, and Kiselev in [35]. Chapter 3 focuses on the homogenized description $\mathcal{A}_{\varepsilon,\text{hom}}$. We investigate three aspects of $\mathcal{A}_{\varepsilon,\text{hom}}$. First, we extract "dispersion functions" $K_{\text{stiff-int}}(\tau, z)$ and $K_{\text{stiff-ls}}(\tau, z)$ from $\mathcal{A}_{\varepsilon,\text{hom}}$. These are meant to capture the effective dispersion relations of an acoustic wave travelling through the composite. Second, we provide formulas for $\mathcal{A}_{\varepsilon,\text{hom}}$ on physical space ($\mathcal{A}_{\varepsilon,\text{hom}}$ was previously defined on frequency space). Third, we perform a spectral analysis on $\mathcal{A}_{\varepsilon,\text{hom}}$. Chapter 4 summarizes what was done in Chapters 2-3, collects the new results, and concludes with open questions and next steps.

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Contents

Chapter 1

Introduction

1.1 Homogenization, high-contrast homogenization, and wave propagation

This thesis lies in the subject of homogenization, which is the study of approximating a highly heterogeneous medium with a homogeneous one. Physically, one is motivated by the desire to understand various properties of composite materials. However, a good composite mixture is highly heterogeneous in space, so one faces serious numerical challenges if one decides solve the corresponding mathematical model directly. Guided by the intuition that the mixture looks as if it is comprised of a single "averaged material" when zoomed out sufficiently far, we may instead look at the mathematical model for this "averaged material", as an approximation of the original composite.

The above describes the idea of homogenization in physical terms. Turning this idea into a mathematically rigorous one has been a subject of intense study since the 1970s, and to date has amassed an extensive literature. We mention for instance, the books [2, 3, 6, 9, 10, 16, 18, 23, 27]. Below, let us give an overview of one such study, from the point of view of a person who wishes to understand the *transport/scattering properties* of a composite material. (This is the point of view that we will take in the thesis.) We will keep the discussion fairly brief, and refer the reader to [10, Chapter 12] for rigorous statements.

(Moderate-contrast periodic) homogenization of the wave equation

Fix dimension $d \geq 2$. For $\varepsilon > 0$, consider the initial-value problem for the wave equation:

$$\begin{cases} \partial_{tt} u_\varepsilon - \operatorname{div}\left(a_\varepsilon \nabla u_\varepsilon\right) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u_\varepsilon(\cdot, 0) = u_0, \quad \partial_t u_\varepsilon(\cdot, 0) = u_1 & \text{on } \mathbb{R}^d \times \{t = 0\}, \end{cases} \tag{1.1}$$

where u_0 and u_1 are given, and we seek a solution $u_\varepsilon(x, t)$ in an appropriate sense.

The coefficient $a_\varepsilon : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a matrix-valued function which encodes physical properties of the material. For simplicity, let us assume that our composite consists of two materials, a "soft" one and a "stiff" one, and that they are combined in a periodic fashion to form a fine mixture. We can encode this kind of structure in a_ε by letting a_ε be $\varepsilon\mathbb{Z}^d$-periodic and take one of two possible values. That is, by writing $a_\varepsilon = \widetilde{a}_c(\frac{\cdot}{\varepsilon})$, where $\widetilde{a}_c$ is a $\mathbb{Z}^d$-periodic matrix, depending on some fixed $c > 0$, let us set

$$\widetilde{a}_c(y) = \begin{cases} cI, & \text{if } y \text{ lies in the soft regions,} \\ I & \text{if } y \text{ lies in the stiff regions.} \end{cases} \tag{1.2}$$

We refer to the choice (1.2) as the "moderate-contrast" setting, describing the fact that the matrix a_ε is bounded and positive definite, uniformly in ε. That is, there exist $c_{\text{ellip}} > 0$ and $C_{\text{bdd}} > 0$ such that

$$a_\varepsilon(x)\xi \cdot \xi \geq c_{\text{ellip}}|\xi|^2, \qquad \text{for almost every } x \text{ and all } \xi \in \mathbb{R}^d, \tag{1.3}$$

$$\|(a_\varepsilon)_{ij}\|_{L^\infty} \leq C_{\text{bdd}}, \qquad \text{for } 1 \leq i, j \leq d. \tag{1.4}$$

Equivalently (in the terminology of [17, Chapter 4]), we express (1.3) by saying that the operator $A_\varepsilon = -\operatorname{div}(a_\varepsilon \nabla \cdot)$ is uniformly (in ε and x) strongly elliptic.

We are interested in the behavior of the solutions u_ε to the Cauchy problem (1.1) for small ε. As noted above, a_ε is highly oscillatory in space, and hence (1.1) is numerically challenging to solve. The basic result of homogenization seeks to answer the following questions in the affirmative (see [10, Theorem 12.6]):

- Do the solutions u_ε converge (and in what sense), as $\varepsilon \downarrow 0$, to some limit u_{hom}?
- Can we characterize u_{hom} as the solution to (1.1), but with a_ε replaced by a constant matrix a_{hom}?

The advantage of such a homogenization result is that u_{hom} now satisfies a problem which is numerically easier to solve, and a_{hom} provides an effective description of the original medium, approximating the composite mixture as a single homogeneous material.

Remark. The case $d = 1$ has been excluded due to the limited possibilities on the arrangement of the two materials. To generate interesting geometries in $d = 1$, one may consider quantum/metric graphs. This is beyond the scope of the thesis. ◦

Extending the basic homogenization result

The basic homogenization result [10, Theorem 12.6], while elegant, is insufficient from a wave propagation perspective. Indeed, it asserts the convergence of u_ε to u_{hom} in the sense of weak-$*$ in $L^\infty([0,T]; H_0^1)$, and is qualitative in nature (i.e. no rate of convergence is given). This prompts us to ask:

> Can we prove a *quantitative* homogenization result, under a mode of convergence that captures the effective behavior of waves in the original composite?

There are many ways to tackle the above question. This thesis adopts an operator-theoretic approach, focusing on the following operator on $L^2(\mathbb{R}^d)$:

$$A_\varepsilon = -\operatorname{div}(a_\varepsilon \nabla \cdot). \tag{1.5}$$

From the perspective of operator theory, the homogenization task at hand requires us to identify the limiting behavior of A_ε as $\varepsilon \downarrow 0$ in an appropriate operator topology. To study the operator A_ε, we consider the resolvent equation

$$(A_\varepsilon - z)u_\varepsilon = f \in L^2(\mathbb{R}^d). \tag{1.6}$$

Solving the resolvent equation helps identify the spectrum $\sigma(A_\varepsilon)$. That is: Find those $z \in \mathbb{C}$ such that (1.6) has a unique solution for every $f \in L^2(\mathbb{R}^d)$. The complement of the set of such values z is the spectrum $\sigma(A_\varepsilon)$.

The spectrum $\sigma(A_\varepsilon)$ contains key information about wave propagation through the composite medium. Therefore, the choice of operator topology should also capture the behavior of $\sigma(A_\varepsilon)$ as $\varepsilon \downarrow 0$. This prompts us to look at *norm-resolvent* convergence/asymptotics of A_ε, a key requirement of the thesis.

Remark. With wave propagation in mind, one intends to use the norm-resolvent asymptotics of A_ε to deduce the effective behavior of the solution $u_\varepsilon(x,t)$ to the initial-value problem for the wave equation (1.1), as $\varepsilon \downarrow 0$. This can be achieved by employing a functional calculus, although we will not perform this step in the thesis. $\circ$

Strengthening the basic homogenization result from a qualitative to a quantitative one remains an active area of research, in part due to the vast number of setups that one could study beyond (1.1). To name a few: differential operators with oscillatory lower order terms, integral functionals (and non-linear problems), almost-periodic or random coefficients. In this thesis, we are interested in extending the setup (1.1) from a moderate-contrast to a “high-contrast” setting. This means that we will let c depend on ε in (1.2), such that the constant $c_{\text{ellip}} > 0$ in (1.3) cannot be chosen independently of ε, thus violating the assumption of uniform strong ellipticity.

The high-contrast setting poses fundamental mathematical challenges. Methods used to tackle the moderate-contrast setting quickly break down in the high-contrast case, and underlying these technical issues is a basic question of identifying the homogenized description. Indeed, does a limit even exist in the first place?

As it will become clear in Section 1.4, the answer to the existence of the limit and its form depends on the choice of convergence and the arrangement of the materials in the composite. This thesis studies a particular high-contrast setup where A_ε does *not* have a norm-resolvent limit. Nonetheless we identify an operator $\mathcal{A}_{\varepsilon,\text{hom}}$ which is asymptotically close to A_ε in the norm-resolvent sense, and serves as a homogenized description of the medium. Let us now provide a brief outline of the problem that we will study in this thesis. (In Section 2.1, we will give a rigorous formulation of the problem and recall all the relevant notation introduced below.)

Problem outline

Consider the problem of homogenization for a high-contrast $\varepsilon\mathbb{Z}^d$-periodic composite on $\mathbb{R}^d$. Our composite will consist of “soft” and “stiff” material components, adopting the terminology of elasticity theory. We think of the "soft" component having small material coefficients relative to the "stiff" ones.

For $\varepsilon > 0$, consider the operator $A_\varepsilon = -\operatorname{div}(a_\varepsilon \nabla \cdot)$, on $L^2(\mathbb{R}^d)$. The coefficient matrix a_ε is defined as $a_\varepsilon(x) := \widetilde{a}_{\varepsilon^2}(\frac{x}{\varepsilon})$, where $\widetilde{a}_{\varepsilon^2}$ is a $\mathbb{Z}^d$-periodic matrix with values given by

$$\widetilde{a}_{\varepsilon^2}(y) = \begin{cases} \varepsilon^2 I, & y \in \bigcup_{n\in\mathbb{Z}^d}(Q_{\text{soft}} + n), \\ I, & y \in \bigcup_{n\in\mathbb{Z}^d}\left((Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}}) + n\right). \end{cases} \tag{1.7}$$

Here, the sets Q_{soft}, $Q_{\text{stiff-int}}$, and $Q_{\text{stiff-ls}}$ partition the reference period cell $Q = [0,1)^d$, and are arranged in a "stiff-soft-stiff" setup as follows: We have a simply connected "stiff-interior" region $Q_{\text{stiff-int}}$, surrounded by an annular shaped "soft" region Q_{soft}, with the remaining region filled by the "stiff-landscape" part $Q_{\text{stiff-ls}}$. See Figure 1-1 for a pictorial description of $\widetilde{a}_{\varepsilon^2}$ when restricted to the period cell Q. We impose transmission boundary conditions on the soft-stiff interfaces Γ_{int} and Γ_{ls}. See Section 2.1 for the precise definition of A_ε.

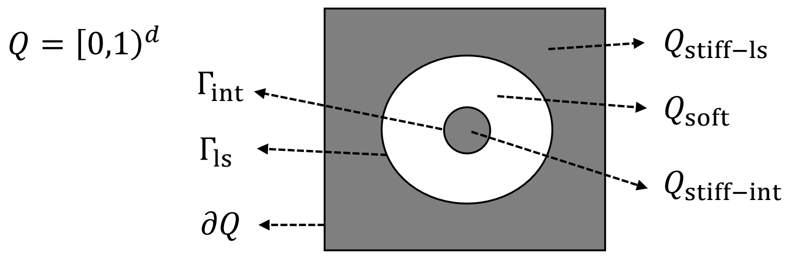

Figure 1-1: The period cell $Q = [0,1)^d$. The subscript "stiff-int" stands for stiff interior, and "stiff-ls" stands for stiff landscape.

We are interested in the limiting behavior as $\varepsilon \downarrow 0$ of A_ε, in the *norm-resolvent* sense. The reason for this particular choice of (operator) topology is that it gives us direct access to the spectrum, in the sense that identifying the norm-resolvent limit/asymptotics of A_ε implies the spectral convergence/asymptotics of $\sigma(A_\varepsilon)$, in the sense of Hausdorff convergence/asymptotics on compact subsets of the real line $\mathbb{R}$ (see Section 1.3.3 for details).

Altogether, the problem of homogenization that we will study in this thesis is new (see Section 1.4 for a discussion of the existing literature), due to a combination of the following features:

- **(Stiff-soft-stiff setup)** Our medium is $\varepsilon\mathbb{Z}^d$-periodic, and consists of a stiff mixture with annular soft inclusions.
- **(High-contrast/resonant inclusions)** We allow c in (1.2) to depend on ε. That is, A_ε depends on ε in two ways, namely, in the periodicity and in the material coefficients. In particular, we will focus on the case $a_\varepsilon = \widetilde{a}_{\varepsilon^2}(\frac{\cdot}{\varepsilon})$.
- **(Mode of convergence)** Identify the *norm-resolvent asymptotics* of A_ε, as $\varepsilon \downarrow 0$.

Remark. The choice $c = \varepsilon^2$ in (1.2), giving us (1.7), is referred to as the "double porosity" scaling [29]. Under this scaling, we also say that the annular soft inclusions act as "resonators" (hence the title of the thesis). This refers to the heuristic that waves propagating through the medium, upon entering the soft inclusions, will have wavelength comparable to the size of the inclusions. We refer the reader to Appendix A for an elaboration of this heuristic. ◦

1.2 Structure of the thesis

This thesis is structured as follows:

Chapter 1 is an introductory chapter. In Section 1.1, we have provided the motivation for and an outline of the problem that will be studied in the thesis. In Section 1.3, we fix some notations, introduce the notion of a periodic Sobolev space following [10], and review some facts on convergence of unbounded operators and its relation to the spectrum. In Section 1.4, we will review the existing literature, with focus on operator norm estimates in homogenization and high-contrast homogenization. In Section 1.5 is a collection of the main results of the thesis.

Chapter 2 embarks on the task of homogenization for our stiff-soft-stiff composite. We will follow the approach proposed by Cherednichenko, Ershova, and Kiselev in [35]. This is an operator framework based on the following key ingredients (see Chapter 2 for precise definitions):

(A) The (rescaled) *Gelfand/Floquet transform* G_ε, which helps take the $\varepsilon\mathbb{Z}^d$-periodic operator A_ε on $L^2(\mathbb{R}^d)$ to a family of operators $A_\varepsilon^{(\tau)}$ on $L^2(Q)$, indexed by $\tau \in Q' = [-\pi, \pi)^d$.

(B) Boundary triples (A_0, Λ, Π) in the sense of Ryzhov [47], to obtain norm-resolvent estimates for each $A_\varepsilon^{(\tau)}$.

(C) Perturbation theory in the sense of Kato [15], and Reed and Simon [20, Chapter XII], to ensure that the estimates in (B) are uniform in τ.

(D) Generalised resolvents, such as the operator $R_\varepsilon^{(\tau)}(z) = P_{\text{soft}}(A_\varepsilon^{(\tau)} - z)^{-1}P_{\text{soft}}$, where P_{soft} is the projection of $L^2(Q)$ onto $L^2(Q_{\text{soft}})$ (see Section 2.4). Here, the norm-resolvent asymptotics of $R_\varepsilon^{(\tau)}(z)$, which we denote as $R_{\varepsilon,\text{hom}}^{(\tau)}(z)$, is identified with a compression of some $(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1}$ (Theorem 2.4.20).

Sections 2.1 and 2.2 involve setting up the "stiff-soft-stiff" problem. In Section 2.1, we define the operator A_ε on $L^2(\mathbb{R}^d)$ and then explain why we can equivalently study the family of operators $\{A_\varepsilon^{(\tau)}\}_{\tau\in[-\pi,\pi)^d}$ on $L^2(Q)$, obtained by the Gelfand transform. Section 2.2 further casts the problem in the language of boundary triples.

Section 2.3 studies the resolvent asymptotics of $A_\varepsilon^{(\tau)}$, and is at the heart of the analysis. Section 2.4 combines the result of Section 2.3 (Theorem 2.3.4) with the boundary triple setup of Section 2.2 to give a self-adjoint operator $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$ that captures the norm-resolvent asymptotics of $A_\varepsilon^{(\tau)}$. Section 2.5 unpacks the notation and summarizes the boundary triple approach for homogenization, giving first the main result of the thesis, Theorem 2.5.3.

Chapter 3 places focus on the limiting operator itself, $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$. We study three aspects of $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$: In Section 3.1 we look at the bottom right entry of the resolvent for $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$ (this is a

2×2 matrix, as we will see). We write each of the four entries in terms of an operator of multiplication on $\mathbb{C}$ by a constant. Moreover, for the two diagonal entries, we are able to express this constant as $(K_{\text{stiff-int}}(\tau, z) - z)^{-1}$ and $(K_{\text{stiff-int}}(\tau, z) - z)^{-1}$, where we will refer to $K_{\text{stiff-int}}(\tau, z)$ and $K_{\text{stiff-ls}}(\tau, z)$ as "dispersion functions". In Section 3.2, we write down the homogenized description on the full space, i.e. the operator $\mathcal{A}_{\varepsilon,\text{hom}} = G^* \left(\int_Q^{\oplus} \mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} d\tau \right) G$. In Section 3.3 we perform a spectral analysis of $\mathcal{A}_{\varepsilon,\text{hom-I}}$ and $\mathcal{A}_{\text{hom-II}}$ (the norm-resolvent asymptotics for Models I and II, see Figure 1-2, as obtained in [35]), with an eye towards treating $\mathcal{A}_{\varepsilon,\text{hom}}$. Section 3.3 leaves with some unfinished tasks, which are collected in Chapter 4.

We wrap up our investigation in **Chapter 4**. We give an overview of what we have done, and state the new results obtained in this thesis (these are also collected in Section 1.5). We end by discussing how one may take forward the work done in this thesis, including a list of short-term unfinished tasks, and a few long-term problems.

1.3 Mathematical preliminaries

1.3.1 Notation, assumptions, abbreviations

Fix the dimension $d \geq 2$.

General notation. $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We will use $\subset$ and $\subseteq$ interchangeably. To denote a strict subset, we will use $\subsetneq$. The indicator function of a set $U \subset \mathbb{R}^d$ will be denoted by $\mathbf{1}_U$. $\oplus$ refers to an orthogonal sum of Hilbert spaces, or of operators on Hilbert spaces. $\dot{+}$ refers to a direct sum of vector spaces. For $a, b \in \mathbb{R}^d$, we write $a \cdot b = \sum_{i=1}^d a_i b_i$ for the inner product on $\mathbb{R}^d$, and $|a| = \sqrt{a \cdot a}$ for the corresponding norm. For $a \in \mathbb{C}$, the real and imaginary components of a are denoted by $\text{Re}(a)$ and $\text{Im}(a)$ respectively.

Spaces. We will assume that our Hilbert spaces $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ are complex, and write $\|\cdot\|_{\mathcal{H}}$ for the corresponding norm. Let $U \subset \mathbb{R}^d$ be open. Denote by $C^\infty(U)$ the space of smooth functions $f : U \to \mathbb{C}$, and $C_c^\infty(U)$ for the vector subspace of functions that have compact support in U. Let $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. We will need the Lebesgue spaces (with respect to the Lebesgue measure) $L^p(U)$, the Sobolev space $W^{k,p}(U)$, and the subspace $W_0^{k,p}(U) := \overline{C_c^\infty(U)}^{W^{k,p}}$, with the important special cases being $H_0^1(U) := W_0^{1,2}(U)$, and $H^k(U) := W^{k,2}(U)$. We will also need the space of locally p-integrable functions $L^p_{\text{loc}}(U)$, locally $W^{k,p}$ functions $W^{k,p}_{\text{loc}}(U)$, and fractional Sobolev spaces $W^{s,p}$, $s > 0$. We will write, for instance, L^p in place of $L^p(U)$, whenever the domain is understood. Finally, we will also need *periodic Sobolev spaces*, which are defined in Section 1.3.2.

Operators. We will mainly follow the notation of [22]. Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. By an (unbounded) operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$, we mean a linear mapping $T : \mathcal{D}(T) \to \mathcal{H}_2$, where $\mathcal{D}(T)$ is a linear subspace of $\mathcal{H}_1$. The set $\mathcal{D}(T)$ is referred to as the domain of T, and we also write $(T, \mathcal{D}(T))$ to mean the operator T, whenever we would like to place an emphasis on the domain. We write $\text{ran}(T)$ for the range of T, and $\ker(T)$ for the kernel of T. $S \subset T$ means that T is an extension of S. If $(T, \mathcal{D}(T))$ is an operator on $\mathcal{H}$ (i.e. from $\mathcal{H}$ to $\mathcal{H}$), the spectrum of T is denoted by $\sigma(T)$, and the resolvent set by $\rho(T)$. For $\lambda \in \rho(T)$, the resolvent $(T - \lambda I)^{-1}$ will be abbreviated as $(T-\lambda)^{-1}$. $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the space of bounded linear operators from $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. The operator norm of $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is denoted either by $\|T\|_{\mathcal{H}_1 \to \mathcal{H}_2}$, or by $\|T\|_{op}$ if the spaces are clear from the context.

Special families of operators. Let $\tau \in \mathbb{R}^d$. The operator $-(\nabla + i\tau)^2$ or $(\frac{1}{i}\nabla + \tau)^2$ (with appropriately defined boundary conditions) are both shorthand for $-\Delta - 2i\tau \cdot \nabla + |\tau|^2$, as opposed to a composition of operators. The multiplication operator on $L^2(\mathbb{R}^d)$ by an almost everywhere finite function f is denoted by M_f. Similarly, multiplication on $\mathbb{C}$ by a constant c is denoted by M_c. Our operators of interest are typically defined through a sesquilinear form. If $(A, \mathcal{D}(A))$ is constructed from a form $(\mathfrak{t}, \mathcal{D}(\mathfrak{t}))$, then we will write $\mathcal{D}[A] := \mathcal{D}(\mathfrak{t})$ to distinguish between the form domain and the operator domain. All our projections will be orthogonal. If H is a subspace of $\mathcal{H}$, then the projection onto H will be denoted either by P_H or $\mathcal{P}_H$.

Abbreviations. We will be using the following abbreviations:

LHS/RHS	left hand side/right hand side (of an equation)
PDE	partial differential equation
BVP/BC	boundary value problem/boundary condition
IBP	integration by parts
a.e.	almost everywhere (with respect to the Lebesgue measure)
w.r.t	with respect to
resp.	respectively

Conventions. We will be dealing with a multitude of projections on two Hilbert spaces, $\mathcal{H}$ and $\mathcal{E}$. The straight font (e.g. P_H) is reserved for projections on $\mathcal{H}$, and the calligraphic font (e.g. $\mathcal{P}_E$) is reserved for projections on $\mathcal{E}$. When writing integrals, we will omit the differential "dx" where it is understood.

Use of colours. Throughout the thesis, we will use coloured text to highlight parallelism in formulae and to help the reader navigate complicated expressions.

1.3.2 Periodic Sobolev spaces

Fix a reference cell $Q = [0,1)^d$.

Definition 1.3.1. A function f, defined a.e. on $\mathbb{R}^d$ is called $\mathbb{Z}^d$-periodic if for all $k \in \mathbb{Z}$ and $i \in \{1, \cdots, d\}$, we have $f(x + ke_i) = f(x)$ a.e. Here $\{e_1, \cdots, e_d\}$ denotes the standard basis of $\mathbb{R}^d$.

We will also require a notion of periodicity up to and including the boundary ∂Q. Since we want to talk about traces of measurable functions on Q, we need at least one weak derivative. This prompts us to make the following definition:

Definition 1.3.2. $C^\infty_{per}(Q) := \{u \in C^\infty(\mathbb{R}^d) : u \text{ is } \mathbb{Z}^d\text{-periodic}\}$. We will identify $u \in C^\infty_{per}(Q)$ with its restriction to $\overline{Q}$.

The key definition of this section is the following Hilbert space:

Definition 1.3.3. $H^1_{per}(Q) := \overline{C^\infty_{per}(Q)}^{\|\cdot\|_{H^1(Q)}}$. We identify this space as a subspace of $L^2(Q)$.

We list here several equivalent characterizations of $H^1_{per}(Q)$:

$$\begin{aligned} H^1_{per}(Q) &= \{u \in H^1_{loc}(\mathbb{R}^d) : u \text{ is } \mathbb{Z}^d\text{-periodic}\} && (1.8)\\ &= \{u \in L^2(Q) : \partial_i u \in L^2(Q), \text{ and } u,\, \partial_i u \text{ have equal trace on} \\ &\qquad\qquad \text{opposite faces of } Q,\, 1 \leq i \leq d.\} && (1.9)\\ &= \left\{ u \in L^2(Q) : \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)|\hat{u}(k)|^2 < \infty \right\}, && (1.10) \end{aligned}$$

where $\hat{u}$ denotes the Fourier transform of u. For an explanation of the equalities, we refer to [27, p. 6] and [10, Proposition 3.50] for the first, [10, Proposition 3.49] for the second, and [8, p. 137] for the third expression. Note that $H^1_{per}(Q) = \overline{C^\infty_{per}(Q)}^{\|\cdot\|_{H^1(Q)}}$ is in general a subspace of $H^1(Q)$, and $\overline{C^\infty_{per}(Q)}^{\|\cdot\|_{L^2(Q)}} = L^2(Q)$.

Remark. At crucial points in Chapter 2, we will use the compactness of $\overline{Q}$ in our arguments. Notably, this is used in Proposition 2.3.3, Theorem 2.3.4, and Proposition 2.3.5. ∘

1.3.3 Convergence of unbounded operators

In this section, we review various notions of convergence of unbounded operators and its relation to the spectrum. Let T_n, T be (unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$.

Definition 1.3.4 (Norm/strong-resolvent convergence)**.** By norm (resp. strong) resolvent convergence of T_n to T, mean that for some (hence all) λ with $\text{Im}\lambda \neq 0$, the resolvents $(T_n - \lambda)^{-1}$ converges in operator norm (resp. strongly) to $(T - \lambda)^{-1}$. We will write $T_n \xrightarrow{\text{nr}} T$ (resp. $T_n \xrightarrow{\text{sr}} T$).

We refer the reader to [21, Section VIII.7], [19, Chapter 10], and [25, Chapter 6.6] for a general discussion on norm and strong resolvent convergences. Here, we focus on the relation between Definition 1.3.4 and the spectrum. To facilitate the discussion, let us first make the following definition:

Definition 1.3.5. Let M_n and M be non-empty and closed subsets of $\mathbb{C}$. We write $M_n \xrightarrow{\text{HC}} M$ or $M = \text{HC} - \lim_{n\to\infty} M_n$ ("Hausdorff on compacts") to mean that

> $d_H(M_n \cap K, M \cap K) \to 0$, for every compact $K \subset \mathbb{C}$ such that $M_n \cap K$ and $M_n \cap K$ are non-empty.

Here, d_H is the Hausdorff distance, which is defined for every non-empty and closed M, $N \subset \mathbb{C}$, by the formula

$$d_H(M,N) := \max\left\{ \sup_{x\in M}\inf_{y\in N}|x-y|, \sup_{y\in N}\inf_{x\in M}|x-y| \right\}.$$

Obtaining convergence/asymptotics of spectra in the sense of Definition 1.3.5 is the focus of the thesis. However, for the purposes of discussing the literature, let us introduce another notion of convergence for sets:

Definition 1.3.6. Let M_n and M be non-empty and closed subsets of $\mathbb{C}$. We write $M_n \xrightarrow{\text{ls}} M$ or $M = \text{ls} - \lim_{n\to\infty} M_n$ ("limit set") to mean that the following two conditions are satisfied:

- If $\lambda \in M$, then there exist a sequence $\lambda_n \in M_n$ such that $\lambda_n \to \lambda$.
- If $\lambda_n \in M_n$ and $\lambda \in \mathbb{C}$ satisfies $\lambda_n \to \lambda$, then $\lambda \in M$.

That is, $\text{ls} - \lim_{n\to\infty} M_n = \{\lambda \in \mathbb{C} : \exists\ \lambda_n \in M_n \text{ and } \lambda_n \to \lambda\}$.

It can be shown that if $\text{HC} - \lim_{n\to\infty} M_n$ exist, then so does $\text{ls} - \lim_{n\to\infty} M_n$, and $\text{ls} - \lim_{n\to\infty} M_n = \text{HC} - \lim_{n\to\infty} M_n$, by an application of [1, Proposition 4.4.14]. See Appendix B for details.

We now relate Definitions 1.3.4, 1.3.5 and 1.3.6.

Theorem 1.3.7. If $T_n \xrightarrow{\text{nr}} T$, then $\sigma(T_n) \xrightarrow{\text{HC}} \sigma(T)$ and $\sigma(T_n) \xrightarrow{\text{ls}} \sigma(T)$. In other words,

$$\text{ls} - \lim_{n\to\infty} \sigma(T_n) = \sigma(T) = \text{HC} - \lim_{n\to\infty} \sigma(T_n). \tag{1.11}$$

Proof. See [57, Section I.3] for a proof of $\sigma(T_n) \xrightarrow{\text{HC}} \sigma(T)$. The result now follows from the remark preceding the theorem. Alternatively, see [21, Theorem VIII.23(a) and Theorem VIII.24(a)] for a direct proof proof of $\sigma(T_n) \xrightarrow{\text{ls}} \sigma(T)$. □

Remark.

- In the case of T_n converging to T in the strong-resolvent sense, we cannot conclude $\sigma(T_n) \xrightarrow{\text{HC}} \sigma(T)$ nor $\sigma(T_n) \xrightarrow{\text{ls}} \sigma(T)$. (E.g. consider the operator $A_n = \frac{1}{n}x$ on $L^2(\mathbb{R})$, which gives $A_n \xrightarrow{\text{sr}} 0$.)
- The notation $\xrightarrow{\text{HC}}$ and $\xrightarrow{\text{ls}}$ are non-standard, but are introduced here to distinguish between various modes of spectral convergence found in the literature.
- In this thesis, we will encounter the setup where T_n is self-adjoint on $\mathcal{H}$, whereas T is self-adjoint as an operator on a subspace $\mathcal{H}_1$ of $\mathcal{H}$. Writing $P_{\mathcal{H}_1}$ for the projection of $\mathcal{H}$ onto $\mathcal{H}_1$, we will show that

$$\|(T_n+i)^{-1} - (T+i)^{-1}P_{\mathcal{H}_1}\|_{\mathcal{H}\to\mathcal{H}} \to 0, \quad \text{as } n\to\infty. \tag{1.12}$$

Inspecting the proof of Theorem 1.3.7 (i.e. [57, Section I.3] and [21, Theorem VIII.23(a) and Theorem VIII.24(a)]), one checks that (1.12) is sufficient to conclude $\sigma(T_n) \xrightarrow{\text{HC}} \sigma(T)$ and $\sigma(T_n) \xrightarrow{\text{ls}} \sigma(T)$. We omit the details for brevity. ○

1.4 Existing literature

This thesis is written under the context of two developments in the subject of homogenization. The first is a push towards turning various qualitative results in [3, 6, 27] into quantitative ones [2, 23]. The second is concerned with extending the techniques of [3, 6, 27] to account for "degenerate" situations, for instance when there is a lack of uniform ellipticity [52]. The purpose of this section is to elaborate on the relevant literature in these two developments.

Operator norm estimates in moderate-contrast homogenization

Let us begin with an overview on existing quantitative results in homogenization, restricting our discussion to operator norm estimates. The first operator norm estimates were obtained by Birman and Suslina in [32], for the resolvent $(A_\varepsilon + I)^{-1}$. More precisely, it was proved that

$$\|(A_\varepsilon + I)^{-1} - (A_{\text{hom}} + I)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C\varepsilon, \quad \text{where } C \text{ is independent of } \varepsilon. \tag{1.13}$$

Here, $A_\varepsilon = -\operatorname{div}(a_\varepsilon \nabla \cdot)$ is uniformly strongly elliptic, and $A_{\text{hom}} = -\operatorname{div}(a_{\text{hom}}\nabla\cdot)$, where the coefficient matrix a_{hom} is constant in space. Equivalently, this can be written in terms of $u_\varepsilon = (A_\varepsilon + I)^{-1}f$ and $u_{\text{hom}} = (A_{\text{hom}} + I)^{-1}f$ for a given $f \in L^2$ by

$$\|u_\varepsilon - u_0\|_{L^2} \leq C\varepsilon\|f\|_{L^2}, \quad \text{where } C \text{ is independent of } \varepsilon \text{ and } f.$$

To obtain (1.13), the authors applied the Floquet transform to A_ε to obtain a family of operators $\{A_\varepsilon^{(\tau)}\}_{[-\pi,\pi)^d}$, and then proceeded with a spectral analysis of $A_\varepsilon^{(\tau)}$ using analytic perturbation theory, with a focus on the behaviour of the resolvent of A_ε near the bottom of the spectrum. The key object here is an auxillary operator referred to as the "spectral germ". Their approach was later extended to include other related setups, for instance, bounded domains [48, 49] and perforated domains [50].

Other methods that appeared thereafter include:

- The periodic unfolding method, introduced by Griso in [40, 41].
- The shift method, introduced by Zhikov and Pastukhova in [54] (see also their survey paper [55]).
- A refinement of the two-scale expansion method by Kenig, Lin, and Shen [43], which directly dealt with the case of bounded domains (see also the recent book by Shen [23]).
- A recent work by Cooper and Waurick [38], proposing an abstract framework under which uniform in τ norm-resolvent estimates for the family $A_\varepsilon^{(\tau)}$ can be achieved.

Let us remark that this list is non-exhaustive, and is growing at the point of writing. These methods work well in the moderate-contrast setting (meaning that $\widetilde{a}(y)$ is positive definite and bounded), but cannot be used in the high-contrast case $\widetilde{a}_\varepsilon(y)$ (see (1.7)), at least without serious modifications. This brings us to the approach of [35].

A boundary triple approach to high-contrast homogenization

As mentioned in Section 1.2, we will use a method proposed by Cherednichenko, Ershova, and Kiselev in [35]. Recall that this is an operator framework based on four key ingredients:

(A) The (rescaled) *Gelfand/Floquet transform* G_ε.

(B) Boundary triples (A_0, Λ, Π) in the sense of Ryzhov [47].

(C) Perturbation theory in the sense of Kato [15], and Reed and Simon [20, Chapter XII].

(D) Generalised resolvents (see Section 2.4 and Theorem 2.4.20).

Let us make a few historical remarks on (A) and (B). The use of Gelfand transform in the mathematical analysis of periodic homogenization problems can be traced back to Zhikov [51], and Conca and Vanninathan [37]. However, they did not pursue the goal of obtaining operator norm estimates. Nonetheless, it is possible to extend the work of Zhikov [51] to obtain operator norm estimates, as explained in the survey paper by Zhikov and Pastukhova in [54, Sections 9-11]. As for ingredient (B), the Ryzhov boundary triple is a generalization of the ("classical") boundary triple introduced independently by Kochubei [45] and Bruk [33] (see also [5], [14, Chapter 3], and [22, Chapter 14]). This generalization is more suited for the PDE setting, as it allows the trace operators to be defined on a smaller set than what is required of a classical boundary triple.

Next, we make a few remarks in connection with the moderate-contrast case. First, we point out that the framework of [35] could in principle, be applied to the moderate-contrast problems. Second, we note that the use of the Gelfand/Floquet transform in periodic homogenization problems is a common first step in operator approaches to homogenization (Birman-Suslina and Cooper-Waurick method). Third, we point out that the use of perturbation theory in homogenization is not new. For instance, it is core to the Birman-Suslina approach. However, the authors of [35] employed (C) in a novel way, by looking at perturbation of objects such as the Dirichlet-to-Neumann operator between the soft-stiff interfaces.

The work [35] has been a culmination of a series of papers attempting to bring boundary triple theory to the asymptotic analysis of high-contrast homogenization problems. We refer the reader to the recent survey by Cherednichenko, Ershova, Kiselev, Ryzhov, and Silva [36] for the full details. Here, let us give a truncated version of the survey: The authors of [35] initially used a simplified version of their framework to study a high-contrast homogenization problem on a periodic quantum graph (ODE on the full space $\mathbb{R}$) [34]. In the quantum graph setting, the classical triple suffices as the new ingredient in (B). In [44], Cherednichenko, Kiselev and Silva demonstrate the use of Ryzhov triples under a PDE setting on a bounded domain. By combining the techniques of [34] and [44], one is able to treat PDE setting on the full space $\mathbb{R}^d$ with periodic coefficients. This is the content of [35].

The operator framework of [35] has proven to be successful in the study of high-contrast composites, at least for simple geometries like those in Figure 1-2. One of the goals of the thesis is to demonstrate how the approach of [35] can be extended to a geometry like Figure 1-1.

High-contrast homogenization for the stiff-soft-stiff composite

Next, let us comment on the choice of our setup, in relation to existing results. The stiff-soft-stiff model (Figure 1-1) is derived from the two auxiliary models studied in [35], referred to as Model I and Model II (Figure 1-2).

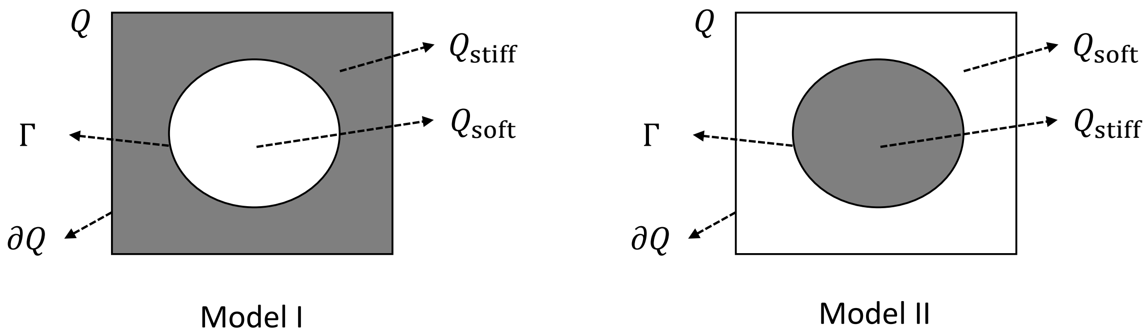

Figure 1-2: Auxiliary models from [35].

As depicted in the figure, we see that the auxiliary models are geometrically identical, having only one inclusion inside the period cell Q, with smooth boundary Γ, and at a positive distance from the boundary of the cube ∂Q. Models I and II differ only in the choice of the soft and stiff components.

Due to the similarity in geometry, one might naively guess that the homogenized descriptions of A_ε are almost identical. This is true to a certain extent. Indeed, it was shown in [35, Section 4.2] that for both Models I and II, the fibres $A_{\text{hom}}^{(\tau)}$ of the homogenized operator A_{hom} is one that, roughly speaking, stays unchanged as $-(\nabla + i\tau)^2$ on $L^2(Q_{\text{soft}})$, and acts only on a $1D$ subspace of $L^2(Q_{\text{stiff}})$. A further study on how the constant of multiplication c_τ in $L^2(Q_{\text{stiff}})$ depends on τ reveals a non-local behavior of A_{hom} on $L^2(\mathbb{R}^d)$, see [35, Section 5.4].

But A_{hom} of Model I and II are different in many respects. For starters, A_{hom} depends on ε for Model I, and does not for Model II. The fact that we only obtain *asymptotics* for Model I is to be expected, because it is known from [42, p. 1447] that A_ε in Model I does not have a norm-resolvent limit. Model I does however possess a strong-resolvent limit $A_{\text{hom,sr}}$ (the Dirichlet Laplacian on the soft parts of $\mathbb{R}^d$ [42, Proposition 2.2]), and also a two-scale strong resolvent limit $A_{\text{hom,2sr}}$ [53, Section 3].

However, the operator $A_{\text{hom,sr}}$ does not capture the spectral information of A_ε, since $\sigma(A_\varepsilon) \not\to \sigma(A_{\text{hom,sr}})$ (in the sense of Definition 1.3.6). Furthermore, the manner in which the limit $A_{\text{hom,sr}}$ was obtained in [42] does not provide us with a rate of convergence. On the other hand, the operator $A_{\text{hom,2sr}}$ satisfies $\sigma(A_\varepsilon) \to \sigma(A_{\text{hom,2sr}})$, and we even know the spectral decomposition of $\sigma(A_{\text{hom,2sr}})$. But the proof is again qualitative in nature, relying on an additional compactness argument to establish spectral convergence. An alternative route taken by [42] is to study $\sigma(A_\varepsilon)$ directly, without characterising the limiting behaviour as the spectrum of some $A_{\text{hom},\varepsilon}$.

As for Model II, it was shown in [52, formulae (5.7) and (7.1), and Theorem 5.1] that the two-scale strong resolvent limit $A_{\text{hom,2sr}}$ exists, using qualitative arguments. But in contrast to Model I, we do not know if there is spectral convergence of A_ε to $A_{\text{hom,2sr}}$, and we do not know the decomposition of $\sigma(A_{\text{hom,2sr}})$. The norm-resolvent limit A_{hom} (which is also the strong-resolvent limit, but different from $A_{\text{hom,2sr}}$) is obtained in [35], together with a rate of convergence.

The methods of [52] pertaining to two-scale strong resolvent limits are quite general, encompassing various configurations (see [52, Sect 5.1] for precise conditions), and various scaling choices ε^{α}, $\alpha > 0$. This is in contrast to norm-resolvent asymptotics, where as mentioned earlier, only the setups in Figure 1-2 have been studied so far, under the double porosity scaling $\alpha = 2$.

Even though it is possible to apply the result of [52] to the stiff-soft-stiff model (with $\alpha = 2$) and obtain a two-scale strong resolvent limit $A_{\text{hom,2sr}}$, there is work to be done. That includes: verifying if there is spectral convergence; finding the spectral decomposition of $A_{\text{hom,2sr}}$; and turning the qualitative arguments into quantitative ones. We will not pursue that route here. We will however apply the methods of [35] to obtain the norm-resolvent asymptotics A_{hom} for the stiff-soft-stiff model. This has been open prior to the writing of this thesis, and is the content of Chapter 2.

One might wonder the sense in which the norm resolvent asymptotics A_{hom}, obtained from [35], provides a simplified description of the high-contrast composite. We attempt to provide an answer in the following context: just as how we may study the dispersion relation of a periodic operator, we could also ask for the limiting dispersion relation of A_{ε}. We will do so by taking a closer look at the non-local part of $A^{(\tau)}_{\text{hom}}$, in particular at how the constant of multiplication c_{τ} in the 1D subspace of $L^2(Q_{\text{stiff}})$ depends on τ. The key object that is extracted from this study is referred to as the "dispersion function" $K(\tau, z)$. As shown in [35, Section 5], $K(\tau, z)$ are very different for Models I and II. In Chapter 3, we will derive $K(\tau, z)$ for stiff-soft-stiff model, and compare it with $K(\tau, z)$ of Models I and II.

1.5 Main results

The main results of the thesis are as follows:

Results from Chapter 2

- A homogenization result for the composite material in Figure 1-1. This is Section 2.5, and in particular, Theorem 2.5.3. We give an effective description of the composite by identifying the norm-resolvent asymptotics of $A^{(\tau)}_{\varepsilon}$, namely the operator $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$.
- Moreover, we supplement the asymptotic argument in [35] with additional details, meant to explain how the estimates obtained are uniform over τ and z. These are Proposition 2.3.3 and large portions of the proof of Theorem 2.3.4.

Results from Chapter 3

- We look at the bottom right entry of the resolvent for $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$. This is a 2×2 matrix, due to the two stiff components. We write each entry in the form of a multiplication by a constant on $\mathbb{C}$. Moreover, for the diagonal entries we are able to express this constant as $(K(\tau, z) - z)^{-1}$, and we refer to $K(\tau, z)$ as the "dispersion function". The precise statements are

- $\mathbb{C}_{\textbf{stiff-int}} \to \mathbb{C}_{\textbf{stiff-int}}$: Theorem 3.1.6,
- $\mathbb{C}_{\textbf{stiff-ls}} \to \mathbb{C}_{\textbf{stiff-ls}}$: Theorem 3.1.10,
- $\mathbb{C}_{\textbf{stiff-int}} \to \mathbb{C}_{\textbf{stiff-ls}}$: Theorem 3.1.12 and Corollary 3.1.13,
- $\mathbb{C}_{\textbf{stiff-ls}} \to \mathbb{C}_{\textbf{stiff-int}}$: Theorem 3.1.15 and Corollary 3.1.16.

- In Section 3.2.3, we provide a formula for the homogenized description of our composite on the full space, i.e. the operator

$$\mathcal{A}_{\varepsilon,\text{hom}} = G^* \left(\int_{Q'}^{\oplus} \mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} d\tau \right) G.$$

- We prove results on the spectrum and spectral decomposition of $\mathcal{A}_{\varepsilon,\text{hom-I}}$ and $\mathcal{A}_{\text{hom-II}}$ (the norm-resolvent asymptotics for Models I and II on the full space $L^2(\mathbb{R}^d)$, as obtained in [35]). The precise statements are
  - **For Model I:** Proposition 3.3.3 (eigenvalues), Corollaries 3.3.6 and 3.3.7 (characterization of $\sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$ in terms of the dispersion function $K_{\text{I}}(\tau, z)$), and Proposition 3.3.8 (when $K_{\text{I}}(\tau, z)$ is undefined).
  - **For Model II:** Proposition 3.3.11 (eigenvalues), Corollaries 3.3.14 and 3.3.15 (characterization of $\sigma(\mathcal{A}_{\text{hom-II}})$ in terms of the dispersion function $K_{\text{II}}(\tau, z)$). We only provided partial results when $K_{\text{II}}(\tau, z)$ is undefined (Propositions 3.3.16 and 3.3.17).

These results are also collected in Chapter 4, where we summarize the work of this thesis.

Chapter 2

Homogenization of the stiff-soft-stiff composite

In this chapter, we detail the process of homogenization of the stiff-soft-stiff composite using the operator framework [35]. This chapter consists of five sections. The first four sections roughly corresponds to the use of the four key ingredients: (A) Gelfand transform, (B) Ryzhov boundary triples, (C) perturbation theory, and (D) generalized resolvents.

In Section 2.1, we provide a rigorous formulation of the problem outlined in Chapter 1. We define the operator A_ε on $L^2(\mathbb{R}^d)$ and explain why we can equivalently study the operator family $\{A_\varepsilon^{(\tau)}\}_{\tau\in[-\pi,\pi)^d}$ on $L^2(Q)$, obtained by the Gelfand transform. In Section 2.2, we recast the problem yet again, this time in the language of boundary triples.

Section 2.3 studies the norm-resolvent asymptotics of $A_\varepsilon^{(\tau)}$, and is at the heart of the analysis. To ensure that the asymptotics are uniform in τ, we use a perturbative argument. In Section 2.4, we combine the result of Section 2.3 (Theorem 2.3.4) with the boundary triple setup of Section 2.2 to give a self-adjoint operator $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$ that captures the norm-resolvent asymptotics of $A_\varepsilon^{(\tau)}$. Finally, Section 2.5 unpacks the notation and summarizes the boundary triple approach for homogenization.

2.1 Problem formulation

This section is structured as follows: In Section 2.1.1, we define the operator A_ε on $L^2(\mathbb{R}^d)$. Next, we introduce the scaled Gelfand transform G_ε in Section 2.1.2, and use it to obtain a family of operators $A_\varepsilon^{(\tau)}$ on $L^2(Q)$ indexed by $\tau \in [-\pi,\pi)^d$. With the help of G_ε, our study of the norm-resolvent asymptotics of A_ε can be restated in terms of the family $A_\varepsilon^{(\tau)}$. This allows to reformulate our problem in terms of $A_\varepsilon^{(\tau)}$, which we will do in Section 2.1.3.

The operator $A_\varepsilon^{(\tau)}$ will be the main object of study in the remainder of the text. We will refer to the setup in Section 2.1.3 as the "main model", and $A_\varepsilon^{(\tau)}$ as the "main model operator".

2.1.1 Operator on the full space

In this section we will define the operator A_ε. On the reference cell $Q = [0,1)^d$, consider the setup as shown in Figure 1-1. That is, Q is split into three connected components: a simply connected “stiff interior” part $Q_{\text{stiff-int}}$, surrounded by an annular “soft” region Q_{soft}, with the remaining region filled by the “stiff landscape” part $Q_{\text{stiff-ls}}$. For the soft-stiff interfaces Γ_{int} and Γ_{ls} we require that

- the boundaries Γ_{int} and Γ_{ls} are smooth, and
- Γ_{int}, Γ_{ls}, and ∂Q are of positive distance from each other.

Recall from (1.7) that our coefficient matrix $\widetilde{a}_{\varepsilon^2} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is given by

$$\widetilde{a}_{\varepsilon^2}(y) = \begin{cases} \varepsilon^2 I, & y \in \cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}} + n), \\ I, & y \in \cup_{n\in\mathbb{Z}^d}\left((Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}}) + n\right), \end{cases} \tag{2.1}$$

where $Q_{\text{soft}} + n = \{y + n : y \in Q_{\text{soft}}\}$, and similarly for $(Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}}) + n$. The matrix $\widetilde{a}_{\varepsilon^2}$ is $\mathbb{Z}^d$-periodic, and thus the matrix

$$a_\varepsilon := \widetilde{a}_{\varepsilon^2}\left(\frac{\cdot}{\varepsilon}\right) \tag{2.2}$$

is $\varepsilon\mathbb{Z}^d$-periodic.

The operator $A_\varepsilon \equiv -\text{div}(a_\varepsilon \nabla \cdot)$ is defined through its sesquilinear form:

$$(u, v) \mapsto \int_{\mathbb{R}^d} \widetilde{a}_{\varepsilon^2}\left(\tfrac{\tilde{x}}{\varepsilon}\right) \nabla u(\tilde{x}) \cdot \overline{\nabla v(\tilde{x})}\, d\tilde{x}, \quad u, v \in \mathcal{D}[A_\varepsilon] := H^1(\mathbb{R}^d). \tag{2.3}$$

A_ε is an unbounded, lower-semibounded self-adjoint operator on $L^2(\mathbb{R}^d)$. Let us emphasize again that A_ε is not uniformly strongly elliptic, in the sense that the coefficient matrices $\widetilde{a}_{\varepsilon^2}(\frac{\cdot}{\varepsilon})$ cannot be bounded away from zero, independently of ε.

If $z \in \rho(A_\varepsilon)$, then the resolvent equation

$$-\,\text{div}(\widetilde{a}_{\varepsilon^2}\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon) - z u_\varepsilon = f, \quad f \in L^2(\mathbb{R}^d), \quad z \in \mathbb{C}, \tag{2.4}$$

has a unique solution u_ε, which can be written as $u_\varepsilon = (A_\varepsilon - z)^{-1} f$. In terms of the weak formulation, the resolvent equation is given by:

$$\int_{\mathbb{R}^d} \left[\widetilde{a}_{\varepsilon^2}\left(\tfrac{\tilde{x}}{\varepsilon}\right) \nabla u(\tilde{x}) \cdot \overline{\nabla v(\tilde{x})} - z u(\tilde{x})\overline{v(\tilde{x})}\right] d\tilde{x} = \int_{\mathbb{R}^d} f(\tilde{x})\overline{v(\tilde{x})}\, d\tilde{x}, \quad \text{for all } v \in H^1(\mathbb{R}^d). \tag{2.5}$$

2.1.2 Passing from the full space to the unit cell

Let $Q' = [-\pi, \pi)^d$. It is customary in the study of periodic differential operators (*Floquet Theory*, see e.g. [20, Section XIII.16]) to begin the analysis of a $\mathbb{Z}^d$-periodic operator T by applying a unitary transformation to T, giving us family of operators $T^{(\tau)}$, $\tau \in Q'$. There are two unitary transforms that we can choose from:

1. The *Floquet transform*, which takes $u \in L^2(\mathbb{R}^d)$ to a function $u_f(x, \chi)$ that is quasiperiodic in x and periodic in τ. This gives rise to the differential operator $(\frac{1}{i}\nabla)^2$ on the unit cube Q subjected to quasiperiodic BCs.

2. The *Gelfand transform*, which takes $u \in L^2(\mathbb{R}^d)$ to a function $u_g(x, \chi)$ that is periodic in x and quasiperiodic in τ. This gives rise to the differential operator $(\frac{1}{i}\nabla + \tau)^2$ on the unit cube Q subjected to periodic BCs.

We use the Gelfand transform, as it will be easier to deal with a varying action as opposed to a varying boundary condition. Let us now summarize the necessary elements from Floquet theory that will be of use here.

First, it would be more convenient to introduce a scaled version of the Gelfand transform since A_ε is $\varepsilon\mathbb{Z}^d$-periodic rather than $\mathbb{Z}^d$-periodic:

Definition 2.1.1. The *scaled Gelfand transform* is the operator G_ε defined first for $u \in C_c^\infty(\mathbb{R}^d)$ by the formula

$$(G_\varepsilon u)(\tilde{x}, \theta) := \left(\frac{\varepsilon}{2\pi}\right)^{d/2} \sum_{n \in \mathbb{Z}^d} u(\tilde{x} + \varepsilon n) e^{-i\theta \cdot (\tilde{x} + \varepsilon n)}, \qquad \tilde{x} \in \varepsilon Q,\ \theta \in \varepsilon^{-1} Q', \tag{2.6}$$

and extended by continuity to an operator from $L^2(\mathbb{R}^d)$ to $L^2(\varepsilon Q \times \varepsilon^{-1} Q')$, which will we still denote by G_ε.

Remark. In fact, G_ε is unitary, with the following inversion formula:

$$u(\tilde{x}) = \left(\frac{\varepsilon}{2\pi}\right)^{d/2} \int_{\varepsilon^{-1} Q'} (G_\varepsilon u)(\tilde{x}, \theta) e^{i\theta \cdot \tilde{x}} d\theta, \qquad \tilde{x} \in \mathbb{R}^d, \tag{2.7}$$

where we have extended $G_\varepsilon u$ in $\tilde{x}$ by $\varepsilon\mathbb{Z}^d$-periodicity to a function on $\mathbb{R}^d \times \varepsilon^{-1} Q'$. ◦

Next, we introduce a new notation for the Bochner spaces $L^2(M, \mu; \mathcal{H}')$, following [20, Section VIII.16]:

Definition 2.1.2. Let (M, μ) be a σ-finite measure space, and $\mathcal{H}'$ a separable Hilbert space. We define the *(constant fiber) direct integral space* $\mathcal{H} = \int_M^\oplus \mathcal{H}' d\mu(m)$ to be the Bochner space $L^2(M, \mu; \mathcal{H}')$. Recall that this is a Hilbert space, equipped with inner product

$$(s, t)_{\mathcal{H}} := \int_M (s(m), t(m))_{\mathcal{H}'} \, d\mu(m). \tag{2.8}$$

Elements of this space $s \in \mathcal{H}$ are called *(measurable cross-)sections*. The space $\mathcal{H}'$ is referred to as *fibers*.

This notation places emphasis on the fibers $\mathcal{H}'$, and therefore on operators on $\mathcal{H}'$ indexed by the set (M, μ), in a measurable way. This requires us to define a notion of measurability. We continue with the notation of Definition 2.1.2 for the remaining definitions of this section:

Definition 2.1.3. We say that $T(\cdot) : M \to \mathcal{L}(\mathcal{H}')$ is measurable if for all $x, y \in \mathcal{H}'$, the mapping

$$M \ni m \mapsto (x, T(m)y)_{\mathcal{H}'} \in \mathbb{C} \tag{2.9}$$

is measurable.

However, we would need to deal with unbounded self-adjoint operators, therefore we make the following definition:

Definition 2.1.4. Let $\{A(m)\}_{m\in M}$ be a collection of unbounded self-adjoint operators on $\mathcal{H}'$. We say that $A(\cdot)$ is measurable if the mapping

$$M \ni m \mapsto (A(m)+i)^{-1} \in \mathcal{L}(\mathcal{H}') \tag{2.10}$$

is measurable.

Remark. This is a simple case of the more general direct integral $\int_M^{\oplus} \mathcal{H}'(m)d\mu(m)$, for which the definition may be found in [11, Section 8.4] or [7, Chapter 7]. See also [20, Section XIII.16] for some useful results concerning the constant fiber direct integral. See [24, Lemma 1.2.2] for equivalent characterizations of measurability for unbounded self-adjoint operators. ∘

We are now ready to introduce the notion of a "continuous direct sum of bounded operators".

Definition 2.1.5 (Decomposable operator)**.** Let T be a bounded operator on $\mathcal{H} = \int_M^{\oplus} \mathcal{H}'d\mu$. Suppose there exists a measurable family $T(\cdot) \in L^\infty(X,\mu;\mathcal{L}(\mathcal{H}'))$ such that for all sections $s \in \mathcal{H}$,

$$(Ts)(x) = T(x)s(x). \tag{2.11}$$

Then we call T *decomposable*, write $T = \int_M^{\oplus} T(m)d\mu(m)$, and call $T(m)$ the *fibers of* T.

Similarly, we will need to extend this notion to the unbounded self-adjoint case:

Definition 2.1.6. Suppose $A(\cdot)$ is a measurable family of unbounded self-adjoint operators on $\mathcal{H}'$. We define the operator $A \equiv \int_M^{\oplus} A(m)d\mu(m)$ by

$$\mathcal{D}(A) = \left\{ u \in \mathcal{H} \;\middle|\; u(m) \in \mathcal{D}\left(A(m)\right) \ \mu\text{-a.e., with } \int_M \|A(m)u(m)\|^2_{\mathcal{H}'}d\mu(m) < \infty \right\}, \tag{2.12}$$

$$(Au)(m) = A(m)u(m). \tag{2.13}$$

This is an unbounded self-adjoint operator, by [20, Theorem XIII.85(a)].

Remark. With the notation of a direct integral space, the Gelfand transform may now be written as

$$G_\varepsilon : L^2(\mathbb{R}^d) \longrightarrow L^2(\varepsilon Q \times \varepsilon^{-1}Q') \cong \int_{\varepsilon^{-1}Q'}^{\oplus} L^2(\varepsilon Q)d\theta,$$
$$(G_\varepsilon u)(\theta) = (G_\varepsilon u)(\cdot,\theta) = G_\varepsilon(\theta)u(\theta) \in L^2(\varepsilon Q).$$

Note also the special case $\int_{\varepsilon^{-1}Q'}^{\oplus} \mathbb{C}d\theta \cong L^2(\varepsilon^{-1}Q')$. ∘

Definition 2.1.7. For $\theta \in \varepsilon^{-1}Q'$, define $A_\varepsilon^{(\theta)}$ to be the operator on $L^2(\varepsilon Q)$ corresponding to the sesquilinear form

$$(u,v) \mapsto \int_{\varepsilon Q} \widetilde{a}_{\varepsilon^2}\left(\tfrac{\tilde{x}}{\varepsilon}\right)(\tfrac{1}{i}\nabla_{\tilde{x}}+\theta)u(\tilde{x}) \cdot \overline{(\tfrac{1}{i}\nabla_{\tilde{x}}+\theta)v(\tilde{x})}d\tilde{x}, \quad u,v \in \mathcal{D}[A_\varepsilon^{(\theta)}] := H^1_{\text{per}}(\varepsilon Q). \tag{2.14}$$

That is, $A_\varepsilon^{(\theta)}$ corresponds to the differential expression $(\frac{1}{i}\nabla_{\tilde{x}}+\theta)\widetilde{a}_{\varepsilon^2}(\frac{\tilde{x}}{\varepsilon})(\frac{1}{i}\nabla_{\tilde{x}}+\theta)$ with periodic BCs on εQ.

Since A_ε (from Section 2.1.1) has $\varepsilon\mathbb{Z}^d$-periodic coefficients, the scaled Gelfand transform sets up a unitary equivalence between A_ε and a family of operators $A_\varepsilon^{(\theta)}$:

Proposition 2.1.8. With A_ε as defined in Section 2.1.1 and $A_\varepsilon^{(\theta)}$ as in Definition 2.1.7, we have the following identity:

$$A_\varepsilon = G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} A_\varepsilon^{(\theta)} d\theta \right) G_\varepsilon. \tag{2.15}$$

Proof. This is just a direct consequence of the product rule, see for example [56, Theorem 2.5] for the short computation. The periodic BC follows from the fact that $G_\varepsilon u(\tilde{x},\theta)$ is $\varepsilon\mathbb{Z}^d$-periodic in $\tilde{x}$. □

While we have shifted our perspective to consider a (θ dependent) operator on a bounded subset of $\mathbb{R}^d$, this is still rather inconvenient as the Hilbert space $L^2(\varepsilon Q)$ varies with ε. Ideally, we would like to have ε only appearing in the domain and action of the operator, keeping the underlying Hilbert space as $L^2(Q)$, for all θ and ε. This motivates us to define:

Definition 2.1.9. For each $\varepsilon > 0$, define the unitary rescaling operators Φ_ε and Ψ_ε

$$\Phi_\varepsilon : L^2(\varepsilon Q) \to L^2(Q) \qquad (\Phi_\varepsilon u)(x) = \varepsilon^{d/2} u(\varepsilon x), \tag{2.16}$$

$$\Psi_\varepsilon : L^2(\varepsilon^{-1}Q') \to L^2(Q') \qquad (\Psi_\varepsilon v)(\tau) = \left(\tfrac{2\pi}{\varepsilon}\right)^{d/2} v(\tfrac{\tau}{\varepsilon}). \tag{2.17}$$

Definition 2.1.10. For $\tau \in Q'$, define $A_\varepsilon^{(\tau)}$ to be the operator on $L^2(Q)$ corresponding to the sesquilinear form

$$(u,v) \mapsto \frac{1}{\varepsilon^2}\int_Q \widetilde{a}_{\varepsilon^2}(x)\,(\tfrac{1}{i}\nabla_x+\theta)u(x)\cdot\overline{(\tfrac{1}{i}\nabla_x+\theta)v(x)}dx, \quad u,v\in\mathcal{D}[A_\varepsilon^{(\tau)}] := H^1_{\mathrm{per}}(Q). \tag{2.18}$$

That is, $A_\varepsilon^{(\tau)}$ corresponds to the differential expression $(\frac{1}{i}\nabla_x+\tau)\frac{1}{\varepsilon^2}\widetilde{a}_{\varepsilon^2}(x)(\frac{1}{i}\nabla_x+\tau)$ with periodic BCs on Q. We will refer to $A_\varepsilon^{(\tau)}$ as the *main model operator.*

Lemma 2.1.11. Let $\tau = \varepsilon\theta$. Then, $A_\varepsilon^{(\tau)} = \Phi_\varepsilon A_\varepsilon^{(\theta)}\Phi_\varepsilon^*$.

Proof. Equivalently, we need to show that $A_\varepsilon^{(\tau)}\Phi_\varepsilon = \Phi_\varepsilon A_\varepsilon^{(\theta)}$ as an operator from $L^2(\varepsilon Q)$ to $L^2(Q)$. It suffices to check this on a form core $C^\infty_{\mathrm{per}}(\varepsilon Q)$. Let $u \in C^\infty_{\mathrm{per}}(\varepsilon Q)$. We use $\tilde{x}$ for the variable on εQ, and x for the variable on Q. First we see that

$$\begin{aligned}\left(A_\varepsilon^{(\theta)}u\right)(\tilde{x}) &= \left(\frac{1}{i}\nabla_{\tilde{x}}+\theta\right)\cdot\left(\widetilde{a}_{\varepsilon^2}\left(\frac{\tilde{x}}{\varepsilon}\right)\left(\frac{1}{i}\nabla_{\tilde{x}}+\theta\right)u(\tilde{x})\right)\\ &= \sum_{j,k=1}^d \left(\frac{1}{i}\frac{\partial}{\partial\tilde{x}_j}+\theta_j\right)\left[\widetilde{a}^{jk}_{\varepsilon^2}\left(\frac{\tilde{x}}{\varepsilon}\right)\left(\frac{1}{i}\frac{\partial}{\partial\tilde{x}_k}+\theta_k\right)u(\tilde{x})\right].\end{aligned} \tag{2.19}$$

Therefore, if $\tau = \varepsilon\theta$ and $x = \frac{\tilde{x}}{\varepsilon}$, then

$$RHS = \left(\Phi_\varepsilon A_\varepsilon^{(\theta)}u\right)(x) = \varepsilon^{d/2}\left(A_\varepsilon^{(\theta)}u\right)(\varepsilon x)$$

$$= \varepsilon^{d/2} \sum_{j,k=1}^{d} \left(\frac{1}{i}\frac{\partial}{\partial \tilde{x}_j} + \theta_j\right) \left[\widetilde{a}^{jk}_{\varepsilon^2}\left(\frac{\varepsilon x}{\varepsilon}\right)\left(\frac{1}{i}\frac{\partial}{\partial \tilde{x}_k} + \theta_k\right) u(\varepsilon x)\right]$$

$$= \varepsilon^{d/2} \sum_{j,k=1}^{d} \left(\frac{1}{i\varepsilon}\frac{\partial}{\partial x_j} + \frac{\tau_j}{\varepsilon}\right) \left[\widetilde{a}^{jk}_{\varepsilon^2}(x)\left(\frac{1}{i\varepsilon}\frac{\partial}{\partial x_k} + \frac{\tau_k}{\varepsilon}\right) u(\varepsilon x)\right]$$

$$= \varepsilon^{d/2} \sum_{j,k=1}^{d} \left(\frac{1}{i}\frac{\partial}{\partial x_j} + \tau_j\right) \left[\frac{1}{\varepsilon^2}\widetilde{a}^{jk}_{\varepsilon^2}(x)\left(\frac{1}{i}\frac{\partial}{\partial x_k} + \tau_k\right) u(\varepsilon x)\right]$$

$$= \varepsilon^{d/2} \left(\frac{1}{i}\nabla_x + \tau\right) \cdot \left(\frac{1}{\varepsilon^2}\widetilde{a}_{\varepsilon^2}(x)\left(\frac{1}{i}\nabla_x + \tau\right) u(\varepsilon x)\right)$$

$$= \left(\frac{1}{i}\nabla_x + \tau\right) \cdot \left(\frac{1}{\varepsilon^2}\widetilde{a}_{\varepsilon^2}(x)\left(\frac{1}{i}\nabla_x + \tau\right) (\Phi_\varepsilon u)(x)\right) = \left(A^{(\tau)}_\varepsilon \Phi_\varepsilon u\right)(x) = LHS. \qquad \square$$

Corollary 2.1.12. A_ε is unitarily equivalent to $\int_{Q'}^{\oplus} A_\varepsilon^{(\tau)} d\tau$.

Proof. Using expressions relating A_ε, $A_\varepsilon^{(\theta)}$, and $A_\varepsilon^{(\tau)}$, $\tau = \varepsilon\theta$, we have

$$A_\varepsilon \overset{\text{Prop 2.1.8}}{=} G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} A_\varepsilon^{(\theta)} d\theta\right) G_\varepsilon \overset{\text{Lemma 2.1.11}}{=} G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon^* A_\varepsilon^{(\varepsilon\theta)} \Phi_\varepsilon d\theta\right) G_\varepsilon$$

$$= G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon^* d\theta\right) \left(\int_{\varepsilon^{-1}Q'}^{\oplus} A_\varepsilon^{(\varepsilon\theta)} d\theta\right) \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon d\theta\right) G_\varepsilon$$

$$= G_\varepsilon^* \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon^* d\theta\right) \left(\int_{Q}^{\oplus} \Psi_\varepsilon^* dx\right) \left(\int_{Q'}^{\oplus} A_\varepsilon^{(\tau)} d\tau\right) \left(\int_{Q}^{\oplus} \Psi_\varepsilon dx\right) \left(\int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon d\theta\right) G_\varepsilon, \qquad (2.20)$$

where in the last equality, we have used $\int_{\varepsilon^{-1}Q'}^{\oplus} L^2(Q) d\theta \cong L^2(Q \times \varepsilon^{-1}Q') \cong \int_{Q}^{\oplus} L^2(\varepsilon^{-1}Q') dx$. Note for instance, that $\| \int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon d\theta \|_{op} = \text{esssup}_\theta \|\Phi_\varepsilon\|_{L^2(\varepsilon Q) \to L^2(Q)} = 1$ [20, Theorem XIII.83]. $\square$

We will therefore turn our attention towards the operator $A_\varepsilon^{(\tau)}$. In the following section, we recast our problem of studying the norm-resolvent asymptotics of A_ε in terms of $A_\varepsilon^{(\tau)}$.

2.1.3 Reformulation in terms of operators on the unit cell

Having established the unitary equivalence between A_ε and $\int_{Q'}^{\oplus} A_\varepsilon^{(\tau)} d\tau$, our goal can now be stated as follows:

Identify, **uniform in τ, norm-resolvent asymptotics** for $A_\varepsilon^{(\tau)}$, as $\varepsilon \downarrow 0$.

Having turned our focus towards $A_\varepsilon^{(\tau)}$, let us collect several ways of describing $A_\varepsilon^{(\tau)}$ that will be useful for purposes of interpretation.

First, we recall from Definition 2.1.10 that $A_\varepsilon^{(\tau)}$ is an operator on $L^2(Q)$ that corresponds to the differential expression $(\frac{1}{i}\nabla_x + \tau)\frac{1}{\varepsilon^2}\widetilde{a}_{\varepsilon^2}(x)(\frac{1}{i}\nabla_x + \tau)$. Recall that the coefficient matrix is given by:

$$\frac{1}{\varepsilon^2}\widetilde{a}_{\varepsilon^2}(x) = \begin{cases} \varepsilon^{-2} I, & x \in Q_{\text{stiff-ls}}, \\ I, & x \in Q_{\text{soft}}, \\ \varepsilon^{-2} I, & x \in Q_{\text{stiff-int}}, \end{cases} \qquad (2.21)$$

where the subscripts "ls" and "int" stands for landscape and interior respectively.

Second, we have obtained $A_\varepsilon^{(\tau)}$ from A_ε through a combination of the Gelfand transform G_ε and rescaling Φ_ε in the previous subsection. Figure 2-1 gives a description of this process, where we pass from the full space $\mathbb{R}^d$ to the unit cell Q.

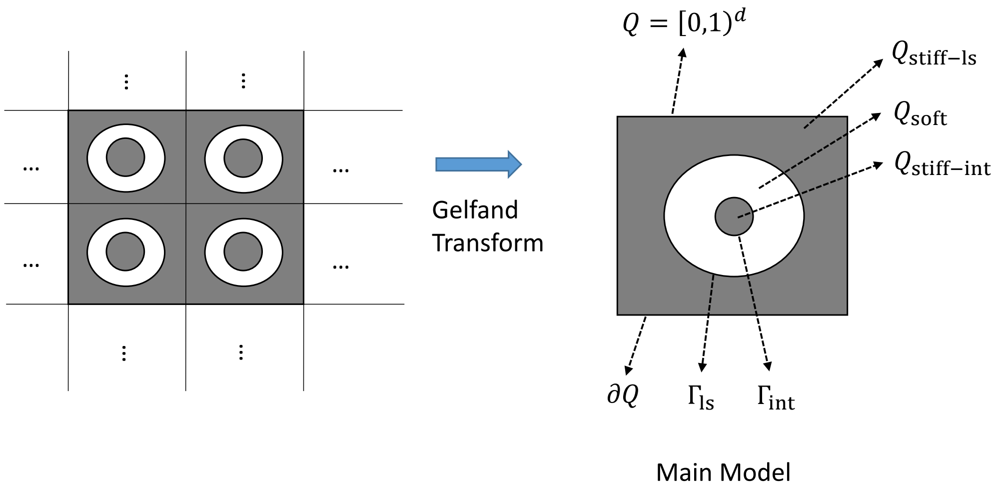

Figure 2-1: Obtaining the main model operator $A_\varepsilon^{(\tau)}$ via Gelfand transform.

Third, we mentioned in Chapter 1, that A_ε (and thus $A_\varepsilon^{(\tau)}$) will have transmission BCs on the soft-stiff interfaces. That Definition 2.1.10 implies transmission BCs on Γ_{ls} and Γ_{int} can be seen from the form domain $\mathcal{D}[A_\varepsilon^{(\tau)}] = H^1_{\text{per}}(Q)$. Alternatively, we can see this by writing out the BVP for the resolvent equation for $A_\varepsilon^{(\tau)}$: The resolvent equation $(A_\varepsilon^{(\tau)} - z)u = f \in L^2(Q)$ has a unique solution $u \equiv u_\varepsilon^{(\tau)} = u_{\text{stiff-ls}} + u_{\text{soft}} + u_{\text{stiff-int}}$ whenever the following BVP can be solved uniquely in the weak sense:

$$\begin{cases} \varepsilon^{-2}\left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{stiff-ls}} - z u_{\text{stiff-ls}} = f, & \text{in } Q_{\text{stiff-ls}}, \\ \left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{soft}} - z u_{\text{soft}} = f, & \text{in } Q_{\text{soft}}, \\ \varepsilon^{-2}\left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{stiff-int}} - z u_{\text{stiff-int}} = f, & \text{in } Q_{\text{stiff-int}}, \\ u_{\text{stiff-ls}} = u_{\text{soft}} & \text{on } \Gamma_{\text{ls}}, \\ \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-ls}}}{\partial n_{\text{stiff-ls}}} + i(\tau\cdot n_{\text{stiff-ls}})u_{\text{stiff-ls}}\right] + \left[\frac{\partial u_{\text{soft}}}{\partial n_{\text{soft}}} + i(\tau\cdot n_{\text{soft}})u_{\text{soft}}\right] = 0 & \text{on } \Gamma_{\text{ls}}, \\ u_{\text{soft}} = u_{\text{stiff-int}} & \text{on } \Gamma_{\text{int}}, \\ \left[\frac{\partial u_{\text{soft}}}{\partial n_{\text{soft}}} + i(\tau\cdot n_{\text{soft}})u_{\text{soft}}\right] + \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-int}}}{\partial n_{\text{stiff-int}}} + i(\tau\cdot n_{\text{stiff-int}})u_{\text{stiff-int}}\right] = 0 & \text{on } \Gamma_{\text{int}}, \\ u_{\text{stiff-ls}} \text{ periodic} & \text{on } \partial Q, \end{cases} \tag{2.22}$$

where $n_\star$ denotes the outward unit normal vector with respect to $Q_\star$, $\star \in \{\text{stiff-int}, \text{soft}, \text{stiff-ls}\}$.

Finally, we conclude this section by introducing the following notation:

Definition 2.1.13. Let $(\star, \bullet) \in \{(\text{soft}, \text{ls}), (\text{stiff-ls}, \text{ls}), (\text{soft}, \text{int}), (\text{stiff-int}, \text{int})\}$, we denote by $n_\star$ the outward pointing unit normal vector with respect to the component $Q_\star$. Also, let

$\partial_{n_\bigstar,\bullet}^{(\tau)} u$ be the trace of the co-normal derivative of u, with respect to $\bigstar$, on the boundary $\Gamma_\bullet$. This is defined for $u \in H^{3/2}(Q)$, by

$$\partial_{n_\bigstar,\bullet}^{(\tau)} u := -\left(\frac{\partial u}{\partial n_\bigstar} + i(\tau \cdot n_\bigstar)u\right)\bigg|_{\Gamma_\bullet}. \tag{2.23}$$

(Note the minus sign convention.)

2.2 Boundary triple theory setup

2.2.1 Preliminaries

To study the operator $A_\varepsilon^{(\tau)}$, it will be helpful to view it as a member of a larger family of operators. In our case, this family will be constructed using the Ryzhov boundary triple [47], for each τ. In this section, we will discuss the three ingredients that make up the boundary triple, namely, the τ-Dirichlet decoupling, the τ-harmonic lift, and the τ-Dirichlet-to-Neumann (τ-DtN) operator. First, we introduce a new notation for the spaces:

Definition 2.2.1. Set $\mathcal{H} := L^2(Q) = L^2(Q_{\text{stiff-int}}) \oplus L^2(Q_{\text{soft}}) \oplus L^2(Q_{\text{stiff-ls}})$ and $\mathcal{E} := L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$. We refer to $\mathcal{E}$ as the *boundary space.*

Remark (On notation)**.** We will view $L^2(Q_{\text{stiff-int}})$, $L^2(Q_{\text{soft}})$, and $L^2(Q_{\text{stiff-ls}})$ as subspaces of $\mathcal{H}$. This means, for instance, that a function $u \in L^2(Q_{\text{soft}})$ may be viewed as an element of $\mathcal{H}$

- by an extension by zero onto $Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}}$, in which we write $0 + u + 0$ or simply u,
- or by an identification with the second component of $(0, u, 0)$.

We will switch between the two notations where convenient. A similar remark applies to $\mathcal{E}$ and its subspaces $L^2(\Gamma_{\text{int}})$ and $L^2(\Gamma_{\text{ls}})$. ○

Definition 2.2.2 (Projections on $\mathcal{H}$ and $\mathcal{E}$)**.** For $\bigstar \in \{\text{stiff-int}, \text{soft}, \text{stiff-ls}\}$, we write $P_\bigstar \in \mathcal{L}(\mathcal{H})$ for the orthogonal projection of $\mathcal{H}$ onto $L^2(Q_\bigstar)$. Similarly, for $\bullet \in \{\text{int}, \text{ls}\}$ we write $\mathcal{P}_\bullet \in \mathcal{L}(\mathcal{E})$ for the orthogonal projection of $\mathcal{E}$ onto $L^2(\Gamma_\bullet)$. (Note the calligraphic font for projections on the boundary space.)

The τ-Dirichlet decoupling

The first ingredient, the τ-Dirichlet decoupling, is constructed using the Dirichlet operators $-(\nabla + i\tau)^2$, appropriately rescaled, on each connected component of Q.

Definition 2.2.3. The τ-*Dirichlet decoupling* is the operator on $\mathcal{H} = L^2(Q)$ defined by

$$A_{\varepsilon,0}^{(\tau)} = A_{\varepsilon,0}^{\text{stiff-int},(\tau)} \oplus A_0^{\text{soft},(\tau)} \oplus A_{\varepsilon,0}^{\text{stiff-ls},(\tau)} \tag{2.24}$$

where,

- $A_{\varepsilon,0}^{\text{stiff-int},(\tau)}$ is the operator $-\varepsilon^{-2}(\nabla + i\tau)^2$ on $L^2(Q_{\text{stiff-int}})$ with Dirichlet BC on Γ_{int}. That is, the operator defined through its sesquilinear form having form domain $\mathcal{D}[A_{\varepsilon,0}^{\text{stiff-int},(\tau)}] = H_0^1(Q_{\text{stiff-int}})$ and action $(u,v) \mapsto \int_{Q_{\text{stiff-int}}} \varepsilon^{-1}(\frac{1}{i}\nabla + \tau)u \cdot \overline{\varepsilon^{-1}(\frac{1}{i}\nabla + \tau)v}$.

- $A_0^{\text{soft},(\tau)}$ is the operator $-(\nabla + i\tau)^2$ on $L^2(Q_{\text{soft}})$ with Dirichlet BCs on $\Gamma_{\text{int}} \cup \Gamma_{\text{ls}}$. That is, $\mathcal{D}[A_0^{\text{soft},(\tau)}] = H_0^1(Q_{\text{soft}})$.
- $A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}$ is the operator $-\varepsilon^{-2}(\nabla + i\tau)^2$ on $L^2(Q_{\text{stiff-ls}})$ with Dirichlet BC on Γ_{ls} and periodic BCs on ∂Q. That is, $\mathcal{D}[A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}] = H_{0,\text{per}}^1(Q_{\text{stiff-ls}})$, the closure of smooth functions that are periodic on ∂Q and with compact support in $\partial Q \cup Q_{\text{stiff-ls}}$, under the H^1 norm.

Write $\widetilde{A}_0^{\text{stiff-int},(\tau)} := \varepsilon^2 A_{\varepsilon,0}^{\text{stiff-int},(\tau)}$ and $\widetilde{A}_0^{\text{stiff-ls},(\tau)} := \varepsilon^2 A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}$ for the unweighted operators.

We record some properties of $A_{\varepsilon,0}^{(\tau)}$ that will be useful to us.

Proposition 2.2.4. For all $\tau \in \overline{Q'} = [-\pi, \pi]^d$, $A_{\varepsilon,0}^{(\tau)}$ is self-adjoint, positive definite, has purely discrete spectrum, and $0 \in \rho(A_{\varepsilon,0}^{(\tau)})$. Moreover, $A_0^{\text{soft},(\tau)}$ and $A_{\varepsilon,0}^{\bigstar,(\tau)}$ are bounded below, *uniformly in* τ *and* ε, assuming ε is small enough, $\bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}$. We also have the following estimates: For some $C > 0$, independent of τ and ε, assuming ε is small enough,

$$\|(A_0^{\text{soft},(\tau)})^{-1}\|_{L^2(Q_{\text{soft}}) \to L^2(Q_{\text{soft}})} \leq C, \tag{2.25}$$

$$\|(A_{\varepsilon,0}^{\bigstar,(\tau)})^{-1}\|_{L^2(Q_\bigstar) \to L^2(Q_\bigstar)} \leq C\varepsilon^2. \tag{2.26}$$

Proof. The self-adjointness, positive-semi-definiteness, and spectral type follows immediately as it is the orthogonal sum of operators with these properties. The positive-definiteness will then follow from $0 \in \rho(A_{\varepsilon,0}^{(\tau)})$. To show this, we first note that the case $\tau = 0$ follows from the Poincaré inequality applied to each of the three operators in $A_{\varepsilon,0}^{(\tau)}$, since the first/lowest eigenvalue λ_1 is related to the optimal Poincaré constant γ by $\lambda_1 = \gamma^{-2} > 0$. ($\gamma$ can be taken to be independent of ε, if we assume ε is small.)

For general τ, the lowest eigenvalue is always greater than or equal to the $\tau = 0$ case. This is due to the diamagnetic inequality $|\nabla |f|(x)| \leq |(\nabla + i\tau) f(x)|$ a.e., $f \in H^1$, and the fact that we can always choose the first Dirichlet eigenfunction (for $\tau = 0$) to be strictly positive. (See [13, Theorem 8.38] or first part of the proof of [8, Theorem 6.34].) This shows the claim on being uniformly bounded below.

Since the norm of $(A_0^{\text{soft},(\tau)})^{-1}$ is bounded above by $\left(\text{dist}(0, \sigma(A_0^{\text{soft},(\tau)}))\right)^{-1}$, the estimate follows. A similar argument applies to the "stiff" decouplings. □

The τ-harmonic lift

The second ingredient, the **τ-harmonic lift**, generalizes the map that takes boundary data to harmonic functions.

Definition 2.2.5. The τ-*harmonic lift* is the operator $\Pi^{(\tau)} : \mathcal{E} \to \mathcal{H}$, defined by

$$\Pi^{(\tau)} = \Pi^{\text{stiff-int},(\tau)} \mathcal{P}_{\text{int}} + \Pi^{\text{soft},(\tau)} + \Pi^{\text{stiff-ls},(\tau)} \mathcal{P}_{\text{ls}}, \tag{2.27}$$

where

- $\Pi^{\text{stiff-int},(\tau)} : L^2(\Gamma_{\text{int}}) \to L^2(Q_{\text{stiff-int}})$ is the operator $\phi \mapsto u_\phi$, where u_ϕ is the unique solution to the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_\phi = 0 & \text{in } Q_{\text{stiff-int}}, \\ u_\phi = \phi & \text{on } \Gamma_{\text{int}}. \end{cases} \tag{2.28}$$

- $\Pi^{\text{soft},(\tau)} : \mathcal{E} \to L^2(Q_{\text{soft}})$ is the operator $(\phi, \varphi) \mapsto u_{\phi,\varphi}$, where $u_{\phi,\varphi}$ is the unique solution to the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_{\phi,\varphi} = 0 & \text{in } Q_{\text{soft}}, \\ u_{\phi,\varphi} = \phi & \text{on } \Gamma_{\text{int}}, \\ u_{\phi,\varphi} = \varphi & \text{on } \Gamma_{\text{ls}}. \end{cases} \tag{2.29}$$

- $\Pi^{\text{stiff-ls},(\tau)} : L^2(\Gamma_{\text{ls}}) \to L^2(Q_{\text{stiff-ls}})$ is the operator $\varphi \mapsto u_\varphi$, where u_ϕ is the unique solution to the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_\phi = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u_\varphi = \varphi & \text{on } \Gamma_{\text{ls}}, \\ u_\varphi \text{ periodic} & \text{on } \partial Q. \end{cases} \tag{2.30}$$

Note that while $\Pi^{(\tau)}$ is not a direct sum of $\Pi^{\text{stiff-int},(\tau)}$, $\Pi^{\text{soft},(\tau)}$, and $\Pi^{\text{soft},(\tau)}$, their lifts into the components $L^2(Q_{\text{stiff-int}})$, $L^2(Q_{\text{soft}})$, and $L^2(Q_{\text{stiff-ls}})$ are mutually orthogonal:

$$\Pi^{(\tau)}(\phi + \varphi) = \left(\Pi^{\text{stiff-int},(\tau)}\phi, \Pi^{\text{soft},(\tau)}(\phi + \varphi), \Pi^{\text{stiff-ls},(\tau)}\varphi \right), \quad \text{for } \phi \in L^2(\Gamma_{\text{int}}) \text{ and } \varphi \in L^2(\Gamma_{\text{ls}}).$$

Below, we give a sketch on how the lifts are constructed, and refer the reader to [17, Theorem 4.25] for the full details. For concreteness, we focus on $\Pi^{\text{stiff-int},(\tau)}$. We remark that the construction applies to $\Pi^{\text{soft},(\tau)}$, as Q_{soft} is connected with Lipschitz domain $\Gamma_{\text{int}} \cup \Gamma_{\text{ls}}$.

> The lift $\Pi^{\text{stiff-int},(\tau)}$ is initially defined as a mapping from $H^{1/2}(\Gamma_{\text{int}})$ to $H^1(Q_{\text{stiff-int}})$. This is possible because the fully homogeneous problem (zero RHS and zero on the boundary) is uniquely solved by $u \equiv 0$, as $A_{\varepsilon,0}^{\text{stiff-int},(\tau)}$ is injective (Proposition 2.2.4).
>
> We then show that $\Pi^{\text{stiff-int},(\tau)}$ admits a continuous extension to $L^2(\Gamma_{\text{int}})$, by verifying an L^2 estimate for $u = \Pi^{\text{stiff-int},(\tau)}\phi$, where $\phi \in H^{1/2}(\Gamma_{\text{int}})$. To do this, consider the adjoint problem "$L^* w = f$" corresponding to our harmonic lift problem "$Lu = 0$". The idea now is to combine both problems with Green's identity, giving
>
> $$0 = (u, f)_{L^2(Q_{\text{stiff-int}})} + (\phi, \partial_n^{(\tau)} w)_{L^2(\Gamma_{\text{int}})}.$$
>
> Pick $f = u$. Bound $\|\partial_n^{(\tau)} w\|_{L^2}$ in terms of $\|f\|_{L^2}$ (this step is tedious, and requires elliptic regularity) and we get the required inequality.

By definition, the lifts do not depend on ε. As for τ, it is natural to ask for $\Pi^{(\tau)}$ to be bounded (in the operator norm) uniformly in τ, since τ comes from a bounded set Q'. This is true, and to show this from scratch would mean revisiting the tedious estimates in [17, Chapter 4], with extra care to be taken if the domain is only Lipschitz. We will not do this here. Rather, let us simply point out that the property that enables the estimate is the ellipticity of the sesquilinear form in the sense of [22, Definition 11.2]:

Definition 2.2.6. A form $\mathfrak{t}$ with domain $\mathcal{D}(\mathfrak{t}) = V$ is called *elliptic* if there exist constants $C > 0$, $\gamma > 0$ and $c \in \mathbb{R}$ such that

$$\text{(Boundedness)} \qquad |\mathfrak{t}[u,v]| \leq C\|u\|_V\|v\|_V, \qquad \text{for } u, v \in V, \tag{2.31}$$

$$\text{(Abstract Gårding inequality)} \qquad (\text{Re } \mathfrak{t})[u] - c\|u\|^2 \geq \gamma\|u\|_V^2, \qquad \text{for } u \in V, \tag{2.32}$$

where $\text{Re } \mathfrak{t} = \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*)$, and the *adjoint form* $\mathfrak{t}^*$ is defined as $\mathcal{D}(\mathfrak{t}^*) = \mathcal{D}(\mathfrak{t})$ and $\mathfrak{t}^*[u,v] = \overline{\mathfrak{t}[v,u]}$.

Picking $C > 0$ to be independent of τ is straightforward. As for Gårding inequality, a sufficient condition is for the coefficient matrix $a(x)$ of the second order terms (*principal part*) to be *uniformly elliptic*, in the sense that if the form $\mathfrak{t}$ is written as

$$\mathfrak{t}[u,v] = \int_\Omega \sum_{k,l=1}^d a_{kl}\partial_l u \cdot \overline{\partial_k v} dx + \int_\Omega \sum_{k=1}^d (b_k \partial_k u \overline{v} + c_k u \overline{\partial_k v}) dx + \int_\Omega q u \overline{v} dx,$$

then Re $\sum_{k,l=1}^d a_{kl}(x)\xi_l\overline{\xi_k} \geq \alpha \sum_{k=1}^d |\xi_k|^2$, for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$ for some constant $\alpha > 0$. See [22, Proposition 11.10] for precise statement. We then observe that the differential expressions $-(\nabla + i\tau)^2$ have the same principal part, $-\Delta$, therefore c and γ in Gårding inequality could also be chosen independently of τ. To summarize,

Proposition 2.2.7. There is some $C > 0$, independent of τ (and ε), such that

$$\|\Pi^{(\tau)}\|_{\mathcal{E}\to\mathcal{H}} < C. \tag{2.33}$$

Remark. Alternatively, one could obtain Proposition 2.2.7 by the continuity of the mapping $\overline{Q'} \ni \tau \to \|\Pi^{(\tau)}\|_{\mathcal{E}\to\mathcal{H}}$. We refer the reader to the proof of Proposition 2.3.5 (the term "$\|u_1 - w\|$") for a proof of the continuity claim. ◦

Let us record two more properties of $\Pi^{(\tau)}$ that are necessary for constructing boundary triples. First, owing to the fact that the decoupling $A_{\varepsilon,0}^{(\tau)}$ has Dirichlet BCs, one has

$$\mathcal{D}(A_{\varepsilon,0}^{(\tau)}) \cap \text{ran}(\Pi^{(\tau)}) = \{0\}. \tag{2.34}$$

Second, the individual lifts $\Pi^{\text{stiff-int},(\tau)}$, $\Pi^{\text{soft},(\tau)}$, and $\Pi^{\text{stiff-ls},(\tau)}$ are injective, and hence

$$\ker(\Pi^{(\tau)}) = \{0\}. \tag{2.35}$$

To prove the injectivity of, say $\Pi^{\text{stiff-int},(\tau)}$, one first observes that $\Pi^{\text{stiff-int},(\tau)}$ can be characterized as the adjoint of the operator

$$L^2(Q_{\text{stiff-int}}) \ni f \mapsto \partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}}(\widetilde{A}_0^{\text{stiff-int},(\tau)})^{-1} f.$$

(This is a prequel to the identity $\Pi^* = \Gamma_1 A_0^{-1}$ of Proposition 2.2.13.) Since Γ_{int} is smooth, an argument using elliptic regularity implies that the range of this operator contains $C^\infty(\Gamma_{\text{int}})$, which is dense in $L^2(\Gamma_{\text{int}})$.

The τ-Dirichlet-to-Neumann operator

The final ingredient of the boundary triple is the **τ-Dirichlet-to-Neumann operator**.

Definition 2.2.8. The τ-Dirichlet-to-Neumann (τ-DtN) operator is the (unbounded) operator $\Lambda_\varepsilon^{(\tau)}$ on $\mathcal{E}$ defined with domain $\mathcal{D}(\Lambda_\varepsilon^{(\tau)}) = H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$ and action

$$\begin{aligned}(\phi,\varphi) \mapsto &- \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-int}}}{\partial n_{\text{stiff-int}}} + i(\tau\cdot n_{\text{stiff-int}})u_{\text{stiff-int}}\right] - \left[\frac{\partial u_{\text{soft}}}{\partial n_{\text{soft}}} + i(\tau\cdot n_{\text{soft}})u_{\text{soft}}\right] \\ &- \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-ls}}}{\partial n_{\text{stiff-ls}}} + i(\tau\cdot n_{\text{stiff-ls}})u_{\text{stiff-ls}}\right] \\ =&\varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}}u_{\text{stiff-int}} + \partial^{(\tau)}_{n_{\text{soft}},\text{int}}u_{\text{soft}} + \partial^{(\tau)}_{n_{\text{soft}},\text{ls}}u_{\text{soft}} + \varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}}u_{\text{stiff-ls}}.\end{aligned} \tag{2.36}$$

where $u_{\phi,\varphi} = u = u_{\text{stiff-int}} + u_{\text{soft}} + u_{\text{stiff-ls}}$ is the solution to the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_{\text{stiff-int}} = 0 & \text{in } Q_{\text{stiff-int}}, \\ -(\nabla + i\tau)^2 u_{\text{soft}} = 0 & \text{in } Q_{\text{soft}}, \\ -(\nabla + i\tau)^2 u_{\text{stiff-ls}} = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u_{\text{stiff-int}} = u_{\text{soft}} = \phi & \text{on } \Gamma_{\text{int}}, \\ u_{\text{stiff-ls}} = u_{\text{soft}} = \varphi & \text{on } \Gamma_{\text{ls}}, \\ u_{\text{stiff-int}} \text{ periodic} & \text{on } \partial Q. \end{cases} \tag{2.37}$$

For convenience, we introduce the following notation for the DtN on the individual components:

- Denote by $\Lambda_\varepsilon^{\text{stiff-int},(\tau)}$ the operator with domain $H^1(\Gamma_{\text{int}})$ and action $\phi \mapsto \varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}}u_\phi$, where $u_\phi = u_{\text{stiff-int}}$ solves the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_{\text{stiff-int}} = 0 & \text{in } Q_{\text{stiff-int}}, \\ u_{\text{stiff-int}} = \phi & \text{on } \Gamma_{\text{int}}. \end{cases} \tag{2.38}$$

- Denote by $\Lambda^{\text{soft},(\tau)}$ the operator with domain $H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$ and action $(\phi,\varphi) \mapsto \partial^{(\tau)}_{n_{\text{soft}},\text{int}}u_{\phi,\varphi} + \partial^{(\tau)}_{n_{\text{soft}},\text{ls}}u_{\phi,\varphi}$, where $u_{\phi,\varphi} = u_{\text{soft}}$ solves the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_{\text{stiff-ls}} = 0 & \text{in } Q_{\text{soft}}, \\ u_{\text{soft}} = \phi & \text{on } \Gamma_{\text{int}}, \\ u_{\text{soft}} = \varphi & \text{on } \Gamma_{\text{ls}}. \end{cases} \tag{2.39}$$

- Denote by $\Lambda_\varepsilon^{\text{stiff-ls},(\tau)}$ the operator with domain $H^1(\Gamma_{\text{ls}})$ and action $\varphi \mapsto \varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}}u_\varphi$, where $u_\varphi = u_{\text{stiff-ls}}$ solves the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u_{\text{stiff-ls}} = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u_{\text{stiff-ls}} = \varphi & \text{on } \Gamma_{\text{ls}}, \\ u_{\text{stiff-ls}} \text{ periodic} & \text{on } \partial Q. \end{cases} \tag{2.40}$$

- Write $\widetilde{\Lambda}^{\text{stiff-int},(\tau)} := \varepsilon^2 \Lambda_\varepsilon^{\text{stiff-int},(\tau)}$ and $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)} := \varepsilon^2 \Lambda_\varepsilon^{\text{stiff-ls},(\tau)}$ for the unweighted operators.

In this way, we may write $\Lambda_\varepsilon^{(\tau)}$ as a sum of self-adjoint operators on $L^2(\Gamma_{\text{int}})$, $\mathcal{E}$, and $L^2(\Gamma_{\text{ls}})$ respectively:

$$\Lambda_\varepsilon^{(\tau)} = \Lambda_\varepsilon^{\text{stiff-int},(\tau)} \mathcal{P}_{\text{int}} + \Lambda^{\text{soft},(\tau)} + \Lambda_\varepsilon^{\text{stiff-ls},(\tau)} \mathcal{P}_{\text{ls}}. \tag{2.41}$$

Remark. We have used the assumption that the boundaries Γ_{int} and Γ_{ls} are at least $C^{1,1}$, so that by [17, Theorem 4.21], the co-normal derivatives are well-defined. ◦

We refer the reader to [17, p. 145] and [30] for general properties of τ-DtN maps. Of note in the construction of the DtN maps, is the requirement that $u \equiv 0$ is the unique solution to the fully homogeneous problem, similarly to the lifts Π. This refers to formulae (4.35) to (4.38) of [17], and Section 3 of [30] (the assumption that 0 belongs to the resolvent set of the Dirichlet Laplacian).

To construct our boundary triple, we require the DtN map to be self-adjoint [47, Assumption 2]. This is immediate for $\Lambda_\varepsilon^{\text{stiff-int},(\tau)}$, $\Lambda^{\text{soft},(\tau)}$, and $\Lambda_\varepsilon^{\text{stiff-ls},(\tau)}$, by [30, Theorem 3.1].

Lemma 2.2.9. For ε small enough, independently of τ, $\Lambda_\varepsilon^{(\tau)}$ is self-adjoint on $\mathcal{E}$.

Sketch of proof. We will outline the idea of [35, Lemma 2.1] and state the modifications needed.

First, it suffices to discuss the $\tau = 0$ case, as general τ can be viewed as a relatively bounded perturbation of the $\tau = 0$ case. We hence omit writing τ for the remainder of the proof.

Second, we note that

$$\Lambda_\varepsilon^{\text{stiff-int}} \mathcal{P}_{\text{int}} + \Lambda_\varepsilon^{\text{stiff-ls}} \mathcal{P}_{\text{ls}} = \Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}}, \tag{2.42}$$

which is an orthogonal sum of self-adjoint operators, hence it is self-adjoint on $\mathcal{E}$ with domain $H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$.

Third, we view Λ_ε as the operator $\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}}$ perturbed by the "soft" DtN operator Λ^{soft}. In fact, we may verify the following estimate: there exist some $\alpha, \beta > 0$, independent of ε, such that for all $(\phi, \varphi) \in \mathcal{D}(\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}})$,

$$\|\Lambda^{\text{soft}}(\phi + \varphi)\|_{\mathcal{E}} \le \alpha \varepsilon^2 \|(\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}})(\phi + \varphi)\|_{\mathcal{E}} + \beta \|\phi + \varphi\|_{\mathcal{E}}.$$

Therefore, if ε is small enough, then Λ^{soft} is relatively $(\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}})$-bounded with bound strictly less than 1, hence the sum $\Lambda_\varepsilon = \Lambda^{\text{soft}} + (\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}})$ is self-adjoint by the Kato-Rellich theorem [22, Theorem 8.5].

Finally, we note that Λ^{soft} and $\Lambda_\varepsilon^{\text{stiff-int}} \oplus \Lambda_\varepsilon^{\text{stiff-ls}}$ have common domain $H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$, so this is also the domain of the sum Λ_ε. □

We will henceforth assume that $\varepsilon > 0$ is small enough to satisfy Proposition 2.2.4 and Lemma 2.2.9.

The DtN operator is an important object for our analysis. Not only is it one of the main ingredients of the boundary triple, it also features prominently in *Krein's formula*, a key result in the boundary triples theory. In particular, of interest to us are spectral properties of the unweighted "stiff" DtN components $\widetilde{\Lambda}^{\text{stiff-int},(\tau)} = \varepsilon^2 \Lambda_\varepsilon^{\text{stiff-int},(\tau)}$ and $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)} = \varepsilon^2 \Lambda_\varepsilon^{\text{stiff-ls},(\tau)}$. Let us collect the required properties in the proposition below.

Proposition 2.2.10. For all $\tau \in \overline{Q'} = [-\pi,\pi]^d$, the DtN operators $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$, $\Lambda^{\text{soft},(\tau)}$, and $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ are unbounded self-adjoint operators on $L^2(\Gamma_{\text{int}})$, $\mathcal{E}$, and $L^2(\Gamma_{\text{ls}})$ respectively. They are semibounded from above (note our sign convention for $\partial_n^{(\tau)}$), and have compact resolvents.

Focusing on $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$ and $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$, if we order their eigenvalues in descending order counting multiplicities, then

- The eigenvalues of $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$ satisfy

$$\text{For all } \tau, \quad 0 = \mu_1^{\text{stiff-int},(\tau)} > \mu_2^{\text{stiff-int},(\tau)} \geq \mu_3^{\text{stiff-int},(\tau)} \geq \cdots \to -\infty. \tag{2.43}$$

  The eigenfunction $\psi_1^{\text{stiff-int},(\tau)}$ corresponding to the first eigenvalue is $\psi_1^{\text{stiff-int},(\tau)}(x) = |\Gamma_{\text{int}}|^{-\frac{1}{2}} e^{-i\tau\cdot x}$. In particular this is a constant when $\tau = 0$.

- The eigenvalues of $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ satisfy

$$\text{If } \tau = 0, \text{ then } 0 = \mu_1^{\text{stiff-ls},(\tau)} > \mu_2^{\text{stiff-ls},(\tau)} \geq \mu_3^{\text{stiff-ls},(\tau)} \geq \cdots \to -\infty. \tag{2.44}$$

$$\text{If } \tau \neq 0, \text{ then } 0 > \mu_1^{\text{stiff-ls},(\tau)} \geq \mu_2^{\text{stiff-ls},(\tau)} \geq \mu_3^{\text{stiff-ls},(\tau)} \geq \cdots \to -\infty. \tag{2.45}$$

  Moreover, $\mu_1^{\text{stiff-ls},(\tau)}$ is simple when $|\tau|$ is small enough.

  The first eigenvalue admits an asymptotic expansion in τ with the leading order being quadratic in τ. That is, there exists a (strictly) negative-definite matrix $\mu_*^{\text{stiff-ls}}$ satisfying

$$\mu_1^{\text{stiff-ls},(\tau)} = \mu_*^{\text{stiff-ls}} \tau\cdot\tau + O\left(|\tau|^3\right). \tag{2.46}$$

  For the case $\tau = 0$, the eigenfunction $\psi_1^{\text{stiff-ls},(\tau)}$ corresponding to the first eigenvalue is constant, $\psi_1^{\text{stiff-ls},(\tau)}(x) = |\Gamma_{\text{ls}}|^{-\frac{1}{2}} \mathbf{1}_{\Gamma_{\text{ls}}}(x)$.

  For general τ, the eigenfunction $\psi_1^{\text{stiff-ls},(\tau)}$ admits an expansion: there exist some $\psi_*^{\text{stiff-ls}} = (\psi_{(1)}^{\text{stiff-ls}}, \cdots, \psi_{(d)}^{\text{stiff-ls}}) \in \left(L^2(\Gamma_{\text{ls}})\right)^d$ such that

$$\psi_1^{\text{stiff-ls},(\tau)} = |\Gamma_{\text{ls}}|^{-\frac{1}{2}} \left(1 + \tau\cdot\psi_*^{\text{stiff-ls}} + O\left(|\tau|^2\right)\right). \tag{2.47}$$

Proof. See [30], which discusses the DtN operator corresponding to $-\Delta$ (the case $\tau = 0$), and with a bounded connected domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. In there, it is proven that the DtN map is self-adjoint, semibounded, and has compact resolvent [30, Theorem 3.1]. The key fact allowing us to conclude the compactness of the resolvent is the compactness of trace operator $H^1(\Omega) \to L^2(\partial\Omega)$. An easy modification to the case $-(\nabla + i\tau)^2$ allows us to conclude the same for $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$, $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$, and $\Lambda^{\text{soft},(\tau)}$.

The simplicity of $\mu_1^{\text{stiff-int},(0)}$ is proved in the follow-up work in [31, Theorem 1.2]. We then note that $(\psi_k^{\text{stiff-int},(0)}, \mu_k^{\text{stiff-int},(0)})$ is an eigenpair for $\widetilde{\Lambda}^{\text{stiff-int},(0)}$ if and only if $(\psi_k^{\text{stiff-int},(0)} e^{-i\tau\cdot x}, \mu_k^{\text{stiff-int},(0)})$ is an eigenpair for $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$.

Since $Q_{\text{stiff-ls}}$ (with edges identified) is connected and Γ_{ls} is smooth, the arguments of [31, Proposition 4.1] can be modified to the setting of $H^1_{\text{per}}(Q_{\text{stiff-ls}})$ to give the simplicity of $\mu_1^{\text{stiff-ls},(0)}$. The claim that $\mu_1^{\text{stiff-ls},(\tau)} < 0$ for $\tau \neq 0$ is a consequence of Corollary C.2. Corollary C.2, combined with the fact that $\mu_2^{\text{stiff-ls},(\tau)}$ can be bounded away from zero uniformly in τ implies the simplicity of $\mu_1^{\text{stiff-ls},(\tau)}$ for small $|\tau|$. We postpone the self-contained argument on $\mu_2^{\text{stiff-ls},(\tau)}$ to the proof of Theorem 2.3.4.

The claims that lowest eigenvalue is zero for $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$ for all τ and for $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ for $\tau = 0$ is a direct check on the expression for the eigenfunctions.

The proof of the asymptotic expansions for $\mu_1^{\text{stiff-ls},(\tau)}$ and $\psi_1^{\text{stiff-ls},(\tau)}$ is postponed to Proposition 2.3.5 (see the term "$\|u_2 - w\|$"). $\square$

To conclude the section, let us write the eigenvalue problem for, say, $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$, in terms of the associated BVP. The eigenvalue problem reads: Find the values $z \in \mathbb{C}$ where $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}\varphi = z\varphi$, $\varphi \in \mathcal{D}(\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}) = H^1(\Gamma_{\text{ls}})$ has a non-trivial solution. In terms of the BVP, this reads: Find $z \in \mathbb{C}$ such that the problem

$$\begin{cases} -(\nabla + i\tau)^2 u = 0 & \text{in } Q_{\text{stiff-ls}}, \\ \partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}} u = zu & \text{on } \Gamma_{\text{ls}}, \\ u \text{ periodic} & \text{on } \partial Q, \end{cases} \tag{2.48}$$

has a non-trivial solution $u \in H^2_{\text{per}}(Q_{\text{stiff-ls}})$. This is also called the *Steklov problem*, and hence the eigenvalues of $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ are also referred to as *Steklov eigenvalues*.

2.2.2 Applying the triple framework. General properties.

We will now use the three ingredients provided in the previous section to define boundary triples and several auxiliary operators. This construction is done for each $\varepsilon > 0$ and $\tau \in Q$.

Definition 2.2.11. ([47].) By a *(Ryzhov) boundary triple*, we mean two separable Hilbert spaces $\mathcal{H}$ and $\mathcal{E}$ ($\mathcal{E}$ is called the *boundary space*), and a triple of operators (A_0, Λ, Π) such that:

- (Dirichlet decoupling) A_0 is an unbounded self-adjoint operator on $\mathcal{H}$, with $0 \in \rho(A_0)$,
- (DtN operator) Λ is an unbounded self-adjoint operator on $\mathcal{E}$,
- (Lift) $\Pi : \mathcal{E} \to \mathcal{H}$ is a bounded linear map such that $\ker(\Pi) = \{0\}$,
- $\mathcal{D}(A_0) \cap \text{ran}(\Pi) = \{0\}$.

When the underlying Hilbert spaces are clear form the context, we will simply refer to (A_0, Λ, Π) as the boundary triple.

We now proceed to define the auxiliary operators $\widehat{A}$, Γ_0, Γ, $S(z)$, and $M(z)$, corresponding to a boundary triple (A_0, Λ, Π).

Definition 2.2.12. Let (A_0, Λ, Π) be a boundary triple with spaces $\mathcal{H}$ and $\mathcal{E}$. Define the following operators:

- $\widehat{A} : \mathcal{H} \supset \mathcal{D}(\widehat{A}) \to \mathcal{H}$, with domain $\mathcal{D}(\widehat{A}) = \mathcal{D}(A_0)\dot{+}\mathrm{ran}(\Pi)$ and action

$$\widehat{A}(A_0^{-1}f + \Pi\phi) = f, \quad f \in \mathcal{H}, \phi \in \mathcal{E}. \tag{2.49}$$

- $\Gamma_0 : \mathcal{H} \supset \mathcal{D}(\Gamma_0) \to \mathcal{E}$, with domain $\mathcal{D}(\Gamma_0) = \mathcal{D}(A_0)\dot{+}\mathrm{ran}(\Pi)$ and action

$$\Gamma_0(A_0^{-1}f + \Pi\phi) = \phi, \quad f \in \mathcal{H}, \phi \in \mathcal{E}. \tag{2.50}$$

(We have used the assumptions $\mathcal{D}(A_0) \cap \mathrm{ran}(\Pi) = \{0\}$, $0 \in \rho(A_0)$, and $\ker(\Pi) = \{0\}$.)

- $\Gamma_1 : \mathcal{H} \supset \mathcal{D}(\Gamma_1) \to \mathcal{E}$, with domain $\mathcal{D}(\Gamma_1) = \mathcal{D}(A_0)\dot{+}\Pi\mathcal{D}(\Lambda)$ and action

$$\Gamma_1(A_0^{-1}f + \Pi\phi) = \Pi^* f + \Lambda\phi, \quad f \in \mathcal{H}, \phi \in \mathcal{D}(\Lambda) \subset \mathcal{E}. \tag{2.51}$$

- (Solution operator) For $z \in \rho(A_0)$, define the bounded linear operator $S(z) : \mathcal{E} \to \mathcal{H}$ by

$$S(z)\phi := \left(I + z(A_0 - z)^{-1}\right)\Pi\phi. \tag{2.52}$$

- (M-operator) For $z \in \rho(A_0)$, we define the operator $M(z) : \mathcal{E} \supset \mathcal{D}(M(z)) \to \mathcal{E}$, with domain $\mathcal{D}(M(z)) = \mathcal{D}(\Lambda)$ (independent of z), and action

$$M(z)\phi := \Gamma_1 S(z)\phi, \quad \phi \in \mathcal{D}(M(z)). \tag{2.53}$$

Remark. Note that $\mathcal{D}(\Gamma_1) \subset \mathcal{D}(\Gamma_0) = \mathcal{D}(A)$. This is one key difference between the Ryzhov triple and the “classical” triple described in [22, Chapter 14]. ◦

Let us now provide a motivation for the operators in Definition 2.2.12. Given $f \in \mathcal{H}$, $\phi \in \mathcal{E}$, and $z \in \rho(A_0)$, we would like to solve the following system of linear equations

$$\begin{cases} (\widehat{A} - z)u = f, \\ \Gamma_0 u = \phi. \end{cases} \tag{2.54}$$

The system bears resemblance to BVPs, with one equation on the main Hilbert space $\mathcal{H}$ and another on the boundary space $\mathcal{E}$. Here, Γ_0 has the interpretation of the (Dirichlet) trace, since by definition $\Gamma_0\Pi\phi = \phi$ and Π will be the harmonic lift in Section 2.2.1. This can also be seen from $\ker(\Gamma_0) = \mathcal{D}(A_0)$, where A_0 will be the operator with Dirichlet BCs in Section 2.2.1.

Choosing Λ to be the DtN map from Section 2.2.1, the identity $\Lambda = \Gamma_1\Pi$ then implies that Γ_1 has the interpretation of the Neumann trace.

As for $S(z)$, [47, Theorem 3.2] says that the system has a unique solution $u_z^{f,\phi} = (A_0 - z)^{-1}f + (I - zA_0^{-1})^{-1}\Pi\phi$, where the two terms solve the following systems respectively:

$$\begin{cases} (\widehat{A} - z)u = f, \\ \Gamma_0 u = 0. \end{cases} \qquad \begin{cases} (\widehat{A} - z)u = 0, \\ \Gamma_0 u = \phi. \end{cases} \tag{2.55}$$

We then set $S(z)$ to be the operator solving the second system, justifying the name “solution operator". One should compare this with the BVP for the harmonic lift in Section 2.2. Notice that $S(z)$ is not merely any generalization of Π from $z = 0$ to $z \in \rho(A_0)$ in the sense that $S(0) = \Pi$, but one with an additional property that the dependence on z is reflected explicitly in the first equation of our system, $(\widehat{A} - z)u = 0$.

Combining the interpretations for Γ_1 and $S(z)$, we hence see that $M(z) = \Gamma_1 S(z)$ could be interpreted as the DtN map with spectral parameter z. Similarly to $S(z)$, we could also check that $M(0) = \Lambda$.

Let us return to our setting. In total we have four triples of interest:

- (Full cube) $(A_{\varepsilon,0}^{(\tau)}, \Lambda_\varepsilon^{(\tau)}, \Pi^{(\tau)})$ with $\mathcal{H} = L^2(Q)$ and $\mathcal{E} = L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$.
- (Stiff interior) $(A_{\varepsilon,0}^{\text{stiff-int},(\tau)}, \Lambda_\varepsilon^{\text{stiff-int},(\tau)}, \Pi^{\text{stiff-int},(\tau)})$ with $L^2(Q_{\text{stiff-int}})$ and boundary space $L^2(\Gamma_{\text{int}})$.
- (Soft annulus) $(A_0^{\text{soft},(\tau)}, \Lambda^{\text{soft},(\tau)}, \Pi^{\text{soft},(\tau)})$ with $L^2(Q_{\text{soft}})$ and boundary space $L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$.
- (Stiff landscape) $(A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}, \Lambda_\varepsilon^{\text{stiff-ls},(\tau)}, \Pi^{\text{stiff-ls},(\tau)})$ with $L^2(Q_{\text{stiff-ls}})$ and boundary space $L^2(\Gamma_{\text{ls}})$.

We then apply Definition 2.2.12 to get the following auxiliary operators

- (Full cube) $\widehat{A}_\varepsilon^{(\tau)}$, $\Gamma_0^{(\tau)}$, $\Gamma_{\varepsilon,1}^{(\tau)}$, $S_\varepsilon^{(\tau)}(z)$, and $M_\varepsilon^{(\tau)}(z)$.
- (Stiff interior) $\widehat{A}_\varepsilon^{\text{stiff-int},(\tau)}$, $\Gamma_0^{\text{stiff-int},(\tau)}$, $\Gamma_{\varepsilon,1}^{\text{stiff-int},(\tau)}$, $S_\varepsilon^{\text{stiff-int},(\tau)}(z)$, and $M_\varepsilon^{\text{stiff-int},(\tau)}(z)$.
- (Soft annulus) $\widehat{A}^{\text{soft},(\tau)}$, $\Gamma_0^{\text{soft},(\tau)}$, $\Gamma_1^{\text{soft},(\tau)}$, $S^{\text{soft},(\tau)}(z)$, and $M^{\text{soft},(\tau)}(z)$.
- (Stiff landscape) $\widehat{A}_\varepsilon^{\text{stiff-ls},(\tau)}$, $\Gamma_0^{\text{stiff-ls},(\tau)}$, $\Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)}$, $S_\varepsilon^{\text{stiff-ls},(\tau)}(z)$, and $M_\varepsilon^{\text{stiff-ls},(\tau)}(z)$.

Remark. Our main model operator $A_\varepsilon^{(\tau)}$ defined in Section 2.1.3 is *not* $\widehat{A}_\varepsilon^{(\tau)}$, but as we see shortly, will coincide with an operator denoted by $\widehat{A}_{\varepsilon,0,I}^{(\tau)}$. $\widehat{A}_{\varepsilon,0,I}^{(\tau)}$ will be derived from $\widehat{A}_\varepsilon^{(\tau)}$. ◦

In the next section we will discuss some extra properties that arise from our specific setup. Here, we collect some properties which are applicable to a general boundary triple (A_0, Λ, Π). Some of these have already been used to motivate the definition of the triple.

Proposition 2.2.13 (Properties of auxiliary operators)**.** Let (A_0, Λ, Π) be a boundary triple with spaces $\mathcal{H}$ and $\mathcal{E}$. Construct the operators $\widehat{A}$, Γ_0, Γ_1, $S(z)$, and $M(z)$. Let $z \in \rho(A_0)$. Then,

1. $\ker(\Gamma_0) = \mathcal{D}(A_0)$, and $\operatorname{ran}(S(z)) = \ker(\widehat{A} - z)$.
2. $\Gamma_0 S(z) = I_{\mathcal{E}}$. In particular, since $S(0) = \Pi$, we have $\Gamma_0 S(0) = \Gamma_0 \Pi = I_{\mathcal{E}}$.
3. $\Lambda = \Gamma_1 \Pi$, and $\Pi^* = \Gamma_1 A_0^{-1}$.
4. $S(z) = (I - zA_0^{-1})^{-1}\Pi$. In particular, $S(\overline{z})^* = \Gamma_1 (A_0 - z)^{-1}$.
5. $M(z) = \Lambda + z\Pi^*(I - zA_0^{-1})^{-1}\Pi$. In particular, $M(0) = \Lambda$, and $M(z)^* = M(\overline{z})$.

6. $\rho(A_0) \ni z \mapsto M(z)$ is an analytic operator-valued function where the operators are closed in $\mathcal{E}$ and have common (z-independent) domain $\mathcal{D}(\Lambda)$.

7. For $z, \zeta \in \rho(A_0)$, $M(z) - M(\zeta)$ is bounded, and $M(z) - M(\zeta) = (z - \zeta)S(\overline{z})^* S(\zeta)$. In particular, $\mathrm{Im}M(z) = \mathrm{Im}(z)S(\overline{z})^* S(\overline{z})$. Here, we define the imaginary part of the unbounded operator $M(z)$ to be the imaginary part of its bounded component, i.e. $\mathrm{Im}M(z) := \mathrm{Im}(M(z) - M(0))$.

8. If $u_z \in \ker(\widehat{A} - z) \cap \mathcal{D}(\Gamma_1)$, then $M(z)\Gamma_0 u_z = \Gamma_1 u_z$.

9. $I_{\ker(\widehat{A}-z)} \subset S(z)\Gamma_0$.

All of the proofs of these claims can be found in [47, Section 3]. Here, let us comment on the statement of these properties. The identity $\Lambda = \Gamma_1 \Pi$ gives the interpretation of Γ_1 as the Neumann trace, and hence $\Pi^* = \Gamma_1 A_0^{-1}$ gives an interpretation for Π^*: it takes $f \in L^2(Q)$ in $A_0 u = f$ to the Neumann trace of u. Point 5 rewrites $M(z)$ as an unbounded self-adjoint operator plus a bounded operator (which is even analytic in $z \in \rho(A_0)$). Point 8 says that $M(z)$ is the DtN map for the problem with spectral parameter z (i.e. with first equation being $(\widehat{A} - z)u_z = 0$). In other words, we have not only generalized Λ in the sense that $M(0) = \Lambda$, but also done so in a structured way that the dependence on z is seen explicitly in the BVPs. (Recall that we have made a similar comment on $S(z)$.) This relies crucially on the property that $I_{\ker(\widehat{A}-z)} \subset S(z)\Gamma_0$, which we record as point 9 for reference.

One more property deserves mention. Because of its importance we put it as a separate result.

Theorem 2.2.14 (Green's second identity)**.** For all $u, v \in \mathcal{D}(\Gamma_1) = \mathcal{D}(A_0)\dot{+}\Pi\mathcal{D}(\Lambda)$, we have

$$(\widehat{A}u, v)_{\mathcal{H}} - (u, \widehat{A}v)_{\mathcal{H}} = (\Gamma_1 u, \Gamma_0 v)_{\mathcal{E}} - (\Gamma_0 u, \Gamma_1 v)_{\mathcal{E}}. \tag{2.56}$$

Proof. See [47, Theorem 3.6]. □

The power of the boundary triple framework starts to be felt once we start considering different "boundary conditions". By "boundary conditions" here, what we really mean is the second equation of the system (2.54), a condition on the boundary space $\mathcal{E}$. Motivated by the classical triple, the goal of the Ryzhov triple here is to construct a family of operators which is parameterized by the BCs, *together with a Krein's resolvent formula.* From the point of view of our homogenization task, employing the triple framework is ideal because

- this family includes all relevant operators that are needed for our analysis,
- these operators have a corresponding BVP interpretation (2.54), albeit slightly abstract,
- Krein's formula provides a way to compute the norm resolvent asymptotics in terms of "nicer" objects like $M(z)$ and $S(z)$.

Let us now outline the key ideas of the construction of the operator $\widehat{A}_{\beta_0,\beta_1}$. (We refer the reader to [47, Section 4-5] for the details.) Given $f \in \mathcal{H}$, $\phi \in \mathcal{E}$, and $z \in \rho(A_0)$, we would like to uniquely solve the following system:

$$\begin{cases} (\widehat{A} - z)u = f, \\ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = \phi. \end{cases} \tag{2.57}$$

As per all constructions involving unbounded operators, we have to address the issue of domains. The biggest set where this would make sense is $u \in \mathcal{D}(\Gamma_1)$. That is, $u = A_0^{-1}g + \Pi\varphi$, with $g \in \mathcal{H}$ and $\varphi \in \mathcal{D}(\Lambda)$. Furthermore, we would like to make sense of β_0, β_1 not only as numbers, but also as *operators* on $\mathcal{E}$, as doing so would allow us to greatly expand our interpretation of a BVP. To figure out reasonable conditions on β_0, β_1, we observe that for u as above,

$$(\beta_0\Gamma_0 + \beta_1\Gamma_1)(A_0^{-1}g + \Pi\varphi) = \beta_1\Pi^*g + (\beta_0 + \beta_1\Lambda)\varphi. \tag{2.58}$$

Therefore, we make the following assumptions:

Definition 2.2.15. We assume that β_0 and β_1 are linear operators on $\mathcal{E}$ such that $\mathcal{D}(\beta_0) \supset \mathcal{D}(\Lambda)$, $\beta_1 \in \mathcal{L}(\mathcal{E})$, and $\beta_0 + \beta_1\Lambda$ is closable.

The closability condition has been added because in what follows, we would like $\overline{\beta_0 + \beta_1\Lambda}$, or equivalently $\overline{\beta_0 + \beta_1 M(z)}$ by Proposition 2.2.13(5), to be boundedly invertible. (Recall that if an operator is not closed, then it cannot be boundedly invertible.) As a byproduct, we have expanded our solution space from $u \in \mathcal{D}(\Gamma_1) = \{u = A_0^{-1}g + \Pi\varphi \mid g \in \mathcal{H}, \varphi \in \mathcal{D}(\Lambda)\}$ to

$$\mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}} := \{u = A_0^{-1}g + \Pi\varphi \mid g \in \mathcal{H}, \varphi \in \mathcal{D}(\overline{\beta_0 + \beta_1\Lambda})\}.$$

The space $\mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}}$ and the closability of $\beta_0 + \beta_1\Lambda$ is what enables the subsequent steps in the construction:

1. It becomes a Hilbert space, equipped with norm $\|u\|^2_{\overline{\beta_0+\beta_1\Lambda}} := \|g\|^2_{\mathcal{H}} + \|\varphi\|^2_{\mathcal{E}} + \|(\overline{\beta_0 + \beta_1\Lambda})\varphi\|^2_{\mathcal{E}}$.
2. It allows $\beta_0\Gamma_0 + \beta_1\Gamma_1$ to be extended to a bounded operator from $(\mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}}, \|\cdot\|_{\overline{\beta_0+\beta_1\Lambda}})$ to $\mathcal{E}$.
3. Consider the operator $\overline{\beta_0 + \beta_1 M(z)}$ which has domain $\mathcal{D}(\overline{\beta_0 + \beta_1 M(z)}) = \mathcal{D}(\overline{\beta_0 + \beta_1\Lambda})$. If we assume that $\overline{\beta_0 + \beta_1 M(z)}$ is boundedly invertible on $\mathcal{E}$, then the system (2.57) has a unique solution in $\mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}}$.
4. There exist an operator $\widehat{A}_{\beta_0,\beta_1}$ constructed from β_0, β_1, and the triple (A_0, Λ, Π):

   Theorem 2.2.16. ([47, Theorem 5.5]) Assume $z \in \rho(A_0)$ is such that $\overline{\beta_0 + \beta_1 M(z)}$ defined on $\mathcal{D}(\overline{\beta_0 + \beta_1\Lambda})$ is boundedly invertible on $\mathcal{E}$. Define

   $$R_{\beta_0,\beta_1}(z) := (A_0 - z)^{-1} + S(z)Q_{\beta_0,\beta_1}(z)S(\overline{z})^*, \tag{2.59}$$

   where $Q_{\beta_0,\beta_1}(z) := -(\overline{\beta_0 + \beta_1 M(z)})^{-1}\beta_1$. Then $R_{\beta_0,\beta_1}(z)$ is the resolvent at z of a closed densely defined operator $\widehat{A}_{\beta_0,\beta_1}$ on $\mathcal{H}$. Its domain satisfies the following inclusion

   $$\mathcal{D}(\widehat{A}_{\beta_0,\beta_1}) \subset \{u \in \mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}} \mid (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = 0\} = \ker(\beta_0\Gamma_0 + \beta_1\Gamma_1). \tag{2.60}$$

Furthermore, we have $\widehat{A}_{\beta_0,\beta_1} \subset \widehat{A}$. That is, $\widehat{A}_{\beta_0,\beta_1} u = \widehat{A} u$ whenever $u \in \mathcal{D}(\widehat{A}_{\beta_0,\beta_1})$.

Remark.

- We refer to the formula for $(\widehat{A}_{\beta_0,\beta_1} - z)^{-1} \equiv R_{\beta_0,\beta_1}(z)$ as *Krein's formula.*
- We did not give a complete description of $\mathcal{D}(\widehat{A}_{\beta_0,\beta_1})$. The best we have is $\mathcal{D}(\widehat{A}_{\beta_0,\beta_1}) = \operatorname{ran}(R_{\beta_0,\beta_1}(z))$, where the RHS can be expressed in the triple (A_0, Λ, Π) by Krein's formula. We also note here that $\mathcal{D}(\widehat{A}_{\beta_0,\beta_1})$ fits into the following chain of inclusions:

$$\ker(\Gamma_0) \cap \ker(\Gamma_1) \subset \mathcal{D}(\widehat{A}_{\beta_0,\beta_1}) \subset \ker(\beta_0\Gamma_0 + \beta_1\Gamma_1) \subset \mathcal{H}_{\overline{\beta_0+\beta_1\Lambda}} \subset \mathcal{D}(A) \subset \mathcal{H}.$$

- We do not claim that $\widehat{A}_{\beta_0,\beta_1}$ is self-adjoint. See [47, Corollary 5.8] for a sufficient condition.
- The construction of the closed operator $\widehat{A}_{\beta_0,\beta_1}$ from resolvents is a general result of "pseudoresolvents" which can be found in [12, Chapter 4, Proposition 1.6].
- For our application, it is important to point out that β_0 and β_1 are allowed to depend on z. Correspondingly, the operator $\widehat{A}_{\beta_0,\beta_1}$ depends on z as well. ◦

The theorem says that for $f \in \mathcal{H}$, the equation $(\widehat{A}_{\beta_0,\beta_1} - z)u = f$ can be solved uniquely if and only if the same holds for the system

$$\begin{cases} (\widehat{A} - z)u = f, \\ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = 0. \end{cases} \tag{2.61}$$

A solution to this system implies a "weak solution" in the sense of [47, Definition 3.8], which coincides with the typical definition of a weak solution. Therefore we can relate the operators constructed in this way to a typical BVP, such as our main model in Section 2.1.3. To be precise, for the case $\beta_0 = 0$ and $\beta_1 = I$, we conclude that

Corollary 2.2.17. $A_\varepsilon^{(\tau)} = \widehat{A}_{\varepsilon,0,I}^{(\tau)}$, and $(A_\varepsilon^{(\tau)} - z)^{-1} = (A_{\varepsilon,0}^{(\tau)} - z)^{-1} - S_\varepsilon^{(\tau)}(z) M_\varepsilon^{(\tau)}(z)^{-1} S_\varepsilon^{(\tau)}(\overline{z})^*$ whenever $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. $A_\varepsilon^{(\tau)}$ is self-adjoint, and so is $\widehat{A}_{\varepsilon,0,I}^{(\tau)}$ by [47, Corollary 5.8]. Now consider the resolvents of both operators at $z = i$. The range of $(\widehat{A}_{\varepsilon,0,I}^{(\tau)} - i)^{-1}$ is $\mathcal{D}(\widehat{A}_{\varepsilon,0,I}^{(\tau)})$, and is also the set of solutions to (2.61) for some $f \in \mathcal{H}$. Similarly, the range of $(A_\varepsilon^{(\tau)} - i)^{-1}$ is $\mathcal{D}(A_\varepsilon^{(\tau)})$, and is also the set of weak solutions to (2.61). By the preceding paragraph, we have $\mathcal{D}(\widehat{A}_{\varepsilon,0,I}^{(\tau)}) \subset \mathcal{D}(A_\varepsilon^{(\tau)})$, and so $\widehat{A}_{\varepsilon,0,I}^{(\tau)} \subset A_\varepsilon^{(\tau)}$. Since self-adjoint operators are maximally symmetric, they must be equal. □

This concludes our discussion of general boundary triples. We now proceed to study boundary triple properties that are unique to our setup.

2.2.3 Properties of the triple arising from our setup

First, let us state the actions of our auxiliary operators in a more convenient form. We will skip $\widehat{A}$ and Γ_0 since they are just null extensions of A_0 and the left inverse of Π respectively. We will also do this only for the the triple on $Q_{\text{stiff-ls}}$, omitting similar statements for $Q_{\text{stiff-int}}$ and Q_{soft} for brevity.

Using the identity $\Lambda_\varepsilon^{\text{stiff-ls},(\tau)} = \Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)}\Pi^{\text{stiff-ls},(\tau)}$, we see that $\Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)}$ takes $u = \Pi^{\text{stiff-ls},(\tau)}\phi \in \Pi^{\text{stiff-ls},(\tau)}\mathcal{D}(\Lambda_\varepsilon^{\text{stiff-ls},(\tau)})$ to $\varepsilon^{-2}\partial_{n_{\text{stiff-ls}},\text{ls}}^{(\tau)} u$, where u solves the BVP

$$\begin{cases} -(\nabla + i\tau)^2 u = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u = \phi & \text{on } \Gamma_{\text{ls}}, \\ u \text{ periodic} & \text{on } \partial Q. \end{cases}$$

Remark. The action of Γ_1 is characterized by two equations, $\Lambda = \Gamma_1\Pi$ and $\Pi^* = \Gamma_1 A_0^{-1}$. The above description only discusses $\Lambda = \Gamma_1\Pi$. ◦

For $S_\varepsilon^{\text{stiff-ls},(\tau)}(z) = (I - z(A_{\varepsilon,0}^{\text{stiff-ls},(\tau)})^{-1})\Pi^{\text{stiff-ls},(\tau)}$, it takes $\phi \in L^2(\Gamma_{\text{ls}})$ to

$$u_\phi \in \mathcal{D}(A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}) \dot{+} \operatorname{ran}(\Pi^{\text{stiff-ls},(\tau)}) \subset L^2(Q_{\text{stiff-ls}}),$$

where $u = u_\phi$ solves the BVP (in the sense of system (2.54))

$$\begin{cases} (-\varepsilon^{-2}(\nabla + i\tau)^2 - z)u = 0, \\ \Gamma_0^{\text{stiff-ls},(\tau)} u = \phi. \end{cases}$$

For $M_\varepsilon^{\text{stiff-ls},(\tau)}(z) = \Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)} S_\varepsilon^{\text{stiff-ls},(\tau)}(z)$, it takes $\phi \in H^1(\Gamma_{\text{ls}}) = \mathcal{D}(\Lambda_\varepsilon^{\text{stiff-ls},(\tau)})$ to $\varepsilon^{-2}\partial_{n_{\text{stiff-ls}},\text{ls}}^{(\tau)} u$, where $u = u_\phi$ solves the BVP

$$\begin{cases} (-\varepsilon^{-2}(\nabla + i\tau)^2 - z)u = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u = \phi & \text{on } \Gamma_{\text{ls}}, \\ u \text{ periodic} & \text{on } \partial Q. \end{cases}$$

In view of keeping our notation compact, we make the following convention:

Remark (On notation)**.** In the remainder of the text, we will often abuse notation and write for instance, the operator "$P_1 A P_2$", for projections P_1 and P_2, to mean any one of the following:

- the composition of the three operators, $P_1 A P_2 : \mathcal{H} \to \mathcal{H}$,
- the compression $rP_1 A|_{P_2\mathcal{H}} : P_2\mathcal{H} \to P_1\mathcal{H}$, where $r : \mathcal{H} \to P_1\mathcal{H}$ is the restriction operator,
- the operator $P_1 A|_{P_2\mathcal{H}} : P_2\mathcal{H} \to \mathcal{H}$, which is equal to the composition of the embedding $i : P_1\mathcal{H} \to \mathcal{H}$ with the compression,
- the operator $rP_1 A|_{P_2\mathcal{H}} + rP_1 0|_{P_2^\perp\mathcal{H}} : \mathcal{H} \to P_1\mathcal{H}$, which is the null extension of the compression to the full space. ◦

Using the projections on $\mathcal{H}$ and $\mathcal{E}$ (Definition 2.2.2), let us now discuss how the auxiliary operators of different triples relate to each other. The first observation follows directly from the definition of the triples:

$$\Pi^{\text{stiff-int},(\tau)} = P_{\text{stiff-int}}\Pi^{(\tau)}\mathcal{P}_{\text{int}}, \quad \Pi^{\text{soft},(\tau)} = P_{\text{soft}}\Pi^{(\tau)}, \quad \text{and } \Pi^{\text{stiff-ls},(\tau)} = P_{\text{stiff-ls}}\Pi^{(\tau)}\mathcal{P}_{\text{ls}}. \tag{2.62}$$

Secondly, by our description of the action of $S(z)$, we have

$$S_\varepsilon^{\text{stiff-int},(\tau)}(z) = P_{\text{stiff-int}} S_\varepsilon^{(\tau)}(z)\mathcal{P}_{\text{int}}, \quad S^{\text{soft},(\tau)}(z) = P_{\text{soft}} S_\varepsilon^{(\tau)}(z), \quad \text{and}$$
$$S_\varepsilon^{\text{stiff-ls},(\tau)}(z) = P_{\text{stiff-ls}} S_\varepsilon^{(\tau)}(z)\mathcal{P}_{\text{ls}}. \tag{2.63}$$

This could be proven directly, for instance for $S^{\text{soft},(\tau)}(z)$, by noting that

$$\begin{aligned} P_{\text{soft}} S_\varepsilon^{(\tau)}(z) &= P_{\text{soft}}(I - z(A_0^{(\tau)} - z)^{-1})\Pi^{(\tau)} && \text{by definition of } S(z), \\ &= P_{\text{soft}}(I - z(A_0^{(\tau)} - z)^{-1})P_{\text{soft}}\Pi^{(\tau)} && L^2(Q_{\text{soft}}) \text{ is an invariant subspace for } A_0^{(\tau)}, \\ &= (I_{L^2(Q_{\text{soft}})} - z(A_0^{\text{soft},(\tau)} - z)^{-1})\Pi^{\text{soft},(\tau)} && \text{by construction of } A_0^{(\tau)} \text{ and by (2.62).} \end{aligned}$$

As for $M(z)$, we have

Proposition 2.2.18. For $z \in \rho(A_{\varepsilon,0}^{(\tau)})$, the following identity holds

$$M_\varepsilon^{(\tau)}(z) = M_\varepsilon^{\text{stiff-int},(\tau)}(z)\mathcal{P}_{\text{int}} + M^{\text{soft},(\tau)}(z) + M_\varepsilon^{\text{stiff-ls},(\tau)}(z)\mathcal{P}_{\text{ls}}. \tag{2.64}$$

Proof. We will drop τ and ε. Let $\phi \in H^1(\Gamma_{\text{int}})$ and $\varphi \in H^1(\Gamma_{\text{ls}})$. We see that

- $M^{\text{stiff-int}}(z) = \Gamma_1^{\text{stiff-int}} S^{\text{stiff-int}}(z)$ takes ϕ to $\varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}} u_\phi$,
- $M^{\text{soft}}(z) = \Gamma_1^{\text{soft}} S^{\text{soft}}(z)$ takes $\phi + \varphi$ to $\partial^{(\tau)}_{n_{\text{soft}},\text{int}} u_{\phi,\varphi} + \partial^{(\tau)}_{n_{\text{soft}},\text{ls}} u_{\phi,\varphi}$,
- $M^{\text{stiff-ls}}(z) = \Gamma_1^{\text{stiff-ls}} S^{\text{stiff-ls}}(z)$ takes φ to $\varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}} u_\varphi$,

where $u = u_{\text{stiff-int}} + u_{\text{soft}} + u_{\text{stiff-ls}} = u_\phi + u_{\phi,\varphi} + u_\varphi$ solves the BVP

$$\begin{cases} (-\varepsilon^{-2}(\nabla + i\tau)^2 - z)u_\phi = 0 & \text{in } Q_{\text{stiff-int}}, \\ (-(\nabla + i\tau)^2 - z)u_{\phi,\varphi} = 0 & \text{in } Q_{\text{soft}}, \\ (-\varepsilon^{-2}(\nabla + i\tau)^2 - z)u_\varphi = 0 & \text{in } Q_{\text{stiff-ls}}, \\ u_\phi = u_{\phi,\varphi} = \phi & \text{on } \Gamma_{\text{int}}, \\ u_\varphi = u_{\phi,\varphi} = \varphi & \text{on } \Gamma_{\text{ls}}, \\ u_\varphi \text{ periodic} & \text{on } \partial Q. \end{cases} \tag{2.65}$$

(Recall that the DtN map and M-operator are related by the identity $M(0) = \Lambda$.) $(\phi, \varphi) \mapsto \varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}} u_\phi + \partial^{(\tau)}_{n_{\text{soft}},\text{int}} u_{\phi,\varphi} + \partial^{(\tau)}_{n_{\text{soft}},\text{ls}} u_{\phi,\varphi} + \varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}} u_\varphi$ is precisely the action of $M(z)$. □

Remark. We will drop τ and ε in this remark. Note that

$$M^{\text{stiff-int}}(z) \neq \mathcal{P}_{\text{int}} M(z)\mathcal{P}_{\text{int}}.$$

The LHS is $\Gamma_1^{\text{stiff-int}} S^{\text{stiff-int}}(z)$, which takes $\phi \in H^1(\Gamma_{\text{int}})$ to $\varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}} u_\phi$. The RHS takes $\phi \in H^1(\Gamma_{\text{int}})$ to $\varepsilon^{-2}\partial^{(\tau)}_{n_{\text{stiff-int}},\text{int}} u_\phi + \partial^{(\tau)}_{n_{\text{soft}},\text{int}} u_{\phi,0}$. Even though we are confronted with this asymmetry, the above proposition assures us that the additive structure of $M(z)$ remains. The additivity is exploited to great effect in [35]. ◦

Finally, we discuss the dependence of the auxiliary operators on ε and τ, and the spectral parameter z. To obtain estimates that are uniform in z, we will restrict our choice of z to the following set.

Definition 2.2.19. Fix $\sigma > 0$ and a compact subset of $K \subset \mathbb{C}$. Let K_σ be the compact subset of K that is at σ distance away from the real line. That is, $K_\sigma = \{z \in \mathbb{C} : z \in K, \operatorname{dist}(z, \mathbb{R}) \geq \sigma\}$.

Lemma 2.2.20. We have $S_\varepsilon^{\text{stiff-int},(\tau)}(z) - \Pi^{\text{stiff-int},(\tau)} = O(\varepsilon^2)$, $S_\varepsilon^{\text{stiff-ls},(\tau)}(z) - \Pi^{\text{stiff-ls},(\tau)} = O(\varepsilon^2)$ and $S^{\text{soft},(\tau)}(z) - \Pi^{\text{soft},(\tau)} = O(1)$ in operator norm. These estimates are uniform in $\tau \in Q'$ and $z \in K_\sigma$.

Proof. This is a direct consequence of the formula $S(z) = (I + z(A_0 - z)^{-1})\Pi$. □

In terms of estimates that are uniform over ε, τ, and $z \in K_\sigma$, recall that we have already provided one for the decoupling A_0 in Proposition 2.2.4, and one for the lift Π in Proposition 2.2.7.

Similarly to the solution operator $S(z)$, we may ask for a simplification of $M(z)$ up to $O(\varepsilon^2)$. Recall the notation for the unweighted decoupling $\widetilde{A}_0^{\text{stiff-int},(\tau)}$, $\widetilde{A}_0^{\text{stiff-ls},(\tau)}$ and unweighted DtN maps $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$, $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$.

Lemma 2.2.21. For $\bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}$, we have

$$M_\varepsilon^{\bigstar,(\tau)}(z) = \varepsilon^{-2}\widetilde{\Lambda}^{\bigstar,(\tau)} + z(\Pi^{\bigstar,(\tau)})^*\Pi^{\bigstar,(\tau)} + O(\varepsilon^2), \tag{2.66}$$

where the estimate is uniform over $\tau \in Q'$ and $z \in K_\sigma$.

Proof. We omit $\bigstar$. Since $\widetilde{A}_0^{(\tau)} = \varepsilon^2 A_{\varepsilon,0}^{(\tau)}$, we get $\varepsilon^2(\widetilde{A}_0^{(\tau)})^{-1} = (A_{\varepsilon,0}^{(\tau)})^{-1}$. Hence

$$\begin{aligned} M_\varepsilon^{(\tau)}(z) &= \varepsilon^{-2}\widetilde{\Lambda}^{(\tau)} + z(\Pi^{(\tau)})^* \left(I - z\varepsilon^2\left(\widetilde{A}_0^{(\tau)}\right)^{-1}\right)^{-1}\Pi^{(\tau)} \\ &= \varepsilon^{-2}\widetilde{\Lambda}^{(\tau)} + z(\Pi^{(\tau)})^*\Pi^{(\tau)} + O(\varepsilon^2). \end{aligned} \tag{2.67}$$

The second equality follows from the Neumann series, which is justified by the uniform in τ bounds for the decoupling and the lift from Propositions 2.2.4 and 2.2.7, and assuming ε is small enough. □

2.3 Norm-resolvent asymptotics

After the long setup, we are now ready to begin the task of homogenization. By "homogenization", we mean that we would like to study the norm-resolvent asymptotics of our main model operator $A_\varepsilon^{(\tau)}$ of Section 2.1.3. We would like to identify an operator $\mathcal{A}_{\text{hom}}^{(\tau)}$ that we will refer to as a *homogenized operator*. To qualify as a "homogenized" operator, we require that

- $\mathcal{A}_{\text{hom}}^{(\tau)}$ be self-adjoint on a possibly smaller subspace $L^2(Q_{\text{soft}}) \oplus \widetilde{\mathcal{H}}$ of $L^2(Q)$.
- The dependence ε is only allowed in the action of $\mathcal{A}_{\text{hom}}^{(\tau)}$, on the stiff component. In particular, the subspace $\widetilde{\mathcal{H}}$, and the domain $\mathcal{D}(\mathcal{A}_{\text{hom}}^{(\tau)})$ must be independent of ε.
- $\mathcal{A}_{\text{hom}}^{(\tau)}$ and $A_\varepsilon^{(\tau)}$ are asymptotically equivalent, as $\varepsilon \downarrow 0$, in some specified topology.

2.3.1 Decomposing the boundary space

In this section, we will decompose the boundary space $\mathcal{E} = L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$ with respect to the spectral subspaces of the DtN map $\Lambda_\varepsilon^{(\tau)}$. This is a key step in [35], so let us explain the underlying rationale.

By the Krein's formula (Theorem 2.2.16), we turn our attention to the solution operator $S_\varepsilon^{(\tau)}(z)$ and M-operator $M_\varepsilon^{(\tau)}(z)$. Lemmas 2.2.20 and 2.2.21 tells us that $M(z)$ is the badly behaved term of the two. Focusing on $M(z)$, problematic region of the resolvent set is located at $z = 0$ and its vicinity. To see this, recall from Corollary 2.2.17 that $A_\varepsilon^{(\tau)}$ has $\beta_0 = 0$ and $\beta_1 = I$, giving us $\beta_0 + \beta_1 M(z) = M(z)$, which the Krein's formula then assumes to be boundedly invertible. This however, becomes increasingly difficult as ε is small, because Lemma 2.2.21 shows that the term $\varepsilon^{-2}\widetilde{\Lambda}^\bigstar$ dominates when ε is small.

This suggests us to break the problem into two in the spectral picture: a compact neighborhood of $z = 0$ and its complement. Thanks to Λ having compact resolvent (Proposition 2.2.10), the spectral subspace of the former could be chosen to be finite dimensional, which greatly simplifies the analysis.

Recall the *unweighted* DtN map in Proposition 2.2.10. We introduce the following notation:

Definition 2.3.1. Let $\bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}$. From now on, we will only consider the first eigenvalue and eigenfunction pair. Therefore, we will drop the indices and write

$$\mu^{\bigstar,(\tau)} := \mu_1^{\bigstar,(\tau)}, \quad \text{and} \quad \psi^{\bigstar,(\tau)} := \psi_1^{\bigstar,(\tau)}.$$

Note that $\psi_1^{\text{stiff-int},(\tau)}$ and $\psi_1^{\text{stiff-ls},(\tau)}$ are mutually orthogonal. Introduce the orthogonal projections

$$\mathcal{P}_\bigstar^{(\tau)} := (\cdot, \psi^{\bigstar,(\tau)})_{\mathcal{E}} \psi^{\bigstar,(\tau)}, \quad \mathcal{P}^{(\tau)} := \mathcal{P}_{\text{stiff-int}}^{(\tau)} \oplus \mathcal{P}_{\text{stiff-ls}}^{(\tau)}, \quad \text{and} \quad \mathcal{P}_\perp^{(\tau)} = I_{\mathcal{E}} - \mathcal{P}^{(\tau)}.$$

Remark.
- (On notation) Note the use of calligraphic font for projections on $\mathcal{E}$. So $\mathcal{P}_{\text{stiff-int}}^{(\tau)}$ should not be confused with $P_{\text{stiff-int}}$, which is a projection on $\mathcal{H}$.
- As Proposition 2.2.10 does not assert the simplicity of $\mu_1^{\text{stiff-ls},(\tau)}$ for large τ, we may for the moment pick any eigenfunction $\psi_1^{\text{stiff-ls},(\tau)}$. In Proposition 2.3.5, we will then show that $\psi_1^{\text{stiff-ls},(\tau)}$ may be chosen in a way that makes $\tau \mapsto \psi_1^{\text{stiff-ls},(\tau)}$ continuous, which we will assume from that point on. ∘

Recall that the unweighted DtN on the stiff-components, $\widetilde{\Lambda}^{\text{stiff-int},(\tau)} \oplus \widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ is self-adjoint with domain $H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$. With respect to the decomposition $\mathcal{E} = \mathcal{P}_{\text{stiff-int}}^{(\tau)}\mathcal{E} \oplus \mathcal{P}_{\text{stiff-ls}}^{(\tau)}\mathcal{E} \oplus \mathcal{P}_\perp^{(\tau)}\mathcal{E}$, we may now write $\widetilde{\Lambda}^{\text{stiff-int},(\tau)} \oplus \widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ as

$$\widetilde{\Lambda}^{\text{stiff-int},(\tau)} \oplus \widetilde{\Lambda}^{\text{stiff-ls},(\tau)} = \begin{pmatrix} \mu^{\text{stiff-int},(\tau)} & 0 & 0 \\ 0 & \mu^{\text{stiff-ls},(\tau)} & 0 \\ 0 & 0 & \mathcal{P}_\perp^{(\tau)} \left(\widetilde{\Lambda}^{\text{stiff-int},(\tau)} \oplus \widetilde{\Lambda}^{\text{stiff-ls},(\tau)} \right) \mathcal{P}_\perp^{(\tau)} \end{pmatrix}. \tag{2.68}$$

As for the (weighted) M-operator $M_\varepsilon^{(\tau)}(z)$, we write its block operator representation with respect to the decompositions $\mathcal{E} = \mathcal{P}^{(\tau)}_{\text{stiff-int}}\mathcal{E} \oplus \mathcal{P}^{(\tau)}_{\text{stiff-ls}}\mathcal{E} \oplus \mathcal{P}^{(\tau)}_{\perp}\mathcal{E}$ and $\mathcal{E} = \mathcal{P}^{(\tau)}\mathcal{E} \oplus \mathcal{P}^{(\tau)}_{\perp}\mathcal{E}$:

$$M_\varepsilon^{(\tau)}(z) = \begin{array}{c} \mathcal{P}^{(\tau)}_{\text{stiff-int}}\mathcal{E} \\ \mathcal{P}^{(\tau)}_{\text{stiff-ls}}\mathcal{E} \\ \mathcal{P}^{(\tau)}_{\perp}\mathcal{E} \end{array} \overset{\begin{array}{ccc} \mathcal{P}^{(\tau)}_{\text{stiff-int}}\mathcal{E} & \mathcal{P}^{(\tau)}_{\text{stiff-ls}}\mathcal{E} & \mathcal{P}^{(\tau)}_{\perp}\mathcal{E} \end{array}}{\begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} & \mathbb{B}_1 \\ \mathbb{A}_{21} & \mathbb{A}_{22} & \mathbb{B}_2 \\ \mathbb{E}_1 & \mathbb{E}_2 & \mathbb{D} \end{pmatrix}} = \begin{array}{c} \mathcal{P}^{(\tau)}\mathcal{E} \\ \mathcal{P}^{(\tau)}_{\perp}\mathcal{E} \end{array} \overset{\begin{array}{cc} \mathcal{P}^{(\tau)}\mathcal{E} & \mathcal{P}^{(\tau)}_{\perp}\mathcal{E} \end{array}}{\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}}. \tag{2.69}$$

Lemma 2.3.2. The components $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{E}$ of $M_\varepsilon^{(\tau)}(z)$ are all extendable to bounded operators on their respective spaces, where $z \in \rho(A^{(\tau)}_{\varepsilon,0})$.

Proof. We will drop ε and τ. We modify the arguments of [35, Section 3.2]. By Proposition 2.2.13(5), it suffices to check the claim for the case $z = 0$. Note that $M(0) = \Lambda$ is not a diagonal matrix, because this is the full DtN map, while the projection $\mathcal{P}^{(\tau)}$ is constructed from $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$ and $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$.

We first check this for the operator $\mathbb{B} = \mathcal{P}\Lambda\mathcal{P}_\perp$. We claim that $\mathcal{P}_\perp\mathcal{D}(\Lambda) \subset \mathcal{D}(\Lambda)$. In other words, $\mathcal{D}(\mathbb{B})$ contains the set $\mathcal{D}(\Lambda)$ which is dense in $\mathcal{E}$. This is because if $(\phi,\varphi) \in \mathcal{D}(\Lambda)$, then $\mathcal{P}(\phi,\varphi) = (\mathcal{P}_{\text{stiff-int}}\phi, \mathcal{P}_{\text{stiff-ls}}\varphi) \in \operatorname{span}\{\psi^{\text{stiff-int}}\} \oplus \operatorname{span}\{\psi^{\text{stiff-ls}}\}$, and the eigenvectors are in H^1 on their respective spaces. Then, notice that $\mathcal{P}_\perp(\phi,\varphi)$ can be written as a linear combination of elements in $\mathcal{D}(\Lambda)$, as $\mathcal{P}_\perp(\phi,\varphi) = (\phi - \mathcal{P}_{\text{stiff-int}}\phi, \varphi - \mathcal{P}_{\text{stiff-int}}\varphi) = (\phi,\varphi) - \mathcal{P}(\phi,\varphi)$.

Now suppose that $(\phi,\varphi) \in \mathcal{D}(\Lambda) = H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}}) \subset \mathcal{D}(\mathbb{B})$, then its image under $\mathbb{B}$ can be written as

$$\begin{aligned} \mathcal{P}\Lambda\mathcal{P}_\perp(\phi,\varphi) &= \Big(\Lambda\mathcal{P}_\perp(\phi,\varphi), (\psi^{\text{stiff-int}}, \psi^{\text{stiff-ls}})\Big)_{\mathcal{E}} (\psi^{\text{stiff-int}}, \psi^{\text{stiff-ls}}) \\ &= \Big(\mathcal{P}_\perp(\phi,\varphi), \Lambda(\psi^{\text{stiff-int}}, \psi^{\text{stiff-ls}})\Big)_{\mathcal{E}} (\psi^{\text{stiff-int}}, \psi^{\text{stiff-ls}}), \end{aligned} \tag{2.70}$$

as Λ is self-adjoint. Then using the Cauchy-Schwarz inequality, $\|\mathcal{P}_\perp\| \leq 1$, and that $\psi^{\text{stiff-int}}$ and $\psi^{\text{stiff-ls}}$ are normalized eigenfunctions, we deduce that

$$\|\mathcal{P}\Lambda\mathcal{P}_\perp(\phi,\varphi)\|_{\mathcal{E}} \leq \sqrt{2}\|\Lambda(\psi^{\text{stiff-int}}, \psi^{\text{stiff-ls}})\|_{\mathcal{E}}\|(\phi,\varphi)\|_{\mathcal{E}}. \tag{2.71}$$

Since $H^1(\Gamma_{\text{int}}) \oplus H^1(\Gamma_{\text{ls}})$ is dense in $\mathcal{E}$, $\mathbb{B}$ admits a continuous extension to an operator $\mathcal{P}_\perp\mathcal{E} \to \mathcal{P}\mathcal{E}$. The same reasoning holds for $\mathbb{A} = \mathcal{P}\Lambda\mathcal{P}$ and $\mathbb{E} = \mathcal{P}_\perp\Lambda\mathcal{P}$. □

Remark.
- We have used: $\mathcal{P}_\perp = I_{\mathcal{E}} - \mathcal{P} = (I_{L^2(\Gamma_{\text{int}})} \oplus I_{L^2(\Gamma_{\text{ls}})}) - (\mathcal{P}_{\text{stiff-int}} \oplus \mathcal{P}_{\text{stiff-ls}}) = (I_{L^2(\Gamma_{\text{int}})} - \mathcal{P}_{\text{stiff-int}}) \oplus (I_{L^2(\Gamma_{\text{ls}})} - \mathcal{P}_{\text{stiff-ls}})$, which follows as our setup has disjoint stiff components.
- The argument does not work for $\mathbb{D} = \mathcal{P}_\perp\Lambda\mathcal{P}_\perp$, because we do not have the *finite* eigenfunction expansion of (2.70) to work with. ◦

We will henceforth write $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{E}$ to mean its continuous extension to the full subspaces $\mathcal{P}^{(\tau)}\mathcal{E}$ and $\mathcal{P}^{(\tau)}_\perp\mathcal{E}$.

Note that $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{E}$ depend on ε, τ, and z, since $M_\varepsilon^{(\tau)}(z)$ does. In light of this, Lemma 2.3.2 is insufficient for our purposes: we would like to argue further why for $\mathbb{B}$ and $\mathbb{E}$, the RHS of (2.71) can be bounded by $C\|(\phi,\varphi)\|$, where C is a uniform constant.

Proposition 2.3.3. The bound on $\|\mathbb{B}\|_{op}$ and $\|\mathbb{E}\|_{op}$ can be chosen independently of $z \in K_\sigma$, $\tau \in Q'$, and $\varepsilon > 0$. The bound on $\|\mathbb{A}\|_{op}$ can be chosen independently of $z \in K_\sigma$ and $\tau \in Q'$.

Proof. Proposition 2.2.18 permits us to address the "soft" and "stiff" parts individually. Proposition 2.2.13(5) allows us to split $M(z)$ into an unbounded part Λ (which depends on ε and τ), and a bounded part $z\Pi^*(I - zA_0^{-1})^{-1}\Pi$ (which depends on ε, τ, and z).

We claim that the bounded part may be bounded uniformly in ε, τ, and z. Indeed, by Lemma 2.2.21, it suffices to work with the "soft" case. The claim then follows from Proposition 2.2.7, and the assumption that K_σ is compact. As the unbounded part does not depend on z, this proves assertion on the **independence on** z.

Next we discuss the **independence on** τ. Without loss of generality, let us consider $\mathbb{B} = \mathcal{P}^{(\tau)}\Lambda_\varepsilon^{(\tau)}\mathcal{P}_\perp^{(\tau)}$. In (2.71), we have shown that $\mathbb{B}$ has operator norm not exceeding

$$\begin{aligned}
&\sqrt{2}\|\Lambda_\varepsilon^{(\tau)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\|_{\mathcal{E}} \\
&\le \sqrt{2}\left(\|\Lambda_\varepsilon^{\text{stiff-int},(\tau)}\psi^{\text{stiff-int},(\tau)}\|_{\mathcal{E}} + \|\Lambda_\varepsilon^{\text{stiff-ls},(\tau)}\psi^{\text{stiff-ls},(\tau)}\|_{\mathcal{E}} + \|\Lambda^{\text{soft},(\tau)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\|_{\mathcal{E}}\right) \\
&= \sqrt{2}\Bigg(|\varepsilon^{-2}\mu^{\text{stiff-int},(\tau)}|\underbrace{\|\psi^{\text{stiff-int},(\tau)}\|_{\mathcal{E}}}_{=1} + |\varepsilon^{-2}\mu^{\text{stiff-ls},(\tau)}|\underbrace{\|\psi^{\text{stiff-ls},(\tau)}\|_{\mathcal{E}}}_{=1} \\
&\qquad + \|\Lambda^{\text{soft},(\tau)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\|_{\mathcal{E}}\Bigg).
\end{aligned} \tag{2.72}$$

(Actually, the first two terms are absent for $\mathbb{B}$ and $\mathbb{E}$, as we see below, but we would like to include $\mathbb{A}$ for this discussion.) We apply a "**(perturbation + compactness) argument**" as follows:

By perturbation theory, we have the continuity of the mapping $\overline{Q'} \ni \tau \mapsto \mu^{\bigstar,(\tau)}$. This implies that $|\varepsilon^{-2}\mu^{\bigstar,(\tau)}|$ is bounded uniformly in τ. Next, we turn to the third term in the RHS of (2.72). By [39, Lemma 2], we may write,

$$\Lambda^{\text{soft},(\tau)} = \Lambda^{\text{soft},(0)} + B_{\text{soft}}, \quad \widetilde{\Lambda}^{\bigstar,(\tau)} = \widetilde{\Lambda}^{\bigstar,(0)} + B_\bigstar, \quad \bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}, \tag{2.73}$$

where $B_\bigstar$, B_{soft} are uniformly (in τ) bounded operators. Also, recall that $\|(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\| = \sqrt{2}$ as the eigenvectors are normalized. We then have

$$\begin{aligned}
\|\Lambda^{\text{soft},(\tau)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\| &\le \|\Lambda^{\text{soft},(0)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\| + \|B_{\text{soft}}\|_{op}\sqrt{2} \\
&\le \|\Lambda^{\text{soft},(0)}(\psi^{\text{stiff-int},(\tau)},\psi^{\text{stiff-ls},(\tau)})\| + C \\
&\le \alpha_1\|\widetilde{\Lambda}^{\text{stiff-int},(0)}\psi^{\text{stiff-int},(\tau)}\| + \alpha_2\|\widetilde{\Lambda}^{\text{stiff-ls},(0)}\psi^{\text{stiff-ls},(\tau)}\| + \beta \\
&\le \alpha_1(\|\widetilde{\Lambda}^{\text{stiff-int},(\tau)}\psi^{\text{stiff-int},(\tau)}\| + \|B_{\text{stiff-int}}\psi^{\text{stiff-int},(\tau)}\|) + \\
&\qquad \alpha_2(\|\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}\psi^{\text{stiff-ls},(\tau)}\| + \|B_{\text{stiff-ls}}\psi^{\text{stiff-ls},(\tau)}\|) + \beta \\
&\le C_1|\mu^{\text{stiff-int},(\tau)}| + C_2|\mu^{\text{stiff-ls},(\tau)}| + C_3,
\end{aligned} \tag{2.74}$$

where the constants are all independent of τ. The second and fourth inequality follows from

(2.73). The third inequality follows by noting that the domains $\mathcal{D}(\Lambda^{\text{soft},(\tau)})$, $\mathcal{D}(\widetilde{\Lambda}^{\bigstar,(\tau)})$ are independent of τ, and then using the observation that $\Lambda^{\text{soft},(0)}$ is relatively $\Lambda^{\bigstar,(0)}$-bounded by [22, Lemma 8.4]. As noted above, $|\mu^{\bigstar,(\tau)}|$ is bounded uniformly in τ.

Finally, for **independence on** ε, we notice further that

$$\mathbb{B} = \mathcal{P}^{(\tau)}(\Lambda^{\text{soft},(\tau)} + \Lambda_\varepsilon^{\text{stiff-int},(\tau)}\mathcal{P}_{\text{ls}} + \Lambda_\varepsilon^{\text{stiff-int},(\tau)}\mathcal{P}_{\text{int}})\mathcal{P}_\perp^{(\tau)} = \mathcal{P}^{(\tau)}\Lambda^{\text{soft},(\tau)}\mathcal{P}_\perp^{(\tau)}, \tag{2.75}$$

since $\mathcal{P}_\perp^{(\tau)}\mathcal{E}$ is an invariant subspace for the stiff DtN maps. (We have a diagonal block matrix.) A similar statement holds for $\mathbb{E}$, but not for $\mathbb{A}$. □

Remark.
- We remark that while [39] does not study the case of annular domains, the arguments of [39, Lemma 2] still applies to give us (2.73), since Q_{soft} is connected with smooth boundary $\Gamma_{\text{int}} \cup \Gamma_{\text{ls}}$.
- Variants of the (perturbation + compactness) argument will be used again in Theorem 2.3.4 (for the term "$\mathbb{S}$"). ◦

2.3.2 Inverting the M-operator

Corollary 2.2.17 suggests that our study of norm-resolvent asymptotics of the main model operator $A_\varepsilon^{(\tau)}$ requires us to estimate $(M_\varepsilon^{(\tau)}(z))^{-1}$ in the operator norm. The goal of this section is to prove

Theorem 2.3.4. We have the following estimate in the operator norm

$$\left(M_\varepsilon^{(\tau)}(z)\right)^{-1} = \begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^2), \tag{2.76}$$

relative to the decomposition $\mathcal{E} = \mathcal{P}^{(\tau)}\mathcal{E} \oplus \mathcal{P}_\perp^{(\tau)}\mathcal{E}$. $\|\mathbb{A}^{-1}\|_{op}$ is bounded uniformly in $\varepsilon > 0$, $\tau \in Q'$ and $z \in K_\sigma$. This estimate is uniform in $\tau \in Q'$ and $z \in K_\sigma$.

Remark. We lay out the details for the estimates on $\mathbb{A}$, outlined in [35, Section 3.2, footnote 11]. These are Claim 1 and (Claim 2 + Proposition 2.3.5) in the proof below. ◦

Proof of Theorem 2.3.4. Since $M_\varepsilon^{(\tau)}(z)$ is closed by Proposition 2.2.13(6), [26, Theorem 2.3.3(i)] implies that its inverse can be written in block operator form as:

$$\left(M_\varepsilon^{(\tau)}(z)\right)^{-1} = \overline{\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}}^{-1} = \begin{pmatrix} \mathbb{A}^{-1} + \overline{\mathbb{A}^{-1}\mathbb{B}}(\overline{\mathbb{S}})^{-1}\mathbb{E}\mathbb{A}^{-1} & -\overline{\mathbb{A}^{-1}\mathbb{B}}(\overline{\mathbb{S}})^{-1} \\ -(\overline{\mathbb{S}})^{-1}\mathbb{E}\mathbb{A}^{-1} & (\overline{\mathbb{S}})^{-1} \end{pmatrix}, \tag{2.77}$$

where the Schur-Frobenius complement $\mathbb{S}$ is given by $\mathbb{S} := \mathbb{D} - \mathbb{E}\mathbb{A}^{-1}\mathbb{B}$. In writing down the above formula, it suffices to check that

(a) $\mathbb{A}$ is boundedly invertible,

(b) $\mathbb{B}$ is bounded,

(c) $\mathbb{D}$ is closed, and

(d) $\mathbb{S}$ is closable, with $\overline{\mathbb{S}}$ being boundedly invertible.

(b) is immediate, since $\mathbb{B}$ has finite range. In the remainder of the proof, we verify (a), (c), and (d), and provide bounds on $\mathbb{A}^{-1}$ and $(\overline{\mathbb{S}})^{-1}$, with dependence on ε, τ, and z shown explicitly.

The term $\mathbb{A} = \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z) \mathcal{P}^{(\tau)}$

As $\mathbb{A}$ is an operator on the finite-dimensional space $\mathcal{P}^{(\tau)}\mathcal{E}$, $\mathbb{A}$ will be boundedly invertible (uniformly in ε, τ, and z) if we can show that $\mathbb{A}$ is bounded below (uniformly in ε, τ, and z), as this implies injectivity. To do so, recall from Proposition 2.2.13(7) that $\operatorname{Im}(M_\varepsilon^{(\tau)}(z)) = \operatorname{Im}(M_\varepsilon^{(\tau)}(z) - M_\varepsilon^{(\tau)}(0))$. Now define the real part of $M_\varepsilon^{(\tau)}(z)$ by

$$\operatorname{Re}(M_\varepsilon^{(\tau)}(z)) := M_\varepsilon^{(\tau)}(0) + \operatorname{Re}\left(M_\varepsilon^{(\tau)}(z) - M_\varepsilon^{(\tau)}(0)\right) = \Lambda_\varepsilon^{(\tau)} + \operatorname{Re}\left(z(\Pi^{(\tau)})^*(I - z(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1}\Pi^{(\tau)}\right). \tag{2.78}$$

Then, by the symmetry of $\Lambda_\varepsilon^{(\tau)}$,

$$(M_\varepsilon^{(\tau)}(z)v, v)_\mathcal{E} = (\operatorname{Re}(M_\varepsilon^{(\tau)}(z))v, v)_\mathcal{E} + i(\operatorname{Im}(M_\varepsilon^{(\tau)}(z))v, v)_\mathcal{E}, \quad v \in \mathcal{D}(M_\varepsilon^{(\tau)}(z)) = \mathcal{D}(\Lambda_\varepsilon^{(\tau)}). \tag{2.79}$$

Therefore, it suffices to show that $\operatorname{Im}(M_\varepsilon^{(\tau)}(z))$ is bounded below on $\mathcal{P}^{(\tau)}\mathcal{E}$. To show this, we recall from Proposition 2.2.13(7) that $\operatorname{Im}(M_\varepsilon^{(\tau)}(z)) = (\operatorname{Im} z) S_\varepsilon^{(\tau)}(\overline{z})^* S_\varepsilon^{(\tau)}(\overline{z})$. Since $z \in K_\sigma$, we may ignore the term $\operatorname{Im} z$. Then, for $v \in \mathcal{E}$, Proposition 2.2.13(4) implies that

$$(\mathcal{P}^{(\tau)} S_\varepsilon^{(\tau)}(\overline{z})^* S_\varepsilon^{(\tau)}(\overline{z}) \mathcal{P}^{(\tau)} v, v)_\mathcal{E} = \|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} \Pi^{(\tau)} \mathcal{P}^{(\tau)} v\|_\mathcal{H}^2. \tag{2.80}$$

Claim 1: $(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1}$ is bounded below in the operator norm, uniformly in ε, τ, and z. By Proposition 2.2.4, $A_{\varepsilon,0}^{(\tau)}$ has compact resolvent, and therefore admits an eigenfunction expansion

$$A_{\varepsilon,0}^{(\tau)} = \sum_{j=1}^{\infty} \left(\cdot, w_{\varepsilon,j}^{(\tau)}\right)_\mathcal{H} \lambda_{\varepsilon,j}^{(\tau)} w_{\varepsilon,j}^{(\tau)}, \tag{2.81}$$

where the eigenvalues $\lambda_{\varepsilon,j}^{(\tau)}$ are real, due to the self-adjointness of $A_{\varepsilon,0}^{(\tau)}$. The idea now is to split the operator in two, in the spectral picture: Since K_σ is compact, there is some $R = R(K_\sigma) > 0$ such that $B(0, R)$ contains K_σ. We then choose (ε and τ dependent) spectral projections $P_1 = P_{\varepsilon,1}^{(\tau)}$ and $P_2 = P_{\varepsilon,2}^{(\tau)}$ on $\mathcal{H} = L^2(Q)$ such that $P_1 = I_\mathcal{H} - P_2$ and

$$P_{\varepsilon,2}^{(\tau)}\mathcal{H} = \operatorname{span}\left\{ w_{\varepsilon,j}^{(\tau)} : j \text{ satisfies } |\lambda_{\varepsilon,j}^{(\tau)}| > 3R (> R \geq |\overline{z}|) \right\}. \tag{2.82}$$

Next we observe that for $f \in \mathcal{H}$,

$$\begin{aligned}
&\|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} f\|_\mathcal{H}^2 = \|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1}(P_1 f + P_2 f)\|_\mathcal{H}^2 \\
&= \|P_1(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} f + P_2(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} f)\|_\mathcal{H}^2 \qquad P_1,\ P_2 \text{ are spectral projections.} \\
&= \|P_1(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} f\|_\mathcal{H}^2 + \|P_2(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} f)\|_\mathcal{H}^2 \quad \text{Pythagoras theorem.}
\end{aligned}$$

$$= \|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} P_1 f\|_{\mathcal{H}}^2 + \|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} P_2 f)\|_{\mathcal{H}}^2. \tag{2.83}$$

If we denote by $J = J_\varepsilon^{(\tau)} \in \mathbb{N}$ the smallest integer that satisfies the condition of $P_2 = P_{\varepsilon,2}^{(\tau)}$, then we may write $P_1 f$ (and similarly for $P_2 f$) as

$$P_{\varepsilon,1}^{(\tau)} f = \sum_{j=1}^{J_\varepsilon^{(\tau)}-1} \left(f, w_{\varepsilon,j}^{(\tau)}\right)_{\mathcal{H}} w_{\varepsilon,j}^{(\tau)} = \sum_{j=1}^{J_\varepsilon^{(\tau)}-1} c_{\varepsilon,j}^{(\tau)} w_{\varepsilon,j}^{(\tau)}. \tag{2.84}$$

With this notation, the first term on the RHS of (2.83) can be estimated below as:

$$\begin{aligned}
\|(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} P_{\varepsilon,1}^{(\tau)} f\|_{\mathcal{H}}^2 &= \|A_{\varepsilon,0}^{(\tau)}(A_{\varepsilon,0}^{(\tau)} - \overline{z})^{-1} P_{\varepsilon,1}^{(\tau)} f\|_{\mathcal{H}}^2 \\
&\geq c_1 \|(A_{\varepsilon,0}^{(\tau)} - \overline{z})^{-1} P_{\varepsilon,1}^{(\tau)} f\|_{\mathcal{H}}^2 && \text{By Proposition 2.2.4.} \\
&= c_1 \left\| \sum_{j=1}^{J_\varepsilon^{(\tau)}-1} \frac{c_{\varepsilon,j}^{(\tau)}}{\lambda_{\varepsilon,j}^{(\tau)} - \overline{z}} w_{\varepsilon,j}^{(\tau)} \right\|_{\mathcal{H}}^2 && \text{By functional calculus.} \\
&= c_1 \sum_{j=1}^{J_\varepsilon^{(\tau)}-1} \frac{|c_{\varepsilon,j}^{(\tau)}|^2}{|\lambda_{\varepsilon,j}^{(\tau)} - \overline{z}|^2} && \text{By Parseval's identity.} \\
&\geq c_1 \left(\min_{1 \leq j \leq J_\varepsilon^{(\tau)}-1} \left\{ \frac{1}{|\lambda_{\varepsilon,j}^{(\tau)} - \overline{z}|^2} \right\} \right) \sum_{j=1}^{J_\varepsilon^{(\tau)}-1} |c_{\varepsilon,j}^{(\tau)}|^2 \\
&= c_1 c_2 \|P_{\varepsilon,1}^{(\tau)} f\|_{\mathcal{H}}^2 && \text{By Parseval's identity,}
\end{aligned} \tag{2.85}$$

where $c_1 > 0$ and $c_2 := 1/(2R)^2$ are constants independent of ε, τ, and z. Observe that although the projection P_1 depends on ε and τ, the constant c_2 does not – it only depends on K_σ through $B(0, R)$.

As for the second term $(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} P_{\varepsilon,2}^{(\tau)} f$, we observe that since $P_{\varepsilon,2}^{(\tau)}$ is a spectral projection for $A_{\varepsilon,0}^{(\tau)}$, the second term equals $(I - \overline{z}(A_{\varepsilon,0}^{(\tau)} P_{\varepsilon,2}^{(\tau)})^{-1})^{-1} f$. Next we recall that P_2 is chosen such that

$$\begin{aligned}
\left\| \overline{z}(A_{\varepsilon,0}^{(\tau)} P_{\varepsilon,2}^{(\tau)})^{-1} \right\|_{\mathcal{H}\to\mathcal{H}} = |\overline{z}| \left\| (A_{\varepsilon,0}^{(\tau)} P_{\varepsilon,2}^{(\tau)})^{-1} \right\|_{\mathcal{H}\to\mathcal{H}} &= |\overline{z}| \frac{1}{\operatorname{dist}\left(0, \lambda_{\varepsilon,J_\varepsilon^{(\tau)}}^{(\tau)}\right)} \\
&= |\overline{z}| \frac{1}{\left|\lambda_{\varepsilon,J_\varepsilon^{(\tau)}}^{(\tau)}\right|} < |\overline{z}| \frac{1}{3R} < \frac{1}{3}.
\end{aligned} \tag{2.86}$$

(Once again, notice that P_2 depends on ε and τ, while this estimate does not.) As a result, we may apply the Neumann series expansion:

$$(I - \overline{z}(A_{\varepsilon,0}^{(\tau)} P_{\varepsilon,2}^{(\tau)})^{-1})^{-1} = I + \overline{z}(A_{\varepsilon,0}^{(\tau)} P_{\varepsilon,2}^{(\tau)})^{-1} + \cdots \tag{2.87}$$

The terms after I have norm not exceeding $\sum_{n=1}^\infty (1/3)^n = 1/2$. Therefore the reverse triangle inequality implies that $(I - \overline{z}(A_{\varepsilon,0}^{(\tau)})^{-1})^{-1} P_{\varepsilon,2}^{(\tau)}$ is bounded below (independently of ε, τ, and z). This proves Claim 1.

Applying Claim 1 to (2.80), we now have some $\tilde{c} > 0$ independent of ε, τ, and z, such that

$$\|(I - \overline{z}(A^{(\tau)}_{\varepsilon,0})^{-1})^{-1}\Pi^{(\tau)}\mathcal{P}^{(\tau)}v\|^2_{\mathcal{H}} \geq \tilde{c}\|\Pi^{(\tau)}\mathcal{P}^{(\tau)}v\|^2_{\mathcal{H}}.$$

Claim 2: There is some $c > 0$, independent of ε, τ, and z, such that

$$\|\Pi^{(\tau)}\mathcal{P}^{(\tau)}v\|^2_{\mathcal{H}} \geq c\|\mathcal{P}^{(\tau)}v\|^2_{\mathcal{E}}, \quad \textbf{where } v \in \mathcal{E}. \tag{2.88}$$

Write $v = \phi + \varphi$, where $\phi \in L^2(\Gamma_{\text{int}})$ and $\varphi \in L^2(\Gamma_{\text{ls}})$. Since $\mathcal{P}^{(\tau)}$ is the projection onto $\text{span}\{\psi^{\text{stiff-int},(\tau)}\} \oplus \text{span}\{\psi^{\text{stiff-ls},(\tau)}\}$, we can write $\mathcal{P}^{(\tau)}v = c_1\psi^{\text{stiff-int},(\tau)} + c_2\psi^{\text{stiff-ls},(\tau)}$ where $c_1, c_2 \in \mathbb{C}$. Now suppose that $\mathcal{P}^{(\tau)}v \neq 0$, then we must have either $c_1 \neq 0$ or $c_2 \neq 0$. Then

$$\begin{cases} \left\|\Pi^{(\tau)}(\mathcal{P}^{(\tau)}v)\right\|^2_{\mathcal{H}} \geq \left\|\Pi^{\text{stiff-int},(\tau)}c_1\psi^{\text{stiff-int},(\tau)}\right\|^2_{L^2(Q_{\text{stiff-int}})} & \text{if } c_1 \neq 0, \\ \left\|\Pi^{(\tau)}(\mathcal{P}^{(\tau)}v)\right\|^2_{\mathcal{H}} \geq \left\|\Pi^{\text{stiff-ls},(\tau)}c_2\psi^{\text{stiff-ls},(\tau)}\right\|^2_{L^2(Q_{\text{stiff-ls}})} & \text{if } c_2 \neq 0. \end{cases} \tag{2.89}$$

The inequality follows by the Pythagoras theorem, as we recall that $\Pi^{(\tau)}(\phi+\varphi) = \Pi^{\text{stiff-int},(\tau)}\phi + \Pi^{\text{soft},(\tau)}(\phi+\varphi) + \Pi^{\text{stiff-ls},(\tau)}\varphi$, and the lifts into the individual components $\Pi^{\text{stiff-int},(\tau)}\phi$, $\Pi^{\text{soft},(\tau)}(\phi+\varphi)$, and $\Pi^{\text{stiff-ls},(\tau)}\varphi$ are orthogonal. Therefore, by the linearity of Π and the homogeneity of norms, it suffices to find some $c > 0$ independent of ε, τ, and z such that

$$\|\Pi^{\bigstar,(\tau)}\psi^{\bigstar,(\tau)}\|_{\mathcal{H}} \geq c\|\psi^{\bigstar,(\tau)}\|_{\mathcal{E}} \overset{\psi \text{ is normalized}}{=} c, \quad \bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}.$$

The proof of this inequality follows from two facts:

(i) For each $\tau \in \overline{Q'}$, $\|\Pi^{\bigstar,(\tau)}\psi^{\bigstar,(\tau)}\|$ is strictly positive, as or else this means the $\tau-$harmonic lift of a non-zero function ψ is zero, which contradicts the injectivity of $\Pi^{(\tau)}$ (see comment after Proposition 2.2.7).

(ii) The mapping $\overline{Q'} \ni \tau \mapsto \|\Pi^{\bigstar,(\tau)}\psi^{\bigstar,(\tau)}\| \in \mathbb{R}_{\geq 0}$ is continuous. (It is important that we consider the closure of Q' here.)

Using (i) and (ii): Suppose we have a sequence $\tau_n \in \overline{Q'}$ with $f(\tau_n) \downarrow 0$, then by closedness of $f(\overline{Q'})$ (due to (ii)), we must have some $\tau \in \overline{Q'}$ with $0 = f(\tau)$, and this contradicts (i).

It remains to prove (ii) and hence complete Claim 2. The proof of the fact will be postponed to Proposition 2.3.5, which concludes our discussion on $\mathbb{A}$.

The term $\mathbb{S} = \mathbb{D} - \mathbb{E}\mathbb{A}^{-1}\mathbb{B}$

We will proceed in four steps. **Step 1.** First, we introduce the notation

$$\mathbb{D}^{(\tau)}_{\varepsilon}(z) = \mathbb{D}^{(\tau)}_{\text{soft}} + \mathbb{D}^{(\tau)}_{\varepsilon,\text{stiff}} + \mathbb{D}^{(\tau)}_{\varepsilon,b}(z) \tag{2.90}$$

where $\mathbb{D}_{\text{soft}} = \mathcal{P}^{(\tau)}_{\perp}\Lambda^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}_{\perp}$, $\mathbb{D}_{\text{stiff}} = \mathcal{P}^{(\tau)}_{\perp}(\Lambda^{\text{stiff-int},(\tau)}_{\varepsilon} \oplus \Lambda^{\text{stiff-ls},(\tau)}_{\varepsilon})\mathcal{P}^{(\tau)}_{\perp}$, and $\mathbb{D}_b$ as what remains of $\mathbb{D}^{(\tau)}_{\varepsilon}(z) = \mathcal{P}^{(\tau)}_{\perp}M^{(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}_{\perp}$. In this way, $\mathbb{D}_b$ is a bounded operator on $\mathcal{P}^{(\tau)}_{\perp}\mathcal{E}$, with operator norm bounded uniformly in ε, τ, and z, by Proposition 2.2.13(5).

Furthermore, we claim that $\mathbb{D}_{\text{soft}}+\mathbb{D}_{\text{stiff}}$ is self-adjoint on $\mathcal{P}_\perp^{(\tau)}\mathcal{E}$ with domain $\mathcal{D}(\mathbb{D}_{\text{soft}}+\mathbb{D}_{\text{stiff}}) = \mathcal{D}(\mathbb{D}_{\text{soft}}) = \mathcal{D}(\mathbb{D}_{\text{stiff}})$: The claim on the domain follows simply by construction. $\mathbb{D}_{\text{stiff}}$ is self-adjoint since $\mathcal{P}^{(\tau)}$ is a spectral projection. $\mathbb{D}_{\text{soft}}$ is symmetric, and is relatively $\mathbb{D}_{\text{stiff}}$-bounded with relative bound strictly less than one, as pointed out in the proof of Lemma 2.2.9. Therefore the claim follows by the Kato-Rellich theorem [22, Theorem 8.5].

Being a sum of a closed $\mathbb{D}_{\text{soft}} + \mathbb{D}_{\text{stiff}}$ and a bounded $\mathbb{D}_b$ operator, it follows that $\mathbb{D}$ is closed, and so $\mathbb{S}$ is closed by the boundedness of $\mathbb{E}\mathbb{A}^{-1}\mathbb{B}$. We therefore drop the closures for $\mathbb{S}$ in (2.77).

Step 2. Next we discuss estimates for $\mathbb{D}$. As mentioned in Step 1, $\mathbb{D}_{\varepsilon,b}^{(\tau)}(z)$ is uniformly bounded in ε, τ and z. As for $\mathbb{D}_{\varepsilon,\text{stiff}}^{(\tau)}$ we claim that $\mathbb{D}_{\varepsilon,\text{stiff}}^{(\tau)}$ is invertible with the following estimate

$$\left\| \left(\mathbb{D}_{\varepsilon,\text{stiff}}^{(\tau)}\right)^{-1} \right\|_{\mathcal{P}_\perp^{(\tau)}\mathcal{E}\to\mathcal{P}_\perp^{(\tau)}\mathcal{E}} \leq C\varepsilon^2, \quad C>0 \text{ is independent of } \varepsilon, \tau \text{ and } z. \tag{2.91}$$

The independence on z is immediate. Invertibility follows from Proposition 2.2.10 and the fact that we have removed the lowest eigenspace using $\mathcal{P}_\perp^{(\tau)}$. Since $\mathcal{P}^{(\tau)}$ is the projection w.r.t the unweighted DtN operator, we can separate out ε and obtain the bound $C\varepsilon^2$, with C independent of ε. It remains to justify the independence of C on τ. For this, we will use a pertrubative argument as follows:

> We need to show that the second eigenvalue $\mu_2^{\text{stiff-int},(\tau)}$ and $\mu_2^{\text{stiff-ls},(\tau)}$ can be bounded away from zero, uniformly in τ. This is certainly true for each τ (by Proposition 2.2.10), and can be extended to a neighbourhood $B(\tau,\delta)$ of τ, as the mapping $\tau \mapsto \mu_2^{(\tau)}$ is continuous, a consequence of perturbation theory.
>
> Now consider a dense set $\{\tau_n\} \subset \overline{Q'}$. With $B(\tau_n,\delta_n)$ obtained as above, $\{B(\tau_n,\delta_n)\}_n$ is now an open cover of $\overline{Q'}$. By compactness of $\overline{Q'}$, we may extract a finite subcover $\{B(\tau_{n_k},\delta_{n_k})\}_{k=1}^K$. Since $\mu_2^{(\tau)}$ is bounded away from zero on each $B_k \equiv B(\tau_{n_k},\delta_{n_k})$, we deduce that $\mu_2^{(\tau)}$ is bounded *above* by $\max_k\{\mu_2^{(\tau)} : \tau \in B_k\}$, the latter being strictly negative (note our convention of the DtN map), and independent of τ.

This concludes the justification of (2.91).

Step 3. Now consider the unweighted stiff DtN operator, denoted by $\widetilde{\mathbb{D}}_{\text{stiff}}^{(\tau)} = \varepsilon^2\mathbb{D}_{\varepsilon,\text{stiff}}^{(\tau)}$. We claim that there exists constants $\alpha,\beta>0$, independent of τ such that

$$\|\mathbb{D}_{\text{soft}}^{(\tau)}u\| \leq \alpha\|\widetilde{\mathbb{D}}_{\text{stiff}}^{(\tau)}u\| + \beta\|u\|, \qquad \forall u \in \mathcal{D}(\mathbb{D}_{\text{soft}}^{(\tau)}) = \mathcal{D}(\widetilde{\mathbb{D}}_{\text{stiff}}^{(\tau)}). \tag{2.92}$$

That is, $\mathbb{D}_{\text{soft}}^{(\tau)}$ is relatively $\widetilde{\mathbb{D}}_{\text{stiff}}^{(\tau)}$−bounded, with uniform constants α, β. To prove this claim, we first verify this without the projections $\mathcal{P}_\perp^{(\tau)}$, that is, for $\Lambda^{\text{soft},(\tau)}$ and $\widetilde{\Lambda}^{\text{stiff-int},(\tau)} \oplus \widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$. This is done by using [22, Lemma 8.4] to show relative boundedness for each τ, and then

then applying perturbation theory to the soft and stiff DtN maps, then using the compactness of $\overline{Q'}$, similarly to what was done for (2.91). We omit the details for brevity.

We then proceed to add back the projections. Pre-composing with $\mathcal{P}_\perp^{(\tau)}$ is trivial. Since $\mathcal{P}^{(\tau)}$ is a spectral projection for the stiff DtN map, post-composing with $\mathcal{P}_\perp^{(\tau)}$ is immediate, giving us the RHS of the inequality. As for $\Lambda^{\text{soft},(\tau)}$, we write

$$\Lambda^{\text{soft},(\tau)}\mathcal{P}_\perp^{(\tau)} = \mathcal{P}^{(\tau)}\Lambda^{\text{soft},(\tau)}\mathcal{P}_\perp^{(\tau)} + \mathcal{P}_\perp^{(\tau)}\Lambda^{\text{soft},(\tau)}\mathcal{P}_\perp^{(\tau)} = \mathcal{P}^{(\tau)}\Lambda^{\text{soft},(\tau)}\mathcal{P}_\perp^{(\tau)} + \mathbb{D}_{\text{soft}}^{(\tau)}. \tag{2.93}$$

The first term is bounded uniformly in τ thanks to Proposition 2.3.3. Hence it can be absorbed into the RHS by picking a bigger β. This shows the claim for (2.92).

Step 4. We omit the short argument combining (2.91) and (2.92) to arrive at

$$\|\mathbb{D}_{\text{soft}}^{(\tau)}(\mathbb{D}_{\varepsilon,\text{stiff}}^{(\tau)})^{-1}\|_{\mathcal{P}_\perp^{(\tau)}\mathcal{E}\to\mathcal{P}_\perp^{(\tau)}\mathcal{E}} \leq C\varepsilon^2, \quad \text{where } C>0 \text{ is independent of } \varepsilon,\ \tau \text{ and } z. \tag{2.94}$$

(See [35, Section 3.2] for details.) As a result, we have found the inverse for $\mathbb{D}$, namely

$$\mathbb{D}^{-1} = \mathbb{D}_{\text{stiff}}^{-1}\left(I_{\mathcal{P}_\perp^{(\tau)}\mathcal{E}} + \mathbb{D}_{\text{soft}}\mathbb{D}_{\text{stiff}}^{-1} + \mathbb{D}_b\mathbb{D}_{\text{stiff}}^{-1}\right)^{-1}. \tag{2.95}$$

Furthermore, thanks to our estimates on $\mathbb{D}_{\text{soft}}\mathbb{D}_{\text{stiff}}^{-1}$ and $\mathbb{D}_{\text{stiff}}^{-1}$ obtained above, we know that the terms after I are of order $O(\varepsilon^2)$. Therefore the Neumann series expansion applies, giving the overall estimate of $\|\mathbb{D}^{-1}\| \leq C\varepsilon^2$. Meanwhile, Proposition 2.3.3 implies that $\|\mathbb{E}\mathbb{A}^{-1}\mathbb{B}\| \leq C$, where C is an independent constant. Therefore, the formula $\mathbb{S}^{-1} = (I - \mathbb{D}^{-1}\mathbb{E}\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{D}^{-1}$ implies that $\|\mathbb{S}^{-1}\| \leq C\varepsilon^2$. That is, $\mathbb{S}$ is boundedly invertible with the mentioned bound, where $C > 0$ is independent of ε, τ, and z. This concludes the discussion on the term $\mathbb{S}$.

We have shown that $\|\mathbb{A}^{-1}\| \leq C$ and $\|\mathbb{S}^{-1}\| \leq C\varepsilon^2$. Together with $\|\mathbb{B}\| \leq C$, $\|\mathbb{E}\| \leq C$ (Proposition 2.3.3), and (2.77), this concludes the proof of the theorem. □

Remark.
- The treatment of $\mathbb{A}$ would be different if $\mathcal{P}^{(\tau)}\mathcal{E}$ were infinite dimensional, due to the lack of rank-nullity theorem: We would have to show that (i) $\mathbb{A}$ is closed, (ii) bounded away from zero, and (iii) has dense range in $\mathcal{P}^{(\tau)}\mathcal{E}$. Point (iii) is the key difficulty in the infinite dimensional case. (iii) could be shown by proving that the adjoint is injective.
- On Step 1 of the term $\mathbb{S}$: We do not show that $\mathbb{D}_{\text{soft}}$ is self-adjoint, but only that it is symmetric. This is in contrast with $\mathcal{P}^{(\tau)}\Lambda^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}$, because $\mathcal{P}^{(\tau)}\mathcal{E}$ is finite dimensional while $\mathcal{P}_\perp^{(\tau)}\mathcal{E}$ is not. ◦

2.3.2.1 Continuous dependence on τ

We conclude the proof of Theorem 2.3.4 with the following result:

Proposition 2.3.5 (Continuous dependence)**.** The mapping $f : \overline{Q'} \to \mathbb{R}_{\geq 0}$ given by

$$f(\tau) = \|\Pi^{\bigstar,(\tau)}\psi^{\bigstar,(\tau)}\|_{L^2(Q)}$$

is continuous, where $(\bullet, \bigstar) \in \{(\text{int}, \text{stiff-int}), (\text{ls}, \text{stiff-ls})\}$.

We remind the reader that $\Pi^{\bigstar,(\tau)}$ is the τ-harmonic lift from $\Gamma_\bullet$ into $Q_\bigstar$, and $\psi^{\bigstar,(\tau)}$ is the eigenfunction corresponding to the smallest (absolute value) eigenvalue of the unweighted τ-DtN operator $\widetilde{\Lambda}^{\bigstar,(\tau)}$. Figure 2-2 below shows a diagram of the lifts $\Pi^{\bigstar,(\tau)}$.

Remark.
- We are taking τ from the larger set $\overline{Q'}$.

- (On notation) We will drop the notation $\bullet$, τ, or $\bigstar$ for notational simplicity. These will be recalled whenever we have to make a distinction in the arguments.

- (On notation) The constants in the estimates below may differ from line to line, but the dependence on the parameters will remain the same unless stated otherwise.

- As we will see below, Proposition 2.3.5 boils down to a proof of the continuity of $\tau \mapsto \Pi^{\bigstar,(\tau)}$ and $\tau \mapsto \psi_1^{\bigstar,(\tau)}$. The bulk of the proof is devoted to the continuity of $\tau \mapsto \psi_1^{\text{stiff-ls},(\tau)}$, and we will show this by the method of asymptotic expansions.

  While this thesis does not prove every continuity claim, the proof below serves to demonstrate how the other continuity claims can be proven. In total, we assert the continuity of: $\tau \mapsto \mu_1^{\bigstar,(\tau)}$, $\tau \mapsto \mu_2^{\bigstar,(\tau)}$, $\tau \mapsto \psi_1^{\bigstar,(\tau)}$, $\tau \mapsto \Lambda^{\bigstar,(\tau)} u$, and $\tau \mapsto \Pi^{\bigstar,(\tau)}\psi_1^{\bigstar,(\tau)}$.

- Implicit in our method by asymptotic expansions, is the use of elliptic regularity: For instance, we have $u = \Pi^{\bigstar,(\tau)}\psi^{\bigstar,(\tau)} \in H^1$, as opposed to the general case, $\Pi^{\bigstar,(\tau)}\phi \in L^2$.

- The proof of [39, Lemma 2] contains a proof of the continuity of $\tau \mapsto \Pi^{\bigstar,(\tau)}$ (as a mapping from $H^{1/2}$ to H^1). We provide an alternative argument (see the term "$\|u_1 - w\|$"). $\circ$

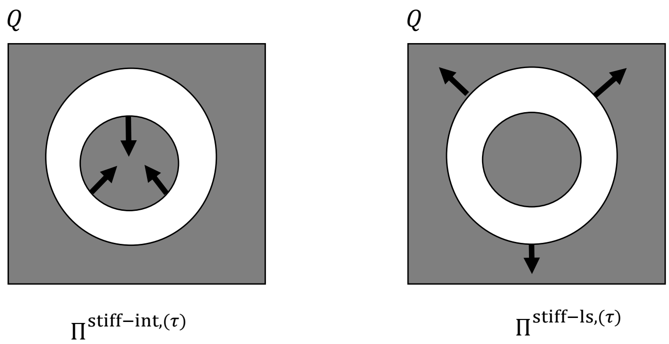

Figure 2-2: Left: τ-harmonic lift into $Q_{\text{stiff-int}}$. Right: τ-harmonic lift into $Q_{\text{stiff-ls}}$.

Proof of Proposition 2.3.5. Let $\tau_1, \tau_2 \in \overline{Q'} = [-\pi, \pi]^d$. We will prove continuity of f at τ_1, that is, given $\varepsilon > 0$, we seek a $\delta = \delta(\tau_1, \varepsilon) > 0$ such that

$$f\big(B(\tau_1, \delta)\big) \subset B\big(f(\tau_1), \varepsilon\big).$$

To this end, we estimate $|f(\tau_2) - f(\tau_1)|$:

$$\begin{aligned} |f(\tau_2) - f(\tau_1)| &\leq \|\Pi^{(\tau_1)}\psi^{(\tau_1)} - \Pi^{(\tau_2)}\psi^{(\tau_2)}\| && (2.96)\\ &\leq \|\Pi^{(\tau_1)}\psi^{(\tau_1)} - \Pi^{(\tau_2)}\psi^{(\tau_1)}\| + \|\Pi^{(\tau_2)}\psi^{(\tau_1)} - \Pi^{(\tau_2)}\psi^{(\tau_2)}\|. && (2.97) \end{aligned}$$

Writing $u_1 = \Pi^{(\tau_1)}\psi^{(\tau_1)}$, $u_2 = \Pi^{(\tau_2)}\psi^{(\tau_2)}$, $w = \Pi^{(\tau_2)}\psi^{(\tau_1)}$, the functions u_1, u_2, and w solves:

$$\begin{cases} -(\nabla + i\tau_1)^2 u_1 = 0, \\ u_1|_{\Gamma_\bullet} = \psi^{(\tau_1)}. \end{cases} \quad \begin{cases} -(\nabla + i\tau_2)^2 u_2 = 0, \\ u_2|_{\Gamma_\bullet} = \psi^{(\tau_2)}. \end{cases} \quad \begin{cases} -(\nabla + i\tau_2)^2 w = 0, & \text{in } Q_\bigstar, \\ w|_{\Gamma_\bullet} = \psi^{(\tau_1)} & \text{on } \Gamma_\bullet, \end{cases}$$

with periodic BCs on ∂Q, if $Q_\bigstar = Q_{\text{stiff-int}}$. We now begin our treatment of each term in (2.97).

The term $\|u_2 - w\|$. We claim that:

$$\begin{aligned}\|u_2 - w\| &\leq C\|\psi^{(\tau_2)} - \psi^{(\tau_1)}\| &(2.98)\\ &\leq C_{\tau_1}|\tau_2 - \tau_1|. &(2.99)\end{aligned}$$

The first inequality follows because the operator $\Pi^{\bigstar,(\tau_2)} : L^2(\Gamma_\bullet) \to L^2(Q_\bigstar)$ is bounded, independently of τ_2, by Proposition 2.2.7.

The second inequality follows from the claim that the mapping $\overline{Q'} \ni \tau \mapsto \psi^{\bigstar,(\tau)} \in L^2(\Gamma_\bullet)$ is locally Lipschitz about τ_1. When $(\bullet, \bigstar) = (\text{int}, \text{stiff-int})$, this is immediate because we have an explicit formula for the eigenfunction by Proposition 2.2.10:

$$\psi^{\text{stiff-int},(\tau)}(x) = e^{-i\tau\cdot x}.$$

The case $(\bullet, \bigstar) = (\mathbf{ls}, \textbf{stiff-ls})$ is complicated as $Q_{\text{stiff-ls}}$ requires periodic BCs on ∂Q. To tackle this, we first note that the first eigenvector-eigenvalue pair for $\widetilde{\Lambda}^\bigstar = \widetilde{\Lambda}^{\text{stiff-ls}}$ is a non-trivial solution to $\widetilde{\Lambda}^{\text{stiff-ls}}\psi = \mu\psi$. Recall from the last part of Section 2.2.1, that this can equivalently be expressed by saying that the following BVP has a non-trivial solution $u = u^{(\tau)}_{\text{stiff-ls}}$:

$$\begin{cases} -(\nabla + i\tau)^2 u = 0 & \text{in } Q_\bigstar = Q_{\text{stiff-ls}}, \\ \partial^{(\tau)}_{n_{\text{stiff-ls}},\text{ls}} u = -\left[\frac{\partial u}{\partial n_{\text{stiff-ls}}} + i(\tau\cdot n_{\text{stiff-ls}})u\right] = \mu u & \text{on } \Gamma_\bullet = \Gamma_{\text{ls}}, \\ u \text{ periodic} & \text{on } \partial Q. \end{cases} \tag{2.100}$$

Note that ψ is the Dirichlet trace of u. Therefore, by the boundedness of the trace operator (independently of τ), it suffices to show that the mapping $\overline{Q'} \ni \tau \mapsto u^{(\tau)}_{\text{stiff-ls}} \in H^1(Q_{\text{stiff-ls}})$ is continuous. To show this, we will employ the method of asymptotic expansion in τ, in polar coordinates, taking inspiration from [32, Section 3.1].

Step 1: Propose a power series expansion for u and μ. Fix $\tau \in Q'$. We begin the method by first writing $\tau = t\theta$, where $t = |\tau|$ and $\theta \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$. We then propose an expansion for $u \equiv u^{(\tau)} \equiv u^{(t\theta)}$:

$$u^{(t\theta)} = u_0 + \big(i|\tau|\big)u_1 + \big(i|\tau|\big)^2 u_2 + \cdots = \sum_{j=0}^{\infty} u_j\big(it\big)^j. \tag{2.101}$$

Here, $u_j : Q_{\text{stiff-ls}} \to \mathbb{C}$. At this stage we do not specify the space for which u_j's belongs to. This will be done in Step 2 by specifying the BVPs that each u_j solves. (It will turn out that the BVPs depends on θ and not t.) Substitute the expansion into the BVP (2.100) and formally compute:

$$\begin{aligned}0 = -(\nabla + i\tau)^2 \sum_j u_j(it)^j &= \sum_j \big(-\Delta - 2it\theta\cdot\nabla \; - (it)^2\big)u_j(it)^j \\ &= -\Delta u_0 \; - 2(it)\theta\cdot\nabla u_0 - (it)^2 u_0\end{aligned}$$

$$
\begin{aligned}
&-(it)\Delta u_1 \quad -2(it)^2\theta\cdot\nabla u_1 \; -(it)^3 u_1\\
&\qquad -(it)^2\Delta u_2 \quad -2(it)^3\theta\cdot\nabla u_2 -(it)^4 u_2\\
&\qquad\qquad -(it)^3\Delta u_3 \quad -2(it)^4\theta\cdot\nabla u_3 -(it)^5 u_3 + \cdots
\end{aligned}
$$

We also propose a similar power series expansion for $\mu \equiv \mu^{(\tau)} \equiv \mu^{(t\theta)}$:

$$
\mu^{(t\theta)} = \sum_{k=1}^{\infty} \alpha_k \big(it\big)^k, \quad \alpha_k \in \mathbb{C}. \tag{2.102}
$$

Note that we are postulating that $\alpha_0 = 0$ (in addition to μ admitting a series expansion), and this will be justified with remainder estimates. Similarly, we substitute this into the BVP (2.100) and formally compute:

$$
\begin{aligned}
-\left[\frac{\partial u}{\partial n} + i(\tau\cdot n)\right]\sum_j u_j (it)^j &= \Big(\sum_k \alpha_k (it)^k\Big)\Big(\sum_j u_j (it)^j\Big)\\
\Leftrightarrow \sum_j -(it)^j\left[\frac{\partial}{\partial n} + (it)(\theta\cdot n)\right]u_j &= \sum_j\sum_k u_j\alpha_k (it)^j (it)^k.
\end{aligned}
$$

Remark (On notation)**.** For ease of notation, we have dropped the subscript "stiff-ls" from $n_{\text{stiff-ls}}$. We will henceforth do the same for $Q_{\text{stiff-ls}}$ and Γ_{ls}. Similarly, the DtN map $\widetilde{\Lambda}^\bigstar = \widetilde{\Lambda}^{\text{stiff-ls}}$ will by denoted by Λ, omitting the weight " $\sim$ ". We will also omit writing the periodic BC on ∂Q. This will apply to the remainder of this case, $(\bullet, \bigstar) = (\text{ls}, \text{stiff-ls})$. ◦

Now equate powers of it, to see that

$$
\begin{aligned}
(it)^0 &: -\frac{\partial}{\partial n}u_0 &&= 0,\\
(it)^1 &: -(it)\frac{\partial}{\partial n}u_1 - (it)(\theta\cdot n)u_0 &&= (it)\alpha_1 u_0 + (it)\alpha_0 u_1,\\
(it)^2 &: -(it)^2\frac{\partial}{\partial n}u_2 - (it)(\theta\cdot n)u_1 &&= (it)^2\alpha_2 u_0 + (it)^2\alpha_1 u_1 + (it)^2\alpha_0 u_2.
\end{aligned}
$$

Step 2: Write down the BVP for each power of it and deduce the coefficients u_j and α_k. The problem for $(it)^0$ is therefore

$$
\begin{cases}
-\Delta u_0 = 0, & \text{in } Q,\\
-\frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma,
\end{cases} \tag{2.103}
$$

which we see to be independent of t, as mentioned earlier. We can use this to obtain information about u_0. Consider its weak formulation with $0 \neq u_0 \in H^1$ as the test function,

$$
0 \le (\nabla u_0, \nabla u_0)_{L^2(Q)} = \left(\frac{\partial u_0}{\partial n}, u_0\right)_{L^2(\Gamma)} = 0. \tag{2.104}
$$

So ∇u_0 is zero a.e., and hence u_0 is a constant a.e.. We will set henceforth $u_0 \equiv 1$ without loss of generality, as the choice of this (universal) constant does not affect the remainder estimates.

Before we move on to higher powers of it, **we further ask that $\int u_j = 0$ for $j \ge 1$.** (This is crucial for the application of the Poincaré inequality.) This can be done again without loss of generality, by absorbing the mean into u_0. Now the problem for $(it)^1$ is then

$$\begin{cases} -\Delta u_1 = 0, & \text{in } Q, \\ -\frac{\partial u_1}{\partial n} - (\theta \cdot n) = \alpha_1 & \text{on } \Gamma. \end{cases} \tag{2.105}$$

We have used the fact that $u_0 \equiv 1$ and $\alpha_0 = 0$. Testing (2.105) with u_0 gives

$$(\nabla u_1, \nabla u_0)_{L^2(Q)} = \left(\frac{\partial u_1}{\partial n}, u_0\right)_{L^2(\Gamma)} = -(\alpha_1 + \theta \cdot n, 1)_{L^2(\Gamma)}. \tag{2.106}$$

Testing (2.103) with u_1 gives:

$$(\nabla u_0, \nabla u_1)_{L^2(Q)} = \left(\frac{\partial u_0}{\partial n}, u_1\right)_{L^2(\Gamma)} = -\alpha_0 (u_0, u_1)_{L^2(\Gamma)} = 0. \tag{2.107}$$

This implies that $(\nabla u_1, \nabla u_0)$ is also zero, in other words, (2.106) implies that

$$-(\alpha_1 + \theta \cdot n, 1)_{L^2(\Gamma)} = 0.$$

Deducing α_1: Fix j, $1 \le j \le d$. Define $g_j : \mathbb{R}^d \to \mathbb{R}$ by $g_j(x) = \theta_j x_j$. Then,

$$0 \overset{\Delta g_j = 0}{=} (1, \Delta g_j)_{L^2(Q)} \overset{\text{IBP}}{=} (\nabla \mathbf{1}, \nabla g_j)_{L^2(Q)} + \left(1, \frac{\partial g_j}{\partial n}\right)_{L^2(\Gamma)} = \left(1, (0, \cdots, \theta_j, \cdots, 0) \cdot n\right)_{L^2(\Gamma)}.$$

Summing over j, we get $(1, \theta \cdot n)_{L^2(\Gamma)} = 0$. Since $\alpha_1 \in \mathbb{C}$, we must have that $\alpha_1 = 0$. With the knowledge that $\alpha_1 = 0$, we now test (2.105) with u_1 to obtain

$$(\nabla u_1, \nabla u_1)_{L^2(Q)} \overset{\text{IBP}}{=} \left(\frac{\partial u_1}{\partial n}, u_1\right)_{L^2(\Gamma)} = -(\theta \cdot n, u_1)_{L^2(\Gamma)}. \tag{2.108}$$

The problem for $(it)^2$ is:

$$\begin{cases} -\Delta u_2 = 2\theta \cdot \nabla u_1 + 1, & \text{in } Q, \\ -\frac{\partial u_2}{\partial n} - (\theta \cdot n) u_1 = \alpha_2 & \text{on } \Gamma. \end{cases} \tag{2.109}$$

We now may test (2.109) with u_0, u_1, u_2. We may also test (2.105) with u_2, and (2.103) with u_2. Of these, the useful ones for our analysis later will be: testing (2.109) with $u_0 \equiv 1$,

$$(2\theta \cdot \nabla u_1 + 1, 1)_{L^2(Q)} = (-\Delta u_2, 1)_{L^2(Q)} \overset{\text{IBP}}{=} (\nabla u_2, \nabla 1)_{L^2(Q)} - \left(\frac{\partial u_2}{\partial n}, 1\right)_{L^2(\Gamma)} = ((\theta \cdot n) u_1 + \alpha_2, 1)_{L^2(\Gamma)}, \tag{2.110}$$

which is useful for obtaining an estimate for α_2, and testing (2.109) with u_2,

$$(2\theta \cdot \nabla u_1 + 1, u_2)_{L^2(Q)} = (-\Delta u_2, 1)_{L^2(Q)} \overset{\text{IBP}}{=} (\nabla u_2, \nabla u_2)_{L^2(Q)} - \left(\frac{\partial u_2}{\partial n}, u_2\right)_{L^2(\Gamma)} \tag{2.111}$$

$$= (\nabla u_2, \nabla u_2)_{L^2(Q)} + \left((\theta \cdot n) u_1 + \alpha_2, u_2\right)_{L^2(\Gamma)}, \tag{2.112}$$

which is useful for obtaining an estimate for ∇u_2.

Remark (Connection to classical results)**.** Observe that since $\alpha_1 = 0$, by picking $\theta_j \in \mathbb{S}^{d-1}$ as the standard basis and writing the solution $u_{1,j}$ as N_j. We obtain the expression $u_1 = \sum_{j=1}^d N_j\theta_j$ for arbitrary $\theta = (\theta_1, \cdots, \theta_d)$. So N_j should be thought of as first order correctors, which are to be compared with the first order term in the asymptotic expansion for periodic homogenization "$N_j(y)\frac{\partial u_0}{\partial x_j}$".

Similarly, for α_2, we write $1 = |\theta|^2 = \theta\cdot\theta$, then putting the BVP for u_1 into (2.110), we obtain $\alpha_2 = \frac{1}{|\Gamma|}\int_Q (\theta\cdot\nabla u_1 + 1)dx$. Again, we can further pick θ to be the standard basis, obtaining a coordinate-wise description of α_2. This is to be compared with α_2 in [34, Appendix B]. ◦

Step 3: Write down the BVP for the "remainder" terms for u and μ. To do so, we first define the remainder $R \in H^1$ and $r \in \mathbb{C}$ by

$$\begin{cases} u = 1 + (it)u_1 + & (it)^2 u_2 + R, \\ \mu = & (it)^2\alpha_2 + r. \end{cases}$$

Substituting this expression for u into the main BVP (2.100), we get, in Q,

$$\begin{aligned} -(\nabla + i\tau)^2 R &= (-\Delta - 2(it)\theta\cdot\nabla - (it)^2)(-1 - (it)u_1 - (it)^2 u_2) \\ &= \qquad + (it)^2 \\ &\qquad +(it)\Delta u_1 + 2(it)^2\theta\cdot\nabla u_1 + (it)^3 u_1 \\ &\qquad \underbrace{}_{-\Delta u_1 = 0} + \underbrace{(it)^2\Delta u_2}_{-\Delta u_2 = 2\theta\cdot\nabla u_1 + 1} + 2(it)^3\theta\cdot\nabla u_2 + (it)^4 u_2 \\ &= (it)^3 u_1 + 2(it)^3\theta\cdot\nabla u_2 + (it)^4 u_2. \end{aligned}$$

On Γ, the main BVP (2.100) gives us

$$-\frac{\partial R}{\partial n} - (it)(\theta\cdot n)R = \underbrace{(r + \alpha_2(it)^2)(1 + (it)u_1 + (it)^2 u_2 + R)}_{\mu u} + \mathbb{X}.$$

To find $\mathbb{X}$, we further compute

$$\begin{aligned} & -\left[\frac{\partial}{\partial n} + (it)(\theta\cdot n)\right]R - \mu u \\ \overset{(2.100)}{=} & -\left[\frac{\partial}{\partial n} + (it)(\theta\cdot n)\right]R + \left[\frac{\partial}{\partial n} + (it)(\theta\cdot n)\right]u \\ = & +\left[\frac{\partial}{\partial n} + (it)(\theta\cdot n)\right](1 + (it)u_1 + (it)^2 u_2) \\ = & \quad + (it)(\theta\cdot n) + (it)^2(\theta\cdot n)u_1 + (it)^3(\theta\cdot n)u_2 \\ & \quad + \underbrace{(it)\frac{\partial u_1}{\partial n}}_{=0 \text{ by } (2.105)} + \underbrace{(it)^2\frac{\partial u_2}{\partial n}}_{=-(it)^2\alpha_2 \text{ by } (2.109)} \\ = & \quad -(it)^2\alpha_2\big(1 + (it)(\theta\cdot n)u_2\big). \end{aligned}$$

The problem for R and r is therefore,

$$\begin{cases} -(\nabla + i\tau)^2 R &= (it)^3(2\theta\cdot\nabla u_2 + u_1) - (it)^4 u_2 & \text{in } Q, \\ -\frac{\partial R}{\partial n} - (it)(\theta\cdot n)R &= (r + \alpha_2(it)^2)\big(1 + (it)u_1 + (it)^2 u_2 + R\big) & \\ & \quad -(it)^2\alpha_2\big(1 + (it)(\theta\cdot n)u_2\big) & \text{on } \Gamma. \end{cases} \tag{2.113}$$

Remark. To see where this is all going, recall that our goal is to obtain an estimate for $u^{(\tau_2)} - u^{(\tau_1)}$ **in** H^1. Write $\tau_1 = t_1\theta_1$ and $\tau_2 = t_2\theta_2$. Using the expression $u^{(\tau)} = 1 + (it)u_1^{(\tau)} + (it)^2 u_2^{(\tau)} + R^{(\tau)}$, we see that

$$\begin{aligned} \|u^{(\tau_2)} - u^{(\tau_1)}\| &\le \big\|(it_2)u_1^{(\tau_2)} - (it_1)u_1^{(\tau_1)}\big\| + \big\|(it_2)^2 u_2^{(\tau_2)} - (it_1)^2 u_2^{(\tau_1)}\big\| + \big\|R^{(\tau_2)} - R^{(\tau_1)}\big\| \\ &\le \big\|(it_2)u_1^{(\tau_2)} - (it_1)u_1^{(\tau_2)}\big\| + \big\|(it_1)u_1^{(\tau_2)} - (it_1)u_1^{(\tau_1)}\big\| \\ &\quad + \big\|(it_2)^2 u_2^{(\tau_2)} - (it_1)^2 u_2^{(\tau_2)}\big\| + \big\|(it_1)^2 u_2^{(\tau_2)} - (it_1)^2 u_2^{(\tau_1)}\big\| \\ &\quad + \big\|R^{(\tau_2)} - R^{(\tau_1)}\big\| \\ &\le |t_2 - t_1|\cdot\big\|u_1^{(\tau_2)}\big\| + |t_1|\cdot\big\|u_1^{(\tau_2)} - u_1^{(\tau_1)}\big\| \\ &\quad + |t_2^2 - t_1^2|\cdot\big\|u_2^{(\tau_2)}\big\| + |t_1|^2\cdot\big\|u_2^{(\tau_2)} - u_2^{(\tau_1)}\big\| \\ &\quad + \big\|R^{(\tau_2)} - R^{(\tau_1)}\big\|. \end{aligned}$$

We are therefore looking to obtain the following estimates **in** H^1:

- $\big\|u_1^{(\tau_2)}\big\|$, $\big\|u_2^{(\tau_2)}\big\| \le C_{\tau_1}$,
- $\big\|u_1^{(\tau_2)} - u_1^{(\tau_1)}\big\|$, $\big\|u_2^{(\tau_2)} - u_2^{(\tau_1)}\big\|$, $\big\|R^{(\tau_2)} - R^{(\tau_1)}\big\| \le C_{\tau_1}|\tau_2 - \tau_1|$ (locally Lipschitz about τ_1.) ○

Step 4: Remainder estimates. (Step 4a: u_1.) As per the remark, we first start with the H^1 estimate for u_1 and $u_1^{(\tau_2)} - u_1^{(\tau_1)}$. Since $\int u_1 = 0$ this is equivalent to an L^2 estimate on its gradient. Continuing from (2.108),

$$\begin{aligned} \|\nabla u_1^{(\tau_2)}\|^2_{L^2(Q)} &\le C\|u_1^{(\tau_2)}\|_{L^2(\Gamma)} && \text{for every } x\in\Gamma,\ |\theta_2\cdot n(x)| \le |\theta_2|\cdot|n(x)| = 1. \\ &\le C\|\nabla u_1^{(\tau_2)}\|_{L^2(Q)} && \text{by the Trace theorem + Poincaré, as } \int u_1 = 0. \end{aligned} \tag{2.114}$$

This gives us $\|\nabla u_1^{(\tau_2)}\|_{L^2(Q)} \le C$, where C is independent of τ_1 and τ_2. As for the difference $u_1^{(\tau_2)} - u_1^{(\tau_1)}$, we first write the BVP that it satisfies:

$$\begin{cases} -\Delta(u_1^{(\tau_2)} - u_1^{(\tau_1)}) = 0, & \text{in } Q, \\ -\frac{\partial}{\partial n}\big(u_1^{(\tau_2)} - u_1^{(\tau_1)}\big) - ((\theta_2 - \theta_1)\cdot n) = \alpha_1 & \text{on } \Gamma. \end{cases} \tag{2.115}$$

Similarly to the estimate above, we test (2.115) with $u_1^{(\tau_2)} - u_1^{(\tau_1)} \in H^1(Q)$, to get:

$$\|\nabla(u_1^{(\tau_2)} - u_1^{(\tau_1)})\|^2_{L^2(Q)} \le C|\theta_2 - \theta_1|\ \|u_1^{(\tau_2)} - u_1^{(\tau_1)}\|_{L^2(\Gamma)} \le C|\theta_2 - \theta_1|\ \|\nabla(u_1^{(\tau_2)} - u_1^{(\tau_1)})\|_{L^2(Q)}. \tag{2.116}$$

This gives us the required estimate $\|\nabla(u_1^{(\tau_2)} - u_1^{(\tau_1)})\|_{L^2(Q)} \le C|\theta_2 - \theta_1|$, where C is independent of τ_1 and τ_2.

(Step 4b: u_2 and α_2.) Next we turn to $u_2^{(\tau_2)}$. As mentioned in Step 3, we use (2.110) to obtain an estimate for $\alpha_2^{(\tau_2)}$. The equation can be rearranged to give:

$$\int_Q (2\theta_2 \cdot \nabla u_1^{(\tau_2)} + 1) - \int_\Gamma (\theta_2 \cdot n) u_1^{(\tau_2)} = |\Gamma| \alpha_2^{(\tau_2)}, \tag{2.117}$$

so that together with Step 4a,

$$|\alpha_2^{(\tau_2)}| \le \frac{1}{|\Gamma|} C \left[\|\nabla u_1^{(\tau_2)}\|_{L^2(Q)} + |Q| \right] < C, \qquad C \text{ is independent of } \tau_1 \text{ and } \tau_2. \tag{2.118}$$

We use (2.111) to obtain an estimate for $u_2^{(\tau_2)}$:

$$\begin{aligned}
\|\nabla u_2^{(\tau_2)}\|^2_{L^2(Q)} &= \int_Q (2\theta_2 \cdot \nabla u_1^{(\tau_2)} + 1) u_2^{(\tau_2)} - \int_\Gamma \left((\theta_2 \cdot n) u_1^{(\tau_2)} + \alpha_2^{(\tau_2)}\right) u_2^{(\tau_2)} \\
&\le C \left[\|\nabla u_1^{(\tau_2)}\|_{L^2(Q)} + 1 \right] \|u_2^{(\tau_2)}\|_{L^2(Q)} + C \left[\|u_1^{(\tau_2)}\|_{L^2(\Gamma)} + |\alpha_2^{(\tau_2)}| \right] \|u_2^{(\tau_2)}\|_{L^2(\Gamma)} \\
&\le C \left[\|\nabla u_1^{(\tau_2)}\|_{L^2(Q)} + 1 \right] \|\nabla u_2^{(\tau_2)}\|_{L^2(Q)} \\
&\le C \|\nabla u_2^{(\tau_2)}\|_{L^2(Q)}, \qquad \text{by Step 4a.}
\end{aligned} \tag{2.119}$$

The second last estimate follows by the Trace theorem, Poincaré inequality ($\int u_1 = \int u_2 = 0$), and the result $|\alpha_2^{(\tau_2)}| < C$. This gives us $\|\nabla u_2^{(\tau_2)}\|_{L^2(Q)} \le C$, where C is independent of τ_1 and τ_2.

As for the difference $u_2^{(\tau_2)} - u_2^{(\tau_1)}$, we first write the BVP that it satisfies:

$$\begin{cases} -\Delta(u_2^{(\tau_2)} - u_2^{(\tau_1)}) = 2\theta_2 \cdot \nabla u_1^{(\tau_2)} - 2\theta_1 \cdot \nabla u_1^{(\tau_1)}, & \text{in } Q, \\ -\frac{\partial}{\partial n}(u_1^{(\tau_2)} - u_1^{(\tau_1)}) - ((\theta_2 \cdot n) u_1^{(\tau_2)} - (\theta_1 \cdot n) u_1^{(\tau_1)}) = \alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)} & \text{on } \Gamma. \end{cases} \tag{2.120}$$

To obtain an estimate for $\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}$, we proceed in the same manner by testing (2.120) with $u_0^{(\tau_2)} \equiv 1$, and rearrange the equation arriving at:

$$\begin{aligned}
&2 \int_Q \left(\theta_2 \cdot \nabla u_1^{(\tau_2)} - \theta_2 \cdot \nabla u_1^{(\tau_1)} + \theta_2 \cdot \nabla u_1^{(\tau_1)} - \theta_1 \cdot \nabla u_1^{(\tau_1)} \right) \\
&+ \int_\Gamma \left((\theta_2 \cdot n) u_1^{(\tau_2)} - (\theta_2 \cdot n) u_1^{(\tau_1)} - (\theta_2 \cdot n) u_1^{(\tau_1)} + (\theta_1 \cdot n) u_1^{(\tau_1)} \right) = |\Gamma| \left(\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)} \right)
\end{aligned} \tag{2.121}$$

(adding and subtracting new terms $\theta_2 \cdot \nabla u_1^{(\tau_1)}$ and $(\theta_2 \cdot n) u_1^{(\tau_1)}$). Of the eight terms on the LHS, they may be grouped together in pairs, and each pair may be estimated by $C|\theta_2 - \theta_1|$. (Details are omitted since they are similar to Step 4a.) Therefore, we deduce that $|\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}| < C|\theta_2 - \theta_1|$.

We now test (2.120) with $u_2^{(\tau_2)} - u_2^{(\tau_1)} \in H^1(Q)$, to get:

$$\|\nabla(u_2^{(\tau_2)} - u_2^{(\tau_1)})\|^2_{L^2(Q)}$$

$$
\begin{aligned}
&= \int_Q \left(2\theta_2 \cdot \nabla u_1^{(\tau_2)} - 2\theta_1 \cdot \nabla u_1^{(\tau_1)}\right)\left(u_2^{(\tau_2)} - u_2^{(\tau_1)}\right) \\
&\quad + \int_\Gamma \left((\theta_2 \cdot n)u_1^{(\tau_2)} - (\theta_1 \cdot n)u_1^{(\tau_1)} + \alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}\right)\left(u_2^{(\tau_2)} - u_2^{(\tau_1)}\right) \\
&\le \int_Q \left[|2\theta_2 \cdot \nabla u_1^{(\tau_2)} - 2\theta_1 \cdot \nabla u_1^{(\tau_2)}| + |2\theta_1 \cdot \nabla u_1^{(\tau_2)} - 2\theta_1 \cdot \nabla u_1^{(\tau_1)}|\right] |u_2^{(\tau_2)} - u_2^{(\tau_1)}| \\
&\quad + \int_\Gamma \left[|(\theta_2 \cdot n)u_1^{(\tau_2)} - (\theta_1 \cdot n)u_1^{(\tau_2)}| + |(\theta_1 \cdot n)u_1^{(\tau_2)} - (\theta_1 \cdot n)u_1^{(\tau_1)}| + |\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}|\right] |u_2^{(\tau_2)} - u_2^{(\tau_1)}| \\
&\le C \int_Q \left[|\theta_2 - \theta_1|\; |\nabla u_1^{(\tau_2)}| + |\nabla(u_1^{(\tau_2)} - u_1^{(\tau_1)})|\right] |u_2^{(\tau_2)} - u_2^{(\tau_1)}| \\
&\quad + C \int_\Gamma \left[|\theta_2 - \theta_1|\; |u_1^{(\tau_2)}| + |u_1^{(\tau_2)} - u_1^{(\tau_1)}| + |\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}|\right] |u_2^{(\tau_2)} - u_2^{(\tau_1)}| \\
&\qquad\qquad \text{by Cauchy-Schwartz on } \mathbb{R}^d. \text{ Note that } |n(x)| \equiv 1. \\
&\le C|\theta_2 - \theta_1| \left[\|u_2^{(\tau_2)} - u_2^{(\tau_1)}\|_{L^2(Q)} + \|u_2^{(\tau_2)} - u_2^{(\tau_1)}\|_{L^2(\Gamma)}\right] \\
&\qquad\qquad \text{since } L^2 \subset L^1 \text{ as } |\Gamma|,\ |Q| < \infty, \text{ use Cauchy-Schwartz on } L^2. \\
&\qquad\qquad \text{Then, apply the estimates from Step 4a and from } |\alpha_2^{(\tau_2)} - \alpha_2^{(\tau_1)}|. \\
&\le C|\theta_2 - \theta_1|\; \|\nabla(u_2^{(\tau_2)} - u_2^{(\tau_1)})\|_{L^2(Q)} \\
&\qquad\qquad \text{By the Trace theorem and Poincaré inequality, as } \int u_2 = 0.
\end{aligned} \tag{2.122}
$$

This gives us the required estimate $\|\nabla(u_2^{(\tau_2)} - u_2^{(\tau_1)})\|_{L^2(Q)} \le C|\theta_2 - \theta_1|$, where C is independent of τ_1 and τ_2.

(Step 4c: R and μ.) We first obtain a uniform bound for $\mu^{(\tau)}$, by applying the min-max principle (to the lower semibounded operator $-\Lambda$):

$$
0 \le -\left(r + \alpha_2(it)^2\right) = -\mu^{(\tau)} = \min_{u \in H^2(Q), u \neq 0} \frac{(-\Lambda u, u)_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} = \min \frac{\int_Q |(\nabla + i\tau)u|^2}{\|u\|^2_{L^2(\Gamma)}} \le \frac{|Q|}{|\Gamma|} t^2, \tag{2.123}
$$

where the final inequality follows by picking $u \equiv 1$.

As for $R \equiv R^{(\tau)}$, testing (2.113) with R gives, on the LHS,

$$
\begin{aligned}
LHS = -((\nabla + i\tau)^2 R, R)_{L^2(Q)} &= \int_Q |(\nabla + i\tau)R|^2 - \int_\Gamma \left[\frac{\partial R}{\partial n} + i(\tau \cdot n)R\right] \bar{R} \\
&= \int_Q |(\nabla + i\tau)R|^2 - \int_\Gamma \left[\mu^{(\tau)}\left(1 + (it)u_1 + (it)^2 u_2 + R\right) - (it)^2 \alpha_2 \left(1 + (it)(\theta \cdot n)u_2\right)\right] \bar{R}.
\end{aligned} \tag{2.124}
$$

Meanwhile, the RHS may be estimated using Step 4b, to give

$$
RHS \le Ct^3 \|R\|_{L^2(Q)}. \tag{2.125}
$$

Rearranging, we get

$$
\|(\nabla + i\tau)R\|^2_{L^2(Q)} + \mu^{(\tau)} \|R\|^2_{L^2(\Gamma)} \le C \int_\Gamma \bar{R} + C\|R\|_{L^2(Q)}
$$

$$\le C\|\nabla R\|_{L^2(Q)} + C\|R\|_{L^2(Q)} \qquad \text{Trace theorem + Poincaré.}$$
$$\le C\|(\nabla + i\tau)R\|_{L^2(Q)} + C\|R\|_{L^2(Q)}$$
$$\le \frac{C}{4\varepsilon} + \varepsilon\|(\nabla + i\tau)R\|^2_{L^2(Q)} + C\|R\|_{L^2(Q)} \qquad \text{Cauchy inequality with } \varepsilon. \tag{2.126}$$

For $\mu^{(\tau)}\|R\|^2_{L^2(\Gamma)}$, this is bounded *below* (note that $\mu^{(\tau)} \le 0$) by $C\mu^{(\tau)}\|\nabla R\|^2_{L^2(Q)}$ by the Trace theorem, which is further bounded below by $C\mu^{(\tau)}\|(\nabla + i\tau)R\|^2_{L^2(Q)}$ by the triangle inequality. Therefore the LHS of the inequality may be replaced by $(1-C\mu^{(\tau)})\|(\nabla+i\tau)R\|^2_{L^2(Q)}$. To prevent a trivial estimate, *we assume that* τ *is small enough*, so that by (2.123), $1 - C\mu^{(\tau)} \le 1 - \varepsilon$.

Combining this with the RHS of the inequality, we arrive at

$$(1 - 2\varepsilon)\|(\nabla + i\tau)R\|^2_{L^2(Q)} \le C\|R\|_{L^2(Q)} + \frac{C}{4\varepsilon}. \tag{2.127}$$

Furthermore, the **Poincaré inequality with** τ (for τ small and $\int R = 0$) gives us an estimate of $\|R\|_{L^2(Q)}$ in terms of $\|(\nabla + i\tau)R\|_{L^2(Q)}$. So by picking $\varepsilon > 0$ small enough, we obtain a quadratic inequality in $\|R\|^2$:

$$C_1\|R\|^2_{L^2(Q)} - C_2\|R\|_{L^2(Q)} - C_3 \le 0, \qquad \text{where } C_1, C_2, C_3 > 0. \tag{2.128}$$

Since the constants are positive, we must have that $\|R\|_{L^2(Q)}$ is bounded. This, (2.127), and the reverse triangle inequality then implies that $\|\nabla R\|_{L^2(Q)}$ is bounded.

(Step 4d: $\mu^{(\tau_2)} - \mu^{(\tau_1)}$.) For this, we appeal to the min-max principle (to $-\Lambda$) once again:

$$\begin{aligned}
0 \le -\mu^{(\tau_2)} &= \min_{u\in H^2(Q),u\neq 0} \frac{(-\Lambda^{(\tau_2)}u, u)_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} = \min_{u\in H^2(Q),u\neq 0} \frac{\int_\Gamma \left[\frac{\partial u}{\partial n} + i(\tau_2 \cdot n)u\right]\bar{u}}{\|u\|^2_{L^2(\Gamma)}} \\
&= \min_{u\in H^2(Q),u\neq 0} \frac{\int_\Gamma \left[\frac{\partial u}{\partial n} + i(\tau_1 \cdot n)u + i((\tau_2 - \tau_1) \cdot n)u\right]\bar{u}}{\|u\|^2_{L^2(\Gamma)}} \\
&\le \min_{u\in H^2(Q),u\neq 0} \frac{\int_\Gamma \left[\frac{\partial u}{\partial n} + i(\tau_1 \cdot n)u\right]\bar{u}}{\|u\|^2_{L^2(\Gamma)}} + C|\tau_2 - \tau_1| \qquad \text{by Cauchy-Schwarz on } \mathbb{R}^d.\ |n(x)| \equiv 1. \\
&= -\mu^{(\tau_1)} + C|\tau_2 - \tau_1|.
\end{aligned} \tag{2.129}$$

Exchanging the roles of τ_2 and τ_1, we arrive at $|\mu^{(\tau_2)}-\mu^{(\tau_1)}| \le C|\tau_2-\tau_1|$, where C is independent of τ_2 and τ_1.

(Step 4e: $R^{(\tau_2)} - R^{(\tau_1)}$.) It will be more convenient to consider the BVP that $R^{(\tau_1)} - R^{(\tau_2)}$

satisfies (as opposed to $R^{(\tau_2)} - R^{(\tau_1)}$):

$$\begin{cases} -(\nabla + i\tau_1)^2(R^{(\tau_1)} - R^{(\tau_2)}) - 2i(\tau_1 - \tau_2)\cdot\nabla R^{(\tau_2)} + (t_1^2 - t_2^2)R^{(\tau_2)} \\ \quad = (it_1)^3(2\theta_1\cdot\nabla u_2^{(\tau_1)} + u_1^{(\tau_1)}) + (it_1)^4 u_2^{(\tau_1)} - (it_2)^3(2\theta_2\cdot\nabla u_2^{(\tau_2)} + u_1^{(\tau_2)}) + (it_2)^4 u_2^{(\tau_2)} \text{ in } Q, \\ -\frac{\partial}{\partial n}\left[R^{(\tau_1)} - R^{(\tau_2)}\right] - \left[(it_1)(\theta_1\cdot n)R^{(\tau_1)} - (it_2)(\theta_2\cdot n)R^{(\tau_2)}\right] \\ \quad = (r^{(\tau_1)} + \alpha_2^{(\tau_1)}(it_1)^2)\left(1 + (it_1)u_1^{(\tau_1)} + (it_1)^2 u_2^{(\tau_1)} + R^{(\tau_1)}\right) - (it_1)^2\alpha_2^{(\tau_1)}\left(1 + (it_1)(\theta_1\cdot n)u_2^{(\tau_1)}\right) \\ \qquad -(r^{(\tau_2)} + \alpha_2^{(\tau_2)}(it_2)^2)\left(1 + (it_2)u_1^{(\tau_2)} + (it_2)^2 u_2^{(\tau_2)} + R^{(\tau_2)}\right) + (it_2)^2\alpha_2^{(\tau_2)}\left(1 + (it_2)(\theta_2\cdot n)u_2^{(\tau_2)}\right) \\ \quad \text{on } \Gamma. \end{cases} \tag{2.130}$$

For brevity, write $v = R^{(\tau_1)} - R^{(\tau_2)}$. Test (2.130) with v. The RHS is then

$$\begin{aligned} &\int_Q \left(2(it_1)^3\theta_1\cdot\nabla u_2^{(\tau_1)} + (it_1)^3 u_1^{(\tau_1)} + (it_1)^4 u_2^{(\tau_1)} - 2(it_2)^3\theta_2\cdot\nabla u_2^{(\tau_2)} + (it_2)^3 u_1^{(\tau_2)} + (it_2)^4 u_2^{(\tau_2)}\right)\bar{v} \\ &\qquad \le \|2A_1 + A_2 + A_3 - 2\tilde{A}_1 - \tilde{A}_2 - \tilde{A}_3\|_{L^2(Q)}\|\nabla v\|_{L^2(Q)} \\ &\qquad\qquad \text{by Cauchy-Schwartz on } L^2 \text{ and Poincaré, as } \int v = 0. \\ &\qquad \le C\left[|t_1 - t_2| + |\theta_1 - \theta_2|\right]\|\nabla v\|_{L^2(Q)}. \end{aligned} \tag{2.131}$$

The final inequality is obtained by using a *cancellation trick*, for example,

$$\begin{aligned} \|A_1 - \tilde{A}_1\|_{L^2(Q)} &= \|(it_1)^3\theta_1\cdot\nabla u_2^{(\tau_1)} - (it_2)^3\theta_1\cdot\nabla u_2^{(\tau_1)}\| \\ &\qquad + \|(it_2)^3\theta_1\cdot\nabla u_2^{(\tau_1)} - (it_2)^3\theta_2\cdot\nabla u_2^{(\tau_1)}\| \\ &\qquad\qquad + \|(it_2)^3\theta_2\cdot\nabla u_2^{(\tau_1)} - (it_2)^3\theta_2\cdot\nabla u_2^{(\tau_2)}\| \\ &\le |t_1^3 - t_2^3|\;|\theta_1|\;\|\nabla u_2^{(\tau_1)}\| + t_2^3|\theta_1 - \theta_2|\;\|\nabla u_2^{(\tau_1)}\| + t_2^3|\theta_2|\;\|\nabla(u_2^{(\tau_1)} - u_2^{(\tau_2)})\| \\ &\qquad \text{by Cauchy-Schwartz on } \mathbb{R}^d. \\ &\le C\left[|t_1^3 - t_2^3| + |\theta_1 - \theta_2| + |\theta_1 - \theta_2|\right] \qquad \text{by Step 4b.} \\ &\le C\left[|t_1 - t_2| + |\theta_1 - \theta_2|\right]. \end{aligned} \tag{2.132}$$

Next, the LHS gives

$$(-(\nabla + i\tau_1)^2 v, v)_{L^2(Q)} - \left(2i(\tau_1 - \tau_2)\cdot\nabla R^{(\tau_2)} + (t_1^2 - t_2^2)R^{(\tau_2)}, v\right)_{L^2(Q)}. \tag{2.133}$$

The second term is brought to the RHS and estimated by

$$C[|t_1^2 - t_2^2| + |\tau_1 - \tau_2|]\;\|\nabla v\|_{L^2(Q)}.$$

We apply integration by parts to the first term to get

$$\|(\nabla + i\tau_1)v\|^2_{L^2(Q)} - \int_\Gamma \left[\frac{\partial v}{\partial n} + i(\tau_1 \cdot n)v\right]\bar{v}. \tag{2.134}$$

For second term (2.134), we write

$$\int_\Gamma \Bigg[\frac{\partial R^{(\tau_1)}}{\partial n} - \frac{\partial R^{(\tau_2)}}{\partial n} + i(\tau_1 \cdot n)R^{(\tau_1)} - i(\tau_2 \cdot n)R^{(\tau_2)} \\ + i(\tau_2 \cdot n)R^{(\tau_2)} - i(\tau_1 \cdot n)R^{(\tau_2)}\Bigg]\bar{v}, \tag{2.135}$$

and bring it over to the RHS. The bottom row is estimated by $C|\tau_1 - \tau_2|\ \|\nabla v\|_{L^2(Q)}$, by the Trace theorem and Poincaré inequality. By the problem (2.130), the top row equals

$$\int_\Gamma \Bigg\{\mu^{(\tau_2)}\Big(1 + (it_2)u_1^{(\tau_2)} + (it_2)^2 u_2^{(\tau_2)} + R^{(\tau_2)}\Big) - (it_2)^2\alpha_2^{(\tau_2)}\Big(1 + i(\tau_2 \cdot n)u_2^{(\tau_2)}\Big) \\ -\mu^{(\tau_1)}\Big(1 + (it_1)u_1^{(\tau_1)} + (it_1)^2 u_2^{(\tau_1)} + R^{(\tau_1)}\Big) + (it_1)^2\alpha_2^{(\tau_1)}\Big(1 + i(\tau_1 \cdot n)u_2^{(\tau_1)}\Big)\Bigg\}\bar{v}. \tag{2.136}$$

Excluding the term $\int\{\mu^{(\tau_2)} - \mu^{(\tau_1)} + \mu^{(\tau_2)}R^{(\tau_2)} - \mu^{(\tau_1)}R^{(\tau_1)}\}\bar{v}$, the rest are once again estimated by $C[|t_1 - t_2| + |\theta_1 - \theta_2|]\ \|\nabla v\|_{L^2(Q)}$, using the cancellation trick. The excluded term may be estimated by the cancellation trick applied to the third and fourth term as follows:

$$\begin{aligned}\int_\Gamma \Big\{\mu^{(\tau_2)} - \mu^{(\tau_1)} + \mu^{(\tau_2)}R^{(\tau_2)} - \mu^{(\tau_1)}R^{(\tau_1)}\Big\}\bar{v} \\ &\leq C|\mu^{(\tau_2)} - \mu^{(\tau_1)}|\ \|\nabla v\|_{L^2(Q)} + Ct_1^2\|\nabla v\|^2_{L^2(Q)} \\ &\leq C\Big[|\theta_2 - \theta_1| + |t_2 - t_1|\Big]\|\nabla v\|_{L^2(Q)} + Ct_1\|\nabla v\|^2_{L^2(Q)}.\end{aligned} \tag{2.137}$$

What is left of the LHS is just $\|(\nabla + i\tau_1)v\|^2_{L^2(Q)}$. Since this equals

$$\|\nabla v\|^2_{L^2(Q)} + \int_Q 2\mathrm{Re}(i\nabla v \cdot \tau_1\bar{v}) + t_1^2\|v\|^2_{L^2(Q)},$$

the second and third term may be absorbed into the term $Ct_1\|\nabla v\|^2_{L^2(Q)}$ on the RHS. Overall, we obtain

$$(1 - Ct_1)\|\nabla v\|^2_{L^2(Q)} \leq C\Big[|\theta_2 - \theta_1| + |t_2 - t_1|\Big]\|\nabla v\|_{L^2(Q)} \tag{2.138}$$

$$\Leftrightarrow \qquad \|\nabla v\|_{L^2(Q)} \leq C_{\tau_1}\Big[|\theta_2 - \theta_1| + |t_2 - t_1|\Big]. \tag{2.139}$$

The equivalence is only valid when $C_{\tau_1} > 0$, which is the case whenever $|\tau_1| = t_1$ is small enough. We have therefore verified the local Lipschitz property (2.99), *for small values of* τ_1.

(Step 4f: $R^{(\tau_2)} - R^{(\tau_1)}$, for τ_1 bounded away from zero.) Fix $t_0 > 0$ and τ_0 with

$|\tau_0| = t_0$. Given $\tau = t\theta$, propose a power series expansion for $u^{(\tau)}$ about τ_0:

$$u^{(t\theta)} = u_0 + \big(i(|\tau| - |\tau_0|)\big)u_1 + \big(i(|\tau| - |\tau_0|)\big)^2 u_2 + \cdots = \sum_{j=0}^{\infty} u_j \big(i(t - t_0)\big)^j. \tag{2.140}$$

And similarly for $\mu^{(\tau)}$. Let us write $\tau_1 = (t_1 - t_0)\theta_1$ and $\tau_2 = (t_2 - t_0)\theta_2$. Note that most of the arguments follows through with minor to no modifications, until (2.137):

- Step 2: No change as we are equating powers of $(it - it_0)$, so the arguments involved do not depend on t.
- Step 3: Just replace t with $t - t_0$.
- Step 4a and 4b: No changes to the estimates, which are estimates by only $|\theta_2 - \theta_1|$.
- Step 4c: No changes to the uniform bound on R and μ, however we now crucially have that
$$0 \leq -\mu^{(\tau)} \leq C(t - t_0)^2,$$
and have assumed henceforth that any $\tau = (t - t_0)\theta$ we work with must be small enough.
- Step 4d: (Global) Lipschitz continuity of $\tau \mapsto \mu^{(\tau)}$ can be obtained with no modifications.

At (2.137) we have to replace t_1 by $t_1 - t_0$. We hence arrive at a version (2.138) centered at τ_0:

$$(1 - C(t_1 - t_0))\|\nabla v\|^2_{L^2(Q)} \leq C\Big[|\theta_2 - \theta_1| + |t_2 - t_1|\Big]\|\nabla v\|_{L^2(Q)}. \tag{2.141}$$

To obtain (2.139), we just have to assume that $|\tau_1| = t_1 - t_0$ is small enough. This means that we have proven the local Lipschitz property for all $\widetilde{\tau}_1 = t_1\theta_1$ residing in a neighbourhood of τ_0, and therefore verifying (2.99). This concludes our discussion on the term $\|u_2 - w\|$ in (2.97).

The term $\|u_1 - w\|$. We shall show that

$$\|u_1 - w\|_{L^2(Q_\star)} \leq C_{\tau_1}|\tau_2 - \tau_1| \qquad \text{(locally Lipschitz at } \tau_1.)$$

Firstly, $u_1 - w$ solves the following BVP

$$\begin{cases} -(\nabla + i\tau_2)^2(u_1 - w) + \Big[(\nabla + i\tau_2)^2 - (\nabla + i\tau_1)^2\Big]u_1 = 0 & \text{in } Q_\star, \\ u_1 - w = 0 & \text{on } \Gamma_\bullet, \end{cases} \tag{2.142}$$

with periodic BCs on ∂Q, if $Q_\star = Q_{\text{stiff-ls}}$. By testing the above BVP against $u_1 - w \in H^1(Q_\star)$, the weak formulation gives

$$\begin{aligned} &\|(\nabla + i\tau_2)(u_1 - w)\|^2_{L^2(Q_\star)} \\ &= \Big((\nabla + i\tau_2)u_1, (\nabla + i\tau_2)(u_1 - w)\Big) - \Big((\nabla + i\tau_1)u_1, (\nabla + i\tau_1)(u_1 - w)\Big) \\ &= (\nabla u_1, -i\tau_2 w + i\tau_1 w + i\tau_2 u_1 - i\tau_1 u_1) \end{aligned}$$

$$
\begin{aligned}
&+ (i\tau_1 u_1 - i\tau_2 u_1, \nabla w) + (i\tau_2 u_1 - i\tau_1 u_1, \nabla u_1) \\
&+ (i\tau_2 u_1, i\tau_2 u_1) - (i\tau_1 u_1, i\tau_1 u_1) - (i\tau_2 u_1, i\tau_2 w) + (i\tau_1 u_1, i\tau_1 w) \\
\le\ & |\tau_2 - \tau_1|\ (|\nabla u_1|, |u_1 - w|) + |\tau_2 - \tau_1|\ (|u_1|, |\nabla(w - u_1)|) + \int_Q \left[(|\tau_2|^2 - |\tau_1|^2)\ |u_1|\ |u_1 - w| \right] \\
& \qquad \text{by Cauchy-Schwarz on } \mathbb{R}^d. \\
\le\ & C|\tau_2 - \tau_1| \left[\|\nabla u_1\|\ \|u_1 - w\| + \|u_1\|\ \|\nabla(w - u_1)\| + \|u_1\|\ \|u_1 - w\| \right] \\
& \qquad \text{by Cauchy-Schwarz on } L^2(Q_\star). \\
\le\ & C|\tau_2 - \tau_1| \left[\|\nabla u\|_{L^2} + \|u\|_{L^2} \right] \|\nabla(u_1 - w)\|_{L^2}, \\
& \qquad \text{by the Poincaré inequality applied to } u_1 - w, \text{ which has zero trace.} \\
\le\ & C_{\tau_1} |\tau_2 - \tau_1|\ \|\nabla(u_1 - w)\|_{L^2} \\
& \qquad \text{as } u_1 \in H^1. \text{ Note that the BVP for } u_1 \text{ depends on } \tau_1. \\
\le\ & C_{\tau_1} |\tau_2 - \tau_1| \left(\|(\nabla + i\tau_1)(u_1 - w)\|_{L^2} + |\tau_1| \cdot \|u_1 - w\|_{L^2} \right) \\
\le\ & \frac{1}{4\varepsilon'} \|(\nabla + i\tau_1)(u_1 - w)\|_{L^2}^2 + \varepsilon' C_{\tau_1}^2 |\tau_2 - \tau_1|^2 + C_{\tau_1} |\tau_2 - \tau_1|\ |\tau_1|\ \|u_1 - w\|_{L^2}. \qquad (2.143)
\end{aligned}
$$

Rearrange,

$$
\left(1 - \frac{1}{4\varepsilon'}\right) \|(\nabla + i\tau_1)(u_1 - w)\|_{L^2(Q)}^2 \le C_{\tau_1} |\tau_2 - \tau_1|\ \|u_1 - w\|_{L^2} + C_{\tau_1} \varepsilon' |\tau_2 - \tau_1|^2. \qquad (2.144)
$$

On the other hand, since $u_1 - w$ has trace zero, the **Poincaré inequality with τ for H_0^1 functions** applies to give a lower bound on the LHS. Therefore, by picking a suitable $\varepsilon' > 0$, we arrive at

$$
C_1 \|u_1 - w\|_{L^2}^2 \le C_2 |\tau_2 - \tau_1|\ \|u_1 - w\|_{L^2} + C_3 |\tau_2 - \tau_1|^2, \quad C_1, C_2, C_3 > 0 \text{ depends on } \tau_1 \text{ only.} \qquad (2.145)
$$

This is a quadratic inequality in $\|u_1 - w\|_{L^2}$, with positive coefficients. We therefore conclude that

$$
\begin{aligned}
0 \le \|u_1 - w\|_{L^2} &\le \frac{1}{2C_1} \left[C_2 |\tau_2 - \tau_1| + \sqrt{(C_2 |\tau_2 - \tau_1|)^2 + 4 C_1 C_3 |\tau_2 - \tau_1|^2} \right] \\
&\le C|\tau_2 - \tau_1|, \qquad \text{where } C \text{ depends on } \tau_1 \text{ but not } \tau_2.
\end{aligned} \qquad (2.146)
$$

This completes the proof. □

Remark. In the proof of Proposition 2.3.5, we have used variants of the Poincaré inequality. In total, we have used a "Poincaré inequality with τ" (i.e. for the operator $-(\nabla + i\tau)^2$) for

- $u \in H_0^1$, for all $\tau \in \overline{Q'}$. This is a consequence of Proposition 2.2.4.
- $u \in H^1$, $\int u = 0$, for small τ. This is the Poincaré-Writinger inequality for $\tau = 0$, and can be extended to a neighborhood of $\tau = 0$ by a continuity argument similar to Step 4d. ◦

2.4 Identifying a suitable homogenized operator

The task now is to identify an operator $\widehat{A}_{\beta_0,\beta_1}$ that is $O(\varepsilon^2)$ close to $A_\varepsilon^{(\tau)} = \widehat{A}_{0,I}$ in the norm-resolvent sense, by using Theorem 2.3.4. To ensure that $\widehat{A}_{\beta_0,\beta_1}$ is well defined, we need to check that (i) β_0 and β_1 satisfies the domain considerations, and (ii) $\beta_0 + \beta_1 M_\varepsilon^{(\tau)}(z)$ is boundedly invertible. Here we record a useful observation that is used for checking (ii):

Lemma 2.4.1. For $z \in K_\sigma$, we have

$$-\left(\overline{\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)}\right)^{-1} \mathcal{P}^{(\tau)} = -\mathcal{P}^{(\tau)} \left(\mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z) \mathcal{P}^{(\tau)}\right)^{-1} \mathcal{P}^{(\tau)}. \tag{2.147}$$

Proof. Note that $\mathcal{P}_\perp^{(\tau)}$ and $\mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)$ are bounded operators, hence the sum is closed. That $\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)$ is boundedly invertible follows from the second equality of (2.77), as

$$-\left(\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)\right)^{-1} = -\begin{pmatrix} \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)} \mathcal{P}^{(\tau)} & \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)} \mathcal{P}_\perp^{(\tau)} \\ 0 & I \end{pmatrix}^{-1} = -\begin{pmatrix} \mathbb{A}^{-1} & -\mathbb{A}^{-1}\mathbb{B} \\ 0 & I \end{pmatrix}, \tag{2.148}$$

which is bounded by Theorem 2.3.4. Now applying $\mathcal{P}^{(\tau)}$ on the right, we obtain

$$-\left(\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)\right)^{-1} \mathcal{P}^{(\tau)} = -\begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.149}$$

This is precisely the RHS of (2.147), completing the proof. □

Remark. We have abused notation when writing $\mathcal{P}^{(\tau)}$ in (2.147). To be precise,

$$-\big(\underbrace{\mathcal{P}_\perp^{(\tau)}}_{\mathcal{E}\to\mathcal{E}} + \underbrace{\mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z)}_{\mathcal{E}\to\mathcal{E}}\big)^{-1} \underbrace{\mathcal{P}^{(\tau)}}_{\mathcal{E}\to\mathcal{E}} = -\underbrace{\mathcal{P}^{(\tau)}}_{\mathcal{P}^{(\tau)}\mathcal{E}\to\mathcal{E}} \big(\underbrace{\mathcal{P}^{(\tau)} M_\varepsilon^{(\tau)}(z) \mathcal{P}^{(\tau)}}_{\mathcal{P}^{(\tau)}\mathcal{E}\to\mathcal{P}^{(\tau)}\mathcal{E}}\big)^{-1} \underbrace{\mathcal{P}^{(\tau)}}_{\mathcal{E}\to\mathcal{P}^{(\tau)}\mathcal{E}}. \qquad \circ$$

Our first attempt on identifying a suitable homogenized operator is

Theorem 2.4.2. There exist $C > 0$, independent of $\varepsilon > 0$ (assumed to be small enough), $z \in K_\sigma$, and $\tau \in Q'$, such that

$$\left\| (A_\varepsilon^{(\tau)} - z)^{-1} - \left(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}_\perp^{(\tau)},\mathcal{P}^{(\tau)}} - z\right)^{-1} \right\|_{\mathcal{H}\to\mathcal{H}} \le C\varepsilon^2. \tag{2.150}$$

The operator $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}_\perp^{(\tau)},\mathcal{P}^{(\tau)}}$ is constructed relative to the triple $(A_{\varepsilon,0}^{(\tau)}, \Lambda_\varepsilon^{(\tau)}, \Pi^{(\tau)})$ with $\mathcal{H} = L^2(Q)$ and boundary space $\mathcal{E} = L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$. Furthermore, $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}_\perp^{(\tau)},\mathcal{P}^{(\tau)}}$ is self-adjoint.

Proof. The inequality follows by Krein's formula (Theorem 2.2.16), the estimate on $M_\varepsilon^{(\tau)}(z)$ (Theorem 2.3.4), and the identity (2.147). Self-adjointness of $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}_\perp^{(\tau)},\mathcal{P}^{(\tau)}}$ follows from [47, Corollary 5.8]. □

Remark. This is exactly the same formula as in [35, Theorem 3.1]. In other words, we have managed to push through the analysis of [35] for our setup, with $\mathcal{P}^{(\tau)}$ now being two-dimensional, corresponding to the two stiff components. ◦

While the operator $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}}$ satisfies the first criterion (self-adjointness) of a homogenized operator, it is unclear what the action of $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}}$ is, since it requires us to convert the boundary condition $\mathcal{P}^{(\tau)}_\perp\Gamma^{(\tau)}_0 + \mathcal{P}^{(\tau)}\Gamma^{(\tau)}_{\varepsilon,1} = 0$ into the action of $\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}}$. The goal now is to build on our result and identify other $O(\varepsilon^2)$-close operators whose actions can be more easily written down.

2.4.1 An "observer" in the soft component

We begin this section by making the following definitions

Definition 2.4.3. $M^{\text{stiff},(\tau)}_\varepsilon(z) := M^{\text{stiff-int},(\tau)}_\varepsilon(z)\mathcal{P}_{\text{int}} + M^{\text{stiff-ls},(\tau)}_\varepsilon(z)\mathcal{P}_{\text{ls}}$.

Definition 2.4.4. For $z \in \rho(A^{(\tau)}_\varepsilon)$, set $R^{(\tau)}_\varepsilon(z) := P_{\text{soft}}(A^{(\tau)}_\varepsilon - z)^{-1}P_{\text{soft}}$.

We will refer to $R^{(\tau)}_\varepsilon(z)$ as the *generalized resolvent* of $A^{(\tau)}_\varepsilon$ at z, with respect to $L^2(Q_{\text{soft}})$. The term "generalized resolvent" refers to the fact that it is the resolvent of some operator on a larger space. This is not to be confused with pseudoresolvents in Theorem 2.2.16. Let us give an interpretation of $R^{(\tau)}_\varepsilon(z)$.

> The resolvent $(A - z)^{-1}$ takes $f \in L^2$ to $u = (A - z)^{-1}f$, which is the unique solution to the BVP $(A - z)u = f$. Since f can be viewed as a forcing term for our system, we may interpret $R^{(\tau)}_\varepsilon(z)$ as an observer living in $L^2(Q_{\text{soft}})$: The goal of the observer is to figure out what happens to the output u of the system, for each input f. However, the observer only has partial information of the system, due to the constraint that the input f must lie in $L^2(Q_{\text{soft}})$, and is only able to observe the part of u which lie in $L^2(Q_{\text{soft}})$.

This point of this section is to demonstrate we can draw conclusions on the full system using the partial information provided by $R^{(\tau)}_\varepsilon(z)$, as the missing pieces can be attributed to "error". Let us begin with an easy but important computation, which says that $R^{(\tau)}_\varepsilon$ is itself a solution operator for some abstract BVP on $L^2(Q_{\text{soft}})$:

Proposition 2.4.5. We have,

$$R^{(\tau)}_\varepsilon(z) = \left(\widehat{A}^{\text{soft},(\tau)}_{M^{\text{stiff},(\tau)}_\varepsilon(z),I} - z\right)^{-1}, \tag{2.151}$$

where $\widehat{A}^{\text{soft},(\tau)}_{M^{\text{stiff},(\tau)}_\varepsilon(z),I}$ is constructed from the triple $(A^{\text{soft},(\tau)}_0, \Lambda^{\text{soft},(\tau)}, \Pi^{\text{soft},(\tau)})$ with $L^2(Q_{\text{soft}})$ and boundary space $L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})$. In other words $R^{(\tau)}_\varepsilon(z)$ is the solution operator of the BVP:

$$\begin{cases} \left(-(\nabla + i\tau)^2 - z\right) u = f & \text{in } Q_{\text{soft}}, \\ \partial^{(\tau)}_{n_{\text{soft}}} u = -M^{\text{stiff-int},(\tau)}_\varepsilon(z)u & \text{on } \Gamma_{\text{int}}, \\ \partial^{(\tau)}_{n_{\text{soft}}} u = -M^{\text{stiff-ls},(\tau)}_\varepsilon(z)u & \text{on } \Gamma_{\text{ls}}, \end{cases} \tag{2.152}$$

which is to be rigorously interpreted in terms of the following system

$$\begin{cases} (\widehat{A}^{\text{soft},(\tau)} - z)u = f, \\ \Gamma_1^{\text{soft},(\tau)} u = -M_\varepsilon^{\text{stiff},(\tau)}(z)\Gamma_0^{\text{soft},(\tau)} u. \end{cases} \tag{2.153}$$

Here, $f \in L^2(Q_{\text{soft}})$. (Note the z-dependent boundary conditions.)

Proof. We have

$$\begin{aligned} R_\varepsilon^{(\tau)}(z) &= P_{\text{soft}}(A_{\varepsilon,0}^{(\tau)} - z)^{-1}P_{\text{soft}} - P_{\text{soft}}S_\varepsilon^{(\tau)}(z)\left(M_\varepsilon^{(\tau)}(z)\right)^{-1}\left(S_\varepsilon^{(\tau)}(\bar{z})\right)^* P_{\text{soft}} \\ &= (A_0^{\text{soft},(\tau)} - z)^{-1} - S^{\text{soft},(\tau)}(z)\left(M_\varepsilon^{(\tau)}(z)\right)^{-1}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^* \\ &= (A_0^{\text{soft},(\tau)} - z)^{-1} - S^{\text{soft},(\tau)}(z)\left(M_\varepsilon^{\text{stiff},(\tau)}(z) + M^{\text{soft},(\tau)}(z)\right)^{-1}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^*. \end{aligned} \tag{2.154}$$

The first equality follows by Corollary 2.2.17. For the second equality, $P_{\text{soft}}(A_{\varepsilon,0}^{(\tau)} - z)^{-1}P_{\text{soft}} = (A_0^{\text{soft},(\tau)} - z)^{-1}$ follows directly by construction, and $P_{\text{soft}}S_\varepsilon^{(\tau)}(z) = S^{\text{soft},(\tau)}(z)$ by (2.63). The final equality follows by Proposition 2.2.18. The assertion on the solution operator then follows by Theorem 2.2.16. □

Remark. In moving from the $L^2(Q)$ to $L^2(Q_{\text{soft}})$, our boundary conditions changed from $(\beta_0, \beta_1) = (0, I)$ to $(\beta_0, \beta_1) = (M^{\text{stiff}}(z), I)$. Using again the analogy of an observer, this means that the observer living in $L^2(Q_{\text{soft}})$ is able to feel the effect of the "stiff" part of the system through the (z-dependent) boundary conditions. ◦

Recall the computations in Lemma 2.4.1: when converting $-(\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)}M_\varepsilon^{(\tau)}(z))^{-1}\mathcal{P}^{(\tau)}$ to $-\mathcal{P}^{(\tau)}(\mathcal{P}^{(\tau)}M_\varepsilon^{(\tau)}(z)\mathcal{P}^{(\tau)})^{-1}\mathcal{P}^{(\tau)}$, we are only interested in the left column of the block matrix. This suggests that we could modify the top right entry to our desire. (The bottom right entry should be kept as I to ensure invertibility of the matrix.) In particular,

$$\begin{aligned} &\begin{pmatrix} \mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z)\mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)} & \mathcal{P}^{(\tau)}M_\varepsilon^{(\tau)}(z)\mathcal{P}_\perp^{(\tau)} \\ 0 & I \end{pmatrix}^{-1}\mathcal{P}^{(\tau)} \\ &= \begin{pmatrix} \mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z)\mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)} & \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}_\perp^{(\tau)} \\ 0 & I \end{pmatrix}^{-1}\mathcal{P}^{(\tau)} = \begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.155}$$

This suggests us to make the following definition:

Definition 2.4.6. $R_{\varepsilon,\text{eff}}^{(\tau)}(z) := \left(\widehat{A}^{\text{soft},(\tau)}_{\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z)\mathcal{P}^{(\tau)}, \mathcal{P}^{(\tau)}} - z\right)^{-1}$.

To check that the choice β_0 and β_1 for $R_{\varepsilon,\text{eff}}^{(\tau)}(z)$ is valid, we note that (i) $\beta_0 + \beta_1 M^{\text{soft}(\tau)}(z)$ is a sum of bounded operators with maximal domain hence it is closed, (ii) the invertibility of the matrix follows because it is upper triangular, (iii) boundedness of the inverse follows from the estimate for $\mathbb{A}^{-1}$ in Theorem 2.3.4. The observation on the equality of matrices is used in the following result:

Proposition 2.4.7. We have the following estimate, uniform in $z \in K_\sigma$ and $\tau \in Q'$:

$$R_\varepsilon^{(\tau)}(z) - R_{\varepsilon,\text{eff}}^{(\tau)}(z) = O(\varepsilon^2).$$

Proof. See Appendix D. □

Remark.
- We have two triples: one on the full space $L^2(Q)$ and one on the soft component $L^2(Q_{\text{soft}})$. It is Krein's formula (Theorem 2.2.16) and our specific setup (in particular the results from Section 2.2.3) that enables us to easily pass between the two sets of triples.
- A byproduct of the proof of Proposition 2.4.7 is that

$$P_{\text{soft}}\left(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}_\perp^{(\tau)},\mathcal{P}^{(\tau)}} - z\right)^{-1} P_{\text{soft}} = \left(\widehat{A}^{\text{soft},(\tau)}_{\mathcal{P}_\perp^{(\tau)}+\mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z),\mathcal{P}^{(\tau)}} - z\right)^{-1}$$
$$= \left(\widehat{A}^{\text{soft},(\tau)}_{\mathcal{P}_\perp^{(\tau)}+\mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z)\mathcal{P}^{(\tau)},\mathcal{P}^{(\tau)}} - z\right)^{-1} = R_{\varepsilon,\text{eff}}^{(\tau)}(z).$$

  We have discussed extensively why we could discard $\mathcal{P}^{(\tau)}M_\varepsilon^{\text{stiff},(\tau)}(z)\mathcal{P}_\perp^{(\tau)}$, but the *reason* for doing so is so that we can work with an operator on a finite-dimensional space $\mathcal{P}^{(\tau)}\mathcal{E}$, this is crucial in our proof of self-adjointness of the operator $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$ (to be defined later). ○

We now turn our attention to discuss dilations of $R_{\varepsilon,\text{eff}}^{(\tau)}(z)$. We would like to guess an operator $\mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(z)$ (note the use of calligraphic font) on the full space $L^2(Q)$ that is $O(\varepsilon^2)$ close to $(A_\varepsilon^{(\tau)} - z)^{-1}$. The hope is that $\mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(z)$ is the resolvent of a self-adjoint operator whose action depends on ε in a clear way. One necessary condition is $\mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(z)^* = \mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z})$. The guess is as follows:

Definition 2.4.8. Let $\mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(z)$ be the operator on $L^2(Q)$ defined by the following formula with respect to the decomposition $\mathcal{H} = L^2(Q_{\text{soft}}) \oplus L^2(Q_{\text{stiff-int}}) \oplus L^2(Q_{\text{stiff-ls}})$:

$$\mathcal{R}_{\varepsilon,\text{eff}}^{(\tau)}(z) = \begin{pmatrix} R_{\varepsilon,\text{eff}}^{(\tau)}(z) & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2.156}$$

where

$$a_{21} = \Pi^{\text{stiff-int},(\tau)}k^{(\tau)}(z)\left[R_{\varepsilon,\text{eff}}^{(\tau)}(z) - (A_0^{\text{soft},(\tau)} - z)^{-1}\right]$$
$$a_{31} = \Pi^{\text{stiff-ls},(\tau)}k^{(\tau)}(z)\left[R_{\varepsilon,\text{eff}}^{(\tau)}(z) - (A_0^{\text{soft},(\tau)} - z)^{-1}\right]$$
$$a_{12} = \left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-int},(\tau)}\right)^*$$
$$a_{22} = \Pi^{\text{stiff-int},(\tau)}k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-int},(\tau)}\right)^*$$
$$a_{32} = \Pi^{\text{stiff-ls},(\tau)}k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-int},(\tau)}\right)^*$$
$$a_{13} = \left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-ls},(\tau)}\right)^*$$
$$a_{23} = \Pi^{\text{stiff-int},(\tau)}k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-ls},(\tau)}\right)^*$$

$$a_{33} = \Pi^{\text{stiff-ls},(\tau)} k^{(\tau)}(z) \left(k^{(\tau)}(\bar{z}) \left[R^{(\tau)}_{\varepsilon,\text{eff}}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1} \right] \right)^* \left(\Pi^{\text{stiff-ls},(\tau)} \right)^*$$

where $k^{(\tau)}(z) := \Gamma_0^{\text{soft},(\tau)}|_{\mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \text{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})}$.

Proposition 2.4.9. We have the following estimate, uniform in $z \in K_\sigma$ and $\tau \in Q'$:

$$(A_\varepsilon^{(\tau)} - z)^{-1} - \mathcal{R}^{(\tau)}_{\varepsilon,\text{eff}}(z) = O(\varepsilon^2).$$

Proof. This is verified entry-wise, and only requires minimal modifications to the proof in [35, Theorem 3.9]. See Appendix D. □

Recall that for $R^{(\tau)}_{\varepsilon,\text{eff}}(z)$ we have $\beta_0 = \mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{\text{stiff},(\tau)}(z) \mathcal{P}^{(\tau)}$. We can further simplify the term $\mathcal{P}^{(\tau)} M_\varepsilon^{\text{stiff},(\tau)}(z) \mathcal{P}^{(\tau)}$ by using Lemma 2.2.21 and (2.62):

$$\begin{aligned}
\mathcal{P}^{(\tau)} M_\varepsilon^{\text{stiff},(\tau)}(z) \mathcal{P}^{(\tau)} &= \mathcal{P}^{(\tau)} M_\varepsilon^{\text{stiff-int},(\tau)}(z) \mathcal{P}_{\text{int}} \mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)} M_\varepsilon^{\text{stiff-ls},(\tau)}(z) \mathcal{P}_{\text{ls}} \mathcal{P}^{(\tau)} \\
&= \mathcal{P}^{(\tau)}_{\text{stiff-int}} M_\varepsilon^{\text{stiff-int},(\tau)}(z) \mathcal{P}^{(\tau)}_{\text{stiff-int}} \oplus \mathcal{P}^{(\tau)}_{\text{stiff-ls}} M_\varepsilon^{\text{stiff-ls},(\tau)}(z) \mathcal{P}^{(\tau)}_{\text{stiff-ls}} \\
&= \left(\mathcal{P}^{(\tau)}_{\text{stiff-int}} \Lambda_\varepsilon^{\text{stiff-int},(\tau)} \mathcal{P}^{(\tau)}_{\text{stiff-int}} + z \mathcal{P}^{(\tau)}_{\text{stiff-int}} (\Pi^{\text{stiff-int},(\tau)})^* \Pi^{\text{stiff-int},(\tau)} \mathcal{P}^{(\tau)}_{\text{stiff-int}} \right) \\
&\quad \oplus \left(\mathcal{P}^{(\tau)}_{\text{stiff-ls}} \Lambda_\varepsilon^{\text{stiff-ls},(\tau)} \mathcal{P}^{(\tau)}_{\text{stiff-ls}} + z \mathcal{P}^{(\tau)}_{\text{stiff-ls}} (\Pi^{\text{stiff-ls},(\tau)})^* \Pi^{\text{stiff-ls},(\tau)} \mathcal{P}^{(\tau)}_{\text{stiff-ls}} \right) + O(\varepsilon^2) \\
&= \mathcal{P}^{(\tau)} \left(\Lambda_\varepsilon^{\text{stiff-int},(\tau)} \oplus \Lambda_\varepsilon^{\text{stiff-ls},(\tau)} \right) \mathcal{P}^{(\tau)} \\
&\quad + z \mathcal{P}^{(\tau)} \left((\Pi^{\text{stiff-int},(\tau)})^* \Pi^{\text{stiff-int},(\tau)} \oplus (\Pi^{\text{stiff-ls},(\tau)})^* \Pi^{\text{stiff-ls},(\tau)} \right) \mathcal{P}^{(\tau)} + O(\varepsilon^2) \\
&= \mathcal{P}^{(\tau)} \left(\Lambda_\varepsilon^{\text{stiff-int},(\tau)} \oplus \Lambda_\varepsilon^{\text{stiff-ls},(\tau)} \right) \mathcal{P}^{(\tau)} \\
&\quad + z \mathcal{P}^{(\tau)} \left((\Pi^{\text{stiff-int},(\tau)} \oplus \Pi^{\text{stiff-ls},(\tau)})^* (\Pi^{\text{stiff-int},(\tau)} \oplus \Pi^{\text{stiff-ls},(\tau)}) \right) \mathcal{P}^{(\tau)} + O(\varepsilon^2). \qquad (2.157)
\end{aligned}$$

This is helpful as it separates the term that depends on ε^{-2} (the stiff DtN maps), from the terms that are uniformly bounded (the stiff harmonic lifts). We therefore define:

Definition 2.4.10. We define $R^{(\tau)}_{\varepsilon,\text{hom}}(z)$ as the following operator on $L^2(Q_{\text{soft}})$:

$$\left(\widehat{A}^{\text{soft},(\tau)}_{\mathcal{P}_\perp^{(\tau)} + \mathcal{P}^{(\tau)} \left[\left(\Lambda_\varepsilon^{\text{stiff-int},(\tau)} \oplus \Lambda_\varepsilon^{\text{stiff-ls},(\tau)} \right) + z (\Pi^{\text{stiff-int},(\tau)} \oplus \Pi^{\text{stiff-ls},(\tau)})^* (\Pi^{\text{stiff-int},(\tau)} \oplus \Pi^{\text{stiff-ls},(\tau)}) \right] \mathcal{P}^{(\tau)}, \mathcal{P}^{(\tau)}} - z \right)^{-1},$$

and set $\mathcal{R}^{(\tau)}_{\varepsilon,\text{hom}}(z)$ as the operator on $L^2(Q)$ defined by (2.156), but with all the terms involving "$R^{(\tau)}_{\varepsilon,\text{eff}}$" to be replaced by $R^{(\tau)}_{\varepsilon,\text{hom}}$.

For the validity of the choice (β_0, β_1), we use: the validity of (β_0, β_1) for $R^{(\tau)}_{\varepsilon,\text{eff}}(z)$, and the observation that $\mathbb{A}$ is bounded below uniformly in ε, τ, and z. (Details are provided in the proof of Theorem 2.4.20 later ("top left entry").)

We conclude this section with the following result:

Theorem 2.4.11. We have the following estimate, uniform in $z \in K_\sigma$ and $\tau \in Q'$:

$$(A_\varepsilon^{(\tau)} - z)^{-1} - \mathcal{R}^{(\tau)}_{\varepsilon,\text{hom}}(z) = O(\varepsilon^2).$$

Proof. This follows from $R^{(\tau)}_{\varepsilon,\mathrm{eff}}(z) - R^{(\tau)}_{\varepsilon,\mathrm{hom}}(z) = O(\varepsilon^2)$, which can be checked, for instance by the resolvent identity applied to $-\overline{(\beta_0 + \beta_1 M(z))}^{-1}$ (this is boundedly invertible by construction). □

2.4.2 Self-adjointness of $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ (Preliminaries)

In the previous section, we have identified a candidate operator $\mathcal{R}^{(\tau)}_{\varepsilon,\mathrm{hom}}(z)$ on $L^2(Q)$ which could serve as the resolvent of some self-adjoint operator which will be denoted later by $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$. However, the term $k^{(\tau)}(z)$ has finite-range, since $\mathcal{P}^{(\tau)}\mathcal{E}$ is finite dimensional. This implies that self-adjointness on $L^2(Q)$ is impossible, as we will be left with non-zero defect indices. (Zero defect indices are a requirement for self-adjointness, see [22, Proposition 3.8].) We may still pursue the question of self-adjointness, but on some subspace of $L^2(Q)$. This motivates us to define:

Definition 2.4.12. Write $\breve{\mathcal{E}}^{(\tau)} := \mathcal{P}^{(\tau)}\mathcal{E}$ for the truncated boundary space. Now introduce the following truncated operators:

$$
\begin{aligned}
\breve{\Pi}^{\mathrm{soft},(\tau)} &:= \Pi^{\mathrm{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}, & \breve{\Lambda}^{\mathrm{soft},(\tau)} &:= \mathcal{P}^{(\tau)}\Lambda^{\mathrm{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}},\\
\breve{\Pi}^{\mathrm{stiff\text{-}int},(\tau)} &:= \Pi^{\mathrm{stiff\text{-}int},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}, & \breve{\Lambda}^{\mathrm{stiff\text{-}int},(\tau)}_{\varepsilon} &:= \mathcal{P}^{(\tau)}\Lambda^{\mathrm{stiff\text{-}int},(\tau)}_{\varepsilon}|_{\breve{\mathcal{E}}^{(\tau)}},\\
\breve{\Pi}^{\mathrm{stiff\text{-}ls},(\tau)} &:= \Pi^{\mathrm{stiff\text{-}ls},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}, & \breve{\Lambda}^{\mathrm{stiff\text{-}ls},(\tau)}_{\varepsilon} &:= \mathcal{P}^{(\tau)}\Lambda^{\mathrm{stiff\text{-}ls},(\tau)}_{\varepsilon}|_{\breve{\mathcal{E}}^{(\tau)}}.
\end{aligned}
$$

Set $\breve{\Pi}^{\mathrm{stiff},(\tau)} := \breve{\Pi}^{\mathrm{stiff\text{-}int},(\tau)} \oplus \breve{\Pi}^{\mathrm{stiff\text{-}ls},(\tau)}$ and $\breve{\Lambda}^{\mathrm{stiff},(\tau)}_{\varepsilon} := \breve{\Lambda}^{\mathrm{stiff\text{-}int},(\tau)}_{\varepsilon} \oplus \breve{\Lambda}^{\mathrm{stiff\text{-}ls},(\tau)}_{\varepsilon}$. By the truncated DtN maps $\breve{\Lambda}$, we mean its continuous extension to the full subspace $\breve{\mathcal{E}}$. (Recall Lemma 2.3.2 and the comment thereafter.)

Remark. As the goal of this section is to prove self-adjointness *for each* ε *and* τ, the dependence on ε and τ is not important here and we will drop them where convenient. ∘

As $\mathcal{P}^{(\tau)}$ is a spectral projection with respect to the stiff DtN maps, we immediately see that $\breve{\Lambda}^{\mathrm{stiff\text{-}int}}$ and $\breve{\Lambda}^{\mathrm{stiff\text{-}ls}}$ are self-adjoint. In fact, $\breve{\Lambda}^{\mathrm{soft}}$ is self-adjoint too, as it is symmetric on the finite dimensional space $\breve{\mathcal{E}}$.

The lifts $\breve{\Pi}^{\mathrm{soft}}$, $\breve{\Pi}^{\mathrm{stiff\text{-}int}}$, and $\breve{\Pi}^{\mathrm{stiff\text{-}ls}}$ are injective and bounded since they are restrictions of operators that are so. We can turn it into a surjective map by restricting its codomain to:

Definition 2.4.13. Introduce the following subspaces of $\mathcal{H} = L^2(Q)$:

$$
\breve{\mathcal{H}}^{\mathrm{soft},(\tau)} := \mathrm{ran}(\breve{\Pi}^{\mathrm{soft},(\tau)}), \quad \breve{\mathcal{H}}^{\mathrm{stiff\text{-}int},(\tau)} := \mathrm{ran}(\breve{\Pi}^{\mathrm{stiff\text{-}int},(\tau)}), \quad \breve{\mathcal{H}}^{\mathrm{stiff\text{-}ls},(\tau)} := \mathrm{ran}(\breve{\Pi}^{\mathrm{stiff\text{-}ls},(\tau)}),
$$

and set $\breve{\mathcal{H}}^{\mathrm{stiff},(\tau)} = \breve{\mathcal{H}}^{\mathrm{stiff\text{-}int},(\tau)} \oplus \breve{\mathcal{H}}^{\mathrm{stiff\text{-}ls},(\tau)}$. (The orthogonality is a consequence of our setup.)

We use the ingredients above to define the following triple on the soft component with its auxiliary operators:

Definition 2.4.14. Consider the $(A_0^{\mathrm{soft},(\tau)}, \breve{\Lambda}^{\mathrm{soft},(\tau)}, \breve{\Pi}^{\mathrm{soft},(\tau)})$ on $L^2(Q_{\mathrm{soft}})$ and boundary space $\breve{\mathcal{E}}^{(\tau)}$. Construct the following operators in accordance with Definition 2.2.12:

$$
\widehat{\breve{A}}^{\mathrm{soft},(\tau)} : \mathcal{D}(A_0^{\mathrm{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\mathrm{soft},(\tau)} \to L^2(Q_{\mathrm{soft}}),
$$

$$\begin{aligned}\breve{\Gamma}_0^{\text{soft},(\tau)} &: \mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\text{soft},(\tau)} \to \breve{\mathcal{E}}^{(\tau)},\\ \breve{\Gamma}_1^{\text{soft},(\tau)} &: \mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \breve{\Pi}^{\text{soft},(\tau)} \mathcal{D}(\breve{\Lambda}^{\text{soft},(\tau)}) \to \breve{\mathcal{E}}^{(\tau)},\\ \breve{S}^{\text{soft},(\tau)}(z) &: \breve{\mathcal{E}}^{(\tau)} \to L^2(Q_{\text{soft}}),\\ \breve{M}^{\text{soft},(\tau)}(z) &: \mathcal{D}(\breve{\Lambda}^{\text{soft},(\tau)}) \to \breve{\mathcal{E}}^{(\tau)}.\end{aligned}$$

Definition 2.4.12 implies $\mathcal{D}(\breve{\Lambda}^{\text{soft},(\tau)}) = \breve{\mathcal{E}}^{(\tau)}$. This means that the domains of $\widehat{A}^{\breve{\text{soft}}}$, $\breve{\Gamma}_0^{\text{soft}}$, and $\breve{\Gamma}_1^{\text{soft}}$ coincide. This is a key assumption of the classical triple, which is not required here.

Remark. The following truncated triple is also available: $(A_{\varepsilon,0}^{\text{stiff-int},(\tau)} \oplus A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}, \breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)}, \breve{\Pi}^{\text{stiff},(\tau)})$ on $L^2(Q_{\text{stiff-int}}) \oplus L^2(Q_{\text{stiff-ls}})$ with boundary space $\breve{\mathcal{H}}^{\text{stiff},(\tau)}$, but we do not need them. ◦

We record some properties of the truncated triple in relation to its original counterpart.

Proposition 2.4.15.

1. $\breve{\Pi}^{\text{soft},(\tau)} : \breve{\mathcal{E}}^{(\tau)} \to \breve{\mathcal{H}}^{\text{soft},(\tau)}$ and $(\breve{\Pi}^{\text{stiff-int},(\tau)} \oplus \breve{\Pi}^{\text{stiff-ls},(\tau)}) : \breve{\mathcal{E}}^{(\tau)} \to \breve{\mathcal{H}}^{\text{stiff},(\tau)}$ are both bounded and boundedly invertible.
2. $\widehat{A}^{\breve{\text{soft}},(\tau)}$ is densely defined and closed.
3. $\breve{S}^{\text{soft},(\tau)}(z) = S^{\text{soft},(\tau)}(z)|_{\breve{\mathcal{E}}}$.
4. $\breve{M}^{\text{soft},(\tau)}(z) = \mathcal{P}^{(\tau)} M^{\text{soft},(\tau)}(z)|_{\breve{\mathcal{E}}}$, that is, $\breve{M}^{\text{soft},(\tau)}(z)$ is the compression of its original operator.
5. $\breve{\Gamma}_0^{\text{soft},(\tau)}$ and $\breve{\Gamma}_1^{\text{soft},(\tau)}$ are surjective mappings from $\mathcal{D}(\widehat{A}^{\breve{\text{soft}},(\tau)})$ to $\breve{\mathcal{E}}^{(\tau)}$. Furthermore, their restrictions to $\mathcal{D}(A_0^{\text{soft},(\tau)})$ are also surjective.
6. $\breve{\Gamma}_0^{\text{soft},(\tau)} = \Gamma_0^{\text{soft},(\tau)}|_{\mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\text{soft},(\tau)}}$ and $\breve{\Gamma}_1^{\text{soft},(\tau)} = \mathcal{P}^{(\tau)} \Gamma_1^{\text{soft},(\tau)}|_{\mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\text{soft},(\tau)}}$.

Proof. (1) We have already discussed the boundedness and invertibility of $\breve{\Pi}^{\text{soft}}$. The boundedness of $(\breve{\Pi}^{\text{soft}})^{-1}$ follows from the fact that $\breve{\mathcal{E}}$ is finite dimensional. The same argument holds for $(\breve{\Pi}^{\text{stiff-int},(\tau)} \oplus \breve{\Pi}^{\text{stiff-ls},(\tau)})$. (2) Density follows from the assumption that A_0^{soft} is densely defined. Closedness follows from the observation that the graph of $\widehat{A}^{\breve{\text{soft}},(\tau)}$ is the union of the graph of A_0^{soft} with $\breve{\mathcal{H}}^{\text{soft},(\tau)} \times \{0\}$, both of which are closed. (3) follows from the formula for $S(z)$ in Proposition 2.2.13(4) and the definition $\breve{\Pi}^{\text{soft}} = \Pi^{\text{soft}}|_{\breve{\mathcal{E}}}$. Similarly, (4) follows from Proposition 2.2.13(5) and the definitions $\breve{\Pi}^{\text{soft}} = \Pi^{\text{soft}}|_{\breve{\mathcal{E}}}$ and $\breve{\Lambda}^{\text{soft}} = \mathcal{P}^{(\tau)} \Lambda^{\text{soft}}|_{\breve{\mathcal{E}}}$. (5) Surjectivity of $\breve{\Gamma}_0^{\text{soft}}$ follows from the observation that $\breve{\Gamma}_0^{\text{soft}}$ is defined as the null extension of $(\breve{\Pi}^{\text{soft}})^{-1}$ (left inverse.) But $(\breve{\Pi}^{\text{soft}})^{-1}$ is in fact a two-sided inverse thanks to (1). Surjectivity of $\breve{\Gamma}_1^{\text{soft}}$ follows from: If $f \in L^2(Q_{\text{soft}})$, $\phi \in \breve{\mathcal{E}}$, then

$$\breve{\Gamma}_1^{\text{soft}}((A_0^{\text{soft}})^{-1} f + \breve{\Pi}^{\text{soft}} \phi) = (\breve{\Pi}^{\text{soft}})^* f + \breve{\Lambda}^{\text{soft}} \phi.$$

Surjectivity of $\breve{\Gamma}_1^{\text{soft}}$ is hence a consequence of surjectivity of $(\breve{\Pi}^{\text{soft}})^*$, which was established in (1). For (6), the claim on $\breve{\Gamma}_0^{\text{soft}}$ is immediate from the definitions. As for $\breve{\Gamma}_1^{\text{soft}}$, we can continue the computation above, to see that

$$\breve{\Gamma}_1^{\text{soft}}((A_0^{\text{soft}})^{-1} f + \breve{\Pi}^{\text{soft}} \phi) = \mathcal{P}^{(\tau)} (\Pi^{\text{soft}})^* f + \mathcal{P}^{(\tau)} \Lambda^{\text{soft}} \mathcal{P}^{(\tau)} \phi = \mathcal{P}^{(\tau)} \left[(\Pi^{\text{soft}})^* f + \Lambda^{\text{soft}} \mathcal{P}^{(\tau)} \phi \right].$$

The latter is precisely the action of $\mathcal{P}\Gamma_1^{\text{soft}}$ on $\mathcal{D}(A_0^{\text{soft}}) \dot{+} \breve{\mathcal{H}}^{\text{soft}}$. This completes the proof. $\square$

Remark. The closedness of $\widehat{\breve{A}}^{\text{soft},(\tau)}$ relies crucially on the fact that $\breve{\mathcal{H}}^{\text{soft},(\tau)} = \text{ran}(\breve{\Pi}^{\text{soft},(\tau)})$ is finite dimensional. For general triples, $\widehat{A}$ is not necessarily closed nor closable. $\circ$

To conclude the section, let us write down $\mathcal{R}_{\varepsilon,\text{hom}}^{(\tau)}(z)$ with respect to the truncated objects. We will do this with respect to the decomposition $\mathcal{H} = L^2(Q_{\text{soft}}) \oplus (L^2(Q_{\text{stiff-int}}) \oplus L^2(Q_{\text{stiff-ls}}))$:

$$\mathcal{R}_{\varepsilon,\text{hom}}^{(\tau)}(z) = \begin{pmatrix} R_{\varepsilon,\text{hom}}^{(\tau)}(z) & \left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{hom}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^* \breve{\Pi}^{\text{stiff},(\tau)*} \\ \breve{\Pi}^{\text{stiff},(\tau)} k^{(\tau)}(z)\left[R_{\varepsilon,\text{hom}}^{(\tau)}(z) - (A_0^{\text{soft},(\tau)} - z)^{-1}\right] & \breve{\Pi}^{\text{stiff},(\tau)} k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{hom}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^* \breve{\Pi}^{\text{stiff},(\tau)*} \end{pmatrix} \tag{2.158}$$

where we recall, $R_{\varepsilon,\text{hom}}^{(\tau)}(z)$ is defined in Definition 2.4.10, $k^{(\tau)}(z) = \Gamma_0^{\text{soft},(\tau)}|_{\mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\text{soft},(\tau)}}$ and $\breve{\Pi}^{\text{stiff},(\tau)} = \breve{\Pi}^{\text{stiff-int},(\tau)} \oplus \breve{\Pi}^{\text{stiff-ls},(\tau)}$. With this in hand, we may view $\mathcal{R}_{\varepsilon,\text{hom}}^{(\tau)}(z)$ as an operator on $L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff},(\tau)} = L^2(Q_{\text{soft}}) \oplus \breve{\Pi}^{\text{stiff},(\tau)}\mathcal{E}$.

Remark. Recall also that $\breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)} = \breve{\Lambda}_\varepsilon^{\text{stiff-int},(\tau)} \oplus \breve{\Lambda}_\varepsilon^{\text{stiff-ls},(\tau)}$. Then by the Krein's formula,

$$\begin{aligned} R_{\varepsilon,\text{hom}}^{(\tau)}(z) &= (A_0^{\text{soft},(\tau)} - z)^{-1} \\ &\quad - \breve{S}^{\text{soft},(\tau)}(z)\left[\breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)} + z\breve{\Pi}^{\text{stiff},(\tau)*}\breve{\Pi}^{\text{stiff},(\tau)} + \breve{M}^{\text{soft},(\tau)}(z)\right]^{-1}\left(\breve{S}^{\text{soft},(\tau)}(\bar{z})\right)^*. \end{aligned} \tag{2.159}$$

Therefore, $R_{\varepsilon,\text{hom}}^{(\tau)}(z) = (\widehat{\breve{A}}_{\beta_0,\beta_1}^{\text{soft},(\tau)} - z)^{-1}$, where $\beta_1 = I$ and

$$\beta_{\varepsilon,0}^{(\tau)}(z) = \breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)} + z\breve{\Pi}^{\text{stiff},(\tau)*}\breve{\Pi}^{\text{stiff},(\tau)}. \tag{2.160}$$

For the validity of the choice (β_0, β_1), we refer to the proof of Theorem 2.4.20 below ("top left entry".) Compare this with Definition 2.4.10. We see that we have two different parameterizations of the boundary conditions (β_0, β_1), arising from two different choices of boundary triples. Formulas (2.158) and (2.159) will serve as quick reference for the subsequent sections. $\circ$

2.4.3 Self-adjointness of $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$

In this section, will use [35, Section 4.1] and in the process supply further details to the arguments provided. Recall the notations for $\breve{\Pi}^{\text{stiff},(\tau)}$ and $\breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)}$. It will be convenient to set:

Definition 2.4.16. $\mathcal{B}_\varepsilon^{(\tau)} := -(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\breve{\Lambda}_\varepsilon^{\text{stiff},(\tau)}(\breve{\Pi}^{\text{stiff},(\tau)})^{-1}$.

Using the truncated "soft" triple $(A_0^{\text{soft},(\tau)}, \breve{\Lambda}^{\text{soft},(\tau)}, \breve{\Pi}^{\text{soft},(\tau)})$ and its auxiliary operators we define

Definition 2.4.17. Let $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ be the operator on $L^2(Q_{\mathrm{soft}}) \oplus \breve{\mathcal{H}}^{\mathrm{stiff},(\tau)}$ defined by

$$\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}) := \left\{ \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} \in L^2(Q_{\mathrm{soft}}) \oplus \breve{\mathcal{H}}^{\mathrm{stiff},(\tau)} : u \in \mathcal{D}(\breve{\widehat{A}}^{\mathrm{soft},(\tau)}),\ \widehat{u} = \breve{\Pi}^{\mathrm{stiff},(\tau)}\breve{\Gamma}^{\mathrm{soft},(\tau)}_0 u \right\}, \tag{2.161}$$

$$\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}} \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} := \begin{pmatrix} \breve{\widehat{A}}^{\mathrm{soft},(\tau)} u \\ -(\breve{\Pi}^{\mathrm{stiff},(\tau)*})^{-1}\breve{\Gamma}^{\mathrm{soft},(\tau)}_1 u + \mathcal{B}^{(\tau)}_{\varepsilon}\widehat{u} \end{pmatrix}. \tag{2.162}$$

Linearity of the subspace $\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}})$ and the operator $\mathcal{A}_{\mathrm{hom}}$ follows from the linearity of all the operators involved.

Let us discuss some basic properties of $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ and $\mathcal{B}^{(\tau)}_{\varepsilon}$, *for each fixed* ε *and* τ (we will therefore drop ε and τ where convenient).

Lemma 2.4.18. $\mathcal{A}_{\mathrm{hom}}$ is densely defined.

Proof. $u \in \mathcal{D}(\breve{\widehat{A}}^{\mathrm{soft}})$ can be expressed as $u = (A^{\mathrm{soft}}_0)^{-1} f + \breve{\Pi}^{\mathrm{soft}}\phi$, for some $f \in L^2(Q_{\mathrm{soft}})$ and $\phi \in \breve{\mathcal{E}}$. But recall that $\mathcal{D}(\breve{\widehat{A}}^{\mathrm{soft},(\tau)}) = \mathcal{D}(A^{\mathrm{soft},(\tau)}_0) \dot{+} \breve{\mathcal{H}}^{\mathrm{soft},(\tau)}$ is a (vector space) direct sum, so we may vary $(A^{\mathrm{soft}}_0)^{-1} f$ independently of $\breve{\Pi}^{\mathrm{soft}}\phi$. Since $\mathcal{D}(A^{\mathrm{soft},(\tau)}_0)$ is dense in $L^2(Q_{\mathrm{soft}})$, and

$$\widehat{u} = \breve{\Pi}^{\mathrm{stiff}}\breve{\Gamma}^{\mathrm{soft}}_0 u = \breve{\Pi}^{\mathrm{stiff}}\phi,$$

ranging through $\phi \in \breve{\mathcal{E}}$ implies that the second component of $\mathcal{D}(\mathcal{A}_{\mathrm{hom}})$ equals (!) $\breve{\mathcal{H}}^{\mathrm{stiff}}$. □

Lemma 2.4.19. $\mathcal{A}_{\mathrm{hom}}$ is symmetric if and only if $\mathcal{B}$ is self-adjoint.

Proof. For ($\Leftarrow$), we use the Green's identity (Theorem 2.2.14): If $(u, \widehat{u})^T$, $(v, \widehat{v})^T \in \mathcal{D}(\mathcal{A}_{\mathrm{hom}})$, then

$$\left(\mathcal{A}_{\mathrm{hom}}\begin{pmatrix} u \\ \widehat{u} \end{pmatrix}, \begin{pmatrix} v \\ \widehat{v} \end{pmatrix}\right) = \left(u, \breve{\widehat{A}}^{\mathrm{soft}} v\right) - \left(\breve{\Gamma}^{\mathrm{soft}}_0 u, \breve{\Gamma}^{\mathrm{soft}}_1 v\right) + \left(\breve{\Pi}^{\mathrm{stiff}}\breve{\Gamma}^{\mathrm{soft}}_0 u, \mathcal{B}^*\breve{\Pi}^{\mathrm{stiff}}\breve{\Gamma}^{\mathrm{soft}}_0 v\right), \tag{2.163}$$

$$\left(\begin{pmatrix} u \\ \widehat{u} \end{pmatrix}, \mathcal{A}_{\mathrm{hom}}\begin{pmatrix} v \\ \widehat{v} \end{pmatrix}\right) = \left(u, \breve{\widehat{A}}^{\mathrm{soft}} v\right) - \left(\breve{\Gamma}^{\mathrm{soft}}_0 u, \breve{\Gamma}^{\mathrm{soft}}_1 v\right) + \left(\breve{\Pi}^{\mathrm{stiff}}\breve{\Gamma}^{\mathrm{soft}}_0 u, \mathcal{B}\breve{\Pi}^{\mathrm{stiff}}\breve{\Gamma}^{\mathrm{soft}}_0 v\right). \tag{2.164}$$

We omit the details as these are exactly the same as [35, Lemma 4.3]. Comparing both equations gives us the result. For ($\Rightarrow$), we use the above equations to arrive at

$$(\widehat{u}, \mathcal{B}\widehat{v}) = (\widehat{u}, \mathcal{B}^*\widehat{v}). \tag{2.165}$$

Since $\widehat{u}$ and $\widehat{v}$ are taken from a dense set of $\breve{\mathcal{H}}^{\mathrm{stiff}}$, $\mathcal{B}^* = \mathcal{B}$. □

By Proposition 2.4.15, $\mathcal{B}$ is a bounded self-adjoint operator on $\breve{\mathcal{H}}^{\mathrm{stiff}}$. As explained in [35, Section 4.1], the point of singling out the operator $\mathcal{B}$ is because the self-adjointness of $\mathcal{B}$ implies the *self-adjointness* of $\mathcal{A}_{\mathrm{hom}}$.

An initial explanation is as follows: $\mathcal{B}$ features in the boundary condition β_0 for $R_{\mathrm{hom}}(z)$, with respect to the truncated triple. To be precise, continuing from (2.160),

$$\beta_0 = (\breve{\Pi}^{\mathrm{stiff}})^*(\mathcal{B} - z)\breve{\Pi}^{\mathrm{stiff}}, \qquad \beta_1 = I. \tag{2.166}$$

Expressing β_0 in this way is helpful as we see that there are two bounded self adjoint operators $(\breve{\Pi}^{\mathrm{stiff}})^*\mathcal{B}\breve{\Pi}^{\mathrm{stiff}}$ (thanks to the self-adjointness of $\mathcal{B}$), and $(\breve{\Pi}^{\mathrm{stiff}})^*\breve{\Pi}^{\mathrm{stiff}}$. Furthermore $(\breve{\Pi}^{\mathrm{stiff}})^*\breve{\Pi}^{\mathrm{stiff}} \geq 0$. The full explanation is contained in the proof of the following result:

Theorem 2.4.20. Fix $\varepsilon > 0$ (small enough) and $\tau \in Q'$. Suppose that $\mathcal{B}$ is self-adjoint. Then $\mathcal{A}_{\mathrm{hom}}$ is self-adjoint. Furthermore its resolvent $(\mathcal{A}_{\mathrm{hom}} - z)^{-1}$ is defined for all $z \in \mathbb{C} \setminus \mathbb{R}$ by the following block matrix decomposition with respect to $L^2(Q_{\mathrm{soft}}) \oplus \breve{\mathcal{H}}^{\mathrm{stiff}}$:

$$\begin{aligned}&(\mathcal{A}_{\mathrm{hom}} - z)^{-1}\\ &= \begin{pmatrix} R(z) & \left(k(\bar{z})\left[R(\bar{z}) - (A_0^{\mathrm{soft}} - \bar{z})^{-1}\right]\right)^* (\breve{\Pi}^{\mathrm{stiff}})^* \\ \breve{\Pi}^{\mathrm{stiff}} k(z)\left[R(z) - (A_0^{\mathrm{soft}} - z)^{-1}\right] & \breve{\Pi}^{\mathrm{stiff}} k(z)\left(k(\bar{z})\left[R(\bar{z}) - (A_0^{\mathrm{soft}} - \bar{z})^{-1}\right]\right)^* (\breve{\Pi}^{\mathrm{stiff}})^* \end{pmatrix},\end{aligned} \tag{2.167}$$

where we define $k(z) := \breve{\Gamma}_0^{\mathrm{soft}} \overset{\text{Prop 2.4.15(6)}}{=} \Gamma_0^{\mathrm{soft}}|_{\mathcal{D}(A_0^{\mathrm{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\mathrm{soft},(\tau)}}$ and

$$R(z) := \left(\widehat{A}^{\breve{\mathrm{soft}},(\tau)}{}_{(\breve{\Pi}^{\mathrm{stiff}})^*(\mathcal{B}-z)\breve{\Pi}^{\mathrm{stiff}}, I} - z\right)^{-1}. \tag{2.168}$$

Remark. By Proposition 2.4.15(6), $k(z)$ as defined in this theorem coincides with the one in (2.156). Thus (2.167) is precisely $\mathcal{R}^{(\tau)}_{\varepsilon,\mathrm{hom}}(z)$. Also, we remind the reader that $\mathrm{ran}(S^{\mathrm{soft},(\tau)}(z)\mathcal{P}^{(\tau)}) \subset \mathcal{D}(A_0^{\mathrm{soft},(\tau)}) \dot{+} \breve{\mathcal{H}}^{\mathrm{soft},(\tau)}$, which we will need in the proof below. ◦

Proof of Theorem 2.4.20. See Appendix D. The proof is taken [35, Theorem 4.4], and we supply it with further details. □

2.5 Homogenization result

This section summarizes the results thus far into a fibre-wise (for each τ) homogenization result. To begin, we collect the key ingredients of Sections 2.3 and 2.4 required for stating the result. We have the following spaces

$$\breve{\mathcal{E}}^{(\tau)} = \mathcal{P}^{(\tau)}\mathcal{E} = \mathcal{P}^{(\tau)}_{\text{stiff-int}}\mathcal{E} \oplus \mathcal{P}^{(\tau)}_{\text{stiff-ls}}\mathcal{E} = \mathrm{span}\{\psi_1^{\text{stiff-int},(\tau)}\} \oplus \mathrm{span}\{\psi_1^{\text{stiff-ls},(\tau)}\}, \tag{2.169}$$

$$\breve{\mathcal{H}}^{\mathrm{stiff},(\tau)} = \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} = \mathrm{ran}(\Pi^{\text{stiff-int},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}) \oplus \mathrm{ran}(\Pi^{\text{stiff-ls},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}). \tag{2.170}$$

We denote by Ψ_1 the lifts of ψ_1 into their respective stiff spaces. That is,

$$\Psi_1^{\bigstar,(\tau)} := \breve{\Pi}^{\bigstar,(\tau)}\psi_1^{\bigstar,(\tau)} = \Pi^{\bigstar,(\tau)}\psi_1^{\bigstar,(\tau)}, \quad (\bullet, \bigstar) \in \{(\mathrm{int}, \text{stiff-int}), (\mathrm{ls}, \text{stiff-ls})\}. \tag{2.171}$$

The homogenized operator $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$ is defined to have domain

$$\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}) = \Bigg\{ \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} \in L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff},(\tau)} : \\ u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \text{ran}(\Pi^{\text{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}), \quad \widehat{u} = \breve{\Pi}^{\text{stiff},(\tau)} \breve{\Gamma}_0^{\text{soft},(\tau)} u \Bigg\}. \tag{2.172}$$

Remark. $\mathcal{D}(A_0^{\text{soft},(\tau)}) = H^2(Q_{\text{soft}}) \cap H_0^1(Q_{\text{soft}})$, which is independent of τ. ◦

Definition 2.5.1. We write $-(\nabla + i\tau)^2$ to mean the operator of $\widehat{\breve{A}}^{\text{soft},(\tau)}$, that is, $-(\nabla + i\tau)^2$ is the magnetic Laplacian on Q_{soft} with (zero) Dirichlet BCs, extended by zero on $\breve{\mathcal{H}}^{\text{stiff},(\tau)}$.

For its action, we first note that a typical $u \in \mathcal{D}(-(\nabla + i\tau)^2)$ may be written as

$$u = (A_0^{\text{soft},(\tau)})^{-1} f + \Pi^{\text{soft},(\tau)}(a\psi_1^{\text{stiff-int},(\tau)} + b\psi_1^{\text{stiff-ls},(\tau)}), \quad f \in L^2(Q_{\text{soft}}), \quad a, b \in \mathbb{C}. \tag{2.173}$$

If we further expand $\widehat{u} \in \breve{\mathcal{H}}^{\text{stiff},(\tau)}$ into $(\widehat{u}_{\text{stiff-int}}, \widehat{u}_{\text{stiff-ls}}) \in \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)}$, then by the definition of Γ_0, the condition on $\widehat{u}$ in (2.172) may be written as

$$\widehat{u} = \begin{pmatrix} \widehat{u}_{\text{stiff-int}} \\ \widehat{u}_{\text{stiff-ls}} \end{pmatrix} = \begin{pmatrix} a\Pi^{\text{stiff-int},(\tau)}\psi_1^{\text{stiff-int},(\tau)} \\ b\Pi^{\text{stiff-ls},(\tau)}\psi_1^{\text{stiff-ls},(\tau)} \end{pmatrix} = \begin{pmatrix} a\Psi_1^{\text{stiff-int},(\tau)} \\ b\Psi_1^{\text{stiff-ls},(\tau)} \end{pmatrix},$$

Therefore, the action of $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$ may be written in with respect to the decomposition $L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)}$ as

$$\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} \begin{pmatrix} u \\ \widehat{u}_{\text{stiff-int}} \\ \widehat{u}_{\text{stiff-ls}} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -(\breve{\Pi}^{\text{stiff-int},(\tau)*})^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-int}} \left[\Gamma_1^{\text{soft},(\tau)} u + \Gamma_{\varepsilon,1}^{\text{stiff-int},(\tau)} \left(a\Psi_1^{\text{stiff-int},(\tau)} \right) \right] \\ -(\breve{\Pi}^{\text{stiff-ls},(\tau)*})^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}} \left[\Gamma_1^{\text{soft},(\tau)} u + \Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)} \left(b\Psi_1^{\text{stiff-ls},(\tau)} \right) \right] \end{pmatrix} \tag{2.174}$$

$$= \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -(\breve{\Pi}^{\text{stiff-int},(\tau)*})^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-int}} \Gamma_1^{\text{soft},(\tau)} u \\ -(\breve{\Pi}^{\text{stiff-ls},(\tau)*})^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}} \left[\Gamma_1^{\text{soft},(\tau)} u + \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)} \left(b\psi_1^{\text{stiff-ls},(\tau)} \right) \right] \end{pmatrix}. \tag{2.175}$$

To deduce this from Definition 2.4.17, we have used: In the first equality, that $\Lambda = \Gamma_1 \Pi$ and Π is boundedly invertible. In the second equality, that $\breve{\Lambda}_\varepsilon^{\text{stiff-ls}}$ acts as multiplication by $\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}$, and similarly for $\breve{\Lambda}_\varepsilon^{\text{stiff-int}}$, but recall that $\mu_1^{\text{stiff-int},(\tau)} = 0$ by Proposition 2.2.10.

Lemma 2.5.2. The action of $(\breve{\Pi}^{\bigstar,(\tau)*})^{-1} : \text{span}\{\psi_1^{\bigstar,(\tau)}\} \to \text{ran}(\Pi^{\bigstar,(\tau)}|_{\text{span}\{\psi_1^{\bigstar,(\tau)}\}})$ is given by

$$\psi_1^{\bigstar,(\tau)} \mapsto \|\Psi_1^{\bigstar,(\tau)}\|^{-2} \Psi_1^{\bigstar,(\tau)}.$$

Proof. We will drop $\bullet$, $\bigstar$, and τ where convenient. It suffices to figure out its action on ψ_1.

Since $(\breve{\Pi}^*)^{-1} = (\breve{\Pi}^{-1})^*$,

$$\Big((\breve{\Pi}^{-1})^*\psi_1, \Psi_1\Big)_{L^2(Q_\star)} = \Big(\psi_1, (\breve{\Pi})^{-1}\Psi_1\Big)_{L^2(Q_\star)} = (\psi_1, \psi_1)_{L^2(\Gamma_\bullet)} = 1.$$

But $(\breve{\Pi}^{-1})^*\psi_1 \in \mathrm{ran}(\Pi|_{\breve{\mathcal{E}}})$ is a multiple of Ψ_1, say $(\breve{\Pi}^{-1})^*\psi_1 = c\Psi_1$. By the above calculation, we must have $c = 1/\|\Psi_1\|^2$. □

We are now in the position to state the homogenization result.

Theorem 2.5.3 (Fibre-wise homogenization result)**.** With the homogenized operator $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ as defined above, we have that:

- $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ is asymptotically close to our main model operator $A^{(\tau)}_\varepsilon$ in the norm-resolvent sense, with an $O(\varepsilon^2)$ estimate. This estimate is uniform in $z \in K_\sigma$ and $\tau \in Q'$.
- The resolvent $(\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}} - z)^{-1}$ is given by $\mathcal{R}^{(\tau)}_{\varepsilon,\mathrm{hom}}(z)$ (see Definition 2.4.10 or (2.158) or (2.167)).
- $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ is self-adjoint on $L^2(Q_{\mathrm{soft}}) \oplus \breve{\mathcal{H}}^{\mathrm{stiff},(\tau)}$, and its null-extension to the full space $L^2(Q) = L^2(Q_{\mathrm{soft}}) \oplus L^2(Q_{\text{stiff-int}}) \oplus L^2(Q_{\text{stiff-ls}})$, which we will still denote by $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$, is symmetric.
- $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ depends on ε only through its action on the third component $\breve{\mathcal{H}}^{\text{stiff-ls},(\tau)}$.

Proof. The only point that remains to be shown is why the operator $\mathcal{R}^{(\tau)}_{\varepsilon,\mathrm{hom}}(z)$, while initially defined for $z \in \mathbb{C} \setminus \mathbb{R}$ (Theorem 2.4.20), can be extended to the whole resolvent set $\rho(\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}})$. This is due to the analyticity of the resolvent $(\mathcal{A}_{\mathrm{hom}} - z)^{-1}$: Given $z_0 \in \rho(\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}) \cap \mathbb{R}$, we can always find an open ball $B(z, \varepsilon_z)$, with $z \in \mathbb{C} \setminus \mathbb{R}$, and $z_0 \in B(z, \varepsilon_z)$, such that the formula (2.167) holds. □

Remark. Explicit expressions for $\breve{\Pi}$ are available for the case $Q_{\text{stiff-int}}$ (for all τ) and for $Q_{\text{stiff-ls}}$ (for $\tau = 0$). See Proposition 2.2.10 for the formulas for the eigenfunction ψ_1. ∘

Chapter 3

The homogenized description

From Chapter 2, we know that $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$ serves as a simplified description of the high-contrast composite, in the sense that it acts only on a $2D$ subspace of $L^2(Q_{\text{stiff-int}} \cup Q_{\text{stiff-ls}})$. However, this is rather unsatisfactory as we have to contend with auxiliary objects arising from boundary triples. Indeed, how do formulas in Section 2.5 inform the effective properties of our composite?

In this chapter, we take a closer look at the homogenized description $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$. We would like to learn about the effective transport/scattering properties of our composite, and from this perspective, we ask the following questions:

- What are the "dispersion functions" $K(\tau, z)$ for the stiff-soft-stiff model? That is, can we show that the resolvent $(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} - z)^{-1}$ is unitarily equivalent to a multiplication operator by some function $(K(\tau, z) - z)^{-1}$?
- How does the norm-resolvent asymptotics look like on the full space? That is, compute $\int^{\oplus}_{Q'} \mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} d\tau$.
- Can we provide a reasonable characterization of the spectrum $\sigma(\int^{\oplus}_{Q'} \mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} d\tau)$ and its decomposition?

While these questions are non-trivial in the high contrast case, they are readily answered in the moderate contrast case: Recall from Sections 1.1 and 1.4, that norm-resolvent limit of $A_\varepsilon = -\operatorname{div}(a_\varepsilon \nabla \cdot)$, when a_ε is positive definite and bounded, is given by $A_{\text{hom}} = -\operatorname{div}(a_{\text{hom}} \nabla \cdot)$, where a_{hom} is positive definite and constant in space. Using the Fourier transform $\mathcal{F}$, we have $A_{\text{hom}} = \mathcal{F}^{-1} M_{a_{\text{hom}}\xi\cdot\xi} \mathcal{F}$, thus $\sigma(A_{\text{hom}}) = [0, \infty)$, and is purely absolutely continuous. Also, $(A_{\text{hom}} - z)^{-1} = \mathcal{F}^{-1} M_{(a_{\text{hom}}\xi\cdot\xi - z)^{-1}} \mathcal{F}$, so the mapping $\xi \mapsto a_{\text{hom}}\xi \cdot \xi$ encodes dispersion of waves: In the context of the wave equation (1.1), we set $z = \omega^2$, and the dispersion relation becomes $a_{\text{hom}}\xi \cdot \xi = \omega^2$.

We will investigate the three bullet points in Sections 3.1, 3.2, and 3.3 respectively. In Section 3.1, we focus on the 2×2 matrix $P_{\breve{\mathcal{H}}^{\text{stiff},(\tau)}} (\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} - z)^{-1} P_{\breve{\mathcal{H}}^{\text{stiff},(\tau)}}$, and express each entry as an operator of multiplication by some constant depending on τ and z. Moreover, we show that for the diagonal entries, this constant can be written in the form $(K(\tau, z) - z)^{-1}$. In Section 3.2, we write down formulas for the full space operator for Models I and II (Figure 1-2), and the stiff-soft-stiff model.

In Section 3.3, we conduct a spectral analysis of $\int_{Q'}^{\oplus} \mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} d\tau$. This section only contains the author's preliminary results, and hence does not completely answer the third bullet point. In particular, we give partial results on the spectrum and its decomposition for Models I and II, and a short discussion on the difficulties of extending these arguments to the stiff-soft-stiff model.

3.1 A closer look at the homogenized operator

In Chapter 2, we have tried as much as possible to separate the computations on $Q_{\text{stiff-int}}$ and $Q_{\text{stiff-ls}}$. But intuitively, we would expect the two stiff components to have influence on each other. Indeed, this is evidenced by the following:

- From (2.174), we see that the second and third components contains the term $\Gamma_1^{\text{soft},(\tau)} u$. This takes u which lives on the annulus Q_{soft}, and then applying the co-normal derivative of u at the boundaries Γ_{int} (for the second component) and Γ_{ls} (for the third component). Clearly the two stiff components are "communicating" through u.
- From (2.167), we see that the bottom right entry of $(\mathcal{A}_{\text{hom}} - z)^{-1}$, when expanded out as a 2×2 matrix with respect to the decomposition $\breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)}$, is not block-diagonal.

In this section, we study the bottom right entry of $(\mathcal{A}_{\text{hom}} - z)^{-1}$, i.e.

$$P_{\breve{\mathcal{H}}^{\text{stiff},(\tau)}} (\mathcal{A}_{\text{hom}} - z)^{-1} P_{\breve{\mathcal{H}}^{\text{stiff},(\tau)}} \overset{(2.156)}{=} \begin{array}{c} \\ \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \\ \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} \end{array} \begin{array}{c} \begin{array}{cc} \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} & \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} \end{array} \\ \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \end{array}, \tag{3.1}$$

with a particular focus on the diagonal entries a_{22} and a_{33}. To begin, we first apply an isomorphism $\breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} \cong \mathbb{C}^2$ so that we do not have to deal with a varying space. Let us define:

Definition 3.1.1. Set $j_{\text{stiff-int}}^{(\tau)} : \text{ran}(\Pi^{\text{stiff-int},(\tau)}|_{\mathcal{P}_{\text{stiff-int}}^{(\tau)} \mathcal{E}}) \to \mathbb{C}$ to be the unitary mapping

$$\Psi_1^{\text{stiff-int},(\tau)} \mapsto \|\Psi_1^{\text{stiff-int},(\tau)}\|_{L^2(Q_{\text{stiff-int}})}.$$

(Note: $\breve{\mathcal{H}}^{\text{stiff-int},(\tau)} = \text{ran}(\Pi^{\text{stiff-int},(\tau)}|_{\mathcal{P}_{\text{stiff-int}}^{(\tau)} \mathcal{E}})$) And similarly for $j_{\text{stiff-ls}}^{(\tau)}$. Set $j^{(\tau)} = j_{\text{stiff-int}}^{(\tau)} \oplus j_{\text{stiff-ls}}^{(\tau)}$.

In this case, the operator $j_{\text{stiff-int}}^{(\tau)} \Pi^{\text{stiff-int}}|_{\breve{\mathcal{E}}^{(\tau)}} : \breve{\mathcal{E}} \mapsto \mathbb{C}$ is a mapping taking $\psi_1^{\text{stiff-int},(\tau)}$ to $(\Psi_1^{\text{stiff-int},(\tau)}$, and then to) $\|\Psi_1^{\text{stiff-int},(\tau)}\|$. For the reader's convenience we compute the inverse of its adjoint:

Lemma 3.1.2. Let $\bigstar \in \{\text{stiff-int}, \text{stiff-ls}\}$. The action of $((j_{\bigstar}^{(\tau)} \breve{\Pi}^{\bigstar,(\tau)})^*)^{-1} : \text{span}\{\psi_1^{\bigstar,(\tau)}\} \to \mathbb{C}$ is given by

$$\psi_1^{\bigstar,(\tau)} \mapsto \|\Psi_1^{\bigstar,(\tau)}\|^{-1}.$$

Proof. We drop ★ and τ. We simply have to note that $((j\breve{\Pi})^*)^{-1} = (j^*)^{-1}(\breve{\Pi}^*)^{-1}$. Since j is unitary, $(j^*)^{-1} = j$. The result now follows from Lemma 2.5.2. □

Under this identification, we may view our homogenized operator as an operator on $L^2(Q_{\text{soft}}) \oplus \mathbb{C}^2$, which we will still denote by $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}$. Let us write $\mathbb{C}_{\text{stiff-int}} \oplus \mathbb{C}_{\text{stiff-ls}}$ to distinguish between the copies of $\mathbb{C}$. In that case, our homogenized operator may be written as

$$\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}) := \{(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}}) \in L^2(Q_{\text{soft}}) \oplus \mathbb{C}_{\text{stiff-int}} \oplus \mathbb{C}_{\text{stiff-ls}} : u \in \mathcal{D}(A_0^{\text{soft}}) \dotplus \operatorname{ran}(\Pi^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}), \\ \beta_{\text{stiff-int}} = j^{(\tau)}_{\text{stiff-int}}\Pi^{\text{stiff-int},(\tau)}\Gamma_0^{\text{soft},(\tau)}u, \quad \beta_{\text{stiff-ls}} = j^{(\tau)}_{\text{stiff-ls}}\Pi^{\text{stiff-ls},(\tau)}\Gamma_0^{\text{soft},(\tau)}u\}, \tag{3.2}$$

$$\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -((j^{(\tau)}_{\text{stiff-int}}\breve{\Pi}^{\text{stiff-int},(\tau)})^*)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-int}}\Gamma_1^{\text{soft},(\tau)}u \\ -((j^{(\tau)}_{\text{stiff-ls}}\breve{\Pi}^{\text{stiff-ls},(\tau)})^*)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}}\left[\Gamma_1^{\text{soft},(\tau)}u + \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}(j^{(\tau)}_{\text{stiff-ls}}\breve{\Pi}^{\text{stiff-ls},(\tau)})^{-1}\beta_{\text{stiff-ls}}\right] \end{pmatrix} \\ =: \begin{pmatrix} -(\nabla + i\tau)^2 u \\ T^{(\tau)}_{\varepsilon,\text{stiff-int}}(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}})^\top \\ T^{(\tau)}_{\varepsilon,\text{stiff-ls}}(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}})^\top \end{pmatrix}. \tag{3.3}$$

Before proceeding with the investigation of the entries a_{ij} of $(\mathcal{A}_{\text{hom}} - z)^{-1}$, we record two facts that we will use without mention throughout the chapter.

Lemma 3.1.3. There exist constants $c, C > 0$, which do not depend on τ, such that $c < \|\Psi_1^{\text{stiff-ls},(\tau)}\|_{L^2(Q_{\text{stiff-ls}})} < C$.

Proof. This follows from two facts: (i) the mapping $\overline{Q'} \ni \tau \mapsto \Psi_1^{\text{stiff-ls},(\tau)} \in L^2(Q_{\text{stiff-ls}})$ is continuous (Proposition 2.3.5), and (ii) $\|\Psi_1^{\text{stiff-ls},(\tau)}\| \neq 0$ (as $\Pi^{\text{stiff-ls},(\tau)}$ is injective). □

While Lemma 3.1.3 is also true for $\Psi_1^{\text{stiff-int},(\tau)}$, the situation is even better:

Lemma 3.1.4. $\|\Psi_1^{\text{stiff-int},(\tau)}\|_{L^2(Q_{\text{stiff-int}})} = \sqrt{\frac{|Q_{\text{stiff-int}}|}{|\Gamma_{\text{int}}|}}$, which does not depend on τ.

Proof. Since $\psi_1^{\text{stiff-ls},(\tau)} = |\Gamma_{\text{int}}|^{-\frac{1}{2}}e^{-i\tau\cdot x}$ (Proposition 2.2.10), we must therefore have $\Psi_1^{\text{stiff-int},(\tau)} = |\Gamma_{\text{int}}|^{-\frac{1}{2}}e^{-i\tau\cdot x}$. □

3.1.1 Stiff-interior to stiff-interior

Let us now figure out the action of $P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\text{hom}} - z)^{-1}P_{\mathbb{C}_{\text{stiff-int}}}$, which is a multiplication by a constant. We will drop ε and τ where convenient. We will also assume that $z \in K_\sigma$.

The operator in question takes $\delta \in \mathbb{C}$, solves the system

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = 0, \\ T_{\text{stiff-int}}(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}})^\top - z\beta_{\text{stiff-int}} = \delta, \\ T_{\text{stiff-ls}}(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}})^\top - z\beta_{\text{stiff-ls}} = 0, \end{cases} \tag{3.4}$$

and then outputs $\beta_{\text{stiff-int}}$. (Recall Definition 2.5.1 for the notation $-(\nabla + i\tau)^2$.) Our goal is to write $T_{\text{stiff-int}}(u, \beta_{\text{stiff-int}}, \beta_{\text{stiff-ls}})^T$ as a constant times $\beta_{\text{stiff-int}}$. To assist us, we define

Definition 3.1.5. $v := \Pi^{\text{soft},(\tau)}(\psi_1^{\text{stiff-int},(\tau)}, 0)$, and $w := \Pi^{\text{soft},(\tau)}(0, \psi_1^{\text{stiff-ls},(\tau)})$.

Observe that if $u = (A_0^{\text{soft},(\tau)})^{-1} f + \Pi^{\text{soft},(\tau)}(a\psi_1^{\text{stiff-int},(\tau)} + b\psi_1^{\text{stiff-ls},(\tau)})$ for some $f \in L^2(Q_{\text{soft}})$ and a, $b \in \mathbb{C}$, then $u = (A_0^{\text{soft}})^{-1} f + av + bw$. That means $\widetilde{u} := u - av - bw \in \mathcal{D}(A_0^{\text{soft}})$. So $(-(\nabla + i\tau)^2 - z)\widetilde{u} = (A_0^{\text{soft}} - z)\widetilde{u}$. In fact,

$$(-(\nabla+i\tau)^2 - z)\widetilde{u} = \underbrace{\cancel{(-(\nabla+i\tau)^2 - z)u}}_{\text{By (3.4)}} - (\underbrace{\cancel{-(\nabla+i\tau)^2}}_{\text{Since } \widehat{A}(\Pi\phi)=0} - z)av - (\underbrace{\cancel{-(\nabla+i\tau)^2}}_{\text{Since } \widehat{A}(\Pi\phi)=0} - z)bw = zav + zbw.$$

This implies that

$$\widetilde{u} = za(A_0^{\text{soft}} - z)^{-1} v + zb(A_0^{\text{soft}} - z)^{-1} w.$$

The key is that a and b are related to $\beta_{\text{stiff-int}}$ and $\beta_{\text{stiff-ls}}$ respectively by

$$\beta_{\text{stiff-int}} = a\|\Psi_1^{\text{stiff-int}}\|, \quad \beta_{\text{stiff-ls}} = b\|\Psi_1^{\text{stiff-ls}}\|.$$

This is a consequence of the computation of $\widehat{u}$ in the previous section, and the definition of the isomorphism j. This allows us to write

$$\begin{aligned}
T_{\text{stiff-int}}\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} &= T_{\text{stiff-int}}\begin{pmatrix} az(A_0^{\text{soft}} - z)^{-1}v + bz(A_0^{\text{soft}} - z)^{-1}w + av + bw \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} \\
&= T_{\text{stiff-int}}\begin{pmatrix} \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} z(A_0^{\text{soft}} - z)^{-1}v + \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} z(A_0^{\text{soft}} - z)^{-1}w + \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} v + \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} w \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} \\
&= T_{\text{stiff-int}}\begin{pmatrix} \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} z(A_0^{\text{soft}} - z)^{-1}v + \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} v \\ \beta_{\text{stiff-int}} \\ 0 \end{pmatrix} + T_{\text{stiff-int}}\begin{pmatrix} \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} z(A_0^{\text{soft}} - z)^{-1}w + \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} w \\ 0 \\ \beta_{\text{stiff-ls}} \end{pmatrix} \\
&= \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} + \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-int}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix}.
\end{aligned} \tag{3.5}$$

Can we write $\beta_{\text{stiff-ls}}$ in terms of $\beta_{\text{stiff-int}}$? Yes: By using the third equation of the system (3.4), and (3.5) with $T_{\text{stiff-int}}$ replaced by $T_{\text{stiff-ls}}$, we obtain

$$\beta_{\text{stiff-ls}} = \frac{1}{z} T_{\text{stiff-ls}}\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix}$$

$$= \frac{\beta_{\text{stiff-int}}}{z\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} + \frac{\beta_{\text{stiff-ls}}}{z\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix}. \tag{3.6}$$

Rearranging and plugging this back into (3.5),

$$T_{\text{stiff-int}} \begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} = \beta_{\text{stiff-int}} \Bigg\{ \frac{1}{\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix}$$

$$+ \frac{1}{z\|\Psi_1^{\text{stiff-int}}\|\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} \times$$

$$\times \left(1 - \frac{1}{z\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} \right)^{-1} \Bigg\} \tag{3.7}$$

$$=: \beta_{\text{stiff-int}} \left\{ K_{a,\text{stiff-int}}(\tau, z) + K_{b,\text{stiff-int}}(\tau, z) \right\} \tag{3.8}$$

$$=: \beta_{\text{stiff-int}} K_{\text{stiff-int}}(\tau, z). \tag{3.9}$$

The derivation above suggests the following:

Theorem 3.1.6. For $\varepsilon > 0$ small enough, independently of $z \in K_\sigma$ and $\tau \in Q'$, $P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-int}}}$ is the operator on $\mathbb{C}_{\text{stiff-int}}$ of multiplication by the number $(K_{\text{stiff-int}}(\tau, z) - z)^{-1}$. In the notation of Section 1.3.1, this means that

$$P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-int}}} = M_{(K_{\text{stiff-int}}(\tau,z)-z)^{-1}}. \tag{3.10}$$

Proof. To ensure that $K_{\text{stiff-int}}(\tau, z)$ is well-defined, we need to show that the denominator of the second term in (3.7) is non-zero. We will do this by showing that it has a non-zero imaginary component. This requires us to uncover the action of $T_{\text{stiff-ls}}$. First, we observe that

$$\begin{aligned} z(A_0^{\text{soft},(\tau)} - z)^{-1}w + w &= (I + z(A_0^{\text{soft},(\tau)} - z)^{-1})w \\ &= (I + z(A_0^{\text{soft},(\tau)} - z)^{-1})\Pi^{\text{soft},(\tau)}(0 + \psi_1^{\text{stiff-ls},(\tau)}) \\ &= S^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}). \end{aligned} \tag{3.11}$$

In the action of $T_{\text{stiff-ls}}$, we need to apply to the above, $\Gamma_1^{\text{soft},(\tau)}$, then $\mathcal{P}_{\text{stiff-ls}}^{(\tau)}$, and then $((j\breve{\Pi})^*)^{-1}$:

$$\begin{aligned} &((j\breve{\Pi})^*)^{-1}\mathcal{P}_{\text{stiff-ls}}^{(\tau)}\Gamma_1^{\text{soft},(\tau)} S^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}) \\ &= ((j\breve{\Pi})^*)^{-1}\mathcal{P}_{\text{stiff-ls}}^{(\tau)} M^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}) \\ &= ((j\breve{\Pi})^*)^{-1} \left\langle M^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}), (0 + \psi_1^{\text{stiff-ls},(\tau)}) \right\rangle_{L^2(\Gamma_{\text{int}}) \oplus L^2(\Gamma_{\text{ls}})} \psi_1^{\text{stiff-ls},(\tau)} \\ &= \left\langle M^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}), (0 + \psi_1^{\text{stiff-ls},(\tau)}) \right\rangle_{\mathcal{E}} \frac{1}{\|\Psi_1^{\text{stiff-ls},(\tau)}\|}. \end{aligned} \tag{3.12}$$

Also we need to apply to $\|\Psi_1^{\text{stiff-ls},(\tau)}\|$, the operator $(j\breve{\Pi})^{-1}$, then a multiplication by $\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}$, then $\mathcal{P}^{(\tau)}_{\text{stiff-ls}}$, and then $((j\breve{\Pi})^*)^{-1}$:

$$\begin{aligned}
&((j\breve{\Pi})^*)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}}\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}(j\breve{\Pi})^{-1}\|\Psi_1^{\text{stiff-ls},(\tau)}\| \\
&\quad = \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}((j\breve{\Pi})^*)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}}\psi_1^{\text{stiff-ls},(\tau)} \\
&\quad = \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}((j\breve{\Pi})^*)^{-1}\left\langle(0+\psi_1^{\text{stiff-ls},(\tau)}),(0+\psi_1^{\text{stiff-ls},(\tau)})\right\rangle_{L^2(\Gamma_{\text{int}})\oplus L^2(\Gamma_{\text{ls}})}\psi_1^{\text{stiff-ls},(\tau)} \\
&\quad = \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}((j\breve{\Pi})^*)^{-1}\psi_1^{\text{stiff-ls},(\tau)} \\
&\quad = \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}\frac{1}{\|\Psi_1^{\text{stiff-ls},(\tau)}\|}.
\end{aligned} \tag{3.13}$$

Using these two computations, we observe that the denominator of the second term in (3.7) is

$$1-\frac{1}{z\|\Psi_1^{\text{stiff-ls},(\tau)}\|}T^{(\tau)}_{\varepsilon,\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}}-z)^{-1}w+w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\|\end{pmatrix}$$

$$= 1+\frac{1}{z\|\Psi_1^{\text{stiff-ls},(\tau)}\|^2}\left[\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}+\left\langle M^{\text{soft},(\tau)}(z)(0+\psi_1^{\text{stiff-ls},(\tau)}),(0+\psi_1^{\text{stiff-ls},(\tau)})\right\rangle\right] \tag{3.14}$$

$$= 1+\frac{(\text{Re }z)-i(\text{Im }z)}{|z|^2\|\Psi_1^{\text{stiff-ls},(\tau)}\|^2}\left[\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}+\left\langle M^{\text{soft},(\tau)}(z)(0+\psi_1^{\text{stiff-ls},(\tau)}),(0+\psi_1^{\text{stiff-ls},(\tau)})\right\rangle\right]. \tag{3.15}$$

Focusing on the imaginary part of (3.15), it suffices to show that the following expression is non-zero for every τ (Note that $|\text{Im }z|>0$, since $z\in K_\sigma$.):

$$\begin{aligned}
&-i(\text{Im }z)\Big[\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}+\left\langle \text{Re }M^{\text{soft},(\tau)}(z)(0+\psi_1^{\text{stiff-ls},(\tau)}),(0+\psi_1^{\text{stiff-ls},(\tau)})\right\rangle \\
&\qquad\qquad -(\text{Re }z)\left\langle S^{\text{soft},(\tau)}(\overline{z})^*S^{\text{soft},(\tau)}(\overline{z})(0+\psi_1^{\text{stiff-ls},(\tau)}),(0+\psi_1^{\text{stiff-ls},(\tau)})\right\rangle\Big] \\
&= -i(\text{Im }z)\left[A+B+C\right]
\end{aligned} \tag{3.16}$$

where we have used the identity $\text{Im }M(z)=(\text{Im }z)S(\overline{z})^*S(\overline{z})$ (Proposition 2.2.13(7)). Recall that $\text{Re }M^{\text{soft},(\tau)}(z)$ was defined in the beginning of the proof of Theorem 2.3.4. The terms B and C are real, and independent of ε. In fact, they can be bounded uniformly in τ:

- For B, use the identity
$$M(z)=\Lambda+z\Pi^*(I-zA_0^{-1})^{-1}\Pi$$
(Proposition 2.2.13(5)). Now apply Proposition 2.2.7 to $\Pi^{\text{soft},(\tau)}$, and apply the arguments of Proposition 2.3.3 (see (2.74)) to $\langle\Lambda^{\text{soft},(\tau)}(0,\psi_1^{\text{stiff-ls},(\tau)}),(0,\psi_1^{\text{stiff-ls},(\tau)})\rangle$.

- For C, apply Lemma 2.2.20 to $S^{\text{soft},(\tau)}(z)$, followed by Proposition 2.2.7.

If $\tau\neq 0$, Proposition 2.2.10 says that $A=\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}$ is a negative real number. If $\tau=0$, we have $A=0$ (Proposition 2.2.10). Nonetheless, the expression (3.14) non-zero. Indeed, we first compute

$$\left\langle M^{\text{soft},(0)}(z)(0+\psi_1^{\text{stiff-ls},(0)}),(0+\psi_1^{\text{stiff-ls},(0)})\right\rangle. \tag{3.17}$$

To do this, we first write down the BVP that $u := S^{\text{soft},(0)}(z)(0+\psi_1^{\text{stiff-ls},(0)})$ solves:

$$\begin{cases} -\Delta u = zu & \text{in } Q_{\text{soft}}, \\ u = 0 & \text{on } \Gamma_{\text{int}}, \\ u = \psi_1^{\text{stiff-ls},(0)} = |\Gamma_{\text{ls}}|^{-\frac{1}{2}}\mathbf{1}_{\Gamma_{\text{ls}}} & \text{on } \Gamma_{\text{ls}}. \end{cases} \tag{3.18}$$

Then, we compute

$$\begin{aligned} &\langle M^{\text{soft},(0)}(z)(0+\psi_1^{\text{stiff-ls},(0)}),(0+\psi_1^{\text{stiff-ls},(0)})\rangle \\ &\quad = -\underbrace{\int_{\Gamma_{\text{int}}}\cdots}_{=0} - \int_{\Gamma_{\text{ls}}}\frac{\partial u}{\partial n}\psi_1^{\text{stiff-ls},(0)} = \int_{Q_{\text{soft}}} z|u|^2 - \int_{Q_{\text{soft}}}|\nabla u|^2. \end{aligned} \tag{3.19}$$

We note that $\|\nabla u\| \neq 0$, or else u will be a constant function on the connected set Q_{soft}, contradicting the fact that u has different traces on Γ_{int} and Γ_{ls}. Since Im $z \neq 0$ and $\|\nabla u\| \neq 0$, (3.19) implies that (3.14) non-zero.

Thus far, we have shown that *for each* τ, we may pick ε small enough such that (3.14) is non-zero. This is not enough, as we would like to pick ε small enough independently of τ. To achieve this, we will have to enhance the above argument argument as follows: Since (3.14) is continuous in τ and non-zero at $\tau = 0$, it must be bounded away from zero in a neighbourhood of $\tau = 0$. Furthermore, the expression (3.14) allows us to pick this neighbourhood independently of ε. Now combine these facts with the arguments of the $\tau \neq 0$ case, which says that $\varepsilon > 0$ can be chosen small enough, independently of $\tau \in Q'$ outside this neighborhood, such that (3.14) is non-zero.

We have therefore shown that for $\varepsilon > 0$ small enough, the mapping $\tau \mapsto K_{\text{stiff-int}}(\tau, z)$ is well-defined. This concludes the proof. □

Definition 3.1.7. We call $K_{\text{stiff-int}}(\tau, z)$ the dispersion function with respect to $Q_{\text{stiff-int}}$.

Remark. To justify $K_{\text{stiff-int}}$ which describe wave propagation on the stiff-interior region, we have relied crucially on the properties of the stiff-landscape region, namely the eigenvalue $\mu_1^{\text{stiff-ls},(\tau)}$. ◦

To conclude the section, let us make a few important observations.

1. The function $K_{\text{stiff-int}}(\tau, z)$ consists of two terms, $K_{a,\text{stiff-int}}(\tau, z)$ and $K_{b,\text{stiff-int}}(\tau, z)$.

   The first term, $K_{a,\text{stiff-int}}(\tau, z)$, is what we would have if there were only one stiff component (See [35, Section 5.3], for Model II). In our case with two stiff components, we have to compensate using the second "correction" term $K_{b,\text{stiff-int}}(\tau, z)$.

2. The dependence of $K_{\text{stiff-int}}(\tau, z)$ on ε falls solely on the $T_{\text{stiff-ls}}$ terms with a non-zero

third component. In particular, we observe that ε appears in the term 'A' of (3.16), and nowhere else. So the correction term becomes small as $\varepsilon \to 0$. To be precise, we have

Corollary 3.1.8. If $\tau \neq 0$, then $K_{b,\text{stiff-int}}(\tau, z) = O(\varepsilon^2)$, uniformly in $z \in K_\sigma$. If we assume further that τ is uniformly bounded away from 0, then the estimate is also uniform in τ.

3.1.2 Stiff-landscape to stiff-landscape

We now turn our attention to $P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}}$. We will use the functions v and w as defined in the previous section. We omit the analogous derivation of $K_{\text{stiff-ls}}(\tau, z)$, and jump straight to the result:

Definition 3.1.9. The dispersion function $K_{\text{stiff-ls}}(\tau, z)$ with respect to $Q_{\text{stiff-ls}}$ is given by

$$K_{\text{stiff-ls}}(\tau, z) := K_{a,\text{stiff-ls}}(\tau, z) + K_{b,\text{stiff-ls}}(\tau, z) \tag{3.20}$$

$$\begin{aligned} &:= \frac{1}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} \\ &+ \frac{1}{z\|\Psi_1^{\text{stiff-int}}\| \|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} \times \\ &\times \left(1 - \frac{1}{z\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} \right)^{-1}. \end{aligned} \tag{3.21}$$

Theorem 3.1.10. For $\varepsilon > 0$ small enough, independently of $z \in K_\sigma$ and $\tau \in Q'$, $P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}}$ is the operator on $\mathbb{C}_{\text{stiff-ls}}$ of multiplication by the number $(K_{\text{stiff-ls}}(\tau, z) - z)^{-1}$. That is,

$$P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}} = M_{(K_{\text{stiff-ls}}(\tau,z)-z)^{-1}}. \tag{3.22}$$

Proof. The first part of the proof proceeds analogously to the proof of Theorem 3.1.6, so we omit this. In place of (3.14), we now have

$$1 + \frac{1}{z\|\Psi_1^{\text{stiff-int},(\tau)}\|^2} \left\langle M^{\text{soft},(\tau)}(z)(\psi_1^{\text{stiff-int},(\tau)} + 0), (\psi_1^{\text{stiff-int},(\tau)} + 0) \right\rangle, \tag{3.23}$$

and now we would like to show that the following expression

$$\left\langle M^{\text{soft},(\tau)}(z)(\psi_1^{\text{stiff-int},(\tau)} + 0), (\psi_1^{\text{stiff-int},(\tau)} + 0) \right\rangle \tag{3.24}$$

is a non-zero constant that does not depend on τ. The argument is a generalization of the case $\tau = 0$ in Theorem 3.1.6. To begin, we write down the BVP that $u := S^{\text{soft},(\tau)}(z)(\psi_1^{\text{stiff-int},(\tau)} + 0)$

solves:

$$\begin{cases} -(\nabla + i\tau)^2 u = zu & \text{in } Q_{\text{soft}}, \\ u = \psi_1^{\text{stiff-int},(\tau)} = e^{-i\tau\cdot x} & \text{on } \Gamma_{\text{int}}, \\ u = 0 & \text{on } \Gamma_{\text{ls}}. \end{cases} \tag{3.25}$$

(Ignore the normalization constant as it will not affect the arguments.) Now define $w(x) = e^{i\tau\cdot x}u(x)$. Then in Q_{soft}, we have that

$$\begin{aligned} -(\nabla + i\tau)^2 u &= -e^{-i\tau\cdot x}\text{div}\left(e^{i\tau\cdot x}(\nabla + i\tau)u\right) \\ &= -e^{-i\tau\cdot x}\text{div}\left(e^{i\tau\cdot x}(\nabla + i\tau)(e^{-i\tau\cdot x}w)\right) \\ &= -e^{-i\tau\cdot x}\text{div}\left(e^{i\tau\cdot x}\left[e^{-i\tau\cdot x}\nabla w + \cancel{w(-i\tau)e^{-i\tau\cdot x}} + \cancel{(i\tau)e^{-i\tau\cdot x}w}\right]\right) = -e^{-i\tau\cdot x}\Delta w. \end{aligned} \tag{3.26}$$

Since $e^{-i\tau\cdot x}$ cannot be zero, we deduce that w solves the BVP:

$$\begin{cases} -\Delta w = zw & \text{in } Q_{\text{soft}}, \\ w = \mathbf{1}_{\Gamma_{\text{int}}} & \text{on } \Gamma_{\text{int}}, \\ w = 0 & \text{on } \Gamma_{\text{ls}}. \end{cases} \tag{3.27}$$

Back to our goal, we compute

$$\begin{aligned} &\langle M^{\text{soft},(\tau)}(z)(\psi_1^{\text{stiff-int},(\tau)} + 0), (\psi_1^{\text{stiff-int},(\tau)} + 0)\rangle \\ &\qquad = -\int_{\Gamma_{\text{int}}}\left[\frac{\partial u}{\partial n} + i(\tau\cdot n)u\right]\psi_1^{\text{stiff-int},(\tau)} - \underbrace{\int_{\Gamma_{\text{ls}}}\cdots}_{=0} \\ &\qquad = \int_{Q_{\text{soft}}} z|u|^2 - \int_{Q_{\text{soft}}} |(\nabla + i\tau)u|^2 \\ &\qquad = \int_{Q_{\text{soft}}} z|e^{-i\tau\cdot x}w|^2 - \int_{Q_{\text{soft}}} |e^{-i\tau\cdot x}\nabla w|^2 = \int_{Q_{\text{soft}}} z|w|^2 - \int_{Q_{\text{soft}}} |\nabla w|^2. \end{aligned} \tag{3.28}$$

Since (3.27) does not depend on τ, the same is true for w, and thus for (3.28). Since $\text{Im}\, z \neq 0$ and $\|\nabla w\| \neq 0$, (3.28) implies that (3.24) is a non-zero constant. □

Similarly to $K_{\text{stiff-int}}(\tau, z)$, we make a few important observations for $K_{\text{stiff-ls}}(\tau, z)$.

1. Again, $K_{\text{stiff-ls}}(\tau, z)$ consists of two terms, $K_{a,\text{stiff-ls}}(\tau, z)$ and $K_{b,\text{stiff-ls}}(\tau, z)$.

   The first term, $K_{a,\text{stiff-ls}}(\tau, z)$, corresponds to the dispersion function for Model I of [35, Section 5.3] (one stiff component). In our case with two stiff components, we have a second "correction" term $K_{b,\text{stiff-ls}}(\tau, z)$.

2. $K_{a,\text{stiff-ls}}(\tau, z)$ depends on ε while $K_{b,\text{stiff-ls}}(\tau, z)$ does not.

3.1.3 Stiff-interior to stiff-landscape

We now turn our attention to $P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\text{hom}} - z)^{-1}P_{\mathbb{C}_{\text{stiff-int}}}$. The operator in question takes $\delta \in \mathbb{C}$, solves the system (3.4), and then outputs $\beta_{\text{stiff-ls}}$.

Once again, replacing $T_{\text{stiff-int}}$ by $T_{\text{stiff-ls}}$ in (3.5) gives us

$$T_{\text{stiff-ls}}\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} = \frac{\beta_{\text{stiff-int}}}{\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} + \frac{\beta_{\text{stiff-ls}}}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix}. \tag{3.29}$$

Using this, along with $\beta_{\text{stiff-int}} = \delta(K_{\text{stiff-int}}(\tau, z) - z)^{-1}$ (Theorem 3.1.6), we see that the third equation of the system (3.4), becomes

$$\left(\frac{1}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} - z \right) \beta_{\text{stiff-ls}}$$
$$= -\frac{\delta}{(K_{\text{stiff-int}}(\tau, z) - z)\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix}. \tag{3.30}$$

This prompts us to make the following definition

Definition 3.1.11. For $z \in K_\sigma$ and $\tau \in Q'$, let $F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z)$ be the number

$$F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z) := -\frac{1}{(K_{\text{stiff-int}}(\tau, z) - z)\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} \times \left(\frac{1}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}}\begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} - z \right)^{-1}. \tag{3.31}$$

Theorem 3.1.12. For $\varepsilon > 0$ small enough, independently of $z \in K_\sigma$ and $\tau \in Q'$, $P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\text{hom}} - z)^{-1}P_{\mathbb{C}_{\text{stiff-int}}} : \mathbb{C}_{\text{stiff-int}} \to \mathbb{C}_{\text{stiff-ls}}$ is the operator of multiplication by the number $F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z)$. That is,

$$P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1}P_{\mathbb{C}_{\text{stiff-int}}} = M_{F_{\text{stiff-int}\to\text{stiff-ls}}(\tau,z)}. \tag{3.32}$$

Proof. We just have to observe that the expression

$$\frac{1}{\|\Psi_1^{\text{stiff-ls},(\tau)}\|} T^{(\tau)}_{\varepsilon,\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} - z \tag{3.33}$$

is non-zero if and only if

$$1 - \frac{1}{z\|\Psi_1^{\text{stiff-ls},(\tau)}\|} T^{(\tau)}_{\varepsilon,\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} \tag{3.34}$$

is non-zero. The latter is shown to be true in the proof of Theorem 3.1.6, for ε small enough, independently of $z \in K_\sigma$ and $\tau \in Q'$. □

By a closer inspection of the proofs of Theorems 3.1.6 and 3.1.10, we can say more about $P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-int}}}$:

Corollary 3.1.13 (The case of large τ)**.** Suppose that τ is uniformly bounded away from 0, i.e. that $|\tau| > c > 0$ for some constant c independent of ε and z, then

$$P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-int}}} = O(\varepsilon^2), \tag{3.35}$$

in the operator norm. This estimate is uniform over all τ with $|\tau| > c$ and $z \in K_\sigma$.

Proof. In the proof of Theorem 3.1.6, we have already shown that if τ is uniformly bounded away from zero, then the denominator of the number $F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z)$, i.e. the expression (3.33), is of order $O(\varepsilon^{-2})$, uniformly over τ and $z \in K_\sigma$.

It remains to show that the numerator of $F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z)$ is of order $O(1)$, uniformly over τ and $z \in K_\sigma$. We check this in two steps:

Step 1: Similarly to the proof of Theorem 3.1.6, we may compute the action of $T_{\text{stiff-ls}}$ on the vector $(z(A_0^{\text{soft}} - z)^{-1}v + v, \|\Psi_1^{\text{stiff-int}}\|, 0)^T$:

$$T_{\text{stiff-ls}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1}v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} = -\frac{1}{\|\Psi_1^{\text{stiff-ls},(\tau)}\|} \left\langle M^{\text{soft},(\tau)}(z)(\psi_1^{\text{stiff-int},(\tau)} + 0), (\psi_1^{\text{stiff-int},(\tau)} + 0) \right\rangle. \tag{3.36}$$

Recall from the proof of Theorem 3.1.10, that this expression does not depend on ε, and may be bounded above by a constant, uniformly in $\tau \in Q'$ and $z \in K_\sigma$.

Step 2: We next compute:

$$\begin{aligned} |(K_{\text{stiff-int}}(\tau, z) - z)^{-1}| &= \|M_{(K_{\text{stiff-int}}(\tau,z)-z)^{-1}}\|_{\text{op}} \\ &\leq \|(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}} - z)^{-1}\|_{\text{op}} && \text{By Theorem 3.1.6.} \\ &= \frac{1}{\text{dist}(z, \sigma(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom}}))} \leq \sigma^{-1} && \text{As } z \in K_\sigma. \end{aligned} \tag{3.37}$$

This completes the proof. $\square$

3.1.4 Stiff-landscape to stiff-interior

We now turn our attention to $P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}}$. Once again, we omit the analogous derivation of $F_{\text{stiff-ls}\to\text{stiff-int}}(\tau, z)$ and jump straight to the result:

Definition 3.1.14. For $z \in K_\sigma$ and $\tau \in Q'$, let $F_{\text{stiff-ls}\to\text{stiff-int}}(\tau, z)$ be the number

$$F_{\text{stiff-ls}\to\text{stiff-int}}(\tau, z) := -\frac{1}{(K_{\text{stiff-ls}}(\tau, z) - z)\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} \times \\ \times \left(\frac{1}{\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} - z \right)^{-1}. \quad (3.38)$$

Theorem 3.1.15. For $\varepsilon > 0$ small enough, independently of $z \in K_\sigma$ and $\tau \in Q'$, $P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\text{hom}} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}} : \mathbb{C}_{\text{stiff-ls}} \to \mathbb{C}_{\text{stiff-int}}$ is the operator of multiplication by the number $F_{\text{stiff-ls}\to\text{stiff-int}}(\tau, z)$. That is,

$$P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}} = M_{F_{\text{stiff-ls}\to\text{stiff-int}}(\tau,z)}. \quad (3.39)$$

Proof. Similarly to Theorem 3.1.12, we note that the expression

$$\frac{1}{\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z(A_0^{\text{soft}} - z)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} - z \quad (3.40)$$

is non-zero if and only if (3.23) is non-zero, which is indeed the case by Theorem 3.1.10. $\square$

Corollary 3.1.16 (The case of large τ)**.** Suppose that τ is uniformly bounded away from 0, i.e. that $|\tau| > c > 0$ for some constant c independent of ε and z, then

$$P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}} = O(\varepsilon^2), \quad (3.41)$$

in the operator norm. This estimate is uniform over all τ with $|\tau| > c$ and $z \in K_\sigma$.

Proof. This follows from the identity

$$P_{\mathbb{C}_{\text{stiff-int}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - z)^{-1} P_{\mathbb{C}_{\text{stiff-ls}}} = \left(P_{\mathbb{C}_{\text{stiff-ls}}}(\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)} - \overline{z})^{-1} P_{\mathbb{C}_{\text{stiff-int}}} \right)^*, \quad (3.42)$$

and an application of Corollary 3.1.13, where we note that the $O(\varepsilon^2)$ estimate remains even after the set K_σ is enlarged to $K_\sigma \cup \{\overline{z} : z \in K_\sigma\}$. $\square$

3.2 The homogenized operator on the full space, $\mathcal{A}_{\varepsilon,\mathrm{hom}}$

The homogenization result that we have provided in Chapter 2 (Theorem 2.5.3) are stated fibre-wise (for each τ). In this section, we pass from the unit cell back to the full space, and provide formulas for the operator

$$\mathcal{A}_{\varepsilon,\mathrm{hom}} := G^* \left(\int_{Q'}^{\oplus} \mathcal{A}_{\varepsilon,\mathrm{hom}}^{(\tau)} d\tau \right) G, \tag{3.43}$$

where $G := G_1$ is the unscaled Gelfand transform. We will begin with Model I in Section 3.2.1, then Model II in Section 3.2.2, and finally discuss the stiff-soft-stiff model in Section 3.2.3.

Remark. In [35, Section 5.4], the authors obtained a full space description of the resolvent of homogenized operator, for Model I, when restricted to the stiff component. To be precise, if we denote the fibre-wise homogenized operator by $A_{\varepsilon,\mathrm{hom\text{-}I}}^{(\tau)}$ (see Section 3.2.1 for definition), then the authors showed that the operator

$$\int_{Q'}^{\oplus} \left(P_{\mathrm{stiff}} (A_{\varepsilon,\mathrm{hom\text{-}I}}^{(\tau)} - z)^{-1} P_{\mathrm{stiff}} \right) d\tau, \quad z \in K_\sigma$$

is unitarily equivalent to a pseudo-differential operator with symbol

$$a(x, \theta) = \mathbf{1}_{\varepsilon^{-1} Q'}(\theta) (K_I(\varepsilon\theta, z) - z)^{-1}.$$

See Definition 3.3.4 for the definition of the dispersion function K_I. In the following subsection (Section 3.2.1), we seek (3.43) for Model I, thus extending the result of [35, Section 5.4]. ◦

3.2.1 Model I

We begin by adapting the notation of Section 2.5 to Model I. In accordance with Figure 1-2, we have $\mathcal{H} = L^2(Q) = L^2(Q_{\mathrm{stiff}}) \oplus L^2(Q_{\mathrm{soft}})$ and $\mathcal{E} = L^2(\Gamma)$. Our boundary triples are

- (Full cube) $(A_{\varepsilon,0}^{(\tau)}, \Lambda_\varepsilon^{(\tau)}, \Pi^{(\tau)})$ w.r.t. $\mathcal{H}$ and $\mathcal{E}$.
- (Stiff component) $(A_{\varepsilon,0}^{\mathrm{stiff},(\tau)}, \Lambda_\varepsilon^{\mathrm{stiff},(\tau)}, \Pi^{\mathrm{stiff},(\tau)})$ w.r.t. $L^2(Q_{\mathrm{stiff}})$ and $\mathcal{E}$.
- (Soft component) $(A_0^{\mathrm{soft},(\tau)}, \Lambda^{\mathrm{soft},(\tau)}, \Pi^{\mathrm{soft},(\tau)})$ w.r.t. $L^2(Q_{\mathrm{soft}})$ and $\mathcal{E}$.

These are defined similarly to Section 2.2.1. For example, we have $\Lambda_\varepsilon^{(\tau)} = \Lambda_\varepsilon^{\mathrm{stiff},(\tau)} + \Lambda^{\mathrm{soft},(\tau)}$, which is a self-adjoint operator on $\mathcal{E}$ with domain $H^1(\Gamma)$. For each of the triples, we introduce auxiliary operators in accordance with Definition 2.2.12, keeping a similar notation to Chapter 2. We omit the details.

Let $(\mu_1^{\mathrm{stiff},(\tau)}, \psi_1^{\mathrm{stiff},(\tau)})$ be the first eigenvalue-eigenfunction pair with respect to $\widetilde{\Lambda}^{\mathrm{stiff},(\tau)} = \varepsilon^2 \Lambda_\varepsilon^{\mathrm{stiff},(\tau)}$. In the current setting, $\widetilde{\Lambda}^{\mathrm{stiff},(\tau)}$ share the same properties as $\widetilde{\Lambda}^{\mathrm{stiff\text{-}ls},(\tau)}$ of Proposition 2.2.10. We set $\Psi_1^{\mathrm{stiff},(\tau)} = \Pi^{\mathrm{stiff},(\tau)} \psi_1^{\mathrm{stiff},(\tau)}$. Our truncated spaces are

$$\check{\mathcal{E}}^{(\tau)} = \mathcal{P}^{(\tau)} \mathcal{E} = \mathrm{span}\{\psi_1^{\mathrm{stiff},(\tau)}\} \quad \text{and} \quad \check{\mathcal{H}}^{\mathrm{stiff},(\tau)} = \mathrm{ran}(\Pi^{\mathrm{stiff},(\tau)}|_{\check{\mathcal{E}}^{(\tau)}}). \tag{3.44}$$

By [35], the fibre-wise homogenized operator $\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}}$ is given by

$$\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}}) = \Bigg\{ \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} \in L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff},(\tau)} : \\ u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \text{ran}(\Pi^{\text{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}), \quad \widehat{u} = \breve{\Pi}^{\text{stiff},(\tau)} \breve{\Gamma}_0^{\text{soft},(\tau)} u \Bigg\}, \tag{3.45}$$

$$\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}} \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1} \mathcal{P}^{(\tau)} \left[\Gamma_1^{\text{soft},(\tau)} u + \varepsilon^{-2} \mu_1^{\text{stiff},(\tau)} \left(c^{(\tau)} \psi_1^{\text{stiff},(\tau)} \right) \right] \end{pmatrix}, \tag{3.46}$$

where $u = (A_0^{\text{soft},(\tau)})^{-1} f + \Pi^{\text{soft},(\tau)}(c^{(\tau)} \psi_1^{\text{stiff},(\tau)})$ for some $f \in L^2(Q_{\text{soft}})$ and $c^{(\tau)} \in \mathbb{C}$.

With these notation at hand, the goal for this section is to find the domain and action of

$$\mathcal{A}_{\varepsilon,\text{hom-I}} := G^* \left(\int_{Q'}^{\oplus} \mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}} d\tau \right) G. \tag{3.47}$$

In the case of Model I, Q_{soft} is of positive distance away from the boundary of the cube ∂Q, and so

$$G^* \left(\int_{Q'}^{\oplus} A_0^{\text{soft},(\tau)} d\tau \right) G = \bigoplus_{n \in \mathbb{Z}^d} A_{0,n}^{\text{soft},(0)}, \tag{3.48}$$

where on the RHS, $A_{0,n}^{\text{soft},(0)}$ refers to the Dirichlet Laplacian on the soft part of $[0,1)^d + n$.

The domain of $\mathcal{A}_{\varepsilon,\text{hom-I}}$

The first component of $\mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom-I}})$ consists of $v \in L^2(\mathbb{R}^d)$ such that the function $Gv \in L^2(Q \times Q')$ satisfies $Gv(\cdot, \tau) \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \text{ran}(\Pi^{\text{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}})$ for almost every τ. By (3.45) and (3.48), we know that

$$v \in \left[\mathcal{D} \left(\bigoplus_{n \in \mathbb{Z}^d} A_{0,n}^{\text{soft},(0)} \right) \dotplus G^* \left(\Pi^{\text{soft},(\tau)} (\text{span}\{\psi_1^{\text{stiff},(\tau)}\}) \right) \right] \bigcap L^2(\mathbb{R}^d). \tag{3.49}$$

That is,

$$v = \sum_{n \in \mathbb{Z}^d} v_n + G^* \left(c^{(\tau)} \Pi^{\text{soft},(\tau)} \psi_1^{\text{stiff},(\tau)} \right), \tag{3.50}$$

subject to the conditions

$$\begin{cases} v_n \in \mathcal{D}(A_{0,n}^{\text{soft},(0)}) = H^2(Q_{\text{soft}} + n) \cap H_0^1(Q_{\text{soft}} + n) \quad \text{and} \quad \| \sum v_n \|_{L^2(\mathbb{R}^d)} < \infty. \\ c \in L^2(Q'). \end{cases} \tag{3.51}$$

The requirement that $Q' \ni \tau \mapsto c^{(\tau)} \in \mathbb{C}$ belongs to $L^2(Q')$ follows from the fact that the mapping $\overline{Q'} \ni \tau \mapsto \Pi^{\text{soft},(\tau)} \psi_1^{\text{stiff},(\tau)}$ is continuous, and hence belongs to $L^\infty(Q')$.

Having determined v, the second component of $\mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom-I}})$ is now fixed by (3.45): In the notation of (3.50), these are the $\widetilde{v}$ such that

$$\begin{aligned}\widetilde{v} &= G^*\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}Gv \\ &= G^*\Pi^{\text{stiff},(\tau)}\cancel{\Gamma_0^{\text{soft},(\tau)}GG^*\Pi^{\text{soft},(\tau)}}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)}) = G^*\left(c^{(\tau)}\Psi_1^{\text{stiff},(\tau)}\right).\end{aligned} \tag{3.52}$$

The action of $\mathcal{A}_{\varepsilon,\text{hom-I}}$

Let us now compute the action of $\mathcal{A}_{\varepsilon,\text{hom-I}}$. We already know its action on the soft component, by (3.48). As for its action on the stiff component, we perform two separate computations.

Computation 1: Fix τ, and let $u^{(\tau)} \in \mathcal{D}(\Gamma_1^{(\tau)}) \subset L^2(Q)$. Then

$$\begin{aligned}-(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\mathcal{P}^{(\tau)}\Gamma_1^{\text{soft},(\tau)}u^{(\tau)} &= -(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\left\langle \partial_n^{(\tau)}u^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)}\psi_1^{\text{stiff},(\tau)} \\ &\overset{\text{Lemma 2.5.2}}{=} -\left\langle \partial_n^{(\tau)}u^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)}\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}.\end{aligned} \tag{3.53}$$

Computation 2: Let $c \in L^2(Q')$. Then

$$\begin{aligned}&-(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\mathcal{P}^{(\tau)}\Gamma_{\varepsilon,1}^{\text{stiff},(\tau)}\left(c^{(\tau)}\Psi_1^{\text{stiff},(\tau)}\right) \\ &= -(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}c^{(\tau)}\cancel{\left\langle\psi_1^{\text{stiff},(\tau)},\psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)}}\psi_1^{\text{stiff},(\tau)} && \text{By } \Lambda = \Gamma_1\Pi. \\ &= -\varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}c^{(\tau)}\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}. && \text{By Lemma 2.5.2.}\end{aligned} \tag{3.54}$$

Conclusion

We summarize our results for Model I. The domain of $\mathcal{A}_{\varepsilon,\text{hom-I}}$ consists of pairs $(v,\widetilde{v}) \in L^2(\cup_n(Q_{\text{soft}}+n)) \oplus L^2(\cup_n(Q_{\text{stiff}}+n))$ such that

$$\begin{cases} v \in \mathcal{D}\left(\bigoplus_{n\in\mathbb{Z}^d} A_{0,n}^{\text{soft},(0)}\right) \dotplus \left\{G^*\left(\Pi^{\text{soft},(\tau)}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)})\right) : c \in L^2(Q')\right\}, \\ \widetilde{v} = G^*\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}Gv. \end{cases} \tag{3.55}$$

Equivalently, $(v,\widetilde{v}) \in \mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom-I}})$ if and only if

- v can be written in the form (3.50), under the conditions (3.51).
- $\widetilde{v}$ is determined by v, through the formula (3.52).

The action of $\mathcal{A}_{\varepsilon,\text{hom-I}}$, with respect to the decomposition $L^2(\cup_n(Q_{\text{soft}}+n)) \oplus L^2(\cup_n(Q_{\text{stiff}}+n))$, and in the notation of (3.50), is given by

$$\mathcal{A}_{\varepsilon,\text{hom-I}}\begin{pmatrix} v \\ \widetilde{v}\end{pmatrix} = \begin{pmatrix} -\Delta\left(\sum v_n\right) \\ G^*\left[\left(-\left\langle\partial_n^{(\tau)}(Gv)^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)} - \varepsilon^{-2}c^{(\tau)}\mu_1^{\text{stiff},(\tau)}\right)\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}\right]\end{pmatrix}. \tag{3.56}$$

$\mathcal{A}_{\varepsilon,\text{hom-I}}$ is symmetric on $L^2(\mathbb{R}^d)$, and is self-adjoint on the subspace

$$L^2(\cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}}+n)) \oplus \left\{G^*\left(\Pi^{\text{soft},(\tau)}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)})\right) : c \in L^2(Q')\right\}. \tag{3.57}$$

3.2.2 Model II

Most of Section 3.2.1 carries over to Model II, so we will keep our discussion fairly brief. We keep the same notation for the spaces $\mathcal{H}$, $\mathcal{E}$, the boundary triples, and the truncated spaces. We note that $\widetilde{\Lambda}^{\text{stiff},(\tau)}$ now share the same properties as $\widetilde{\Lambda}^{\text{stiff-int},(\tau)}$ of Proposition 2.2.10, and in particular $\mu_1^{\text{stiff},(\tau)} \equiv 0$. As a result, the fibre-wise homogenized operator $\mathcal{A}_{\text{hom-II}}^{(\tau)}$ for Model II, does not depend on ε, and by [35] is given by

$$\mathcal{D}(\mathcal{A}_{\text{hom-II}}^{(\tau)}) = \left\{ \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} \in L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff},(\tau)} : \right.$$
$$\left. u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \operatorname{ran}(\Pi^{\text{soft},(\tau)}|_{\breve{\mathcal{E}}^{(\tau)}}), \quad \widehat{u} = \breve{\Pi}^{\text{stiff},(\tau)}\breve{\Gamma}_0^{\text{soft},(\tau)}u \right\}, \tag{3.58}$$

$$\mathcal{A}_{\text{hom-II}}^{(\tau)} \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -(\breve{\Pi}^{\text{stiff},(\tau)*})^{-1}\mathcal{P}^{(\tau)}\Gamma_1^{\text{soft},(\tau)}u \end{pmatrix}, \tag{3.59}$$

where $u = (A_0^{\text{soft},(\tau)})^{-1}f + \Pi^{\text{soft},(\tau)}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)})$ for some $f \in L^2(Q_{\text{soft}})$ and $c^{(\tau)} \in \mathbb{C}$. Now set

$$\mathcal{A}_{\text{hom-II}} := G^*\left(\int_{Q'}^{\oplus} \mathcal{A}_{\text{hom-II}}^{(\tau)} d\tau\right) G. \tag{3.60}$$

In the case of Model II, we have

$$G^*\left(\int_{Q'}^{\oplus} A_0^{\text{soft},(\tau)} d\tau\right) G = -\Delta_D \tag{3.61}$$

where $-\Delta_D$ denotes the Dirichlet Laplacian on $L^2(\cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}}+n))$. In contrast to Model I, $\cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}}+n)$ is now a connected set.

The domain of $\mathcal{A}_{\text{hom-II}}$ consists of pairs $(v,\widetilde{v}) \in L^2(\cup_n(Q_{\text{soft}}+n)) \oplus L^2(\cup_n(Q_{\text{stiff}}+n))$ such that

$$\begin{cases} v \in \mathcal{D}(-\Delta_D) \dotplus \left\{G^*\left(\Pi^{\text{soft},(\tau)}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)})\right) : c \in L^2(Q')\right\}. \\ \widetilde{v} = G^*\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}Gv. \end{cases} \tag{3.62}$$

Equivalently, $(v,\widetilde{v}) \in \mathcal{D}(\mathcal{A}_{\text{hom-II}})$ if and only if

$$\begin{cases} v = v_D + G^*\left(c^{(\tau)}\Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}\right), \quad \text{where} \quad v_D \in \mathcal{D}(-\Delta_D) \quad \text{and} \quad c \in L^2(Q). \\ \widetilde{v} = G^*\left(c^{(\tau)}\Psi_1^{\text{stiff},(\tau)}\right). \end{cases} \tag{3.63}$$

The action of $\mathcal{A}_{\text{hom-II}}$, with respect to the decomposition $L^2(\cup_n(Q_{\text{soft}}+n)) \oplus L^2(\cup_n(Q_{\text{stiff}}+$

$n)$), and in the notation of (3.63), is given by

$$\mathcal{A}_{\text{hom-II}}\begin{pmatrix} v \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} -\Delta v_D \\ G^*\left[-\left\langle \partial_n^{(\tau)}(Gv)^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}\right] \end{pmatrix}. \tag{3.64}$$

$\mathcal{A}_{\text{hom-II}}$ is symmetric on $L^2(\mathbb{R}^d)$, and is self-adjoint on the subspace

$$L^2(\cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}}+n)) \oplus \left\{G^*\left(\Pi^{\text{soft},(\tau)}(c^{(\tau)}\psi_1^{\text{stiff},(\tau)})\right) : c \in L^2(Q')\right\}. \tag{3.65}$$

3.2.3 Stiff-soft-stiff model

Finally, we compute the domain and action of $\mathcal{A}_{\varepsilon,\text{hom}}$, as defined in (3.43). In the present case, the annulus Q_{soft} is of positive distance away from ∂Q, thus

$$G^*\left(\int_{Q'}^{\oplus} A_0^{\text{soft},(\tau)} d\tau\right) G = \bigoplus_{n\in\mathbb{Z}^d} A_{0,n}^{\text{soft},(0)}, \tag{3.66}$$

where $A_{0,n}^{\text{soft},(0)}$ refers to the Dirichlet Laplacian on the soft part of $[0,1)^d + n$.

The domain of $\mathcal{A}_{\varepsilon,\text{hom}}$ consists of triples

$$(v, \widetilde{v}_{\text{stiff-int}}, \widetilde{v}_{\text{stiff-ls}}) \in L^2(\cup_n(Q_{\text{soft}}+n)) \oplus L^2(\cup_n(Q_{\text{stiff-int}}+n)) \oplus L^2(\cup(Q_{\text{stiff-ls}}+n))$$

such that

$$\begin{cases} v \in \mathcal{D}\left(\bigoplus_{n\in\mathbb{Z}^d} A_{0,n}^{\text{soft},(0)}\right) \\ \qquad\qquad \dot{+}\left\{G^*\left(\Pi^{\text{soft},(\tau)}(a^{(\tau)}\psi_1^{\text{stiff-int},(\tau)}, b^{(\tau)}\psi_1^{\text{stiff-ls},(\tau)})\right) : a, b \in L^2(Q')\right\}, \\ \widetilde{v}_{\text{stiff-int}} = G^*\Pi^{\text{stiff-int},(\tau)}\Gamma_0^{\text{soft},(\tau)}Gv, \\ \widetilde{v}_{\text{stiff-ls}} = G^*\Pi^{\text{stiff-ls},(\tau)}\Gamma_0^{\text{soft},(\tau)}Gv. \end{cases} \tag{3.67}$$

Remark. While the condition $v \in L^2$ only asks that $a^{(\tau)} + b^{(\tau)} \in L^2(Q')$, we have to impose the stronger condition $a^{(\tau)}, b^{(\tau)} \in L^2(Q')$, as we also want $\widetilde{v}_{\text{stiff-int}} \in L^2$ and $\widetilde{v}_{\text{stiff-ls}} \in L^2$. ◦

Equivalently, $(v, \widetilde{v}_{\text{stiff-int}}, \widetilde{v}_{\text{stiff-ls}}) \in \mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom}})$ if and only if

- v can be written in the form

$$v = \sum_{n\in\mathbb{Z}^d} v_n + G^*\left(\Pi^{\text{soft},(\tau)}(a^{(\tau)}\psi_1^{\text{stiff-int},(\tau)}, b^{(\tau)}\psi_1^{\text{stiff-ls},(\tau)})\right), \tag{3.68}$$

  subject to the conditions

$$\begin{cases} v_n \in H^2(Q_{\text{soft}}+n) \cap H_0^1(Q_{\text{soft}}+n) \quad \text{and} \quad \|\sum v_n\|_{L^2(\mathbb{R}^d)} < \infty. \\ a, b \in L^2(Q'). \end{cases} \tag{3.69}$$

- $\widetilde{v}_{\text{stiff-int}}$ and $\widetilde{v}_{\text{stiff-ls}}$ are determined by v, through

$$\widetilde{v}_{\text{stiff-int}} = G^* \left(a^{(\tau)} \Psi_1^{\text{stiff-int},(\tau)} \right) \quad \text{and} \quad \widetilde{v}_{\text{stiff-ls}} = G^* \left(b^{(\tau)} \Psi_1^{\text{stiff-ls},(\tau)} \right). \tag{3.70}$$

The action of $\mathcal{A}_{\varepsilon,\text{hom}}$, with respect to the decomposition

$$L^2(\cup_n (Q_{\text{soft}} + n)) \oplus L^2(\cup_n (Q_{\text{stiff-int}} + n)) \oplus L^2(\cup_n (Q_{\text{stiff-ls}} + n)),$$

and in the notation of (3.68), is given by

$$\mathcal{A}_{\varepsilon,\text{hom}} \begin{pmatrix} v \\ \widetilde{v}_{\text{stiff-int}} \\ \widetilde{v}_{\text{stiff-ls}} \end{pmatrix}$$
$$= \begin{pmatrix} -\Delta \left(\sum v_n \right) \\ G^* \left[-\left\langle \partial_n^{(\tau)} (Gv)^{(\tau)}, \psi_1^{\text{stiff-int},(\tau)} \right\rangle_{L^2(\Gamma)} \dfrac{1}{\| \Psi_1^{\text{stiff-int},(\tau)} \|^2} \Psi_1^{\text{stiff-int},(\tau)} \right] \\ G^* \left[\left(-\left\langle \partial_n^{(\tau)} (Gv)^{(\tau)}, \psi_1^{\text{stiff-ls},(\tau)} \right\rangle_{L^2(\Gamma)} - \varepsilon^{-2} b^{(\tau)} \mu_1^{\text{stiff-ls},(\tau)} \right) \dfrac{1}{\| \Psi_1^{\text{stiff-ls},(\tau)} \|^2} \Psi_1^{\text{stiff-ls},(\tau)} \right] \end{pmatrix}. \tag{3.71}$$

$\mathcal{A}_{\varepsilon,\text{hom}}$ is symmetric on $L^2(\mathbb{R}^d)$, and is self-adjoint on the subspace

$$L^2(\cup_{n \in \mathbb{Z}^d} (Q_{\text{soft}} + n)) \oplus \left\{ G^* \left(\Pi^{\text{soft},(\tau)} (a^{(\tau)} \psi_1^{\text{stiff-int},(\tau)}, b^{(\tau)} \psi_1^{\text{stiff-ls},(\tau)}) \right) : a, b \in L^2(Q') \right\}. \tag{3.72}$$

3.3 Spectral analysis of $\mathcal{A}_{\varepsilon,\text{hom}}$, first steps

In this section, we embark on the task of identifying the spectrum and spectral decomposition of $\mathcal{A}_{\varepsilon,\text{hom}}$ (defined in (3.43)). We follow the approach of the two-scale strong resolvent case [53], proceeding in three steps:

1. Find the eigenvalues of $\mathcal{A}_{\varepsilon,\text{hom}}$.
2. Characterize $\sigma(\mathcal{A}_{\varepsilon,\text{hom}})$ in terms of the dispersion functions $K_{\text{stiff-int}}(\tau, z)$ and $K_{\text{stiff-ls}}(\tau, z)$.
3. Prove the absence of singular continuous spectrum.

As mentioned at the start of the chapter, we do not complete the program in this thesis. We will only discuss steps 1 and 2 for $\mathcal{A}_{\varepsilon,\text{hom-I}}$ (Model I) and $\mathcal{A}_{\text{hom-II}}$ (Model II), and include a short discussion on the stiff-soft-stiff case $\mathcal{A}_{\varepsilon,\text{hom}}$. A list of unfinished tasks can be found in Chapter 4.

3.3.1 Model I

We use the notation of Section 3.2.1. We will write $c(\tau)$ to also mean the function $c^{(\tau)}$ in (3.50).

The eigenvalues of $\mathcal{A}_{\varepsilon,\text{hom-I}}$

We begin with a preparatory lemma.

Lemma 3.3.1 (Model I)**.** Let $c \in L^2(Q')$. Suppose that

$$\left(\langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)}\rangle_{L^2(\Gamma)} + \varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}\right) c^{(\tau)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2} \Psi_1^{\text{stiff},(\tau)}(x) = 0 \tag{3.73}$$

for a.e. $x \in Q$ and $\tau \in Q'$, then $c^{(\tau)} = 0$ for almost every τ.

Proof. In the case of Model I, we have

$$\begin{cases} \langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)}\rangle \le 0 & \text{for every } \tau \in Q', \\ \mu_1^{\text{stiff},(0)} = 0, & \\ \mu_1^{\text{stiff},(\tau)} < 0 & \text{for every } \tau \in Q' \setminus \{0\}. \end{cases} \tag{3.74}$$

Moreover, by the same arguments as Lemma 3.1.3, we also know that

$$0 < c < \|\Psi_1^{\text{stiff},(\tau)}\| < C < \infty, \quad \text{for constants } c,\, C \text{ which do not depend on } \tau. \tag{3.75}$$

Let us abbreviate the LHS of (3.73) as $f(\tau)c(\tau)\|\Psi_1^{\text{stiff},(\tau)}\|^{-2}\Psi_1^{\text{stiff},(\tau)}(x)$, and compute the square of its norm in $L^2(Q \times Q')$:

$$\begin{aligned} 0 &= \int_{Q'}\int_Q \left| f(\tau)c(\tau)\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}(x)\right|^2 dx d\tau \\ &= \int_{Q'} \left| f(\tau)c(\tau)\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\right|^2 \int_Q \left|\Psi_1^{\text{stiff},(\tau)}(x)\right|^2 dx d\tau = \int_{Q'} |f(\tau)|^2|c(\tau)|^2 \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2} d\tau. \end{aligned} \tag{3.76}$$

The result now follows from (3.75) and (3.74). □

Let us now find the eigenvalues of $\mathcal{A}_{\varepsilon,\text{hom-I}}$.

Proposition 3.3.2. 0 is not an eigenvalue of $\mathcal{A}_{\varepsilon,\text{hom-I}}$.

Proof. Let $(v, \widetilde{v}) \in \mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom-I}})$ and suppose that

$$\mathcal{A}_{\varepsilon,\text{hom-I}} \begin{pmatrix} v \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.77}$$

This is a system of two equations. The first equation implies that v must be of the form

$$v = G^*\left(c^{(\tau)}\Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}\right), \quad \text{for some } c \in L^2(Q'), \tag{3.78}$$

as the Dirichlet Laplacian $A_0^{\text{soft},(0)}$ on the soft component of Q has trivial kernel. Substituting this into the second equation of the system (3.77), and we have

$$\left(\left\langle \Gamma_1^{\text{soft},(\tau)}(Gv)^{(\tau)}\right\rangle + \varepsilon^{-2}c^{(\tau)}\mu_1^{\text{stiff},(\tau)}\right)\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}(x) = 0, \tag{3.79}$$

for a.e. x and τ. Equivalently,

$$\left(\langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)}\rangle + \varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}\right) c^{(\tau)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}(x) = 0, \tag{3.80}$$

for a.e. x and τ. By Lemma 3.3.1, we obtain $c = 0$. Thus $(v, \widetilde{v}) = (0, 0)$. □

Let us now turn to the non-zero eigenvalues of $\mathcal{A}_{\varepsilon,\text{hom-I}}$. Let $\lambda \neq 0$, and consider the eigenvalue equation

$$\mathcal{A}_{\varepsilon,\text{hom-I}} \begin{pmatrix} v \\ \widetilde{v} \end{pmatrix} = \lambda \begin{pmatrix} v \\ \widetilde{v} \end{pmatrix}. \tag{3.81}$$

We enumerate the possibilities where $(v, \widetilde{v})$ could be an eigenfunction for $\mathcal{A}_{\varepsilon,\text{hom-I}}$ w.r.t eigenvalue λ.

Case 1: v is of the form

$$v = G^*\left(c^{(\tau)}\Pi^{\textbf{soft},(\tau)}\psi_1^{\textbf{stiff},(\tau)}\right), \quad \textbf{for some non-zero } c \in L^2(Q')\textbf{.} \tag{3.82}$$

Then, by definition of $\mathcal{A}_{\varepsilon,\text{hom-I}}$, the first component of $\mathcal{A}_{\varepsilon,\text{hom-I}}(v, \widetilde{v})^T$ is zero. This implies that the first equation of the system (3.81) is $0 = \lambda v$. Since $\lambda \neq 0$, we get $0 = v$, and thus $\widetilde{v} = 0$. In other words, it is not possible to have eigenfunctions of the form (3.82).

Case 2: v is of the form

$$v = \sum v_n \neq 0, \quad \textbf{where } v_n \textbf{ satisfies (3.45).} \tag{3.83}$$

Then $\widetilde{v} = 0$, and the eigenvalue equation (3.81) becomes

$$\begin{pmatrix} -\Delta(\sum v_n) \\ G^*\left[-\left\langle \partial_n^{(\tau)}(G(\sum v_n))^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)}\right] \end{pmatrix} = \lambda \begin{pmatrix} \sum v_n \\ 0 \end{pmatrix}. \tag{3.84}$$

By the first equation of the system (3.84), this is only possible if $\lambda \in \sigma(A_0^{\text{soft},(0)})$. Moreover, $v = \sum v_n$ must be constructed from eigenfunctions of $A_0^{\text{soft},(0)}$ with respect to the eigenvalue λ of $A_0^{\text{soft},(0)}$. We will now investigate how the eigenfunctions of $A_0^{\text{soft},(0)}$ can be used to create an eigenfunction v of $\mathcal{A}_{\varepsilon,\text{hom-I}}$.

Assume for the moment, that λ is a simple eigenvalue of $A_0^{\text{soft},(0)}$, corresponding to the eigenfunction $w \in L^2(Q_{\text{soft}})$. If we set $v = w$ (i.e. take $v = w$ on Q, and extend by zero to the whole of $\mathbb{R}^d$), then a direct computation gives us

$$(Gv)(x, \tau) = \frac{1}{(2\pi)^{d/2}} w(x) e^{-i\tau\cdot x}.$$

Thus, if $\sum v_n = c_1 w(\cdot + n_1) + c_2 w(\cdot + n_2)$ for some c_1, $c_2 \in \mathbb{C}$ and n_1, $n_2 \in \mathbb{Z}^d$, then

$$(Gv)(x, \tau) = \frac{1}{(2\pi)^{d/2}} w(x) e^{-i\tau\cdot x}\left(c_1 e^{-i\tau\cdot n_1} + c_2 e^{-i\tau\cdot n_2}\right).$$

In general, we can take

$$v = \sum_{n\in\mathbb{Z}^d} v_n = \sum_{n\in\mathbb{Z}^d} c_n w(\cdot + n), \qquad \text{where } c_n \in \mathbb{C}, \text{ and } v \in L^2(\mathbb{R}^d). \tag{3.85}$$

This is equivalent to

$$(Gv)(x,\tau) = \frac{1}{(2\pi)^{d/2}} w(x)e^{-i\tau\cdot x}a(\tau), \qquad \text{for some } a \in L^2(Q'). \tag{3.86}$$

Let us put (3.86) into the second equation of the system (3.84). We note that G is unitary, and we take the $L^2(Q\times Q')$ norm for the expression in the square brackets (see the argument of Lemma 3.3.1). This gives us the condition for λ to be an eigenvalue for $\mathcal{A}_{\varepsilon,\text{hom-I}}$:

$$\left\langle \Gamma_1^{\text{soft},(\tau)}(w(x)e^{-i\tau\cdot x}), \psi_1^{\text{stiff},(\tau)} \right\rangle a(\tau) = 0, \quad \text{for some non-zero } a \in L^2(Q').$$

Equivalently, if

$$\left| \left\{ \tau \in Q' : \left\langle \Gamma_1^{\text{soft},(\tau)}(w(x)e^{-i\tau\cdot x}), \psi_1^{\text{stiff},(\tau)} \right\rangle = 0 \right\} \right| > 0. \tag{3.87}$$

Remark. The condition (3.87) should be thought of as the norm-resolvent analogue of the two-scale strong resolvent case in [53]. For instance, when $\tau = 0$, the equation in (3.87) becomes

$$\left\langle -\frac{\partial w}{\partial n_{\text{soft}}}\bigg|_\Gamma, \mathbf{1}_\Gamma \right\rangle_{L^2(\Gamma)} = 0.$$

(Recall that $\psi_1^{\text{stiff},(0)} = |\Gamma|^{-\frac{1}{2}}\mathbf{1}_\Gamma$ by Proposition 2.2.10.) That is, the Neumann trace of the Dirichlet eigenfunction (on Q_{soft}) has zero mean. Compare this with [53], where the author looked at whether the eigenspace of $A_0^{\text{soft},(0)}$ w.r.t λ contains eigenfunctions of zero mean, i.e. $\langle w, \mathbf{1}\rangle_{L^2(Q_{\text{soft}})} = 0$. ◦

The above argument may be enhanced to include the case when the eigenvalue λ of $A_0^{\text{soft},(0)}$ has multiplicity $K > 1$. We omit the details and jump straight to the criterion: Write the K linearly independent eigenfunctions as $w_1, \cdots, w_K$. Then λ is an eigenvalue for $\mathcal{A}_{\varepsilon,\text{hom-I}}$ if

$$\left| \left\{ \tau \in Q' : \left\langle \Gamma_1^{\text{soft},(\tau)}(w_k(x)e^{-i\tau\cdot x}), \psi_1^{\text{stiff},(\tau)} \right\rangle = 0 \right\} \right| > 0 \quad \text{for some } k \in \{1,\cdots,K\}. \tag{3.88}$$

Case 3: v is of the form

$$v = \sum v_n + G^*\left(c^{(\tau)}\Pi^{\textbf{soft},(\tau)}\psi_1^{\textbf{stiff},(\tau)} \right), \tag{3.89}$$

where where c and v_n satisfies (3.45), and both c and $\sum v_n$ are non-zero.

Then, the first component of eigenvalue equation is

$$\sum -\Delta v_n = \sum \lambda v_n + \lambda G^*\left(c^{(\tau)}\Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)} \right), \tag{3.90}$$

and the second component of eigenvalue equation is

$$\begin{aligned}&\left\langle \Gamma_1^{\text{soft},(\tau)}(G(\textstyle\sum v_n))^{(\tau)}, \psi_1^{\text{stiff},(\tau)} \right\rangle \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)} \\ &+ \left[\left\langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)} \right\rangle + \varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}\right] c^{(\tau)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2}\Psi_1^{\text{stiff},(\tau)} = \lambda c^{(\tau)}\Psi_1^{\text{stiff},(\tau)}. \end{aligned} \tag{3.91}$$

To keep the notation compact, let us set

$$\begin{cases} g(\tau) = \left\langle \Gamma_1^{\text{soft},(\tau)}(G(\sum v_n))^{(\tau)}, \psi_1^{\text{stiff},(\tau)} \right\rangle, \\ f(\tau) = \left\langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)} \right\rangle + \varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}, \end{cases} \tag{3.92}$$

so that after rearranging (3.91), we obtain

$$\left[g(\tau) + \left(f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2\right) c(\tau)\right] \Psi^{\text{stiff},(\tau)}(x) = 0, \tag{3.93}$$

for a.e. x and τ. By taking the $L^2(Q \times Q')$ norm (see the argument of Lemma 3.3.1) we arrive at the condition

$$g(\tau) + \left(f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2\right) c(\tau) = 0, \quad \text{for almost every } \tau. \tag{3.94}$$

To summarize the present case:

> v taking the form (3.89) is an eigenfunction for $\mathcal{A}_{\varepsilon,\text{hom-I}}$ w.r.t eigenvalue λ, if (3.90) and (3.94) are satisfied.

Unfortunately, (3.90) and (3.94) are a pretty unwieldy set of conditions. At the time of writing, the author is unsure if these conditions could be simplified further. Nonetheless, we attempt to address this concern by unpacking aspects of (3.90) and (3.94) (The following bullet points are not critical to our discussion.):

- We focus on $\lambda > 0$, since $\mathcal{A}_{\varepsilon,\text{hom-I}}$ is asymptotically close to the main model operator A_ε, which are non-negative. So let $\lambda > 0$. Then the expression $f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2$ is zero at $\tau = 0$, and is strictly negative if $\tau \neq 0$. This allows us to rearrange (3.94) into

$$-\frac{g(\tau)}{f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2} = c(\tau). \tag{3.95}$$

  (Note that $f(\tau)-\lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2$ does not depend on v.) Moreover, by a continuity argument, we know that $\tau \mapsto f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2$ belongs to $L^\infty(Q')$. So if $c \in L^2(Q')$, then $g \in L^2(Q')$. The converse does not hold, as $f(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2$ is not bounded away from zero.

  Note that to construct v (3.89), we make a choice on v_n and c, which in turn determines $g(\tau)$ and $c(\tau)$ respectively (colored in blue in (3.95)). Thus, (3.95) says that once v_n has been picked, c is also fixed.

- As for the first condition, let us rearrange (3.90) into

$$\left(\oplus A_{0,n}^{\mathrm{soft},(0)} - \lambda\right)\left(\sum v_n\right) = \lambda G^* \left(c^{(\tau)} \Pi^{\mathrm{soft},(\tau)} \psi_1^{\mathrm{stiff},(\tau)}\right). \tag{3.96}$$

  This shows that if v takes form (3.89) and is an eigenfunction for $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$, then $\sum v_n \notin C_c^\infty(\cup(Q_{\mathrm{soft}}+n))$. Indeed, if $\sum v_n$ were so, then the LHS of (3.96) lies in $\mathcal{D}(\oplus A_{0,n}^{\mathrm{soft},(0)})$, which has trivial intersection with the subspace $\{G^*(\Pi^{\mathrm{soft},(\tau)} c^{(\tau)} \psi_1^{\mathrm{stiff},(\tau)}) : c \in L^2(Q')\}$.

- In fact, it is rather unlikely that a positive $\lambda \in \rho(A_0^{\mathrm{soft},(0)}) = \rho(\oplus A_{0,n}^{\mathrm{soft},(0)})$ is an eigenvalue for $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$. This is because $\oplus A_{0,n}^{\mathrm{soft},(0)} - \lambda$ is now a bijection, and combining (3.95) with (3.96) and (3.48), we require that $\sum v_n$ satisfy

$$\left(\int_{Q'}^{\oplus} (A_0^{\mathrm{soft},(\tau)} - \lambda) d\tau\right) G(\sum v_n) = \frac{\lambda \left\langle \Gamma_1^{\mathrm{soft},(\tau)} (G(\sum v_n))^{(\tau)}, \psi_1^{\mathrm{stiff},(\tau)} \right\rangle}{f(\tau) - \lambda \|\Psi_1^{\mathrm{stiff},(\tau)}\|^2} \Pi^{\mathrm{soft},(\tau)} \psi_1^{\mathrm{stiff},(\tau)}. \tag{3.97}$$

  In other words, a positive $\lambda \in \rho(A_0^{\mathrm{soft},(\tau)})$ is an eigenvalue for $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$ if and only if there exists v of the form (3.89) (Case 3), where the LHS of (3.95) belongs to $L^2(Q')$, and $\sum v_n$ satisfies (3.97). While unlikely, it remains to be proven that this cannot happen.

- We do not exclude the possibility that eigenfunctions of the form (3.83) (Case 2) and (3.89) (Case 3) contribute to the same eigenvalue λ.

We summarize our findings in the proposition below:

Proposition 3.3.3 (Eigenvalues of $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$)**.** $\lambda \in \mathbb{R}$ is an eigenvalue for $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$ if and only if $\lambda \neq 0$ and if either one of the following (not mutually exclusive) criterion is satisfied:

- $\lambda \in \sigma(A_0^{\mathrm{soft},(0)}) = \sigma(\oplus A_{0,n}^{\mathrm{soft},(0)})$ is such that (3.88) holds.
- There exist v of the form (3.89) satisfying (3.90) and (3.94).

The dispersion function for Model I

Before we proceed to Step 2 of our analysis (locating $\sigma(\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}})$), we have to recall the dispersion function $K_I(\tau, z)$ for Model I defined in [35].

Identify $\check{\mathcal{H}}^{\mathrm{stiff},(\tau)} = \mathrm{ran}(\Pi^{\mathrm{stiff},(\tau)}|_{\mathcal{P}^{(\tau)}\mathcal{E}})$ with $\mathbb{C}$ using the isomorphism

$$\begin{aligned} j^{(\tau)} &: \mathrm{ran}(\Pi^{\mathrm{stiff},(\tau)}|_{\mathcal{P}^{(\tau)}\mathcal{E}}) \longrightarrow \mathbb{C} \\ j^{(\tau)} \Psi_1^{\mathrm{stiff},(\tau)} &= \|\Psi_1^{\mathrm{stiff},(\tau)}\|_{L^2(Q_{\mathrm{stiff}})}, \end{aligned} \tag{3.98}$$

so that our homogenized operator may now be viewed as an operator on $L^2(Q_{\mathrm{soft}}) \oplus \mathbb{C}$, which we will still denote by $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}^{(\tau)}$ (and similarly for $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}$). In that case, $\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}^{(\tau)}$ is now given by

$$\mathcal{D}(\mathcal{A}_{\varepsilon,\mathrm{hom\text{-}I}}^{(\tau)}) = \{(u,\beta) \in L^2(Q_{\mathrm{soft}}) \oplus \mathbb{C} :$$

$$u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \operatorname{ran}(\Pi^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}), \quad \beta = j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}u\}, \tag{3.99}$$

$$\begin{aligned}\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)}\begin{pmatrix} u \\ \beta \end{pmatrix} &= \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -((j^{(\tau)}\breve{\Pi}^{\text{stiff},(\tau)})^*)^{-1}\mathcal{P}^{(\tau)}\left[\Gamma_1^{\text{soft},(\tau)}u + \varepsilon^{-2}\mu_1^{\text{stiff},(\tau)}(j^{(\tau)}\breve{\Pi}^{\text{stiff},(\tau)})^{-1}\beta\right] \end{pmatrix} \\ &=: \begin{pmatrix} -(\nabla + i\tau)^2 u \\ T_{\varepsilon,I}^{(\tau)}(u,\beta)^\top \end{pmatrix}. \end{aligned} \tag{3.100}$$

Definition 3.3.4. The dispersion function for Model I, is the mapping $K_{I,\varepsilon} \equiv K_I$ given by

$$\begin{aligned} &K_I : Q' \times K_\sigma \longrightarrow \mathbb{C} \\ &K_I(\tau, z) = \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|} T_{\varepsilon,I}^{(\tau)} \begin{pmatrix} z(A_0^{\text{soft},(\tau)} - z)^{-1}w_I + w_I \\ \|\Psi_1^{\text{stiff},(\tau)}\| \end{pmatrix}, \end{aligned} \tag{3.101}$$

where $w_I = \Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}$.

We make some comments on $K_I(\tau, z)$ in relation to the dispersion functions of the stiff-soft-stiff model in Section 3.1. First, it was proven in [35, Sect 5.3], that

$$P_{\mathbb{C}}(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z)^{-1}P_{\mathbb{C}} = M_{(K_I(\tau,z)-z)^{-1}}, \quad \text{for all } \tau \in Q' \text{ and } z \in K_\sigma. \tag{3.102}$$

Second, we recall an earlier remark, that $K_{a,\text{stiff-ls}}(\tau, z)$ of the stiff-soft-stiff model is simply the function $K_I(\tau, z)$ extended by zero on the complementary $1D$ subspace $\mathbb{C}_{\text{stiff-int}}$.

Finally, the most important point: We note that the resolvent equation for $\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)}$, when the RHS is restricted to $(f, \delta) \in 0 \oplus \mathbb{C}$, is

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = 0, \\ T_{\varepsilon,I}^{(\tau)}(u,\beta)^\top - z\beta = \delta, \end{cases} \tag{3.103}$$

and this can be written in terms of $K_I(\tau, z)$ as

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = 0, \\ (K(\tau, z) - z)\beta = \delta, \end{cases} \tag{3.104}$$

as long as $z \in \rho(A_0^{\text{soft},(\tau)})$. In other words, we may extend K_I to include real-valued z, provided that z does not lie in $\sigma(A_0^{\text{soft},(\tau)})$ for some τ, or equivalently (by (3.48)), if $z \notin \sigma(A_0^{\text{soft},(0)})$.

Locating the spectrum of $\mathcal{A}_{\varepsilon,\text{hom-I}}$

Motivated by the two-scale strong resolvent case [53], we characterize $\sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$ (for a fixed, small ε) in terms of $K_I(\tau, z)$. The key ingredient for this step is the following result:

Proposition 3.3.5 (A partial decoupling for Model I)**.** Fix $\tau \in Q'$. Suppose that $z \in \rho(A_0^{\text{soft},(0)})$, so that $K_I(\tau, z)$ is well-defined. Then the resolvent equation for $\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)}$

$$\left(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z\right)\begin{pmatrix} u \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}, \tag{3.105}$$

has a unique solution in $\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}})$ for every $f \in L^2(Q_{\text{soft}})$, $\widetilde{f} \in \mathbb{C}$ if and only if the system

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = g, \\ (K_I(\tau, z) - z)\widetilde{u} = \widetilde{g}, \end{cases} \tag{3.106}$$

has a unique solution in $\mathcal{D}(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}})$ for every $g \in L^2(Q_{\text{soft}})$, $\widetilde{g} \in \mathbb{C}$.

Remark.
- The significance of this result is that we have partially "decoupled" the system (3.105), so that the operator in the second equation (3.106), namely $K_I(\tau, z) - z$, does not depend on u. Note that u and $\widetilde{u}$ are still connected by the identity

$$\widetilde{u} = j^{(\tau)} \Pi^{\text{stiff},(\tau)} \Gamma_0^{\text{soft},(\tau)} u. \tag{3.107}$$

- To simplify the notation, we will write a typical $u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dot{+} \text{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$ as $u = u_a + u_b$, where $u_a \in \mathcal{D}(A_0^{\text{soft},(\tau)})$ and $u_b \in \text{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$. Furthermore, we write

$$\widetilde{u}_b = j^{(\tau)} \Pi^{\text{stiff},(\tau)} \Gamma_0^{\text{soft},(\tau)} u_b, \tag{3.108}$$

and note that $\widetilde{u}_a = j^{(\tau)} \Pi^{\text{stiff},(\tau)} \Gamma_0^{\text{soft},(\tau)} u_a = 0$. ◦

Proof of Proposition 3.3.5. ($\Rightarrow$) Let $g \in L^2(Q_{\text{soft}})$ and $\widetilde{g} \in \mathbb{C}$ be given. Since $z \in \rho(A_0^{\text{soft},(0)})$, $(A_0^{\text{soft},(\tau)} - z)$ is invertible. So take $u_a = (A_0^{\text{soft},(\tau)} - z)^{-1} g$. Then $\widetilde{u}_a = j^{(\tau)} \Pi^{\text{stiff},(\tau)} \Gamma_0^{\text{soft},(\tau)} u_a = 0$, and we have

$$\left(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}} - z\right) \begin{pmatrix} u_a \\ 0 \end{pmatrix} = \begin{pmatrix} g \\ T_I(u_a, 0)^\top - z \cdot 0 \end{pmatrix} = \begin{pmatrix} g \\ T_I(u_a, 0)^\top \end{pmatrix}, \tag{3.109}$$

and $(u_a, 0)$ solves (3.109) uniquely by our assumption.

Next, pick $f = 0$ and $\widetilde{f} = \widetilde{g}$ in (3.105). Let $(u_b, \widetilde{u}_b)$ be the solution to

$$\left(\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}} - z\right) \begin{pmatrix} u_b \\ \widetilde{u}_b \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{g} \end{pmatrix}, \tag{3.110}$$

which exists and is unique by our assumption. Note that the first equation of the system (3.110) implies that $u_b \in \text{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$, since $(A_0^{\text{soft},(\tau)} - z)$ is invertible. Note also, that by the definition of $K_I(\tau, z)$, the second equation of (3.110) is

$$(K_I(\tau, z) - z)\widetilde{u}_b = \widetilde{g}. \tag{3.111}$$

Therefore, $u = u_a + u_b$ and $\widetilde{u} = \widetilde{u_a + u_b} = \widetilde{u}_b$ solves

$$\begin{cases} (-(\nabla + i\tau)^2 - z)u = (-(\nabla + i\tau)^2 - z)u_a = g, \\ (K_I(\tau, z) - z)\widetilde{u} = (K_I(\tau, z) - z)\widetilde{u}_b = \widetilde{g}. \end{cases} \tag{3.112}$$

It is clear that the choice $u_a \in \mathcal{D}(A_0^{\text{soft},(0)})$ has to be unique. Similarly, $(u_b, \widetilde{u}_b)$ must also be unique.

($\Leftarrow$) Let $f \in L^2(Q_{\text{soft}})$ and $\widetilde{f} \in \mathbb{C}$ be given. As $(A_0^{\text{soft},(\tau)} - z)$ is invertible by assumption, we can take $u_a = (A_0^{\text{soft},(\tau)} - z)^{-1} f$. Then $\widetilde{u}_a = j^{(\tau)} \Pi^{\text{stiff},(\tau)} \Gamma_0^{\text{soft},(\tau)} u_a = 0$, and we have

$$\left(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z\right) \begin{pmatrix} u_a \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ T_I(u_a, 0)^\top - z \cdot 0 \end{pmatrix} =: \begin{pmatrix} f \\ \widetilde{f}_{u_a} \end{pmatrix}. \tag{3.113}$$

We now show that there exist a unique $u_b \in \operatorname{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$ solving

$$\left(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z\right) \begin{pmatrix} u_b \\ \widetilde{u}_b \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{f} - \widetilde{f}_{u_a} \end{pmatrix}. \tag{3.114}$$

Indeed, by definition of $K_I(\tau, z)$, the second equation of (3.114) is

$$(K_I(\tau, z) - z)\widetilde{u}_b = \widetilde{f} - \widetilde{f}_{u_a}, \tag{3.115}$$

so the system (3.114) is the same as (3.106) with RHS $(0, \widetilde{f} - \widetilde{f}_{u_a})$, which we know has a unique solution $(u_b, \widetilde{u}_b)$, by our assumption on (3.106). Here, u_b and $\widetilde{u}_b$ are related by (3.108). Moreover the first equation of (3.114) implies that $u_b \in \operatorname{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$, since $(A_0^{\text{soft},(\tau)} - z)$ is invertible.

Back to the proof of the proposition, if we set $u = u_a + u_b$, then $\widetilde{u} = 0 + \widetilde{u}_b$, and $(u, \widetilde{u})$ solve

$$\left(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z\right) \begin{pmatrix} u_a + u_b \\ 0 + \widetilde{u}_b \end{pmatrix} = \begin{pmatrix} f \\ \widetilde{f}_{u_a} + \widetilde{f} - \widetilde{f}_{u_a} \end{pmatrix} = \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}. \tag{3.116}$$

Once again, it is clear that the choice $u_a \in \mathcal{D}(A_0^{\text{soft},(0)})$ here is unique. This in turn fixes $\widetilde{f}_{u_a}$, and as a consequence $(u_b, \widetilde{u}_b)$ must also be unique. This completes the proof. □

Remark. While the (3.108) says that fixing u_b determines $\widetilde{u}_b$, we can proceed in the reverse direction: If we know $\widetilde{u}_b$ and we assume that $u_b \in \operatorname{ran}(\Pi^{\text{soft},(\tau)} \mathcal{P}^{(\tau)})$, then u_b is completely determined. Indeed, since $u_b = \Pi^{\text{soft}}(c\psi_1^{\text{stiff},(\tau)})$ for some c, the identity (3.108) implies that $\widetilde{u}_b = c\|\Psi_1^{\text{stiff},(\tau)}\|$. Hence

$$u_b = 0 + \Pi^{\text{soft},(\tau)} \left(\frac{\widetilde{u}_b}{\|\Psi_1^{\text{stiff},(\tau)}\|} \psi_1^{\text{stiff},(\tau)} \right). \tag{3.117}$$

This is useful for the proof below. ◦

With the "decoupling" result (Proposition 3.3.5), we can now characterize $\sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$:

Corollary 3.3.6. If $z \in \rho(A_0^{\text{soft},(0)})$ is such that $K_I(\tau, z) - z \neq 0$ for all τ, then $z \notin \sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$.

Proof. By definition, $(\mathcal{A}_{\varepsilon,\text{hom-I}} - z)$ is unitarily equivalent to $\int_{Q'}^{\oplus} \left(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)} - z\right) d\tau$, so we can equivalently look at the resolvent equation for $\mathcal{A}_{\varepsilon,\text{hom-I}}^{(\tau)}$, i.e. (3.105). Since $z \notin \sigma(A_0^{\text{soft},(0)})$, we may apply Proposition 3.3.5, and check that the system

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = g, \\ (K_I(\tau, z) - z)\widetilde{u} = \widetilde{g}, \end{cases} \tag{3.118}$$

has a unique solution in $\mathcal{D}(\mathcal{A}_{\varepsilon,\text{hom-I}})$ for every $g \in L^2(Q_{\text{soft}})$ and $\widetilde{g} \in \mathbb{C}$. We remind the reader that $\widetilde{u} = j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}u$.

In fact, under the hypothesis of this corollary, we can show that the system (3.118) is solved uniquely by

$$u = (A_0^{\text{soft},(\tau)} - z)^{-1}g + \Pi^{\text{soft},(\tau)}\left(\frac{(K_I(\tau,z) - z)^{-1}\widetilde{g}}{\|\Psi_1^{\text{stiff},(\tau)}\|}\psi_1^{\text{stiff},(\tau)}\right) =: u_a + u_b. \tag{3.119}$$

Indeed, it is clear that u_a is the unique element in $\mathcal{D}(A_0^{\text{soft},(\tau)})$ that gives $(-(\nabla+i\tau)^2 - z)u_a = g$, due to the assumption $z \notin \sigma(A_0^{\text{soft},(0)})$. Meanwhile, by setting

$$\widetilde{u}_a = j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}u_a = 0, \quad \text{and} \quad \widetilde{u}_b = j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}u_b, \tag{3.120}$$

we see that

$$\begin{aligned}(K_I(\tau,z) - z)\widetilde{u} &= (K_I(\tau,z) - z)\widetilde{u}_b \\ &= \cancel{(K_I(\tau,z) - z)}\frac{\cancel{(K_I(\tau,z) - z)^{-1}}\,\widetilde{g}}{\cancel{\|\Psi_1^{\text{stiff},(\tau)}\|}}\cancel{j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}\Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}} = \widetilde{g}.\end{aligned} \tag{3.121}$$

Hence u does indeed solve the second equation of (3.118). We argue further that the choice u_b is unique: By the assumption $(K_I(\tau,z) - z) \neq 0$, $\widetilde{u}_b$ is uniquely determined by $\widetilde{g}$ by the formula $(K_I(\tau,z) - z)^{-1}\widetilde{g} = \widetilde{u}_b$. Having fixed $\widetilde{u}_b$, we observe that u_b as defined in (3.119) is the unique element in $\operatorname{ran}(\Pi^{\text{soft},(\tau)}\mathcal{P}^{(\tau)})$ satisfying (3.120) (see (3.117)). $\square$

Corollary 3.3.7. If $z \in \rho(A_0^{\text{soft},(0)})$ is such that $K_I(\tau,z) - z = 0$ some τ, then $z \in \sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$.

Proof. This means that there is some τ such that the system

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = g, \\ (K_I(\tau,z) - z)\widetilde{u} = \widetilde{g}, \end{cases} \tag{3.122}$$

does not have a unique solution for every $g \in L^2(Q_{\text{soft}})$ and $\widetilde{g} \in \mathbb{C}$. (Just consider a RHS $(g,\widetilde{g})$ with $\widetilde{g} \neq 0$.) The result now follows from Proposition 3.3.5. $\square$

Proposition 3.3.8. $\sigma(A_0^{\text{soft},(0)}) \subset \sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$.

Proof. Let $z \in \sigma(A_0^{\text{soft},(0)})$. So $z \neq 0$. Let $u_a \in \mathcal{D}(A_0^{\text{soft},(0)})$ be an eigenfunction for $A_0^{\text{soft},(0)}$ w.r.t. the eigenvalue z. We will construct a non-zero $u \in \mathcal{D}(A_0^{\text{soft},(0)}) \dotplus \operatorname{ran}(\Pi^{\text{soft},(0)}\mathcal{P}^{(0)})$ such that $(u,\widetilde{u}) \in \ker(\mathcal{A}_{\varepsilon,\text{hom-I}}^{(0)} - z)$. We remind the reader that $\widetilde{u} = j^{(0)}\Pi^{\text{stiff},(0)}\Gamma_0^{\text{soft},(0)}u$.

Let $u_b \in \operatorname{ran}(\Pi^{\text{soft},(0)}\mathcal{P}^{(0)})$, so that we may write $u_b = c\Pi^{\text{soft},(0)}\psi_1^{\text{stiff},(0)}$ for some $c \in \mathbb{C}$. Then, with $\widetilde{u}_a$ and $\widetilde{u}_b$ defined by the formula (3.120), we have

$$\widetilde{u}_a = 0, \quad \text{and} \quad \widetilde{u}_b = c\|\Psi_1^{\text{stiff},(0)}\|. \tag{3.123}$$

With u_a and $\widetilde{u}_a = 0$ at hand, we can compute

$$\left(\mathcal{A}^{(0)}_{\varepsilon,\text{hom-I}} - z\right)\begin{pmatrix} u_a \\ \widetilde{u}_a \end{pmatrix} = \begin{pmatrix} 0 \\ T_I(u_a, 0)^\top \end{pmatrix} =: \begin{pmatrix} 0 \\ \widetilde{f}_{u_a} \end{pmatrix}. \tag{3.124}$$

Let us show that we can always find $(u_b, \widetilde{u}_b)$ such that

$$\left(\mathcal{A}^{(0)}_{\varepsilon,\text{hom-I}} - z\right)\begin{pmatrix} u_b \\ \widetilde{u}_b \end{pmatrix} = \begin{pmatrix} 0 \\ T_I(u_b, \widetilde{u}_b)^\top - z\widetilde{u}_b \end{pmatrix} = \begin{pmatrix} 0 \\ -\widetilde{f}_{u_a} \end{pmatrix}. \tag{3.125}$$

To see this, we first note that the setup of Model I gives us

$$\langle \Lambda^{\text{soft},(0)} \psi_1^{\text{stiff},(0)}, \psi_1^{\text{stiff},(0)} \rangle = 0, \quad \text{and} \quad \varepsilon^{-2}\mu_1^{\text{stiff},(0)} = 0. \tag{3.126}$$

This means that $T_I(u_b, \widetilde{u}_b)^\top \equiv T^{(0)}_{\varepsilon,I}(u_b, \widetilde{u}_b)^\top = 0$, and thus

$$T_I(u_b, \widetilde{u}_b)^\top - z\widetilde{u}_b = -z\widetilde{u}_b = -zc\|\Psi_1^{\text{stiff},(0)}\|. \tag{3.127}$$

Since z and $\|\Psi_1^{\text{stiff},(0)}\|$ are non-zero, we can always pick $c \in \mathbb{C}$ so that (3.125) holds.

In other words, by setting $u := u_a + u_b$, which gives $\widetilde{u} = \widetilde{u}_a + \widetilde{u}_b = \widetilde{u}_b$, the pair $(u, \widetilde{u})$ belongs to the kernel of $(\mathcal{A}^{(0)}_{\varepsilon,\text{hom-I}} - z)$, hence

$$z \in \sigma(\mathcal{A}^{(0)}_{\varepsilon,\text{hom-I}}) \subset \sigma(\mathcal{A}_{\varepsilon,\text{hom-I}}).$$

□

Remark. Proposition 3.3.8 does not imply that $\lambda \in \sigma(A_0^{\text{soft},(0)})$ is an eigenvalue for $\sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$. λ is an eigenvalue for $\mathcal{A}_{\varepsilon,\text{hom-I}}$ if and only if $|\{\tau : \lambda \text{ is an eigenvalue for } \mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-I}}\}| > 0$ (see [20, Theorem XIII.85(e)]). ∘

3.3.2 Model II

We will use the notation of Section 3.2.2. We first modify Lemma 3.3.1 to the case of Model II.

Lemma 3.3.9 (Model II)**.** Let $c \in L^2(Q')$. Suppose that

$$\langle \Lambda^{\text{soft},(\tau)} \psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)} \rangle_{L^2(\Gamma)} c^{(\tau)} \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2} \Psi_1^{\text{stiff},(\tau)}(x) = 0 \tag{3.128}$$

for a.e. $x \in Q$ and $\tau \in Q'$, then $c^{(\tau)} = 0$ for almost every τ.

Proof. In the case of Model II, $\|\Psi_1^{\text{stiff},(\tau)}\|$ is a non-zero constant that does not depend on τ (Lemma 3.1.4). Similarly to the proof of Lemma 3.3.1, we compute the $L^2(Q \times Q')$ norm of the LHS of (3.128), and it remains to show that

$$\langle \Lambda^{\text{soft},(\tau)} \psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)} \rangle_{L^2(\Gamma)} < 0 \tag{3.129}$$

for almost every τ (note the strict inequality). For this, we note that $\Lambda^{\text{soft},(\tau)}$ (of Model II) satisfies the same properties as $\widetilde{\Lambda}^{\text{stiff-ls},(\tau)}$ (of the stiff-soft-stiff model). Recall from Proposition 2.2.10, that this means $\Lambda^{\text{soft},(\tau)}$ is strictly negative when $\tau \neq 0$. □

Proposition 3.3.10. 0 is not an eigenvalue of $\mathcal{A}_{\text{hom-II}}$.

Proof. We just have to note that $-\Delta_D$, the Dirichlet Laplacian on the (connected) set $\cup_{n\in\mathbb{Z}^d}(Q_{\text{soft}}+n)$, has trivial kernel. This follows from (3.61). The rest of the proof proceeds in the same manner as Proposition 3.3.2, using Lemma 3.3.9 in place of Lemma 3.3.1. $\square$

For the rest of the section, we will keep the discussion brief, since most of the details are similar to Model I in Section 3.3.1.

For the non-zero eigenvalues $\lambda \neq 0$ of $\mathcal{A}_{\text{hom-II}}$, consider the eigenvalue equation

$$\mathcal{A}_{\text{hom-II}}\begin{pmatrix} v \\ \widetilde{v}\end{pmatrix} = \lambda\begin{pmatrix} v \\ \widetilde{v}\end{pmatrix}. \tag{3.130}$$

Once again, enumerating the possibilities for an eigenfunction $(v,\widetilde{v})$, we have that

- **Case 1:** $v = G^*(c^{(\tau)}\Pi^{\textbf{soft},(\tau)}\psi_1^{\textbf{stiff},(\tau)})$**, where** $c\in L^2(Q_{\textbf{soft}})$ **and** $c\neq 0$, cannot give rise to an eigenfunction for $\mathcal{A}_{\text{hom-II}}$.
- **Case 2:** $v = v_D \in \mathcal{D}(-\Delta_D)$**, where** $v_D \neq 0$**.** This is only possible if λ is an eigenvalue of $-\Delta_D$, and there exist an eigenfunction $v_{D,\lambda}$ of $-\Delta_D$ corresponding to the eigenvalue λ such that

$$\left\langle \Gamma_1^{\text{soft},(\tau)}(Gv_{D,\lambda})^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle_{L^2(\Gamma)} = 0 \quad \text{for almost every } \tau. \tag{3.131}$$

  (It is possible that $-\Delta_D$ has no eigenvalues, and in that case $(v_D,0)$ cannot be an eigenfunction for $\mathcal{A}_{\text{hom-II}}$.)

- **Case 3:** v **is of the form**

$$v = v_D + G^*(c^{(\tau)}\Pi^{\textbf{soft},(\tau)}\psi_1^{\textbf{stiff},(\tau)}), \tag{3.132}$$

  where c **and** v_D **satisfies** (3.63)**, and both** c **and** v_D **are non-zero.** Then, the first component of (3.130) is

$$-\Delta v_D = \lambda v_D + \lambda G^*\left(c^{(\tau)}\Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}\right). \tag{3.133}$$

  Setting

$$\begin{cases} g_2(\tau) = \left\langle \Gamma_1^{\text{soft},(\tau)}(Gv_D)^{(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle, \\ f_2(\tau) = \left\langle \Lambda^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}, \psi_1^{\text{stiff},(\tau)}\right\rangle, \end{cases} \tag{3.134}$$

  the second component of (3.130) gives us the condition

$$g_2(\tau) + \left(f_2(\tau) - \lambda\|\Psi_1^{\text{stiff},(\tau)}\|^2\right)c(\tau) = 0, \quad \text{for almost every } \tau. \tag{3.135}$$

To summarize,

Proposition 3.3.11 (Eigenvalues of $\mathcal{A}_{\text{hom-II}}$)**.** $\lambda \in \mathbb{R}$ is an eigenvalue for $\mathcal{A}_{\text{hom-II}}$ if and only if $\lambda \neq 0$ and if either one of the following (not mutually exclusive) criterion is satisfied:

- λ is an eigenvalue of $-\Delta_D$ is such that (3.131) holds.
- There exist v of the form (3.132) satisfying (3.133) and (3.135).

The dispersion function for Model II

Next, let us quickly recall the dispersion function $K_{\text{II}}(\tau, z)$ for Model II, defined in [35]. Introduce the isomorphism $j^{(\tau)} : \text{ran}(\Pi^{\text{stiff},(\tau)}|_{\mathcal{P}^{(\tau)}\mathcal{E}}) \to \mathbb{C}$ in the same way as we did for Model I. Now the fibre-wise homogenized operator may be viewed as an operator on $L^2(Q_{\text{soft}}) \oplus \mathbb{C}$, which we will still denote by $\mathcal{A}^{(\tau)}_{\text{hom-II}}$ (and similarly for $\mathcal{A}_{\text{hom-II}}$). $\mathcal{A}^{(\tau)}_{\text{hom-II}}$ is now given by

$$\mathcal{D}(\mathcal{A}^{(\tau)}_{\text{hom-II}}) = \{(u, \beta) \in L^2(Q_{\text{soft}}) \oplus \mathbb{C} : \\ u \in \mathcal{D}(A_0^{\text{soft},(\tau)}) \dotplus \text{ran}(\Pi^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}), \quad \beta = j^{(\tau)}\Pi^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)}u\}, \tag{3.136}$$

$$\mathcal{A}^{(\tau)}_{\varepsilon,\text{hom-II}} \begin{pmatrix} u \\ \beta \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -((j^{(\tau)}\breve{\Pi}^{\text{stiff},(\tau)})^*)^{-1}\mathcal{P}^{(\tau)}\Gamma_1^{\text{soft},(\tau)}u \end{pmatrix} =: \begin{pmatrix} -(\nabla + i\tau)^2 u \\ T^{(\tau)}_{\text{II}}(u, \beta)^\top \end{pmatrix}. \tag{3.137}$$

Definition 3.3.12. The dispersion function for Model II, is the mapping K_{II} given by

$$\begin{aligned} &K_{\text{II}} : Q' \times K_\sigma \longrightarrow \mathbb{C} \\ &K_{\text{II}}(\tau, z) = \frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|} T^{(\tau)}_{\text{II}} \begin{pmatrix} z(A_0^{\text{soft},(\tau)} - z)^{-1} v_{\text{II}} + v_{\text{II}} \\ \|\Psi_1^{\text{stiff},(\tau)}\| \end{pmatrix}, \end{aligned} \tag{3.138}$$

where $v_{\text{II}} = \Pi^{\text{soft},(\tau)}\psi_1^{\text{stiff},(\tau)}$.

Once again we make three remarks: First, in [35, Sect 5.3], the authors proved that

$$P_{\mathbb{C}}(\mathcal{A}^{(\tau)}_{\text{hom-II}} - z)^{-1}P_{\mathbb{C}} = M_{(K_{\text{II}}(\tau,z)-z)^{-1}}, \quad \text{for all } \tau \in Q' \text{ and } z \in K_\sigma. \tag{3.139}$$

Second, we recall an earlier remark, that $K_{a,\text{stiff-int}}(\tau, z)$ of the stiff-soft-stiff model is simply the function $K_{\text{II}}(\tau, z)$ extended by zero on the complementary $1D$ subspace $\mathbb{C}_{\text{stiff-ls}}$. Finally, we emphasize that $K_{\text{II}}(\tau, z)$ may be extended to real-valued z, provided $z \notin \sigma(A_0^{\text{soft},(\tau)})$ for any τ, or equivalently (by (3.61)), if $z \notin \sigma(-\Delta)$.

Locating the spectrum of $\mathcal{A}_{\text{hom-II}}$

Much of our discussion here follows with minimal modifications to the case of Model I. We remind the reader that by (3.61), we have

$$\sigma(-\Delta_D) = \bigcup_{\tau \in Q'} \sigma(A_0^{\text{soft},(\tau)}).$$

We also have a decoupling result for Model II:

Proposition 3.3.13 (A partial decoupling for Model II)**.** Fix $\tau \in Q'$. Suppose that $z \in \rho(-\Delta_D)$, so that $K_{\rm II}(\tau, z)$ is well-defined. Then the resolvent equation for $\mathcal{A}^{(\tau)}_{\text{hom-II}}$

$$\left(\mathcal{A}^{(\tau)}_{\text{hom-II}} - z\right)\begin{pmatrix} u \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix}, \tag{3.140}$$

has a unique solution in $\mathcal{D}(\mathcal{A}^{(\tau)}_{\text{hom-II}})$ for every $f \in L^2(Q_{\text{soft}})$, $\widetilde{f} \in \mathbb{C}$ if and only if the system

$$\begin{cases} -(\nabla + i\tau)^2 u - zu = g, \\ (K_{\rm II}(\tau, z) - z)\widetilde{u} = \widetilde{g}. \end{cases} \tag{3.141}$$

has a unique solution in $\mathcal{D}(\mathcal{A}^{(\tau)}_{\text{hom-II}})$ for every $g \in L^2(Q_{\text{soft}})$, $\widetilde{g} \in \mathbb{C}$.

In the same vein, Proposition 3.3.13 gives us the following two results:

Corollary 3.3.14. If $z \in \rho(-\Delta_D)$ is such that $K_{\rm II}(\tau, z) - z \neq 0$ for all τ, then $z \notin \sigma(\mathcal{A}_{\text{hom-II}})$.

Corollary 3.3.15. If $z \in \rho(-\Delta_D)$ is such that $K_{\rm II}(\tau, z) - z = 0$ some τ, then $z \in \sigma(\mathcal{A}_{\text{hom-II}})$.

We state some partial results on whether an element λ of $\sigma(-\Delta_D)$ belongs to $\sigma(\mathcal{A}_{\text{hom-II}})$:

Proposition 3.3.16. $\sigma(A_0^{\text{soft},(0)}) \subset \sigma(\mathcal{A}_{\text{hom-II}})$.

Proposition 3.3.17. If $\tau \neq 0$ and $z \in \sigma(A_0^{\text{soft},(\tau)})$ is such that $-\frac{|\Gamma|}{|Q_{\text{stiff}}|} f_2(\tau) \neq z$, where f_2 is defined in (3.134), then $z \in \sigma(\mathcal{A}^{(\tau)}_{\text{hom-II}}) \subset \sigma(\mathcal{A}_{\text{hom-II}})$.

The first proposition follows by the same argument as Proposition 3.3.8. For the second proposition, we follow the proof of Proposition 3.3.8, and a computation on $T_{\rm II}(u_b, \widetilde{u}_b)^\top - z\widetilde{u}_b$ will give us the criterion $-\frac{|\Gamma|}{|Q_{\text{stiff}}|} f_2(\tau) \neq z$. We omit the details of this short computation. Note that $\frac{1}{\|\Psi_1^{\text{stiff},(\tau)}\|^2} = \frac{|\Gamma|}{|Q_{\text{stiff}}|}$ (Lemma 3.1.4).

3.3.3 Stiff-soft-stiff model

In the final section of this chapter, we highlight some differences in the spectral analysis of Models I (Section 3.3.1) and II (Section 3.3.2) with the stiff-soft-stiff setting.

Regarding the point spectrum, we can adapt Lemma 3.3.1 to the stiff-soft-stiff case, and we will encounter the following two expressions:

$$\left\langle \Lambda^{\text{soft},(\tau)}(a^{(\tau)}\psi_1^{\text{stiff-int},(\tau)} + b^{(\tau)}\psi_1^{\text{stiff-ls},(\tau)}), (a^{(\tau)}\psi_1^{\text{stiff-int},(\tau)} + b^{(\tau)}\psi_1^{\text{stiff-ls},(\tau)}) \right\rangle, \tag{3.142}$$

$$\varepsilon^{-2} b^{(\tau)} \mu_1^{\text{stiff-ls},(\tau)}. \tag{3.143}$$

We see that by picking $a^{(\tau)} = -b^{(\tau)}$, and letting supp a be away from a neighbourhood of $\tau = 0$, we may construct non-zero a, $b \in L^2(Q')$ such that the two expressions above sum to zero. As a consequence, 0 is an eigenvalue of $\mathcal{A}_{\varepsilon,\text{hom}}$ (compare this with Proposition 3.3.2 of Model I). The analysis of the non-zero eigenvalues of $\mathcal{A}_{\varepsilon,\text{hom}}$, should closely follow Section 3.3.1 (Model I), since the annulus Q_{soft} is at a positive distance from the boundary of the cube.

Key difficulties arise when we discuss attempt to characterize the spectrum of $\mathcal{A}_{\varepsilon,\text{hom}}$ in terms of the dispersion functions obtained in Section 3.1. The first is to prove a decoupling result in the sense of Proposition 3.3.5: What would the system (3.106) be for the stiff-soft stiff model? Should this be a system of three equations (involving $-(\nabla + i\tau)^2$, $K_{\text{stiff-int}}(\tau, z)$, and $K_{\text{stiff-ls}}(\tau, z)$), or five equations (involving the previous three, plus $F_{\text{stiff-int}\to\text{stiff-ls}}(\tau, z)$ and $F_{\text{stiff-ls}\to\text{stiff-int}}(\tau, z)$)?

The second and most critical point, is with regards to the extension of $K_{\text{stiff-int}}(\tau, z)$ and $K_{\text{stiff-ls}}(\tau, z)$ to real-valued z: Just like in Models I and II, we have to exclude the case

$$z \in \bigcup_{\tau \in Q'} \sigma(A_0^{\text{soft},(\tau)}) = \sigma(A_0^{\text{soft},(0)}),$$

so that $K_{\text{stiff-int}}(\tau, z)$ and $K_{\text{stiff-ls}}(\tau, z)$ are well-defined, and deal with this case afterwards. In the present setting, there are more that needs to be excluded (see Section 3.1.1). For $K_{\text{stiff-int}}(\tau, z)$, these are:

- The case $z = 0$. This is actually a rather harmless case, as the derivation in Section 3.1.1 may be suitably modified, and in many instances simplified (e.g. the terms $z\beta_{\text{stiff-int}}$ and $z(A_0^{\text{soft}} - z)^{-1} = 0$ are absent). Note that 0 is always an eigenvalue, as discussed above, but it might also belong to the absolutely continuous spectrum. That is, the intersection of the closed sets $\sigma_{ac}(\mathcal{A}_{\varepsilon,\text{hom}})$ and $\sigma_p(\mathcal{A}_{\varepsilon,\text{hom}})$ may be non-empty due to embedded eigenvalues.
- The case $z \in \mathbb{R} \setminus \{0\}$, where z satisfies

$$1 + \frac{1}{z\|\Psi_1^{\text{stiff-ls},(\tau)}\|^2}\left[\varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)} + \left\langle M^{\text{soft},(\tau)}(z)(0 + \psi_1^{\text{stiff-ls},(\tau)}), (0 + \psi_1^{\text{stiff-ls},(\tau)})\right\rangle\right] = 0 \tag{3.144}$$

  for some τ (see (3.14)). In this case we do not have $K_{\text{stiff-int}}(\tau, z)$, but we do have an extra equation (3.144). Nonetheless this splits into further sub-cases, and in each instance we have to check if the system (3.4) is uniquely solvable.

Lastly, we point out that if $K_{\text{stiff-int}}(\tau, z)$ is undefined at some $z \in \mathbb{R}$, it does not necessarily mean that $z \in \sigma(\mathcal{A}_{\varepsilon,\text{hom}})$ — it just means that the second equation of (3.4) cannot be written in the form $(K_{\text{stiff-int}}(\tau, z) - z)\beta = \delta$, and we have to deal with (3.4) as it is.

Chapter 4

Conclusion and next steps

In this thesis, we looked at a high-contrast $\varepsilon\mathbb{Z}^d$ periodic composite, consisting of a "soft" and a "stiff" material, and arranged in a stiff-soft-stiff setup (Figure 2-1). Under the Gelfand transform and rescaling operators, we arrived at a family of operators $\{A_\varepsilon^{(\tau)}\}_{\tau\in Q'}$ on $L^2(Q)$, where the period cell Q and $A_\varepsilon^{(\tau)}$ are roughly described by Figure 4-1 below.

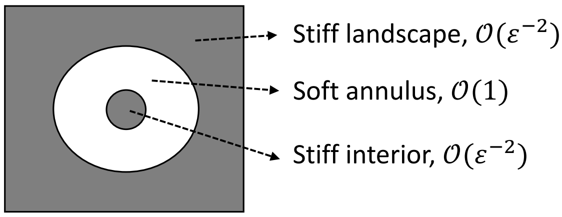

Figure 4-1: The period cell $Q = [0,1)^d$.

Our investigation on the stiff-soft-stiff composite was broken into two stages:

Summary of Chapter 2

In Chapter 2, we homogenized the composite. More precisely, we identified the (uniform in τ) *norm-resolvent asymptotics* of the family $\{A_\varepsilon^{(\tau)}\}$. As we saw in the thesis, the analysis of high-contrast composites depends greatly on how the two materials are configured. We perform the analysis using the method of boundary triples, following [35]. This is a novel approach introduced by Cherednichenko, Ershova, and Kiselev in the context of homogenization. In short, this is a tool that brings the problem on the unit cell Q to a problem on the soft-stiff interfaces, and in turn we may rely on the spectral properties of the Dirichlet-to-Neumann operators on these interfaces. By adapting the framework of [35] to our setting, we obtained:

- A homogenization result for the stiff-soft-soft composite. This is Section 2.5. We give an effective description of the composite by identifying the norm-resolvent asymptotics of $A_\varepsilon^{(\tau)}$, namely the operator $\mathcal{A}_{\varepsilon,\text{hom}}^{(\tau)}$.
- Moreover, we supplement the asymptotic argument in [35] with additional details. These are meant to explain how the estimates obtained are uniform over τ and z. (See Proposition 2.3.3 and Theorem 2.3.4.)

Summary of Chapter 3

The second stage of our investigation is detailed in Chapter 3. We passed to the norm-resolvent asymptotics, and studied the family $\{\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}\}_{\tau\in Q'}$. As our homogenized operator $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$ is defined using objects from boundary triples, there is some work to be done to uncover the effective transport/scattering properties of our composite. We explored three aspects of the homogenized description:

- In Section 3.1, we focused on the bottom right entry of the resolvent for $\mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}}$. This is a 2×2 matrix, due to the two stiff components. We wrote each entry in terms of an operator of multiplication on $\mathbb{C}$ by a constant. For the diagonal entries, we were able to express this constant as $(K(\tau,z)-z)^{-1}$, in which we refer to $K(\tau,z)$ as the "dispersion function". In particular, our results are

  - $\mathbb{C}_{\textbf{stiff-int}} \to \mathbb{C}_{\textbf{stiff-int}}$: Theorem 3.1.6,
  - $\mathbb{C}_{\textbf{stiff-ls}} \to \mathbb{C}_{\textbf{stiff-ls}}$: Theorem 3.1.10,
  - $\mathbb{C}_{\textbf{stiff-int}} \to \mathbb{C}_{\textbf{stiff-ls}}$: Theorem 3.1.12 and Corollary 3.1.13,
  - $\mathbb{C}_{\textbf{stiff-ls}} \to \mathbb{C}_{\textbf{stiff-int}}$: Theorem 3.1.15 and Corollary 3.1.16.

- In Section 3.2.3, we wrote down the homogenized description of our composite on the full space, i.e. the operator

$$\mathcal{A}_{\varepsilon,\mathrm{hom}} = G^*\left(\int_{Q'}^{\oplus} \mathcal{A}^{(\tau)}_{\varepsilon,\mathrm{hom}} d\tau\right) G.$$

- In Section 3.3, we performed a spectral analysis of $\mathcal{A}_{\varepsilon,\text{hom-I}}$ and $\mathcal{A}_{\text{hom-II}}$ (the norm-resolvent asymptotics for Models I and II on the full space $L^2(\mathbb{R}^d)$, as obtained in [35]), with an eye towards treating the stiff-soft-stiff case $\mathcal{A}_{\varepsilon,\mathrm{hom}}$. Our results are

  - **For Model I:** Proposition 3.3.3 (eigenvalues), Corollaries 3.3.6 and 3.3.7 (characterization of $\sigma(\mathcal{A}_{\varepsilon,\text{hom-I}})$ in terms of the dispersion function $K_{\mathrm{I}}(\tau,z)$), and Proposition 3.3.8 (when $K_{\mathrm{I}}(\tau,z)$ is undefined).
  - **For Model II:** Proposition 3.3.11 (eigenvalues), Corollaries 3.3.14 and 3.3.15 (characterization of $\sigma(\mathcal{A}_{\text{hom-II}})$ in terms of the dispersion function $K_{\mathrm{II}}(\tau,z)$). We only provided partial results when $K_{\mathrm{II}}(\tau,z)$ is undefined (Propositions 3.3.16 and 3.3.17).

Next steps

We mention some open problems and possible directions for future work.

First, we have some unfinished tasks from Chapter 3, namely Section 3.3, the spectral analysis of $\mathcal{A}_{\varepsilon,\text{hom}}$, $\mathcal{A}_{\varepsilon,\text{hom-I}}$, and $\mathcal{A}_{\text{hom-II}}$. Here are a list of them:

- On eigenfunctions of Case 3, (3.89) and (3.132): Can we simplify the conditions (3.90) and (3.95) for Model I? (And similarly for Model II.)
- Find the singularly continuous spectrum of $\mathcal{A}_{\varepsilon,\text{hom}}$, $\mathcal{A}_{\varepsilon,\text{hom-I}}$, and $\mathcal{A}_{\text{hom-II}}$. We expect that these are all empty, since they arise from the dilation of an operator that is asymptotically close to $P_{\text{soft}}(A_\varepsilon^{(\tau)} - z)^{-1}P_{\text{soft}}$.
- In the case of Model II, what is $\sigma(-\Delta_D)$? Does $\sigma(-\Delta_D)$ contain eigenvalues? When does $\lambda \in \sigma(-\Delta_D)$ belong to the spectrum of $\mathcal{A}_{\text{hom-II}}$?
- Extend the arguments of Sections 3.3.1 (Model I) and 3.3.2 (Model II) to $\mathcal{A}_{\varepsilon,\text{hom}}$. See Section 3.3.3 for a discussion on the difficulties of this task.

Here are some directions for future work:

- Investigate the norm-resolvent asymptotics with respect to other scaling choices ε^α, $\alpha > 0$, starting with Models I and II.
- Adapt the boundary triple approach to unbounded domains. This will be a step towards treating non-periodic and even random high-contrast composites.
- An example of a random high-contrast composite: Consider the stiff-soft-stiff setup (Figure 1-1), where the coefficient matrix $\widetilde{a}_{\varepsilon^2} = \widetilde{a}_{\varepsilon^2,\omega}$ is random on the soft annulus, and equals $\varepsilon^{-2}I$ on the stiff regions. What is the *norm-resolvent* asymptotic of the corresponding operator $A_{\varepsilon,\omega} = -\operatorname{div}(a_{\varepsilon,\omega}\cdot)$? What is its spectrum?
- Establish a precise connection between the dispersion relation of the pre-limit (i.e. A_ε for very small ε) and the dispersion functions $K(\tau, z)$.

  This is not immediate from norm-resolvent asymptotic equivalence, as $\sigma(A_\varepsilon)$ is purely absolutely continuous, while $\sigma(A_{\varepsilon,\text{hom}})$ may contain eigenvalues. It is expected that $K(\tau, z)$ captures the absolutely continuous part of the spectrum (e.g. Corollaries 3.3.6 and 3.3.7), but as mentioned above, this has yet to be shown.

Appendix A

A heuristic explanation of the phrase "resonant inclusions"

In this appendix, we provide an informal justification of the term "resonant inclusions", which we use to describe the soft annular regions of the stiff-soft-stiff model (Figure 2-1).

We do this in two steps. **Step 1:** Consider the matrix $A(x) \equiv A_\varepsilon(x)$, given by

$$A(x) = \begin{cases} \varepsilon^2 I & \text{if } x \text{ lies in the soft regions,} \\ I & \text{if } x \text{ lies in the stiff regions.} \end{cases} \tag{A.1}$$

For a fixed wavenumber $k \in \mathbb{R}^d$, let us find $u = u(x)$, made up of standing waves, such that

$$-\nabla \cdot A(x)\nabla u(x) = |k|^2 u(x). \tag{A.2}$$

To find u, we have a useful computation

$$\begin{cases} -\nabla \cdot \varepsilon^2 \nabla e^{i\frac{k}{\varepsilon}\cdot x} & = |k|^2 e^{i\frac{k}{\varepsilon}\cdot x}, \\ -\nabla \cdot \nabla e^{ik\cdot x} & = |k|^2 e^{ik\cdot x}, \end{cases} \tag{A.3}$$

so that u is given by

$$u(x) = \begin{cases} e^{i\frac{k}{\varepsilon}\cdot x} & \text{if } x \text{ lies in the soft regions,} \\ e^{ik\cdot x} & \text{if } x \text{ lies in the stiff regions.} \end{cases} \tag{A.4}$$

Thus, when $u(x)$ enters the soft region, it has wavelength of the order $O(\varepsilon)$, which coincides with the size of the (width of the) soft inclusions. In this case we say that the soft inclusions act as "resonators".

Step 2: Returning to the wave equation

$$(\partial_{tt} - \nabla \cdot A\nabla)U(x,t) = 0, \tag{A.5}$$

we would like to find plane wave solutions that oscillates at a specified frequency ω. Moreover, we ask that the wave keeps its direction of propagation as it passes through the soft and stiff regions. In this case, we have

$$U(x,t) = u(x)e^{-i\omega t}. \tag{A.6}$$

Here, ω is fixed, and $u(x)$ is made up of standing waves. For instance, let us write

$$u(x) = \begin{cases} e^{ik_{\text{soft}}\cdot x} & \text{if } x \text{ lies in the soft regions,} \\ e^{ik_{\text{stiff}}\cdot x} & \text{if } x \text{ lies in the stiff regions.} \end{cases} \tag{A.7}$$

Our requirement for a constant direction of travel gives us $ck_{\text{soft}} = k_{\text{stiff}}$.

Since $U(x,t)$ solves (A.5), we require that $-\nabla \cdot A\nabla u(x) = \omega^2 u(x)$. And in this case, it is clear from Step 1 that we must have $c = \frac{1}{\varepsilon}$. In other words, our solution is

$$U(x,t) = \begin{cases} e^{i\frac{k_{\text{stiff}}}{\varepsilon}\cdot x}e^{-i\omega t} & \text{if } x \text{ lies in the soft regions,} \\ e^{ik_{\text{stiff}}\cdot x}e^{-i\omega t} & \text{if } x \text{ lies in the stiff regions,} \end{cases} \tag{A.8}$$

where we can pick any $k_{\text{stiff}} \in \mathbb{R}^d$ so long as $|k_{\text{stiff}}|^2 = \omega^2$ (the dispersion relation for the wave equation $(\partial_{tt} - \Delta)U = 0$).

To summarize, we call the soft inclusions in Figure 2-1 "resonators", because of the heuristic that if $U(x,t)$ is a plane wave propagating through the full medium, solves (A.5), oscillates at a specific frequency ω, and travels in a single direction, then as U enters the soft region, its wavelength will be comparable to the size of (the width) of the inclusion.

Figure A-1 below provides a sketch of $U(x,t)$, in the setup of Model I (Figure 1-2, soft inclusions in a stiff medium).

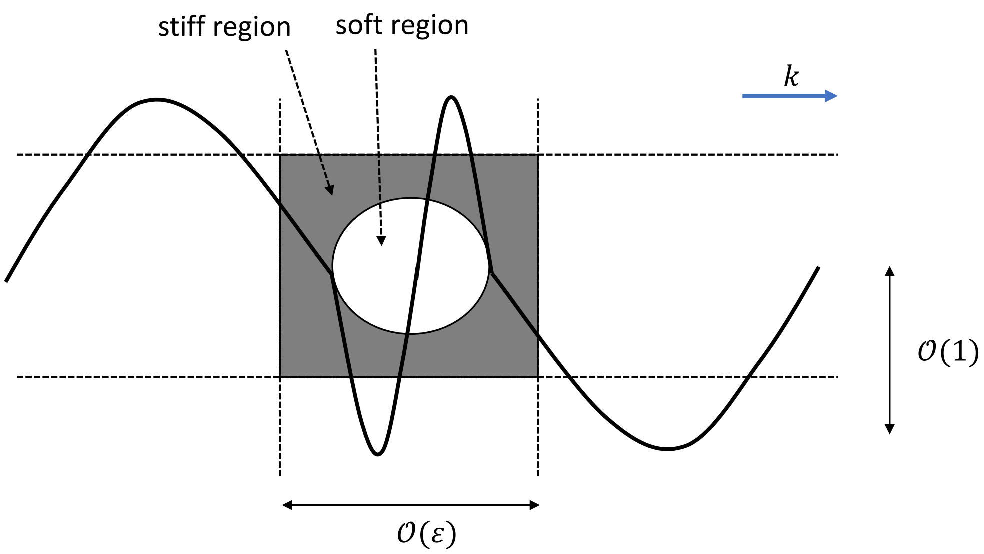

Figure A-1: A picture of $U(x,t)$, in the $2D$ case, for Model I. The wave is oscillating into and out of the paper with frequency ω, and is travelling in the direction $\frac{k}{|k|} \in S^1$ (to the right). The wavelength is $O(1)$ in the stiff regions, and becomes $O(\varepsilon)$ as it enters the soft inclusions. $U(x,t)$ has amplitude $O(1)$ throughout the medium.

Appendix B

Comparing various notions of convergence for sets

In this appendix, we will look at various notions of convergence for sets on a metric space X. The main example we have in mind is the case $X = \mathbb{C}$.

First, we look at convergence with respect to the Hausdorff distance d_H. We repeat the definition of d_H here for the reader's convenience:

$$d_H(M,N) := \max\left\{\sup_{x\in M}\inf_{y\in N}|x-y|, \sup_{y\in N}\inf_{x\in M}|x-y|\right\} \tag{B.1}$$

$$= \inf\left\{\varepsilon > 0 : M \subset U_\varepsilon(N) \text{ and } N \subset U_\varepsilon(M)\right\}, \tag{B.2}$$

where $U_\varepsilon(M) = \{x \in X : \inf_{y\in N}|x-y| \leq \varepsilon\}$ is the ε-fattening of the set M. It is assumed here that the sets M and N are non-empty. For simplicity, let us assume further that M and N are closed.

Definition B.1. [1, Definition 4.4.11] Let M_n and M be non-empty and closed subsets of X. We write $M_n \xrightarrow{\text{H}} M$ or $M = \text{H} - \lim_{n\to\infty} M_n$ ("Hausdorff") to mean that

$$d_H(M_n, M) \to 0, \quad \text{as } n \to \infty. \tag{B.3}$$

In this case, we say that M_n converges to M in the Hausdorff metric.

Second, we introduce a new notion of convergence:

Definition B.2. [1, Definition 4.4.13] Let M_n and M be non-empty and closed subsets of X. We write $M_n \xrightarrow{\text{K}} M$ or $M = \text{K} - \lim_{n\to\infty} M_n$ ("Kuratowski") if the following two conditions are satisfied:

(i) If $x = \lim_{k\to\infty} x_{n_k}$ for some subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ such that $x_n \in M_n$, then $x \in M$.

(ii) If $x \in M$, then there exist a full sequence $\{x_n\}$, with $x_n \in M_n$, such that $\lim_{n\to\infty} x_n = x$.

In this case, we say that M_n convergence to M in the sense of Kuratowski.

The two conditions in Definition B.2 can be written in terms of "upper closed limits" and "lower closed limits" respectively. For this we follow [4, Section 5.2], restricting ourselves to the case of a metric space X.

Definition B.3. Let X be a metric space and $\{M_n\}$ be a sequence of subsets of X. Define

$$\text{Li } M_n := \{x \in X : \text{for all open neighbourhoods } U \text{ of } x, U \cap M_n \neq \emptyset \text{ for } n \text{ large enough}\},$$
$$\text{Ls } M_n := \{x \in X : \text{for all open neighbourhoods } U \text{ of } x, U \cap M_n \neq \emptyset \text{ for infinitely many } n\}.$$

We call the set Li M_n the *lower closed limit* of M_n, and Ls M_n the *upper closed limit* of M_n.

In other words, Li M_n is the set of limit points for $\{M_n\}$, and Ls M_n is the set of cluster points for $\{M_n\}$. As the name suggests, we always have Li $M_n \subset$ Ls M_n, and Li M_n and Ls M_n are closed. Using upper and lower closed limits, we can write the two conditions in Definition B.2 as

(i) Ls $M_n \subset M$,

(ii) $M \subset$ Li M_n.

We can also do the same for limit set convergence (Definition 1.3.6). This gives us an equivalent way of defining limit set convergence:

Definition B.4. Let M_n and M be non-empty and closed subsets of X. We write $M_n \xrightarrow{\text{ls}} M$ or $M = \text{ls} - \lim_{n\to\infty} M_n$ ("limit set") if the following two conditions are satisfied:

(i) Li $M_n \subset M$,

(ii) $M \subset$ Li M_n

That is, if $M =$ Li M_n.

Remark. Clearly, M_n always has a limit in the sense of Definition B.4, namely Li M_n. But it is another question if a candidate set M equals this limit. For example, we are interested in the case $M_n = \sigma(A_n)$ and $M = \sigma(A)$ for self-adjoint operators A_n and A. We know that $\text{ls} - \lim \sigma(A_n)$ exist, but it is another question if $\sigma(A) = \text{ls} - \lim \sigma(A_n)$ holds. ◦

In this way, we can now easily compare the four notions of set convergence. To remind the reader, the four notions are:

- Hausdorff convergence (Definition B.1),
- Hausdorff convergence on compact subsets of X (Definition 1.3.5),
- Kuratowski convergence (Definition B.2),
- limit set convergence (Definition 1.3.6).

Clearly, if M_n converges in the sense of Hausdorff, then M_n must also converge in the sense of Hausdorff on compact subsets of X, and the two limits must coincide.

As for Hausdorff convergence on compact subsets of X and Kuratowski convergence, we quote the following result:

Proposition B.5. [1, Proposition 4.4.14] Assume that M_n and M are non-empty and closed subsets of X, then $M_n \xrightarrow{\text{K}} M$ if and only if $M_n \xrightarrow{\text{HC}} M$.

Finally, for Kuratowski convergence and limit set convergence, we have

Lemma B.6. Assume that M_n are non-empty and closed subsets of X. If the Kuratowski limit $\text{K} - \lim_{n\to\infty} M_n$ exists, then it must coincide with $\text{ls} - \lim_{n\to\infty} M_n$, i.e.

$$\text{K} - \lim_{n\to\infty} M_n = \text{ls} - \lim_{n\to\infty} M_n. \tag{B.4}$$

Proof. The conditions (i) and (ii) for the Kuratowski limit $\text{K} - \lim_{n\to\infty} M_n$ means that

$$\text{Ls } M_n \subset \text{K} - \lim_{n\to\infty} M_n \subset \text{Li } M_n. \tag{B.5}$$

Together with $\text{Li } M_n \subset \text{Ls } M_n$, this means that

$$\text{K} - \lim_{n\to\infty} M_n = \text{Ls } M_n = \text{Li } M_n. \tag{B.6}$$

Meanwhile, we recall that $\text{ls} - \lim_{n\to\infty} M_n$ is just another notation for $\text{Li } M_n$. □

Remark. In fact, the author of [4] defines Kuratowski convergence by (B.6). As $\text{Li } M_n \subset \text{Ls } M_n$ is always true, this is equivalent to Definition B.2 (which we took from [1]). ◦

To summarize, Hausdorff convergence implies Hausdorff convergence on compact subsets of X, the latter is equivalent to Kuratowski convergence, and Kuratowski convergence implies limit set convergence. Moreover, if one limit exist, so does the next, and the limits must coincide (as an equality of sets).

To make things concrete, let us provide several examples to show that these notions of set convergence are indeed distinct. We focus on the case $X = \mathbb{C}$.

Example B.7 (Limit set vs Kuratowski)**.**

$$M_n = \begin{cases} \{0, 1\} & \text{if } n \text{ is odd,} \\ \{0, -1\} & \text{if } n \text{ is even.} \end{cases} \tag{B.7}$$

Then $\text{Li } M_n = \{0\}$ and $\text{Ls } M_n = \{-1, 0, 1\}$. So $\text{ls} - \lim_{n\to\infty} M_n = \text{Li } M_n = \{0\}$. Since $\text{Li } M_n \neq \text{Ls } M_n$, $\text{K} - \lim_{n\to\infty} M_n$ does not exist. We may use Proposition B.5 to conclude that $\text{HC} - \lim_{n\to\infty} M_n$ does not exist (alternatively a direct check would suffice). Similarly, it is clear that M_n does not convergence with respect to the Hausdorff metric d_H. ◦

Example B.8 (Kuratowski vs Hausdorff, with unbounded sets)**.**

$$M = \mathbb{N} \quad \text{and} \quad M_n = \mathbb{N} \cup \{n + \tfrac{1}{2}\}. \tag{B.8}$$

Then $d_H(M_n, M) = \frac{1}{2}$ for all n, and thus M_n does not converge to M in the Hausdorff sense. In fact, $d_H(M_n, M_m) \geq 1$ if $n \neq m$, so M_n does not have a Hausdorff limit. On the other hand, we may check directly from the definitions, that $M_n \xrightarrow{\text{HC}} M$, $M_n \xrightarrow{\text{K}} M$, and $M_n \xrightarrow{\text{ls}} M$. ◦

As discussed in Section 1.3.3, if T_n and T are self-adjoint operators on a Hilbert space $\mathcal{H}$, then $T_n \xrightarrow{\text{nr}} T$ implies $\sigma(T_n) \xrightarrow{\text{HC}} \sigma(T)$. The following example shows that this cannot be upgraded to "$\sigma(T_n) \xrightarrow{\text{H}} \sigma(T)$":

Example B.9. Let $\mathcal{H} = L^2([0,\infty);\mathbb{C})$. Define the functions f_n, $f : [0,\infty) \to \mathbb{C}$ by

$$\begin{aligned} f(x) &:= \sum_{m=1}^{\infty} m\mathbf{1}_{[m-1,m)}(x), \\ f_n(x) &:= \sum_{m=1}^{\infty} m\mathbf{1}_{[m-1,m)}(x) + \frac{1}{2}\mathbf{1}_{[n+\frac{1}{2},n+1)}(x) \\ &= \sum_{m=1}^{n} m\mathbf{1}_{[m-1,m)}(x) + n\mathbf{1}_{[n,n+\frac{1}{2})}(x) + \left(n+\frac{1}{2}\right)\mathbf{1}_{[n+\frac{1}{2},n+1)}(x) + \sum_{m=n+2}^{\infty} m\mathbf{1}_{[m-1,m)}(x). \end{aligned}$$

(The second line of f_n views the function as sum of four terms with disjoint support.) Now consider the multiplication operators M_{f_n} and M_f on $\mathcal{H}$. We have

$$\sigma(M_f) = \overline{\operatorname{ran} f} = \mathbb{N} \quad \text{and} \quad \sigma(M_{f_n}) = \overline{\operatorname{ran} f_n} = \mathbb{N} \cup \{n + \tfrac{1}{2}\}.$$

In particular 0 lies in the resolvent sets of M_f and M_{f_n}. Let us compute the difference of the resolvents for M_f and M_{f_n} at the point $z = 0$:

$$(M_f)^{-1} - (M_{f_n})^{-1} = M_{f^{-1}} - M_{f_n^{-1}} = M_{f^{-1}-f_n^{-1}}.$$

Figure B-1 shows a graph of $f^{-1} - f_n^{-1}$.

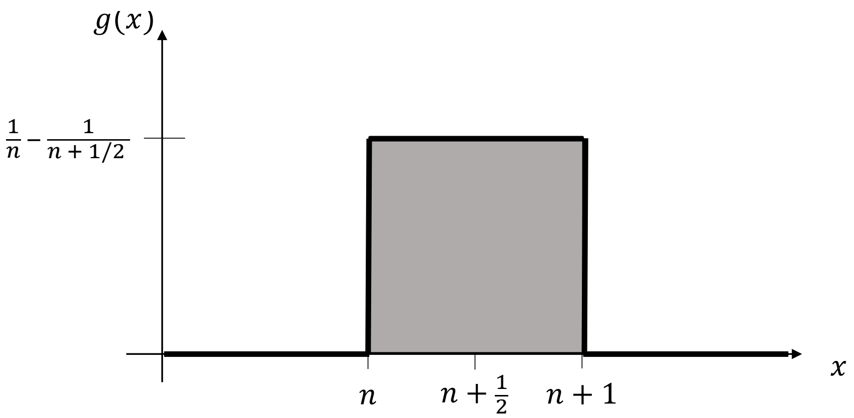

Figure B-1: The graph of $g = f^{-1} - f_n^{-1}$.

We may now compute

$$\|M_{f^{-1}-f_n^{-1}}\|_{op} = \|f^{-1} - f_n^{-1}\|_\infty = \frac{1}{n} - \frac{1}{n+\frac{1}{2}} = \frac{\cancel{n} + \frac{1}{2} - \cancel{n}}{n(n+\frac{1}{2})} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Thus, $M_{f_n} \xrightarrow{\text{nr}} M_f$. Meanwhile, Example B.8 tells us that

$$d_H(\sigma(M_{f_n}), \sigma(M_f)) \not\to 0, \quad \text{while} \quad \sigma(M_{f_n}) \xrightarrow{\text{HC}} \sigma(M_f). \qquad \circ$$

Appendix C

Preparatory results for estimating the first Steklov eigenvalue with respect to the stiff landscape

The following result is a slight modification of [46, Proposition A.7]:

Proposition C.1. There exists a constant $C > 0$ such that for every $u \in H^1_{\mathrm{per}}(Q)$, $\tau \in Q'$ we have the following estimates:

$$|\tau| \, \|u\|_{L^2(Q)} \leq C \, \|(\nabla + i\tau) \, u\|_{L^2(Q;\mathbb{C}^d)} \,, \tag{C.1}$$

$$\|\nabla u\|_{L^2(Q;\mathbb{C}^d)} \leq C \, \|(\nabla + i\tau) \, u\|_{L^2(Q;\mathbb{C}^d)} \,, \tag{C.2}$$

$$\left\| u - \int_Q u \right\|_{L^2(Q)} \leq C \, \|(\nabla + i\tau) \, u\|_{L^2(Q;\mathbb{C}^d)} \,. \tag{C.3}$$

Proof. Fix $u \in H^1_{\mathrm{per}}(Q)$ and consider its Fourier series decomposition:

$$u = \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k \cdot y}, \quad \nabla u = \sum_{k \in \mathbb{Z}^d} (2\pi i k) a_k e^{2\pi i k \cdot y}, \quad u - \int_Q u = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} a_k e^{2\pi i k \cdot y}.$$

Plancherel's formula yields

$$\|u\|^2_{L^2(Q)} = \sum_{k \in \mathbb{Z}^d} |a_k|^2, \quad \|\nabla u\|^2_{L^2(Q;\mathbb{C}^d)} = \sum_{k \in \mathbb{Z}^d} |2\pi|^2 |a_k|^2 |k|^2, \quad \left\| u - \int_Q u \right\|^2_{L^2(Q)} = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |a_k|^2. \tag{C.4}$$

Furthermore, we have

$$(\nabla + i\tau) u = \sum_{k \in \mathbb{Z}^d} (2\pi i k + i\tau) \, a_k e^{2\pi i k \cdot y}. \tag{C.5}$$

Now we calculate

$$\|(\nabla + i\tau) u\|^2_{L^2(Q;\mathbb{C}^d)} = \sum_{k \in \mathbb{Z}^d} |a_k \left(2\pi i k + i\tau\right)|^2 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |a_k|^2 |2\pi i k + i\tau|^2 + |a_0|^2 |\tau|^2. \tag{C.6}$$

Since $\tau \in Q' = [-\pi, \pi)^d$, if at least one $(k)_j \geq 1$, it is clear that

$$|2\pi i k + i\tau|^2 \geq C, \tag{C.7}$$

where the constant $C > 0$ does not depend on τ and $k \in \mathbb{Z}^d \setminus \{0\}$. This gives us

$$\begin{aligned} \|(\nabla + i\tau)u\|^2_{L^2(Q;\mathbb{C}^d)} &= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |a_k|^2 |2\pi i k + i\tau|^2 + |a_0|^2 |\tau|^2 \\ &\geq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} C|a_k|^2 = C \left\| u - \int_Y u \right\|^2_{L^2(Q)}. \end{aligned} \tag{C.8}$$

Moreover, we have

$$\|(\nabla + i\tau)u\|^2_{L^2(Q;\mathbb{C}^d)} = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |a_k|^2 |2\pi i k + i\tau|^2 + |a_0|^2 |\tau|^2 \geq \sum_{k \in \mathbb{Z}^d} C|\tau|^2 |a_k|^2 = C|\tau|^2 \|u\|^2_{L^2(Q)}, \tag{C.9}$$

and also

$$\|(\nabla + i\tau)u\|^2_{L^2(Q;\mathbb{C}^d)} \geq \sum_{k \in \mathbb{Z}^d} C|a_k|^2 |2\pi|^2 |k|^2 = C \|\nabla u\|^2_{L^2(Q;\mathbb{C}^d)}. \tag{C.10}$$

This concludes the proof. □

We are interested in using the above inequalities to estimate the eigenvalue $\mu_1^{\text{stiff-ls},(\tau)}$.

Corollary C.2. There exist constants $C_1, C_2 > 0$ independent of τ such that

$$C_1 |\tau|^2 \leq -\mu_1^{\text{stiff-ls},(\tau)} \leq C_2 |\tau|^2. \tag{C.11}$$

Proof. Fix $u \in H^1_{\text{per}}(Q_{\text{stiff-ls}})$, since $\partial Q_{\text{stiff-ls}} = \Gamma_{\text{ls}}$ is smooth, u may be extended to a function in $H^1_{\text{per}}(Q)$, which we will still denote as u, such that

$$\|(\nabla + i\tau)u\|_{L^2(Q;\mathbb{C}^d)} \leq C \|(\nabla + i\tau)u\|_{L^2(Q_{\text{stiff-ls}};\mathbb{C}^d)}, \tag{C.12}$$

where the constant $C > 0$ only depends on $Q_{\text{stiff-ls}}$ [28]. Combine this with (C.1) and (C.2), and we get

$$|\tau| \|u\|_{H^1(Q_{\text{stiff-ls}})} \leq C|\tau| \|u\|_{H^1(Q)} \leq C \|(\nabla + i\tau)u\|_{L^2(Q;\mathbb{C}^d)} \leq C \|(\nabla + i\tau)u\|_{L^2(Q_{\text{stiff-ls}};\mathbb{C}^d)}. \tag{C.13}$$

By the Trace theorem, this implies that

$$|\tau| \|u|_{\Gamma_{\text{ls}}}\|_{L^2(\Gamma_{\text{ls}})} \leq C \|(\nabla + i\tau)u\|_{L^2(Q_{\text{stiff-ls}};\mathbb{C}^d)}, \tag{C.14}$$

which in turn gives us the lower bound of (C.11) by the min-max principle. The upper bound follows by testing $\mathbf{1}_{Q_{\text{stiff-ls}}}$ in the variational characterization of $\mu_1^{\text{stiff-ls},(\tau)}$ (similar to Step 4c in the proof of Proposition 2.3.5). □

Appendix D

Proofs for Section 2.4

Proof of Proposition 2.4.7. Consider the generalized resolvent $P_{\text{soft}}(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}}-z)^{-1}P_{\text{soft}}$, which we know is $O(\varepsilon^2)$ close to $R^{(\tau)}_\varepsilon(z)=P_{\text{soft}}(A^{(\tau)}_\varepsilon - z)^{-1}P_{\text{soft}}$ by Theorem 2.4.2. This can be expressed as

$$
\begin{aligned}
&P_{\text{soft}}(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}} - z)^{-1}P_{\text{soft}}\\
&= P_{\text{soft}}(A^{(\tau)}_{\varepsilon,0} - z)^{-1}P_{\text{soft}} - P_{\text{soft}}S^{(\tau)}_\varepsilon(z)\left(\overline{\mathcal{P}^{(\tau)}_\perp + \mathcal{P}^{(\tau)}M^{(\tau)}_\varepsilon(z)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{(\tau)}_\varepsilon(\bar{z})\right)^* P_{\text{soft}}\\
&= (A^{\text{soft},(\tau)}_0 - z)^{-1} - S^{\text{soft},(\tau)}(z)\left(\overline{\mathcal{P}^{(\tau)}_\perp + \mathcal{P}^{(\tau)}M^{(\tau)}_\varepsilon(z)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^*\\
&= (A^{\text{soft},(\tau)}_0 - z)^{-1} - S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_\varepsilon(z)\mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^*.
\end{aligned}
\tag{D.1}
$$

The second equality follows by the same reasoning as Proposition 2.4.5. The final equality uses Lemma 2.4.1. On the other hand, by the Krein's formula, we have

$$
\begin{aligned}
R^{(\tau)}_{\varepsilon,\text{eff}}(z) &= (A^{\text{soft},(\tau)}_0 - z)^{-1}\\
&\quad - S^{\text{soft},(\tau)}(z)\left(\overline{\mathcal{P}^{(\tau)}_\perp + \mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_\varepsilon(z)\mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^*\\
&= (A^{\text{soft},(\tau)}_0 - z)^{-1}\\
&\quad - S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_\varepsilon(z)\mathcal{P}^{(\tau)} + \mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^*.
\end{aligned}
\tag{D.2}
$$

The second equality follows from the observation made before the proposition. □

Proof of Proposition 2.4.9. We will verify this entry-wise. The top left entry is done in Proposition 2.4.7. For the remaining entries, we will compare this with $(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_\perp,\mathcal{P}^{(\tau)}} - z)^{-1}$ since it is $O(\varepsilon^2)$ close to $(A^{(\tau)}_\varepsilon - z)^{-1}$ by Theorem 2.4.2. In the computations below, we will use

$$
\operatorname{ran}\left(S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right) \subset \mathcal{D}(A^{\text{soft},(\tau)}_0)\dot{+}\operatorname{ran}(\Pi^{\text{soft},(\tau)}\mathcal{P}^{(\tau)}),
\tag{D.3}
$$

which is a consequence of the identity $S(z)=(I+z(A_0-z)^{-1})\Pi$. The argument for a_{21} and a_{31} are the same. For a_{21}, we have

$$P_{\text{stiff-int}}(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_{\perp},\mathcal{P}^{(\tau)}}-z)^{-1}P_{\text{soft}}$$
$$=-S^{\text{stiff-int},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}+\mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^{*}$$

By a similar argument to (D.1).

$$=-S^{\text{stiff-int},(\tau)}_{\varepsilon}(z)\Gamma^{\text{soft},(\tau)}_{0}S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}+\mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^{*}$$

By Proposition 2.2.13(2), $\Gamma^{\text{soft},(\tau)}_{0}S^{\text{soft},(\tau)}(z)=I$.

$$=-S^{\text{stiff-int},(\tau)}_{\varepsilon}(z)k^{(\tau)}(z)\left[S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}+\mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^{*}\right]$$
$$=S^{\text{stiff-int},(\tau)}_{\varepsilon}(z)k^{(\tau)}(z)\left[R^{(\tau)}_{\varepsilon,\text{eff}}(z)-(A^{\text{soft},(\tau)}_{0}-z)^{-1}\right]$$

By (D.2).

$$=\Pi^{\text{stiff-int},(\tau)}k^{(\tau)}(z)\left[R^{(\tau)}_{\varepsilon,\text{eff}}(z)-(A^{\text{soft},(\tau)}_{0}-z)^{-1}\right]+O(\varepsilon^2)$$

By Lemma 2.2.20. The remaining terms equals $\mathcal{P}^{(\tau)}\mathbb{A}^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(\bar{z})\right)^{*}$, which is $O(1)$.

The argument for a_{12} and a_{13} are the same. For a_{12}, we have

$$P_{\text{soft}}(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_{\perp},\mathcal{P}^{(\tau)}}-z)^{-1}P_{\text{stiff-int}}$$
$$=-S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{\text{stiff},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}+\mathcal{P}^{(\tau)}M^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{stiff-int},(\tau)}_{\varepsilon}(\bar{z})\right)^{*}$$

Similarly to (D.1). The decoupling term $(A^{(\tau)}_{\varepsilon,0}-z)^{-1}$ vanishes as $L^2(Q_{\text{soft}})$ and $L^2(Q_{\text{stiff-int}})$ are orthogonal, and are invariant subspaces for $A^{(\tau)}_{\varepsilon,0}$.

$$=\left(-\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{(\tau)}_{\varepsilon}(\bar{z})\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(z)\right)^{*}\right)^{*}\left(S^{\text{stiff-int},(\tau)}_{\varepsilon}(\bar{z})\right)^{*}$$

Take adjoint twice. Use Proposition 2.2.13(5).

$$=\left(k^{(\tau)}(\bar{z})S^{\text{soft},(\tau)}(\bar{z})\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{(\tau)}_{\varepsilon}(\bar{z})\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft},(\tau)}(z)\right)^{*}\right)^{*}\left(S^{\text{stiff-int},(\tau)}_{\varepsilon}(\bar{z})\right)^{*}$$

By Proposition 2.2.13(2), $\Gamma^{\text{soft},(\tau)}_{0}S^{\text{soft},(\tau)}(\bar{z})=I$, and $\Gamma^{\text{soft},(\tau)}_{0}$ may be restricted to $k^{(\tau)}(\bar{z})$.

$$=\left(k^{(\tau)}(\bar{z})\left[R^{(\tau)}_{\varepsilon,\text{eff}}(\bar{z})-(A^{\text{soft},(\tau)}_{0}-\bar{z})^{-1}\right]\right)^{*}\left(S^{\text{stiff-int},(\tau)}_{\varepsilon}(\bar{z})\right)^{*}$$
$$=\left(k^{(\tau)}(\bar{z})\left[R^{(\tau)}_{\varepsilon,\text{eff}}(\bar{z})-(A^{\text{soft},(\tau)}_{0}-\bar{z})^{-1}\right]\right)^{*}\left(\Pi^{\text{stiff-int},(\tau)}\right)^{*}+O(\varepsilon^2).$$

The second last equality follows by (D.2). The final equality follows by Lemma 2.2.20.

The argument for a_{22} is the same as a_{33} are the same. For a_{22}, we have

$$P_{\text{stiff-int}}(\widehat{A}^{(\tau)}_{\varepsilon,\mathcal{P}^{(\tau)}_{\perp},\mathcal{P}^{(\tau)}}-z)^{-1}P_{\text{stiff-int}}$$
$$=(A^{\text{stiff-int},(\tau)}_{\varepsilon,0}-z)^{-1}-S^{\text{stiff-int},(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M^{(\tau)}_{\varepsilon}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{stiff-int},(\tau)}_{\varepsilon}(\bar{z})\right)^{*}$$

Similarly to (D.1).

$$= (A_{\varepsilon,0}^{\text{stiff-int},(\tau)} - z)^{-1} - S_\varepsilon^{\text{stiff-int},(\tau)}(z)k^{(\tau)}(z)S^{\text{soft},(\tau)}(z)\mathcal{P}^{(\tau)}\left(\mathcal{P}^{(\tau)}M_\varepsilon^{(\tau)}(z)\mathcal{P}^{(\tau)}\right)^{-1}\mathcal{P}^{(\tau)}\left(S_\varepsilon^{\text{stiff-int},(\tau)}(\bar{z})\right)^*$$

By Proposition 2.2.13(2), $\Gamma_0^{\text{soft},(\tau)}S^{\text{soft},(\tau)}(z) = I$, and $\Gamma_0^{\text{soft},(\tau)}$ may be restricted to $k^{(\tau)}(z)$.

$$= (A_{\varepsilon,0}^{\text{stiff-int},(\tau)} - z)^{-1} + S_\varepsilon^{\text{stiff-int},(\tau)}(z)k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(S_\varepsilon^{\text{stiff-int},(\tau)}(\bar{z})\right)^*$$

By the arguments of a_{12}.

$$= \Pi^{\text{stiff-int},(\tau)}(z)k^{(\tau)}(z)\left(k^{(\tau)}(\bar{z})\left[R_{\varepsilon,\text{eff}}^{(\tau)}(\bar{z}) - (A_0^{\text{soft},(\tau)} - \bar{z})^{-1}\right]\right)^*\left(\Pi^{\text{stiff-int},(\tau)}(\bar{z})\right)^* + O(\varepsilon^2)$$

By Lemma 2.2.20 and Proposition 2.2.4.

Finally, the argument for a_{23} and a_{32} is similar to that of a_{22}, the only difference being that the decoupling term $(A_0 - z)^{-1}$ is now absent. This completes the proof. □

Proof of Theorem 2.4.20. We will start with the **top left entry** of (2.167). To qualify as a resolvent of $\mathcal{A}_{\text{hom}}$ at z, the operator on the top left entry must take any given $f \in L^2(Q_{\text{soft}})$ to u, where u is the first entry of $(u, \widehat{u})^T \in \mathcal{D}(\mathcal{A}_{\text{hom}})$, and $(u, \widehat{u})$ is the unique solution to the problem

$$(\mathcal{A}_{\text{hom}} - z)\begin{pmatrix} u \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \iff \begin{cases} \breve{\widehat{A}}^{\text{soft}}u - zu = f, \\ -(\breve{\Pi}^{\text{stiff}\,*})^{-1}\breve{\Gamma}_1^{\text{soft}}u + \mathcal{B}\widehat{u} - z\widehat{u} = 0. \end{cases} \tag{D.4}$$

But we may rearrange the second line of the latter system:

$$\begin{aligned} &(\breve{\Pi}^{\text{stiff}\,*})^{-1}\breve{\Gamma}_1^{\text{soft}}u - (\mathcal{B} - z)\widehat{u} = 0 \\ \Leftrightarrow\ &\breve{\Gamma}_1^{\text{soft}}u - (\breve{\Pi}^{\text{stiff}})^*(\mathcal{B} - z)\widehat{u} = 0 \\ \Leftrightarrow\ &\underbrace{I}_{\beta_1 :=}\breve{\Gamma}_1^{\text{soft}}u \underbrace{-(\breve{\Pi}^{\text{stiff}})^*(\mathcal{B} - z)\breve{\Pi}^{\text{stiff}}}_{\beta_0 :=}\breve{\Gamma}_0^{\text{soft}}u = 0. \end{aligned} \tag{D.5}$$

That is, the mapping $f \mapsto u$ as described above, is precisely that of $(\breve{\widehat{A}}_{\beta_0,\beta_1}^{\text{soft}} - z)^{-1} \equiv R(z)$, *provided it exist.* This means to check the conditions on $\beta_0 + \beta_1\breve{M}^{\text{soft}}(z)$ so that Theorem 2.2.16 applies: The condition on the domains of β_0 and β_1 are immediate as these are bounded operators. The boundedness also implies that $\beta_0 + \beta_1\breve{M}^{\text{soft}}(z)$ (with its maximal domain) is closed.

As for the boundedness of the inverse, it suffices to check that it is bounded below, since we are working on a finite dimensional space $\breve{\mathcal{H}}^{\text{stiff}}$. Just like in Theorem 2.3.4, it suffices to show that the imaginary part is bounded below: Let $\phi \in \breve{\mathcal{E}}$, then

$$\left|\left(\text{Im}\left[-(\breve{\Pi}^{\text{stiff}})^*(\mathcal{B} - z)\breve{\Pi}^{\text{stiff}} + \breve{M}^{\text{soft}}(z)\right]\phi, \phi\right)_{\breve{\mathcal{E}}}\right|$$

$$= \left|\left(\text{Im}\left[z(\breve{\Pi}^{\text{stiff}})^*\breve{\Pi}^{\text{stiff}} + \breve{M}^{\text{soft}}(z)\right]\phi, \phi\right)_{\breve{\mathcal{E}}}\right| \qquad \text{as } (\breve{\Pi}^{\text{stiff}})^*\mathcal{B}\breve{\Pi}^{\text{stiff}} \text{ is self-adjoint.}$$

$$= \left|\text{Im}z\left((\breve{S}^{\text{soft}}(\bar{z}))^*\breve{S}^{\text{soft}}(\bar{z})\phi + (\breve{\Pi}^{\text{stiff}})^*\breve{\Pi}^{\text{stiff}}\phi, \phi\right)_{\breve{\mathcal{E}}}\right| \qquad \text{by Proposition 2.2.13(7).}$$

$$= |\text{Im}z|\left((\breve{S}^{\text{soft}}(\bar{z}))^*\breve{S}^{\text{soft}}(\bar{z})\phi, \phi\right)_{\breve{\mathcal{E}}} + |\text{Im}z|\left((\breve{\Pi}^{\text{stiff}})^*\breve{\Pi}^{\text{stiff}}\phi, \phi\right)_{\breve{\mathcal{E}}} \qquad \text{both operators are positive.}$$

$$\geq |\text{Im}z|\ \|\breve{\Pi}^{\text{stiff}}\phi\|_{\breve{\mathcal{H}}^{\text{stiff}}}^2 \qquad \text{both operators are positive.}$$

$$\geq |\mathrm{Im} z| \; \|(\breve{\Pi}^{\mathrm{stiff}})^{-1}\|^{-2}_{\breve{\mathcal{H}}^{\mathrm{stiff}} \to \breve{\mathcal{E}}} \|\phi\|^2_{\breve{\mathcal{E}}}. \tag{D.6}$$

We have shown that $R(z)$ exists, and equals $P_{\mathrm{soft}}(\mathcal{A}_{\mathrm{hom}} - z)^{-1} P_{\mathrm{soft}}$. Next, we check the **bottom left entry** of (2.167). This is the mapping $f \mapsto \widehat{u}$, where $(u, \widehat{u})$ solves the system (D.5). But we defined $\widehat{u} = \breve{\Pi}^{\mathrm{stiff}} \breve{\Gamma}_0^{\mathrm{soft}} u$ and have just shown that $u = R(z) f$, therefore

$$\begin{aligned}
\widehat{u} &= \breve{\Pi}^{\mathrm{stiff}} \breve{\Gamma}_0^{\mathrm{soft}} R(z) f \\
&= \breve{\Pi}^{\mathrm{stiff}} \breve{\Gamma}_0^{\mathrm{soft}} \left[R(z) - (A_0^{\mathrm{soft}} - z)^{-1} \right] f \quad \text{as } \mathcal{D}(A_0^{\mathrm{soft}}) = \ker(\breve{\Gamma}_0^{\mathrm{soft}}) \text{ by definition.} \\
&= \breve{\Pi}^{\mathrm{stiff}} k(z) \left[R(z) - (A_0^{\mathrm{soft}} - z)^{-1} \right] f.
\end{aligned}$$

The final equality holds by exactly the same argument as Proposition 2.4.9 (the term a_{21}), since the Krein's formula is now applicable to $R(z)$. We have shown that $P_{\breve{\mathcal{H}}^{\mathrm{stiff}}}(\mathcal{A}_{\mathrm{hom}} - z)^{-1} P_{\mathrm{soft}}$ equals the bottom left entry of (2.167).

Next, we discuss the **top right entry** of (2.167). Similarly to $R(z)$, this must take any given $\widehat{f} \in \breve{\mathcal{H}}^{\mathrm{stiff}}$ to u, where u is the first entry of $(u, \widehat{u})^T \in \mathcal{D}(\mathcal{A}_{\mathrm{hom}})$, and $(u, \widehat{u})$ is the unique solution to the problem

$$(\mathcal{A}_{\mathrm{hom}} - z) \begin{pmatrix} u \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{f} \end{pmatrix} \iff \begin{cases} \widehat{\breve{A}}^{\mathrm{soft}} u - z u = 0, \\ \underbrace{I}_{\beta_1 =} \breve{\Gamma}_1^{\mathrm{soft}} u = \underbrace{(\breve{\Pi}^{\mathrm{stiff}})^* (\mathcal{B} - z) \breve{\Pi}^{\mathrm{stiff}}}_{-\beta_0 =} \breve{\Gamma}_0^{\mathrm{soft}} u - (\breve{\Pi}^{\mathrm{stiff}})^* \widehat{f}. \end{cases} \tag{D.7}$$

We would like to put the system into the form (2.61), forcing us address the term $(\breve{\Pi}^{\mathrm{stiff}})^* \widehat{f}$. Using Proposition 2.4.15(5) ("furthermore" part), we find some $v_{\widehat{f}} \in \mathcal{D}(A_0^{\mathrm{soft}})$ satisfying

$$\breve{\Gamma}_1^{\mathrm{soft}} v_{\widehat{f}} = (\breve{\Pi}^{\mathrm{stiff}})^* \widehat{f}.$$

Now consider the function $v = R(z)(A_0^{\mathrm{soft}} - z) v_{\widehat{f}}$. By applying Theorem 2.2.16 to $R(z)$, we know that $v \in \mathcal{D}(\widehat{\breve{A}}^{\mathrm{soft},(\tau)})$, and furthermore v solves the following system uniquely:

$$\begin{cases} (\widehat{\breve{A}}^{\mathrm{soft}} - z) v = (A_0^{\mathrm{soft}} - z) v_{\widehat{f}}, \\ \beta_0 \breve{\Gamma}_0^{\mathrm{soft}} v + \beta_1 \breve{\Gamma}_1^{\mathrm{soft}} v = 0. \end{cases}$$

Using the first line of the system, $v_{\widehat{f}} \in \mathcal{D}(A_0^{\mathrm{soft}})$, and $A_0^{\mathrm{soft}} \subset \widehat{\breve{A}}^{\mathrm{soft}}$,

$$(\widehat{\breve{A}}^{\mathrm{soft}} - z)(v - v_{\widehat{f}}) = (A_0^{\mathrm{soft}} - z) v_{\widehat{f}} - (\widehat{\breve{A}}^{\mathrm{soft}} - z) v_{\widehat{f}} = 0.$$

Using the second line of the system, $v_{\widehat{f}} \in \mathcal{D}(A_0^{\mathrm{soft}})$, and $\mathcal{D}(A_0^{\mathrm{soft}}) = \ker(\breve{\Gamma}_0^{\mathrm{soft}})$,

$$\beta_0 \breve{\Gamma}_0^{\mathrm{soft}} (v - v_{\widehat{f}}) + \breve{\Gamma}_1^{\mathrm{soft}} (v - v_{\widehat{f}}) = -\beta_0 \breve{\Gamma}_0^{\mathrm{soft}} v_{\widehat{f}} - \breve{\Gamma}_1^{\mathrm{soft}} v_{\widehat{f}} = -\breve{\Gamma}_1^{\mathrm{soft}} v_{\widehat{f}} = -(\breve{\Pi}^{\mathrm{stiff}})^* \widehat{f}. \tag{D.8}$$

In other words, if we define $u \in \mathcal{D}(\widehat{\breve{A}}^{\mathrm{soft},(\tau)})$ by $u := v - v_{\widehat{f}} = R(z)(A_0^{\mathrm{soft}} - z) v_{\widehat{f}} - v_{\widehat{f}}$, then u is a solution to the system (D.7). To show that the solution is unique, we consider the fully homogeneous case, $f = 0$ and $\widehat{f} = 0$. But this can be viewed as a special case of (D.4), for which uniqueness has been established by Theorem 2.2.16. We further compute

$$\begin{aligned} u &= \left[R(z) - (A_0^{\text{soft}} - z)^{-1}\right](A_0^{\text{soft}} - z)v_{\widehat{f}} \\ &= -S^{\text{soft}}(z)\mathcal{P}^{(\tau)}\left(\beta_0 + \breve{M}^{\text{soft}}(z)\right)^{-1}\mathcal{P}^{(\tau)}\left(S^{\text{soft}}(\bar{z})\right)^*(A_0^{\text{soft}} - z)v_{\widehat{f}} && \text{By Krein's formula.} \\ &= -S^{\text{soft}}(z)\mathcal{P}^{(\tau)}\left(\beta_0 + \breve{M}^{\text{soft}}(z)\right)^{-1}\mathcal{P}^{(\tau)}\breve{\Gamma}_1^{\text{soft}}v_{\widehat{f}} && \text{By Proposition 2.2.13(4)} \\ &= -S^{\text{soft}}(z)\mathcal{P}^{(\tau)}\left(\beta_0 + \breve{M}^{\text{soft}}(z)\right)^{-1}\mathcal{P}^{(\tau)}(\breve{\Pi}^{\text{stiff}})^*\widehat{f} && \text{Definition of } v_{\widehat{f}}. \\ &= \left(k(\bar{z})\left[R(\bar{z}) - (A_0^{\text{soft}} - \bar{z})^{-1}\right]\right)^*(\breve{\Pi}^{\text{stiff}})^*\widehat{f}, \end{aligned} \tag{D.9}$$

where the last equality is proven in the same way as in Proposition 2.4.9(the term a_{12}). Therefore the expression for u does not depend on the choice $v_{\widehat{f}}$, and coincides with the top right entry of (2.167).

Next, we discuss the **bottom right entry** of (2.167). Similarly to the bottom left entry, we use $\widehat{u} = \breve{\Pi}^{\text{stiff}}\breve{\Gamma}_0^{\text{soft}}u$, and that u is given by (D.9) to see that

$$\begin{aligned} \widehat{u} &= \breve{\Pi}^{\text{stiff}}\breve{\Gamma}_0^{\text{soft}}\left(k(\bar{z})\left[R(\bar{z}) - (A_0^{\text{soft}} - \bar{z})^{-1}\right]\right)^*(\breve{\Pi}^{\text{stiff}})^*\widehat{f} \\ &= \breve{\Pi}^{\text{stiff}}k(z)\left(k(\bar{z})\left[R(\bar{z}) - (A_0^{\text{soft}} - \bar{z})^{-1}\right]\right)^*(\breve{\Pi}^{\text{stiff}})^*\widehat{f} \quad \text{as } \mathcal{D}(A_0^{\text{soft}}) = \ker(\breve{\Gamma}_0^{\text{soft}}). \end{aligned}$$

Finally the show the **self-adjointness of $\mathcal{A}_{\text{hom}}$.** Notice that the arguments provided above implies that $(\mathcal{A}_{\text{hom}} - z)$ is surjective for all $z \in \mathbb{C} \setminus \mathbb{R}$. As $\mathcal{A}_{\text{hom}}$ is symmetric by Lemma 2.4.19, the conclusion follows from [22, Proposition 3.11]. ☐

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