Observability inequalities for transport equations through Carleman estimates

Piermarco Cannarsa, Giuseppe Floridia and Masahiro Yamamoto

Abstract We consider the transport equation $\partial_t u(x,t) + H(t)\cdot\nabla u(x,t) = 0$ in $\Omega\times(0,T)$, where $T>0$ and $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$. First, we prove a Carleman estimate for solutions of finite energy with piecewise continuous weight functions. Then, under a further condition which guarantees that the orbits of H intersect $\partial\Omega$, we prove an energy estimate which in turn yields an observability inequality. Our results are motivated by applications to inverse problems.

Key words: Carleman estimates, transport equation, observability inequality.

1 Introduction

Let $d\in\mathbb{N}$ and $\Omega\subset\mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, $\nu=\nu(x)$ be the unit outward normal vector at x to $\partial\Omega$, and let $x\cdot y$ and $|x|$ denote the scalar product of $x,y\in\mathbb{R}^d$ and the norm of $x\in\mathbb{R}^d$, respectively. We set $Q:=\Omega\times(0,T)$, and we consider

$$Pu(x,t) := \partial_t u + H(t)\cdot\nabla u = 0 \quad \text{in } Q, \tag{1}$$

where $H(t) := (H_1(t),\ldots,H_d(t)) : [0,T]\to\mathbb{R}^d,\ H\in C^1([0,T];\mathbb{R}^d)$.

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Equation (1) is called a transport equation and $H(t)$ describes the velocity of the flow, which is here assumed to be independent of the spatial variable x.

Problem formulation

We assume

$$H_0 := \min_{t\in[0,T]} |H(t)| > 0, \tag{2}$$

and, without loss of generality, we suppose that $\mathbf{0} = (0,\dots,0) \in \overline{\Omega}$.

Let us recall the following definition.

Definition 1.1 *A partition $\{t_j\}_0^m$ of $[0,T]$ is a strictly increasing finite sequence $t_0,t_1,\dots,t_m$ (for some $m\in\mathbb{N}$) of real numbers starting from the initial point $t_0 = 0$ and arriving at the final point $t_m = T$.*

Hereafter, we will call $\{t_j\}_0^m$ a uniform partition of $[0,T]$ when the length of the intervals $[t_j,t_{j+1}]$ is constant for $j = 0,\dots,m-1$, that is, $t_j = \frac{T}{m} j$, $j = 0,\dots,m$.

Lemma 1.2 below ensures that any vector-valued function $H(t)$, satisfying (2), admits a partition $\{t_j\}_0^m$ of $[0,T]$ such that the angles of oscillations of the vector $H(t)$ are less than $\frac{\pi}{2}$ in any time interval $[t_j,t_{j+1}]$, $j = 0,\dots,m-1$ (see Figure 1).

Given a partition $\{t_j\}_0^m$ of $[0,T]$, let us set

$$\eta_j := \frac{H(t_j)}{|H(t_j)|}, \quad j = 0,\dots,m-1. \tag{3}$$

Lemma 1.2 *Let $S_* \in (1/\sqrt{2},1)$. For any given $H \in Lip([0,T];\mathbb{R}^d)$, satisfying condition (2), there exist $m\in\mathbb{N}$ and a partition $\{t_j\}_0^m$ of $[0,T]$ such that*

$$\frac{H(t)}{|H(t)|}\cdot\eta_j \geq S_*, \quad \forall t\in[t_j,t_{j+1}], \ \forall j = 0,\dots,m-1, \tag{4}$$

where η_j are defined in (3).

Lemma 1.2 is proved in the Appendix.

Remark 1.3 Condition (4) means that there exist m cones in $\mathbb{R}^d$ such that the axis of every cone, that is, the straight line passing through the apex about which the whole cone has a circular symmetry, is the line between $\mathbf{0} = (0,\dots,0)$ and η_j, $j = 0,\dots,m-1$. Moreover, a straight line passing through the apex is contained in the cone if the angle between this line and the axis of the cone is less than $\pi/4$. Indeed, the inequality (4), that is $H(t)\cdot\eta_j > \cos\vartheta^*|H(t)|$ for some $\vartheta^* \in (0,\frac{\pi}{4})$, is equivalent to the fact that the angle between $H(t)$ and η_j is less than $\pi/4$. Thus, $H(t)$ is contained in the same cone $\forall t \in [t_j,t_{j+1}]$. Let us note that it can occur that $\eta_i = \eta_j$, for $i \neq j$.

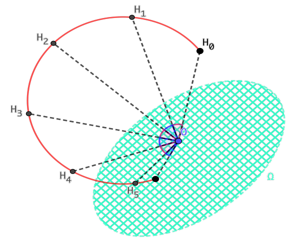

Fig. 1 In this picture $S_* = \cos\frac{\pi}{6}$, $m = 6$ and $H_j := H(t_j)$, $j = 0, \ldots, 5$.

Let $\delta_\Omega = diam(\Omega) = \sup_{x,y\in\overline{\Omega}} |x-y|$. Let us fix $S_* \in (1/\sqrt{2}, 1)$, $r > 0$ and define

$$x_j := -R_j \eta_j, \quad j = 0, \ldots, m-1, \tag{5}$$

where η_j is defined in (3) and

$$\begin{cases} R_j = 2^j R_0 + (2^j - 1)(\delta_\Omega + r), \\ R_0 = \frac{1+S_*}{1-S_*}\delta_\Omega. \end{cases} \tag{6}$$

We note that from (6) it follows that

$$x_j \notin \overline{\Omega}, \qquad j = 0, \ldots, m-1.$$

For every $j = 0, \ldots, m-1$, let us define

$$M_\Omega(x_j) := \max_{x\in\overline{\Omega}} |x - x_j| \qquad \text{and} \qquad d_\Omega(x_j) := \min_{x\in\overline{\Omega}} |x - x_j|. \tag{7}$$

Remark 1.4 The choice of the R_j's in (6) (see Lemma 2.2 below and Figure 2) guarantees that the points x_j's are located sufficiently far away from Ω and at increasing distances from the origin.

By the choice of the finite sequence $R_j = |x_j|$ in (6) (R_j sufficiently large compared with δ_Ω) we deduce in Lemma 2.1 below that

$$(x + R_j\eta_j)\cdot\eta_j \geq S_*|x + R_j\eta_j|, \ \ \forall x \in \overline{\Omega}.$$

In other words, the apex angle of the minimum cone with the apex x_j which includes Ω is less than $2\arccos S_*(< \pi/2)$ (see Figure 3).

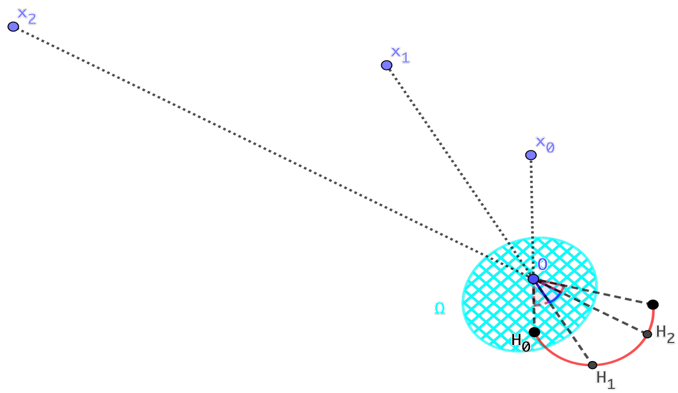

Fig. 2 In this picture $S_* = \cos\frac{\pi}{6}$, $m = 3$ and $H_j := H(t_j)$, $j = 0, 1, 2$.

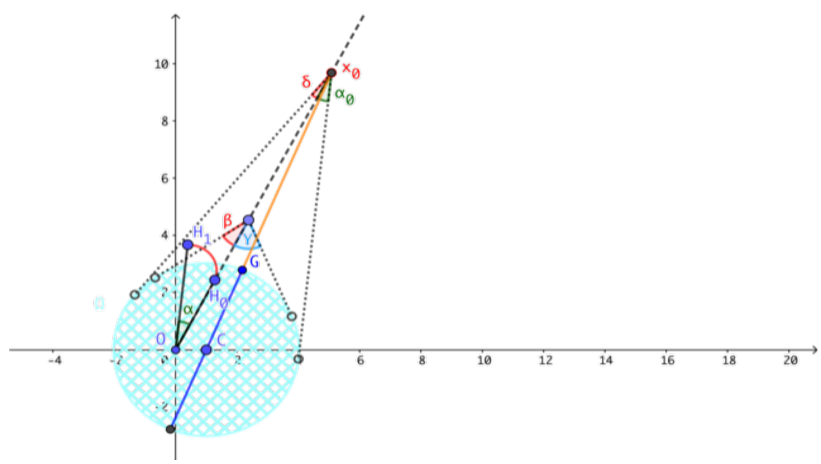

Fig. 3 In this picture: $\Omega := \{(x, y) \in R^2 : |(x, y) - (1, 0)| < 3\}$, $C = (1, 0)$, $S_* = \cos\alpha \in (\frac{1}{\sqrt{2}}, 1)$, $m = 1$, $H_j := H(t_j)$, $j = 0, 1$, and $\beta, \gamma > \alpha$, $\alpha_0 = \alpha, \delta \le \alpha$. We note that $d_\Omega(x_0) = \operatorname{dist}(x_0, G)$ and $M_\Omega(x_0) = d_\Omega(x_0) + 6$.

We now introduce the weight function $\varphi(x,t)$ to be used in our Carleman estimate, as follows. First, we define φ on $\overline{\Omega}\times[0,T)$ setting, for every $x\in\overline{\Omega}$,

$$\varphi(x,t)=\varphi_j(x,t):=-\beta(t-t_j)+|x-x_j|^2,\ \ t\in[t_j,t_{j+1}),\ j=0,\dots,m-1, \tag{8}$$

where

$$\beta:=(2S_*^2-1)H_0d_\Omega(x_0), \tag{9}$$

with H_0 and $d_\Omega(x_0)$ defined by (2) and (7), respectively. Then we extend φ to $\overline{\Omega}\times[0,T]$ by continuity. Observe that φ is piecewise smooth in t and smooth in x.

Main results

In this article, under condition (2), we establish an observability inequality for (1) which estimates the L^2-norm of $u(x,0)$ by lateral boundary data $u|_{\partial\Omega\times(0,T)}$ under some conditions on $H(t)$ (see Theorem 1.6). This observability inequality is a consequence of the following Carleman estimate.

Theorem 1.5 *Let $u\in H^1(Q)$ be a solution of equation (1), where $H\in C^1([0,T];\mathbb{R}^d)$ satisfies (2). Let $\{t_j\}_0^m$ be a partition of $[0,T]$ satisfying (4). Then, there exist constants $s_0,C_0,C>0$ such that for all $s>s_0$ we have*

$$\begin{aligned}&s^2\int_Q|u|^2e^{2s\varphi}dxdt+se^{-C_0s}\sum_{j=0}^{m-1}\int_\Omega|u(x,t_j)|^2dx\\&\le C\int_Q|Pu|^2e^{2s\varphi}dxdt+Cse^{Cs}\int_\Sigma|u|^2d\gamma dt+Cse^{Cs}\int_\Omega|u(x,T)|^2\,dx,\end{aligned}$$

where $\varphi(x,t):Q\longrightarrow\mathbb{R}$ is the weight function defined in (8), and

$$\Sigma=\{(x,t)\in\partial\Omega\times(0,T):\ H(t)\cdot\nu(x)\ge0\} \tag{10}$$

is the subboundary of all exit points for H.

We now give the observability inequality for the equation (1).

Theorem 1.6 *Let $g\in L^2(\partial\Omega\times(0,T))$ and let us consider the following problem*

$$\begin{cases}\partial_tu+H(t)\cdot\nabla u=0 & \text{in } Q:=\Omega\times(0,T),\\ u|_{\partial\Omega\times(0,T)}=g.\end{cases} \tag{11}$$

Let us suppose that there exists a partition $\{t_j\}_0^m$ of $[0,T]$ associated to $H(t)$ satisfying (4) such that the following condition holds

$$\max_{0\le j\le m-1}\frac{(t_{j+1}-t_j)d_\Omega(x_j)}{M_\Omega^2(x_j)}>\frac{1}{H_0(2S_*^2-1)}, \tag{12}$$

where $M_\Omega(x_j), d_\Omega(x_j)$ and H_0 are defined in (7) *and* (2), *respectively. Then, there exists a constant $C > 0$ such that the following inequality holds*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\|g\|_{L^2(\partial\Omega\times(0,T))}, \quad 0 \le t \le T,$$

for any $u \in H^1(Q)$ satisfying (11).

Assumption (12) is meant to guarantee that the orbit $\{H(t) \in \mathbb{R}^d \ : \ t \in [0,T]\}$ intersects $\partial\Omega$. In the following example, we show that this or a similiar condition is indeed necessary: observability fails without some extra assumption.

In the following, for $\eta > 0$ we consider $\Omega_\eta := \{z \in \mathbb{R}^2 : |z| < \eta\}$.

Example 1 Let $\sigma > 0$ and $\rho \in (0, 2\sigma/3)$. Let $\Omega := \Omega_\rho$ and let $f \in C^1(\overline{\Omega}_\sigma; \mathbb{R})$ satisfy $\operatorname{supp}(f) \subset \Omega_{\rho/2} \subseteq \overline{\Omega}_\sigma$ and let $\alpha(t) = (\rho\cos t, \rho\sin t)$, $t \in [0, 2\pi]$. We set

$$v(x,y,t) = f(x - \rho\cos t, y - \rho\sin t).$$

Thus, v satisfies (1), where $H(t) = \alpha'(t)$, $0 \le t \le T$, and v vanishes at the boundary of Ω_σ. So,

$$\begin{cases} \partial_t v + \alpha'(t)\cdot\nabla v = 0 & \text{in } \Omega_\sigma \times (0,T), \\ v|_{\partial\Omega_\sigma\times(0,T)} = g, \end{cases} \tag{13}$$

with $g \equiv 0$. We note that $|\alpha'(t)| = \rho > 0$ and, for $t \in [0,T]$, the support of $v(\cdot,\cdot,t)$ is

$$\operatorname{supp}(v(\cdot,\cdot,t)) = \left\{(x,y) \in \mathbb{R}^2 : |(x - \rho\cos t, y - \rho\sin t)| < \frac{\rho}{2}\right\}. \tag{14}$$

Then, from (13) and (14) it follows that observability fails. □

We conclude this introduction with some comments on our main results.

1. One could establish an estimate similar to the one in Theorem 1.6 with the maximum norm by the method of characteristics. Our proof is based on Carleman estimates, which naturally provide L^2-estimates for solutions over $\Omega \times \{t\}$. The method of characteristics does not yield such global L^2-estimates directly. L^2-estimates, not estimates in the maximum norm, are related to exact controllability and are more flexibly applied to other problems such as inverse problems, although we discuss no such aspects in this paper.
2. Although, due to the simplicity of equation (1), the method of characteristics can be easily applied to explain the validity of observability results, the one point we would like to stress is the fact that, in this paper, we intend to derive a Carleman estimate under minimal assumptions. Essentially, we want to give an explicit construction of the weight function that only depends on the lower bound (2) and the modulus of continuity of H.
3. It is worth noting that Theorem 1.6 aims at the determination of the solution u on the whole cylinder $\Omega \times [0,T]$, not only of $u(\cdot,0)$ in Ω. For this reason, in Theorem 1.6, we have to measure data on the whole lateral boundary $\partial\Omega \times (0,T)$,

not just on a subboundary as we did for the Carleman estimate in Theorem 1.5—where, however, the norm of $u(\cdot,T)$ in Ω is included. The fact that measurements on the whole boundary are necessary to majorize u on $\Omega \times [0,T]$ can be easily understood by looking at the representation solutions given by characteristics.

4. Another purpose of this paper is to single out an assumption which suffices to derive observability from a Carleman estimate. We do so with condition (12), which has a clear geometric meaning: one requires $H(t)$ not to oscillate too much for enough time, giving an explicit evaluation of such a time. We do not pretend our method to provide the optimal evaluation of the observability time. On the other hand, Example 1 shows that some assumption is needed for observability: (12) is an example of a sufficient quantitative condition for the observability of solutions on $\Omega \times [0,T]$.

Main references and outline of the paper

Carleman estimates for transport equations are proved in Gaitan and Ouzzane [5], Gölgeleyen and Yamamoto [6], Cannarsa, Floridia, Gölgeleyen and Yamamoto [4], Klibanov and Pamyatnykh [7], Machida and Yamamoto [8] to be applied to inverse problems of determining spatially varying coefficients, where coefficients of the first-order terms in x are assumed not to depend on t. In order to improve results for inverse problems by the application of Carleman estimates, we need a better choice of the weight function in the Carleman estimate. The works [5] and [7] use one weight function which is very conventional for a second-order hyperbolic equation but seems less useful to derive analogous results for a time-dependent function $H(t)$. Our choice is more similar to the one in [8] and [6], but even these papers allow no time dependence for H. Although it is very difficult to choose the best possible weight function for the partial differential equation under consideration, our choice (8) of the weight function seems more adapted for the nature of the transport equation (1).

In [4] we consider the transport equation $\partial_t u(x,t) + (H(x) \cdot \nabla u(x,t)) + p(x)u(x,t) = 0$ in $\Omega \times (0,T)$ ($\Omega \subset \mathbb{R}^n$ bounded domain), and discuss two inverse problems which consist of determining a vector-valued function $H(x)$ or a real-valued function $p(x)$ by initial values and data on a subboundary of Ω. In particular in [4] we obtain conditional stability of Hölder type in a subdomain D provided that the outward normal component of $H(x)$ is positive on $\partial D \cap \partial\Omega$. The proofs are based also on a Carleman estimate where the weight function depends on H.

As it is commented above, the method of characteristics is applicable to inverse problems for first-order hyperbolic systems as well as transport equations and we refer for example to Belinskij [2] and Chapter 5 in Romanov [9], which discuss an inverse problem of determining an $N \times N$-matrix $C(x)$ in

$$\partial_t U(x,t) + \Lambda \partial_x U(x,t) + C(x)U(x,t) = F(x,t), \quad 0 < x < \ell,\ t > 0$$

with a suitably given matrix Λ and vector-valued function F. The works [2] and [9] apply the method of characteristics to prove the uniqueness and the existence of $C(x)$ realizing extra data of U provided that $\ell > 0$ is sufficiently small.

The method by Carleman estimates for establishing both energy estimates like Theorem 1.6 and inverse problems of determining spatial varying functions is well-known for hyperbolic and parabolic equations and we refer to Beilina and Klibanov [1], Bellassoued and Yamamoto [3], Yamamoto [10].

The plan of the paper is the following. In Section 2, we prove the Carleman estimate (Theorem 1.5). In Section 3, we obtain the observability inequality (Theorem 1.6). Finally, in Appendix we put the proof of Lemma 1.2.

2 Proof of the Carleman estimate

Let $S_* \in \left(\frac{1}{\sqrt{2}}, 1\right)$ and $\{t_j\}_0^m$ a partition of $[0,T]$ associated to $H(t)$ such that (4) is satisfied.

2.1 Some preliminary lemmas

Lemma 2.1 *Given R_j, $j = 0,\dots,m-1$, as in (6), then*

$$(x+R_j\eta_j)\cdot\eta_j \geq S_*|x+R_j\eta_j|, \;\; \forall x \in \overline{\Omega}, \tag{15}$$

where η_j are defined in (3).

Proof. For every $x \in \overline{\Omega}$, we have $|x| = |x-\mathbf{0}| \leq \delta_\Omega$ since $\mathbf{0} \in \overline{\Omega}$, and

$$S_*|x+R_j\eta_j| \leq S_*\left(|x|+R_j|\eta_j|\right) = S_*\left(|x|+R_j\right) \leq S_*\left(\delta_\Omega+R_j\right), \tag{16}$$

and, since $-x\cdot\eta_j \leq |x\cdot\eta_j| \leq |x||\eta_j| = |x| \leq \delta_\Omega$,

$$(x+R_j\eta_j)\cdot\eta_j = x\cdot\eta_j + R_j\eta_j\cdot\eta_j = x\cdot\eta_j + R_j \geq R_j - |x| \geq R_j - \delta_\Omega. \tag{17}$$

From (16) and (17) it follows that a sufficient condition for the inequality (15) is the following

$$R_j - \delta_\Omega \geq S_*(\delta_\Omega + R_j),$$

that is, $R_j \geq \frac{1+S_*}{1-S_*}\delta_\Omega$. For every $j = 1,\dots,m-1$, the last condition is verified by R_j defined as in (6). $\square$

By the definition (6) of the sequence $\{R_j\}$ the following Lemma 2.2 follows.

Lemma 2.2 *Let $x_j = -R_j\eta_j$, $j = 0,\dots,m-1$, with R_j defined as in (6). Then*

$$M_\Omega(x_j) = \max_{x\in\overline{\Omega}} |x-x_j| < \min_{x\in\overline{\Omega}} |x-x_{j+1}| = d_\Omega(x_{j+1}), \qquad j=0,\dots,m-2. \quad (18)$$

By Lemma 2.2 (see also Figure 2) we deduce

$$\max_{j=0,\dots,m-1} M_\Omega(x_j) = M_\Omega(x_{m-1}) \qquad \text{and} \qquad \min_{j=0,\dots,m-1} d_\Omega(x_j) = d_\Omega(x_0). \quad (19)$$

Lemma 2.3 *Let* $x_j = -R_j\eta_j$, $j=0,\dots,m-1$, *with* R_j *defined as in (6). Then,*

$$H(t)\cdot(x-x_j) \ge C_* H_0\, d_\Omega(x_0), \quad t_j \le t \le t_{j+1},\ j=0,\dots,m-1,\ x\in\overline{\Omega},$$

where $C_* = 2S_*^2 - 1 > 0$ *and* $H_0 = \min_{t\in[0,T]} |H(t)| > 0$.

Proof. Let $\vartheta^* \in (0,\pi/4)$ satisfy $\cos\vartheta^* = S_*$. For $t\in[t_j,t_{j+1}]$, $j=0,\dots,m-1$, from (15) and Remark 1.3 we deduce that

$$H(t)\cdot(x-x_j) \ge \cos 2\vartheta^* H_0 d_\Omega(x_j) \ge (2S_*^2-1)H_0 d_\Omega(x_0), \quad x\in\overline{\Omega}$$

which is our conclusion. □

2.2 Derivation of the Carleman estimate

After introducing the previous lemmas in Section 2.1, we are able to prove Theorem 1.5. In this section, for simplicity of notations, for $j=0,\dots,m-1$ let us set

$$M_j := M_\Omega(x_j) \qquad \text{and} \qquad \mu_j := d_\Omega(x_j), \quad (20)$$

see (7) for the definitions of $M_\Omega(x_j)$ and $d_\Omega(x_j)$.

Proof. (of Theorem 1.5). We derive a Carleman estimate on

$$Q_j := \Omega\times(t_j,t_{j+1}),\ 0\le j\le m-1.$$

Let $w_j := e^{s\varphi_j}u$, where φ_j is defined in (8), and

$$L_j w_j := e^{s\varphi_j} P(e^{-s\varphi_j} w_j). \quad (21)$$

By direct calculations, we obtain

$$L_j w_j = \partial_t w_j + H(t)\cdot\nabla w_j - s(P\varphi_j)w_j \quad \text{in} \quad Q_j, \quad (22)$$

where, keeping in mind (8) and the definition of the operator P contained in (1),

$$P\varphi_j(x,t) = \partial_t\varphi_j + H(t)\cdot\nabla\varphi_j = -\beta + 2H(t)\cdot(x-x_j), \quad 0\le j\le m-1.$$

By Lemma 2.3 and (9), since $\beta = (2S_*^2-1)H_0\mu_0 \in \big(0, 2(2S_*^2-1)H_0\mu_0\big)$ we have

$$P\varphi_j = -\beta + 2H(t)\cdot(x-x_j) \geq C_* H_0 \mu_0, \tag{23}$$

where $C_* = 2S_*^2 - 1$. Therefore, by (23) we obtain

$$\begin{aligned}\int_{Q_j} |L_j w_j|^2 dxdt &\geq -2s\int_{Q_j} (P\varphi_j) w_j(\partial_t w_j + H(t)\cdot\nabla w_j) dxdt \\ &+ s^2 \int_{Q_j} |2H(t)\cdot(x-x_j) - \beta|^2 |w_j|^2 dxdt \\ &\geq I_1 + I_2 + C_*^2 H_0^2 \mu_0^2 s^2 \int_{Q_j} |w_j|^2 dxdt, \end{aligned} \tag{24}$$

where

$$I_1 := -2s\int_{Q_j} (P\varphi_j) w_j \partial_t w_j dxdt \quad \text{and} \quad I_2 := -2s\int_{Q_j} (P\varphi_j) H(t)\cdot(w_j \nabla w_j) dxdt.$$

We have

$$\begin{aligned} I_1 &= -2s\int_{Q_j} (P\varphi_j) w_j \partial_t w_j dxdt = -s\int_{t_j}^{t_{j+1}} \int_\Omega (P\varphi_j)\partial_t(w_j^2) dxdt \\ &= s\int_\Omega \left[P\varphi_j(x,t)|w_j(x,t)|^2\right]_{t=t_{j+1}}^{t=t_j} dx + s\int_{Q_j} \partial_t(P\varphi_j(x,t))|w_j|^2 dxdt. \end{aligned} \tag{25}$$

Recalling (20), we obtain

$$\partial_t(P\varphi_j(x,t)) = 2(x-x_j)\cdot H'(t) \geq -2M_{m-1} \max_{t\in[0,T]} |H'(t)| =: -H_0'.$$

Consequently, from (25) we deduce

$$I_1 \geq s\int_\Omega \left[P\varphi_j(x,t)|w_j(x,t)|^2\right]_{t=t_{j+1}}^{t=t_j} dx - sH_0' \int_{Q_j} |w_j|^2 dxdt. \tag{26}$$

Then, for I_2 we deduce

$$\begin{aligned} I_2 &= -2s\int_{Q_j} (P\varphi_j) H(t)\cdot(w_j\nabla w_j) dxdt = -s\int_{t_j}^{t_{j+1}} \int_\Omega P\varphi_j \sum_{k=1}^d H_k(t)\partial_k(w_j^2) dxdt \\ &= s\int_{t_j}^{t_{j+1}} \int_\Omega \sum_{k=1}^d (\partial_k(P\varphi_j)) H_k(t)|w_j|^2 dxdt - s\int_{t_j}^{t_{j+1}} \int_{\partial\Omega} P\varphi_j (H(t)\cdot\nu(x))|w_j|^2 d\gamma dt. \end{aligned}$$

We note that

$$H(t)\cdot(x-x_j) \leq |H(t)||x-x_j| \leq H_* M_*, \tag{27}$$

where we set (see (19))

$$M_* = M_{m-1} \qquad \text{and} \qquad H_* := \max_{t\in[0,T]} |H(t)| > 0.$$

Therefore, since $P\varphi_j > 0$ by (23) and $\partial_k(P\varphi_j) = 2H_k(t)$, we estimate I_2 in the following way:

$$
\begin{aligned}
I_2 &= 2s\int_{t_j}^{t_{j+1}}\int_\Omega \sum_{k=1}^{d} H_k^2(t)|w_j|^2 dxdt - s\int_{t_j}^{t_{j+1}}\int_{\partial\Omega} P\varphi_j(H(t)\cdot \nu(x))|w_j|^2 d\gamma dt \\
&\geq 2s\int_{t_j}^{t_{j+1}}\int_\Omega |H(t)|^2|w_j|^2 dxdt \\
&- s\int_{\Sigma_j}(-\beta + 2H(t)\cdot(x-x_j))(H(t)\cdot\nu(x))|w_j|^2 d\gamma dt \\
&\geq 2H_0^2 s\int_{t_j}^{t_{j+1}}\int_\Omega |w_j|^2 dxdt - 2s\int_{\Sigma_j}(H(t)\cdot(x-x_j))(H(t)\cdot\nu(x))|w_j|^2 d\gamma dt \\
&\geq 2H_0^2 s\int_{Q_j}|w_j|^2 dxdt - 2H_* M_* s\int_{\Sigma_j}|H(t)||\nu(x)||w_j|^2 d\gamma dt \\
&\geq 2H_0^2 s\int_{Q_j}|w_j|^2 dxdt - 2H_*^2 M_* s\int_{\Sigma_j}|w_j|^2 d\gamma dt,
\end{aligned} \tag{28}
$$

where

$$
\Sigma_j = \{(x,t) \in \partial\Omega \times (t_j, t_{j+1}) : H(t)\cdot\nu(x) \geq 0\}.
$$

Hence, by (24), (26) and (28), we obtain

$$
\begin{aligned}
\int_{Q_j}|L_j w_j|^2 dxdt \geq\; & s\int_\Omega \left[P\varphi_j(x,t)|w_j(x,t)|^2\right]_{t=t_{j+1}}^{t=t_j} dx \\
&- H_0' s\int_{Q_j}|w_j|^2 dxdt + C_1 s^2\int_{Q_j}|w_j|^2 dxdt \\
&- 2H_*^2 M_* s\int_{\Sigma_j}|w_j|^2 d\gamma dt,
\end{aligned}
$$

for some positive constant C_1. Since $w_j := e^{s\varphi_j}u$, from the previous inequality, for $j = 0,\ldots,m-1$, by (21) we deduce that there exists also a positive constant C_2 such that

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}}\int_\Omega |Pu|^2 e^{2s\varphi_j} dxdt \geq\; & s\int_\Omega \psi_j(x)dx + (C_1 s^2 - H_0' s)\int_{Q_j} e^{2s\varphi_j}|u|^2 dxdt \\
&- C_2 s e^{C_2 s}\int_{\Sigma_j}|u|^2 d\gamma dt,
\end{aligned} \tag{29}
$$

where C_1, C_2 are positive constants and

$$
\psi_j(x) := \left[P\varphi_j(x,t)e^{2s\varphi_j(x,t)}\,|u(x,t)|^2\right]_{t=t_{j+1}}^{t=t_j}.
$$

By (8) and (23) we obtain

$$\begin{aligned}\psi_j(x) &= \left[(2H(t)\cdot(x-x_j)-\beta)\, e^{2s(-\beta(t-t_j)+|x-x_j|^2)}|u(x,t)|^2\right]_{t=t_{j+1}}^{t=t_j}\\ &= (2H(t_j)\cdot(x-x_j)-\beta)e^{2s|x-x_j|^2}|u(x,t_j)|^2\\ &- (2H(t_{j+1})\cdot(x-x_j)-\beta)e^{2s(-\beta(t_{j+1}-t_j)+|x-x_j|^2)}|u(x,t_{j+1})|^2. \end{aligned} \tag{30}$$

Therefore, summing in j from 0 to $m-1$ and keeping in mind that $t_0=0$ and $t_m=T$ by (9) and (27) we have

$$\begin{aligned}\sum_{j=0}^{m-1}\psi_j(x) &\geq (2H(0)\cdot(x-x_0)-\beta)e^{2s(|x-x_0|^2)}|u(x,0)|^2+\sum_{j=1}^{m-1}q_j(x)|u(x,t_j)|^2\\ &- (2H(T)\cdot(x-x_{m-1})-\beta)e^{2s(-\beta(T-t_{m-1})+|x-x_{m-1}|^2)}|u(x,T)|^2\\ &\geq \mu_0H_0e^{2s\mu_0^2}|u(x,0)|^2-2M_*H_*e^{2sM_*^2}|u(x,T)|^2+\sum_{j=1}^{m-1}q_j(x)|u(x,t_j)|^2, \end{aligned}$$

where, for $j=1,\dots,m-1$, we set

$$q_j(x) := (2H(t_j)\cdot(x-x_j)-\beta)e^{2s|x-x_j|^2}-\big(2H(t_j)\cdot\big(x-x_{j-1}\big)-\beta\big)\, e^{2s\left|x-x_{j-1}\right|^2}.$$

Thus, by (7), (20), (23) and (27), we obtain the following estimate

$$q_j(x)\geq \tilde{C}\mu_0H_0e^{2s\mu_j^2}-H_*M_*e^{2sM_{j-1}^2}=\tilde{C}\mu_0H_0e^{2s\mu_j^2}\left(1-\frac{M_*H_*}{\tilde{C}\mu_0H_0}e^{-2s\left(\mu_j^2-M_{j-1}^2\right)}\right).$$

Thanks to (18) (see Lemma 2.2), the choice of the points x_j permits to have $\mu_j-M_{j-1}>0$, and we deduce that there exist $s_j>0$ enough large, that is $s_j>\frac{1}{2\left(\mu_j^2-M_{j-1}^2\right)}\log\left(\frac{2H_*M_*}{\tilde{C}\mu_0H_0}\right)$, $j=1,\dots,m-1$, such that, for every $s>s_0:=\max\limits_{j=1,\dots,m-1}s_j$, we have

$$q_j(x)\geq\frac{\mu_0H_0}{2}e^{2s\mu_j^2}\geq\frac{\mu_0H_0}{2}e^{2s\mu_0^2}\geq C_0e^{C_0s}, \tag{31}$$

for some positive constant $C_0=C_0(s)$. Thus, by (29), (30), and (31) we have that

$$
\begin{aligned}
\int_Q |Pu|^2 e^{2s\varphi} dxdt &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_\Omega |Pu|^2 e^{2s\varphi_j} dxdt \\
&\geq s \sum_{j=0}^{m-1} \int_\Omega \psi_j(x) dx + (C_1 s^2 - H_0' s) \sum_{j=0}^{m-1} \int_{Q_j} e^{2s\varphi_j} |u|^2 dxdt \\
&- C_2 s e^{C_2 s} \sum_{j=0}^{m-1} \int_{\Sigma_j} |u|^2 d\gamma dt \\
&\geq C_3 s^2 \int_Q e^{2s\varphi_j} |u|^2 dxdt - C_2 s e^{C_2 s} \sum_{j=0}^{m-1} \int_{\Sigma_j} |u|^2 d\gamma dt \\
&+ C_0 s e^{C_0 s} \sum_{j=0}^{m-1} \int_\Omega |u(x,t_j)|^2 dx - C_2 s e^{C_2 s} \int_\Omega |u(x,T)|^2 dx
\end{aligned}
$$

for any $0 < C_3 < C_1$ and all s sufficiently large. The last estimate completes the proof of Theorem 1.5. ☐

3 Proof of the observability inequality

Let us give in Section 3.1 two lemmas and in Section 3.2 the proof of Theorem 1.6.

3.1 Energy estimates

Let us give the following energy estimates.

Lemma 3.1 *Let $g \in L^2(\partial\Omega \times (0,T))$ and let us consider the problem*

$$
\begin{cases} \partial_t u + H(t) \cdot \nabla u = 0 \quad \text{in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega \times (0,T)} = g. \end{cases} \tag{11}
$$

Then, for every $t \in [0,T]$, the following energy estimates hold

$$
\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq \|u(\cdot,0)\|^2_{L^2(\Omega)} + H_* \|g\|^2_{L^2(\partial\Omega \times (0,T))}, \tag{32}
$$

$$
\|u(\cdot,0)\|^2_{L^2(\Omega)} \leq \|u(\cdot,t)\|^2_{L^2(\Omega)} + H_* \|g\|^2_{L^2(\partial\Omega \times (0,T))}, \tag{33}
$$

for any $u \in H^1(Q)$ satisfying (11), where $H_* := \max_{\xi \in [0,T]} |H(\xi)|$.

Proof. Let $H(t) = (H_1(t), \ldots, H_d(t))$, $t \in [0,T]$. Multiplying the equation in (11) by $2u$ and integrating over Ω, we have

$$\int_\Omega 2u\partial_t u dx + \int_\Omega \sum_{k=1}^d H_k(t) 2u\partial_k u dx = 0,$$

then,

$$\partial_t \left(\int_\Omega |u(x,t)|^2 dx \right) + \sum_{k=1}^d \int_\Omega H_k(t)\partial_k(|u(x,t)|^2) dx = 0.$$

So, integrating by parts, for every $t \in [0,T]$, we obtain

$$\partial_t \left(\int_\Omega |u(x,t)|^2 dx \right) = -\sum_{k=1}^d \int_{\partial\Omega} H_k |u|^2 \nu_k d\gamma = -\int_{\partial\Omega} (H\cdot\nu)|g|^2 d\gamma, \tag{34}$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit normal vector outward to the boundary $\partial\Omega$. Setting

$$E(t) := \int_\Omega |u(x,t)|^2 dx, \qquad t \in [0,T],$$

by (34), integrating on $[0,t]$ we deduce

$$|E(t) - E(0)| = \left| -\int_0^t \int_{\partial\Omega} (H(\xi)\cdot\nu(x))|g(x,\xi)|^2 d\gamma d\xi \right| \le H_* \|g\|^2_{L^2(\partial\Omega\times(0,T))}$$

where $H_* = \max_{\xi\in[0,T]} |H(\xi)|$. Thus, for all $t \in [0,T]$, we have

$$E(t) \le E(0) + H_* \|g\|^2_{L^2(\partial\Omega\times(0,T))},$$

and

$$E(0) \le E(t) + H_* \|g\|^2_{L^2(\partial\Omega\times(0,T))}.$$

□

Lemma 3.2 *Let $0 \le s_1 < s_2 \le T$, $g \in L^2(\partial\Omega \times (0,T))$. Let us assume that there exists a positive constant $C = C(s_1,s_2)$ such that for every $t \in [s_1,s_2]$ the following observability inequality holds*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C\|g\|_{L^2(\partial\Omega\times(0,T))}, \quad \textit{for all } u \in H^1(Q) \textit{ solution to } (11). \tag{35}$$

Then, there exists a positive constant $C = C(s_1,s_2,T)$ such that the inequality (35) holds for every $t \in [0,T]$.

Proof. Let $E(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)}$, $t \in [0,T]$. For every $t \in [0,s_1]$, keeping in mind Lemma 3.1, by (32), (33) and (35) we obtain

$$\begin{aligned} \|u(\cdot,t)\|^2_{L^2(\Omega)} = E(t) &\le E(0) + H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))} \le E(s_1) + 2H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))} \\ &\le (C^2 + 2H_*)\|g\|^2_{L^2(\partial\Omega\times(0,T))}. \end{aligned} \tag{36}$$

For every $t \in [s_2,T]$, using again Lemma 3.1, by (32) and (35) we deduce

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} = E(t) \le E(s_2) + H_* \|g\|^2_{L^2(\partial\Omega\times(0,T))} \le (C^2 + H_*)\|g\|^2_{L^2(\partial\Omega\times(0,T))}. \quad (37)$$

From (36) and (37) the conclusion follows. □

3.2 The Proof

Proof. (of Theorem 1.6).
Let φ be the weight function given in (8). By the assumption (12) it follows that there exists $j^* \in \{0,\dots m-1\}$ such that

$$\frac{(t_{j^*+1} - t_{j^*})d_\Omega(x_{j^*})}{M^2_\Omega(x_{j^*})} > \frac{1}{H_0(2S^2_* - 1)}. \quad (38)$$

By the definition of the weight function $\varphi(x,t)$ (see (8)), it follows that, for every $x \in \overline{\Omega}$, we have

$$\varphi(x,t_{j^*}) = \varphi_{j^*}(x,t_{j^*}) = |x - x_{j^*}|^2 > 0$$

and, since (38) holds, keeping in mind that $\beta = (2S^2_* - 1)H_0 d_\Omega(x_0)$,

$$\lim_{t\to(t_{j^*+1})^-} \varphi_{j^*}(x,t) = |x - x_{j^*}|^2 - \beta(t_{j^*+1} - t_{j^*}) < 0.$$

Therefore, there exist $\varepsilon \in \left(0, \dfrac{t_{j^*+1} - t_{j^*}}{2}\right)$ and $\delta > 0$ such that

$$\begin{cases} \varphi(x,t) = \varphi_{j^*}(x,t) > \delta, & t \in [t_{j^*}, t_{j^*} + \varepsilon], x \in \overline{\Omega}, \\ \varphi(x,t) = \varphi_{j^*}(x,t) < -\delta, & t \in [t_{j^*+1} - 2\varepsilon, t_{j^*+1}), x \in \overline{\Omega}. \end{cases} \quad (39)$$

Let $u \in H^1(Q)$ satisfy (11) on $Q = \Omega \times (0,T)$. Let us consider $Q^* := \Omega \times (t_{j^*}, t_{j^*+1}) \subseteq Q$. Now we define a cut-off function $\chi \in C^\infty([t_{j^*}, t_{j^*+1}])$ such that $0 \le \chi \le 1$ and

$$\chi(t) = \begin{cases} 1, & t \in [t_{j^*}, t_{j^*+1} - 2\varepsilon], \\ 0, & t \in [t_{j^*+1} - \varepsilon, t_{j^*+1}]. \end{cases}$$

We set

$$v(x,t) = \chi(t)u(x,t), \ \ (x,t) \in Q^*, \quad (40)$$

and keeping in mind (11) and (40), we deduce

$$\begin{cases} \partial_t v + H(t)\cdot\nabla v = u(\partial_t \chi) & \text{in } Q^*, \\ v|_{\partial\Omega\times(t_{j^*},t_{j^*+1})} = \chi g, & \\ v(x,t_{j^*+1}) = 0, & x \in \Omega. \end{cases} \quad (41)$$

Applying Theorem 1.5 to the problem (41), since $|v(x,t)| \le |u(x,t)|$ for every $(x,t) \in Q^*$ (see (40)), we obtain

$$s^2\int_{Q^*}|v|^2e^{2s\varphi}dxdt \le C\int_{Q^*}|u|^2|\partial_t\chi|^2e^{2s\varphi}dxdt + Ce^{Cs}\int_{\Sigma}|u|^2d\gamma dt, \tag{42}$$

for all large $s>0$ and for some positive constant C.
Therefore, by (40) and (39) we have

$$s^2\int_{Q^*}|v|^2e^{2s\varphi}dxdt \ge s^2\int_{t_{j^*}}^{t_{j^*}+\varepsilon}\int_{\Omega}|u|^2e^{2s\varphi_0}dxdt \ge s^2e^{2s\delta}\int_{t_{j^*}}^{t_{j^*}+\varepsilon}\int_{\Omega}|u|^2dxdt \tag{43}$$

and, since $\chi\in C^\infty([t_{j^*},t_{j^*+1}])$, we also deduce

$$\begin{aligned}\int_{Q^*}|u|^2|\partial_t\chi|^2e^{2s\varphi}dxdt &= \int_{t_{j^*+1}-2\varepsilon}^{t_{j^*+1}-\varepsilon}\int_{\Omega}|u|^2|\partial_t\chi|^2e^{2s\varphi_{j^*}}dxdt\\ &\le K_1e^{-2s\delta}\int_{t_{j^*+1}-2\varepsilon}^{t_{j^*+1}-\varepsilon}\int_{\Omega}|u|^2dxdt \le K_1\|u\|^2_{L^2(Q^*)}e^{-2s\delta}, \end{aligned}\tag{44}$$

for all large $s>0$ and for some positive constant K_1.
From (42), by (43) and (44) we obtain

$$s^2e^{2s\delta}\int_{t_{j^*}}^{t_{j^*}+\varepsilon}\int_{\Omega}|u|^2dxdt \le C_1\|u\|^2_{L^2(Q^*)}e^{-2s\delta} + C_1e^{C_1s}\|g\|^2_{L^2(\partial\Omega\times(0,T))}, \tag{45}$$

for all large $s>0$ and for some positive constant C_1.
Setting

$$E(t) := \int_{\Omega}|u(x,t)|^2dx, \qquad t\in[t_{j^*},t_{j^*+1}],$$

by the energy estimate (33) of Lemma 3.1 we deduce

$$\begin{aligned}\int_{t_{j^*}}^{t_{j^*}+\varepsilon}\int_{\Omega}|u|^2dxdt &= \int_{t_{j^*}}^{t_{j^*}+\varepsilon}E(t)dt \ge \int_{t_{j^*}}^{t_{j^*}+\varepsilon}(E(t_{j^*}) - H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))})dt\\ &= \varepsilon\left(E(t_{j^*}) - H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))}\right)\end{aligned}\tag{46}$$

and, by the energy estimate (32) of Lemma 3.1 we obtain

$$\begin{aligned}\|u\|^2_{L^2(Q^*)} &= \int_{t_{j^*}}^{t_{j^*+1}}E(t)dt = \int_{t_{j^*}}^{t_{j^*+1}}\left(E(t_{j^*}) + H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))}\right)dt\\ &\le E(t_{j^*})T + H_*T\|g\|^2_{L^2(\partial\Omega\times(0,T))}.\end{aligned}\tag{47}$$

Substituting (46) and (47) into (45), we have

$$\begin{aligned} s^2e^{2s\delta}\varepsilon\left(E(t_{j^*})-H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))}\right) &\le s^2e^{2s\delta}\int_{t_{j^*}}^{t_{j^*}+\varepsilon}\int_\Omega |u|^2dxdt \\ &\le C_1\|u\|^2_{L^2(Q^*)}e^{-2s\delta}+C_1e^{C_1s}\|g\|^2_{L^2(\partial\Omega\times(0,T))} \\ &\le C_1e^{-2s\delta}\left(E(t_{j^*})T+H_*T\|g\|^2_{L^2(\partial\Omega\times(0,T))}\right) \\ &+ C_1e^{C_1s}\|g\|^2_{L^2(\partial\Omega\times(0,T))}, \end{aligned}$$

for all large $s>0$. Hence, for all s large enough,

$$(s^2e^{2s\delta}\varepsilon-C_1Te^{-2s\delta})E(t_{j^*})\le\left(C_1e^{C_1s}+s^2e^{2s\delta}\varepsilon H_*+C_1e^{-2s\delta}H_*T\right)\|g\|^2_{L^2(\partial\Omega\times(0,T))}$$

But, for $s>0$ enough large, $s^2e^{2s\delta}\varepsilon-C_1Te^{-2s\delta}>0$. Thus, using again (32), for every $t\in[t_{j^*},t_{j^*+1}]$, we obtain

$$\|u(\cdot,t)\|_{L^2(\Omega)}=E(t)\le E(t_{j^*})+H_*\|g\|^2_{L^2(\partial\Omega\times(0,T))}\le C_2\|g\|_{L^2(\partial\Omega\times(0,T))},$$

for some positive constant C_2. The conclusion of the proof of Theorem 1.6 follows from the above inequality, using Lemma 3.2 to extend the above observability inequality from $[t_{j^*},t_{j^*+1}]$ to $[0,T]$. □

Remark 3.3 By adapting the above proof, one could easily obtain an observability inequality for $u(\cdot,0)$ on Ω, requiring measurements just on the subboundary Σ defined in (10).

Appendix

In this appendix we prove Lemma 1.2.

Proof. (*of Lemma 1.2*). Since $H\in Lip([0,T];\mathbb{R}^d)$ there exists $L>0$ such that

$$|H(t)-H(s)|\le L|t-s|,\ \forall t,s\in[0,T].$$

Let us consider, for simplicity, a uniform partition $\{t_j\}_0^m$ of $[0,T]$. Let us set

$$\eta_j:=\frac{H(t_j)}{|H(t_j)|},\ j=0\ldots,m-1.$$

For $t\in[t_j,t_{j+1}]$, $j=0,...,m-1$, we deduce

$$\begin{aligned} H(t)\cdot\eta_j &= (H(t)-H(t_j))\cdot\eta_j+H(t_j)\cdot\eta_j\ge-|H(t)-H(t_j)|+|H(t_j)| \\ &\ge -L|t-t_j|+|H(t_j)|\ge -L\frac{T}{m}+|H(t_j)|, \end{aligned} \tag{48}$$

and, since $|H(t)|\le\left|H(t)-H(t_j)\right|+|H(t_j)|$,

$$\left|H(t_j)\right| \geq |H(t)| - |H(t) - H(t_j)| \geq |H(t)| - L|t - t_j| \geq |H(t)| - L\frac{T}{m}. \qquad (49)$$

From (48) and (49), if we choose the uniform partition with $m \geq \frac{2LT}{H_0(1-S_*)}$, where we recall that $H_0 = \min_{t\in[0,T]} |H(t)|$, we obtain the conclusion, that is,

$$H(t) \cdot \frac{H(t_j)}{|H(t_j)|} \geq |H(t)| - 2L\frac{T}{m} \geq S_*|H(t)|, \quad \forall t \in [t_j, t_{j+1}], \ \forall j = 0, \ldots, m-1.$$

□

Acknowledgements This work was partially supported by Grant-in-Aid for Scientific Research (S) 15H05740 and A3 Foresight Program Modeling and Computation of Applied Inverse Problemsh by Japan Society for the Promotion of Science. The first and second authors were visitors at The University of Tokyo in February 2018, supported by the above grant.
The third author was a Visiting Scholar at Rome in April 2018 supported by the University of Rome "Tor Vergata". The third author was also visitor in July 2017 at the University of Naples Federico II, supported by the Department of Mathematics and Applications "R. Caccioppoli" of that University.
This work was supported also by the Istituto Nazionale di Alta Matematica (INdAM), through the GNAMPA Research Project 2017 "Comportamento asintotico e controllo di equazioni di evoluzione non lineari" (the coordinator: C. Pignotti). Moreover, this research was performed within the framework of the GDRE CONEDP (European Research Group on "Control of Partial Differential Equations") issued by CNRS, INdAM and Université de Provence. This work was also supported by the research project of the University of Naples Federico II: "Spectral and Geometrical Inequalities".

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