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Hamiltonian/Stroh formalism for anisotropic media with microstructure

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Moving from variational principles, we develop the Hamiltonian formalism for generally anisotropic microstructured materials, in an attempt to extend the celebrated Stroh formulation. Microstructure is expressed through the indeterminate (or Mindlin-Tiersten's) theory of couple stress elasticity. The resulting canonical formalism appears in the form of a Differential Algebraic system of Equations (DAE), which is then recast in purely differential form. This structure is due to the internal constraint that relates the micro- to the macro- rotation. The special situations of plain and anti-plane deformations are also developed and they both lead to a 7-dimensional coupled linear system of differential equations. In particular, the antiplane problem shows remarkable similarity with the theory of anisotropic plates, with which it shares the Lagrangian. Yet, unlike for plates, a classical Stroh formulation cannot be obtained, owing to the difference in the constitutive assumptions. Nonetheless, the canonical formalism brings new insight into the problem's structure and highlights important symmetry properties.

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1. Introduction

The celebrated Stroh formalism [27] is a reformulation of the equations of elasticity which proves particularly useful for solving problems in plane anisotropic elastostatics [28]. These are reduced to a six-dimensional eigenvalue problem, of which they share all the relevant features. Besides, the method may be readily extended to steady-state elastodynamics [29]. In particular, the formalism is especially suited for discussing travelling wave propagation and it has gained considerable attention since it allows to prove existence of surface waves in generally anisotropic materials, a result that has eluded early researchers dealing with leaky waves [26]. As an example, in [10] it is illustrated how to derive the Stroh form of the governing equations for localized edge vibration modes in a circular isotropic Kirchhoff-Love shell, and then use the impedance matrix to efficiently compute the real roots of the frequency equation. It is now established that the secular equation governing surface waves is always real and that, whenever a surface wave exists, it is unique [3].

Only recently, it could be recognized that the essence of the formalism lies in its Hamiltonian nature, thereby a space variable is treated as a time-like coordinate [8]. Despite this knowledge having been already noted in passing [2], the realization of its full potential is a recent progress, which has been put to advantage to systematically generalise the formalism to more complicated situations. For example, it could bring constrained elasticity [5] and laminated plates [7,9] in Stroh form. Also, it provided a basis to develop asymptotic reduced models for near-resonance disturbances in anisotropic media [11]. Indeed, in the absence of such a guidance, the right first-order formulation may only be developed by trial and error, such as it occurred for plates, see [8] and references therein. Furthermore, to the best of the authors' knowledge, no similar attempts appear in the literature in the direction of applying the Stroh formalism to complex media. As a remarkable exception, we mention the extension of the Stroh formalism to piezoelectricity in the form of a 8-dimensional eigenvalue problem [4], and to piezo-magneto-elastic or magneto-electro-elastic media, as a 10-dimensional eigenproblem [1]. Similarly, anisotropic micropolar elasticity is considered in [15], where a 14-dimensional system is found for generalized plane strain and 6-dimensional for plane strain. It is worth emphasizing that all such papers develop the Stroh formalism through ad-hoc assumptions, in a trial and error approach, and it is not entirely clear how conjugate variable have been selected (that is whether they are the conjugate momenta of the variational principle or a linear combination thereof). In similar fashion, we mention the extension of the Stroh formalism to self similar problems in elastodynamics by the Smirnov-Sobolev method [30]. Although moving from a different perspective, that is directed at the problem's solution rather than at elucidating the underlying variational structure, the paper reveals that a Stroh-like formalism still holds in dynamics.

In this paper, we apply the Hamiltonian formalism to systematically develop the canonical form of the governing equations of elastostatics for a microstructured medium. Microstructure is described in the spirit of the indeterminate (or Mindlin-Tiersten's) couple stress theory, which is a Cosserat theory wherein the couple stress is related to the gradient of the continuum (or macro) rotation [14,17,24]. Introduction of the microstructure has important downfalls on fracture mechanics [18,21] as well as body [12], Rayleigh [20,25], Stoneley [23] and Love [6] wave propagation, with important potential for applications [22]. It is therefore natural to investigate the variational structure of the associated Hamiltonian system. We show that the internal constraint relating the micro to the macro behaviour prevents reaching a classical Stroh formalism. This is especially surprising for antiplane problems, whose variational structure parallels that of anisotropic plates, which are amenable to the Stroh formalism. Still, the canonical system provides new insight into the fundamental structure of the equations.

2. Couple stress elasticity

Let us consider a Cartesian co-ordinate system (O, x_1, x_2, x_3), with the triad of unit vectors $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ directed alongside the relevant axes, attached to an elastic couple stress (CS) material. This is a polar material, for which, alongside the classical Cauchy force stress tensor $\boldsymbol{s}$, we define the couple stress tensor $\boldsymbol{\mu}$ (a table of symbols is presented in the Appendix). Across any surface of unit normal $\boldsymbol{n}$, an internal reduced couple vector acts, determined by the couple stress tensor as $\boldsymbol{q_n^e} = \boldsymbol{\mu n}$. It is expedient to decompose the force stress tensor $\boldsymbol{s}$ into its symmetric and skew-symmetric parts, respectively

$$\boldsymbol{\sigma} = \mathrm{Sym}\, \boldsymbol{s} = \tfrac{1}{2}\left(\boldsymbol{s} + \boldsymbol{s}^T\right), \quad \text{and} \quad \boldsymbol{\tau} = \mathrm{Skw}\, \boldsymbol{s} = \tfrac{1}{2}\left(\boldsymbol{s} - \boldsymbol{s}^T\right),$$

where the superscript T denotes the transposed tensor. Componentwise, we have $s_{ij} = \sigma_{ij} + \tau_{ij}$, with

$$\sigma_{ij} = s_{(ij)} = \tfrac{1}{2}\left(s_{ij} + s_{ji}\right), \quad \tau_{ij} = s_{<ij>} = \tfrac{1}{2}\left(s_{ij} - s_{ji}\right).$$

In addition, the couple stress tensor $\boldsymbol{\mu}$ is split into its deviatoric and spherical parts

$$\boldsymbol{\mu} = \boldsymbol{\mu}^D + \boldsymbol{\mu}^S, \quad \boldsymbol{\mu}^S = \tfrac{1}{3}(\boldsymbol{\mu} \cdot \mathbf{1})\mathbf{1},$$

where $\mathbf{1}$ is the rank 2 identity tensor and $\cdot$ denotes the scalar product, i.e. componentwise, $1_{ij} = \delta_{ij}$, with δ_{ij} indicating Kronecker's delta symbol, while $\boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{ij}$ and Einstein's summation convention over twice repeated subscripts is assumed.

In terms of kinematics, we introduce the displacement vector $\boldsymbol{u}$ and the rotation vector $\boldsymbol{\varphi}$. Unlike Cosserat micro-polar theories, for which displacements and micro-rotations are independent fields, CS theory relates one to the other, through [14, Eqs.(4.9)]

$$\boldsymbol{\varphi} = \tfrac{1}{2}\,\mathrm{curl}\, \boldsymbol{u}, \quad \Leftrightarrow \quad \varphi_i = \tfrac{1}{2} e_{ijk} u_{k,j}, \tag{2.1}$$

where E is the rank-3 permutation (Levi-Civita) tensor, whose components are denoted by e_{ijk}, and a subscript comma denotes partial differentiation, e.g. $(\mathrm{grad}\, u)_{kj} = u_{k,j} = \partial u_k / \partial x_j$. It follows that $\boldsymbol{\varphi}$ is solenoidal

$$\mathrm{div}\, \boldsymbol{\varphi} = 0, \quad \Leftrightarrow \quad \varphi_{j,j} = 0. \tag{2.2}$$

As in CE, we define the linear strain tensor

$$\boldsymbol{\varepsilon} = \mathrm{Sym}\,\mathrm{grad}\, \boldsymbol{u}, \quad \Leftrightarrow \quad \varepsilon_{ij} = u_{(i,j)}.$$

For a linear elastic anisotropic material, we have

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon},$$

where $\mathbb{C}$ is the rank 4 tensor of elastic moduli, i.e. $\mathbb{C}_{ijkl} = c_{ijkl}$, possessing the major and the minor symmetry property, i.e. $c_{ijkl} = c_{klij}$ and $c_{ijkl} = c_{jikl} = c_{ijlk}$, respectively. By $\mathbb{C}\boldsymbol{\varepsilon}$ we mean the rank 2 tensor obtained by double summation over the last pair of indices: $c_{ijkl}\varepsilon_{kl}$. For isotropic materials, we have

$$\mathbb{C} = 2G\mathbb{I} + \lambda \mathbf{1} \otimes \mathbf{1}, \quad \Leftrightarrow \quad c_{ijkl} = 2G\delta_{ik}\delta_{jl} + \lambda\delta_{kl}\delta_{ij},$$

where $\mathbb{I}$ is the rank 4 identity tensor and Λ and $G > 0$ are Lamé moduli. Here, the dyadic product is introduced for rank 2 tensors such that $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{C} = (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$, for any triple of rank 2 tensors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

We introduce the *torsion-flexure (wryness or curvature) tensor* as the gradient of the rotation field

$$\boldsymbol{\chi} = \mathrm{grad}\, \boldsymbol{\varphi}, \quad \Leftrightarrow \quad \chi_{ij} = \varphi_{i,j}. \tag{2.3}$$

In light of the connection (2.2) and recalling that $\mathrm{tr}\ \mathrm{grad} \equiv \mathrm{div}$, it is seen that $\boldsymbol{\chi}$ is purely deviatoric, i.e. $\boldsymbol{\chi} = \boldsymbol{\chi}^D$. Here, the divergence operator on rank 2 tensors operates on the *second* component, i.e. $(\mathrm{div}\, \boldsymbol{s})_i = s_{ij,j}$.

For a linear elastic CS material, we have the constitutive law

$$\boldsymbol{\mu} = \ell^2 \mathbb{G} \boldsymbol{\chi},$$

where $\ell > 0$ is a characteristic length related to the microstructure and $\mathbb{G}$ is the rank 4 tensor of couple stress moduli possessing the major symmetry property $g_{ijkl} = g_{klij}$. Immediately, it appears that, to any effect, $\boldsymbol{\mu}$ may be replaced by $\boldsymbol{\mu}^D$ and in fact, the CS theory is named indeterminate after the observation that the first invariant of the couple stress tensor, i.e. $\mathrm{tr}\ \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \mathbf{1} = \mu_{11} + \mu_{22} + \mu_{33}$, rests indeterminate. Therefore, we are free to set it equal to zero without any loss of generality, i.e. $g_{iikl} = g_{klii} = 0$. In the following, for the sake of brevity, we shall write $\boldsymbol{\mu}$ with the understanding that $\boldsymbol{\mu}^D$ is meant. For isotropic materials, we have

$$\boldsymbol{\mu} = 2G\ell^2 \left(\boldsymbol{\chi} + \eta \boldsymbol{\chi}^T \right) \quad \Leftrightarrow \quad g_{ijkl} = 2G \left(\delta_{jl}\delta_{ik} + \eta \delta_{jk}\delta_{il} \right), \tag{2.4}$$

where $-1 < \eta < 1$ is a dimensionless number similar to Poisson's ratio.

The equilibrium equations, in the absence of body forces, read [19, Eq.(2)]

$$\mathrm{div}\, \boldsymbol{s} = \boldsymbol{o}, \tag{2.5a}$$

$$\mathrm{axial}\, \boldsymbol{\tau} - \tfrac{1}{2}\, \mathrm{div}\, \boldsymbol{\mu} = \boldsymbol{o}, \tag{2.5b}$$

where $\mathrm{axial}\, \boldsymbol{\tau} = \frac{1}{2} \mathsf{E} \boldsymbol{\tau}$, i.e. $(\mathrm{axial}\, \boldsymbol{\tau})_i = \frac{1}{2} e_{ijk} \tau_{jk}$, denotes the axial vector attached to a skew-symmetric tensor. It is observed that, introducing the curl of a tensor as $(\mathrm{curl}\, \boldsymbol{W})_{ij} = e_{jkl} W_{il,k}$, it can be easily proved that

$$2\, \mathrm{axial}\, \mathrm{curl}\, \boldsymbol{W} = \mathrm{div} \left[(\mathrm{tr}\ \boldsymbol{W}) \mathbf{1} - \boldsymbol{W}^T \right].$$

Consequently, Eqs.(2.5) admit a solution in terms of the Günther stress tensors $\boldsymbol{W}$ and $\boldsymbol{Z}$ [13,15]

$$\boldsymbol{s} = \mathrm{curl}\, \boldsymbol{W}, \quad \boldsymbol{\mu} = \mathrm{curl}\, \boldsymbol{Z} + (\mathrm{tr}\ \boldsymbol{W}) \mathbf{1} - \boldsymbol{W}^T. \tag{2.6}$$

However, as pointed out in [15], this representation leads to a formalism that is not closed. Through the inverse formula

$$\boldsymbol{\tau} = \mathsf{E}\, \mathrm{axial}\, \boldsymbol{\tau}, \quad \Leftrightarrow \quad \tau_{ij} = e_{ijk} (\mathrm{axial}\, \boldsymbol{\tau})_k, \tag{2.7}$$

Eq.(2.5b) may be solved for $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = \tfrac{1}{2} \mathsf{E}\, \mathrm{div}\, \boldsymbol{\mu} = -\, \mathrm{Skw}\, \mathrm{curl} \boldsymbol{\mu}^T, \tag{2.8}$$

that in components read $\tau_{ij} = \frac{1}{2} e_{ijk} \mu_{kl,l}$. Hence, the skew-symmetric part of the force stress tensor $\boldsymbol{s}$ is determined by rotational equilibrium. Clearly, CE is retrieved taking $\ell = 0$, for then $\boldsymbol{\mu} = \boldsymbol{\tau} = \mathbf{0}$.

We now write the total energy in the sense of Eshelby [8]

$$\mathcal{L} = \int_{\mathcal{B}} \left[\tfrac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} + \tfrac{1}{2} \boldsymbol{\mu} \cdot \boldsymbol{\chi} - \boldsymbol{p} \cdot \left(\boldsymbol{\varphi} - \tfrac{1}{2}\, \mathrm{curl}\, \boldsymbol{u} \right) \right] \mathrm{d}V - \int_{\partial \mathcal{B}} \left(\boldsymbol{t_n} \cdot \boldsymbol{u} + \boldsymbol{q_n} \cdot \boldsymbol{\varphi} \right) \mathrm{d}A, \tag{2.9}$$

having introduced the Lagrangian multiplier vector $\boldsymbol{p} = [p_i]$ to account for the constraint (2.1) and being $\boldsymbol{n}$ the unit normal, in the outwards direction, to the surface element $\mathrm{d}A$. Besides, we let the normal tensor $\mathfrak{N} = \boldsymbol{n} \otimes \boldsymbol{n}$, the projector tensor $\mathfrak{P} = \mathbf{1} - \mathfrak{N}$, and the skew tensor $\boldsymbol{P} = \mathsf{E}\boldsymbol{p}$ associated with the vector $\boldsymbol{p}$ thought of as an axial vector. The prescribed boundary force and couple vector are given by

$$\boldsymbol{t_n} = \boldsymbol{t_n^e} + \boldsymbol{\tau n} - \tfrac{1}{2} \boldsymbol{n} \times \mathrm{grad}\, \mu_{nn}, \quad \boldsymbol{q_n} = \mathfrak{P} \boldsymbol{q_n^e} = \boldsymbol{q_n^e} - \mu_{nn} \boldsymbol{n},$$

being $\boldsymbol{t_n^e} = \boldsymbol{\sigma n}$ and $\boldsymbol{q_n^e} = \boldsymbol{\mu n}$ the "elastic" part of the force and couple stress vector and $\mu_{nn} = \mathfrak{N} \cdot \boldsymbol{\mu} = \boldsymbol{n} \cdot \boldsymbol{\mu n}$ the normal part of the couple stress. We observe that the surface integral in (2.9) may be equivalently rewritten as

$$-\int_{\partial\mathcal{B}}\left[\left(\boldsymbol{t}^{e}_{\boldsymbol{n}}+\boldsymbol{\tau}\boldsymbol{n}\right)\cdot\boldsymbol{u}+\boldsymbol{q}^{e}_{\boldsymbol{n}}\cdot\boldsymbol{\varphi}\right]\mathrm{d}A.$$

Indeed, recalling the vector identities

$$\operatorname{div}\left(\boldsymbol{a}\times\boldsymbol{b}\right)=\boldsymbol{b}\cdot\operatorname{curl}\boldsymbol{a}-\boldsymbol{a}\cdot\operatorname{curl}\boldsymbol{b}, \tag{2.10a}$$

$$\operatorname{div}\left(\phi\boldsymbol{b}\right)=\operatorname{grad}\phi\cdot\boldsymbol{b}+\phi\operatorname{div}\boldsymbol{b}, \tag{2.10b}$$

$$\operatorname{curl}\operatorname{grad}\phi=\boldsymbol{o}, \tag{2.10c}$$

and making use of the divergence theorem, it is easily proved that

$$\begin{aligned}-\tfrac{1}{2}\int_{\partial\mathcal{B}}\left(\boldsymbol{n}\times\operatorname{grad}\mu_{nn}\cdot\boldsymbol{u}\right)\mathrm{d}A&=\tfrac{1}{2}\int_{\partial\mathcal{B}}\left(\boldsymbol{u}\times\operatorname{grad}\mu_{nn}\cdot\boldsymbol{n}\right)\mathrm{d}A\\ &=\tfrac{1}{2}\int_{\mathcal{B}}\operatorname{div}\left(\boldsymbol{u}\times\operatorname{grad}\mu_{nn}\right)\mathrm{d}V=\tfrac{1}{2}\int_{\mathcal{B}}\left(\operatorname{grad}\mu_{nn}\cdot\operatorname{curl}\boldsymbol{u}\right)\mathrm{d}V\\ &=\int_{\mathcal{B}}\left(\operatorname{grad}\mu_{nn}\cdot\boldsymbol{\varphi}\right)\mathrm{d}V=\int_{\mathcal{B}}\operatorname{div}\left(\mu_{nn}\boldsymbol{\varphi}\right)\mathrm{d}V=\int_{\partial\mathcal{B}}\left(\boldsymbol{\varphi}\cdot\mu_{nn}\boldsymbol{n}\right)\mathrm{d}A,\end{aligned}$$

having made use of Eq.(2.2). Therefore, $-\frac{1}{2}\boldsymbol{n}\times\operatorname{grad}\mu_{nn}$ in $\boldsymbol{t}_{\boldsymbol{n}}$ cancels out the term $\mu_{nn}\boldsymbol{n}$ in $\boldsymbol{q}_{\boldsymbol{n}}$.

By the divergence theorem and making use of the equilibrium equations (2.5), we get

$$\mathcal{L}=-\int_{\mathcal{B}}L\mathrm{d}V,$$

having introduced the Lagrangian density function L

$$L(\operatorname{grad}\boldsymbol{u},\boldsymbol{\varphi},\operatorname{grad}\boldsymbol{\varphi},\boldsymbol{p})=\tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+\tfrac{1}{2}\boldsymbol{\mu}\cdot\boldsymbol{\chi}+\boldsymbol{p}\cdot\left(\boldsymbol{\varphi}-\tfrac{1}{2}\operatorname{curl}\boldsymbol{u}\right), \tag{2.11}$$

that, component-wise, reads

$$L(\operatorname{grad}\boldsymbol{u},\boldsymbol{\varphi},\operatorname{grad}\boldsymbol{\varphi},\boldsymbol{p})=\tfrac{1}{2}\sigma_{ij}u_{(i,j)}+\tfrac{1}{2}\mu_{ij}\varphi_{i,j}+p_i\left(\varphi_i-\tfrac{1}{2}e_{ijk}u_{k,j}\right).$$

The Euler-Lagrange (E-L) equations are

$$-\frac{\partial}{\partial x_j}\frac{\partial L}{\partial u_{i,j}}+\frac{\partial L}{\partial u_i}=-\sigma_{ij,j}+\tfrac{1}{2}p_{m,j}e_{mji}=0, \tag{2.12a}$$

$$-\frac{\partial}{\partial x_j}\frac{\partial L}{\partial \varphi_{i,j}}+\frac{\partial L}{\partial \varphi_i}=-\mu_{ij,j}+p_i=0, \tag{2.12b}$$

$$\frac{\partial L}{\partial p_i}=\varphi_i-\tfrac{1}{2}e_{ijk}u_{k,j}=0, \tag{2.12c}$$

that, recalling (2.7), amount to

$$-\operatorname{div}\left(\boldsymbol{\sigma}+\tfrac{1}{2}\boldsymbol{P}\right)=0, \tag{2.13a}$$

$$\boldsymbol{p}-\operatorname{div}\boldsymbol{\mu}=0, \tag{2.13b}$$

and the constraint (2.1). In particular, looking at (2.5*b*,2.13*b*), we are able to give a physical meaning to the Lagrange multiplier $\boldsymbol{p}$

$$\boldsymbol{p}=2\operatorname{axial}\boldsymbol{\tau}=2\left(\tau_{23},\tau_{31},\tau_{12}\right) \tag{2.14}$$

wherefrom, $\frac{1}{2}\boldsymbol{P}=\boldsymbol{\tau}$ and Eqs.(2.13) correspond to (2.5).

Special care is required when dealing with the Lagrange multiplier $\boldsymbol{p}$. Indeed, acknowledging (2.2), it can be proved that $\operatorname{div}\boldsymbol{p}=0$, whence

$$\boldsymbol{p}=-\operatorname{curl}\boldsymbol{h},$$

with the gauge relation $\operatorname{div}\boldsymbol{h}=0$. Consequently, making use of the vector identities (2.10) alongside

$$\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{u}=2\operatorname{grad}u\cdot\operatorname{Skw}\operatorname{grad}\boldsymbol{h},$$

we may equivalently write

$$L(\operatorname{grad}\boldsymbol{u},\boldsymbol{\varphi},\operatorname{grad}\boldsymbol{\varphi},\boldsymbol{h})=\tfrac{1}{2}\sigma_{ij}u_{(i,j)}+\tfrac{1}{2}\mu_{ij}\varphi_{i,j}-e_{kji}h_k\varphi_{i,j}+\tfrac{1}{2}u_{i,j}\left(h_{i,j}-h_{j,i}\right),$$

up to boundary terms. This observation will be used in Sec.4 when seeking expressions for the Lagrange multiplier.

3. Plane strain

We now consider plane-strain conditions [24]

$$u_3=\varphi_1=\varphi_2=0,$$

by which there is no dependence of the deformation on x_3. Thus, $\boldsymbol{\varepsilon}=\operatorname{grad}_2\boldsymbol{u}$, and grad_2 is the gradient operator restricted to the co-ordinates x_α, Greek subscripts ranging in the set $\{1,2\}$. Similarly, the constraint (2.1) reduces to the single component

$$\varphi_3=\tfrac{1}{2}\left(u_{2,1}-u_{1,2}\right)=\tfrac{1}{2}\left(\boldsymbol{u}_{,1}\cdot\boldsymbol{e_2}-\boldsymbol{u}_{,2}\cdot\boldsymbol{e_1}\right), \tag{3.1}$$

that immediately satisfies (2.2), while the wryness tensor (2.3) becomes

$$\boldsymbol{\chi}=\operatorname{grad}_2\varphi_3=\varphi_{3,1}\boldsymbol{e}_3\otimes\boldsymbol{e}_1+\varphi_{3,2}\boldsymbol{e}_3\otimes\boldsymbol{e}_2,$$

having introduced the dyadic operator for vectors such that $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$, for any triple of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Within a Stroh formalism, we define the usual rank 2 matrices

$$Q_{\alpha\beta}=c_{\alpha 1\beta 1},\quad R_{\alpha\beta}=c_{\alpha 1\beta 2},\quad T_{\alpha\beta}=c_{\alpha 2\beta 2}, \tag{3.2}$$

where $\boldsymbol{Q}$ and $\boldsymbol{T}$ are *symmetric*, alongside the symmetric matrix

$$U_{\alpha\beta}=\ell^2 g_{3\alpha 3\beta}. \tag{3.3}$$

Through these, we can define the elastic part of the reduced traction vectors (in the plane of strain)

$$\boldsymbol{t_1^e}=\mathbf{Q}\boldsymbol{u}_{,1}+\mathbf{R}\boldsymbol{u}_{,2},\quad \boldsymbol{t_2^e}=\mathbf{R}^T\boldsymbol{u}_{,1}+\mathbf{T}\boldsymbol{u}_{,2}, \tag{3.4}$$

and the *out-of-plane component* of the reduced couple-stress vectors $\boldsymbol{q_1^e}$ and $\boldsymbol{q_2^e}$, respectively

$$q_{13}^e=\boldsymbol{q_1^e}\cdot\boldsymbol{e_3}=U_{11}\varphi_{3,1}+U_{12}\varphi_{3,2},\quad q_{23}^e=\boldsymbol{q_2^e}\cdot\boldsymbol{e_3}=U_{21}\varphi_{3,1}+U_{22}\varphi_{3,2}. \tag{3.5}$$

We observe that

$$\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}=\boldsymbol{t_1^e}\cdot\boldsymbol{u}_{,1}+\boldsymbol{t_2^e}\cdot\boldsymbol{u}_{,2},$$

and

$$\boldsymbol{\mu}\cdot\boldsymbol{\chi}=\boldsymbol{\mu}\cdot\operatorname{grad}_2\varphi_3=\mu_{31}\chi_{31}+\mu_{32}\chi_{32}=q_{13}^e\varphi_{3,1}+q_{23}^e\varphi_{3,2}.$$

Besides, specializing (2.13*b*), the Lagrangian multiplier $\boldsymbol{p}$ has the single non-zero component

$$p_3=\mu_{31,1}+\mu_{32,2}=q_{13,1}^e+q_{23,2}^e,$$

and from (2.7,2.14) we get the single non-zero component in the skew part of the force stress tensor

$$\tau_{12}=\tfrac{1}{2}P_{12}=\tfrac{1}{2}p_3=-\tau_{21}. \tag{3.6}$$

Then, the Lagrangian density (2.11) becomes

$$L(\boldsymbol{u}_{,1},\boldsymbol{u}_{,2},\varphi_3,\varphi_{3,1},\varphi_{3,2},p_3)=\tfrac{1}{2}\left[\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{1}}\cdot\boldsymbol{u}_{,1}+\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{2}}\cdot\boldsymbol{u}_{,2}+q^e_{13}\varphi_{3,1}+q^e_{23}\varphi_{3,2}\right]\\ +p_3\varphi_3-\tfrac{1}{2}p_3\left(\boldsymbol{u}_{,1}\cdot\boldsymbol{e}_{\boldsymbol{2}}-\boldsymbol{u}_{,2}\cdot\boldsymbol{e}_{\boldsymbol{1}}\right),$$

where it is understood that the scalar product now carries over 2 components only. The conjugate momenta are

$$\frac{\partial L}{\partial \boldsymbol{u}_{,1}}=\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{1}}-\tfrac{1}{2}p_3\boldsymbol{e}_2=\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{1}}+\tau_{21}\boldsymbol{e}_2=\boldsymbol{s}_{\boldsymbol{1}}, \tag{3.7a}$$

$$\frac{\partial L}{\partial \boldsymbol{u}_{,2}}=\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{2}}+\tfrac{1}{2}p_3\boldsymbol{e}_1=\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{2}}+\tau_{12}\boldsymbol{e}_1=\boldsymbol{s}_{\boldsymbol{2}}, \tag{3.7b}$$

$$\frac{\partial L}{\partial \varphi_{3,1}}=q^e_{13}, \tag{3.7c}$$

$$\frac{\partial L}{\partial \varphi_{3,2}}=q^e_{23}, \tag{3.7d}$$

whereupon the Lagrangian may be rewritten as

$$L(\boldsymbol{u}_{,1},\boldsymbol{u}_{,2},\varphi_3,\varphi_{3,1},\varphi_{3,2},p_3)=\tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\mathbf{Q}\boldsymbol{u}_{,1}+\boldsymbol{u}_{,1}\cdot\mathbf{R}\boldsymbol{u}_{,2}+\tfrac{1}{2}\boldsymbol{u}_{,2}\cdot\mathbf{T}\boldsymbol{u}_{,2}\\ +\tfrac{1}{2}U_{11}\varphi_{3,1}^2+U_{12}\varphi_{3,1}\varphi_{3,2}+\tfrac{1}{2}U_{22}\varphi_{3,2}^2+p_3\varphi_3+\tfrac{1}{2}p_3\left(\boldsymbol{u}_{,2}\cdot\boldsymbol{e}_{\boldsymbol{1}}-\boldsymbol{u}_{,1}\cdot\boldsymbol{e}_{\boldsymbol{2}}\right).$$

We now come to an important juncture and treat either co-ordinate as a time-like variable, say x_2 to fix ideas. Consequently, we introduce the Legendre transformation

$$H(\boldsymbol{u}_{,1},\boldsymbol{s}_{\boldsymbol{2}},\varphi_3,\varphi_{3,1},q^e_{23},p_3)=\boldsymbol{s}_{\boldsymbol{2}}\cdot\boldsymbol{u}_{,2}+q^e_{23}\varphi_{3,2}-L\\ =\tfrac{1}{2}\boldsymbol{s}_{\boldsymbol{2}}\cdot\boldsymbol{u}_{,2}+\tfrac{1}{2}q^e_{23}\varphi_{3,2}-\tfrac{1}{2}\boldsymbol{t}^{\boldsymbol{e}}_{\boldsymbol{1}}\cdot\boldsymbol{u}_{,1}-\tfrac{1}{2}q^e_{13}\varphi_{3,1}-p_3\varphi_3-\tfrac{1}{4}p_3\left(\boldsymbol{u}_{,2}\cdot\boldsymbol{e}_{\boldsymbol{1}}-2\boldsymbol{u}_{,1}\cdot\boldsymbol{e}_{\boldsymbol{2}}\right),$$

provided that we write $\boldsymbol{u}_{,2}$ in terms of $\boldsymbol{s}_{\boldsymbol{2}}$ by (3.7*b*) and $\varphi_{3,2}$ in terms of q^e_{23} by (3.7*d*). For the former, making use of (3.4), we get

$$\boldsymbol{u}_{,2}=\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right), \tag{3.8}$$

assuming that $\mathbf{T}$ is invertible, while for the latter

$$\varphi_{3,2}=U_{22}^{-1}\left(q^e_{23}-U_{21}\varphi_{3,1}\right), \tag{3.9}$$

assuming $U_{22}\neq 0$. Therefore, we can write the Hamiltonian density function (whose arguments are omitted for brevity)

$$H=\tfrac{1}{2}\boldsymbol{s}_{\boldsymbol{2}}\cdot\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right)+\tfrac{1}{2}q^e_{23}\frac{q^e_{23}-U_{21}\varphi_{3,1}}{U_{22}}-\tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\mathbf{Q}\boldsymbol{u}_{,1}\\ -\tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\mathbf{R}\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right)-\tfrac{1}{2}U_{11}\varphi_{3,1}^2-\tfrac{1}{2}U_{12}\varphi_{3,1}\frac{q^e_{23}-U_{21}\varphi_{3,1}}{U_{22}}\\ -p_3\varphi_3-\tfrac{1}{4}p_3\left[\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right)\cdot\boldsymbol{e}_{\boldsymbol{1}}-2\boldsymbol{u}_{,1}\cdot\boldsymbol{e}_{\boldsymbol{2}}\right],$$

which reduces to

$$H=\tfrac{1}{2}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right)\cdot\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s}_{\boldsymbol{2}}\right)\\ +\tfrac{1}{2}\frac{\left(q^e_{23}-U_{21}\varphi_{3,1}\right)^2}{U_{22}}-\tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\mathbf{Q}\boldsymbol{u}_{,1}-\tfrac{1}{2}U_{11}\varphi_{3,1}^2-p_3\left(\varphi_3-\tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\boldsymbol{e}_{\boldsymbol{2}}\right). \tag{3.10}$$

Indeed, letting the generalized co-ordinate vector $\bar{\boldsymbol{q}}=(\boldsymbol{u},\varphi_3)$ and the conjugate momenta $\bar{\boldsymbol{p}}=(\boldsymbol{s_2},q^e_{23})$, the first set of canonical equations

$$\frac{\delta H}{\delta \bar{\boldsymbol{p}}}=\dot{\bar{\boldsymbol{q}}} \tag{3.11}$$

is

$$\frac{\delta H}{\delta \boldsymbol{s_2}}=\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s_2}\right)=\boldsymbol{u}_{,2},$$

corresponding to (3.8), and

$$\frac{\delta H}{\delta q^e_{23}}=\frac{(q^e_{23}-U_{21}\varphi_{3,1})}{U_{22}}=\varphi_{3,2},$$

that matches (3.9). Likewise, the second set of canonical equations

$$\frac{\delta H}{\delta \bar{\boldsymbol{q}}}=-\dot{\bar{\boldsymbol{p}}}, \tag{3.12}$$

yields

$$\frac{\delta H}{\delta \boldsymbol{u}}=-\frac{\partial}{\partial x_1}\frac{\partial H}{\partial \boldsymbol{u}_{,1}}=\left[\mathbf{R}\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s_2}\right)+\mathbf{Q}\boldsymbol{u}_{,1}-\tfrac{1}{2}p_3\boldsymbol{e_2}\right]_{,1}=-\boldsymbol{s_2}_{,2} \tag{3.13}$$

and

$$\frac{\delta H}{\delta \varphi_3}=-\frac{\partial}{\partial x_1}\frac{\partial H}{\partial \varphi_{3,1}}+\frac{\partial H}{\partial \varphi_3}=\left[\frac{U_{21}}{U_{22}}(q^e_{23}-U_{21}\varphi_{3,1})+U_{11}\varphi_{3,1}\right]_{,1}-p_3=-q^e_{23,2}. \tag{3.14}$$

Indeed, in light of Eqs.(3.6,3.7*a*), Eq.(3.13) is simply

$$\left(\boldsymbol{t^e_1}+\tau_{21}\boldsymbol{e_2}\right)_{,1}+\boldsymbol{s_2}_{,2}=\boldsymbol{s_1}_{,1}+\boldsymbol{s_2}_{,2}=\boldsymbol{o},$$

that corresponds to (2.5*a*). Similarly, making use of (3.5,3.6) and (3.9), Eq.(3.14) may be rewritten as

$$q^e_{13,1}+q^e_{23,2}-2\tau_{12}=0,$$

which amounts to the rotational equilibrium (2.8). Thus, for a homogeneous material, we get

$$\boldsymbol{u}_{,2}=-\mathbf{T}^{-1}\mathbf{R}^T\boldsymbol{u}_{,1}+\mathbf{T}^{-1}\boldsymbol{\phi}_{,1}-\Lambda_{,1}\mathbf{T}^{-1}\boldsymbol{e}_1, \tag{3.15a}$$

$$\boldsymbol{\phi}_{,2}=\left(\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T-\mathbf{Q}\right)\boldsymbol{u}_{,1}-\mathbf{R}\mathbf{T}^{-1}\boldsymbol{\phi}_{,1}+\Lambda_{,1}\left(\boldsymbol{e_2}+\mathbf{R}\mathbf{T}^{-1}\boldsymbol{e}_1\right), \tag{3.15b}$$

$$\varphi_{3,2}=-\frac{U_{21}}{U_{22}}\varphi_{3,1}+U^{-1}_{22}\Phi_{,1}, \tag{3.15c}$$

$$\Phi_{,2}=\left(\frac{U^2_{21}}{U_{22}}-U_{11}\right)\varphi_{3,1}-\frac{U_{21}}{U_{22}}\Phi_{,1}+2\Lambda, \tag{3.15d}$$

having let the stress functions $\boldsymbol{\phi}=\int^{x_1}\boldsymbol{s_2}\mathrm{d}\xi_1$, $\Lambda=\frac{1}{2}\int^{x_1}p_3\mathrm{d}\xi_1=\int^{x_1}\tau_{12}\mathrm{d}\xi_1$ and $\Phi=\int^{x_1}q^e_{23}\mathrm{d}\xi_1$, that are defined up to an arbitrary function of x_2. In the spirit of considering x_2 a time-like variable, this is *a system of ODEs in canonical form*. We also observe a connection with Günther tensor potentials (2.6), namely $\phi_\alpha=-W_{\alpha 3}$ and $\Phi_{,1}=-Z_{33,1}+\phi_2$, cf. [15, Eqs(2.38-40)].

Now, we only need to dispose of the so-far undetermined Lagrange multiplier Λ. For this, we take the scalar product of (3.8) with $\boldsymbol{e_1}$ and make use of the constraint (3.1)

$$\boldsymbol{u}_{,2}\cdot\boldsymbol{e_1}=-2\varphi_3+\boldsymbol{u}_{,1}\cdot\boldsymbol{e_2}=\mathbf{T}^{-1}\left(-\tfrac{1}{2}p_3\boldsymbol{e}_1-\mathbf{R}^T\boldsymbol{u}_{,1}+\boldsymbol{s_2}\right)\cdot\boldsymbol{e_1},$$

whereupon we find

$$\Lambda_{,1}=\zeta\left(\boldsymbol{\phi}_{,1}\cdot\boldsymbol{f_1}-\boldsymbol{u}_{,1}\cdot\boldsymbol{f_2}+2\varphi_3\right), \tag{3.16}$$

having let the shorthand vectors

$$\boldsymbol{f_1}=\mathbf{T}^{-1}\boldsymbol{e_1},\quad \text{and}\quad \boldsymbol{f_2}=\boldsymbol{e_2}+\mathbf{R}\boldsymbol{f_1},$$

and $\zeta^{-1}=\mathbf{T}^{-1}\boldsymbol{e}_1\cdot\boldsymbol{e_1}=\boldsymbol{f_1}\cdot\boldsymbol{e_1}$. The connection (3.16) shows that, similarly to classical incompressible elasticity, the Lagrange multiplier is determined by an *algebraic* relation where no x_2 derivative appears. Consequently, the governing equations (3.15,3.16) form a system of Differential Algebraic Equations (DAEs) in *semi-explicit form*. However, in contrast to incompressible elasticity, Eqs.(3.15*d*) and (3.16) indicate that a Stroh classical formulation, where the unknown vectors only appear in differential form, is not accessible.

We now show that this system of DAEs has *differential index 1*. For the sake of convenience, we let the matrices

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \tag{3.17a}$$

$$\mathbf{N}_2 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} = -\mathbf{R}\mathbf{N_1} - \mathbf{Q} = \mathbf{N}_2^T, \tag{3.17b}$$

$$\mathbf{N}_3 = U_{22}^{-1}\begin{bmatrix} -U_{21} & 1 \\ U_{21}^2 - U_{22}U_{11} & -U_{21} \end{bmatrix}. \tag{3.17c}$$

Differentiating (3.16) with respect to x_2 and then making use of (3.15*c*), we get

$$\zeta^{-1}\Lambda_{,2} = \boldsymbol{\phi}_{,2}\cdot\boldsymbol{f_1} - \boldsymbol{u}_{,2}\cdot\boldsymbol{f_2} + 2U_{22}^{-1}\left(-U_{21}\varphi_3 + \Phi\right),$$

and, by (3.15*a*,3.15*b*),

$$\zeta^{-1}\Lambda_{,2} = \left(\mathbf{N}_2\boldsymbol{u}_{,1} + \mathbf{N}_1^T\boldsymbol{\phi}_{,1} + \Lambda_{,1}\boldsymbol{f}_2\right)\cdot\boldsymbol{f_1} \\ - \left(\mathbf{N}_1\boldsymbol{u}_{,1} + \mathbf{T}^{-1}\boldsymbol{\phi}_{,1} - \Lambda_{,1}\boldsymbol{f}_1\right)\cdot\boldsymbol{f_2} + 2U_{22}^{-1}\left(-U_{21}\varphi_3 + \Phi\right),$$

that provides the evolution equation for Λ

$$\zeta^{-1}\Lambda_{,2} = \left(\mathbf{N}_2\boldsymbol{f_1} - \mathbf{N}_1^T\boldsymbol{f_2}\right)\cdot\boldsymbol{u}_{,1} + \left(\mathbf{N}_1\boldsymbol{f_1} - \mathbf{T}^{-1}\boldsymbol{f_2}\right)\cdot\boldsymbol{\phi}_{,1} + 2\Lambda_{,1}\boldsymbol{f_1}\cdot\boldsymbol{f_2} \\ + 2U_{22}^{-1}\left(-U_{21}\varphi_3 + \Phi\right).$$

For better understanding, we adopt the convention that vectors are columns and their transpose are rows. Thus, letting the 7-component row vector

$$\boldsymbol{\xi}^T = \left[\boldsymbol{u}^T,\ \boldsymbol{\phi}^T,\ \Lambda,\ \varphi_3,\ \Phi\right],$$

we finally obtain the system of first order linear PDEs

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}x_2} = \mathbf{N}\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}x_1} + \boldsymbol{b}, \tag{3.18}$$

where we have let the 7 by 7 Stroh matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{5\times5} & \mathbf{O} \\ \mathbf{O} & \mathbf{N_3} \end{bmatrix}, \tag{3.19}$$

with the 5 by 5 matrix

$$\mathbf{N}_{5\times5} = \begin{bmatrix} \mathbf{N_1} & \mathbf{T}^{-1} & -\boldsymbol{f_1} \\ \mathbf{N_2} & \mathbf{N}_1^T & \boldsymbol{f_2} \\ \zeta\left(\mathbf{N}_2\boldsymbol{f_1} - \mathbf{N}_1^T\boldsymbol{f_2}\right)^T & \zeta\left(\mathbf{N}_1\boldsymbol{f_1} - \mathbf{T}^{-1}\boldsymbol{f_2}\right)^T & 2\zeta\boldsymbol{f_1}\cdot\boldsymbol{f_2} \end{bmatrix}.$$

and the right hand side is a linear function of $\boldsymbol{\xi}$

$$\mathbf{b} = \begin{bmatrix} \boldsymbol{o} \\ \boldsymbol{o} \\ 2\zeta U_{22}^{-1}\left(-U_{21}\varphi_3 + \Phi\right) \\ 0 \\ 2\Lambda \end{bmatrix}. \tag{3.20}$$

Clearly, the Stroh (or *fundamental elasticity*) matrix (3.19) has block structure and coupling of the unknowns $\boldsymbol{\xi}_1^T = [\boldsymbol{u}^T,\ \boldsymbol{\phi}^T,\ \Lambda]$ and $\boldsymbol{\xi}_2^T = [\varphi_3,\ \Phi]$ only occurs through the right hand side (3.20).

Indeed, we can write the coupled system

$$\frac{\mathrm{d}\boldsymbol{\xi}_1}{\mathrm{d}x_2} = \mathbf{N}_{5\times 5}\frac{\mathrm{d}\boldsymbol{\xi}_1}{\mathrm{d}x_1} + \mathbf{L}_{5\times 2}\boldsymbol{\xi}_2, \tag{3.21a}$$

$$\frac{\mathrm{d}\boldsymbol{\xi}_2}{\mathrm{d}x_2} = \mathbf{N_3}\frac{\mathrm{d}\boldsymbol{\xi}_2}{\mathrm{d}x_2} + \mathbf{L}_{2\times 5}\boldsymbol{\xi}_1, \tag{3.21b}$$

with

$$\mathbf{L}_{5\times 2} = 2\zeta U_{22}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -U_{21} & 1 \end{bmatrix}, \quad \mathbf{L}_{2\times 5} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.22}$$

(a) Isotropic material

We now show that the Hamiltonian/Stroh formulation so far developed correctly represents the governing equations of plane isotropic CS elasticity. These are [12,19]

$$(G+\lambda)\,\mathrm{grad}_2\,\mathrm{div}_2\,\boldsymbol{u} + G\triangle_2\left[\boldsymbol{u} - \tfrac{1}{2}\ell^2\,\mathrm{curl}_2\,\mathrm{curl}_2\,\boldsymbol{u}\right] = 0, \tag{3.23}$$

where $\triangle_2 \equiv \mathrm{div}_2\,\mathrm{grad}_2$, while $(\mathrm{curl}_2\,\boldsymbol{u})_\alpha = e_{\alpha\beta}u_{\beta,\alpha}$, having let the rank 2 alternating tensor $e_{\alpha\gamma}$ such that $e_{11} = e_{22} = 0$ and $e_{12} = -e_{21} = 1$. Besides, the sharing force is connected to the rotation through

$$\tau_{12} = G\ell^2\triangle_2\varphi_3.$$

Upon introducing the potentials ω, H such that

$$u_1 = \omega_{,1} + H_{,2}, \quad u_2 = \omega_{,2} - H_{,1}, \tag{3.24}$$

the governing equations (3.23) decouple as [19, Eqs.(14)]

$$(2G+\lambda)\triangle_2\omega = 0, \quad \text{and} \quad G\triangle_2\left(1 - \tfrac{1}{2}\ell^2\triangle_2\right)H = 0.$$

Indeed, $\varphi_3 = -\frac{1}{2}\triangle_2 H$ and

$$\mu_{3\alpha} = 2G\ell^2\varphi_{3,\alpha} = -G\ell^2\triangle_2 H_{,\alpha}. \tag{3.25}$$

whence we get the physical meaning of H, whose bilaplacian is related to the shearing force τ_{12},

$$\tau_{12} = \Lambda_{,1} = G\ell^2\triangle_2\varphi_3 = -\tfrac{1}{2}G\ell^2\triangle_2^2 H. \tag{3.26}$$

Finally, the scalar potential is related to displacement flux

$$\mathrm{div}_2\,\boldsymbol{u} = \triangle_2\omega.$$

We let the matrices (3.2,3.3)

$$\mathbf{U} = 2G\ell^2\mathbf{1}_2, \qquad \mathbf{Q} = \begin{bmatrix} 2G+\lambda & 0 \\ 0 & G \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} 0 & \lambda \\ G & 0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} G & 0 \\ 0 & 2G+\lambda \end{bmatrix},$$

where $\mathbf{1}_2$ is the rank 2 identity matrix. It easily follows that

$$\mathbf{N}_1 = -\begin{bmatrix} 0 & 1 \\ \frac{\lambda}{2G+\lambda} & 0 \end{bmatrix}, \quad \mathbf{N}_2 = -4G\begin{bmatrix} \frac{G+\lambda}{2G+\lambda} & 0 \\ 0 & 0 \end{bmatrix},$$

while $\zeta = G$,

$$\boldsymbol{f_1} = G^{-1}\boldsymbol{e_1}, \quad \boldsymbol{f_2} = 2\boldsymbol{e_2}.$$

The system (3.18) becomes

$$u_{1,2} = -u_{2,1} + \frac{1}{G}\phi_{1,1} - \frac{1}{G}\Lambda_{,1} \tag{3.27a}$$

$$u_{2,2} = -\frac{\lambda}{2G+\lambda}u_{1,1} + \frac{1}{2G+\lambda}\phi_{2,1}, \tag{3.27b}$$

$$\phi_{1,2} = -4G\frac{G+\lambda}{2G+\lambda}u_{1,1} - \frac{\lambda}{2G+\lambda}\phi_{2,1}, \tag{3.27c}$$

$$\phi_{2,2} = -\phi_{1,1} + 2\Lambda_{,1}, \tag{3.27d}$$

$$\Lambda_{,2} = -2Gu_{1,1} - \phi_{2,1} + \ell^{-2}\Phi, \tag{3.27e}$$

$$\varphi_{3,2} = \frac{1}{2G\ell^2}\Phi_{,1}, \tag{3.27f}$$

$$\Phi_{,2} = -2G\ell^2\varphi_{3,1} + 2\Lambda. \tag{3.27g}$$

Differentiation of Eq.(3.27d) with respect to x_1 gives

$$s_{12,1} + s_{22,2} = 2\tau_{12,1},$$

that immediately corresponds to (2.5a) in consideration of the connection $s_{21} = \sigma_{21} + \tau_{21} = s_{12} - 2\tau_{12}$. Similarly, differentiation of (3.27c) lends

$$s_{12,2} + \frac{\lambda}{2G+\lambda}s_{22,1} + 4G\frac{G+\lambda}{2G+\lambda}u_{1,11} = 0,$$

which, with a bit of algebra, corresponds to the first of Eqs.(3.23). Cross differentiation of (3.27f) and (3.27g) allows eliminating $\Phi_{,12}$ to give

$$\Lambda_{,1} = G\ell^2\triangle_2\varphi_3,$$

that matches Eq.(3.26). Besides, plugging this result in either equation lends

$$\Phi_{,12} = q^e_{23,2} = \mu_{32,2} = 2G\ell^2\varphi_{3,22},$$

which corresponds to (3.25).

4. Antiplane deformations

Under antiplane shear deformations, the displacement field $\boldsymbol{u} = (u_1, u_2, u_3)$ is completely defined by the out-of-plane component $u_3 = w(x_1, x_2)$. Thus we have

$$u_1 = u_2 = \varphi_3 = 0,$$

and again no dependence of the deformation on x_3. Thus, Eq.(2.1) lends the rotation $\boldsymbol{\varphi} = \frac{1}{2}\,\mathrm{curl}_2\, w$ (see [19] for the definition of curl operating on a scalar field)

$$\varphi_\alpha = \tfrac{1}{2}e_{\alpha\gamma}w_{,\gamma}. \tag{4.1}$$

Thus, we define the 2D rotation vector

$$\boldsymbol{\varphi}^T = \begin{bmatrix}\varphi_1, & \varphi_2\end{bmatrix},$$

whence the curvature tensor (2.3) is immediately obtained and it is deviatoric

$$\chi_{\alpha\beta} = \varphi_{\alpha,\beta} = \tfrac{1}{2}e_{\alpha\gamma}w_{,\gamma\beta}, \quad \Leftrightarrow \quad \chi = \frac{1}{2}\begin{bmatrix} w_{,12} & w_{,22} \\ -w_{,11} & -w_{,12}\end{bmatrix}.$$

Furthermore, from (2.8) and (2.13b), we get the non-zero components of the skew force stress tensor

$$\tau_{13} = -\tfrac{1}{2}\mu_{2\beta,\beta} = -\tfrac{1}{2}p_2, \quad \tau_{23} = \tfrac{1}{2}\mu_{1\beta,\beta} = \tfrac{1}{2}p_1. \tag{4.2}$$

The Lagrangian density (2.11) becomes

$$L(\operatorname{grad}_2 w, \boldsymbol{\varphi}, \operatorname{grad}_2 \boldsymbol{\varphi}, \boldsymbol{p}) = \tfrac{1}{2}\left[\sigma_{3\alpha} w_{,\alpha} + \boldsymbol{q}_{\boldsymbol{1}}^{\boldsymbol{e}} \cdot \boldsymbol{\varphi}_{,1} + \boldsymbol{q}_{\boldsymbol{2}}^{\boldsymbol{e}} \cdot \boldsymbol{\varphi}_{,2}\right] + p_1\left(\varphi_1 - \tfrac{1}{2} w_{,2}\right) + p_2\left(\varphi_2 + \tfrac{1}{2} w_{,1}\right). \tag{4.3}$$

The Euler-Lagrange equation associated with the variation of w reads

$$-\sigma_{3\alpha,\alpha} + \tfrac{1}{2} p_{1,2} - \tfrac{1}{2} p_{2,1} = 0,$$

that, by (4.2), reduces to (2.13*a*). Similarly, through varying $\boldsymbol{\varphi}$, we get the vector E-L equation,

$$-\boldsymbol{q}_{\boldsymbol{1},1} - \boldsymbol{q}_{\boldsymbol{2},2} + \boldsymbol{p} = \boldsymbol{o}, \tag{4.4}$$

that corresponds to (2.13*b*).

We now try to relate antiplane problems in CS elasticity with the theory of anisotropic Kirchhoff plates, which admits a classical Stroh formalism. If we identify the Lagrange multiplier $\boldsymbol{p}$ with the shearing force for plates, by assuming $\boldsymbol{p} = -\operatorname{curl}_2 h$, we immediately obtain the plate equilibrium equation $\operatorname{div}_2 \boldsymbol{p} = 0$. Besides, employing the divergence theorem, (4.3) attains the alternative form

$$L_p(\operatorname{grad}_2 w, \operatorname{grad}_2 \boldsymbol{\varphi}, h, \operatorname{grad}_2 h) = \tfrac{1}{2}\left[\left(\sigma_{3\alpha} + h_{,\alpha}\right) w_{,\alpha} + \mu_{\alpha\beta}\varphi_{\alpha,\beta}\right] + h\left(\varphi_{1,2} - \varphi_{2,1}\right), \tag{4.5}$$

that is formally equivalent to the Lagrangian density adopted for anisotropic plates [8, Eq.(4.15)], provided we identify $\boldsymbol{\varphi}$ with $\boldsymbol{\theta}$, $\mu_{\alpha\beta}$ with the bending moment in the plate, $M_{\alpha\beta}$, and $\sigma_{3\alpha} + h_{,\alpha}$ with either normal force, $N_{1\alpha}$ or $N_{2\alpha}$, acting in the plane of the plate. Indeed, such an assumption allows to express the constraint associated with the Lagrange multiplier h in terms of "time" derivatives, i.e. they become *rheonomic*. However, to allow for a classical Stroh formulation, one feature is missing: namely, unlike in plate problems, the term $\sigma_{3\alpha} + h_{,\alpha}$ is not constitutively defined.

We proceed with the Lagrangian density (4.3) and, making use of (2.14), obtain the conjugate momenta

$$\frac{\partial L}{\partial w_{,2}} = \sigma_{32} - \tfrac{1}{2} p_1 = \sigma_{32} - \tau_{23} = s_{32}, \tag{4.6a}$$

$$\frac{\partial L}{\partial \boldsymbol{\varphi}_{,2}} = \boldsymbol{q}_{\boldsymbol{2}}^{\boldsymbol{e}}. \tag{4.6b}$$

In analogy with (3.3), we let the symmetric matrix

$$\hat{U}_{\alpha\beta} = c_{3\alpha3\beta}, \tag{4.7}$$

whence

$$\sigma_{3\alpha} = \hat{U}_{\alpha\beta} w_{,\beta}, \tag{4.8}$$

and Eq.(4.6*a*) may be easily solved for $w_{,2}$

$$w_{,2} = \hat{U}_{22}^{-1}\left(s_{32} + \tfrac{1}{2} p_1 - \hat{U}_{21} w_{,1}\right). \tag{4.9}$$

For (4.6*b*) we need to let, in analogy with (3.2),

$$\hat{Q}_{\alpha\beta} = \ell^2 g_{\alpha1\beta1}, \quad \hat{R}_{\alpha\beta} = \ell^2 g_{\alpha1\beta2}, \quad \hat{T}_{\alpha\beta} = \ell^2 g_{\alpha2\beta2}, \tag{4.10}$$

so that, paralleling (3.4),

$$\boldsymbol{q}_{\boldsymbol{1}}^{\boldsymbol{e}} = \hat{\mathbf{Q}}\boldsymbol{\varphi}_{,1} + \hat{\mathbf{R}}\boldsymbol{\varphi}_{,2}, \quad \boldsymbol{q}_{\boldsymbol{2}}^{\boldsymbol{e}} = \hat{\mathbf{R}}^T\boldsymbol{\varphi}_{,1} + \hat{\mathbf{T}}\boldsymbol{\varphi}_{,2}, \tag{4.11}$$

we can write

$$\boldsymbol{\varphi}_{,2} = \hat{\mathbf{T}}^{-1}\left(\boldsymbol{q}_{\boldsymbol{2}}^{\boldsymbol{e}} - \hat{\mathbf{R}}^T\boldsymbol{\varphi}_{,1}\right). \tag{4.12}$$

Besides, from (4.4) and the constitutive law (4.11), we get

$$\boldsymbol{p} = \hat{\mathbf{Q}}\boldsymbol{\varphi}_{,11} + \left(\hat{\mathbf{R}} + \hat{\mathbf{R}}^T\right)\boldsymbol{\varphi}_{,12} + \hat{\mathbf{T}}\boldsymbol{\varphi}_{,22}, \tag{4.13}$$

which shows that indeed $\mathrm{div}_2\, \boldsymbol{p} = 0$, inasmuch as (2.2) holds, as discussed in Sec.2. Recalling (2.14), this implies

$$\tau_{31,2} - \tau_{32,1} = 0, \tag{4.14}$$

which is in fact satisfied by Eqs.(2.14) of [20].

We define the Hamiltonian density function $H = H(s_{32}, w_{,1}, \boldsymbol{\varphi}, \boldsymbol{\varphi}_{,1}, \boldsymbol{q_2^e}, \boldsymbol{p})$

$$\begin{aligned} H &= s_{32} w_{,2} + \boldsymbol{q_2^e} \cdot \boldsymbol{\varphi}_{,2} - L = \tfrac{1}{2}\hat{U}_{22}^{-1}\left(s_{32} + \tfrac{1}{2}p_1 - \hat{U}_{21} w_{,1}\right)^2 \\ &+ \tfrac{1}{2}\left(\boldsymbol{q_2^e} - \hat{\mathbf{R}}^T \boldsymbol{\varphi}_{,1}\right) \cdot \hat{\mathbf{T}}^{-1}\left(\boldsymbol{q_2^e} - \hat{\mathbf{R}}^T \boldsymbol{\varphi}_{,1}\right) - \tfrac{1}{2}\hat{U}_{11} w_{,1}^2 - \tfrac{1}{2}\boldsymbol{\varphi}_{,1} \cdot \hat{\mathbf{Q}} \boldsymbol{\varphi}_{,1} - \boldsymbol{p} \cdot \boldsymbol{\varphi} - \tfrac{1}{2} p_2 w_{,1}, \end{aligned} \tag{4.15}$$

whence, from (3.11), we retrieve (4.9)

$$\frac{\delta H}{\delta s_{32}} = \hat{U}_{22}^{-1}\left(s_{32} + \tfrac{1}{2}p_1 - \hat{U}_{21} w_{,1}\right) = w_{,2},$$

and (4.12)

$$\frac{\delta H}{\delta \boldsymbol{q_2^e}} = \hat{\mathbf{T}}^{-1}\left(\boldsymbol{q_2^e} - \hat{\mathbf{R}}^T \boldsymbol{\varphi}_{,1}\right) = \boldsymbol{\varphi}_{,2}.$$

The canonical equation (3.12) gives

$$\frac{\delta H}{\delta \boldsymbol{\varphi}} = \left[\hat{\mathbf{Q}} \boldsymbol{\varphi}_{,1} + \hat{\mathbf{R}} \hat{\mathbf{T}}^{-1}\left(\boldsymbol{q_2^e} - \hat{\mathbf{R}}^T \boldsymbol{\varphi}_{,1}\right)\right]_{,1} - \boldsymbol{p} = -\boldsymbol{q_2^e}_{,2},$$

and

$$\frac{\delta H}{\delta w} = \left[\hat{U}_{21}\hat{U}_{22}^{-1}\left(s_{32} + \tfrac{1}{2}p_1 - \hat{U}_{21} w_{,1}\right) + \hat{U}_{11} w_{,1} + \tfrac{1}{2}p_2\right]_{,1} = -s_{32,2},$$

corresponding to (2.13*b*) and (2.13*a*), respectively. We introduce the stream functions, which are defined up to a function of x_2,

$$\phi = \int^{x_1} s_{32}\, \mathrm{d}\xi_1, \quad \boldsymbol{\Phi} = \int^{x_1} \boldsymbol{q_2^e}\, \mathrm{d}\xi_1, \quad \boldsymbol{\Lambda} = \tfrac{1}{2}\int^{x_1} \boldsymbol{p}\, \mathrm{d}\xi_1,$$

and write the first order system

$$\boldsymbol{\varphi}_{,2} = \hat{\mathbf{N}}_1 \boldsymbol{\varphi}_{,1} + \hat{\mathbf{T}}^{-1} \boldsymbol{\Phi}_{,1}, \tag{4.16a}$$

$$\boldsymbol{\Phi}_{,2} = \hat{\mathbf{N}}_2 \boldsymbol{\varphi}_{,1} + \hat{\mathbf{N}}_1^T \boldsymbol{\Phi}_{,1} + 2\boldsymbol{\Lambda}, \tag{4.16b}$$

$$w_{,2} = -\frac{\hat{U}_{21}}{\hat{U}_{22}} w_{,1} + \frac{1}{\hat{U}_{22}} \phi_{,1} + \frac{1}{\hat{U}_{22}} \Lambda_{1,1} \tag{4.16c}$$

$$\phi_{,2} = \left(\frac{\hat{U}_{21}^2}{\hat{U}_{22}} - \hat{U}_{11}\right) w_{,1} - \frac{\hat{U}_{21}}{\hat{U}_{22}} \phi_{,1} - \frac{\hat{U}_{21}}{\hat{U}_{22}} \Lambda_{1,1} - \Lambda_{2,1}, \tag{4.16d}$$

having let

$$\hat{\mathbf{N}}_1 = -\hat{\mathbf{T}}^{-1} \hat{\mathbf{R}}^T, \quad \hat{\mathbf{N}}_2 = -\hat{\mathbf{R}} \hat{\mathbf{N}}_1 - \hat{\mathbf{Q}}.$$

It only remains to determine an expression for the Lagrange multiplier $\boldsymbol{p}$, which amounts to acknowledging the constraint (2.1). In fact, Eq.(4.1) allows to solve (4.16*c*) for $\Lambda_{1,1}$ and to dispense with $w_{,1}$ and $w_{,2}$

$$\Lambda_{1,1} = 2\hat{U}_{22}\varphi_1 - 2\hat{U}_{21}\varphi_2 - \phi_{,1}. \tag{4.17}$$

In light of (4.1) and (4.8), this *algebraic* condition simply states that $\tau_{23} = \sigma_{32} - s_{32}$. When we plug this result into (4.16*d*) and use (4.8), we find

$$\phi_{,2} = 2\hat{U}_{11}\varphi_2 - 2\hat{U}_{12}\varphi_1 - \Lambda_{2,1} = -\sigma_{31} - \tau_{31} = -s_{31} \tag{4.18}$$

that, differentiated with respect to x_1, gives the equilibrium equation (2.5*a*). To get an equation for Λ_2, we cannot directly employ the connection $2\varphi_2 = -w_{,1}$, for it is *algebraic*. Instead, we take advantage of $\mathrm{div}_2\, \boldsymbol{p} = 0$, whereby

$$\mathrm{div}_2\, \boldsymbol{\Lambda} = 0.$$

Mathematically, this amounts to exploiting (2.2), which is obtained cross differentiating and adding (4.1), whence a differentiation index 2 is implied. Thus, Eq.(4.17) immediately lends the *evolution equation*

$$\Lambda_{2,2} = -2\hat{U}_{22}\varphi_1 + 2\hat{U}_{21}\varphi_2 + \phi_{,1},$$

that corresponds to (4.14), integrated with respect to x_1. In this form, the problem's variables are $\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$, ϕ, $\boldsymbol{\Lambda}$, and they are governed by a semi-explicit system of first order DAE, the single algebraic relation being (4.17). To obtain a pure system of ODEs, an evolution equation for Λ_1 is demanded. This is obtained differentiating (4.17) with respect to x_2 and then integrating with respect to x_1

$$\Lambda_{1,2} = 2\left[\hat{\mathbf{N}}_1\boldsymbol{\varphi} + \hat{\mathbf{T}}^{-1}\boldsymbol{\Phi}\right]\cdot\left(\hat{U}_{22}\boldsymbol{e_1} - \hat{U}_{21}\boldsymbol{e_2}\right) - \phi_{,2} = 2\boldsymbol{\varphi}\cdot\left[\hat{\mathbf{N}}_1^T\boldsymbol{f_2} + \boldsymbol{f_1}\right] + 2\boldsymbol{\Phi}\cdot\hat{\mathbf{T}}^{-1}\boldsymbol{f_2} + \Lambda_{2,1},$$

having made use of (4.16*a*,4.18) and let

$$\boldsymbol{f_1} = \hat{U}_{12}\boldsymbol{e_1} - \hat{U}_{11}\boldsymbol{e_2}, \quad \boldsymbol{f_2} = \hat{U}_{22}\boldsymbol{e_1} - \hat{U}_{21}\boldsymbol{e_2}.$$

Consequently, the system of DAEs has *differentiation order 3*, that is typical of constrained mechanical systems. Also, we note that

$$\sigma_{31} = 2\boldsymbol{\varphi}\cdot\boldsymbol{f_1}, \quad \sigma_{32} = 2\boldsymbol{\varphi}\cdot\boldsymbol{f_2}. \tag{4.19}$$

We thus obtain the linear system in the variables $(\boldsymbol{\varphi}, \boldsymbol{\Phi}, \phi, \boldsymbol{\Lambda})$

$$\boldsymbol{\varphi}_{,2} = \hat{\mathbf{N}}_1\boldsymbol{\varphi}_{,1} + \hat{\mathbf{T}}^{-1}\boldsymbol{\Phi}_{,1}, \tag{4.20a}$$

$$\boldsymbol{\Phi}_{,2} = \hat{\mathbf{N}}_2\boldsymbol{\varphi}_{,1} + \hat{\mathbf{N}}_1^T\boldsymbol{\Phi}_{,1} + 2\boldsymbol{\Lambda}, \tag{4.20b}$$

$$\phi_{,2} = -\Lambda_{2,1} - 2\boldsymbol{\varphi}\cdot\boldsymbol{f_1}, \tag{4.20c}$$

$$\Lambda_{1,2} = \Lambda_{2,1} + 2\boldsymbol{\varphi}\cdot\left[\hat{\mathbf{N}}_1^T\boldsymbol{f_2} + \boldsymbol{f_1}\right] + 2\boldsymbol{\Phi}\cdot\hat{\mathbf{T}}^{-1}\boldsymbol{f_2}, \tag{4.20d}$$

$$\Lambda_{2,2} = \phi_{,1} - 2\boldsymbol{\varphi}\cdot\boldsymbol{f_2}, \tag{4.20e}$$

Cross-differentiating Eqs.(4.20*a*,4.20*b*) to eliminate $\boldsymbol{\Phi}_{,12}$ yields (4.13). Besides, multiplying (4.20*a*) by $-\mathbf{R}$ and substituting in (4.20*b*) gives (4.4). In light of Eqs.(4.19), Eq.(4.20*c*) gives the equilibrium equation (2.5*a*), while (4.20*e*) amounts to (4.14), both having being integrated along x_1. Finally, adding (4.20*c*) and (4.20*d*) and differentiating lends (4.6*a*), while cross-differentiating (4.20*c*,4.20*e*) and adding lends the second order connection for ϕ

$$\triangle_2\phi = 2\boldsymbol{\varphi}_{,1}\cdot\boldsymbol{f_2} - 2\boldsymbol{\varphi}_{,2}\cdot\boldsymbol{f_1} = \sigma_{32,1} - \sigma_{31,2}, \tag{4.21}$$

that supports the interpretation of ϕ as a stress function for the problem.

Thus, letting $\hat{\boldsymbol{\xi}}^T = [\boldsymbol{\varphi}^T, \boldsymbol{\Phi}^T, \phi, \boldsymbol{\Lambda}^T]$, we have

$$\frac{\mathrm{d}\hat{\boldsymbol{\xi}}}{\mathrm{d}x_2} = \hat{\mathbf{N}}\frac{\mathrm{d}\hat{\boldsymbol{\xi}}}{\mathrm{d}x_1} + \hat{\boldsymbol{b}}, \tag{4.22}$$

where we have let the Stroh matrix

$$\hat{\mathbf{N}} = \begin{bmatrix} \hat{\mathbf{N}}_1 & \hat{\mathbf{T}}^{-1} & \boldsymbol{o} & \boldsymbol{o} & \boldsymbol{o} \\ \hat{\mathbf{N}}_2 & \hat{\mathbf{N}}_1^T & \boldsymbol{o} & \boldsymbol{o} & \boldsymbol{o} \\ \boldsymbol{o}^T & \boldsymbol{o}^T & 0 & 0 & -1 \\ \boldsymbol{o}^T & \boldsymbol{o}^T & 0 & 0 & 1 \\ \boldsymbol{o}^T & \boldsymbol{o}^T & 1 & 0 & 0 \end{bmatrix}, \tag{4.23}$$

and the right hand side is a linear function of $\hat{\boldsymbol{\xi}}$

$$\hat{\boldsymbol{b}} = 2\begin{bmatrix} \mathbf{O} & \mathbf{O} & \boldsymbol{o} & \boldsymbol{o} & \boldsymbol{o} \\ \mathbf{O} & \mathbf{O} & \boldsymbol{o} & \boldsymbol{e_1} & \boldsymbol{e_2} \\ -\boldsymbol{f_1}^T & \boldsymbol{o}^T & 0 & 0 & 0 \\ (\hat{\mathbf{N}}_1^T \boldsymbol{f_2} + \boldsymbol{f_1})^T & (\mathbf{T}^{-1}\boldsymbol{f_2})^T & 0 & 0 & 0 \\ \boldsymbol{f_2}^T & \boldsymbol{o}^T & 0 & 0 & 0 \end{bmatrix} \hat{\boldsymbol{\xi}}. \tag{4.24}$$

(a) Isotropic anti-plane deformations

We now show that the above canonical formulation correctly reproduces the governing equations for antiplane deformations in isotropic CS media. Such framework demands

$$\boldsymbol{\epsilon} = \tfrac{1}{2} w_{,1} \boldsymbol{e_3} \otimes \boldsymbol{e_1} + \tfrac{1}{2} w_{,2} \boldsymbol{e_3} \otimes \boldsymbol{e_2},$$

whence

$$\sigma_{31} = G w_{,1} = -2G\varphi_2, \quad \sigma_{32} = G w_{,2} = 2G\varphi_1. \tag{4.25}$$

In particular,

$$\sigma_{32,1} - \sigma_{31,2} = 2G \operatorname{div} \boldsymbol{\varphi} = 0, \tag{4.26}$$

and, by (4.21), ϕ turns harmonic. From (2.4), we get the curvature tensor

$$\mu_{11} = 2G\ell^2(1+\eta)\varphi_{1,1} = G\ell^2(1+\eta)w_{,12} = -\mu_{22}, \tag{4.27a}$$

$$\mu_{21} = 2G\ell^2\left(\varphi_{2,1} + \eta\varphi_{1,2}\right) = -G\ell^2\left(w_{,11} - \eta w_{,22}\right), \tag{4.27b}$$

$$\mu_{12} = 2G\ell^2\left(\eta\varphi_{2,1} + \varphi_{1,2}\right) = -G\ell^2\left(\eta w_{,11} - w_{,22}\right), \tag{4.27c}$$

whereby, from (2.13*b*,2.14), we have [20, Eq.(2.14)]

$$\tau_{13} = -G\ell^2 \triangle_2 \varphi_2, \quad \tau_{23} = G\ell^2 \triangle_2 \varphi_1, \tag{4.28}$$

which clearly satisfy (4.14) in light of (2.2). The equilibrium equation (2.5*a*) reads [20, Eq.(2.15)]

$$G\left(1 - \tfrac{1}{2}\ell^2 \triangle_2\right)\triangle_2 w = 0, \tag{4.29}$$

or, equivalently, given that $\operatorname{curl}_2 \boldsymbol{\varphi} = -\frac{1}{2}\triangle_2 w$,

$$2G \operatorname{curl}_2 \boldsymbol{\varphi} - \tau_{31,1} - \tau_{32,2} = 0. \tag{4.30}$$

We let the vectors

$$\boldsymbol{f_1}^T = [0, -G], \quad \boldsymbol{f_2}^T = [G, 0],$$

alongside the matrices (4.7,4.10)

$$\hat{\mathbf{U}} = G\mathbf{1}_2, \qquad \hat{\mathbf{Q}} = 2G\ell^2\begin{bmatrix} 1+\eta & 0 \\ 0 & 1 \end{bmatrix},$$

$$\hat{\mathbf{R}} = 2G\ell^2\begin{bmatrix} 0 & 0 \\ \eta & 0 \end{bmatrix}, \qquad \hat{\mathbf{T}} = 2G\ell^2\begin{bmatrix} 1 & 0 \\ 0 & 1+\eta \end{bmatrix},$$

where $\mathbf{1}_2$ is the rank 2 identity matrix. It easily follows that

$$\hat{\mathbf{N}}_1 = -\begin{bmatrix} 0 & \eta \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = -2G\ell^2\begin{bmatrix} 1+\eta & 0 \\ 0 & 1-\eta^2 \end{bmatrix}.$$

Eq.(4.20) gives the first order system

$$\varphi_{1,2} = -\eta\varphi_{2,1} + \frac{1}{2G\ell^2}\Phi_{1,1}, \tag{4.31a}$$

$$\varphi_{2,2} = \frac{1}{2G\ell^2(1+\eta)}\Phi_{2,1} \tag{4.31b}$$

$$\Phi_{1,2} = -2G\ell^2(1+\eta)\varphi_{1,1} + 2\Lambda_1 \tag{4.31c}$$

$$\Phi_{2,2} = -2G\ell^2\left(1-\eta^2\right)\varphi_{2,1} - \eta\Phi_{1,1} + 2\Lambda_2 \tag{4.31d}$$

$$\phi_{,2} = 2G\varphi_2 - \Lambda_{2,1} \tag{4.31e}$$

$$\Lambda_{1,2} = -2G(1+\eta)\varphi_2 + \Lambda_{2,1} + \ell^{-2}\Phi_1 \tag{4.31f}$$

$$\Lambda_{2,2} = \phi_{,1} - 2G\varphi_1. \tag{4.31g}$$

Cross-differentiating and adding Eqs.(4.31*e*) and (4.31*g*) shows that ϕ is harmonic inasmuch as (2.2) holds, which result is in line with (4.21). Consequently, letting the harmonic conjugate function ϕ^*, upon recalling that $\phi_{,2} = -\phi^*_{,1}$, we get, from (4.31*e*),

$$\phi^* = Gw + \Lambda_2 = \int^{x_1} s_{31}\,\mathrm{d}\xi_1,$$

which gives to the harmonic conjugate function the role of the stress function for s_{31}. Eqs.(4.31*a*,4.31*b*) correspond to Eqs.(4.27*c*) and (4.27*a*), respectively. Eqs.(4.31*c*,4.31*d*) represent rotational equilibrium (2.13*b*), provided that we use (4.31*a*) to eliminate $\Phi_{1,1}$. Similarly, in light of (4.25) and of (2.14), Eq.(4.31*e*) lends translational equilibrium (4.18). Eq.(4.31*g*) amounts to (4.14), while (4.31*f*) is (4.30), having differentiated and used (4.31*a*) to eliminate $\ell^{-2}\Phi_{1,1}$.

5. Conclusions

We derived the Hamiltonian formalism associated with the indeterminate couple stress theory of elasticity for general anisotropic media. The Hamiltonian framework is known to lead to the celebrated Stroh formalism in classical elasticity. This canonical rewriting of the governing equations is of great theoretical and practical value, because it lends fundamental existence and uniqueness results, as well as providing a powerful tool for solving problems in generally anisotropic media. For such reasons, we extend the formalism to the couple stress theory. This is a strain gradient theory that incorporates microstructural effects in a fashion similar to lattice elasticity [16]. We show that, unlike classical and constrained elasticity, the theory does not allow for a standard Stroh formalism, owing to the nature of the internal constraint on the micro-rotation vector. Indeed, the constraint is *algebraic* and it cannot be eliminated. The resulting canonical formulation is a differential algebraic system of equations (DAE), which may be rewritten in purely differential terms by developing suitable evolution equations. However, the simple structure of classical elasticity cannot be reproduced.

The developed canonical system is then specialized to the case of plane and antiplane strain for couple stress anisotropic media. The antiplane framework is especially noteworthy because it admits a Lagrangian formulation that exactly matches that of flexural/extensional Kirchhoff anisotropic plates, which are amenable to a Stroh formalism. Nonetheless, the corresponding canonical system in couple stress elasticity still lacks the features of a classical Stroh formulation, because the term corresponding to the normal force in the plate is not determined constitutively, owing to the presence of tangential stresses. This notwithstanding, the Hamiltonian formalism still provides a wealth of informations, including unexpected connections which are not apparent from the standard treatment.

Competing Interests. The authors declare that they have no competing interests

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Table of symbols

Symbol	Description	Symbol	Description
$\boldsymbol{s}$	Cauchy stress tensor	$\boldsymbol{\mu}$	Couple-stress tensor
$\boldsymbol{\epsilon}$	Strain tensor	$\boldsymbol{\chi}$	Curvature tensor
$\boldsymbol{\sigma}$	Sym part of the stress tensor	$\boldsymbol{\tau}$	Skew-sym part of the stress tensor
$\boldsymbol{u}$	Displacement field	$\boldsymbol{\varphi}$	Micro-rotation field
L	Lagrangian density	H	Hamiltonian density
$\mathbf{N}$	Stroh matrix	$\mathbf{U}$	Microstructure matrix (symmetric)
$\mathbf{Q}, \mathbf{T}$	Diagonal blocks in $\mathbf{N}$ (symmetric)	$\mathbf{R}, \mathbf{R}^T$	Off-diagonal blocks in $\mathbf{N}$
e_{ijk}	Rank 3 permutation tensor	δ_{ij}	Kroneker delta tensor
$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$	Orthonormal basis vectors	Λ, G, ℓ, η	Constitutive parameters

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