COMPLETE SINGULAR COLLINEATIONS AND QUADRICS

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Abstract. We construct wonderful compactifications of the spaces of linear maps, and symmetric linear maps of a given rank as blow-ups of secant varieties of Segre and Veronese varieties. Furthermore, we investigate their birational geometry and their relations with some spaces of degree two stable maps.

Contents

1. Introduction

We construct the wonderful compactification of the space of linear maps of rank h, between two vector spaces of dimensions $n+1$ and $m+1$, as a sequence of blow-ups of secant varieties of Segre varieties. This generalizes a construction, due to I. Vainsencher, for complete collineations that is maps of maximal rank [Vai84, Theorem 1].

Complete collineations have been widely studied from the algebraic, enumerative and birational viewpoint since the 19th-century [Cha64], [Gia03], [Hir75], [Hir77], [Sch86], [Seg84], [Sem48], [Sem51], [Sem52], [Tyr56], [Vai82], [Vai84], [KT88], [LLT89], [Tha99], [Hue15], [Mas20a], [Mas20b].

Spaces of complete collineations are examples of wonderful compactifications. The *wonderful compactification* of a symmetric space was introduced by C. De Concini and C. Procesi in [DCP83]. Later on, D. Luna gave a more general definition of wonderful variety and then he proved that, according to his definition, all wonderful varieties are spherical [Lun96].

Let $\mathscr{G}$ be a reductive group, and $\mathscr{B}\subset\mathscr{G}$ a Borel subgroup. A *spherical variety* is a variety admitting an action of $\mathscr{G}$ with an open dense $\mathscr{B}$-orbit. For *wonderful varieties* we require in addition the existence of an open orbit whose complementary set is a simple normal crossing divisor, $E_1\cup\dots\cup E_r$, where the E_i are the $\mathscr{G}$-invariant prime divisors in the variety X.

Let $\mathcal{S}^{n,m}$ be the image of the Segre embedding $\mathbb{P}^n\times\mathbb{P}^m\to\mathbb{P}^N$, and $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ the h-secant variety of $\mathcal{S}^{n,m}$, that is the subvariety of $\mathbb{P}^N$ obtained as the closure of the union of all $(h-1)$-planes spanned by h general points of $\mathcal{S}^{n,m}$. We summarize the main results in Theorem 2.14 and Propositions 3.4, 3.10.

Theorem 1.1. *Consider the following sequence of blow-ups*

$$\mathcal{C}(n,m,h):=\mathbb{S}ec_h^{(h-1)}(\mathcal{S}^{n,m})\to\mathbb{S}ec_h^{(h-2)}(\mathcal{S}^{n,m})\to\dots\to\mathbb{S}ec_h^{(1)}(\mathcal{S}^{n,m})\to\mathbb{S}ec_h^{(0)}(\mathcal{S}^{n,m}):=\mathbb{S}ec_h(\mathcal{S}^{n,m})$$

where $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})\to\mathbb{S}ec_h^{(k-1)}(\mathcal{S}^{n,m})$ *is the blow-up of* $\mathbb{S}ec_h^{(k-1)}(\mathcal{S}^{n,m})$ *along the strict transform of* $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ *for* $k=1,\dots,h-1$. *Denote by* $E_k\subset\mathcal{C}(n,m,h)$ *the exceptional divisor over* $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ *for* $k=1,\dots,h-1$.

Date: September 22, 2022.
2020 *Mathematics Subject Classification.* Primary 14M27, 14E30; Secondary 14J45, 14N05, 14E07, 14M27.
Key words and phrases. Wonderful compactifications; Mori dream spaces; Cox rings; Spherical varieties; Stable maps.

The $(SL(n+1)\times SL(m+1))$-action

$$\begin{array}{ccc}(SL(n+1)\times SL(m+1))\times\mathbb{P}^N & \longrightarrow & \mathbb{P}^N\\ ((A,B),Z) & \longmapsto & AZB^t\end{array}$$

induces an $(SL(n+1)\times SL(m+1))$-action on $\mathcal{C}(n,m,h)$, and $\mathcal{C}(n,m,h)$ is wonderful.

Assume that $h<n+1$ and fix homogeneous coordinates $[z_{0,0}:\dots:z_{n,n}]$ on $\mathbb{P}^N$. For $i=1,\dots,h$ we define the divisors $D_i^{\mathcal{C}}$ as the strict transforms in $\mathcal{C}(n,m,h)$ of the divisor given by the intersection of

$$\det\begin{pmatrix} z_{0,0} & \dots & z_{0,i-1}\\ \vdots & \ddots & \vdots\\ z_{i-1,0} & \dots & z_{i-1,i-1}\end{pmatrix}=0$$

with $\mathcal{C}(n,m,h)$. The divisor $D_h^{\mathcal{C}}$ in $\mathcal{C}(n,m,h)$ has two irreducible components $H_1^{\mathcal{C}},H_2^{\mathcal{C}}$, and the Picard rank of $\mathcal{C}(n,m,h)$ is $\rho(\mathcal{C}(n,m,h))=h+1$. Moreover, the effective cone $\mathrm{Eff}(\mathcal{C}(n,m,h))$ is generated by $E_1,\dots,E_{h-1},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$ and the nef cone $\mathrm{Nef}(\mathcal{C}(n,m,h))$ is generated by $D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$.

In the case $h=n+1$ we present similar results. Furthermore, we extend the construction in Theorem 1.1, by replacing $\mathcal{S}^{n,m}$ with the Veronese variety $\mathcal{V}^n$, to the space $\mathcal{Q}(n,h)$ of rank h symmetric complete collineations.

Note that both $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ and $\mathbb{S}ec_h(\mathcal{V}^n)$ are singular, the wonderful varieties $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ are examples of the process producing a wonderful compactification from a conical one in [MP98].

Spherical varieties are Mori dream spaces. Roughly, a *Mori dream space* is a projective variety X whose cone of effective divisors $\mathrm{Eff}(X)$ admits a well-behaved decomposition into convex sets, called Mori chamber decomposition, and these chambers are the nef cones of the birational models of X [HK00].

In Propositions 3.17 and 3.18 we give a detailed description of the Mori chamber decompositions of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ when their Picard rank is at most three. Moreover, in Section 4 we investigate the connection of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ with some Kontsevich spaces of degree two maps.

Kontsevich moduli spaces are denoted by $\overline{M}_{g,n}(X,\beta)$ where X is a projective scheme and $\beta\in H_2(X,\mathbb{Z})$ is the homology class of a curve in X. A point in $\overline{M}_{g,n}(X,\beta)$ corresponds to a holomorphic map α from an n-pointed genus g curve C to X such that $\alpha_*([C])=\beta$. When X is a projective space or a Grassmannians the class β is completely determined by its degree, similarly when X is the product of two projective spaces we identify the class β with its the bidegree. By Propositions 4.1, 4.6, 4.8, 4.12, and Corollary 4.11 we have the following:

Theorem 1.2. *There are isomorphisms*

$$\mathcal{C}(n,m,2)\xrightarrow{\sim}\overline{M}_{0,0}(\mathbb{P}^n\times\mathbb{P}^m,(1,1))$$

and

$$\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)\xrightarrow{\sim}\overline{M}_{0,0}(\mathbb{P}^n,2).$$

Furthermore, there is a 2-to-1 morphism

$$\overline{M}_{0,0}(\mathbb{G}(1,n),2)\to\mathbb{S}ec_4^{(2)}(\mathcal{V}^n).$$

For the automorphism groups we have that

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n\times\mathbb{P}^m,(1,1)))\cong\begin{cases} PGL(n+1)\times PGL(m+1) & \text{if } n<m;\\ S_2\ltimes(PGL(n+1)\times PGL(n+1)) & \text{if } n=m\geqslant 2;\end{cases}$$

and $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1)))\cong PGL(4)$.

Furthermore, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n,2))\cong PGL(n+1)$ *for* $n\geqslant 3$, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^2,2))\cong PGL(3)\rtimes S_2$, *and* $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1,2))\cong PGL(3)$.

Finally,

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))\cong\begin{cases} S_2\ltimes PGL(n+1) & \text{if } n>3;\\ S_2\ltimes(S_2\ltimes PGL(n+1)) & \text{if } n=3.\end{cases}$$

The Mori theory of the spaces $\overline{M}_{0,n}(X,\beta)$, especially when the target variety is a projective space or a Grassmannian, has been widely investigated in a series of papers [CS06], [Che08], [CHS08], [CHS09], [CC10], [CC11], [CM17]. As an application of Theorem 1.2 we recover some of these results in Propositions 4.2, 4.6, and Remark 4.4. In particular, Theorem 1.2 gives an explicit description of the birational contraction of $\overline{M}_{0,0}(\mathbb{P}^n,2)$ in [CHS09, Theorem 1.2] as the blow-down $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)\to\mathbb{S}ec_3(\mathcal{V}^n)$.

Organization of the paper. Throughout the paper we work over an algebraically closed field K of characteristic zero. In Section 2, we construct the spaces of complete singular collineations and quadrics, $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$. In Section 3, we study their Picard rank, their effective and nef cones, and compute the Mori chamber decomposition of $\mathcal{C}(n,m,2)$ and $\mathcal{Q}(n,3)$. Finally, in Section 4 we investigate the relation of the space of complete singular collineations and quadrics with Kontsevich moduli spaces of conics.

Acknowledgments. The first and the third named authors are members of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of the Istituto Nazionale di Alta Matematica "F. Severi" (GNSAGA-INDAM). The first named author is supported by Fondo PRIN-MIUR "Moduli Theory and Birational Classification" 2017. We thank the referee for many helpful comments that allowed us to improve the paper.

2. Complete rank h collineations

Let V,W be K-vector spaces of dimension respectively $n+1$ and $m+1$ with $n\leqslant m$, and let $\mathbb{P}^N$ with $N=nm+n+m$ be the projective space parametrizing collineations from V to W that is non-zero linear maps $V\to W$ up to a scalar multiple.

The line bundle $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^m}(1,1)=\mathcal{O}_{\mathbb{P}(V)}(1)\boxtimes\mathcal{O}_{\mathbb{P}(W)}(1)$ induces an embedding

$$\begin{array}{cccc}\sigma: & \mathbb{P}(V)\times\mathbb{P}(W) & \longrightarrow & \mathbb{P}(V\otimes W)=\mathbb{P}^N\\ & ([u],[v]) & \longmapsto & [u\otimes v].\end{array}$$

The image $\mathcal{S}^{n,m}=\sigma(\mathbb{P}^n\times\mathbb{P}^m)\subset\mathbb{P}^N$ is the *Segre variety*. Let $[x_0,\dots,x_n],[y_0,\dots,y_m]$ be homogeneous coordinates respectively on $\mathbb{P}^n$ and $\mathbb{P}^m$. Then the morphism σ can be written as

$$\sigma([x_0,\dots,x_n],[y_0,\dots,y_m])=[x_0y_0:\dots:x_0y_m:x_1y_0:\dots:x_ny_m].$$

We will denote by $[z_{0,0}:\dots:z_{n,m}]$ the homogeneous coordinates on $\mathbb{P}^N$, where $z_{i,j}$ corresponds to the product x_iy_j.

A point $p\in\mathbb{P}^N=\mathbb{P}(\mathrm{Hom}(W,V))$ can be represented by an $(n+1)\times(m+1)$ matrix Z. The Segre variety $\mathcal{S}^{n,m}$ is the locus of rank one matrices. More generally, $p\in\mathbb{S}ec_h(\mathcal{S}^{n,m})$ if and only if Z can be written as a linear combination of h rank one matrices that is if and only if $\mathrm{rank}(Z)\leqslant h$. If $p=[z_{0,0}:\dots:z_{n,m}]$ then we may write

$$Z=\begin{pmatrix} z_{0,0} & \dots & z_{0,m}\\ \vdots & \ddots & \vdots\\ z_{n,0} & \dots & z_{n,m}\end{pmatrix}. \tag{2.1}$$

Therefore, the ideal of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ is generated by the $(h+1)\times(h+1)$ minors of Z.

2.1. Spherical and Wonderful varieties.

Let X be a normal projective $\mathbb{Q}$-factorial variety. We denote by $N^1(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\mathrm{Nef}(X)\subset N^1(X)$ generated by classes of nef divisors.

The stable base locus $\mathbf{B}(D)$ of a $\mathbb{Q}$-divisor D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$\mathbf{B}(D)=\bigcap_{s>0}B(sD). \tag{2.2}$$

The *movable cone* of X is the convex cone $\mathrm{Mov}(X)\subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X. The *effective cone* of X is the convex cone $\mathrm{Eff}(X)\subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions

$\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}$. We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

Definition 2.3. A *spherical variety* is a normal variety X together with an action of a connected reductive affine algebraic group $\mathscr{G}$, a Borel subgroup $\mathscr{B} \subset \mathscr{G}$, and a base point $x_0 \in X$ such that the $\mathscr{B}$-orbit of x_0 in X is a dense open subset of X.

Let $(X, \mathscr{G}, \mathscr{B}, x_0)$ be a spherical variety. We distinguish two types of $\mathscr{B}$-invariant prime divisors: a *boundary divisor* of X is a $\mathscr{G}$-invariant prime divisor on X, a *color* of X is a $\mathscr{B}$-invariant prime divisor that is not $\mathscr{G}$-invariant. We will denote by $\mathcal{B}(X)$ and $\mathcal{C}(X)$ respectively the set of boundary divisors and colors of X.

Definition 2.4. A *wonderful variety* is a smooth projective variety X with the action of a semi-simple simply connected group $\mathscr{G}$ such that:

- there is a point $x_0 \in X$ with open $\mathscr{G}$ orbit and such that the complement $X \backslash \mathscr{G} \cdot x_0$ is a union of prime divisors $E_1, \cdots, E_r$ having simple normal crossing;
- the closures of the $\mathscr{G}$-orbits in X are the intersections $\bigcap_{i \in I} E_i$ where I is a subset of $\{1, \dots, r\}$.

As proven by D. Luna in [Lun96] wonderful varieties are in particular spherical.

2.4. **Complete singular forms.** For $n = m$, let $\mathbb{P}^{N_+} \subset \mathbb{P}^N$ be the subspace of symmetric matrices. Then $\mathbb{S}ec_h(\mathcal{S}^{n,m}) \cap \mathbb{P}^{N_+} = \mathbb{S}ec_h(\mathcal{V})$ for any $h \geqslant 1$, where $\mathcal{V}^n \subset \mathbb{P}^{N_+}$ is the image of the degree two Veronese embedding of $\mathbb{P}^n$.

Definition 2.5. The space of *complete rank h collineations* is the variety $\mathcal{C}(n, m, h)$ obtained by blowing-up $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ along the strict transforms of the secant varieties $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ for $k < h$ in order of increasing dimension. When $n = m$ we will denote $\mathcal{C}(n, n, h)$ simply by $\mathcal{C}(n, h)$. Furthermore, we will denote by $E_1, \dots, E_{h-1}$ the exceptional divisors.

Similarly, for $n = m$ the space of *complete rank h quadrics* is the variety $\mathcal{Q}(n, h)$ obtained by blowing-up $\mathbb{S}ec_h(\mathcal{V}^n)$ along the strict transforms of the secant varieties $\mathbb{S}ec_k(\mathcal{V}^n)$ for $k < h$ in order of increasing dimension. We will denote by $E_1^{\mathcal{Q}}, \dots, E_{h-1}^{\mathcal{Q}}$ its exceptional divisors.

Remark 2.6. The case $\mathcal{C}(n, m, n+1)$ and $\mathcal{Q}(n, n+1)$ are respectively the space of complete collineations from V to W and the space of complete quadrics of V. By [Vai84, Theorem 1] and [Vai82, Theorem 6.3] they are wonderful varieties and their birational geometry has been studied in [Mas20a].

Notation 2.7. For $k \leqslant h$, we will denote by $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ the blow-up of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ along the strict transforms of the secant varieties $\mathbb{S}ec_i(\mathcal{S}^{n,m})$ for $i = 1, \dots, k$, and by $\mathbb{S}ec_h^{(k)}(\mathcal{V}^n)$ the blow-up of $\mathbb{S}ec_h(\mathcal{V}^n)$ along the strict transforms of the secant varieties $\mathbb{S}ec_i(\mathcal{V}^n)$ for $i = 1, \dots, k$.

Note that there is an embedding

$$i : \mathcal{Q}(n, h) \hookrightarrow \mathcal{C}(n, h). \tag{2.8}$$

The following $(SL(n+1) \times SL(m+1))$-action

$$\begin{array}{ccc} (SL(n+1) \times SL(m+1)) \times \mathbb{P}^N & \longrightarrow & \mathbb{P}^N \\ ((A, B), Z) & \longmapsto & AZB^t \end{array} \tag{2.9}$$

induces an $(SL(n+1) \times SL(m+1))$-action on $\mathcal{C}(n, m, h)$. Similarly, when $n = m$ the $SL(n+1)$-action

$$\begin{array}{ccc} SL(n+1) \times \mathbb{P}^{N_+} & \longrightarrow & \mathbb{P}^{N_+} \\ (A, Z) & \longmapsto & AZA^t \end{array} \tag{2.10}$$

induces an $SL(n+1)$-action on $\mathcal{Q}(n, h)$.

Remark 2.11. Since $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ can be identified with the variety of $(n+1) \times (m+1)$ matrices modulo scalar of rank at most h, [Har95, Example 12.1], [HT84, Proposition 12(a)] give

$$\dim(\mathbb{S}ec_h(\mathcal{S}^{n,m})) = h(m+n+2-h) - 1, \quad \deg(\mathbb{S}ec_h(\mathcal{S}^{n,m})) = \prod_{i=0}^{n-h} \frac{\binom{m+1+i}{n-i}}{\binom{m+1-h+i}{n-h-i}}.$$

Similarly, $\mathbb{S}ec_h(\mathcal{V}^n)$ identifies with the variety parametrizing $(n+1)\times(n+1)$ symmetric matrices modulo scalar of rank at most h and

$$\dim(\mathbb{S}ec_h(\mathcal{V}^n)) = \frac{2nh-h^2+3h-2}{2},\quad \deg(\mathbb{S}ec_h(\mathcal{V}^n)) = \prod_{i=0}^{n-h}\frac{\binom{n+1+i}{n+1-h-i}}{\binom{2i+1}{i}}.$$

Proposition 2.12. *The tangent cone of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ at a point $p\in\mathbb{S}ec_k(\mathcal{S}^{n,m})\backslash\mathbb{S}ec_{k-1}(\mathcal{S}^{n,m})$ for $k\leqslant h$ is a cone with vertex of dimension $nm+n+m-(m+1-k)(n+1-k)$ over $\mathbb{S}ec_{h-k}(\mathcal{S}^{n-k,m-k})$.*

The tangent cone of $\mathbb{S}ec_h(\mathcal{V}^n)$ at a point $p\in\mathbb{S}ec_k(\mathcal{V}^n)\backslash\mathbb{S}ec_{k-1}(\mathcal{V}^n)$ for $k\leqslant h$ is a cone with vertex of dimension $\binom{n+2}{2}-1-\frac{(n-k+1)(n-k+2)}{2}$ over $\mathbb{S}ec_{h-k}(\mathcal{V}^{n-k})$.

Proof. We compute the tangent cones of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$. The symmetric case can be worked out similarly. It is enough to compute the tangent cone of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ at

$$p_k = \begin{pmatrix} I_{k,k} & 0_{k,m+1-k}\\ 0_{n+1-k,k} & 0_{n+1-k,m+1-k}\end{pmatrix}$$

where $I_{k,k}$ is the $k\times k$ identity matrix. Consider the affine chart $z_{0,0}\neq 0$ and the change of coordinates $z_{i,i}\mapsto z_{i,i}-1$ for $i=1,\dots,k-1$, $z_{i,j}\mapsto z_{i,j}$ otherwise. Then the matrix Z in (2.1) takes the following form

$$\begin{pmatrix}
1 & z_{0,1} & \dots & z_{0,k-1} & z_{0,k} & \dots & z_{0,m}\\
z_{1,0} & z_{1,1}-1 & \dots & z_{1,k-1} & z_{1,k} & \dots & z_{1,m}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
z_{k-1,0} & z_{k-1,1} & \dots & z_{k-1,k-1}-1 & z_{k-1,k} & \dots & z_{k-1,m}\\
z_{k,0} & z_{k,1} & \dots & z_{k,k-1} & z_{k,k} & \dots & z_{k,m}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
z_{n,0} & z_{n,1} & \dots & z_{n,k-1} & z_{n,k} & \dots & z_{n,m}
\end{pmatrix}.$$

Recall that $\mathbb{S}ec_h(\mathcal{S}^{n,m})\subseteq\mathbb{P}^N$ is cut out by the $(h+1)\times(h+1)$ minors of Z. Now, the lowest degree terms of these minors are given by the $(h+1-k)\times(h+1-k)$ minors of the following matrix

$$\begin{pmatrix} z_{k,k} & \dots & z_{k,m}\\ \vdots & \ddots & \vdots\\ z_{n,k} & \dots & z_{n,m}\end{pmatrix}.$$

Therefore, the tangent cone $TC_{p_k}\mathbb{S}ec_h(\mathcal{S}^{n,m})$ is contained in the cone C over $\mathbb{S}ec_{h-k}(\mathcal{S}^{n-k,m-k})$ with vertex the linear subspace of $\mathbb{P}^N$ given by $\{z_{k,k}=\dots=z_{k,m}=z_{k+1,k}=\dots=z_{k+1,m}=\dots=z_{n,k}=\dots=z_{n,m}=0\}$. Finally, by Remark 2.11 we conclude that $TC_{p_k}\mathbb{S}ec_h(\mathcal{S}^{n,m})=C$. □

We will need the following result on fibrations with smooth fibers on a smooth base.

Proposition 2.13. *Let $f:X\to Y$ be a surjective morphism of varieties over an algebraically closed field with equidimensional smooth fibers. If Y is smooth then X is smooth as well.*

Proof. By [Sch10, Theorem 3.3.27] the morphism $f:X\to Y$ is flat. Finally, since all the fibers of $f:X\to Y$ are smooth and of the same dimension [Mum99, Theorem 3', Chapter III, Section 10] yields that X is smooth. □

Theorem 2.14. *The variety $\mathcal{C}(n,m,h)$ is smooth and the divisors $E_1,\dots,E_{h-1}$ are smooth and intersect transversally. The closures of the orbits of the $SL(n+1)\times SL(m+1)$-action on $\mathcal{C}(n,m,h)$ induced by (2.9) are given by all the possible intersections of $E_1,\dots,E_{h-1}$ and $\mathcal{C}(n,m,h)$. Furthermore, the analogous statements hold for $\mathcal{Q}(n,h)$. Hence $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ are wonderful.*

Proof. We will proceed as follows. For $h=1$ we will prove the statement for any n and m. Then we will prove that if for $h<j$ the statement holds for any n and m then it also holds for $h=j$ and any n and m. This will prove the statement for any n,m and $h=0,\dots,n+1$.

For $h=1$ we have $\mathcal{C}(n,m,1)=\mathcal{S}^{n,m}$. Hence, the statements holds for any n and m. Assume that for any $h<j$ the statement holds for any n and m and consider $\mathcal{C}(n,m,j)$.

In order to understand the geometry of our construction it is more useful to focus on a specific case. For instance take $n=m=3$. We have $\mathcal{S}^{3,3}\subset\mathbb{S}ec_2(\mathcal{S}^{3,3})\subset\mathbb{S}ec_3(\mathcal{S}^{3,3})\subset\mathbb{P}^{15}$. Let X_1 be the blow-up of $\mathbb{P}^{15}$ along $\mathcal{S}^{3,3}$ with exceptional divisor $\overline{E}_1$. Then $\overline{E}_1$ is a $\mathbb{P}^8$-bundle over $\mathcal{S}^{3,3}$. The strict transform $\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})$ intersects the fiber $\overline{E}_{1,p}$ of $\overline{E}_1$ over a point $p\in\mathcal{S}^{3,3}$ along the base of the tangent cone of $\mathbb{S}ec_2(\mathcal{S}^{3,3})$ at p which by Proposition 2.12 is $\mathcal{S}^{2,2}$. Similarly, $\mathbb{S}ec_3(\mathcal{S}^{3,3})$ intersects $\overline{E}_{1,p}$ along $\mathbb{S}ec_2(\mathcal{S}^{2,2})$. Hence, the fibers of $E_1\rightarrow\mathcal{S}^{3,3}$ are secant varieties $\mathbb{S}ec_2(\mathcal{S}^{2,2})$. Now, let X_2 be the blow-up of X_1 along $\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})$ with exceptional divisor $\overline{E}_2$. Then $\overline{E}_2\rightarrow\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})$ is a $\mathbb{P}^3$-bundle. Fix a point $p\in\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})\backslash(E_1\cap\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3}))$. By Proposition 2.12 $\mathbb{S}ec_3^{(2)}(\mathcal{S}^{3,3})$ intersects $\overline{E}_{2,p}$ along $\mathcal{S}^{1,1}$. If $p\in\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})\cap E_1$ then the projective tangent cone of $\mathbb{S}ec_3^{(1)}(\mathcal{S}^{3,3})$ at p coincides with the projective tangent cone of $\mathbb{S}ec_3^{(1)}(\mathcal{S}^{3,3})\cap E_{1,p}=\mathbb{S}ec_2(\mathcal{S}^{2,2})$ at $p\in\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})\cap E_{1,p}=\mathcal{S}^{2,2}$ which in turn by Proposition 2.12 is $\mathcal{S}^{1,1}$. Hence, the fibers of $E_2\rightarrow\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})$ are isomorphic to $\mathcal{S}^{1,1}$. Summing up after the two blow-ups the fibers of $E_1\rightarrow\mathcal{S}^{3,3}$ are isomorphic to $\mathcal{C}(2,2,2)$, that is the blow-up of $\mathbb{S}ec_2(\mathcal{S}^{2,2})$ along $\mathcal{S}^{2,2}$, and the fibers of $E_2\rightarrow\mathbb{S}ec_2^{(1)}(\mathcal{S}^{3,3})$ are isomorphic to $\mathcal{C}(1,1,1)$ that is $\mathcal{S}^{1,1}$.

Arguing in the same way we see that for any $i=1,\dots,j-1$, Proposition 2.12 gives a fibration $E_i\rightarrow\mathbb{S}ec_i^{(i-1)}(\mathcal{S}^{n,m})=\mathcal{C}(n,m,i)$ whose fibers are isomorphic to $\mathbb{S}ec_{j-i}^{(j-i-1)}(\mathcal{S}^{n-i,m-i})=\mathcal{C}(n-i,m-i,j-i)$. Then, by the induction hypothesis and Proposition 2.13 the exceptional divisors $E_1,\dots,E_{j-1}$ in $\mathcal{C}(n,m,j)$ are smooth. Moreover, by Proposition 2.12, $\mathcal{C}(n,m,j)$ is smooth away from $E_1,\dots,E_{j-1}$ and for $i=1,\dots,j-1$ there is a fibration $\mathcal{C}(n,m,j)\cap E_i\rightarrow\mathcal{C}(n,m,i)$ whose fibers are isomorphic to $\mathcal{C}(n-i,m-i,j-i)$. Hence, by induction and Proposition 2.13 we get that $\mathcal{C}(n,m,j)\cap E_i$ is smooth and

$$\dim(\mathcal{C}(n,m,j)\cap E_i)=i(n+m-i)-1+(j-i)(n-i+m-i-j+i)-1=\dim(\mathcal{C}(n,m,j))-1.$$

So $\mathcal{C}(n,m,j)$ is smooth and the intersection $\mathcal{C}(n,m,j)\cap E_i$ is transversal for any $i=1,\dots,j-1$.

Now, consider an intersection of the following form $E_{j_1}\cap\dots\cap E_{j_t}$. By Proposition 2.12 the restriction of the blow-down morphism

$$E_{j_1}\cap\dots\cap E_{j_t}\rightarrow E_{j_1}\cap\dots\cap E_{j_{t-1}}\cap\mathcal{C}(n,m,j_t)$$

has fibers isomorphic to $\mathcal{C}(n-j_t,m-j_t,j-j_t)$. Again by the induction hypothesis and Proposition 2.13 $E_{j_1}\cap\dots\cap E_{j_t}$ is smooth of dimension

$$(j-j_t)(n-j+m-j-j+j_t)-1+j_t(n+m-j_t)-1-(t-1)=\dim(\mathcal{C}(n,m,j))-t$$

and hence the intersection is transversal.

The claim about the orbit closures follows from [Vai84, Theorem 1] and the fact that the $SL(n+1)\times SL(m+1)$ action on $\mathcal{C}(n,m,h)$ is given by the restriction of the action (2.9) on the space of complete collineations. With an analogous proof we get the result for $\mathcal{Q}(n,h)$. ☐

3. Divisors on $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$

In the section we study the Picard groups and the cones of effective and nef divisors of the wonderful varieties introduces in Section 2. We will denote by $\mathcal{C}(n,m,h)^o$ and $\mathcal{Q}(n,h)^o$ the orbits of the matrix

$$J_h=\begin{pmatrix} I_{h,h} & 0\\ 0 & 0\end{pmatrix} \tag{3.1}$$

where $I_{h,h}$ is the $h\times h$ identity matrix, under the actions (2.9) and (2.10) respectively.

Proposition 3.2. *The Picard groups of $\mathcal{C}(n,m,h)^o$ and $\mathcal{Q}(n,h)^o$ are given by*

$$\operatorname{Pic}(\mathcal{C}(n,m,h)^o)\cong\begin{cases}\mathbb{Z} & \text{if } h=n+1<m+1;\\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } h<n+1;\\ \frac{\mathbb{Z}}{(n+1)\mathbb{Z}} & \text{if } h=n+1=m+1;\end{cases}$$

and

$$\operatorname{Pic}(\mathcal{Q}(n,h)^o)\cong\begin{cases}\mathbb{Z} & \text{if } h<n+1 \text{ is odd};\\ \frac{\mathbb{Z}}{2\mathbb{Z}}\oplus\mathbb{Z} & \text{if } h<n+1 \text{ is even};\\ \frac{\mathbb{Z}}{(n+1)\mathbb{Z}} & \text{if } h=n+1.\end{cases}$$

Proof. Let G_h be the stabilizer of the matrix J_h in (3.1) under the action (2.9). Since the Picard group and the character group of $SL(n+1)\times SL(m+1)$ are trivial [ADHL15, Theorem 4.5.1.2] yields that $\mathrm{Pic}(\mathcal{C}(n,m,h)^o)$ is isomorphic to the character group $\mathbb{X}(G_h)$ of G_h. Write an element $(A,B)\in SL(n+1)\times SL(m+1)$ as

$$A=\left(\begin{array}{cc} A_{h,h} & A_{h,n+1-h}\\ A_{n+1-h,h} & A_{n+1-h,n+1-h}\end{array}\right),\quad B=\left(\begin{array}{cc} B_{h,h} & B_{h,m+1-h}\\ B_{m+1-h,h} & B_{m+1-h,m+1-h}\end{array}\right). \tag{3.3}$$

Then $(A,B)\in G_h$ if and only if $A_{n+1-h,h}=0$, $B_{m+1-h,h}=0$ and $A_{h,h}B_{h,h}^T=\lambda I_{h,h}$. Assume that $h<n+1$ and $h<m+1$. Then $\mathbb{X}(G_h)$ is generated by the characters

$$d_{A_h}:=\det(A_{h,h}),d_{B_h}:=\det(B_{h,h}),d_{A_{n+1-h}}:=\det(A_{n+1-h,n+1-h}),d_{B_{m+1-h}}:=\det(B_{m+1-h,m+1-h}),\lambda$$

with the following relations

$$d_{A_h}+d_{A_{n+1-h}}=d_{B_h}+d_{B_{m+1-h}}=0,d_{A_h}+d_{B_h}=h\lambda.$$

Hence, $\mathbb{X}(G_h)$ is the free abelian group generated by d_{A_h} and λ.

Now, assume that $h=n+1<m+1$. Then $d_{A_{n+1-h}}=0$ and so $d_{A_h}=0$. Therefore, $\mathbb{X}(G_h)$ is the free abelian group generated by λ.

If $h=n+1=m+1$ then $d_{A_{n+1-h}}=d_{B_{m+1-h}}=0$. So $d_{A_h}=d_{B_h}=0$, and hence $\mathbb{X}(G_h)$ is the abelian group generated by λ with the relation $(n+1)\lambda=0$.

Now, we consider the symmetric case. We will keep denoting by G_h the stabilizer of the matrix J_h in (3.1) under the action (2.10). Write an element $A\in SL(n+1)$ as in (3.3). Then $A\in G_h$ if and only if $A_{n+1-h,h}=0$ and $A_{h,h}A_{h,h}^T=\lambda I_{h,h}$. Therefore, $\mathbb{X}(G_h)$ is generated by

$$d_{A_h}:=\det(A_{h,h}),\lambda$$

with the relation $2d_{A_h}-h\lambda=0$.

Assume $h<n+1$. If $h=2k+1$ then $(2,-h)\in\mathbb{Z}^2$ is primitive. Considering the basis $u=2d_{A_h}-h\lambda, v=d_{A_h}-k\lambda$ of $\mathbb{Z}^2$ we get that $\mathbb{X}(G_h)\cong\mathbb{Z}^2/\langle u\rangle\cong\mathbb{Z}$. If $h=2k$ then $(2,-h)=2(1,-k)$, and considering the basis $u=2d_{A_h}-k\lambda, v=\lambda$ of $\mathbb{Z}^2$ we get that $\mathbb{X}(G_h)\cong\mathbb{Z}^2/\langle 2u\rangle\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$. Finally, if $h=n+1$ we have $d_{A_h}=0$, and hence $(n+1)\lambda=0$. So $\mathbb{X}(G_h)\cong\mathbb{Z}/(n+1)\mathbb{Z}$. □

Proposition 3.4. *The Picard rank of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ is given by*

$$\rho(\mathcal{C}(n,m,h))=\begin{cases} h-1 & \text{if } h=n+1=m+1;\\ h+1 & \text{if } h<n+1;\\ h & \text{if } h=n+1<m+1;\end{cases}$$

and

$$\rho(\mathcal{Q}(n,h))=\begin{cases} h & \text{if } h<n+1;\\ h-1 & \text{if } h=n+1.\end{cases}$$

Proof. Assume that $h<n+1$. Since, by Theorem 2.14 the variety $\mathcal{C}(n,m,h)$ is wonderful with boundary divisors $E_1,\dots,E_{h-1}$, [Bri07, Proposition 2.2.1] yields an exact sequence

$$0\to\mathbb{Z}^{h-1}\to\mathrm{Pic}(\mathcal{C}(n,m,h))\to\mathrm{Pic}(\mathcal{C}(n,m,h)^o)\to 0$$

where $\mathbb{Z}^{h-1}$ is the free abelian group generated by the boundary divisors. To conclude it is enough to use Proposition 3.2. The proof in the symmetric case is similar. □

For $i=1,\dots,h$, we define the divisor $D_i^{\mathcal{C}}$ in $\mathcal{C}(n,m,h)$ as the strict transform of the divisor given by the intersection of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ with

$$\det\begin{pmatrix} z_{0,0} & \dots & z_{0,i-1}\\ \vdots & \ddots & \vdots\\ z_{i-1,0} & \dots & z_{i-1,i-1}\end{pmatrix}=0.$$

We will keep the same notation for the corresponding divisors in the intermediate blow-ups $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$.

Similarly, for $i = 1, \dots, h$ we define the divisor $D_i^{\mathcal{Q}}$ in $\mathcal{Q}(n,h)$ as the strict transform of the divisor given by the intersection of $\mathbb{S}ec_h(\mathcal{V}^n)$ with

$$\det\begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{0,i-1} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0.$$

Again we will keep the same notation for the corresponding divisors in the intermediate blow-ups $\mathbb{S}ec_h^{(k)}(\mathcal{V}^n)$.

Lemma 3.5. *Let Z be an $(n+1)\times(m+1)$ matrix of rank $k < \min\{n+1, m+1\}$ such that the determinant of the top left $k \times k$ minor Z_k of Z vanishes. Then, either the first k rows of Z are linearly dependent or the the first k columns of Z are linearly dependent.*

Proof. Assume that both the first k rows and the first k columns of Z are linearly independent. We will then prove that either $\det(Z_k) \neq 0$ or $\operatorname{rank}(Z) > k$. If $\det(Z_k) \neq 0$ the claim follows. So, assume $\det(Z_k) = 0$. We will write $e_1, \dots, e_{m+1}$ for the canonical basis of K^{m+1} and $\bar{e}_1, \dots \bar{e}_{n+1}$ for the canonical basis of K^{n+1}. Since the first k columns of Z are linearly independent, up to a change of coordinates, we may assume that these columns are the vectors $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{k-1}, \bar{e}_{k+1}$. The first $k+1$ rows of the matrix Z are of the following form

$$\begin{cases} Z_{0,-} & = e_1^t + a_{k+1}^0 e_{k+1}^t + \cdots + a_{m+1}^0 e_{m+1}^t; \\ Z_{1,-} & = e_2^t + a_{k+1}^1 e_{k+1}^t + \cdots + a_{m+1}^1 e_{m+1}^t; \\ & \vdots \\ Z_{k-2,-} & = e_{k-1}^t + a_{k+1}^{k-2} e_{k+1}^t + \cdots + a_{m+1}^{k-2} e_{m+1}^t; \\ Z_{k-1,-} & = a_{k+1}^{k-1} e_{k+1}^t + \cdots + a_{m+1}^{k-1} e_{m+1}^t; \\ Z_{k,-} & = e_k^t + a_{k+1}^k e_{k+1}^t + \cdots + a_{m+1}^k e_{m+1}^t; \end{cases}$$

for some $a_i^j \in K$. By assumption, the first k rows are linearly independent and so we must have $a_i^{k-1} \neq 0$ for at least one $i \in \{k+1, \dots m+1\}$. Hence, the $k+1$ rows $Z_{0,-}, \dots, Z_{k,-}$ are linearly independent, and $\operatorname{rank}(Z) \geqslant k+1$. $\square$

Corollary 3.6. *For $k < \min\{n+1, m+1\}$, the divisor cut out on $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ by the top left $k \times k$ minor of the matrix in (2.1) has two components H_1 and H_2, where H_1 is cut out by the $k \times k$ minors of the first k rows of Z, and H_2 is cut out by the $k \times k$ minors of the first k columns of Z.*

Proof. The claim follows immediately from Lemma 3.5. $\square$

Remark 3.7. In $\mathbb{S}ec_k(\mathcal{V}^n)$ the divisor associated to D_k is irreducible. Indeed, in the symmetric case the divisors H_1, H_2 in Corollary 3.6 coincide.

In order to further clarify this we explicitly work out the case of 3×3 matrices. The hypersurface $D_2 = \{z_{0,0}z_{1,1} - z_{0,1}z_{1,0} = 0\}$ cuts out on $\mathbb{S}ec_2(\mathcal{S}^{2,2}) \subset \mathbb{P}^8$ a divisor with two irreducible components:

$$H_1 = \{z_{0,1}z_{1,0} - z_{0,0}z_{1,1} = z_{0,2}z_{1,1} - z_{0,1}z_{1,2} = z_{0,2}z_{1,0} - z_{0,0}z_{1,2} = 0\};$$

$$H_2 = \{z_{0,1}z_{1,0} - z_{0,0}z_{1,1} = z_{0,1}z_{2,0} - z_{0,0}z_{2,1} = z_{1,1}z_{2,0} - z_{1,0}z_{2,1} = 0\}.$$

In the symmetric case the divisor $\{z_{0,0}z_{1,1} - z_{0,1}^2 = 0\}$ cuts out on $\mathbb{S}ec_2(\mathcal{V}^2) \subset \mathbb{P}^5$ the irreducible divisor

$$\{z_{0,2}z_{1,1} - z_{0,1}z_{1,2} = z_{0,1}z_{0,2} - z_{0,0}z_{1,2} = z_{0,1}^2 - z_{0,0}z_{1,1} = 0\}$$

with multiplicity two.

Notation 3.8. We will denote by $H_1^{\mathcal{C}}, H_2^{\mathcal{C}}$ the strict transforms of H_1, H_2 in $\mathcal{C}(n,m,h)$.

Proposition 3.9. *The set of colors of $\mathcal{C}(n,m,h)$ is given by*

$$\begin{array}{ll} \{D_1^{\mathcal{C}}, \dots, D_n^{\mathcal{C}}\} & \textit{if } h = n+1 = m+1; \\ \{H_1^{\mathcal{C}}, H_2^{\mathcal{C}}, D_1^{\mathcal{C}}, \dots, D_{h-1}^{\mathcal{C}}\} & \textit{if } h < n+1; \\ \{D_1^{\mathcal{C}}, \dots, D_{n+1}^{\mathcal{C}}\} & \textit{if } h = n+1 < m+1; \end{array}$$

while for $\mathcal{Q}(n,h)$ the set of colors is given by

$$\begin{array}{ll} \{D_1^{\mathcal{Q}},\dots,D_h^{\mathcal{Q}}\} & \text{if } h<n+1;\\ \{D_1^{\mathcal{Q}},\dots,D_n^{\mathcal{Q}}\} & \text{if } h=n+1. \end{array}$$

Proof. The claim for $\mathcal{C}(n,m)$ and $\mathcal{Q}(n)$ follows from [Mas20a, Proposition 3.6]. In particular, the divisors listed in the statement are stabilized by the action of the Borel subgroups in (2.9) and (2.10) respectively. Moreover, $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ and $\mathbb{S}ec_h(\mathcal{V}^n)$ are stabilized respectively by the action (2.9) and (2.10). Then, $D_1^{\mathcal{C}},\dots,D_h^{\mathcal{C}}$ are stabilized by the restriction of the action (2.9), and similarly the strict transform in $\mathcal{Q}(n,h)$ of $D_1^{\mathcal{Q}},\dots,D_h^{\mathcal{Q}}$ are stabilized by the restriction of the action (2.10).

The groups acting are connected, so any reducible divisor which is stabilized must be stabilized component wise. In particular, since by Corollary 3.6 in $\mathcal{C}(n,m,h)$ for $h<n+1$ we have $D_h^{\mathcal{C}}=H_1^{\mathcal{C}}\cup H_2^{\mathcal{C}}$ and since $D_h^{\mathcal{C}}$ is stabilized, we have that both $H_1^{\mathcal{C}}$ and $H_2^{\mathcal{C}}$ are stabilized.

As noticed in [ADHL15, Remark 4.5.5.3], if $(X,\mathscr{G},\mathscr{B},x_0)$ is a spherical wonderful variety with colors $D_1,\dots,D_s$ the big cell $X\backslash(D_1\cup\cdots\cup D_s)$ is an affine space. Therefore, it admits only constant invertible global functions and $\mathrm{Pic}(X)=\mathbb{Z}[D_1,\dots,D_s]$.

Now, for $h<n+1$ in $\mathcal{C}(n,m,h)$ we have $h+1$ colors and since by Proposition 3.4 the Picard rank of $\mathcal{C}(n,m,h)$ is $h+1$, these divisors $D_1,\dots,D_{h-1},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$ must be all the colors. Similarly, for $h=n+1<m+1$ we found the divisors $D_1^{\mathcal{C}},\dots,D_{n+1}^{\mathcal{C}}$, and since in this case $\rho(\mathcal{C}(n,m,h))=h$, they are all the colors. Note that when $h=n+1=m+1$, the divisor $D_{n+1}^{\mathcal{C}}$ is not a color, since it is stabilized by the whole group. In this case $\rho(\mathcal{C}(n,m,h))=h-1$ and then $D_1^{\mathcal{C}},\dots,D_n^{\mathcal{C}}$ are the colors. With a similar argument we can compute the colors of $\mathcal{Q}(n,h)$. □

Proposition 3.10. *For the effective and the nef cone of $\mathcal{C}(n,m,h)$ we have*

$$\mathrm{Eff}(\mathcal{C}(n,m,h))=\begin{cases} \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}}\rangle & \text{if } h=n+1=m+1;\\ \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}\rangle & \text{if } h<n+1;\\ \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},D_{n+1}^{\mathcal{C}}\rangle & \text{if } h=n+1<m+1; \end{cases}$$

$$\mathrm{Nef}(\mathcal{C}(n,m,h))=\begin{cases} \langle D_1^{\mathcal{C}},\dots,D_n^{\mathcal{C}}\rangle & \text{if } h=n+1=m+1;\\ \langle D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}\rangle & \text{if } h<n+1;\\ \langle D_1^{\mathcal{C}},\dots,D_{n+1}^{\mathcal{C}}\rangle & \text{if } h=n+1<m+1; \end{cases}$$

and for the effective and the nef cone of $\mathcal{Q}(n,h)$ we have

$$\mathrm{Eff}(\mathcal{Q}(n,h))=\begin{cases} \langle E_1^{\mathcal{Q}},\dots,E_{h-1}^{\mathcal{Q}},D_h^{\mathcal{Q}}\rangle & \text{if } h<n+1;\\ \langle E_1^{\mathcal{Q}},\dots,E_{h-1}^{\mathcal{Q}}\rangle & \text{if } h=n+1; \end{cases}$$

$$\mathrm{Nef}(\mathcal{Q}(n,h))=\begin{cases} \langle D_1^{\mathcal{Q}},\dots,D_h^{\mathcal{Q}}\rangle & \text{if } h<n+1;\\ \langle D_1^{\mathcal{Q}},\dots,D_n^{\mathcal{Q}}\rangle & \text{if } h=n+1. \end{cases}$$

Proof. The statement for $\mathcal{C}(n,m)$ and $\mathcal{Q}(n)$ follows from [Mas20a, Theorem 3.13]. We consider now the case $h<n+1$.

Consider $\mathcal{C}(n,m,h)$. By [ADHL15, Proposition 4.5.4.4], Theorem 2.14 and Proposition 3.10 the effective cone of $\mathcal{C}(n,m,h)$ is generated by $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$. By [Mas20a, Section 5] the divisor $D_i^{\mathcal{C}}$ induces a birational morphism that contracts the exceptional divisor $E_i^{\mathcal{C}}$. Therefore $D_i^{\mathcal{C}}$ lies in the interior of the effective cone for any $i=1,\dots,h-1$. In particular, since by Proposition 3.4 $\rho(\mathcal{C}(n,m,h))=h+1$, we conclude that the extremal rays of the effective cone are $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$.

Furthermore, by [Bri89, Section 2.6] the nef cone is generated by $D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$. A similar argument gives the generators for the effective and nef cone of $\mathcal{Q}(n,h)$. □

3.10. **Birational geometry of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$.** Let X be a normal $\mathbb{Q}$-factorial variety. We say that a birational map $f:X\dashrightarrow X'$ to a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small $\mathbb{Q}$-factorial modification* if X' is $\mathbb{Q}$-factorial and f is an isomorphism in codimension one. If $f:X\dashrightarrow X'$ is a small $\mathbb{Q}$-factorial modification then the natural pullback map $f^*:N^1(X')\to N^1(X)$ sends $\mathrm{Mov}(X')$ and $\mathrm{Eff}(X')$ isomorphically onto $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$ respectively. In particular, we have $f^*(\mathrm{Nef}(X'))\subset\overline{\mathrm{Mov}(X)}$.

Now, assume that the divisor class group $\mathrm{Cl}(X)$ is free and finitely generated, and fix a subgroup G of the group of Weil divisors on X such that the canonical map $G \to \mathrm{Cl}(X)$, mapping a divisor $D \in G$ to its class $[D]$, is an isomorphism. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where $D \in G$ represents $[D] \in \mathrm{Cl}(X)$, and the multiplication in $\mathrm{Cox}(X)$ is defined by the standard multiplication of homogeneous sections in the field of rational functions on X.

Definition 3.11. A normal projective $\mathbb{Q}$-factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\mathrm{Pic}\,(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$,
- $\mathrm{Nef}\,(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and $\mathrm{Mov}\,(X) = \bigcup_i f_i^*(\mathrm{Nef}\,(X_i))$.

The collection of all faces of all cones $f_i^*(\mathrm{Nef}\,(X_i))$ above forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}\,(X_i))$ and $f_j^*(\mathrm{Nef}\,(X_j))$, meet along a facet, then there exist a normal projective variety Y, a small modification $\varphi : X_i \dashrightarrow X_j$, and $h_i : X_i \to Y$, $h_j : X_j \to Y$ small birational morphisms of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 3.12. Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form $\mathcal{C}_i = \langle g_i^*\big(\mathrm{Nef}(Y_i)\big), \mathrm{Exc}(g_i)\rangle$. We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^{\circ}$ a *maximal chamber* of $\mathrm{Eff}(X)$. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details.

Remark 3.13. By the work of M. Brion [Bri93] we have that $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [Per14, Section 4]. In particular, by Theorem 2.14 $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ are Mori dream spaces.

Remark 3.14. Recall that by [HK00, Proposition 2.11] given a Mori Dream Space X there is an embedding $i : X \to \mathcal{T}_X$ into a simplicial projective toric variety $\mathcal{T}_X$ such that $i^* : \mathrm{Pic}(\mathcal{T}_X) \to \mathrm{Pic}(X)$ is an isomorphism inducing an isomorphism $\mathrm{Eff}(\mathcal{T}_X) \to \mathrm{Eff}(X)$. Furthermore, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{T}_X)$ is a refinement of the Mori chamber decomposition of $\mathrm{Eff}(X)$. Indeed, if $\mathrm{Cox}(X) \cong \frac{K[T_1,\dots,T_s]}{I}$ where the T_i are homogeneous generators with non-trivial effective $\mathrm{Pic}(X)$-degrees then $\mathrm{Cox}(\mathcal{T}_X) \cong K[T_1,\dots,T_s]$.

Since the variety $\mathcal{T}_X$ is toric, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{T}_X)$ can be computed by means of the Gelfand–Kapranov–Zelevinsky, GKZ for short, decomposition [ADHL15, Section 2.2.2]. Let us consider the family $\mathcal{W}$ of vectors in $\mathrm{Pic}(\mathcal{T}_X)$ given by the generators of $\mathrm{Cox}(\mathcal{T}_X)$, and let $\Omega(\mathcal{W})$ be the set of all convex polyhedral cones generated by some of the vectors in $\mathcal{W}$. By [ADHL15, Construction 2.2.2.1] the GKZ chambers of $\mathrm{Eff}(\mathcal{T}_X)$ are given by the intersections of all the cones in $\Omega(\mathcal{W})$ containing a fixed divisor in $\mathrm{Eff}(\mathcal{T}_X)$.

Remark 3.15. Let $(X, \mathscr{G}, \mathscr{B}, x_0)$ be a projective spherical variety. Consider a divisor D on X, and let f_D be the, unique up to constants, section of $\mathcal{O}_X(D)$ associated to D. We will denote by $\mathrm{lin}_K(\mathscr{G}\cdot D) \subseteq \mathrm{Cox}(X)$ the finite-dimensional vector subspace of $\mathrm{Cox}(X)$ spanned by the orbit of f_D under the action of $\mathscr{G}$ that is the smallest linear subspace of $\mathrm{Cox}(X)$ containing the $\mathscr{G}$-orbit of f_D.

By [ADHL15, Theorem 4.5.4.6] if $\mathscr{G}$ is a semi-simple and simply connected algebraic group and $(X, \mathscr{G}, \mathscr{B}, x_0)$ is a spherical variety with boundary divisors $E_1, \dots, E_r$ and colors $D_1, \dots, D_s$ then $\mathrm{Cox}(X)$ is generated as a K-algebra by the canonical sections of the E_i and the finite dimensional vector subspaces $\mathrm{lin}_K(\mathscr{G} \cdot D_i) \subseteq \mathrm{Cox}(X)$ for $1 \leqslant i \leqslant s$.

Next, we study the birational geometry of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ when the Picard rank is small. We begin with $\mathcal{Q}(n,h)$. The varieties $\mathcal{Q}(1,2)$ and $\mathcal{Q}(2,3)$ are covered by [Mas20a, Section 6]. So, the first case to consider is that of $\mathcal{Q}(n,3)$ for $n \geqslant 3$.

Lemma 3.16. *For the variety $\mathcal{Q}(n,3)$ we have that $D_1^{\mathcal{Q}} \sim H$, $D_2^{\mathcal{Q}} \sim 2H - E_1^{\mathcal{Q}}$, $D_3^{\mathcal{Q}} \sim 3H - 2E_1^{\mathcal{Q}} - E_2^{\mathcal{Q}}$.*

Proof. Consider the strict transform $L \subset \mathcal{Q}(n,3)$ of the line $L_{\mu,\lambda} = \{\mu x_0^2 + \lambda(x_1^2 + x_2^2) = 0\}$. This line intersects $\mathcal{V}^n$ at a point p, $\mathbb{S}ec_2(\mathcal{V}^n)\backslash\mathcal{V}^n$ at a point q, and it is not contained neither in the tangent cone of $\mathbb{S}ec_2(\mathcal{V}^n)$ at p nor in the tangent space of $\overline{H}_2 = \{z_{0,0}z_{1,1} - z_{0,1}^2 = 0\}$ at p.

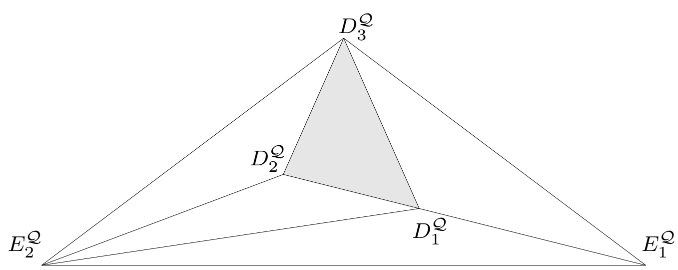
First, consider the blow-up $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ of $\mathbb{S}ec_3(\mathcal{V}^n)$ along $\mathcal{V}^n$ and keep the same notation for the push-forward to $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ of L, $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$. Recall that $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ is singular along the strict transform of $\mathbb{S}ec_2(\mathcal{V}^n)$. However, $D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$ are Cartier on $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ since they are restrictions to $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ of divisors in the blow-up of $\mathbb{P}^{N_+}$ along $\mathcal{V}^n$.

Write $D_2^{\mathcal{Q}} = 2H - aE_1$. Note that $\overline{H}_2$ intersect $L_{\mu,\lambda}$ at p. Since $L_{\mu,\lambda}$ is not contained in the tangent space of $\overline{H}_2$ at p in $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ the strict transforms L and $D_2^{\mathcal{Q}}$ intersect just in one point. Then $1 = D_2^{\mathcal{Q}} \cdot L = 2 - a$ yields $a = 1$.

Similarly, $L_{\mu,\lambda}$ intersects the cubic hypersurface $\overline{H}_3$ given be the top left 3×3 minor of (2.1) at p with multiplicity two and at q. Moreover, $L_{\mu,\lambda}$ is not contained in the tangent cone of $\overline{H}_3$ at p and hence in $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ the strict transforms L and $D_3^{\mathcal{Q}}$ intersect just in one point. Then $1 = D_3^{\mathcal{Q}} \cdot L$. Writing $D_3^{\mathcal{Q}} \sim 3H - bE_1$ we get $1 = D_3^{\mathcal{Q}} \cdot L = 3 - b$ and hence $b = 2$.

Now, we consider $\mathcal{Q}(n,3)$. Since $D_2^{\mathcal{Q}}$ does not contain the strict transform of $\mathbb{S}ec_2(\mathcal{V}^n)$ its expression remains unvaried after the last blow-up. On the other hand, E_2 must appear in the expression of $D_3^{\mathcal{Q}}$. Let us write $D_3^{\mathcal{Q}} \sim 3H - 2E_1 - cE_2$ and keep denoting by L its strict transform in $\mathcal{Q}(n,3)$. Note that $L_{\mu,\lambda}$ is not contained in the tangent space of $\overline{H}_3$ at q. So $0 = D_3^{\mathcal{Q}} \cdot L = 3 - 2 - c$ and hence $c = 1$. $\square$

Proposition 3.17. *For $n \geqslant 3$, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{Q}(n,3))$ has five chambers as displayed in the following 2-dimension section of $\mathrm{Eff}(\mathcal{Q}(n,3))$*



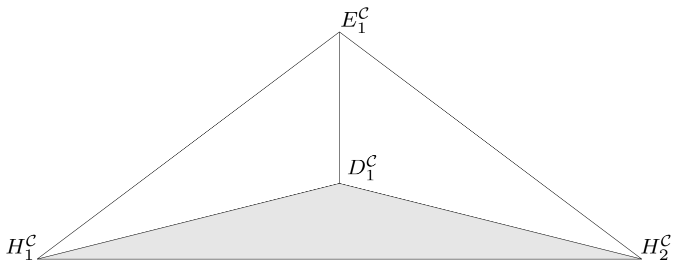
where $\mathrm{Mov}(\mathcal{Q}(n,3))$ coincides with $\mathrm{Nef}(\mathcal{Q}(n,3))$ and is generated by $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$.

Proof. By Theorem 2.14, Proposition 3.9, Remarks 3.14, 3.15, and Lemma 3.16 the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{Q}(n,3))$ is a possibly trivial coarsening of the decomposition in the statement.

Since by Proposition 3.10 $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$ are the generators of the nef cone of $\mathcal{Q}(n,3)$, these rays can not be removed. Furthermore, since Mori chamber are convex the walls between $E_2^{\mathcal{Q}}, D_2^{\mathcal{Q}}$ and $E_1^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ can not be removed. Finally, to see that the wall between $E_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ can not be removed it is enough to observe that the stable base locus of a divisor in the chamber delimited by $E_2^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ is $E_2^{\mathcal{Q}}$, while the stable base locus of a divisor in the chamber delimited by $E_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}, E_1^{\mathcal{Q}}$ is $E_1^{\mathcal{Q}} \cup E_2^{\mathcal{Q}}$. $\square$

We will study the decomposition of the effective cone of $\mathcal{C}(n,m,2)$. For $n = m = 1$ we have $\mathcal{C}(1,1,2) \cong \mathbb{P}^3$. Hence, the first interesting cases occur for $n = 1$ and $m > 1$. The case $\mathcal{C}(1,m,2)$ is in [Mas20a, Page 1606].

Proposition 3.18. *For $n > 1$ and $m > 1$ the Mori chamber decomposition of* $\mathrm{Eff}(\mathcal{C}(n,m,2))$ *has three chambers as displayed in the following* 2*-dimensional section of* $\mathrm{Eff}(\mathcal{C}(n,m,2))$

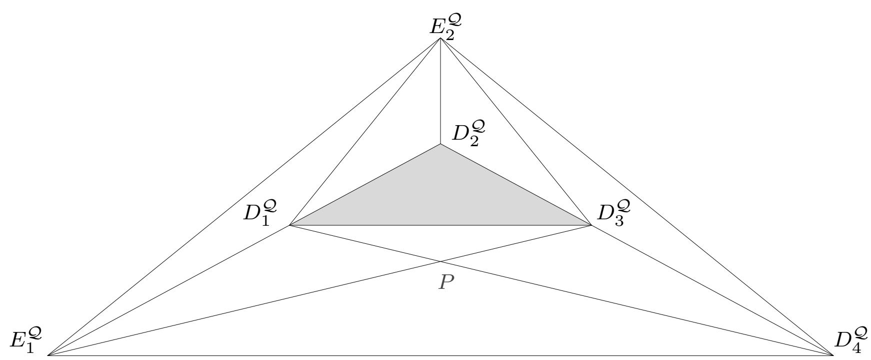


where $\mathrm{Mov}(\mathcal{C}(n,m,2))$ *coincides with* $\mathrm{Nef}(\mathcal{C}(n,m,2))$ *and is generated by* $H_1^{\mathcal{C}}, H_2^{\mathcal{C}}, D_1^{\mathcal{C}}$.

Proof. It is enough to argue as in the proof of Proposition 3.17, and to observe that since Mori chambers are convex in the case $n > 1, m > 1$ the wall between $E_1^{\mathcal{C}}, D_1^{\mathcal{C}}$ can not be removed. $\square$

In the following we consider the spherical variety $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ obtained by blowing-up $\mathbb{S}ec_4(\mathcal{V}^n)$ along $\mathcal{V}^n$ and then along the strict transform of $\mathbb{S}ec_2(\mathcal{V}^n)$. We will keep denoting by $D_i^{\mathcal{Q}}, E_j^{\mathcal{Q}}$ the push-forward of the corresponding divisors via the blow-down $\mathcal{Q}(n,4) \to \mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$.

Proposition 3.19. *The Mori chamber decomposition of* $\mathrm{Eff}(\mathbb{S}ec_4^{(2)}(\mathcal{V}^n))$ *has nine chambers as displayed in the following* 2*-dimensional section of* $\mathrm{Eff}(\mathbb{S}ec_4^{(2)}(\mathcal{V}^n))$



where $\mathrm{Nef}(\mathbb{S}ec_4^{(2)}(\mathcal{V}^n))$ *is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$*, and the movable cone* $\mathrm{Mov}(\mathbb{S}ec_4^{(2)}(\mathcal{V}^n))$ *is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}, P$ *with* $P \sim 6D_1^{\mathcal{Q}} - 3E_1^{\mathcal{Q}} - 2E_2^{\mathcal{Q}}$.

Proof. Note that the $SL(n+1)$-actions on $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ and $\mathcal{Q}(n,4)$ are equivariant with respect to the blow-down morphism $\mathcal{Q}(n,4) \to \mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$. Hence, by Proposition 3.9 the colors of $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ are $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}, D_4^{\mathcal{Q}}$, and its boundary divisors are $E_1^{\mathcal{Q}}, E_2^{\mathcal{Q}}$. Arguing as in the proof of Lemma 3.16 we have that $D_4^{\mathcal{Q}} \sim 4H - 3E_1^{\mathcal{Q}} - 2E_2^{\mathcal{Q}}$. Note that $D_4^{\mathcal{Q}}$ is also a boundary divisor when $n = 3$. Now, the claim on the movable cone follows from Remark 3.15 and [ADHL15, Proposition 3.3.2.3]. Finally, to conclude it is enough to argue as in the proof of Proposition 3.17. $\square$

We conclude this section by computing the automorphism groups of the varieties $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ and $\mathbb{S}ec_h^{(k)}(\mathcal{V}^n)$.

Proposition 3.20. *For all $h \leqslant n$ we have*

$$\mathrm{Aut}(\mathbb{S}ec_h(\mathcal{S}^{n,m})) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m; \end{cases}$$

and $\mathrm{Aut}(\mathbb{S}ec_h(\mathcal{V}^n)) \cong PGL(n+1)$.

Proof. Let ϕ be an automorphism of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$. By the stratification of the singular locus of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ described in Proposition 2.12 ϕ must stabilize $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ for all $k \leqslant h$. In particular, ϕ induces an automorphism $\phi_{|\mathcal{S}^{n,m}} \in \mathrm{Aut}(\mathcal{S}^{n,m})$, and by [Mas20a, Lemma 7.4] we have that $\mathrm{Aut}(\mathcal{S}^{n,m}) \cong PGL(n+1) \times PGL(m+1)$ if $n < m$, and $\mathrm{Aut}(\mathcal{S}^{n,n}) \cong S_2 \ltimes (PGL(n+1) \times PGL(n+1))$.

Note that in the case $n = m$ also the involution in S_2 switching the two factors comes from an automorphism of the ambient projective space $\mathbb{P}^N$ and so it induces an automorphism of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$. Let us proceed by induction on h. So $\mathrm{Aut}(\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})) \cong \mathrm{Aut}(\mathcal{S}^{n,m})$, and we have a surjective morphism of groups

$$\begin{array}{cccc} \chi: & \mathrm{Aut}(\mathbb{S}ec_h(\mathcal{S}^{n,m})) & \longrightarrow & \mathrm{Aut}(\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})) \\ & \phi & \longmapsto & \phi_{|\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})}. \end{array}$$

Recall that $\mathbb{S}ec_h(\mathcal{S}^{n,m}) = \mathrm{Join}(\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m}), \mathcal{S}^{n,m})$. Assume that $\phi_{|\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})} = Id_{\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})}$. Then $\phi_{|\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})}$ fixes $\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})$ and hence $\mathcal{S}^{n,m}$. Let $p \in \mathbb{S}ec_h(\mathcal{S}^{n,m})$ be a general point. By Remark 2.11 the actual dimension of $\mathrm{Join}(\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m}), \mathcal{S}^{n,m})$ is smaller than the expected one. So there are infinitely many lines intersecting $\mathcal{S}^{n,m}$ and $\mathbb{S}ec_{h-1}(\mathcal{S}^{n,m})$ through p. Any two of these lines are stabilized by ϕ and intersect at p, so $\phi(p) = p$. Hence $\phi = Id_{\mathbb{S}ec_h(\mathcal{S}^{n,m})}$ and χ is an isomorphism. The same proof, with the obvious variations, works in the symmetric case as well. $\square$

Theorem 3.21. *For all $h \leqslant n$ and $k = 1, \dots, h-1$ we have*

$$\mathrm{Aut}(\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m; \end{cases}$$

$$\mathrm{Aut}(\mathbb{S}ec_h^{(k)}(\mathcal{V}^n)) \cong PGL(n+1);$$

and for $h = n+1$ we have

$$\mathrm{Aut}(\mathcal{C}(n,m,n+1)) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ (S_2 \ltimes (PGL(n+1) \times PGL(n+1))) \rtimes S_2 & \text{if } n = m \geqslant 2; \end{cases}$$

$$\mathrm{Aut}(\mathcal{Q}(n,n+1)) \cong PGL(n+1) \rtimes S_2;$$

$\mathrm{Aut}(\mathcal{C}(1,1,2)) \cong PGL(4)$, *and* $\mathrm{Aut}(\mathcal{Q}(1,2)) \cong PGL(3)$.

Proof. When $h = n+1$ the statement follows from [Mas20a, Theorem 7.5]. Hence we consider the case $h \leqslant n$. We will prove the claim for $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$. The argument in the symmetric case is completely analogous.

First, take $k = h-1$. Hence $\mathbb{S}ec_h^{(h-1)}(\mathcal{S}^{n,m}) \cong \mathcal{C}(n,m,h)$. An automorphism $\phi \in \mathrm{Aut}(\mathcal{C}(n,m,h))$ acts on the extremal rays of $\mathrm{Eff}(\mathcal{C}(n,m,h))$ as a permutation. If it acts non trivially then it must act non trivially also on the generators of $\mathrm{Nef}(\mathcal{C}(n,m,h))$ in Proposition 3.10. However, this is not possible since for instance these nef divisors have spaces of global sections of different dimensions. Hence, ϕ stabilizes all the exceptional divisors in Definition 2.5, and therefore it induces an automorphism $\widetilde{\phi} \in \mathrm{Aut}(\mathbb{S}ec_h(\mathcal{S}^{n,m}))$. The morphism of groups

$$\begin{array}{cccc} \widetilde{\chi}: & \mathrm{Aut}(\mathcal{C}(n,m,h)) & \longrightarrow & \mathrm{Aut}(\mathbb{S}ec_h(\mathcal{S}^{n,m})) \\ & \phi & \longmapsto & \widetilde{\phi} \end{array}$$

is clearly an isomorphism, and we conclude by Proposition 3.20.

Now, consider the case $k < h-1$. Recall that $\mathcal{C}(n,m,h)$ is obtained from $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ by blow-ups centered at subvarieties of $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ that are stabilized by all $\phi \in \mathrm{Aut}(\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m}))$. Hence, $\phi \in \mathrm{Aut}(\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m}))$ lifts two an automorphism $\overline{\phi}$ of $\mathcal{C}(n,m,h)$, and we get a morphism of groups

$$\begin{array}{cccc} \overline{\chi}: & \mathrm{Aut}(\mathbb{S}ec_h(\mathcal{S}^{n,m})) & \longrightarrow & \mathrm{Aut}(\mathcal{C}(n,m,h)) \\ & \phi & \longmapsto & \overline{\phi} \end{array}$$

which again is an isomorphism. Finally, we conclude by the computation of $\mathrm{Aut}(\mathcal{C}(n,m,h))$ in the first part of the proof. $\square$

4. Kontsevich spaces of conics and complete singular forms

An n-pointed rational pre-stable curve $(C,(x_1,...,x_n))$ is a projective, connected, reduced curve with at most nodal singularities of arithmetic genus zero, with n distinct and smooth marked points $x_1,...,x_n \in C$. We will refer to the marked and the singular points of C as special points.

Let X be a homogeneous variety. A map $(C,(x_1,...,x_n),\alpha)$, where $\alpha : C \to X$ is a morphism from an n-pointed rational pre-stable curve to X is stable if any component $E \cong \mathbb{P}^1$ of C contracted by α contains at least three special points.

Now, let us fix a class $\beta \in H_2(X,\mathbb{Z})$. By [FP97, Theorem 2] there exists a smooth, proper, and separated Deligne-Mumford stack $\overline{\mathcal{M}}_{0,n}(X,\beta)$ parametrizing isomorphism classes of stable maps $[C,(x_1,...,x_n),\alpha]$ such that $\alpha_*[C]=\beta$.

Furthermore, by [KP01, Corollary 1] the coarse moduli space $\overline{M}_{0,n}(X,\beta)$ associated to the stack $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is a normal, irreducible, projective variety with at most finite quotient singularities of dimension

$$\dim(\overline{M}_{0,n}(X,\beta)) = \dim(X) + \int_\beta c_1(T_X) + n - 3.$$

The variety $\overline{M}_{0,n}(X,\beta)$ is called the *moduli space of stable maps*, or the *Kontsevich moduli space* of stable maps of class β from a rational pre-stable n-pointed curve to X. The boundary $\partial\overline{M}_{0,n}(X,\beta) = \overline{M}_{0,n}(X,\beta)\backslash M_{0,n}(X,\beta)$ is a simple normal crossing divisor in $\overline{M}_{0,n}(X,\beta)$ whose points parametrize isomorphism classes of stable maps $[C,(x_1,...,x_n),\alpha]$ where C is a reducible curve. When $X=\mathbb{P}^N$, we will write $\overline{M}_{0,n}(\mathbb{P}^N,d)$ for $\overline{M}_{0,n}(\mathbb{P}^N,d[L])$, where $L\subseteq\mathbb{P}^N$ is a line.

For details on moduli spaces parametrizing curves in projective spaces, and in particular conics, we refer to [EH16, Section 8.4].

4.0. **Conics in $\mathbb{P}^n$.** Let $\overline{M}_{0,0}(\mathbb{P}^n,2)$ be the Kontsevich space of conics in $\mathbb{P}^n$. We will denote by $\Delta\subset\overline{M}_{0,0}(\mathbb{P}^n,2)$ the boundary divisor parametrizing maps with reducible domain, and by $\Gamma\subset\overline{M}_{0,0}(\mathbb{P}^n,2)$ the locus of maps of degree two onto a line. Note that Γ is a $\mathbb{P}^2$-bundle over the Grassmannian $\mathbb{G}(1,n)$ parametrizing lines in $\mathbb{P}^n$. In $\overline{M}_{0,0}(\mathbb{P}^n,2)$ we consider the following divisor classes:

- $\mathcal{H}$ of conics intersecting a fixed codimension two linear subspace of $\mathbb{P}^n$;
- $\mathcal{T}$ of conics which are tangent to a fixed hyperplane in $\mathbb{P}^n$;
- D_{deg} of conics spanning a plane that intersects a fixed linear subspace of dimension $n-3$ in $\mathbb{P}^n$.

It is well-known that $\overline{M}_{0,0}(\mathbb{P}^2,2)$ is isomorphic to the space of complete conics $\mathcal{Q}(2,3)$ [FP97, Section 0.4]. The following result generalizes this fact.

Proposition 4.1. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n,2)$ is isomorphic to the blow-up $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ of $\mathbb{S}ec_3(\mathcal{V}^n)$ along $\mathcal{V}^n$.*

Proof. We may associate to a rank three quadric $Q\subset\mathbb{P}^n$ its dual conic $C_Q\subset\mathbb{P}^{n*}$. Conversely, given a smooth conic $C_Q\in\overline{M}_{0,0}(\mathbb{P}^n,2)$ we can consider the cone swept by the duals of the tangent lines of C_Q and whose vertex is the dual of the plane spanned by C_Q. This yields a morphism

$$\begin{array}{cccc} \phi^o: & M_{0,0}(\mathbb{P}^n,2) & \longrightarrow & \mathbb{S}ec_3(\mathcal{V}^n)\subset\mathbb{P}^{N_+} \\ & C_Q & \longmapsto & Q. \end{array}$$

Consider the hyperplane $H=\{z_{0,0}=0\}\subset\mathbb{P}^{N_+}$. The points of $H\cap\mathbb{S}ec_3(\mathcal{V}^n)$ correspond to the rank three quadrics $Q\subset\mathbb{P}^n$ passing through $p=[1:0:\dots:0]$. These quadric in turn correspond via the morphism ϕ^o to the conics $C_Q\subset\mathbb{P}^{n*}$ that are tangent the the hyperplane $H_p\subset\mathbb{P}^{n*}$ which is dual to $p\in\mathbb{P}^n$. Hence, ϕ^o is induced by the divisor class $\mathcal{T}$. Now, [CHS09, Theorem 1.2] yields that ϕ^o extends to a morphism

$$\phi:\overline{M}_{0,0}(\mathbb{P}^n,2)\to\mathbb{S}ec_3(\mathcal{V}^n)$$

restricting to an isomorphism on $M_{0,0}(\mathbb{P}^n,2)$ and contracting the boundary divisor Δ.

Fix a rank two conic in $C_Q\subset\mathbb{P}^{n*}$. Up to an automorphism of $\mathbb{P}^{n*}$ we may assume that $C_Q=\{x_0=\dots=x_{n-3}=x_{x-2}x_{n-1}=0\}$. Consider the family of smooth conics $C_{Q,t}=\{x_0=\dots=x_{n-3}=$

$x_{x-2}x_{n-1} - tx_n^2 = 0\}$, with $t \neq 0$, degenerating to C_Q. Then

$$\phi(C_{Q,t}) = \{x_n^2 - 4tx_{n-2}x_{n-1} = 0\}$$

where we keep denoting by $[x_0 : \cdots : x_n]$ the homogeneous coordinates on the dual projective space. Hence

$$\phi(C_Q) = \lim_{t \mapsto 0} \phi(C_{Q,t}) = \{x_n^2 = 0\}$$

and so Δ gets contracted onto the Veronese variety $\mathcal{V}^n \subset \mathbb{S}ec_3(\mathcal{V}^n)$. Now, by [Har77, Proposition 7.14] ϕ yields a morphism

$$\psi : \overline{M}_{0,0}(\mathbb{P}^n, 2) \to \mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$$

mapping Δ onto $E_1^{\mathcal{Q}}$. Hence, ψ restricts to a morphism $\psi_{|\Gamma} : \Gamma \to \mathbb{S}ec_2^{(1)}(\mathcal{V}^n)$ associating to a double cover $\mathbb{P}^1 \to L$ ramified at $p, q \in L$ the rank two quadric $H_p \cup H_q$, where H_p, H_q are the hyperplanes dual to p and q. Moreover, associating to a rank two quadric $H_1 \cup H_2$ the 2-to-1 cover $\mathbb{P}^1 \to (H_1 \cap H_2)^*$ ramified at H_1^*, H_2^* we get a birational inverse of $\psi_{|\Gamma}$. Note that $\psi_{|\Gamma}$ can not contract any divisor in Γ since both Γ and $\mathbb{S}ec_2^{(1)}(\mathcal{V}^n)$ have Picard rank two. Furthermore, $\psi_{|\Gamma}$ can not contract any locus of codimension greater than one in Γ either since $\mathbb{S}ec_2^{(1)}(\mathcal{V}^n)$ is smooth.

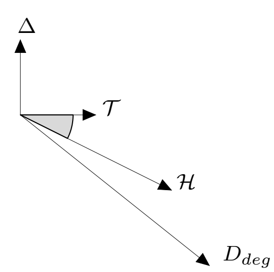
Hence, $\psi : \overline{M}_{0,0}(\mathbb{P}^n, 2) \to \mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ is a finite and birational morphism. Finally, since $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ and $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ are normal [Mum99, Chapter 3, Section 9] yields that ψ is an isomorphism. $\square$

As an application of Proposition 4.1 we have the following result.

Proposition 4.2. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is a spherical variety with respect to the following $SL(n+1)$-action:*

$$\begin{array}{ccc} SL(n+1) \times \overline{M}_{0,0}(\mathbb{P}^n, 2) & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, 2) \\ (A, [C, \alpha]) & \longmapsto & [C, A \circ \alpha]. \end{array} \tag{4.3}$$

The effective cone of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is generated by Δ and D_{deg}, and the nef cone of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is generated by $\mathcal{T}$ and $\mathcal{H}$. Furthermore, the following



is the Mori chamber decomposition of $\mathrm{Eff}(\overline{M}_{0,0}(\mathbb{P}^n, 2))$*, where $\mathcal{H} \sim 2\mathcal{T} - \Delta$ and $D_{deg} \sim 3\mathcal{T} - 2\Delta$.*

Proof. The effective and the nef cone of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ had already been computed in [CHS08, Theorem 1.5, Corollary 1.6] and [CHS09, Theorem 1.1] respectively.

The $SL(n+1)$-action on $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ in (4.3) corresponds to the $SL(n+1)$-action on $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ induced by (2.10) via the isomorphism in Proposition 4.1. Note that with respect to this action $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ is spherical but not wonderful. However, we can deduce its boundary divisors and colors from those of $\mathcal{Q}(n, 3)$ via the blow-down $\mathcal{Q}(n, 3) \to \mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ of $E_2^{\mathcal{Q}}$. Since boundary divisors and colors of $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ lift to boundary divisors and colors of $\mathcal{Q}(n, 3)$ by Proposition 3.9 we get that $E_1^{\mathcal{Q}}$ is the only boundary divisor of $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$, and that its colors are $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$, where we kept the same notation for divisors on $\mathcal{Q}(n, 3)$ and $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$. Hence, arguing as in the proof of Proposition 3.17 we get that $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}$ generate the nef cone of $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$, $D_3^{\mathcal{Q}}, E_1^{\mathcal{Q}}$ generate it effective cone, and the Mori chamber decomposition of $\mathrm{Eff}(\mathbb{S}ec_3^{(1)}(\mathcal{V}^n))$ has three chambers delimited respectively by the divisors $D_3^{\mathcal{Q}}, D_2^{\mathcal{Q}}$, the divisors $D_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ and the divisors $D_1^{\mathcal{Q}}, E_1^{\mathcal{Q}}$.

Now, by the proof of Proposition 4.1 we have that $E_1^{\mathcal{Q}}$ gets mapped to Δ by the isomorphism $\psi^{-1}: \mathbb{S}ec_3^{(1)}(\mathcal{V}^n) \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$. Moreover, a straightforward computation shows that $\psi^{-1*}\mathcal{T} = D_1^{\mathcal{Q}}$, $\psi^{-1*}\mathcal{H} = D_2^{\mathcal{Q}}$ and $\psi^{-1*}D_{deg} = \frac{1}{2}D_3^{\mathcal{Q}}$. Finally, the statement follows from Lemma 3.16, Proposition 4.1 and the description of the Mori chamber decomposition of $\mathrm{Eff}(\mathbb{S}ec_3^{(1)}(\mathcal{V}^n))$ in the first part of the proof. $\square$

Remark 4.4. We sum up the birational models of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ in the following diagram:

$$\begin{tikzcd}
 & \mathcal{Q}(n,3) \arrow[dl] \arrow[dr] & \\
\mathbb{S}ec_3^{(1)}(\mathcal{V}^n) \cong \overline{M}_{0,0}(\mathbb{P}^n, 2) \arrow[rr, dashed, "\chi"] \arrow[dr, "\mathcal{H}"] \arrow[dd, "\mathcal{T}"'] & & \mathrm{Hilb}_2(\mathbb{P}^n) \arrow[dl] \arrow[dd, "\widetilde{D}_{deg}"] \\
 & \mathrm{Chow}_2(\mathbb{P}^n) & \\
\mathbb{S}ec_3(\mathcal{V}^n) & & \mathbb{G}(2,n)
\end{tikzcd}$$

where $\mathrm{Hilb}_2(\mathbb{P}^n)$ and $\mathrm{Chow}_2(\mathbb{P}^n)$ are respectively the Hilbert scheme and the Chow variety of conics in $\mathbb{P}^n$, χ is the flip of $\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$, $\mathbb{G}(2,n)$ is the Grassmannians of planes in $\mathbb{P}^n$, and $\widetilde{D}_{deg}$ is the strict transform of D_{deg} via χ. The morphism induced by $\widetilde{D}_{deg}$ associates to a conic in $\mathrm{Hilb}_2(\mathbb{P})^n$ the unique plane of $\mathbb{P}^n$ in which it is contained. We would like to stress that the modular interpretation of the flip of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ as a Hilbert scheme was well-know, see for instance [Kie11, Section 3].

4.4. **Conics in $\mathbb{P}^n \times \mathbb{P}^m$.** Let $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ be the Kontsevich space parametrizing conics in $\mathbb{P}^n \times \mathbb{P}^m$. Denote by $\pi : \overline{M}_{0,1}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ the forgetful morphism, and by $ev : \overline{M}_{0,1}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \mathbb{P}^n \times \mathbb{P}^m$ the evaluation morphism.

Let H_n and H_m be the hyperplane sections of $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, and $H_{n,m} \cong \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^n \times \mathbb{P}^m$. Consider the divisors

$$\mathcal{K}^n := \pi_* ev^* H_n^2,\ \mathcal{K}^m := \pi_* ev^* H_m^2,\ \mathcal{K}^{n,m} := \pi_* ev^* \mathcal{H}_{n,m}$$

and let Δ be the boundary divisor of maps with reducible domain.

By the proof of [Opr05, Lemma 1, Section 2.1], the Picard group of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is generated by $\Delta, \mathcal{K}^n, \mathcal{K}^m$. In particular, since $H_1^2 = 0$, the Picard rank of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is:

$$\rho(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))) = \begin{cases} 1 & \text{if } n = m = 1; \\ 2 & \text{if } n = 1, m \geqslant 2; \\ 3 & \text{if } n, m \geqslant 2. \end{cases} \tag{4.5}$$

Proposition 4.6. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is isomorphic to the space $\mathcal{C}(n,m,2)$ of rank two complete collineations on $\mathbb{P}^n \times \mathbb{P}^m$.*

Proof. First consider the case $n = m = 1$. We have that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1)) \cong \mathbb{P}^3$. Indeed, we may embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ as a smooth quadric Q, and the conics in Q are in bijection with the hyperplanes in $\mathbb{P}^3$.

Furthermore, $\mathcal{C}(1,1,2) \cong \mathbb{P}^3$ as well, and we may write down explicitly as isomorphism $\mathcal{C}(1,1,2) \to \overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ as follows: write a point of $\mathcal{C}(1,1,2)$ as a 2×2 matrix Z, fix homogeneous coordinates $([x_0 : x_1], [y_0 : y_1])$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and associate to Z the conic $C_Z = \{(x_0, x_1) \cdot Z \cdot (y_0, y_1)^t = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Now, let $Z \in \mathbb{S}ec_2(\mathcal{S}^{n,m}) \backslash \mathcal{S}^{n,m}$ an $(n+1) \times (m+1)$ matrix of rank two. The image of Z yields a line L_Z in $\mathbb{P}^n$, and the dual of the kernel of Z gives a line R_Z in $\mathbb{P}^{m*}$. Hence, we get a morphism

$$\begin{array}{cccc} \gamma^o: & \mathcal{C}(n,m,2)^o & \longrightarrow & \mathbb{G}(1,n) \times \mathbb{G}(1,m) \\ & Z & \longmapsto & (L_Z, R_Z). \end{array}$$

The fiber of γ^o over (L_Z, R_Z) can be identified with the collineations on $L_Z \times R_Z$. To see this we argue as follows. Acting with $SL(n+1) \times SL(m+1)$ on $\mathbb{G}(1,n) \times \mathbb{G}(1,m)$ we may assume that $L_Z =$

$\{x_2 = \cdots = x_n = 0\}$ and $R_Z = \{y_2 = \cdots = y_m = 0\}$. Hence, in $(\gamma^o)^{-1}(L_Z, R_Z)$ we have the matrices annihilating the vectors $(0,0,1,0,\dots,0),\dots,(0,0,\dots,0,1)$ and whose image is generated by the vectors $(1,0,0,\dots,0),(0,1,0,\dots,0)$ that is matrices of the following form

$$Z = \left(\begin{array}{cc} \overline{Z} & 0_{2,m-1} \\ 0_{n-1,2} & 0_{n-1,m-1} \end{array}\right), \text{ with } \overline{Z} = \left(\begin{array}{cc} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{array}\right).$$

By the first part of the proof the collineations on $L_Z \times R_Z$ are in bijection with $\overline{M}_{0,0}(L_Z \times R_Z, (1,1)) \subseteq \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$. This yields an isomorphism

$$\begin{array}{cccc} \delta^o: & \mathcal{C}(n,m,2)^o & \longrightarrow & M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \\ & Z & \longmapsto & C_Z. \end{array}$$

Now, consider the embedding $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \subset \overline{M}_{0,0}(\mathbb{P}^N, 2)$. We will show that the inverse of δ^o is induced by the restriction to $M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ of the divisor $\mathcal{T}$ on $\overline{M}_{0,0}(\mathbb{P}^N, 2)$. Since $\mathcal{T}$ restricts on $\overline{M}_{0,0}(L_Z \times R_Z, (1,1))$ to the corresponding tangency divisor it is enough to show the claim for $\overline{M}_{0,0}(L_Z \times R_Z, (1,1))$. By the first part of the proof the correspondence between $\mathcal{C}(1,1,2)$ and $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ is defined by mapping a matrix $Z = (z_{i,j})_{0\leqslant i,j\leqslant 1}$ to the divisor $C_Z = \{z_{0,0}x_0y_0 + z_{0,1}x_0y_1 + z_{1,0}x_1y_0 + z_{1,1}x_1y_1\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ which in turn is mapped by the Segre embedding to the conic

$$\overline{C}_Z = \{z_{0,0}X + z_{0,1}Y + z_{1,0}Z + z_{1,1}W = XW - YZ = 0\} \subset \mathbb{P}^3_{(X,Y,Z,W)}.$$

Now, considering the intersection of $\overline{C}_Z$ with the plane $\{W = 0\}$ we get the points $[z_{1,0} : 0 : -z_{0,0} : 0]$ and $[z_{0,1} : -z_{0,0} : 0 : 0]$. Therefore $\overline{C}_Z$ is tangent to $\{W = 0\}$ if and only if $z_{0,0} = 0$ that is if only if the matrix Z lies on the hyperplane section $\{z_{0,0} = 0\}$ of $\mathbb{P}^N$.

By [CHS09, Theorem 1.2] the divisor $\mathcal{T}$ is base point free and hence it restricts to a base point free divisor on $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)$. Therefore, the inverse of δ^o is indeed a morphism

$$\begin{array}{cccc} \eta: & \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) & \longrightarrow & \mathcal{C}(n,m,2) \\ & C_Z & \longmapsto & Z \end{array}$$

mapping the boundary divisor Δ to $E_1^{\mathcal{C}}$. Moreover, by Propositions 3.4 and (4.5) we get that η does not contract any divisor. Finally, since $\mathcal{C}(n,m,2)$ is smooth we conclude, by [Mum99, Chapter 3, Section 9], that η is an isomorphism. $\square$

Remark 4.7. Via the isomorphism

$$\eta^{-1} : \mathcal{C}(n,m,2) \to \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$$

we have

$$\eta^{-1*}(\Delta) = E_1^{\mathcal{C}},\ \eta^{-1*}(\mathcal{K}^n) = H_1^{\mathcal{C}},\ \eta^{-1*}(\mathcal{K}^m) = H_2^{\mathcal{C}},\ \eta^{-1*}(\mathcal{K}^{n,m}) = D_1^{\mathcal{C}}.$$

These equalities together with Proposition 3.18 give that for $n = 1 < m$, the Mori chamber decomposition of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ has two chambers delimited by $\Delta, \mathcal{K}^{n,m}$ and $\mathcal{K}^{n,m}, \mathcal{K}^m$, while for $1 < n \leqslant m$ the Mori chamber decomposition of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ has three chambers delimited respectively by the divisors $\mathcal{K}^n, \mathcal{K}^m, \mathcal{K}^{n,m}$, the divisors $\mathcal{K}^n, \mathcal{K}^{n,m}, \Delta$ and the divisors $\mathcal{K}^m, \mathcal{K}^{n,m}, \Delta$.

Recall that a divisor of class $\mathcal{K}^n$ parametrizes stable maps $\alpha : \mathbb{P}^1 \to \mathbb{P}^n \times \mathbb{P}^m$ intersecting a codimension two cycle of class H_n^2. These curves are mapped via the projection onto $\mathbb{P}^n$ to lines intersecting a fixed codimension two linear subspace of $\mathbb{P}^n$. Call L_α the line corresponding to the stable map α. Note that these lines correspond in turn to a hyperplane section of the Grassmannian $\mathbb{G}(1,n)$ in its Plücker embedding. Hence, the semi-ample divisor $\mathcal{K}^n$ induces a morphism $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \mathbb{G}(1,n)$ associating to a map $[\mathbb{P}^1, \alpha] \in M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ the line L_α. Then by the proof of Proposition 4.6 $H_1^{\mathcal{C}}$ yields a morphism $\mathcal{C}(n,m,2) \to \mathbb{G}(1,n)$ associating to a matrix $Z \in \mathcal{C}(n,m,2)^o$ the projectivization of its image.

Similarly, $\mathcal{K}^m$ induces a morphism $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \mathbb{G}(1,m)$ associating to a map $[\mathbb{P}^1, \alpha] \in M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ the line R_α given by projecting the image of α to $\mathbb{P}^m$, and $H_2^{\mathcal{C}}$ yields a morphism $\mathcal{C}(n,m,2) \to \mathbb{G}(1,m)$ associating to a matrix $Z \in \mathcal{C}(n,m,2)^o$ the projectivization of the dual of its kernel.

4.7. **Conics in $\mathbb{G}(1,n)$.** Let $\mathbb{G}(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Following [CC10, Section 2] we describe divisor classes on $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$. Fix projective subspaces $\Pi^{n-1},\Pi^{n-3}\subset\mathbb{P}^n$ of dimension $n-1$ and $n-3$, and consider the Schubert cycles

$$\begin{array}{rcl}\sigma_{1,1}^{1,n} & = & \{W\in\mathbb{G}(1,n)\mid \dim(W\cap\Pi^{n-1})\geqslant 1\};\\ \sigma_{2}^{1,n} & = & \{W\in\mathbb{G}(1,n)\mid \dim(W\cap\Pi^{n-3})\geqslant 0\}.\end{array}$$

Let $\pi:\overline{M}_{0,1}(\mathbb{G}(1,n),2)\rightarrow\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ be the forgetful morphism and $ev:\overline{M}_{0,1}(\mathbb{G}(1,n),2)\rightarrow\mathbb{G}(1,n)$ the evaluation morphism. We define

$$H_{\sigma_{1,1}}^{1,n}=\pi_*ev^*\sigma_{1,1},\ H_{\sigma_2}^{1,n}=\pi_*ev^*\sigma_2.$$

Furthermore, we will denote by $T^{1,n}$ the class of the divisor of conics that are tangent to a fixed hyperplane section of $\mathbb{G}(1,n)$.

Let $D_{deg}^{1,n}$ be the class of the divisor of maps $[C,\alpha]\in\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ such that the projection of the span of the linear spaces parametrized by $\alpha(C)$ from a fixed subspace of dimension $n-4$ has dimension less than three.

Next we define the divisor class $D_{unb}^{1,n}$. A stable map $\alpha:\mathbb{P}^1\rightarrow\mathbb{G}(1,n)$ induces a rank two subbundle $\mathcal{E}_\alpha\subset\mathcal{O}_{\mathbb{P}^1}\otimes\mathbb{C}^{n+1}$. We define $D_{unb}^{1,n}$ as the closure of the locus of maps $[\mathbb{P}^1,\alpha]\in\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ such that $\mathcal{E}_\alpha\neq\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

Finally, we denote by $\Delta^{k,n}$ the boundary divisor parametrizing stable maps with reducible domain.

Proposition 4.8. *There is a finite* 2*-to-*1 *morphism*

$$\varphi:\overline{M}_{0,0}(\mathbb{G}(1,n),2)\longrightarrow\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$$

mapping a stable map $[\mathbb{P}^1,\alpha]\in M_{0,0}(\mathbb{G}(1,n),2)$ *to the rank four quadric* $Q_C^*=\bigcup_{p\in Q_C}(T_pQ)^*\subset\mathbb{P}^{n*}$, *where* $Q_C=\bigcup_{[L]\in\alpha(\mathbb{P}^1)}L$.

Proof. By [CM17, Proposition 4.10, Theorem 5.1, Corollary 5.4] there is a birational morphism $f:\overline{M}_{0,0}(\mathbb{G}(1,n),2)\rightarrow\mathcal{T}_4^n$, contracting $D_{deg}^{1,n}$ and $\Delta^{1,n}$, where $\mathcal{T}_4^n$ is the double symmetric determinantal locus of rank at most four constructed in [HT15, Section 2.2]. By [HT15, Proposition 2.3] there is a finite 2-to-1 morphism $\rho:\mathcal{T}_4^n\rightarrow\mathbb{S}ec_4(\mathcal{V}^n)$ branched along $\mathbb{S}ec_3(\mathcal{V}^n)$.

Now, consider the morphism $\rho\circ f:\overline{M}_{0,0}(\mathbb{G}(1,n),2)\rightarrow\mathbb{S}ec_4(\mathcal{V}^n)$. By [Har77, Proposition 7.14] there is a morphism $\varphi:\overline{M}_{0,0}(\mathbb{G}(1,n),2)\rightarrow\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ such that $\pi\circ\varphi=\rho\circ f$, where $\pi:\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)\rightarrow\mathbb{S}ec_4(\mathcal{V}^n)$ is the blow-down.

Hence φ is 2-to-1 and by [HT15, Theorem 1.1] on $M_{0,0}(\mathbb{G}(1,n),2)$ it is defined by

$$\begin{array}{rccc}\varphi_{|M_{0,0}(\mathbb{G}(1,n),2)}: & M_{0,0}(\mathbb{G}(1,n),2) & \longrightarrow & \mathbb{S}ec_4^{(2)}(\mathcal{V}^n)\\ & [\mathbb{P}^1,\alpha] & \mapsto & Q_C^*\end{array}$$

where $Q_C^*=\bigcup_{p\in Q_C}(T_pQ)^*\subset\mathbb{P}^{n*}$, and $Q_C=\bigcup_{[L]\in\alpha(\mathbb{P}^1)}L$. Note that Q_C^* is indeed a quadric hypersurface of rank four, and since Q_C can be constructed from either of its two rulings $\varphi_{|M_{0,0}(\mathbb{G}(1,n),2)}$ is 2-to-1. □

Remark 4.9. For $n=3$ the double cover in Proposition 4.8 had been constructed in [Hue15, Section 5].

Remark 4.10. As an application of Propositions 3.19, 4.8 we recover some results of [CC10]. Indeed, on $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ there is an $SL(n+1)$-action given by

$$\begin{array}{ccc}SL(n+1)\times\overline{M}_{0,0}(\mathbb{G}(1,n),2) & \longrightarrow & \overline{M}_{0,0}(\mathbb{G}(1,n),2)\\ (M,[C,\alpha]) & \longmapsto & [C,\wedge^2M\circ\alpha]\end{array}$$

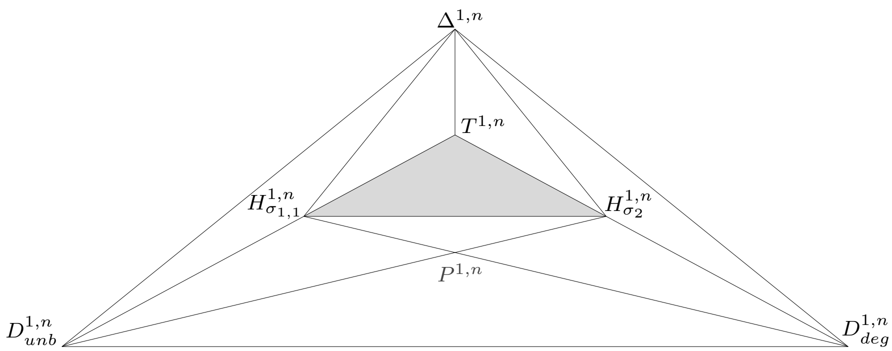
inducing on $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ a structure of spherical variety.

Considering the subspace $H=\{x_4=\dots=x_n=0\}\subset\mathbb{P}^n$ we get an embedding $i:\mathbb{G}(1,H)\hookrightarrow\mathbb{G}(1,n)$ which in turn induces an embedding $j:\overline{M}_{0,0}(\mathbb{G}(1,3),2)\rightarrow\overline{M}_{0,0}(\mathbb{G}(1,n),2)$. Furthermore, the pull-back map $j^*:\mathrm{Pic}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))\rightarrow\mathrm{Pic}(\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ is an isomorphism. This reduces the study of the birational geometry of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ to that of $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$.

By Proposition 3.19 and the 2-to-1 morphism in Proposition 4.8 we get that the divisor classes $\Delta^{1,n}, D^{1,n}_{deg}, D^{1,n}_{unb}$ and the divisor classes $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$ are respectively the classes of the boundary divisors and the colors of the spherical variety $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$.

Furthermore, the divisors classes $D^{1,n}_{unb}, D^{1,n}_{deg}, \Delta^{1,n}$ generate the effective cone of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$. The Cox ring of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ is generated by the global sections of the divisors $\Delta^{1,n}, D^{1,n}_{deg}, D^{1,n}_{unb}$ and $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$.

The nef cone of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ is generated by $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$. Moreover, the following is a 2-dimensional section of the Mori chamber decomposition of $\mathrm{Eff}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))$



where $P^{1,n} \sim \frac{1}{4}(3H^{1,n}_{\sigma_{1,1}} + 3H^{1,n}_{\sigma_2} - \Delta^{1,n})$, and $\mathrm{Mov}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))$ is generated by $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}, P^{1,n}$.

We have the following result on the automorphisms of Kontsevich spaces of conics.

Corollary 4.11. *We have that*

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m,(1,1))) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m \geqslant 2; \end{cases}$$

and $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1,(1,1))) \cong PGL(4)$.

Furthermore, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n,2)) \cong PGL(n+1)$ *for* $n \geqslant 3$*,* $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^2,2)) \cong PGL(3) \rtimes S_2$*, and* $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1,2)) \cong PGL(3)$.

Proof. The first claim on $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m,(1,1)))$ follows from Proposition 4.6 and Theorem 3.21. For the second claim recall that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1,(1,1)) \cong \mathbb{P}^3$ since curves of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are in bijection with the hyperplane sections of a smooth quadric surface in $\mathbb{P}^3$.

The automorphism group of $\overline{M}_{0,0}(\mathbb{P}^n,2)$ for $n \geqslant 3$ follows from Proposition 4.1 and Theorem 3.21. The automorphism group of $\overline{M}_{0,0}(\mathbb{P}^2,2)$ has been computed in [Mas20a, Remark 7.6]. Finally, to get the claim on $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1,2))$ notice that $\overline{M}_{0,0}(\mathbb{P}^1,2) \cong \mathbb{P}^2$. Indeed, a 2-to-1 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is determined by its branch locus, and so $\overline{M}_{0,0}(\mathbb{P}^1,2)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ mod out by the involution switching the factors. $\square$

Finally, we compute the automorphism group of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$. Since the cases $n = 2$ has been covered in Corollary 4.11 we assume that $n \geqslant 3$.

Proposition 4.12. *The automorphism group of* $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ *is given by*

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,n),2)) \cong \begin{cases} S_2 \ltimes PGL(n+1) & \text{if } n > 3; \\ S_2 \ltimes (S_2 \ltimes PGL(n+1)) & \text{if } n = 3. \end{cases}$$

Proof. First, consider the case $n = 3$. An automorphism of $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ must either preserve or switch the extremal rays $D^{1,3}_{unb}$ and $D^{1,3}_{deg}$. Indeed, there is an automorphism $\tau : \overline{M}_{0,0}(\mathbb{G}(1,3),2) \to \overline{M}_{0,0}(\mathbb{G}(1,3),2)$ switching them, namely the automorphism induced by the involution of $\mathbb{G}(1,3)$ given by projective duality. This yields a surjective morphism of groups

$$\begin{array}{rccc} \Psi : & \mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,3),2)) & \longrightarrow & S_2 \\ & \varphi & \mapsto & \sigma_\varphi \end{array}$$

where σ_φ is the permutation of the extremal rays of $\mathrm{Eff}(\overline{M}_{0,0}(\mathbb{G}(1,3),2))$ induced by φ. Now, assume that σ_φ is trivial. Then φ descends to an automorphism $\overline{\varphi}$ of the variety $\mathcal{T}_4^3$ in the proof of Proposition 4.8. By [HT15, Proposition 2.5 (3)] $\mathcal{T}_4^3$ is Fano and the morphism $\rho : \mathcal{T}_4^3 \to \mathbb{S}ec_4(\mathcal{V}^3) = \mathbb{P}^9$ in the proof of Proposition 4.8 is induced by a multiple of $-K_{\mathcal{T}_4^3}$. Hence, $\overline{\varphi}$ in turn descends to an automorphism of $\mathbb{S}ec_4(\mathcal{V}^3) = \mathbb{P}^9$ stabilizing the branch locus $\mathbb{S}ec_3(\mathcal{V}^3)$. Since the group of automorphisms of $\mathbb{P}^9$ stabilizing $\mathbb{S}ec_3(\mathcal{V}^3)$ is isomorphic to $PGL(4)$ we get an exact sequence

$$1 \to S_2 \to \mathrm{Aut}(\mathcal{T}_4^3) \to PGL(4) \to 1.$$

Note that $PGL(4)$ acts on $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ and hence on $\mathcal{T}_4^3$. So the last morphism in the sequence has a section, and $\mathrm{Aut}(\mathcal{T}_4^3) \cong PGL(4) \rtimes S_2$.

Now, the morphism Ψ yields the exact sequence

$$1 \to \mathrm{Aut}(\mathcal{T}_4) \to \mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,3),2)) \to S_2 \to 1$$

and since the last morphism in this sequence has a section we get the claim.

When $n > 3$ it is enough to argue as in the case $n = 3$ noticing that in this case $D_{unb}^{1,n}$ and $D_{deg}^{1,n}$ can not be switched and applying Proposition 3.20. $\square$

4.12. **On the anti-canonical divisor.** In this last section we study the positivity of the anti-canonical divisor of the varieties in Propositions 3.17, 3.18 and 3.19. Recall that a normal and $\mathbb{Q}$-factorial projective variety X is

- Fano if $-K_X$ is ample;
- weak Fano if $-K_X$ is nef and big;
- log Fano if there exists an effective divisor $D \subset X$ such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal.

Clearly, Fano implies weak Fano which in turn implies log Fano. As a consequence of Kodaira's lemma [Laz04, Proposition 2.2.6] X is log Fano if and only if there exists an effective divisor $D \subset X$ such that $-(K_X + D)$ is nef and big and the pair (X, D) is Kawamata log terminal. Moreover, if X and Y are normal and $\mathbb{Q}$-factorial projective varieties which are isomorphic in codimension one then X is log Fano if and only if Y is so. We refer to [GOST15] for further information on these notions. Finally, by [BCHM10, Corollary 1.3.2] if X is log Fano then it is a Mori dream space.

4.12.1. *The anti-canonical divisor of* $\mathcal{Q}(n,3)$. If $n = 2$ then $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ is the space of complete conics that is the blow-up of $\mathbb{P}^5$ along $\mathcal{V}^2$. So

$$-K_{\mathbb{S}ec_3^{(1)}(\mathcal{V}^2)} = 6D_1^{\mathcal{Q}} - 2E_1^{\mathcal{Q}} = 2(D_1^{\mathcal{Q}} + D_2^{\mathcal{Q}}).$$

Assume $n \geqslant 3$. By [dJS17, Theorem 1.1] we have that

$$-K_{\overline{M}_{0,0}(\mathbb{P}^n,2)} = \frac{3(n+1)}{4}\mathcal{H} - \frac{n-7}{4}\Delta$$

and hence Proposition 4.1 yields

$$-K_{\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)} = \frac{3(n+1)}{2}D_1^{\mathcal{Q}} - (n-1)E_1^{\mathcal{Q}} = \frac{7-n}{2}D_1^{\mathcal{Q}} + (n-1)D_2^{\mathcal{Q}} = 3D_1^{\mathcal{Q}} + \frac{n-1}{2}D_3^{\mathcal{Q}}.$$

Therefore, $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ is Fano if and only if $1 \leqslant n < 7$, weak Fano for $n = 7$ and log Fano for $n \geqslant 8$.

Now, note that by Proposition 2.12 the tangent cone of $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ at a point of $\mathbb{S}ec_2^{(1)}(\mathcal{V}^n)\backslash(\mathbb{S}ec_2^{(1)}(\mathcal{V}^n) \cap E_1^{\mathcal{Q}})$ is a cone with vertex of dimension $2n$ over $\mathcal{V}^{n-2}$. Hence, $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ looks, locally around a point of $\mathbb{S}ec_2^{(1)}(\mathcal{V}^n)\backslash(\mathbb{S}ec_2^{(1)}(\mathcal{V}^n) \cap E_1^{\mathcal{Q}})$, as the weighted projective space $\mathbb{P}(1^{n-1}, 2^{2n+1})$. Therefore, $\mathbb{S}ec_3^{(1)}(\mathcal{V}^n)$ has quotient singularities of type $\frac{1}{2}(1^{n-1})$ along $\mathbb{S}ec_2^{(1)}(\mathcal{V}^n)\backslash(\mathbb{S}ec_2^{(1)}(\mathcal{V}^n) \cap E_1^{\mathcal{Q}})$ and the discrepancy of the canonical divisor of $\mathcal{Q}(n,3)$ with respect to $E_2^{\mathcal{Q}}$ is $\frac{n-3}{2}$. Summing up we have

$$-K_{\mathcal{Q}(n,3)} = \frac{7-n}{2}D_1^{\mathcal{Q}} + (n-1)D_2^{\mathcal{Q}} - \frac{n-3}{2}E_2^{\mathcal{Q}} = 2D_1^{\mathcal{Q}} + 2D_2^{\mathcal{Q}} + \frac{n-3}{2}D_3^{\mathcal{Q}}. \tag{4.13}$$

Hence, by Proposition 3.17 and (4.13) we get that $\mathcal{Q}(n,3)$ if Fano for $n \geqslant 4$ and weak Fano for $n = 3$.

4.13.2. *The anti-canonical divisor of* $\mathcal{C}(n,m,2)$. The first two Chern classes of the tangent bundle $T_{\mathbb{P}^n\times\mathbb{P}^m}$ of $\mathbb{P}^n\times\mathbb{P}^m$ are given by

$$c_1=(n+1)H_n+(m+1)H_m,\ c_2=\binom{n+1}{2}H_n^2+(n+1)(m+1)H_nH_m+\binom{m+1}{2}H_m^2.$$

Hence, by [dJS17, Theorem 1.1] we have

$$-K_{\overline{M}_{0,0}(\mathbb{P}^n\times\mathbb{P}^m,(1,1))}=\frac{(n+1)(2n+m+3)}{2n+2m+4}\mathcal{K}^n+\frac{(n+1)(m+1)}{(n+m+2)}\mathcal{K}^{n,m}+\frac{(m+1)(2m+n+3)}{2n+2m+4}\mathcal{K}^m-\frac{nm-3n-3m-7}{2n+2m+4}\Delta$$

and plugging in the relation $\Delta=2\mathcal{K}^{n,m}-\mathcal{K}^n-\mathcal{K}^m$ from [Opr05, Section 2.2] we get

$$-K_{\overline{M}_{0,0}(\mathbb{P}^n\times\mathbb{P}^m,(1,1))}=(n-1)\mathcal{K}^n+4\mathcal{K}^{n,m}+(m-1)\mathcal{K}^m. \tag{4.14}$$

As a consequence of Propositions 3.18, 4.6 and (4.14) we see that $\mathcal{C}(n,m,2)$ is Fano for all $n,m\geqslant 1$.

4.14.3. *The anti-canonical divisor of* $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$. By Proposition 4.8 there is a 2-to-1 morphism

$$\varphi:\overline{M}_{0,0}(\mathbb{G}(1,n),2)\to\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$$

branched along $E_1^{\mathcal{Q}}$ and $\mathbb{S}ec_3^{(2)}(\mathcal{V}^n)$. Note that $\mathbb{S}ec_3^{(2)}(\mathcal{V}^n)$ is a divisor in $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ if and only if $n=3$. In this case $\mathbb{S}ec_4^{(2)}(\mathcal{V}^3)$ is the space of complete quadrics of $\mathbb{P}^3$. So its anti-canonical divisor is given by

$$-K_{\mathbb{S}ec_4^{(2)}(\mathcal{V}^3)}=10D_1^{\mathcal{Q}}-5E_1^{\mathcal{Q}}-2E_2^{\mathcal{Q}}$$

and by Proposition 3.19 $\mathbb{S}ec_4^{(2)}(\mathcal{V}^3)$ is Fano.

Assume that $n\geqslant 4$. Then $\mathbb{S}ec_3^{(2)}(\mathcal{V}^n)$ has codimension greater than one in $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ and so it does not play any role in the Riemann-Hurwitz formula relating the canonical divisors of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ and $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$. By [CC10, Remark 2.4] we have that

$$-K_{\overline{M}_{0,0}(\mathbb{G}(1,n),2)}=\frac{11-n}{4}H^{1,n}_{\sigma_{1,1}}+\frac{3n-1}{4}H^{1,n}_{\sigma_2}+\frac{7-n}{4}\Delta^{1,n}.$$

Write $-K_{\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)}=aD_1^{\mathcal{Q}}+bE_1^{\mathcal{Q}}+cE_2^{\mathcal{Q}}$. Since $\varphi^*D_1^{\mathcal{Q}}=H^{1,n}_{\sigma_{1,1}}$, $\varphi^*D_2^{\mathcal{Q}}=T^{1,n}$, $\varphi^*D_3^{\mathcal{Q}}=H^{1,n}_{\sigma_2}$, $\varphi^*E_1^{\mathcal{Q}}=2D^{1,n}_{unb}$, $\varphi^*E_2^{\mathcal{Q}}=\Delta^{1,n}$ we have that

$$-K_{\overline{M}_{0,0}(\mathbb{G}(1,n),2)}=\varphi^*(-K_{\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)})-D^{1,n}_{unb}=\frac{4a+6b-3}{4}H^{1,n}_{\sigma_{1,1}}+\frac{1-2b}{4}H^{1,n}_{\sigma_2}+\frac{4c-2b+1}{4}\Delta^{1,n}$$

where we used the relation $D^{1,n}_{unb}=\frac{1}{4}(3H^{1,n}_{\sigma_{1,1}}-H^{1,n}_{\sigma_2}-\Delta^{1,n})$ in [CC10, Section 3]. Finally,

$$-K_{\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)}=(2n+2)D_1^{\mathcal{Q}}-\frac{3n-2}{2}E_1^{\mathcal{Q}}-(n-2)E_2^{\mathcal{Q}}=2D_1^{\mathcal{Q}}+\frac{6-n}{2}D_2^{\mathcal{Q}}+(n-3)D_3^{\mathcal{Q}}. \tag{4.15}$$

By Proposition 3.19 and (4.15) we get that $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ is Fano for $3\leqslant n\leqslant 5$ and weak Fano for $n=6$. Furthermore, writing

$$-K_{\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)}=(8-n)D_1^{\mathcal{Q}}+3D_3^{\mathcal{Q}}+(n-6)P$$

we see that $\mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ is log Fano for $n\leqslant 8$.

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