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Existence of Planar Curves Minimizing Length and Curvature

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Abstract—We consider the problem of reconstructing a curve that is partially hidden or corrupted by minimizing the functional $\int\sqrt{1+K_\gamma^2}\,ds$, depending both on the length and curvature K. We fix starting and ending points as well as initial and final directions. For this functional we discuss the problem of existence of minimizers on various functional spaces. We find nonexistence of minimizers in cases in which initial and final directions are considered with orientation. In this case, minimizing sequences of trajectories may converge to curves with angles. We instead prove the existence of minimizers for the "time-reparametrized" functional $\int\|\dot\gamma(t)\|\sqrt{1+K_\gamma^2}\,dt$ for all boundary conditions if the initial and final directions are considered regardless of orientation. In this case, minimizers may present cusps (at most two) but not angles.

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1. PROBLEMS STATEMENTS AND MAIN RESULTS

Consider a smooth function $\gamma_0\colon [a,b]\cup[c,d]\to\mathbb{R}^2$ (with $a<b<c<d$) representing a curve that is partially hidden or deleted in (b,c). We want to find a curve $\gamma\colon[b,c]\to\mathbb{R}^2$ that completes γ_0 in the deleted part and that minimizes a cost depending both on the length $\mathcal{L}(\gamma)$ and curvature K_γ.

The fact that γ completes γ_0 means that $\gamma(b)=\gamma_0(b)$ and $\gamma(c)=\gamma_0(c)$. It is also reasonable to require that the directions of tangent vectors (with orientation) coincide, i.e. $\dot\gamma(b)\sim\dot\gamma_0(b)$ and $\dot\gamma(c)\sim\dot\gamma_0(c)$ where

$$v_1\sim v_2 \qquad \text{if there exists } \alpha\in\mathbb{R}^+ \text{ such that } v_1=\alpha v_2. \tag{1}$$

We call these conditions *boundary conditions with orientation.* Throughout this paper we assume that the starting and ending points never coincide, i.e. $\gamma_0(b)\neq\gamma_0(c)$, and that the initial and final directions are nonvanishing.

In the literature this problem has been extensively studied for its application to problems of segmentation of images (see, e.g., [2, 5, 9, 10]) and for the construction of spiral splines [7].

The cost studied in [5, 7, 9] is the total squared curvature $E_1[\gamma]=\int_0^{\mathcal{L}(\gamma)}|K_\gamma(s)|^2\,ds$ where s is the arclength. In [7, 9] the boundary conditions differ from our boundary conditions with orientation. In particular, the starting and ending directions are fixed with angles measured in $\mathbb{R}$ (while we identify α and $\alpha+2k\pi$). In this framework, nonexistence of minimizers is proved if the starting and ending angles θ_0 and θ_1 satisfy $|\theta_1-\theta_0|>\pi$.

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The cost studied in [2] is $E_2[\gamma] = \int_0^{\mathcal{L}(\gamma)} (1 + |K_\gamma(s)|^2)\,ds$, while in [10] the cost is $E_3[\gamma] = \int_0^{\mathcal{L}(\gamma)} (\eta + |K_\gamma(s)|^2)\,ds$ with $\eta \to 0$. Depending on the cost, minimizers may present angles and the curvature becomes a measure.

The cost $E_4[\gamma] = \int_0^{\mathcal{L}(\gamma)} \sqrt{1 + |K_\gamma(s)|^2}\,ds$ naturally arises in problems of geometry of vision [6, 13, 14]. In this paper we study the following cost:

$$J[\gamma] = \int\limits_b^c \sqrt{\|\dot\gamma(t)\|^2 + \|\dot\gamma(t)\|^2 K_\gamma^2(t)}\,dt, \tag{2}$$

which is an extension of $E_4[\gamma]$ (see Remarks 2 and 3 below). Using this cost, one can study the existence of minimizers with angles without involving sophisticated functional spaces. Moreover, this problem has also been studied in [4], where it is defined on the sphere S^2 instead of the plane.

Remark 1. The cost J is invariant both under rototranslation and reparametrization of the curve.

Define the cost $J_\beta[\gamma] := \int_b^c \sqrt{\|\dot\gamma(t)\|^2 + \beta^2\|\dot\gamma(t)\|^2 K_\gamma^2(t)}\,dt$ with a fixed $\beta \neq 0$. Consider a homothety $(x, y) \mapsto (\beta x, \beta y)$ and the corresponding transformation of a curve $\gamma = (x(t), y(t))$ into $\gamma_\beta = (\beta x(t), \beta y(t))$. It is easy to prove that $J_\beta[\gamma_\beta] = \beta^2 J[\gamma]$. Hence the problem of minimization of J_β is equivalent to the minimization of J with a suitable change of boundary conditions. Thus, the results about J given in this paper also hold for the cost J_β.

The first question we address in this paper is the choice of a set of smooth curves on which this cost is well-defined. We want $\dot\gamma(t)$ and $K_\gamma(t) = \frac{\dot x\ddot y - \dot y\ddot x}{(\dot x^2 + \dot y^2)^{3/2}}$ to be well-defined; thus it is reasonable to look for minimizers in

$$\mathcal{D}_1 := \left\{ \gamma \in C^2([b,c], \mathbb{R}^2) \;\middle|\; \dot\gamma(t) \neq 0 \ \forall\, t \in [b,c], \ \begin{array}{l} \gamma(b) = \gamma_0(b),\ \gamma(c) = \gamma_0(c), \\ \dot\gamma(b) \sim \dot\gamma_0(b),\ \dot\gamma(c) \sim \dot\gamma_0(c) \end{array} \right\}.$$

Moreover, $\dot\gamma(b)$ and $\dot\gamma(c)$ are well-defined in this case.

Remark 2. The cost $J[\gamma]$ on the set $\mathcal{D}_1$ coincides with the cost $E_4[\gamma]$. To prove this, reparametrize a curve in $\mathcal{D}_1$ by the arclength and observe that in this case we have $\|\dot\gamma\| = 1$.

Under this assumption, one of the main results of the paper is the nonexistence of minimizers for J.

Proposition 1. *There exist boundary conditions $\gamma_0(b), \gamma_0(c) \in \mathbb{R}^2$ with $\gamma_0(b) \neq \gamma_0(c)$ and $\dot\gamma_0(b), \dot\gamma_0(c) \in \mathbb{R}^2 \setminus \{0\}$ such that the cost* (2) *does not admit a minimum over the set $\mathcal{D}_1$.*

To get the existence of minimizers for this cost, one can choose to enlarge the set of admissible curves. In this paper we consider the simplest generalization, by taking the space

$$\mathcal{D}_2 := \left\{ \gamma \in C^2([b,c], \mathbb{R}^2) \;\middle|\; \|\dot\gamma(t)\|^2 + \|\dot\gamma(t)\|^2 K_\gamma^2(t) \in L^1([b,c], \mathbb{R}), \ \begin{array}{l} \gamma(b) = \gamma_0(b),\ \gamma(c) = \gamma_0(c), \\ \dot\gamma(b) \sim \dot\gamma_0(b),\ \dot\gamma(c) \sim \dot\gamma_0(c) \end{array} \right\},$$

on which the cost $J[\gamma]$ is defined and always finite.

Remark 3. Notice that the cost $E_4[\gamma]$ is not well defined on $\mathcal{D}_2$, since it is not possible in general to perform an arclength parametrization. Then $J[\gamma]$ is an extension of $E[\gamma]$, since they coincide on $\mathcal{D}_1$.

On $\mathcal{D}_2$ we also have the nonexistence of minimizers for J.

Proposition 2. *There exist boundary conditions $\gamma_0(b), \gamma_0(c) \in \mathbb{R}^2$ with $\gamma_0(b) \neq \gamma_0(c)$ and $\dot\gamma_0(b), \dot\gamma_0(c) \in \mathbb{R}^2 \setminus \{0\}$ such that the cost* (2) *does not admit a minimum over the set $\mathcal{D}_2$.*

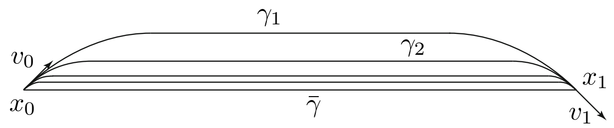

Fig. 1. Minimizing sequence converging to a nonadmissible curve (angles at the beginning/end).

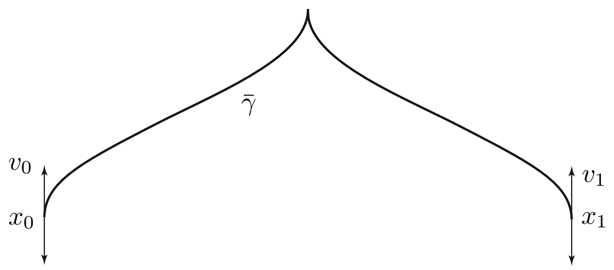

Fig. 2. A minimizer with a cusp.

The basic problem is that we may have a sequence of minimizing curves converging to a nonadmissible curve. In particular, we may have angles at the beginning and/or at the end, i.e. each curve γ_n satisfies given boundary conditions with orientation but the limit curve $\bar{\gamma}$ does not satisfy them (see Fig. 1).

The main result of the paper is the existence of minimizers for the cost (2) in the case when we take again curves for which $\|\dot{\gamma}(t)\|^2 + \|\dot{\gamma}(t)\|^2 K_\gamma^2(t)$ is integrable, but change boundary conditions. We only impose conditions on the direction of $\dot{\gamma}$ regardless of its orientation.

As before, fix a starting point x_0 with a direction v_0 and an ending point x_1 with a direction v_1. Consider planar curves satisfying the following boundary conditions: $\gamma(0) = x_0$, $\dot{\gamma}(0) \approx v_0$, $\gamma(T) = x_1$, and $\dot{\gamma}(T) \approx v_1$, where the identification rule $\approx$ is

$$v_1 \approx v_2 \qquad \text{if there exists } \alpha \in \mathbb{R} \setminus \{0\} \text{ such that } v_1 = \alpha v_2. \tag{3}$$

We call them *projective boundary conditions.* As already stated, we have the following existence result:

Proposition 3. *For all boundary conditions $x_0, x_1 \in \mathbb{R}^2$ with $x_0 \neq x_1$ and $v_0, v_1 \in \mathbb{R}^2 \setminus \{0\}$, the cost* (2) *has a minimizer over the set*

$$\mathcal{D}_3 := \left\{ \gamma \in C^2([b,c], \mathbb{R}^2) \;\middle|\; \|\dot{\gamma}(t)\|^2 + \|\dot{\gamma}(t)\|^2 K_\gamma^2(t) \in L^1([b,c], \mathbb{R}), \begin{array}{l} \gamma(b) = x_0,\ \gamma(c) = x_1, \\ \dot{\gamma}(b) \approx v_0,\ \dot{\gamma}(c) \approx v_1 \end{array} \right\}.$$

Observe that we can have minimizers with cusps, as $\bar{\gamma}$ in Fig. 2. Indeed, the limit direction (regardless of orientation) is well defined at the cusp point, while the limit direction with orientation is undefined.

All the previous results are obtained as consequences of the study of two similar mechanical problems. As regards problems with boundary conditions with orientation, we consider a car on the plane that can move only forwards and rotate on itself (it is Dubins' car, see [8]). Fix two points (x_0, y_0), (x_1, y_1) and two angles θ_0, θ_1 at these points measured with respect to the positive x-semiaxis. Consider all trajectories $q(\cdot)$ steering the car from the point (x_0, y_0) rotated through an angle θ_0 to the point (x_1, y_1) rotated through an angle θ_1. Our goal is to find the cheapest trajectory with respect to a cost depending both on the length of displacement on the plane and on the angle of rotation on itself.

The dynamics can be written as the following control system on the group of motions of the plane $\mathrm{SE}(2) := \{(x, y, \theta) \mid (x, y) \in \mathbb{R}^2,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

where x, y are coordinates on the plane and θ represents the angle of rotation of the car. Since we forbid backward displacements, we impose $u_1 \geq 0$. We want to minimize the cost

$$\mathcal{C}[q(t)] = \int_0^T \sqrt{u_1^2(t) + u_2^2(t)}\, dt \tag{5}$$

with the following boundary conditions: $x(0) = x_0$, $y(0) = y_0$, $\theta(0) = \theta_0$, $x(T) = x_1$, $y(T) = y_1$, and $\theta(T) = \theta_1$.

Remark 4. Any smooth planar curve can be naturally transformed into an admissible trajectory of this control system. Indeed, given $\gamma(t) = (x(t), y(t))$, we set $q(t) = (x(\cdot), y(\cdot), \theta(\cdot))$ where $\theta(t)$ is the angle of the tangent vector with respect to the positive x-semiaxis. This construction is called the *lift* of the curve γ.

We then find suitable controls u_i corresponding to $q(t)$ defined above. In this framework u_1 plays the role of $\|\dot{\gamma}\|$, while u_2 is $\|\dot{\gamma}\| K_\gamma$. Hence, the cost (5) coincides with $J[\gamma]$ defined in (2). Moreover, boundary conditions with orientation can be easily translated to boundary conditions on $(x, y, \theta) \in \mathrm{SE}(2)$.

Notice that, on the contrary, not all trajectories of (4) are lifts of planar curves. Indeed, consider a trajectory of the system with $u_1 \equiv 0$. It represents the rotation of the car on itself. If we consider its projection to the plane $\Pi\colon (x, y, \theta) \mapsto (x, y)$, the curve is reduced to a point; thus $\dot{\gamma} = 0$ and the curvature is undefined.

We will prove that for the optimal control problem (4), (5) on SE(2) we have the existence of minimizers with L^1 controls. Starting from a minimizer of this problem, we will find counterexamples to the existence of minimizers of J on $\mathcal{D}_1$ and $\mathcal{D}_2$.

As regards problems with projective boundary conditions, we study the dynamics given by (4) where we admit also backward displacements (it is the Reeds–Shepp car, see [16]). In this case we do not have to impose $u_1 \geq 0$ and we identify $(x, y, \theta) \simeq (x, y, \theta + \pi)$. Hence this dynamics is naturally defined on the quotient space $\mathrm{SE}(2)/\simeq$. We choose the same cost (5). Also in this case, it is possible to lift planar curves to curves on $\mathrm{SE}(2)/\simeq$. Projective boundary conditions can be easily translated to conditions on $(x, y, \theta) \in \mathrm{SE}(2)/\simeq$.

For the optimal control problem (4), (5) on $\mathrm{SE}(2)/\simeq$ we have the existence of minimizers with L^1 optimal controls. Its consequence for the problem of planar curves is the existence of minimizers of J in $\mathcal{D}_3$.

For both optimal control problems, the basic tool we use to compute a minimizer is the Pontryagin maximum principle (PMP in the following, see [15]). It gives a necessary first-order condition for minimizers. Solutions of the PMP are called extremals; hence minimizers have to be found among extremals. For details, see, e.g., [1].

The structure of the paper is the following. In Section 2 we introduce the group SE(2) and the space $\mathrm{SE}(2)/\simeq$, and we define the optimal control problems on these spaces corresponding to the ones defined in Section 1 on the plane. We then study the optimal control problems and find some properties of minimizers.

Section 3 contains the main results of the paper: we prove Propositions 1–3 using the properties of minimizers of the problems studied in Section 2.

2. SOLUTION OF OPTIMAL CONTROL PROBLEMS

In this section we recall the definition of the two optimal control problems given above. In the first we consider Dubins' car [8]: it can both move forwards and rotate on itself. In the second we have the Reeds–Shepp car [16] that can move forwards, backwards and rotate on itself. Nevertheless, the problems we study are different from the ones studied in [8, 16]. We do not have constraints on the velocity and curvature. We want instead to minimize (in both cases) a cost depending both on the velocity and curvature.

2.1. Dubins' car with length–curvature cost. Dubins' car is a car that can move both forwards and rotate on itself. The dynamics of the car is given by the following control system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = u_1\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + u_2\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}, \qquad u_1 \geq 0, \tag{6}$$

with $u_1, u_2 \in L^1([0,T],\mathbb{R})$. We impose $u_1 \geq 0$ to forbid backward displacements. Observe that u_1 is the planar velocity of the car and u_2 is its angular velocity. The controllability of this system can be checked by hand, and we omit the proof.

We fix a starting point $q_0 = (x_0, y_0, \theta_0)$ and an ending point $q_1 = (x_1, y_1, \theta_1)$. We want to minimize the cost

$$\mathcal{C}[q(\cdot)] = \int_0^T \sqrt{u_1^2 + u_2^2}\, dt \tag{7}$$

over all trajectories of (6) steering q_0 to q_1. Here the end time T is fixed.

Remark 5. This problem is a left-invariant problem on the group of motions of the plane

$$\mathrm{SE}(2) := \left\{ \begin{pmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, (x,y)\in\mathbb{R}^2,\ \theta\in\mathbb{R}/2\pi\mathbb{Z} \right\},$$

where the group operation is the standard matrix operation. Indeed, in this case the dynamics is given by $\dot{g} = u_1 g p_1 + u_2 g p_2$ with

$$p_1 = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad p_2 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

If the constraint $u_1 \geq 0$ is removed, we have a minimal length problem on the sub-Riemannian manifold $(\mathrm{SE}(2), \Delta, \mathbf{g})$ where Δ is the left-invariant distribution generated by p_1 and p_2 at the identity and $\mathbf{g}$ is the metric on Δ defined at g by the condition $\mathbf{g}_g(gp_i, gp_j) = \delta_{ij}$. For details about sub-Riemannian geometry on Lie groups see, e.g., [3]. For a complete study of this sub-Riemannian problem on SE(2) see [12, 17, 18].

As a consequence, the problem of minimization of $\mathcal{C}$ from q_0 to q_1 is equivalent to the same problem from Id to $q_0^{-1}q_1$. For this reason, from now on we will study only problems starting from Id.

2.1.1. *Existence of minimizers and reduction to L^∞ controls.* In this subsection we apply the Filippov existence theorem to the optimal control problem (6), (7) on SE(2), which provides the existence of a minimum. We then prove that it is equivalent to solve this problem with controls $u_i \in L^1$ or with controls $u_i \in L^\infty$. This result permits us to verify that minimizers found via the

PMP, which works in the framework of L^∞ controls, are also minimizers in the larger class of L^1 controls.

We first transform problem (6), (7) into a minimum time problem. It is a standard procedure to transform problem (6), (7) with fixed final time T into a problem in which the dynamics is given again by (6), the cost is the time (which is free) and the constraints on the controls are $u_1 \geq 0$ and $u_1^2 + u_2^2 \leq 1$.

We apply the Filippov existence theorem for minimum time problems (see, e.g., [1, Corollary 10.2]), which gives a minimizer and hence L^1 optimal controls.

We now prove that we can restrict ourselves to L^∞ optimal controls. This generalization cannot be proved in general. Indeed, the Lavrentiev phenomenon may occur for more general dynamics and costs; i.e., there may exist a trajectory with L^1 controls such that its cost is strictly less than the cost of all trajectories with L^∞ controls, in particular solutions of the PMP. For details see, e.g., [11].

We thus restrict ourselves to minimal length problems on a trivializable sub-Riemannian manifold with constraints on the values of controls.

Lemma 1. *Consider a minimal length problem on a trivializable sub-Riemannian manifold* $(M, \Delta, \mathbf{g})$ *with constraints on the values of the control, i.e.*

$$\dot{q} = \sum_{i=1}^{m} u_i F_i(q), \qquad u(t) \in V \subset \mathbb{R}^n, \qquad \mathcal{C}[q(\cdot)] = \int_0^T \left(\sum_{i=1}^{m} u_i^2 \right)^{1/2} dt \to \min, \tag{8}$$

where $\Delta(q) = \operatorname{span}\{F_1, \dots, F_m\}$ *and* $\mathbf{g}_q(F_i(q), F_j(q)) = \delta_{ij}$. *Assume that the set* V *satisfies*

$$aV = \{av \mid v \in V\} \subset V \qquad \text{for all } a \in \mathbb{R}^+ \cup \{0\}.$$

If there exists a minimizer $\bar{q}(t)$ *of this problem with optimal controls* $\bar{u}_i \in L^1$, *then there exist other optimal controls* $\widehat{u}_i \in L^\infty$ *such that the corresponding trajectory is a minimizer that is a reparametrization of* $\bar{q}$.

Proof. Let $\bar{q}\colon [0,T] \to M$ be a minimizer of problem (8) with optimal controls in L^1. Define

$$f(t) := \int_0^t \sqrt{\mathbf{g}(\dot{\bar{q}}(\tau), \dot{\bar{q}}(\tau))}\, d\tau = \int_0^t \left(\sum_{i=1}^{m} u_i^2(\tau) \right)^{1/2} d\tau,$$

which is a function from $[0,T]$ to $[0,L]$, with $L = \mathcal{C}[\bar{q}]$. The function f is absolutely continuous and nondecreasing; hence the set R of its regular values is of full measure in $[0,L]$. We also define $g\colon [0,L] \to [0,T]$ by

$$g(s) := \inf\bigl\{t \in [0,T] \mid f(t) = s\bigr\}.$$

One can easily check that $f(g(s)) = s$ for all $s \in [0,L]$.

Moreover, if g is discontinuous at s_0, then $f^{-1}(s_0)$ is a closed interval of the form $[t_0, t_1]$ and for any $t \in [t_0, t_1]$ one has $\int_{t_0}^{t} \sqrt{\mathbf{g}(\dot{\bar{q}}(\tau), \dot{\bar{q}}(\tau))}\, d\tau = 0$; hence $\bar{q}(t) = \bar{q}(t_0) = \bar{q}(g(s_0))$. This also proves that $\bar{q} \circ g$ is continuous.

We also find that $\bar{q} \circ g$ is a 1-Lipschitzian function (hence absolutely continuous) since

$$d(\bar{q}(g(s_0)), \bar{q}(g(s_1))) \leq \int_{g(s_0)}^{g(s_1)} \sqrt{\mathbf{g}(\dot{\bar{q}}(\tau), \dot{\bar{q}}(\tau))}\, d\tau = |s_0 - s_1|,$$

where $d(\cdot,\cdot)$ is the sub-Riemannian distance

$$d(q_0, q_1) := \inf\bigl\{\mathcal{C}[q(\cdot)] \bigm| q(\cdot) \text{ satisfies (8) and steers } q_0 \text{ to } q_1\bigr\}.$$

For $s \in R$, g is differentiable at s and its derivative is $\dot{g}(s) = \frac{1}{f(g(s))}$. We also find that $\bar{q}$ is differentiable at $g(s)$ because s is a regular value of f, which implies that $\dot{\bar{q}}$ is defined. Hence one can easily compute $\mathbf{g}(\dot{\bar{q}}(\tau), \dot{\bar{q}}(\tau)) = 1$. Moreover, for $s \in R$ we have

$$\frac{d(\bar{q} \circ g)}{ds}(s) = \dot{g}(s)\dot{\bar{q}}(g(s)) = \frac{1}{\sqrt{\sum_{i=1}^{m} u_i^2(g(s))}}\dot{\bar{q}}(g(s));$$

hence $\bar{q} \circ g$ is an admissible curve corresponding to the controls

$$\widetilde{u}_i(s) = \frac{\bar{u}_i(g(s))}{\sqrt{\sum_{i=1}^{m} u_i^2(g(s))}},$$

which are L^∞ controls.

Once these L^∞ controls on $[0, L]$ are found, make a linear reparametrization of the time $s \mapsto \frac{sT}{L}$ and a corresponding rescaling of the controls $\widetilde{u}_1 \mapsto \widehat{u}_i := \frac{\widetilde{u}_i L}{T}$. We have now a reparametrization of the same trajectory $\bar{q}$ with controls $\widehat{u}_i$ bounded by $\frac{L}{T}$, hence L^∞, on the interval $[0, T]$. □

Remark 6. Our problem (6), (7) satisfies the hypotheses of Lemma 1, with $V = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 \geq 0\}$.

Remark 7. A consequence of Lemma 1 is that it is equivalent to look for L^1 or for L^∞ optimal controls. Indeed, if we have a minimizer with controls in L^1, then we reparametrize them and find a minimizer with controls in L^∞. On the other hand, if we have a minimizer $\bar{q}(t)$ over the set of controls in L^∞, it is also a minimizer over the set of controls in L^1. We prove this by contradiction. Let $\widetilde{q}(t)$ be a trajectory with controls in L^1 and with cost less than $\mathcal{C}[\bar{q}(t)]$. We reparametrize $\widetilde{q}(t)$ and find a trajectory with controls in L^∞ and with the same cost; hence $\bar{q}(t)$ is not a minimizer. Contradiction.

2.1.2. *Computation of extremals.* We now apply the PMP to problem (6), (7) transformed into a minimum time problem. For the expression of the PMP for minimum time problems see, e.g., [1, Ch. 12]. The control-dependent Hamiltonian of the system is

$$H(q, \lambda, u) = \langle \lambda, \dot{q} \rangle = u_1 h_1 + u_2 h_2 \tag{9}$$

where $h_1 = \lambda_x \cos\theta + \lambda_y \sin\theta$, $h_2 = \lambda_\theta$, and λ_x, λ_y, λ_θ are the components of the covector λ in the dual basis with respect to the coordinates (x, y, θ). Notice that H can be seen as the scalar product $(u_1, u_2) \cdot (h_1, h_2)$ in $\mathbb{R}^2$.

We do not give a complete synthesis of the problem, since we only need to find a particular minimizer to use it in the proofs of Propositions 1 and 2.

We first consider normal extremals, for which we can choose $H = 1$. Let us denote by α and ρ an angle and a positive number in such a way that $\lambda_x = \rho\cos\alpha$ and $\lambda_y = \rho\sin\alpha$.

Let us assume that at $t = t_0$ we have $h_1(t_0) > 0$. Then the PMP gives the controls $u_1 = h_1$ and $u_2 = h_2$, after having normalized $\|(h_1, h_2)\| = 1$. The dynamics is given by

$$\begin{cases} \dot{x} = h_1 \cos\theta, \\ \dot{y} = h_1 \sin\theta, \\ \dot{\lambda}_x = \dot{\lambda}_y = 0, \\ \dot{\theta} = h_2, \\ \dot{\lambda}_\theta = h_1(-\lambda_x \sin\theta + \lambda_y \cos\theta). \end{cases} \tag{10}$$

We have $|\theta(t_0)-\alpha|<\frac{\pi}{2}$ in $\mathbb{R}/2\pi\mathbb{Z}$, and the equation for θ is

$$2\ddot{\theta}=\rho^2\sin(2(\theta-\alpha)).$$

It is the equation of the pendulum with $\theta=\alpha$ being the unstable equilibrium. This implies that when starting with $|\theta(t_0)-\alpha|<\frac{\pi}{2}$ in $\mathbb{R}/2\pi\mathbb{Z}$, θ will reach a value such that $|\theta(t_0)-\alpha|>\frac{\pi}{2}$ in $\mathbb{R}/2\pi\mathbb{Z}$. Hence the corresponding extremal will have a time $t_1>t_0$ for which $h_1(t_1)<0$.

Let us assume now that at $t=t_0$ we have $h_1(t_0)\le 0$. Then the PMP gives the controls $u_1=0$ and $u_2=\operatorname{sign}(h_2)$. In this case, the extremal corresponds to a rotation on itself. Indeed, the dynamics is given by

$$\begin{cases}\dot{x}=\dot{y}=\dot{\lambda}_x=\dot{\lambda}_y=\dot{\lambda}_\theta=0,\\ \dot{\theta}=\operatorname{sign}(h_2).\end{cases}\tag{11}$$

Since λ_x and λ_y are constant, either they are both vanishing along the whole extremal (that is, $h_1\equiv 0$) or at least one of them is nonvanishing. In this case, there exists $t_1>t_0$ such that $h_1(t_1)>0$.

As already stated, for an extremal satisfying $h_1(t_0)>0$ (respectively, $h_1(t_0)<0$) there exists a time $t_1>t_0$ such that $h_1(t_1)<0$ (respectively, $h_1(t_1)>0$). Thus an extremal is the concatenation of trajectories satisfying (11), i.e. pure rotations, and trajectories satisfying (10), which are arcs of a pendulum in θ.

Consider an arc $\gamma([t_0,t_1])$ satisfying (11) between two arcs satisfying (10). Then the variation of θ along this arc should be π because $\theta(t_0)=\alpha+\frac{\pi}{2}\bmod\pi$ and we should come back to $\theta(t_1)=\alpha+\frac{\pi}{2}\bmod\pi$ at the end with the dynamics $\dot{\theta}=1$. Moreover, one can prove that a concatenation of dynamics (10), (11) and (10) cannot be optimal.

Remark 8. A consequence of this study for the planar problem of minimization of J on $\mathcal{D}_2$ is that planar curves with cusps are extremal, but never minimizers. Indeed, a curve with cusp is the projection of a curve $q(\cdot)$ containing an interval in which $\dot{x}=\dot{y}=0$, while θ has a variation of π. Hence, the nonoptimality of q implies the nonoptimality of the planar curve with cusp.

We finally consider abnormal extremals, for which we have $H=0$. We have two possibilities:

- either $h_1=h_2=0$, for which the trajectory is a straight line (i.e. $\dot{\theta}=0$);
- or $h_2=0$, $h_1<0$, and thus $u_1=0$, for which the trajectory is a pure rotation (i.e. $\dot{x}=\dot{y}=0$).

Abnormal extremals can also be concatenations of these two kinds of trajectories.

2.1.3. *An example of a minimizer.* In this subsection we give an example of a minimizer $\mathbf{q}(\cdot)$ defined on a small interval $[0,2\xi]$ and satisfying (11) on $[0,\xi]$ and (10) on $[\xi,2\xi]$. This trajectory is the basic example that we will use to prove the nonexistence of minimizers of cost (2) both on $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e. Propositions 1 and 2.

Consider a trajectory $q^1(t)$ starting from Id, with given $\lambda_x=-\frac{1}{\sqrt{2}}$, $\lambda_y=0$, and $\lambda_\theta=\frac{1}{\sqrt{2}}$. All quantities related to this trajectory are marked with superscript 1. Since $h_1(0)<0$, we follow the dynamics given by (11) on an interval $[0,t^1]$ and have

$$x^1(t^1)=y^1(t^1)=\lambda^1_y(t^1)=0,\qquad \lambda^1_x(t^1)=-\frac{1}{\sqrt{2}},\qquad \theta^1(t^1)=t^1,\qquad \lambda^1_\theta(t^1)=\frac{1}{\sqrt{2}}.$$

We choose $t^1=\frac{\pi}{2}$ and observe that $h^1_1(t^1)=0$. Recall that on this interval the controls are $u^1_1=0$ and $u^1_2=1$.

Then the dynamics is given by (10) on an interval $[t^1,t^1+s^1]$. Since λ^1_θ is continuous, so is u^1_2. Then $\theta^1(t)=\frac{\pi}{2}+(t-t^1)+o(t-t^1)$ on (t^1,t^1+s^1); thus $h^1_1(t)>0$ on (t^1,t^1+s^1) for a sufficiently

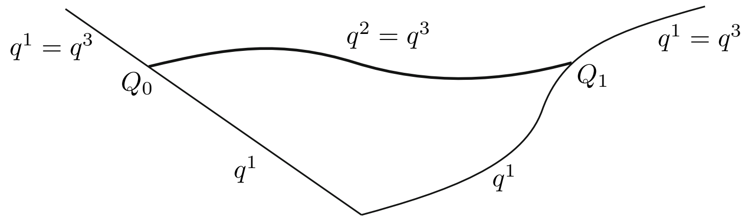

Fig. 3. Construction of the trajectory q^3.

small choice of s^1. As a consequence, we have $x(t) \neq 0$ and $y(t) \neq 0$ for all $t \in (t^1, t^1 + s^1)$, eventually choosing a smaller s^1.

Recall now that all normal extremals are local minimizers; i.e., for each extremal $q(t)$ and time t_0 there exists ε such that $q(\cdot)$ defined on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ is a minimizer between $q(t_0 - \varepsilon)$ and $q(t_0 + \varepsilon)$. For details, see, e.g., [1, Corollary 17.1]. We apply this result to $q^1(t)$ at t^1 and find a corresponding ε^1. Hence we have the minimizer $q^1(t)$ over the interval $[t^1 - \varepsilon^1, t^1 + \varepsilon^1]$. Notice that this minimizer is C^2 but not C^3 at t^1.

We now prove that for a small $\xi < \varepsilon^1$ the trajectory $q^1(t)$ is not only a minimizer, but a unique normal minimizer steering $Q_0 = q^1(t^1 - \xi)$ to $Q_1 = q(t^1 + \xi)$. We prove this by contradiction. Assume that there exists another minimizer q^2 steering Q_0 to Q_1. In what follows all quantities related to this minimizer are marked with superscript 2. As a consequence of the existence of q^2, we have another minimizer $q^3(\cdot)$ steering $q^1(t^1 - \varepsilon^1)$ to $q^1(t^1 + \varepsilon^1)$, given by the concatenation of q^1 on $[t^1 - \varepsilon^1, t^1 - \xi]$, then q^2 on $[t^1 - \xi, t^1 + \xi]$, then again q^1 on $[t^1 + \xi, t^1 + \varepsilon^1]$. See Fig. 3. Since q^3 is a minimizer, it is a solution of the PMP. As a consequence, its tangent covector is continuous. For this reason, we have $\lambda^1(t^1 - \xi) = \lambda^2(t^1 - \xi)$. Since this covector satisfies $h_1 < 0$, the trajectory q^3 satisfies the dynamics given by (11) on a neighborhood of $t^1 - \xi$; hence q^1 and q^2 coincide on this neighborhood due to the uniqueness of a solution for (11). We can prove in the same way that q^1 and q^2 coincide on the whole interval $[t^1 - \xi, t^1)$. Similarly, we have $\lambda^1(t^1 + \xi) = \lambda^2(t^1 + \xi)$; hence q^1 and q^2 coincide in the whole interval $(t^1, t^1 + \xi]$ due to the uniqueness of a solution for (10). Finally, they also coincide at t^1 due to continuity. Hence $q^2 = q^1$ on the interval $[t^1 - \xi, t^1 + \xi]$. Contradiction.

We now define the trajectory $\mathbf{q}$ in SE(2), using q^1 defined on the interval $[t^1 - \xi, t^1 + \xi]$. We first perform a left multiplication of q^1 in order to have $q^1(t^1) = \mathrm{Id}$, and then a time shift $[t^1 - \xi, t^1 + \xi] \mapsto [0, 2\xi]$. The resulting trajectory is $\mathbf{q}(t) := (q^1(t^1))^{-1} q^1(t + t^1 - \xi)$. We recall some properties of this trajectory that we will use in the following:

- $\mathbf{q}(\xi) = \mathrm{Id}$;
- $\mathbf{q}$ is a unique minimizer steering $\mathbf{q}(0)$ to $\mathbf{q}(2\xi)$;
- $\mathbf{q}$ satisfies dynamics (11) on $[0, \xi]$ and (10) on $[\xi, 2\xi]$.

2.2. Projective Reeds–Shepp car with length–curvature cost. The Reeds–Shepp car is a car that can move forwards, backwards and rotate on itself. The set of configurations can be identified with a quotient of the group of motions of the plane $\mathrm{SE}(2)/\simeq$ where $(x, y, \theta) \simeq (x, y, \theta + \pi)$. For a better comprehension we use the same notation as for SE(2), omitting the identification. We also omit verification of good definitions of dynamics and cost given below.

The dynamics of the car is given by the following control system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

where $u_1, u_2 \in L^1([0, T], \mathbb{R})$.

Fix a starting point $q_0 = (x_0, y_0, \theta_0)$ and an ending point $q_1 = (x_1, y_1, \theta_1)$. We want to minimize the cost

$$\mathcal{C}[q(\cdot)] = \int_0^T \sqrt{u_1^2 + u_2^2}\, dt \tag{13}$$

over all trajectories of (12) steering q_0 to q_1. Here the end time T is fixed.

Also in this case, due to the invariance under rototranslation of both the dynamics and the cost, we can study only problems starting from $\mathrm{Id} = (0, 0, 0) = (0, 0, \pi)$ and we will do so throughout the paper.

Controllability is a direct consequence of the Rashevsky–Chow theorem (see, e.g., [1]) for this problem, since the distribution $\mathrm{span}\{(\cos\theta, \sin\theta, 0), (0, 0, 1)\}$ is bracket-generating.

2.2.1. *Computation of extremals.* In this subsection we compute the minimizers for the optimal control problem (12), (13). We follow the procedure presented in Subsections 2.1.1 and 2.1.2.

First transform it into a minimal time problem where the dynamics is given again by (12) and the controls are bounded by $u_1^2 + u_2^2 \le 1$. Following Subsection 2.1.1, we prove that the problem admits a minimum for all pairs of starting and ending points and we restrict ourselves to L^∞ optimal controls. We then apply the PMP, using its expression for a minimal time problem. Since dynamics (12) on $\mathrm{SE}(2)/{\simeq}$ coincides locally with dynamics (6) on SE(2), we have the same control-depending Hamiltonian

$$H(q, \lambda, u) = \langle \lambda, \dot{q} \rangle = u_1 h_1 + u_2 h_2 \tag{14}$$

where $h_1 = \lambda_x \cos\theta + \lambda_y \sin\theta$, $h_2 = \lambda_\theta$, and λ_x, λ_y, λ_θ are the components of the covector λ in the dual basis with respect to the coordinates (x, y, θ).

We can neglect abnormal extremals, since in this case they are trajectories reduced to a point.

We fix $H = 1$ and observe that we do not have the condition $u_1 \ge 0$ in this case. Hence the solutions of the PMP are given by the choice $u_1 = h_1$, $u_2 = h_2$, which corresponds to pendulum oscillations presented in Subsection 2.1.2. The corresponding dynamical system is

$$\begin{cases} \dot{x} = h_1 \cos\theta, \\ \dot{y} = h_1 \sin\theta, \\ \dot{\lambda}_x = \dot{\lambda}_y = 0, \\ \dot{\theta} = h_2, \\ \dot{\lambda}_\theta = h_1(-\lambda_x \sin\theta + \lambda_y \cos\theta). \end{cases} \tag{15}$$

The explicit solution of this problem is given in [12, 17, 18] in the case of SE(2). For our treatment, it is sufficient to observe some properties of extremals. First of all, they are completely determined by the initial covector λ. Moreover, the solution is analytic. As a consequence, we have only one of these possibilities:

- either $h_1 \equiv 0$, and the corresponding extremals are $q(t) = (0, 0, \pm t)$,
- or h_1 has only a finite number of times $t_1, \dots, t_n$ at which it vanishes; hence the corresponding trajectory $q(\cdot)$ has only a finite number of points at which both $\dot{x}$ and $\dot{y}$ vanish.

Notice that trajectories of the second kind can be "well projected" to the plane, i.e. the following holds.

Lemma 2. *Let $q(t) = (x(t), y(t), \theta(t))$ be an extremal for the optimal control problem* (12), (13) *for which h_1 vanishes only for a finite number of times $t_1, \dots, t_n$. Let $p(t) = \Pi(q(t))$ be the projection of q on the plane via $\Pi\colon (x, y, \theta) \mapsto (x, y)$. Then for each time $t \in [0, T]$ we have either $\dot{p}(t) \approx (\cos\theta(t), \sin\theta(t))$ or $\dot{p}(t) = 0$ and $\lim_{\tau\to t^-} \frac{\dot{p}(\tau)}{\|\dot{p}(\tau)\|} \approx \lim_{\tau\to t^+} \frac{\dot{p}(\tau)}{\|\dot{p}(\tau)\|} \approx (\cos\theta(t), \sin\theta(t))$.*

Proof. First notice that $\dot{p} = (u_1 \cos\theta, u_1 \sin\theta)$ since q satisfies (12). Hence it is clear that $\dot{p}(t) \approx (\cos\theta(t), \sin\theta(t))$ if $u_1(t) \neq 0$.

If instead $u_1(t) = 0$, then there exists an interval $(t - \varepsilon, t + \varepsilon)$ on which $u_1(\tau) \neq 0$ for all $\tau \neq t$. Thus $\frac{\dot{p}(\tau)}{\|\dot{p}(\tau)\|} = \frac{(u_1(\tau)\cos\theta(\tau), u_1(\tau)\sin\theta(\tau))}{|u_1(\tau)|} \approx (\cos\theta(\tau), \sin\theta(\tau))$. A passage to limit provides the result at t. $\square$

Remark 9. An interesting property (see [12, 17, 18]) of this second family of extremals is that there are minimizers with one or two points at which $u_1 = 0$, but trajectories with three or more points at which $u_1 = 0$ are never minimizers. Thus minimizers for J over the set $\mathcal{D}_3$ may present one or two cusps, but not more than two.

We will use in the following these properties to prove the existence of a minimizer of J over all curves in $\mathcal{D}_3$. Notice that Lemma 2 does not hold for minimizers of the problem on SE(2) defined in Subsection 2.1, since there are minimizers (like $\mathbf{q}$) such that their projection satisfies $\dot{\mathbf{p}} = 0$ on an interval and $\dot{\mathbf{p}} \neq 0$ on another interval.

3. SOLUTION OF PROBLEMS AND EXISTENCE OF MINIMIZERS

This section contains proofs of the main results of the paper. We first prove Propositions 1 and 2, i.e. the nonexistence of minimizers for the problem of minimization of J in $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. On the other hand, we prove Proposition 3, that is, the existence of a minimizer for the problem of minimization of J in $\mathcal{D}_3$.

3.1. Boundary conditions with orientation: nonexistence of minimizers. In this subsection we give a counterexample to the existence of minimizers of J for boundary conditions with orientation. We prove it both in the case in which curves are chosen to be in $\mathcal{D}_1$ and in $\mathcal{D}_2$. The basic tools we use are the lift of a planar curve to SE(2) (see Remark 4) and the trajectory $\mathbf{q}(t)$ on SE(2) defined in Subsection 2.1.3, which is a solution of the optimal control problem (6), (7) studied in Subsection 2.1.

The basic idea is that we can lift the planar problem to the problem on SE(2), then solve the problem on SE(2) and finally project it again on the plane. But this last step does not work well, since in the case we present below the projection of the solution of the problem on SE(2) does not satisfy the boundary conditions with orientation fixed at the beginning.

Consider the trajectory $\mathbf{q}(t) = (\mathbf{x}(t), \mathbf{y}(t), \boldsymbol{\theta}(t))$ on SE(2) defined in Subsection 2.1.3 on the interval $[0, 2\xi]$. Define its projection $\mathbf{p}(t) := \Pi(\mathbf{q}(t))$ on the plane $\mathbb{R}^2$ via the map $\Pi\colon (x, y, \theta) \mapsto (x, y)$. As already stated, notice that $\dot{\mathbf{p}} = 0$ on $(0, \xi)$ and $\dot{\mathbf{p}} \neq 0$ on $(\xi, 2\xi)$. Then define a sequence of planar curves p^n on the same interval, satisfying the following conditions:

- Each of them satisfies the following boundary conditions with orientation:

$$p^n(0) = \mathbf{p}(0), \qquad p^n(2\xi) = \mathbf{p}(2\xi), \qquad \dot{p}^n(0) \sim (\cos\boldsymbol{\theta}(0), \sin\boldsymbol{\theta}(0)), \qquad \dot{p}^n(2\xi) \sim \dot{\mathbf{p}}(2\xi).$$

- The sequence converges to $\mathbf{p}$.
- The cost $J[p^n]$ converges to $\mathcal{C}[\mathbf{q}]$.

Now, notice that if p^n exists, then it is an example of the fact that each p^n satisfies some boundary conditions with orientation but the limit trajectory $\mathbf{p}$ does not, since $\dot{\mathbf{p}} = 0$ on $(0, \xi)$ and $\dot{\mathbf{p}}(\xi^+) \sim (1, 0)$.

We define the curve p^n with a geometric construction (see Fig. 4). First define p^n on $\left[\xi + \frac{\xi}{n}, 2\xi\right]$ coinciding with $\mathbf{p}$. Then define the point $C := \mathbf{p}(\xi + \frac{\xi}{n})$ and draw the line r that is the tangent to $\mathbf{p}$ or p^n at C. Then draw the line s passing through the origin $O = (0, 0)$ and $(\cos\boldsymbol{\theta}(0), \sin\boldsymbol{\theta}(0))$. Since $\boldsymbol{\theta}(\xi + t) = \boldsymbol{\theta}(\xi) + t + o(t) = t + o(t)$, it follows that $\boldsymbol{\theta}\big(\xi + \frac{\xi}{n}\big) > 0$, while $\boldsymbol{\theta}(0) < 0$; hence r and s are not parallel; thus they have an intersection point B.

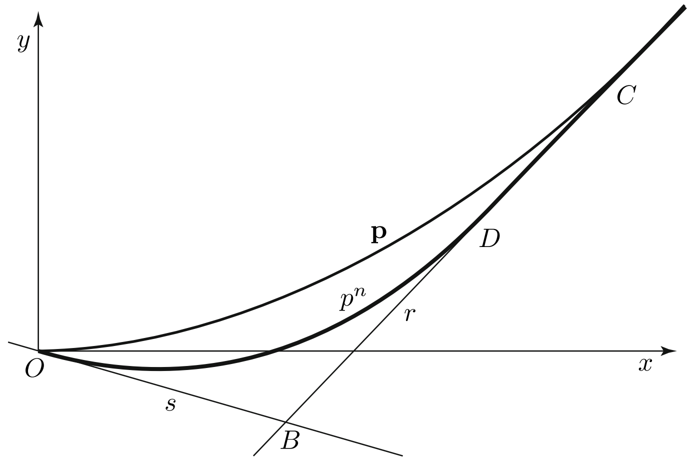

Fig. 4. Construction of the trajectory p^n (case $\mathcal{L}(OB) \le \mathcal{L}(BC)$).

Then we have two cases:

- If $\mathcal{L}(OB) \le \mathcal{L}(BC)$, fix a point D on BC such that $\mathcal{L}(OB) = \mathcal{L}(BD)$ and define the arc $\widehat{OD}$ that is tangent to OB at O and to BC at D. In this case, define p^n on $\left(0, \xi + \frac{\xi}{n}\right)$ as the concatenation of the arc $\widehat{OD}$ and the segment DC.
- If instead $\mathcal{L}(OB) \ge \mathcal{L}(BC)$, fix D on OB satisfying $\mathcal{L}(BD) = \mathcal{L}(BC)$ and make the construction of the arc $\widehat{DC}$. In this case p^n on $\left(0, \xi + \frac{\xi}{n}\right)$ is the concatenation of the segment OD and the arc $\widehat{DC}$.

Notice that all p^n satisfy boundary conditions with orientation and that the sequence converges to $\mathbf{p}$. Moreover, $J[p^n]$ restricted to the interval $\left[\xi + \frac{\xi}{n}, 2\xi\right]$ coincides with $J[\mathbf{p}]$ on the same interval, which in turn coincides with $\mathcal{C}[\mathbf{q}]$ on the same interval, since $\mathbf{q}$ is the lift of $\mathbf{p}$ (see Remark 4).

Concerning the interval $\left[0, \varepsilon + \frac{\varepsilon}{n}\right]$, we have $C \to B \to O$ as $n \to \infty$; hence $J[DC]$ or $J[OD]$ tends to 0. On the other hand, the cost of the arc $\widehat{OD}$ or $\widehat{DC}$ tends to $-\boldsymbol{\theta}(0)$. Indeed, assume that $\mathcal{L}(OB) \le \mathcal{L}(BC)$ and compute $J[\widehat{OD}]$ with an arclength parametrization. Recall that in this case $\int_a^b K_\gamma\, ds = \alpha_\gamma(b) - \alpha_\gamma(a)$ where $\alpha_\gamma(t)$ is the angle of the tangent vector $\dot\gamma$; thus

$$\alpha_{\widehat{OD}}(D) - \alpha_{\widehat{OD}}(O) = \int_0^{\mathcal{L}(\widehat{OD})} K_\gamma\, ds \le J[\widehat{OD}] \le \int_0^{\mathcal{L}(\widehat{OD})} (1 + K_\gamma)\, ds = \mathcal{L}(\widehat{OD}) + \alpha_{\widehat{OD}}(D) - \alpha_{\widehat{OD}}(O).$$

The result follows by recalling that $\alpha_{\widehat{OD}}(D) \to \boldsymbol{\theta}(\xi + \frac{\xi}{n}) \to \boldsymbol{\theta}(\xi) = 0$. The case $\mathcal{L}(OB) \ge \mathcal{L}(BC)$ can be treated similarly.

We have thus defined a sequence of curves $p^n \in \mathcal{D}_1 \subset \mathcal{D}_2$ minimizing the cost J but such that the limit curve $\mathbf{p}$ does not satisfy boundary conditions with orientation; hence it is not in $\mathcal{D}_2$. We prove that it implies the nonexistence of a minimizer for these boundary conditions. By contradiction, assume that a minimizer of J exists in $\mathcal{D}_2$. Thus its lift $\bar q$ to SE(2) is a minimizer for $\mathcal{C}$ between $\mathbf{q}(0)$ and $\mathbf{q}(2\varepsilon)$. Since $\mathbf{q}$ is a unique normal minimizer between the two points, $\bar q$ is abnormal. Since $(\mathbf{x}(0), \mathbf{y}(0)) \ne (\mathbf{x}(2\varepsilon), \mathbf{y}(2\varepsilon))$ and $\boldsymbol{\theta}(0) \ne \boldsymbol{\theta}(2\varepsilon)$, $\bar q$ is neither a straight line nor a pure rotation; hence it is a concatenation of straight lines and rotations. Its projection is thus a curve with angles, i.e. it is not in $\mathcal{D}_2$. Contradiction.

3.2. Projective boundary conditions with orientation: existence of minimizers. In this subsection we prove the existence of a minimizing curve in $\mathcal{D}_3$ for all choices of projective boundary conditions, i.e. we prove Proposition 3. The basic idea is that in this case we can also lift the problem of planar curves to the problem on SE(2)/$\simeq$ defined above, solve it and then project the solution to the plane. But in this case the whole procedure works well, since the projection of a solution of the problem on SE(2)/$\simeq$ is always a solution of the planar problem. In particular, it satisfies projective boundary conditions.

Fix projective boundary conditions, i.e. fix a starting point (x_0, y_0) with direction v_0 and an ending point (x_1, y_1) with direction v_1. Assume that $(x_0, y_0) \neq (x_1, y_1)$ and v_0, v_1 are nonvanishing vectors. Recall that we want to find a curve $\gamma \in \mathcal{D}_2$ such that $\gamma(0) = (x_0, y_0)$, $\dot{\gamma}(0) \approx v_0$, $\gamma(T) = (x_1, y_1)$, $\dot{\gamma}(T) \approx v_1$ and that is a minimizer of J.

Consider the optimal control defined on SE(2)/$\simeq$ presented in Subsection 2.2 with the following starting and ending points: $q_0 = (x_0, y_0, \theta_0)$ and $q_1 = (x_1, y_1, \theta_1)$ where each θ_i is the angle formed by the vector v_i with respect to the x-axis. Then solve the problem and call $\mathbf{q}(\cdot)$ the minimizing trajectory (which is not necessary unique). The basic remark is that $\mathbf{q}$ is of the second kind (see Subsection 2.2.1), since $(x_0, y_0) \neq (x_1, y_1)$. As proved in Lemma 2, in this case $\dot{\mathbf{p}} \approx (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$ except for a discrete set of points $t_1, \dots, t_n$ on which we have the weaker property $\lim_{\tau\to t_i} \frac{\dot{\mathbf{p}}}{\|\dot{\mathbf{p}}\|} \approx (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$. If we have $\dot{\mathbf{p}}(0) \neq 0$ at the starting point, then $\mathbf{p}$ satisfies projective boundary conditions at the beginning. Otherwise reparametrize $\mathbf{p}$ by the arclength in an interval $[0, \varepsilon]$, which is possible since 0 is a unique point in the interval at which $\dot{\mathbf{p}} = 0$. As a consequence, now $\mathbf{p}$ satisfies boundary conditions at the beginning, since we have $\dot{\mathbf{p}}(0) = \lim_{\tau\to 0} \frac{\dot{\mathbf{p}}}{\|\dot{\mathbf{p}}\|} \approx (\cos\boldsymbol{\theta}(0), \sin\boldsymbol{\theta}(0))$. The same result can be proved for the ending point. Hence $\mathbf{p}$ satisfies projective boundary conditions.

We now prove that $\mathbf{p}$ is a minimizer of J, by contradiction. Assume that there exists $\bar{p}$ satisfying the same projective boundary conditions and such that $J[\bar{p}] < J[\mathbf{p}]$. Thus its lift $\bar{q}$ steers q_0 to q_1 and satisfies $\mathcal{C}[\bar{q}] < \mathcal{C}[\mathbf{q}]$; hence $\mathbf{q}$ is not a minimizer. Contradiction.

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