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Theory and Optimization of Generalized Maximum Intuitionistic Fuzzy Entropy Methods

Mohammad Al-Talib*[ab], Amjad Al-Nasser†[a], Bara' Al-Juneidi‡[a], and Nihal Ince§[c]

[a]*Department of statistics, Yarmouk University, Irbid-Jordan 21163.*
[b]*Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahed University, 31952 AlKhobar, Kingdom of Saudi Arabia.*
[c]*Eskişehir Teknik Üniversitesi,Eskişehir-Turkey 26555.*

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The studying and formulating of the generalized maximum fuzzy entropy methods for Intuitionistic Fuzzy Entropy is the focus of this manuscript. The methods were constructed by finding two generalized maximum fuzzy entropy distributions as MinMaxFE and MaxMaxFE, which gives the least and the greatest values of the entropy based on membership function values. We define the optimization problem and study the existence of the solution subject to moment constraints through Lagrange multiplier method. Real life application of data sets in medical fields and in image processing is studied to show whether the developed method can be applied successfully in fuzzy data analysis, and the performance of these distributions is measured using chi-square, RMSE, MFE criteria.

keywords: Maximum fuzzy entropy, Fuzzy set theory, Intuitionistic fuzzy entropy, Entropy optimization distributions, Lagrange multipliers.

*Corresponding author: m.altalib@yu.edu.jo; maltalib@pmu.edu.sa
†amjadn@yu.edu.jo
‡bara.jun20@gmail.com
§nihalyilmaz@eskisehir.edu.tr

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1 Introduction

In fuzzy set theory, the determination of uncertainty distributions of fuzzy values is an important problem and needs to be estimated with available information about fuzzy values. Maximum Entropy Method (MaxEnt) can successfully solve this problem by maximizing the Shannon entropy measure subject to moment constraints. Following that, the idea was introduced and generalized to fuzzy entropy measures which maximize the value of fuzzy entropy subject to some moment constraints.
The methods of Generalized Maximum Fuzzy Entropy studies the membership function based on moments constraints. This methodology introduces special functions on the given compact set of moment vector functions in the form of two distributions, referred to in the literature as MinMaxFE and MaxMaxFE, which gives the least and the greatest values of the entropy based on membership function values.
Maximum entropy principle (classical) was proposed by Jaynes (1957),Jaynes (2003) to determine the distribution for information and determine the probability constraint, he called this distribution "maximum entropy probability distribution" or "least unbiased probability distribution".
The Maximum entropy principle approach did not have clear mechanism for taking this uncertainty in consideration. Jaynes (1957) found this problem and spoke about it in his paper, and he proposed three ways to solve it. The first is to ignore this problem. The second is to generalize the maximum entropy approach. The third solution is adding extra variance constraints. It appears that the first and last solutions cannot be proven, while the second solution follows the laws of probability. (Cheeseman and Stutz (2005))
With no information about the distribution, the best candidate is the distribution that maximizes uncertainty. However, if some information is available of the moments of the distribution and using the mathematical expression of the entropy of the distribution, it is obvious to generalize it and pick the distribution with the highest uncertainty.
In this manuscript, we focus on studying and formulating the generalized maximum fuzzy entropy methods for Intuitionistic Fuzzy Entropy (IFE); and we introduce what we called Generalized Maximum Intuitionistic Fuzzy Entropy (GMIFE) methods.
The center of attention in this field and s special concern of such entropies is the one introduced by Vlachos and Sergiadis (2007). In section 2 we present an introduction of entropy optimization methods. In section 3, we introduced and studied the generalized maximum methods of fuzzy entropy and intuitionistic fuzzy entropies and studied the existence of the solution subject to moment constraints through Lagrange multiplier method. In section 4, we set two real life data examples to the application of the methods. And finally we setup our conclusions and finding remarks.

2 Entropy Optimization Methods

In entropy optimization theory, the method of generalized maximum fuzzy entropy (GMFE) consists of finding the distribution that maximizes the entropy based on some

constraints. Jaynes (2003) stated that the natural choice would be to pick the distribution with the highest entropy, called the MFE distribution. Jaynes's exact words were: "If the information incorporated into the maximum-entropy analysis includes all the constraints actually operating in the random experiment, then the distribution predicted by maximum entropy is overwhelmingly the most likely to be observed experimentally." Computing the MFE distribution has applications in randomized rounding and the design of approximation algorithms. More precisely, it has been shown how to improve the approximation ratio for the symmetric and the asymmetric traveling salesman problem via MFE distributions (Asadpour and Saberi (2007)). Often, it is important to efficiently compute the MFE distribution; for example, the zero-information moment closure method (Smadbeck and Kaznessis (2013)), a recent approximate dynamic programming method for constrained Markov decision processes, as well as the approximation of the channel capacity of a large class of memoryless channels. Sutter et al. (2019) dealt with iterative algorithms that require the numerical computation of the MFE distribution in each iteration step. Also, the applications spread to cover modeling of stochastic data (Sutter et al., 2019), Statistical Physics, Biology (Uddin et al., 2019) and psychology (Ciavolino et al., 2014) and regression models (Ciavolino and Calcagnì, 2016, 2014).
Shannon (1948) stated that a straightforward application of calculus can be used to derive maximum entropy (ME) distributions. Maximizing the fuzzy entropy measure subject to some conditions is studied in the literature as a method for deriving the forms of minimal information prior distributions or as know as well by Generalized Maximum Fuzzy Entropy (GMFE) methods. Among others, Jaynes (1957), Zellner and Highfield (1988) and Jaynes (2003) has studied examples of discrete entropies. On the other hand, Lisman et al. (1972), Al-Talib and Al-Nasser (2018) considered continuous cases. Akaike (1983) studied the difference between the entropies of the product of two random variables and their mutual entropy; they showed that the minimum information prior distribution that maximizes such criterion will maximize the dependence between these variables. Sutter et al. (2019) considered the problem of estimating a probability distribution that maximizes the entropy while satisfying a finite number of moment constraints, possibly corrupted by noise.

3 Generalized Maximum Fuzzy Entropy Methods

The aim of GMFE method is to maximize the entropy of fuzzy sets by finding two distributions of membership functions. For illustration, Nihal and SHAMİLOV (2017); a newer version of a three article series (Shamilov et al. (2016), Shamilov and İnce (2016),ŞAMİLOV et al. (2017)) studied De Luca and Termini (1972) fuzzy entropy (denoted below by $E_{DT}(A)$) and they followed these steps:
1. Obtain MFE measure which maximizes the value of fuzzy entropy, which is subject to moment constraints, by Lagrange multipliers method.

Maximize

$$E_{DT}(A) = -\frac{1}{n}\sum_{i=1}^{n}\Big[\mu_A(x_i)\ln\mu_A(x_i) + \big(1-\mu_A(x_i)\big)\ln\big(1-\mu_A(x_i)\big),\Big]. \quad (1)$$

subject to

$$\sum_{i=1}^{n}\mu_A(x_i)g_j(x_i) = \mu_j \quad , j = 0, 1, ..., m. \quad (2)$$

where, μ_k are moments values of $\mu_A(x_i)$ with respect to moments functions $g_k(x_i)$, and $m \leq n$.

2. Generate two distributions denoted by MinMaxFE and MaxMaxFE, which are the moment functions giving the least and the greatest value of MFE, respectively. The process of obtaining the distributions is defined in Step 2, is referred to as the Generalized Maximum Fuzzy Entropy Methods. The solution of De Luca and Termini measure exists and was found given that several conditions are satisfied (Shamilov et al. (2016),ŞAMİLOV et al. (2017)).

3.1 Generalized Maximum Intuitionistic Fuzzy Entropy

The investigation of the IFE under the main purpose; to reduce the uncertainty of IFS by finding two bivariate distributions of membership functions and non-membership functions.

In fuzzy theory, Maximum Fuzzy Entropy (MFE) problem consists of maximizing the fuzzy entropy measure that is subject to its constraints. Vlachos and Sergiadis (2007) introduced their version of intuitionistic fuzzy entropy, and it has been the center of attention in any articles. Denoted by E_{VS} and given by,

$$\begin{aligned} E_{VS}(A) = -\frac{1}{n\ln 2}\sum_{i=0}^{n}\Big[&\mu_A(x_i)\ln\mu_A(x_i) + v_A(x_i)v_A(x_i) \\ &+ \big(1-\pi_A(x_i)\big)\ln\big(1-\pi_A(x_i)\big) - \pi_A(x_i)\ln 2\Big] \end{aligned} \quad (3)$$

where, $\pi_A(x_i) = 1 - \mu_A(x_i) - v_A(x_i)$.

Maximization Vlachos and Sergiadis's intuitionistic fuzzy entropy (E_{VS}) consists of maximizing the entropy measure with respect to membership function $\mu_A(x)$ with finite number of the fuzzy values $\mu_A(x_i), i = 0, 1, ..., n$ and the non-membership function $v_A(x)$ with finite number of the fuzzy values $v_A(x_i), i = 0, 1, ..., n$. In more details, we consider maximizing E_{VS} subject to the constraints:

$$\begin{aligned} &\sum_{i=0}^{n}\mu_A(x_i)g_j(x_i) = \mu_j \quad , j = 0, 1, ..., m. \\ &\sum_{i=0}^{n}v_A(x_i)h_j(x_i) = v_j \quad , j = 0, 1, ..., m. \end{aligned} \quad (4)$$

where, $g_0(x) = h_0(x) = 1; \mu_0 = v_0 = 1$; μ_j and v_j are the expected value of the fuzzy values $\mu_A(x_i)$ and $v_A(x_i)$ with respect to the linearly independent moment functions $g_j(x_i) = h_j(x_i)$, respectively for $j = 1, ..., m; i = 0, 1, ..., n; m < n$.
Using Lagrange multiplier method the optimization function is given by

$$U = -\frac{1}{n \ln 2} \times \sum_{i=1}^{n} \Big[\mu_A(x_i) \ln \mu_A(x_i) + v_A(x_i) \ln v_A(x_i) + \big(1 - \pi_A(x_i)\big) \ln \big(1 - \pi_A(x_i)\big) - \pi_A(x_i) \ln 2 \Big] - \sum_{j=0}^{m} \lambda_j \left[\sum_{i=0}^{n} \mu_A(x_i) g_j(x_i) - \mu_j \right] - \sum_{j=0}^{m} \delta_j \left[\sum_{i=0}^{n} v_A(x_i) h_j(x_i) - v_j \right] \tag{5}$$

This can be rewritten as

$$U = -\frac{1}{n \ln 2} \sum_{i=1}^{n} \Big[\mu_A(x_i) \ln \mu_A(x_i) + v_A(x_i) \ln v_A(x_i) + \big(\mu_A(x_i) + v_A(x_i)\big) \ln \big(\mu_A(x_i) + v_A(x_i)\big) - (1 - \mu_A(x_i) - v_A(x_i)) \ln 2 \Big] - \sum_{j=0}^{m} \lambda_j \left[\sum_{i=0}^{n} \mu_A(x_i) g_j(x_i) - \mu_j \right] - \sum_{j=0}^{m} \delta_j \left[\sum_{i=0}^{n} v_A(x_i) h_j(x_i) - v_j \right] \tag{6}$$

To start with finding the optimal values of the optimization function, we derive U with respect to $\mu_A(x_i)$, we get

$$\frac{\partial U}{\partial \mu_A(x_i)} = \left[- \ln \mu_A(x_i) - 1 + \ln \big(\mu_A(x_i) + v_A(x_i)\big) + 1 - \ln 2 - \sum_{j=0}^{m} \lambda_j g_j(x_i) \right].$$

Simplifying the derivative and equating it to zero, we get

$$\ln \left(\frac{\mu_A(x_i) + v_A(x_i)}{\mu_A(x_i)} \right) - \ln 2 - \sum_{j=0}^{m} \lambda_j g_j(x_i) = 0$$

so,

$$\frac{\mu_A(x_i) + v_A(x_i)}{\mu_A(x_i)} = 2 + e^{\sum_{j=0}^{m} \lambda_j g_j(x_i)},$$

hence,

$$\mu_A(x_i) + v_A(x_i) = 2\mu_A(x_i) + \mu_A(x_i) e^{\sum_{j=0}^{m} \lambda_j g_j(x_i)},$$

which directly follows

$$\mu_A(x_i) = \frac{v_A(x_i)}{1 + e^{\sum_{j=0}^{m} \lambda_j g_j(x_i)}}, i = 0, 1, ..., n. \tag{7}$$

The second derivative is given as

$$\frac{\partial U^2}{\partial \mu_A^2(x_i)} = \frac{-v_A(x_i)}{\mu_A(x_i)(\mu_A(x_i) + v_A(x_i))} \tag{8}$$

as, $\mu_A(x_i)$ and $v_A(x_i)$ are always greater than zero, then the second derivation is negative; that is mean $\mu_A(x_i)$ in Equation 7 is what makes the E_{VS} as large as possible. Deriving the optimization function with respect to λ we get,

$$\frac{\partial U}{\partial \lambda} = -\sum_{i=0}^{n} \mu_A(x_i) g_j(x_i) - \mu_j \ \ , j = 0, 1, ..., m. \tag{9}$$

Now, deriving the optimization function with respect to $v_A(x_i)$;

$$\frac{\partial U}{\partial v_A(x_i)} = \ln(v_A(x_i)) - \ln\big(\mu_A(x_i) + v_A(x_i)\big) + \ln 2 - \sum_{j=0}^{m} \delta_j h_j(x_i),$$

simplifying it and equating to zero, we get

$$\ln\left(\frac{\mu_A(x_i) + v_A(x_i)}{\mu_A(x_i)}\right) - \ln 2 + \sum_{j=0}^{m} \delta_j h_j(x_i) = 0,$$

hence,

$$v_A(x_i) = \frac{\mu_A(x_i)}{1 - e^{\sum_{j=0}^{m} \delta_j h_j(x_i)}}, i = 0, 1, ..., n. \tag{10}$$

Now, the second derivative follows to be

$$\frac{\partial U^2}{\partial v_A^2(x_i)} = \frac{-\mu_A(x_i)}{v_A(x_i)(\mu_A(x_i) + v_A(x_i))}, \tag{11}$$

$\mu_A(x_i)$ and $v_A(x_i)$ are always greater than zero, then the second derivation is negative. Deriving the optimization function with respect to δ we get ,

$$\frac{\partial U}{\partial \delta} = -\sum_{i=0}^{n} v_A(x_i) h_j(x_i) - v_j \ \ , j = 0, 1, ..., m. \tag{12}$$

Also,

$$\frac{\partial U^2}{\partial \mu_A(x_i) \partial v_A(x_i)} = \frac{\partial U^2}{\partial v_A(x_i) \partial \mu_A(x_i)} = \frac{-1}{\big(\mu_A(x_i) + v_A(x_i)\big)^2} < 0.$$

Depending on Equations 9 and 12, the eigenvalues of the Hessian matrix for$\mu_A(x_i)$ and $v_A(x_i)$ negative, where the Hessian matrix is a square matrix of second order partial derivatives of a scalar-valued function, which means that matrix is negative defined. The direct conclusion of this result is that the IFE of Vlachos and Sergiadis is convex and reaches its maximum value at a critical point.

The produced solution of the maximization problem is $\Big(\big(\mu_A(x_0), v_A(x_0)\big), \big(\mu_A(x_1), v_A(x_1)\big), ..., \big(\mu_A(x_n), v_A(x_n)\big)\Big)$, such that

$$\begin{aligned} \mu_A(x_i) &= \frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}} \quad , i=0,1,...,n. \\ v_A(x_i) &= \frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}} \quad , i=0,1,...,n. \end{aligned} \tag{13}$$

where, $\mu_A^*(x_i)$ and $v_A^*(x_i)$ are initial values of the membership and non-membership function.

Substituting the solution in the optimization function presented in equation (3.5), we get the MFE.

$$\begin{aligned} U(g,h) &= \\ &-\frac{1}{n\ln 2}\sum_{i=1}^{n}\Bigg[\frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}\ln\frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}} \\ &+\frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}}\ln\frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}} \\ &+\left(\frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}+\frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}}\right)\ln\left(\frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}+\frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}}\right) \\ &-\left(1-\frac{v_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\lambda_j g_j(x_i)}}-\frac{\mu_A^*(x_i)}{1+e^{\sum_{j=0}^{m}\delta_j h_j(x_i)}}\right)\ln 2\Bigg]+\sum_{j=0}^{m}\lambda_j\mu_j+\sum_{j=0}^{m}\delta_j v_j. \end{aligned} \tag{14}$$

The formula of the MFE depends on the moment functions $g_j(x), h_j(x); j=0,1,...,m$, the Lagrange multipliers $\lambda_j, \delta_j; j=0,1,...,m$ and the moment fuzzy values μ_j, v_j , hence the maximum value of the IFE measure is considered to be a joint function of g and h.

3.2 Moment Vector Function and GMIFE Methods

Based on the literature of entropy optimization distribution, the estimated values of $\mu_A(x_i)$ and $v_A(x_i)$which produce the minimum of 3 generates a distribution which is the closest to the membership and non-membership functions, such distribution is referred to MinMaxFE distribution. On the other hand, the values which generates the furthest distribution to the membership and non-membership functions is referred to MaxMaxFE. Both of these distributions depend on finding the moment vector functions $g_j(x), h_j(x); j=0,1,...,m$, which minimize or maximize the function $U(g,h)$.
To clarify, the MinMaxFE and MaxMaxFE are distributions giving minimum and maximum values of $U(g,h)$ respectively among all moment vector functions $g_j(x), h_j(x); j=0,1,...,m$.

Entropy optimization distributions methods to model statistical data depends on the suitable choice of moment functions and the decision of their elements are very important. It is also important to have criteria to compare various moment sets for suitability. As we said briefly earlier, the choice of the moment functions is left to the researcher as it could define some known statistical distributions such as normal, beta, gamma, exponential and Laplace distributions, which maximize the entropy subject to certain constraints. Also, Nihal and SHAMİLOV (2017), presented the concept of relative suitability of moment function sets via MinMaxFE and MaxMaxFE distributions. It is important to state that their findings and restrictions are taking into account in our choice of moment functions.
Now, we present the Entropy Optimization Problems (EOP) and the Generalized Entropy Optimization Problems (GEOP) as defined in the literature,

EOP: Let $p(x)$ be a probability distribution function of a random variable X, L be an entropy optimization measure and g is a given moment vector function generating 'm' of moment constraints. It is required to obtain the distribution corresponding to g, which gives extremum value to L.

GEOP: Let $p(x)$ be a probability distribution function of a random variable X, L be an entropy optimization measure and K be a set of given moment vector functions. It is required to choose moment vector functions $g^{(0)}, g^{(1)} \in K$ such that $g^{(0)}$ defines entropy optimization distribution $f^{(0)}(x)$ closest to $p(x)$.
$g^{(1)}$ defines entropy optimization distribution $f^{(1)}(x)$ furthest from $p(x)$ with respect to entropy optimization measure L.

Considering that the optimization function presented in 14 has a solution 13 (say, $p_i = \Big(\mu_A(x_i), v_A(x_i)\Big), i = 0, 1, ..., n)$ subject to the constraints and depending on the moment vector function $(g, h) = \big((1, 1), (g_1, h_1), ..., (g_m, h_m)\big)$.
If the distribution $p^{(0)} = \big(p_1^{(0)}, p_2^{(0)}, ..., p_n^{(0)}\big)$ is given, then the moment values $\mu_j, v_j, j = 0, 1, ..., m$ will be found using moment vector function (g, h).
So, Let us define compact set $K_r = (g_0, h_0), (g_1, h_1), ..., (g_m, h_m)$, then the optimization function $U(g, h)$ reaches its optimum values in this set, because of the continuity property. (Nihal and SHAMİLOV (2017),Shamilov et al. (2016),ŞAMİLOV et al. (2017)).

Define,

$$\min_{(g,h)\in K} U(g, h) = U(g^{(1)}, h^{(1)}) \qquad and \qquad \max_{(g,h)\in K} U(g, h) = U(g^{(2)}, h^{(2)})$$

It is straightforward to state

$$U(g^{(1)}, h^{(1)}) \leq U(g^{(2)}, h^{(2)}).$$

The moment vector function $g^{(1)}, h^{(1)}$ which gives the minimal value of $U(g, h)$, generates the distribution

$$\left(\mu^{(1)}, v^{(1)}\right) = \left(\left(\mu^{(1)}(x_0), v^{(1)}(x_0)\right), \left(\mu^{(1)}(x_1), v^{(1)}(x_1)\right), ..., \left(\mu^{(1)}(x_n), v^{(1)}(x_n)\right)\right)$$

Such distribution is the MinMaxFE$_m$ distribution. Note that, the subscript m to denote the length of the moment vector function.
Similarly,

$$\left(\mu^{(2)}, v^{(2)}\right) = \left(\left(\mu^{(2)}(x_0), v^{(2)}(x_0)\right), \left(\mu^{(2)}(x_1), v^{(2)}(x_1)\right), ..., \left(\mu^{(2)}(x_n), v^{(2)}(x_n)\right)\right)$$

is the MaxMaxFE$_m$ distribution provided that it corresponds to the moment vector function $g^{(2)}, h^{(2)}$ that gives the maximal values of $U(g,h)$.
Obtaining these two distributions are to be called Generalized Maximum Intuitionistic Fuzzy Entropy Methods (GMIFEM).

The following theorem summarizes these findings in what we call the existence theorem.

Theorem 3.1 *The possibility of maximizing 3 with respect to its constraints 4 requires the fulfillment of the following conditions:*

- *Moment functions $g_j(x); j = 0, 1, ..., m$ are linearly independent;*
- *Moment functions $h_j(x); j = 0, 1, ..., m$ are linearly independent;*
- *The inequality $m < n$ is satisfied;*
- *Moment values $\tilde{\mu}_j, j = 0, 1, ..., m$ are given in the form of equalities*

$$\sum_{i=0}^{n} \tilde{\mu}_A(x_i) g_j(x_i) = \tilde{\mu}_j \quad , j = 0, 1, ..., m.$$

- *Moment values $\tilde{v}_j, j = 0, 1, ..., m$ are given in the form of equalities*

$$\sum_{i=0}^{n} \tilde{v}_A(x_i) h_j(x_i) = \tilde{v}_j \quad , j = 0, 1, ..., m.$$

3.3 Finding the Solution

In order to calculate MinMaxFE$_m$ and MaxMaxFE$_m$ distributions for fuzzy data, it is required to following the steps:

1. Determine the moments vector function and $(g, h) = \left((1, 1), (g_1, h_1), ..., (g_m, h_m)\right)$. according to fuzzy data.
2. Calculate the moment values μ_j and v_j subject to each moments vector function $E(g_0(x)), E(g_1(x)), ..., E(g_m(x))$ and $E(h_0(x)), E(h_1(x)), ..., E(h_m(x))$.

3. Obtain the values of the Lagrange multipliers λ_j, δ_j by substituting the solution of $\mu_A(xi)$ and $v_A(xi)$ found in 13 into constraints 4, using initial points of the Lagrange multipliers $\lambda_j, \delta_j; j = 0, 1, ..., n$.

4. Determine MinMaxFE$_m$ and MaxMaxFE$_m$ distributions corresponding to selected MFE characterizing moments vector functions.

5. Among obtained distributions choose the accepted Generalized Maximum Intuitionistic Fuzzy Entropy distributions.

It is noted that the selection of moment functions set is important in the application of MFE method. In our investigation, MFE characterizing moments $E\{x\}, E\{\ln x\}, E\{(\ln x)^2\}$, $E\{\ln(1+x)\}, E\{\ln(1+x^2)\}$ are acquired by experimental way (see, Shamilov et al. (2016)).
In order to obtain MinMaxFE$_m$ and MaxMaxFE$_m$ $(m = 1, 2, 3)$ distributions, we should choose the moment vector functions giving the maximum and minimum values to the MFE functional $U(g, h)$.

4 Real data applications of life

In this chapter we discuss MinMaxFE$_m$ and MaxMaxFE$_m$ distributions, results are obtained for the membership function and non-membership function values in fuzzy data sets of three real life applications on the IFE and we will study the GMIFE and their results accordingly. It should be noted that mentioned distributions are calculated by using MATLAB program.
In this section we illustrate the methods of GMIFE to two real life applications, for similar choices of moment vector functions (g, h). Here, as explained in a Section (3.3); we used the moment functions

$$g_0(x) = 1, g_1(x) = x, g_2(x) = \ln x, g_3(x) = (\ln x)^2, g_4(x) = \ln(1+x), g_5(x) = \ln(1+x^2),$$

and

$$h_0(x) = 1, h_1(x) = x, h_2(x) = \ln x, h_3(x) = (\ln x)^2, h_4(x) = \ln(1+x), h_5(x) = \ln(1+x^2).$$

According to suggested method,

$$K_0 = \begin{Bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \end{Bmatrix}$$

all combinations of r elements of K_0 taken m elements at a time are denoted by $K_{0,m}$.
In each of the following applications, the performance of MinMaxFE$_m$ and MaxMaxFE$_m$ distributions is tested using various criterias such as Root Mean Square Error (RMSE), Chi-Square (χ^2), and MFE measure (MFE). The best distribution function can be determined according to the lowest values of $RMSE$, χ^2 and MFE measure. It is obtained through comparison of the MinMaxFE$_m$ and MaxMaxFE$_m$ distributions by using the given criteria.

4.1 Defuzzification of Intuitionistic Fuzzy Sets Example

Radhika and Parvathi (2016) studied the process of converting a fuzzy quantity to precise quantity, such process is referred to as defuzzification. A special concern in their article is intuitionistic defuzzification (converting membership and non-membership values to crisp), and they listed seven types to such conversions. In one application, Radhika and Parvathi (2016) considered the gray levels extracted from images to form a two-dimensional gray matrix. In image processing, the statistical method of examining image texture that considers the spatial relationship of pixels is the gray matrix (also known as gray-level co-occurrence matrix (GLCM) and known as the gray-level spatial dependence matrix).

$A_{3\times 3}$ gray matrix extracted from an image whose gray values vary from 0 to 256 is

$$A = \begin{bmatrix} 50 & 120 & 192 \\ 202 & 220 & 166 \\ 256 & 32 & 64 \end{bmatrix}$$

With corresponding fuzzy values; $\Big(\mu_A(x_i), v_A(x_i)\Big)$

$$A = \begin{bmatrix} (0.3896, 0.6094) & (0.9990, 0.0000) & (0.4990, 0.5000) \\ (0.4206, 0.5781) & (0.2803, 0.7188) & (0.7021, 0.2969) \\ (0.0000, 0.9990) & (0.2490, 0.7500) & (0.4990, 0.5000) \end{bmatrix}$$

MFE values subject to moment constraints generated by elements of $K_{0,m}$, $m = 1, 2, 3$ are studied and investigated briefly. In the table below, we present MinMaxFE$_m$ and MaxMaxFE$_m$ in the case when $m = 1$.

Table 1: Entropy of calculated MFE values subject to two moment functions

Moment Functions $(g_1(x), h_1(x))$	Fuzzy Entropy
$\big((1,1),(x,x)\big)$	0.9902
$\big((1,1),(\ln x,\ln x)\big)$	0.9909
$\big((1,1),((\ln x)^2,(\ln x)^2)\big)$	0.9903
$\big((1,1),(\ln(1+x^2),\ln(1+x^2))\big)$	0.9908

Tables 2 and 3 show that in the sense of $RMSE$ and χ^2 criteria each of MinMaxFE$_m$, $(m = 1, 2, 3)$ distribution is better than each of MaxMaxFE$_m$,$(m = 1, 2, 3)$ distribution. Also, from these tables, MinMaxFE$_1$ distribution show better fitting in terms of almost all criteria than other MinMaxFE$_m$ and all MaxMaxFE$_m$ $(m = 1, 2, 3)$ distributions.

4.2 Medical Diagnosis

Let us consider the example discussed by Vlachos and Sergiadis (2007), (same data as in De et al. (2001); Szmidt and Kacprzyk (2001)). Similarity and dissimilarity between symptoms and diseases in medical diagnosis brought the attention of many researchers in medical fields to fuzzy set theory and in particular to intuitionistic fuzzy set and its entropy.

The application studied here consists of a set of diagnoses D = {Viral fever, Malaria, Typhoid, Stomach problem, Chest pain}, and a set of symptoms S = {Temperature, Headache, Stomach pain, Cough, Chest pain}. Each element of the tables, is given in the form of a pair of numbers corresponding to the membership and non-membership values.(Data are presented in Table 4).

Table 2: The obtained results for MinMaxFE_m ,$m = 1, 2, 3$ distributions

Distributions of MinMaxFE_m	Moment Constraints	MFE	$RMSE$	χ^2
MinMaxFE_1	$\big((1,1),(x,x)\big)$	0.9902	0.8325	6.4868
MinMaxFE_2	$\big((1,1),(\ln x,\ln x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.7469	0.4523	2.2856
MinMaxFE_3	$\big((1,1),(x,x),(\ln x,\ln x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.7683	0.3931	2.4806

Table 3: The obtained results for MaxMaxFE_m , $m = 1, 2, 3$ distributions

Distributions of MaxMaxFE_m	Moment Constraints	MFE	$RMSE$	χ^2
MaxMaxFE_1	$\big((1,1),(\ln x,\ln x)\big)$	0.9909	0.8319	6.6636
MaxMaxFE_2	$\big((1,1),(x,x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.9750	0.8325	6.4643
MaxMaxFE_3	$\big((1,1),(x,x),(\ln x,\ln x),((\ln x)^2,(\ln x)^2)\big)$	0.9753	0.8334	6.6463

Table 4: Entropy of calculated MFE values subject to two moment functions

Symptoms	Diagnosis	$\big(\mu_A(x_i), v_A(x_i)\big)$
	Viral fever	(0.4, 0.0)
	Malaria	(0.7, 0.0)
Temperature	Typhoid	(0.3, 0.3)
	Stomach problem	(0.1, 0.7)
	Chest problem	(0.1, 0.8)
	Viral fever	(0.3, 0.5)
	Malaria	(0.2, 0.6)
Headache	Typhoid	(0.6, 0.1)
	Stomach problem	(0.2, 0.4)
	Chest problem	(0.0, 0.8)
	Viral fever	(0.1, 0.7)
	Malaria	(0.0, 0.9)
Stomach pain	Typhoid	(0.2, 0.7)
	Stomach problem	(0.8, 0.0)
	Chest problem	(0.2, 0.8)
	Viral fever	(0.4, 0.3)
	Malaria	(0.7, 0.0)
Cough	Typhoid	(0.2, 0.6)
	Stomach problem	(0.2, 0.7)
	Chest problem	(0.2, 0.8)
	Viral fever	(0.1, 0.7)
	Malaria	(0.1, 0.8)
Chest pain	Typhoid	(0.1, 0.9)
	Stomach problem	(0.2, 0.7)
	Chest problem	(0.8, 0.1)

Next, MFE values subject to moment constraints generated by elements of $K_{0,m}, m = 1,2,3$. For illustration, we present fuzzy entropy of calculated MFE values subject to two moment functions. In these tables, corresponding to minimum and maximum values of MFE measure MinMaxFE_m and $\text{MaxMaxFE}_m, m = 1,2,3$ distributions are represented with bold font.

Table 5 shows that $(g_0, h_0), (g^{(1)}, h^{(1)}) = ((1,1), (\ln x, \ln x)) \in K_{0,1}$ gives to least value to $U(g, h)$, consequently corresponding distribution is MinMaxFE_1 and $(g_0, h_0), (g^{(2)}, h^{(2)}) =$

$((1,1),(x,x)) \in K_{0,1}$ gives to greatest value to $U(g,h)$, consequently corresponding distribution is MaxMaxFE_1.

Table 5: Entropy of calculated MFE values subject to two moment functions

Moment Functions $(g_1(x), h_1(x))$	Fuzzy Entropy
$\big((1,1),(x,x)\big)$	**0.7844**
$\big((1,1),(\ln x, \ln x)\big)$	**0.7158**
$\big((1,1),((\ln x)^2,(\ln x)^2)\big)$	0.7684
$\big((1,1),(\ln(1+x^2),\ln(1+x^2)\big)$	0.7258

Tables 6 and 7 show that the $RMSE$ and χ^2 of $\text{MinMaxFE}_m, (m = 1,2,3)$ distribution is better than each of $\text{MaxMaxFE}_m, (m = 1,2,3)$ distribution. Moreover, MinMaxFE_3 distribution shows better fitting in terms of almost all criteria MinMaxFE_m and all MaxMaxFE_m distributions.

5 Conclusion

In the present study, we proved the convexity property of new MIFE measure, then it is formulated a Maximum Fuzzy Entropy Problem and proposed sufficient conditions for existence of its solution. A special functional $U(g,h)$ depended on moment vector functions g and h is defined by applying Lagrange multipliers method.

According to obtained results, for this fuzzy data in applications, MinMaxFE_m and $\text{MaxMaxFE}_m, m = 1,2,3$ distributions are compared by using different criterias in terms of modeling data. It is shown that each of $\text{MinMaxFE}_m, m = 1,2,3$ distribution is more suitable in modeling fuzzy data than each of $\text{MaxMaxFE}_m, m = 1,2,3$ distributions in the sense of $RMSE$, χ^2 and MFE criteria.
Consequently, the obtained results are shown that developed methods can be applied successfully in fuzzy data analysis.

Table 6: The obtained results for MinMaxFE_m ,$m = 1, 2, 3$ distributions

Distributions of MinMaxFE_m	Moment Constraints	MFE	$RMSE$	χ^2
MinMaxFE_1	$\big((1,1),(x,x)\big)$	0.7158	0.54877	10.1630
MinMaxFE_2	$\big((1,1),(\ln x,\ln x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.7625	0.2666	4.1588
MinMaxFE_3	$\big((1,1),(x,x),(\ln x,\ln x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.7549	0.2411	3.6142

Table 7: The obtained results for MaxMaxFE_m , $m = 1, 2, 3$ distributions

Distributions of MaxMaxFE_m	Moment Constraints	MFE	$RMSE$	χ^2
MaxMaxFE_1	$\big((1,1),(\ln x,\ln x)\big)$	0.7844	0.6378	11.8349
MaxMaxFE_2	$\big((1,1),(x,x),(\ln(1+x^2),\ln(1+x^2))\big)$	0.8024	0.5195	7.7378
MaxMaxFE_3	$\big((1,1),(x,x),(\ln x,\ln x),((\ln x)^2,(\ln x)^2)\big)$	0.7962	0.4588	6.8401

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