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A Novel Image Similarity Measure Based on Greatest and Smallest Eigen Fuzzy Sets

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Abstract: A novel image similarity index based on the greatest and smallest fuzzy set solutions of the max–min and min–max compositions of fuzzy relations, respectively, is proposed. The greatest and smallest fuzzy sets are found symmetrically as the min–max and max–min solutions, respectively, to a fuzzy relation equation. The original image is partitioned into squared blocks and the pixels in each block are normalized to [0, 1] in order to have a fuzzy relation. The greatest and smallest fuzzy sets, found for each block, are used to measure the similarity between the original image and the image reconstructed by joining the squared blocks. Comparison tests with other well-known image metrics are then carried out where source images are noised by applying Gaussian filters. The results show that the proposed image similarity measure is more effective and robust to noise than the PSNR and SSIM-based measures.

Keywords: image similarity; IQA; GEFS; SEFS; fuzzy relation

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Citation: Di Martino, F.; Sessa, S. A Novel Image Similarity Measure Based on Greatest and Smallest Eigen Fuzzy Sets. *Symmetry* **2023**, *15*, 1104. https://doi.org/10.3390/sym15051104

Academic Editor: Calogero Vetro

Received: 30 March 2023
Revised: 5 May 2023
Accepted: 16 May 2023
Published: 18 May 2023

1. Introduction

One of the main computer vision goals is to check the similarities between images to detect if images have been copied, altered, or degraded.

There are numerous types of image degradation and manipulation processes producing variations in a grey or color image. Simple manipulations occur by re-encoding or resizing the image and generating near-exact copies, while more complex changes include cropping, color variations, and collages with other images. Similarity measures can be used to analyze the degree of similarity between images and evaluate how much an image has been modified or degraded.

A first class of similarity measures between images used in Image Quality Assessment (IQA) is given from comparisons of the corresponding pixel intensities. The Mean Square Error (MSE) and Peak Signal-to-Noise Ratio (PSNR) [1] are applied as the pixel intensity measuring the similarity between two grey-level images. Two images are identical if the corresponding MSE is null.

To include the spatial correlations with neighboring pixels, in [2], the Structural Similarity Index Measure (SSIM) is introduced. The SSIM compares two images by evaluating the brightness and contrast in the windows around each pixel.

Some variations of SSIM are used to improve the image quality assessment. The Multiscale Structural Similarity Index Measure (MS-SSIM) is proposed in [3] to consider details with different resolutions. In [4], a three-component weighted SSIM method, called 3-SSIM, is proposed, consisting of the distinction between three categories of regions in the original image: edges, textures, and smooth. Then, a weighted SSIM measure is computed in which weights are assigned to each category of region.

Some authors propose image similarity measures based on features. In [5], an index based on the Phase Congruency (PC) [6], an invariant property of image features, is proposed as well. It is able to detect both intensity variations in the image and the presence of outliers or noise. The image Gradient Magnitude (GM) [7], a contrast information measure sensitive to noise, has also been used as a feature-based IQA similarity index [8–10]. In [9], the authors show that the GM similarity index is more efficient and more robust than the PSNR and SSIM.

In [11], a new feature-based IQA measure was proposed called the Feature-based Structural Similarity Index Measure (FSIM) which combines both the PC and GM indices (the contrast and the gradient have independent roles in the characterization of the image quality). The experiments showed that the FSIM is more efficient than the PSNR and SSIM. However, it is necessary to fix some parameters which depend particularly on the image resolution and the viewing distance. A wavelet-based SIM measure, called the Complex Wavelet Structural Similarity (CW-SSIM), is proposed in [12] to capture the phase changes in the image. The CW-SSIM is robust to small rotations and translations, but it strongly depends on many parameters, such as the size of the window and the robustness of the similarity measure in the case of a low local signal-to-noise ratio.

In order to increase IQA performances, some researchers have recently proposed image similarity measures based on the Convolutional Neural Network (CNN) since it provides a complete description of the visual content of the image. A deep CNN model is applied in [13] for image retrieval. In [14,15], a Siamese CNN model is proposed in which the two images are represented as neural network-based feature vectors and the Euclidean distance is used to assess the similarity between them. To match images having any size without necessarily scaling, [16] proposed a hybrid image similarity measure combining a triplet deep CNN with spatial pyramid pooling. A new deep CNN image similarity measure combining spatial and feature characteristics is proposed in [17]. Lastly, a review of the deep learning techniques applied to image similarity is given in [18].

Deep CNN-based image similarity measures are more efficient and noise-robust than PSNR and SSIM-based measures but they require massive learning image datasets, as well as high CPU time.

The main limitation of the feature-based image similarity measures is their computational efficiency; in addition, CNN-based measures require many parameters to be set, and, furthermore, a long time is spent in the training phase.

Fuzzy-based methods are presented by some authors in order to find the similarities between images. In [19], each image was coded as a fuzzy set and a similarity measure between two images based on the fuzzy inclusion between the corresponding fuzzy sets was proposed. A set of fuzzy similarity indices applied in color image retrieval was proposed in [20]. A new fuzzy image similarity index based on the solutions of bilinear fuzzy relation equations was also proposed in [21]; this image similarity measure improves other fuzzy-based image similarity indices but its implementation is unsuccessful for huge images due to the algorithm's requirement for non-negligible execution times.

In this paper, we propose a new image similarity measure based on the definitions of the Greatest Eigen Fuzzy Set (for short, GEFS) and Smallest Eigen Fuzzy Set (for short, SEFS) of fuzzy relations [22–24].

Let X be a set and R be a fuzzy relation defined on $X \times X$. The GEFS and SEFS are fuzzy sets of X that represent the greatest and the smallest solutions of the fuzzy relation equations: $R \circ A = A$ and $R \bullet A = A$, respectively, where A is a fuzzy set of X and the operators $\circ$ and $\bullet$ are the max–min and min–max compositions, respectively. The greatest and smallest fuzzy sets are found symmetrically as the solutions of these fuzzy relation equations with respect to the max–min and min–max composition, respectively.

The GEFS and SEFS have been applied in image retrieval [25–27], in image reconstruction [28,29], and in decision problems [30,31]. In [32], an image retrieval method based on the fuzzy-transform image compression technique was applied to image retrieval and compared with the GEFS and SEFS image retrieval methods. It was found that this method provides good performances for image retrieval, but it cannot be used for image similarity because the reduced quality of compressed images can affect the similarity measure.

The idea of this research is to compare two normalized images by treating them as fuzzy relations. The two images are compared by measuring their Euclidean distance from the differences between their GEFS and SEFS.

The results of image similarity tests performed in [25–27] showed that the GEFS and SEFS image similarity measures are more efficient and robust to noise than pixel intensity measures, though their use is limited to square images because the calculation of GEFS and SEFS is performed only for square fuzzy relations.

To apply the GEFS and SEFS image similarity measure to any image, we partition the image into square blocks (sub-images). The N × M original image is partitioned in $n \times n$ blocks where n < N and n < M.

Each block is normalized and transformed in an $n \times n$ fuzzy relation; then, the GEFS and SEFS of each block are calculated. The image similarity between two N × M images is measured by calculating the mean Euclidean distance between the GEFS and SEFS of the corresponding sub-images.

The main benefits of the proposed GEFS and SEFS approach are the following:

- It is computationally faster to calculate the iterative algorithm to compute SEFS and GEFS as it converges quickly;
- It provides a very efficient image similarity measure as it improves the image similarity of PSNR and SSIM-based indices; moreover, it is more robust to image noise than PSNR and SSIM-based similarity measures;
- Unlike other image quality measures, it does not depend on specific parameters that must be set beforehand and does not need massive learning image datasets to run.

The remainder of the paper is organized as follows: in Section 2, the concepts of the GEFS and SEFS of a fuzzy relation are recalled as well as the algorithm to find the GEFS and SEFS in a fuzzy relation; in Section 3, we present our image similarity measure; in Section 4, the experimental results are shown and discussed; and the conclusions are reported in Section 5.

2. Preliminaries

Let X be a finite set, let R be a fuzzy relation defined on $X \times X$, and let F(X) be the family of all fuzzy sets defined on X. We searched the fuzzy sets $A \in F(X)$ that are solutions of the fuzzy equation:

$$R \circ A = A \tag{1}$$

where the symbol ∘ denotes the max–min composition.

Fuzzy set A, the solution of (1), is called an eigen fuzzy set of R with respect to the max–min composition. The GEFS of R with respect to the max–min composition is the greatest fuzzy set, the solution of Equation (1).

Equation (1) can be written explicitly as:

$$A(y) = \max_{x \epsilon X}\{\min(A(x), R(x, y))\}, \quad x, y \in X \tag{2}$$

Where $A_1(y) = \max_{x \epsilon X}\{R(x, y)\}$. Indeed, it is easy to prove that A_1 is the solution to (1).

We now iteratively construct the following fuzzy sets: $A_2 = R \circ A_1$, $A_3 = R \circ A_2, \ldots,$ $A_n = R \circ A_{n-1}, \ldots$ We enunciate the following theorem proved in [22–24]:

Theorem 1. *There exists p ∈ { 1,2, ..., card(X)} such that A_p is the GEFS of R with respect to the max–min composition; moreover* $A_p \subseteq \ldots \subseteq A_2 \subseteq A_1$.

A_p is obtained by finding iteratively the smallest index p for which holds:

$$A_{p+1}(y) = \max_{x \epsilon X}\{\min(A_p(x), R(x, y))\} = A_p(y) \quad \forall y \in X, \tag{3}$$

The steps to find the GEFS of R with respect to the max–min composition (1) are described Algorithm 1:

Algorithm 1: Find the GEFS of R with respect to the max–min composition	
1. Calculate $A_p(y) = \max_{x\in X}\{R(x,y)\} y X.$	A_p is initialized to A_1
2. $A = R \circ A_p$	
3. **While** $A \neq A_p$	
4. $A_p = A$	
5. $A = R \circ A_p$	
6. **End while**	
7. **Return** A_p	

To show an applicational example, we consider the following fuzzy relation:

$$R = \begin{pmatrix} 0.7 & 0.9 & 0.3 & 0.4 & 0.6 \\ 0.5 & 0.7 & 0.5 & 0.7 & 0.7 \\ 0.4 & 0.6 & 0.4 & 0.8 & 0.5 \\ 0.5 & 0.4 & 0.2 & 1.0 & 0.4 \\ 0.6 & 0.6 & 0.1 & 0.7 & 0.2 \end{pmatrix}$$

We obtain:
A_1 = $(0.7\ 0.9\ 0.5\ 1.0\ 0.7)^{-1}$
A_2 = $(0.7\ 0.7\ 0.5\ 1.0\ 0.7)^{-1}$
A_3 = $(0.7\ 0.7\ 0.5\ 1.0\ 0.7)^{-1}$ = A_2
Then, A_2 is the GEFS of R with respect to the max–min composition. The symmetric equation to (1), with respect to the min–max composition, is the following:

$$R \bullet B = B \tag{4}$$

where the symbol • denotes the min–max composition.

Fuzzy set B, the solution of (4), is called an eigen fuzzy set of R with respect to the min–max composition. The SEFS of R with respect to the min–max composition is the least fuzzy set solution of Equation (4).

In an explicit form, Equation (4) becomes:

$$B(y) = \min_{x\epsilon X}\{\max(B(x), R(x,y))\}\ x, y \in X \tag{5}$$

Where $B_1(y) = \min_{x\epsilon X}\{R(x,y)\}$. B_1 is easily seen to be the solution of (4). We can construct iteratively the following fuzzy sets of R: $B_2 = R \bullet B_1, \ldots, B_{n+1} = R \bullet B_n, \ldots$

The following theorem then holds [23]:

Theorem 2. *There exists q ∈ { 1,2, ..., card(X)} such that B_q is the SEFS of R with respect to the min–max composition; moreover* $B_0 \supseteq \ldots B_q \supseteq \ldots \supseteq B_2 \supseteq B_1$.

B_q is obtained by finding iteratively the smallest index q for which holds:

$$B_{q+1}(y) = \min_{x\epsilon X}\{\max(B_q(x), R(x,y))\} = B_q(y)\ \forall y \in X, \tag{6}$$

The steps to find the SEFS of R are described below:

Now, we apply Algorithm 2 to the fuzzy relation in the previous example to find the SEFS of R with respect to the min–max composition:
B_1 = $(0.4\ 0.4\ 0.1\ 0.4\ 0.2)^{-1}$
B_2 = $(0.4\ 0.4\ 0.2\ 0.4\ 0.2)^{-1}$
B_3 = $(0.4\ 0.4\ 0.2\ 0.4\ 0.2)^{-1}$ = B_2

Then, B_2 is the SEFS of R with respect to the min–max composition.

Algorithm 2: Find the SEFS of R with respect to the min–max composition

1. Calculate $B_q(y) = \min_{x \in X}\{R(x, y)\}$ y ∈ X. B_q is initialized to B_1
2. $B = R \bullet B_q$
3. **While** $B \neq B_q$
4. $B_q = B$
5. $B = R \bullet B_q$
6. **End while**
7. **Return** B_q

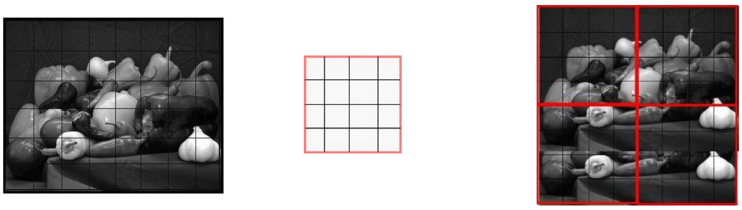
3. The GEFS–SEFS Image Similarity Measure

We apply the two algorithms to find the GEFS and SEFS of a fuzzy relation described previously to measure the similarity between two images.

Let I_1 and I_2 be two N × M images with L gray levels. We partition each of the two images into $n \times n$ square blocks, where n < N and n < M.

If the number N of rows or (and) the number M columns are not divisible by n, in the blocks intersecting the image boundary, the values of the pixels in the rows outside the image are assigned as equal to the values of the corresponding pixels belonging to the last row.

This procedure is highlighted in Figure 1, where the image consists of 6 rows and 8 columns and each block has a size n = 4.



(**a**) Original 6 × 8 image (**b**) Block 4 × 4 (**c**) Image partitioned into 4 × 4 blocks

Figure 1. Example of a 6 × 8 image partitioned into 4 × 4 blocks.

The image in the example is partitioned into four 4 × 4 blocks. Since the number of rows, which is 4, in the blocks does not exactly divide the number of rows in the image, 6, the two blocks below have two rows that do not cover the image; in these blocks, the last two rows replicate the previous two rows and are assigned the values of the corresponding pixels.

Formally, let σ_N be the integer obtained by dividing N by n and let $\rho_N = 0$ if N is divisible by n, otherwise $\rho_N = 1$. Furthermore, let σ_M be the integer obtained by dividing M by n and let $\rho_M = 0$ if M is divisible by n, otherwise $\rho_M = 1$. The number of blocks, n_B, is given by the formula:

$$n_B = \sigma_N \cdot \sigma_M + \rho_N \cdot \sigma_N + \rho_M \cdot \sigma_M + \rho_N \cdot \rho_M \quad (7)$$

In the example in Figure 1, we have that $\sigma_N = 1$, $\rho_N = 1$, $\sigma_M = 2$, and $\rho_M = 0$; then $n_B = 4$.

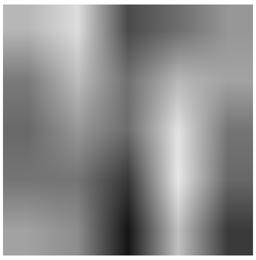
The pixel values in each block are normalized as [0, 1]; if $i_{h,k}$ is the pixel value in the cell (h,k) of the block, it is normalized by the formula:

$$x_{h,k} = \frac{i_{h,k}}{L-1} \quad h, k = 1, \ldots, n \quad (8)$$

Let $R_{1,i}$ and $R_{2,i}$ be the i*th* blocks of the images I_1 and I_2. $R_{1,i}$ and $R_{2,i}$ can be treated as two fuzzy relations, and Algorithms 1 and 2 can be executed to find their GEFS and SEFS.

Let $GEFS_{1,i}$ and $SEFS_1$,i be the GEFS and SEFS of $R_{1,i,}$ and let $GEFS_{2,i}$ and $SEFS_{2,i}$ be the GEFS and SEFS of $R_{2,i}$.

For example, the fuzzy relation R in the previous example is obtained by normalizing the 5 × 5 image block in Figure 2 and having 256 grey levels.



$$= \begin{pmatrix} 179 & 230 & 77 & 102 & 153 \\ 128 & 179 & 128 & 179 & 179 \\ 102 & 153 & 102 & 204 & 128 \\ 128 & 102 & 51 & 255 & 102 \\ 153 & 153 & 26 & 179 & 51 \end{pmatrix}$$

Figure 2. The 5 × 5 image block normalized in the fuzzy relation R in the previous example.

Following (8), the fuzzy relation R is given by dividing the pixel values in the image block by the value 255.

We measure the distance D_i between the two blocks $R_{1,i}$ and $R_{2,i}$ using the Euclidean metric:

$$D_i = \frac{\sqrt{\sum_{h=1}^{n} \sum_{k=1}^{n} (x_{1,i}(h,k) - x_{2,i}(h,k))^2}}{n^2} \tag{9}$$

The similarity S_i between the two i*th* blocks $R_{1,i}$ and $R_{2,i}$ is given by the following formula:

$$S_i = 1 - \frac{D_i}{n^2} \tag{10}$$

where the term $\frac{D_i}{n^2}$ is the mean distance between two corresponding cells of blocks R_1 and R_2.

S_i varies in the range [0, 1]. It is equal to 1 if and only if the two blocks are equal.

The similarity between the two images is given by the average of the similarity between the corresponding blocks.

$$S = \frac{\sum_{i=1}^{n_B} s_i}{n_B} \tag{11}$$

Below, in Algorithm 3, the pseudocode of our algorithm used to measure the similarity between the two N × M in images I_1 and I_2 is shown.

Algorithm 3: GEFS–SEFS image similarities
Input: N × M images I_1 and I_2
Sizes of the blocks *n*
Output: Similarity S between the two images I_1 and I_2

1. Partition the two images in n_B nxn blocks
2. S:= 0 //
3. **For** i = 1 to n_B
4. Normalize $R_{i,1}$ by (8)
5. Normalize $R_{i,2}$ by (8)
6. Execute Algorithm 1 to find the GEFS of $R_{i,1}$
7. Execute Algorithm 2 to find the SEFS of $R_{i,1}$
8. Execute Algorithm 1 to find the GEFS of $R_{i,2}$
9. Execute Algorithm 2 to find the SEFS of $R_{i,2}$
10. Compute S_i by (10)
11. S:= S + S_i
12. **Next** i
13. S:= S/n_B
14. **Return** S

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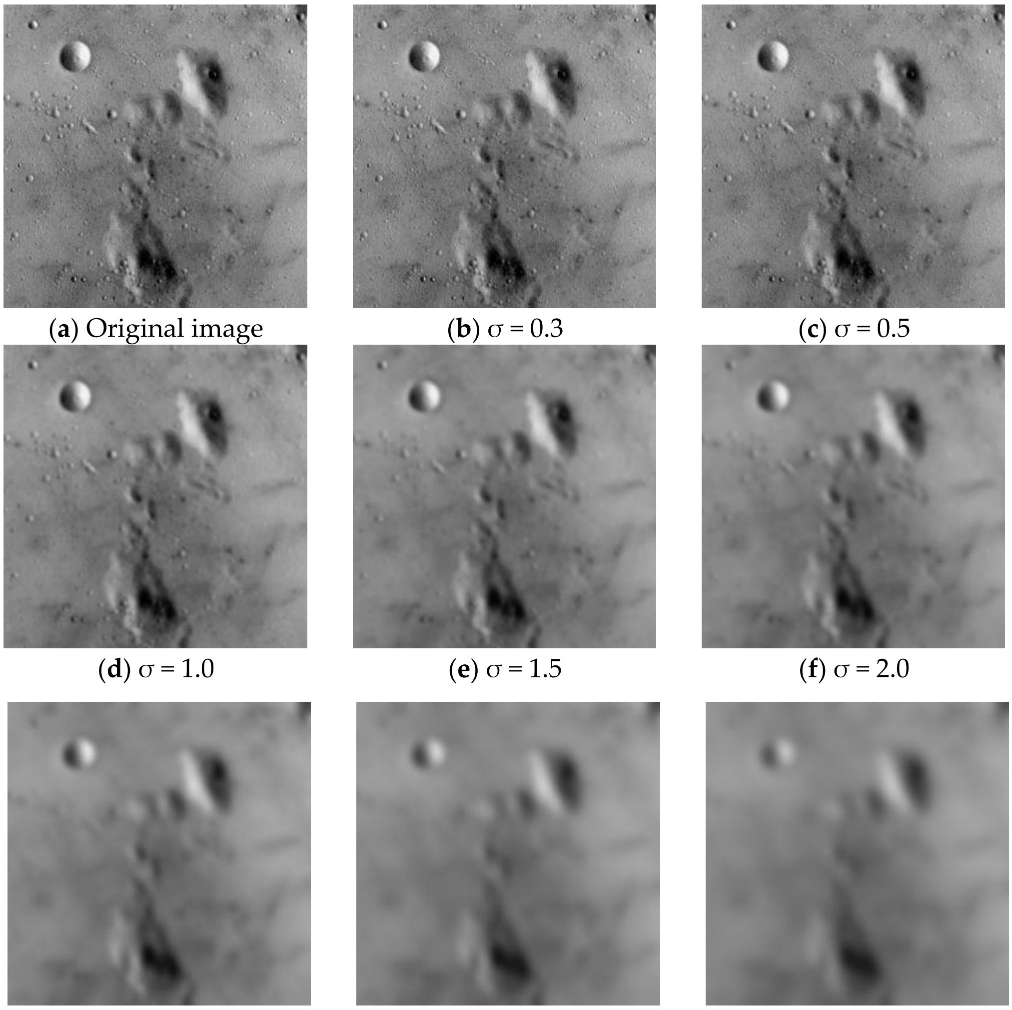
4. Discussion and Results

To compare the performances of our method with other IQA similarity measures, an image dataset of over 100 images from the Signal Image Processing Institute (SIPI) image database (https://sipi.usc.edu/database (accessed on 1 February 2023)) was used.

The source image was blurred using a Gaussian filter; the standard deviation σ in the Gaussian distribution increased as the noise increased. We applied our method to measure the similarity between the original and the blurred images by setting $n = 5$ and afterward $n = 7$. In addition, we compared our index with the PSNR, SSIM, MS-SSIM, and FSIM indices. For brevity, we show the results obtained for two images: the 256 × 256 image 5.1.09 and the 512 × 512 image 7.1.07.

In Figure 3, the original image 5.01.09 and the blurred images using σ = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0, 4.0, and 5.0, are shown.



(**a**) Original image (**b**) σ = 0.3 (**c**) σ = 0.5
(**d**) σ = 1.0 (**e**) σ = 1.5 (**f**) σ = 2.0
(**g**) σ = 3.0 (**h**) σ = 4.0 (**i**) σ = 5.0

Figure 3. Image 5.1.09 blurred using Gaussian filters with different values of the parameter σ.

Table 1 shows the similarity indices between the original and the blurred images obtained for image 5.1.09. The PSNR is normalized by dividing its value by the PSNR obtained for σ = 0.3.

Table 1. Image similarity measures for the grey image 5.1.09.

σ	PSNR	PSNR Normalized	SSIM	MS-SSIM	FSIM	GEFS-SEFS $n = 5$	GEFS–SEFS $n = 7$
0.3	68.85	1.00	1.00	1.00	1.00	1.00	1.00
0.5	40.81	0.59	0.97	0.98	0.98	0.99	0.99
1.0	32.26	0.47	0.82	0.84	0.85	0.89	0.88
1.5	30.45	0.44	0.71	0.75	0.76	0.82	0.80
2.0	29.10	0.42	0.65	0.68	0.68	0.74	0.71
3.0	27.81	0.40	0.55	0.57	0.57	0.68	0.66
4.0	26.85	0.39	0.46	0.47	0.48	0.61	0.60
5.0	26.11	0.38	0.40	0.41	0.41	0.52	0.51

Therefore, Table 1 shows that the GEFS–SEFS-based similarity indices are more robust to noise than the PSSNR, SSIM, MS-SSIM, and FSIM indices. Indeed, compared to the other indices, the GEFS–SEFS measures do not decrease as quickly as the standard deviation of the Gaussian filter increases. Figure 4 shows the related trends.

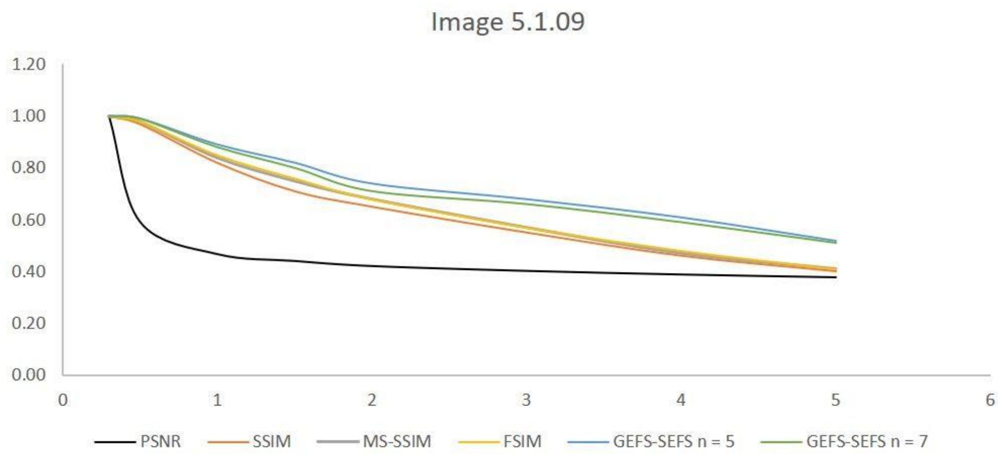


Figure 4. Similarity index trends for image 5.1.09.

The PSNR decreases quickly starting at σ = 0.5. The SSIM, MS-SSIM, and FSIM decrease below the value of 0.60 starting at σ = 3. Instead, both of the GEFS–SEFS measures obtained for $n = 5, 7$ slowly decrease and they have no variations, independent of block sizes.

In Figure 5, we show the original image 7.1.07 and the blurred images using σ = 0.3, 0.5, 1.0, 1.5, 2.0, 3.0, 4.0, and 5.0.

Table 2 shows the similarity indices between the original and the blurred images obtained for image 7.1.07.

Table 2. Image similarity measures for the grey image 7.1.07.

σ	PSNR	PSNR Normalized	SSIM	MS-SSIM	FSIM	GEFS-SEFS $n = 5$	GEFS–SEFS $n = 7$
0.3	68.41	1.00	1.00	1.00	1.00	1.00	1.00
0.5	40.01	0.58	0.98	0.98	0.98	0.99	0.99
1.0	31.89	0.47	0.82	0.83	0.83	0.88	0.88
1.5	29.35	0.43	0.70	0.72	0.73	0.79	0.78
2.0	28.26	0.41	0.58	0.58	0.60	0.69	0.68
3.0	26.67	0.39	0.43	0.45	0.46	0.57	0.56
4.0	25.71	0.38	0.33	0.36	0.37	0.49	0.48
5.0	25.10	0.37	0.26	0.28	0.29	0.41	0.41

The GEFS–SEFS measures are more robust to noise than the PSSNR, SSIM, MS-SSIM, and FSIM indices, slowly decreasing with respect to the other image similarity indices. Figure 6 shows the trends for σ varying between 0.3 and 5.

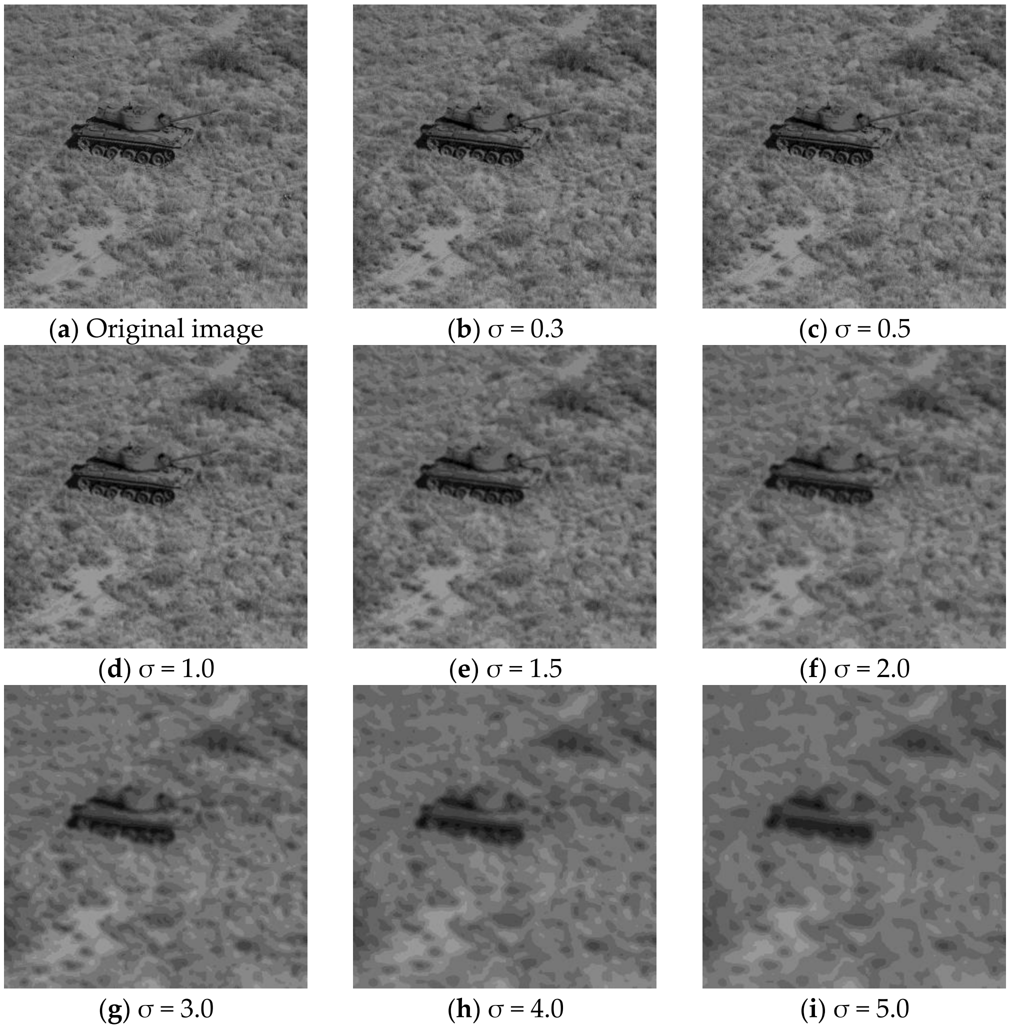


Figure 5. Image 7.1.07 blurred using Gaussian filters with different values of the parameter σ.

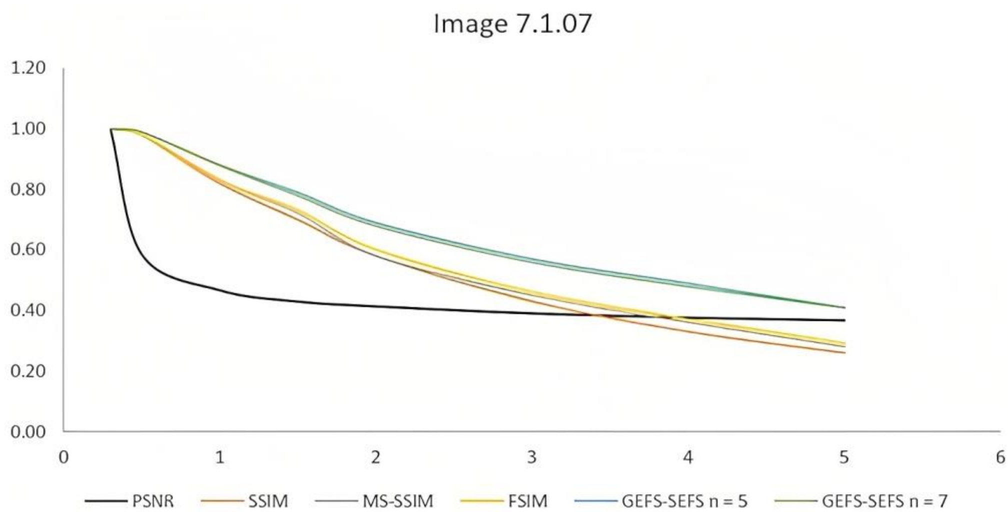


Figure 6. Similarity indices trends for image 7.1.07.

Figure 6 shows that the trend of the two GEFS–SEFS indices slowly decrease with respect to the similarity indices; moreover, the two GEFS–SEFS measures have the same trend; this confirms that the proposed similarity index is independent of the choice of block sizes. Figure 7 shows the trend of the mean values of the similarity indices obtained for all the images.

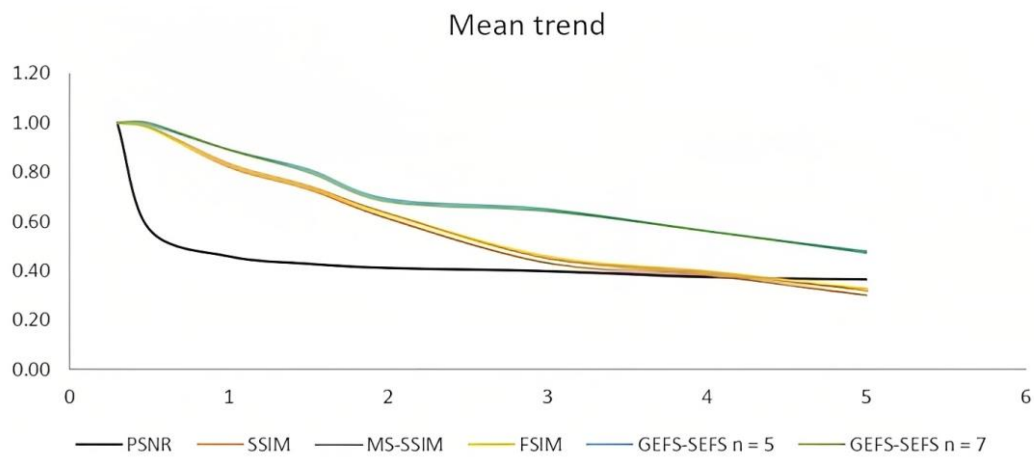


Figure 7. Mean similarity index trends.

The mean trends show a trend similar to the ones obtained for the two images 5.1.09 and 7.1.07. The two SEFS–GEFS indices obtained by setting $n = 5$ and $n = 7$ slowly decrease as soon as the Gaussian noise increases in the image.

These results show, in general, that the GEFS and SEFS-based image similarity measures perform better than the PSNR and SSIM-based measures and that they can be used even in the presence of noise in the images, whereas the performance of the PSNR and SSIM measures decrease quickly.

5. Conclusions

In this paper, we present new image similarity metrics based on the GEFS and SEFS of fuzzy relations; each of the two images to be compared is partitioned in squared blocks, normalized, and treated as squared fuzzy relations. For each block, the GEFS and SEFS are found and the Euclidean distance between the GEFS and SEFS of the corresponding blocks is used to compute the similarity index between the two images. This measure does not require particular parameters to be set beforehand; moreover, the algorithm is computationally fast due to the rapid convergence of the extraction methods of the SEFS and GEFS. The results of the comparative tests carried out on a sample of more than 100 images show that the GEFS–SEFS-based image similarity measure is more efficient and robust to noise than the PSNR and SSIM-based measures.

A critical point of the similarity measure metrics based on the GEFS and SEFS of fuzzy relations is the choice of the size n of the image blocks. In the tests carried out, it was shown that similar results were obtained by setting $n = 5$ and $n = 7$, but it is necessary to carry out further tests on a larger sample of images and on different types of problems to analyze if and how the choice of block size affects the performance of this image similarity measure.

In the future, we intend to apply this image similarity to various problems, such as image reconstruction and image tamper detection, and to perform comparative tests by other image similarity measures.

Author Contributions: This article was written in full collaboration by both authors. The final manuscript was read and approved by both authors. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: On request, the data used to support the findings of this study can be obtained from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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