

LEFSCHETZ PROPERTIES OF JACOBIAN ALGEBRAS AND JACOBIAN MODULES

ALEXANDRU DIMCA¹ AND GIOVANNA ILARDI

ABSTRACT. Let $V : f = 0$ be a hypersurface of degree $d \geq 3$ in the complex projective space \mathbb{P}^n , $n \geq 3$, having only isolated singularities. Let $M(f)$ be the associated Jacobian algebra and $H : \ell = 0$ be a hyperplane in \mathbb{P}^n avoiding the singularities of V , but such that $V \cap H$ is singular. We related the Lefschetz type properties of the linear maps $\ell : M(f)_k \rightarrow M(f)_{k+1}$ induced by the multiplication by linear form ℓ to the singularities of the hyperplane section $V \cap H$. Similar results are obtained for the Jacobian module $N(f)$.

1. INTRODUCTION

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded polynomial ring in $n+1$ variables with complex coefficients, with $n \geq 2$. We denote by S_k the degree k homogeneous component of S . Let $f \in S_d$ be a homogeneous polynomial such that the hypersurface $V : f = 0$ in the projective space \mathbb{P}^n is smooth. Then the Jacobian algebra $M(f) = S/J(f)$ is a standard graded Artinian Gorenstein algebra with socle degree $T = (n+1)(d-2)$, where we set

$$f_j = \frac{\partial f}{\partial x_j}$$

for $j = 0, \dots, n$ and $J(f) = (f_0, \dots, f_n) \subset S$ is the Jacobian ideal of f . This algebra is also known as the Milnor algebra of f . In fact, the hypersurface $V : f = 0$ is smooth if and only if $M(f)$ is an Artinian algebra.

Definition 1.1. Let $M = \bigoplus_{i=0}^T M_i$ be an Artinian graded \mathbb{C} -algebra with $M_T \neq 0$.

- (1) The algebra M is said to have the Weak Lefschetz Property in degree i , for $i < T$, for short WLP_i , if there exists an element $L \in M_1$ such that the multiplication map $L : M_i \rightarrow M_{i+1}$ is of maximal rank. We say that the algebra M has WLP if it has WLP_i for all $0 \leq i \leq T-1$.
- (2) We say that M has the Strong Lefschetz Property in degree $k < T/2$, for short SLP_k , if there is $L \in M_1$ such that the linear map $L^{T-2k} : M_k \rightarrow M_{T-k}$ is an isomorphism. We say that the algebra M has SLP if it has SLP_k in degree k for all $k < T/2$.

2010 *Mathematics Subject Classification.* Primary 32S25; Secondary 14B05, 13A02, 13A10, 13D02.

Key words and phrases. hypersurface, dual hypersurface, Lefschetz properties, hyperplane section, singularities.

¹This work has been partially supported by the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, grant PN-III-P4-ID-PCE-2020-0029, within PNCDI III.

Example 1.2. This example is due to Stanley [28] and Watanabe [30] and it is considered to be the starting point of the research area of Lefschetz properties for graded algebras. Consider the standard graded Artinian Gorenstein algebra

$$M = \frac{\mathbb{C}[x_0, \dots, x_n]}{(x_0^{a_0}, \dots, x_n^{a_n})}$$

with integers $a_i > 0$, for all $i = 0, \dots, n$. Then M has *SLP*.

In particular, for a Fermat type polynomial

$$f_F = x_0^d + \dots + x_n^d,$$

the Jacobian algebra $M(f_F)$ has *SLP*. By obvious semi-continuity properties, it follows that the Jacobian algebra $M(f)$ has *SLP* for a *generic* polynomial $f \in S_d$. However, it seems that we have no control on the meaning of the word 'generic' in this claim. In fact, one has the following.

Conjecture 1.3. The Jacobian algebra $M(f)$ has *SLP* for any polynomial $f \in S_d$ such that the associated hypersurface $V : f = 0$ is smooth.

Remark 1.4. We list here the known results related to this conjecture.

- (1) When $n = 2$, for any smooth curve $V : f = 0$, the associated Jacobian algebra $M(f)$ has *WLP*, as follows from the more general results in [22]. Moreover, when $d = 2d'$ is even, then the multiplication by the square of a generic linear form $\ell \in S_1$ induces an isomorphism

$$\ell^2 : M(f)_{3d'-4} \rightarrow M(f)_{3d'-2}.$$

- In particular, when $d = 4$, the Jacobian algebra $M(f)$ has the *SLP*, see [13].
- (2) When $n = 3$, if $V : f = 0$ is any smooth cubic surface in \mathbb{P}^3 , then $M(f)$ has the *SLP*, see [13]. In fact, this result is a consequence of the classical Hesse-Gordan-Noether's Theorem, see for instance [21, 25]. Moreover, for any smooth surface $V(f)$, the *SLP* holds in degree 1, see for instance [3, Theorem B]. For any smooth surface in \mathbb{P}^3 of degree $d \in \{4, 5, 6\}$, the *WLP* holds in all degrees, see [2, Corollary 7.2]. Moreover, for degrees $d > 6$, the *WLP* holds in all degrees up to $(3d - 2)/2$, see [2, Corollary 7.3].
- (3) For $n = 4$, if $V : f = 0$ is any smooth cubic 3-fold in \mathbb{P}^4 , then $M(f)$ has the *SLP*, [3, Theorem C].
- (4) For arbitrary dimension n and degree d , one knows that $M(f)$ has the *WLP* in degree $\leq d - 2$, see [24]. The *SLP*₀ also holds for obvious reasons. Indeed, if $\ell^T \in J(f)$ for all $\ell \in S_1$, then the hessian polynomial $\text{hess}(f) \notin J(f)$ would have no Waring decomposition.
- (5) In [1] *WLP* is proven for some complete intersection algebras presented in degree 2, improving some previous bounds. Finally, in [4], *SLP*₁ for the Jacobian Algebra of a smooth cubic fourfold is proved.

Remark 1.5. Using the duality of the Jacobian algebra $M(f)$, in order to prove the *WLP* for $M(f)$, it is enough to prove the *WLP* _{i} for $i < T/2$.

In fact, one can define *WLP* for more general graded objects as follows.

Definition 1.6. Let $M = \bigoplus_{i \geq 0} M_i$ be a graded \mathbb{C} -algebra (resp. a graded S -module) with $\dim M_i < +\infty$ for any $i \geq 0$. The algebra (resp. module) M is said to have the Weak Lefschetz Property in degree i , for short WLP_i , if there exists an element $L \in M_1$ (resp. $L \in S_1$) such that the multiplication map $L : M_i \rightarrow M_{i+1}$ is of maximal rank. We say that M has WLP if it has WLP_i for all $i \geq 0$.

When the hypersurface $V : f = 0$ in \mathbb{P}^n is singular, then one can consider the Jacobian algebra $M(f) = S/J(f)$ and the Jacobian module $N(f) \subset M(f)$ given by $N(f) = I(f)/J(f)$, where $I(f)$ is the saturation of $J(f)$ with respect to the maximal ideal $\mathfrak{m} = (x_0, \dots, x_n)$, see [26]. Recall that, for any homogeneous ideal I in S , we define its saturation \widehat{I} with respect to \mathfrak{m} as the set of all elements $s \in S$ such that for any $i = 0, \dots, n$ there is a positive integer m_i such that $x_i^{m_i} s \in I$. One has $N(f) = M(f)$ if and only if $V : f = 0$ is smooth, so $N(f)$, which is always an Artinian module, can be thought of as a replacement of $M(f)$ when V is singular.

Example 1.7. Let $C : f = 0$ be a reduced degree d curve in \mathbb{P}^2 . Then the corresponding Jacobian algebra $M(f)$ satisfies WLP_i for any

$$i < \frac{3(d-2)}{2}$$

and the corresponding mappings $M(f)_i \rightarrow M(f)_{i+1}$ are injective. Moreover, the Jacobian algebra $M(f)$ satisfies WLP_i for any

$$i \geq i_0 = \left\lfloor \frac{3(d-2)}{2} \right\rfloor$$

and the corresponding mappings $M(f)_i \rightarrow M(f)_{i+1}$ are surjective if and only if

$$(1.1) \quad ct(C) \geq 3(d-2) - i_0,$$

see [16, Corollary 4.4]. Here

$$ct(C) = \max\{k \in \mathbb{Z} : \dim M(f)_j = \dim M(f_F)_j \text{ for all } j \leq k\},$$

where f_F denotes the Fermat degree d polynomial as in Example 1.2. As explained in [16, Remark 4.5], many curves satisfy the condition 1.1, in particular all nodal curves. Moreover, it has been shown in [16, Corollary 4.3] that the Jacobian module $N(f)$ satisfies WLP for any reduced plane curve.

Example 1.8. Let $V : f = 0$ be a degree d hypersurface in \mathbb{P}^n having only isolated singularities. Then the corresponding Jacobian algebra $M(f)$ satisfies WLP_i for any

$$i > n(d-2)$$

and the corresponding mappings $M(f)_i \rightarrow M(f)_{i+1}$ are surjective. For $i > (n+1)(d-2)$ these mappings are isomorphisms, see [9, Corollary 8].

In this note, we fix a hypersurface $V : f = 0$ in \mathbb{P}^n having only isolated singularities and a hyperplane $H : \ell = 0$ in \mathbb{P}^n avoiding the singularities of V , with $n, d \geq 3$. The hyperplane section $V(f, \ell) = V(f) \cap H$ has then at most isolated singularities, as follows from the following.

Lemma 1.9. *Let V be a hypersurface in \mathbb{P}^n of degree d , with $n, d \geq 2$. Assume that H is a hyperplane such that $H \cap V_s = \emptyset$, where V_s denotes the singular set of V . Then V and $V \cap H$ have at most isolated singularities. Moreover, when V has at most isolated singularities, then there are hyperplanes H such that $H \cap V_s = \emptyset$ and $V \cap H$ is singular, except when V is a cone.*

As usual, V is a cone if one can choose the coordinates x such that $f_0 = 0$, hence f does not depend on the variable x_0 . If $V : f = 0$ is a cone with an isolated singularity, then up to a linear change of coordinates, we have that f is a polynomial $g \in \mathbb{C}[x_1, \dots, x_n]$ and the corresponding hypersurface $g = 0$ is smooth in $H : x_0 = 0$. Then obviously the multiplication by x_0 is injective in any range, since

$$M(f) = M(g) \otimes \mathbb{C}[x_0].$$

Remark 1.10. Note that any hypersurface $W \subset H$ with only isolated singularities may occur as a section $W = V \cap H$ for a certain smooth hypersurface V , see [10, Proposition (11.6)].

In this setting, and under the assumption that $V(f, \ell)$ is singular, we investigate the injectivity of the multiplication maps $\ell : M(f)_k \rightarrow M(f)_{k+1}$. Our main result is stated in terms of some numerical invariants of the hyperplane section $V(f, \ell)$, which we define now. Clearly $V(f, \ell)$ is a hypersurface in $H = \mathbb{P}^{n-1}$. If we choose a system of coordinates $y = (y_1, \dots, y_n)$ on H , then $V(f, \ell)$ is given by an equation $g = 0$, hence it has a Jacobian ideal $J(g)$ in the polynomial ring $R = \mathbb{C}[y_1, \dots, y_n]$. Let $I(g)$ be the saturation of the Jacobian ideal $J(g)$ with respect to the maximal ideal (y_1, \dots, y_n) and let $s(g)$ be the initial degree of the graded ideal $I(g)$, namely

$$(1.2) \quad s(g) = \min\{j \in \mathbb{N} : I(g)_j \neq 0\} \leq d - 1,$$

where $d = \deg f = \deg g$. Let g_j denote the partial derivative of g with respect to y_j and consider the graded R -module $Syz(g)$ of first order syzygies of g_1, \dots, g_n , namely

$$(1.3) \quad Syz(g) = \{a = (a_1, \dots, a_n) \in R^n : a_1 g_1 + \dots + a_n g_n = 0\}.$$

Let $r(g)$ be the initial degree of the graded module $Syz(g)$, namely

$$(1.4) \quad r(g) = \min\{j \in \mathbb{N} : Syz(g)_j \neq 0\} \leq d - 1.$$

It is clear that both invariants $r(g)$ and $s(g)$ do not depend on the choice of the linear coordinates y on $H = \mathbb{P}^{n-1}$. Note that $V(f, \ell)$ singular implies $s(g) > 0$. On the other hand, $r(g) = 0$ if and only if $V(f, \ell)$ is a cone. With this notation, our main result is the following improvement of the second author result recalled in Remark 1.4 (4).

Theorem 1.11. *Let $V : f = 0$ be a hypersurface in \mathbb{P}^n of degree d , with $n, d \geq 2$. Assume that $H : \ell = 0$ is a hyperplane such that $H \cap V_s = \emptyset$, where V_s denotes the singular set of V , and that $V(f, \ell) = V \cap H$ is singular. Then the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for any*

$$k \leq \min\{d - 3 + r(g), d - 3 + s(g)\}.$$

Theorem 1.11 can give the *WLP* for the Jacobian algebra $M(f)$ (in some degrees or in all degrees) for some generic classes of hypersurfaces, *with a precise meaning of the word 'generic'*, as well as for many new *non generic classes* of smooth hypersurfaces.

In terms of the Jacobian module $N(f)$, this result implies the following.

Corollary 1.12. *Let $V : f = 0$ be a hypersurface in \mathbb{P}^n of degree d , with $n, d \geq 3$. Assume that $H : \ell = 0$ is a hyperplane such that $H \cap V_s = \emptyset$, where V_s denotes the singular set of V , and that $V(f, \ell) = V \cap H$ is singular. Then the multiplication map $\ell : N(f)_k \rightarrow N(f)_{k+1}$ is injective for any*

$$k \leq k_0 = \min\{d - 3 + r(g), d - 3 + s(g)\}.$$

Moreover, the multiplication map $\ell : N(f)_k \rightarrow N(f)_{k+1}$ is surjective for any

$$k \geq (n + 1)(d - 2) - k_0 - 1.$$

In section 2 we prove Lemma 1.9, Theorem 1.11, and Corollary 1.12. In section 3 we apply Theorem 1.11 to surfaces S in \mathbb{P}^3 with isolated singularities. The surfaces of degree 4 are considered in Example 3.1 and Example 3.5. The surfaces having a section $C = S \cap H$ which is a nodal curve and such that all irreducible components of C are rational (resp. C is a free or a nearly free curve) are considered in Proposition 3.2 (resp. Proposition 3.4). When the surface S is smooth, all the results in this section except Proposition 3.2 are obtained in stronger forms in [2].

In section 4, we consider higher dimensional hypersurfaces V , having a nodal hyperplane section $Y = V \cap H$ with many singularities, e.g. Y is a Kummer surface in Proposition 4.1, respectively a Chebyshev hypersurface in Proposition 4.2. Finally, an application to *WLP* of a recent result of the authors, see Theorem 4.3 below quoted from [15], is given in Corollary 4.4 and Corollary 4.5. We state all the results in this section for a smooth hypersurface V , but all of them, except Theorem 4.3, have obvious versions for a hypersurface V having only isolated singularities.

We would like to thank Arnaud Beauville for very useful discussions related to this paper. We also thank the referee for the very careful reading of our manuscript and for his suggestions to improve the presentation.

2. THE PROOFS OF LEMMA 1.9, THEOREM 1.11 AND COROLLARY 1.12

2.1. The proof of Lemma 1.9. Since $H \cap V_s = \emptyset$, it is clear that $\dim V_s \leq 0$. Assume that $V : f = 0$ and $H : x_0 = 0$. The singular set of $V \cap H$ is then given by

$$Z : f_1(0, x_1, \dots, x_n) = f_2(0, x_1, \dots, x_n) = \dots = f_n(0, x_1, \dots, x_n) = 0.$$

If $\dim Z > 0$, then any point in the non-empty intersection of Z with the hypersurface $f_0(0, x_1, \dots, x_n) = 0$ would give rise to a point in $V_s \cap H$. This contradiction shows that $\dim Z \leq 0$.

The second claim in Lemma 1.9 is more subtle. Assume from now on that V is not a cone. For any singular point $p \in V_s$, denote by H_p the hyperplane in the dual space $(\mathbb{P}^n)^\vee$ corresponding to all the hyperplanes in \mathbb{P}^n passing through p . Note that the dual hypersurface V^\vee is irreducible, since V is so since $n \geq 3$. It follows that if V^\vee is contained in the union of all hyperplanes H_p , for $p \in V_s$, there is a point $q \in V_s$

such that $V^\vee \subset H_q$. We can assume that $q = (1 : 0 : \dots : 0)$, and then this condition means

$$f_0(x) = 0 \text{ for all } x \in V \setminus V_s.$$

But this implies that f_0 is divisible by f , which is possible only if $f_0 = 0$, that is when V is a cone. This contradiction shows that there is a hyperplane $H : \ell = 0$ such that $H \cap V_s = \emptyset$ and H is tangent to V . In fact H corresponds to a point in V^\vee not in any H_p , for $p \in V_s$. This implies that the section $V \cap H$ is singular.

2.2. The proof of Theorem 1.11. Without loss of generality, we may take $\ell = x_0$ and $y_j = x_j$ for all $j = 1, \dots, n$. Then one may write

$$f(x_0, y) = g(y) + x_0 h(y) + x_0^2 p_2(y) + \dots + x_0^d p_d(y),$$

where $p_j \in R_{d-j}$. Assume that $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is not injective. Then there is a homogeneous polynomial $q \in S_k$ such that

$$q \notin J(f)_k \text{ and } x_0 q \in J(f)_{k+1}.$$

It follows that one has a relation

$$x_0 q = b_0 f_0 + \dots + b_n f_n,$$

with homogeneous polynomials $b_j \in S_{k+2-d}$, not all divisible by x_0 . If we set $x_0 = 0$, this yields

$$(2.1) \quad c_0 h + c_1 g_1 + \dots + c_n g_n = 0,$$

where not all the polynomials $c_j(y) = b_j(0, y) \in R_{k+2-d}$ are zero. There are two cases to discuss.

Case 1. $c_0 \neq 0$. Then note that the singular set of the hypersurface V on the hyperplane $x_0 = 0$ is given by the solutions of the system

$$h = g_1 = \dots = g_n = 0.$$

Since $V_s \cap H = \emptyset$, and $V(f, \ell)$ is assumed to be singular, it follows that h does not vanish on the singular set of the section $V(f, \ell)$, which is given by the solutions of the system

$$g_1 = \dots = g_n = 0.$$

In other words, if q is a singular point of $V(f, \ell)$, and $u \in R_{d-1}$ is a homogeneous polynomial not vanishing at q , the regular function germ of h/u at q is an invertible element in the corresponding local ring $\mathcal{O}_{H,q}$. This observation and the equation (2.1) imply that $c_0 \in I(g)$, see the discussion at the beginning of Section 2 in [11]. Indeed, equation (2.1) implies that the germ of rational function associated to c_0 at each singular point q of $V(f, \ell)$ belongs to the stalk at p of the sheaf associated to the Jacobian ideal at g , which is an ideal in the local ring $\mathcal{O}_{H,q}$. This property implies that $c_0 \in I(g)$. Hence $k + 2 - d = \deg c_0 \geq s(g)$ which gives $k \geq d - 2 + s(g)$.

Case 2. $c_0 = 0$. Then the equation (2.1) becomes a non-zero homogeneous element $\rho \in Syz(g)$. It follows that $k + 2 - d = \deg \rho \geq r(g)$. In other words, we have $k \geq d - 2 + r(g)$.

This ends the proof of Theorem 1.11.

2.3. The proof of Corollary 1.12. The first claim follows from Theorem 1.11, since $N(f)$ is a submodule in $M(f)$. The second claim follows from the duality property of the Jacobian module $N(f)$, namely

$$N(f)_j^\vee \cong N(f)_{T-j},$$

for any integer $j = 0, \dots, T$, where $T = (n + 1)(d - 2)$, see either [26, Theorem 3.4] or [29, Theorem 4.7].

3. SURFACES OF DEGREE $d \geq 4$ IN \mathbb{P}^3

In this case the plane section $V(f, \ell)$ is a plane curve of degree d , with isolated singularities. We discuss several possibilities.

Example 3.1. For smooth surfaces of degree 4 *WLP* holds in all degrees, see [2, Corollary 7.2]. Let $V : f = 0$ be a surface of degree 4 in \mathbb{P}^3 having isolated singularities and containing no line, and let $H : \ell = 0$ be a plane such that $V(f)_s \cap H = \emptyset$ and the plane curve $V(f, \ell) = V \cap H$ has at least 3 singular points with total Tjurina number $\tau(V(f, \ell))$ at most 5. Then clearly $s(g) \geq 2$, since any element in $I(g)$ vanishes at the singular points and $V(f, \ell)$ has no line components. Using the lower bound on $\tau(V(f, \ell))$ given in [20], we see that $r(g) \geq 2$. It follows from Theorem 1.11 and Corollary 1.12 that the multiplication map $\ell : M(f)_i \rightarrow M(f)_{i+1}$ is injective for $i \leq 3$ and that the Jacobian module $N(f)$ has *WLP*. Other quartic surfaces are discussed in Example 3.5.

Proposition 3.2. *If the surface $V : f = 0$ with isolated singularities admits a plane section $V(f, \ell) = V \cap H$, where $H : \ell = 0$, such that $V_s \cap H = \emptyset$ and $V(f, \ell)$ is a nodal curve, all irreducible components of $V(f, \ell)$ being rational curves, then the corresponding Jacobian algebra $M(f)$ has the WLP_k for $k < 2d - 4$ and the Jacobian module $N(f)$ has *WLP*.*

Proof. Using [18, Theorem 4.1], it follows that $r(g) \geq d - 2$. On the other hand, we have $s(g) \geq d - 2$, using [7, Theorem 3.2]. It follows from Theorem 1.11 that the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for $k < 2d - 4 = T/2$. We conclude using Corollary 1.12. \square

Remark 3.3. Any generic quartic surface in \mathbb{P}^3 admits a section which is a rational nodal curve, see [8, Theorem 1.2]. On the other hand, for $d \geq 5$, a generic degree d surface in \mathbb{P}^3 does not admit a section which is a rational nodal curve. More precisely, in this case any irreducible component C of a section $V(f, \ell)$ of a generic surface has geometric genus satisfying the inequality

$$g(C) > \frac{d(d-3)}{2} - 3,$$

see [31, Theorem 1]. In the same paper, Xu shows that the list of singularities on the section $V(f, \ell)$ is one of the following, see [31, Proposition 3].

- (1) $A_1, 2A_1, 3A_1$;
- (2) A_2, A_1A_2
- (3) A_3 .

In other words, only list of singularities with total Tjurina number $\tau(V(f, \ell)) \leq 3$ may occur for a generic surface of degree $d \geq 5$. For a similar result, see also [5].

For a curve $C : g = 0$, we set $n(f)_k = \dim N(g)_k$ for any integer k and also $\nu(C) = \max_j \{n(g)_j\}$. Recall that the curve C is free if and only if $N(g) = 0$ and hence $\nu(C) = 0$, and C is nearly free if and only if $\nu(C) = 1$, see for instance [7, 19]. For a free (resp. nearly free) curve, the exponents are $d_1 = r(g)$, the initial degree of the graded R -module $Syz(g)$ and $d_2 = d - 1 - d_1$ (resp. $d_2 = d - d_1$).

Proposition 3.4. *If the surface $V : f = 0$ with isolated singularities admits a plane section $V(f, \ell) = V \cap H$, where $H : \ell = 0$, such that $V_s \cap H = \emptyset$ and $V(f, \ell)$ is a free (resp. nearly free) curve with exponents (d_1, d_2) , then the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for any*

$$k \leq d + d_1 - 3.$$

In the case of smooth surfaces V , this result is weaker than [2, Corollary 7.2].

Proof. For a free curve $V(f, \ell) : g = 0$ one has $J(g) = I(g)$ and hence $s(g) = d - 1$. By definition one has $r(g) = d_1$ and it is known that $2d_1 \leq d - 1$ which implies $d_1 \leq d - 1 = s(g)$. This proves our claim for a free curve.

For a nearly free curve $V(f, \ell) : g = 0$ one has $s(g) = \min\{d - 1, d + d_1 - 3\}$, see [19, Corollary 2.17]. By definition $r(g) = d_1$ and it is known that $d_1 + d_2 = d$. It follows that $d_1 \leq d - 1$ and also $d_1 \leq d + d_1 - 3$ since we suppose $d \geq 3$. \square

In view of the point (4) in Remark 1.4, the above result is useful only when $d_1 \geq 2$.

Example 3.5. Let V be a surface of degree 4 with isolated singularities in \mathbb{P}^3 and let $H : \ell = 0$ be a plane avoiding the singularities of $V(f)$ and such that the plane curve $V(f, \ell) = V(f) \cap H$ has one of the following lists of singularities, not covered by the discussion in Example 3.1.

- (1) $3A_2$;
- (2) A_2A_4
- (3) A_6
- (4) $6A_1$
- (5) A_1A_5 .

In the first 3 cases the curve $V(f, \ell)$ is irreducible and rational, while in case (4) the curve $V(f, \ell)$ is a union of 4 lines in general position. In particular, the case (4) can be treated using also Proposition 3.2. In case (5), the curve $V(f, \ell)$ is a union of two conics, meeting at 2 points, with intersection multiplicities 1, and respectively 3. All these curves are shown to be nearly free with exponents $(2, 2)$, see [19, Example 2.13] and [14, Proposition 5.5]. It follows from Proposition 3.4 that the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for $k \leq 3$.

Remark 3.6. All the possibilities of the singularities of a plane section of a smooth quartic surface in \mathbb{P}^3 are listed in [6]. We have discussed above only some of these possibilities, assuming that they occur also as sections of quartic surfaces with isolated singularities.

4. HYPERSURFACES IN \mathbb{P}^n , $n \geq 4$, HAVING SECTIONS WITH MANY NODES

As already said in Introduction, all the results in this section are stated for a smooth hypersurface V , but all of them, except Theorem 4.3, have obvious versions for a hypersurface V having only isolated singularities. In this section the hyperplane section $V(f, \ell) \in H = \mathbb{P}^{n-1}$ is assumed to have only nodes, namely ordinary double points, also known as A_1 -singularities. In this case, it is known that

$$(4.1) \quad r(g) = d - 1,$$

when $d \geq 3$, see [12]. To get lower bounds on $s(g)$ we impose a large number of singularities on the hyperplane section $V(f, \ell)$. We discuss several cases.

Proposition 4.1. *If the smooth 3-fold $V : f = 0$ in \mathbb{P}^4 of degree 4 admits a hyperplane section $V(f, \ell) : g = 0$ that is a nodal surface, with at least 10 nodes, not all on a quadric, then the corresponding Jacobian algebra $M(f)$ has the WLP. In particular, this occurs when the hyperplane section $K = V(f, \ell) : g = 0$ is a Kummer surface with 16 nodes.*

Proof. Using the equality (4.1), it follows that $r(g) = 3$. In the first claim, we assume that $s(g) \geq 3$. This claim follows now from Theorem 1.11 and Remark 1.5, since $T = 10$ in this case.

We show now that $s(g) \geq 3$ in the case of a section being a Kummer surface. Let X be the minimal resolution of K . Denote by H' the pull-back on X of a plane section of K and let E_1, \dots, E_{16} be the exceptional divisors. Assume there is a quadric Q in the hyperplane $H = \mathbb{P}^3$, passing through all the 16 nodes of K . Then the pull-back on X of this quadric Q gives an effective divisor D in the linear system

$$|2H' - \sum_{i=1}^{16} E_i|.$$

On the other hand, there are 16 planes in H , classically called tropes, each of them tangent to K along a conic passing through 6 of the singularities of K , see [23, Chapter 1]. Hence the proper transform C_j of one of these conics satisfies

$$2C_j = H' - \sum_{i \in I_j} E_i,$$

with $|I_j| = 6$. It follows that

$$C_j \cdot D = \frac{1}{2}(H' - \sum_{i \in I_j} E_i) \cdot (2H' - \sum_{i=1}^{16} E_i) = -2.$$

Since C_j is irreducible, this implies that D contains the curve C_j . Note that $D \cdot H' = 8$, $C_j \cdot H' = 2$ for all $j = 1, \dots, 16$, and hence D cannot contain all the conics C_j . This contradiction shows that $I(g)_2 = 0$, and hence $s(g) \geq 3$. \square

There is a family of nodal hypersurfaces $X : g = 0$, in any dimension n and degree d , for which the subtle invariant $s(g)$ is known. They are the Chebyshev

hypersurfaces $\mathcal{C}(n, d) \subset \mathbb{P}^n$, defined as follows. For more details see [17]. Consider the d -th Chebyshev polynomial

$$T_d(x) = \cos(d \arccos(x)).$$

Then the affine part of Chebyshev hypersurface $\mathcal{C}(n, d)$, that is before homogenization with respect to x_0 , is defined by the affine equation

$$g(n, d) = T_d(x_1) + \cdots + T_d(x_n) = 0$$

when n is even, and by

$$g(n, d) = T_d(x_1) + \cdots + T_d(x_n) + 1 = 0$$

when n is odd. The equality

$$(4.2) \quad s(g) = d - 2,$$

follows from [17, Proposition 3.1].

Proposition 4.2. *If the smooth hypersurface $V : f = 0$ in \mathbb{P}^n , $n \geq 4$, of degree $d \geq 3$ admits a hyperplane section $V(f, \ell) : g = 0$ which is a Chebyshev hypersurface, then the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for any*

$$k \leq 2d - 5.$$

Proof. It is enough to use Theorem 1.11 and the formulas (4.1) and (4.2). \square

We recall the following result, see [15].

Theorem 4.3. *Generic hypersurfaces of degree d in \mathbb{P}^n , with $d, n \geq 3$, have hyperplane sections with n nodes in general position.*

Using this result, we can improve by one, in the case of generic hypersurfaces, the result of the second author, mentioned above in Remark 1.4, (4).

Corollary 4.4. *If the smooth hypersurface $V : f = 0$ in \mathbb{P}^n , $n \geq 3$, of degree $d \geq 3$ admits a nodal hyperplane section $V(f, \ell) = V \cap H$, which has exactly n singularities in general position, then the multiplication map $\ell : M(f)_k \rightarrow M(f)_{k+1}$ is injective for any $k \leq d - 1$. In particular, this property holds for a generic hypersurface V in \mathbb{P}^n , $n \geq 3$, of degree $d \geq 3$.*

Proof. It is enough to use Theorem 1.11, the formulas (4.1) and the obvious fact that $s(g) \geq 2$ in this situation, since the nodes are in general position. \square

When $n = 3$, Corollary 4.4 is much weaker than [2, Corollary 7.3]. The following is a special case of the previous result, for $n = 5$.

Corollary 4.5. *If the smooth 4-fold $V : f = 0$ in \mathbb{P}^5 of degree 3 admits a hyperplane section $V(f, \ell) = V \cap H : g = 0$ that is a nodal 3-fold, with at least 5 nodes, not all on a hyperplane in H , then the corresponding Jacobian algebra $M(f)$ has the WLP. In particular, this occurs for a generic 4-fold $V(f) : f = 0$ in \mathbb{P}^5 of degree 3 and for a smooth 4-fold $V : f = 0$ in \mathbb{P}^5 of degree 3 having a hyperplane section $V(f, \ell) : g = 0$ which is a Segre 3-fold with 10 nodes.*

Proof. Using the inequality (4.1), it follows that $r(g) \geq 2$. The assumptions in the first claim imply that $s(g) \geq 2$. This claim follows now from Theorem 1.11 and Remark 1.5, since $T = 6$ in this case. The fact that a this situation occurs for a generic 4-fold $V(f) : f = 0$ in \mathbb{P}^5 of degree 3 follows from Corollary 4.4

We treat next the case of the Segre 3-fold section. The Segre 3-fold is unique up-to a projective transformation, and can be given by the equation

$$X : g = x_1^3 + x_2^3 + x_3^2 + x_4^3 + x_5^3 - (x_1 + x_2 + x_3 + x_4 + x_5)^3 = 0,$$

where x_1, x_2, x_3, x_4, x_5 are the coordinates on $H = \mathbb{P}^4$. The 10 nodes are located at $(1 : 1 : 1 : -1 : -1)$ and the other 9 points obtained by permuting the coordinates. Using this description of the singular set of X , it follows that $I(g)_1 = 0$, and hence $s(g) \geq 2$. We conclude as for the first claim above. \square

REFERENCES

- [1] A. Alzati, R. Re, Complete intersections of quadrics and the Weak Lefschetz Property. *Collect. Math.* 70(2019), 283–294. 5
- [2] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel, On the Weak Lefschetz Property for height four equigenerated complete intersections, arXiv:2212.09890. 2, 1, 3.1, 3, 4
- [3] D. Bricalli, F. F. Favale, G.P. Pirola, A theorem of Gordan and Noether via Gorenstein rings, *Sel. Math. New Ser.* 29, 74 (2023). <https://doi.org/10.1007/s00029-023-00882-7>. 2, 3
- [4] D. Bricalli D., F. F. Favale, Lefschetz properties for jacobian rings of cubic fourfolds and other Artinian algebras, *Collect. Math.* (2022), <https://doi.org/10.1007/s13348-022-00382-5>. 5
- [5] J. W. Bruce, The duals of generic hypersurfaces. *Math. Scand.*, 49 (1981), 36–60. 3.3
- [6] J. W. Bruce, P. J. Giblin, A Stratification of the Space of Plane Quartic Curves, *Proceedings of the London Mathematical Society*, 42 (1981), 270–298. 3.6
- [7] A. Cerminara, A. Dimca, G. Ilardi, On the Hilbert vector of the Jacobian module of a plane curve, *Portugal. Math.* **76** (2020), 311–325. 3, 3
- [8] X. Chen, Rational curves on K3 surfaces, *J. Algebraic Geom.* 8 (1999), no. 2, 245–278. 3.3
- [9] A.D.R. Choudary, A. Dimca, Koszul complexes and hypersurface singularities, *Proc. Amer. Math. Soc.* 121 (1994), 1009–1016. 1.8
- [10] A. Dimca, Topics on Real and Complex Singularities, *Vieweg Advanced Lecture in Mathematics*, Friedr. Vieweg und Sohn, Braunschweig, 1987. 1.10
- [11] A. Dimca, Syzygies of Jacobian ideals and defects of linear systems, *Bull. Math. Soc. Sci. Math. Roumanie Tome 56(104) No. 2* (2013), 191–203. 2.2
- [12] A. Dimca, On the syzygies and Hodge theory of nodal hypersurfaces, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* 63 (2017), 87–101. 4
- [13] A. Dimca, R. Gondim, G. Ilardi, Higher order Jacobians, Hessians and Milnor algebras, *Collectanea Mathematica* 71(2019), 407–425. 1, 2
- [14] A. Dimca, G. Ilardi, G. Sticlaru, Addition-deletion results for the minimal degree of a Jacobian syzygy of a union of two curves, *J. Algebra*, DOI 10.1016/j.jalgebra.2022.10.019. 3.5
- [15] A. Dimca, G. Ilardi, On the duals of generic projective hypersurfaces, *Publicacions Matemàtiques* (to appear). 1, 4
- [16] A. Dimca, D. Popescu, Hilbert series and Lefschetz properties of dimension one almost complete intersections, *Comm. Algebra* 44 (2016), 4467–4482. 1.7
- [17] A. Dimca, G. Sticlaru, On the syzygies and Alexander polynomials of nodal hypersurfaces, *Math. Nachrichten* 285 (2012), 2120–2128. 4, 4
- [18] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, *Proc. Edinburg. Math. Soc.* **58** (2015), 333–354. 3

- [19] A. Dimca and G. Sticlaru, Free and Nearly Free Curves vs. Rational Cuspidal Plane Curves. *Publ. Res. Inst. Math. Sci.* **54** (2018) 163 – 179. 3, 3, 3.5
- [20] A.A. du Plessis, C.T.C. Wall, Application of the theory of the discriminant to highly singular plane curves, *Math. Proc. Camb. Phil. Soc.*, **126** (1999), 259-266. 3.1
- [21] R. Gondim, On higher Hessians and the Lefschetz properties, *J. Algebra* **489** (2017), 241–263. 2
- [22] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The weak and strong Lefschetz properties for artinian K -algebras*, *J. Algebra* 262 (2003), 99–126. 1
- [23] R. W. H. T. Hudson, *Kummer’s quartic surface*, Cambridge University Press, Cambridge 1905. 4
- [24] G. Ilardi, *Jacobian ideals, Arrangements and Lefschetz properties*, *J. Algebra* 508 (2018), 418–430. 4
- [25] F. Russo On the Geometry of Some Special Projective Varieties, *Lecture Notes of the Unione Matematica Italiana*, vol. 18, Springer (2016). 2
- [26] E. Sernesi, The local cohomology of the jacobian ring, *Documenta Mathematica*, 19 (2014), 541-565. 1, 2.3
- [27] A. Simis, S.O. Tohăneanu, Homology of homogeneous divisors, *Israel J. Math.* 200 (2014), 449-487.
- [28] R. Stanley, *Hilbert functions of graded algebras*, *Adv. in Math.* 28 (1978), 57–83. 1.2
- [29] van Straten, D., Warmt, T. . Gorenstein-duality for one-dimensional almost complete intersections-with an application to non-isolated real singularities. *Math. Proc. Cambridge Philos. Soc.* 158 (2015), 249–268. 2.3
- [30] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*. In: *Commutative algebra and combinatorics* (pp. 303-312). Mathematical Society of Japan. 1987. 1.2
- [31] Geng Xu, Subvarieties of general hypersurfaces in projective space, *J. Diff. Geom.* 39(1994), 139–172. 3.3

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE AND SIMION STOILOW INSTITUTE OF MATHEMATICS, P.O. BOX 1-764, RO-014700 BUCHAREST, ROMANIA

Email address: dimca@unice.fr

DIPARTIMENTO MATEMATICA ED APPLICAZIONI “R. CACCIOPOLI” UNIVERSITÀ DEGLI STUDI DI NAPOLI “FEDERICO II” VIA CINTIA - COMPLESSO UNIVERSITARIO DI MONTE S. ANGELO 80126 - NAPOLI - ITALIA

Email address: giovanna.ilardi@unina.it