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Short communication

A representation of a class of quasi-arithmetic means using a unary modifier operator

József Dombi [a,b], Tamás Jónás [c,*]

[a] ELKH-SZTE Research Group on Artificial Intelligence, Szeged, Tisza Lajos körút, Szeged H-6720, Hungary
[b] Institute of Informatics, University of Szeged, Árpád tér 2, Szeged H-6720, Hungary
[c] Faculty of Economics, Eötvös Loránd University, Egyetem tér 1-3, Budapest H-1053, Hungary

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ABSTRACT

In this short communication, we will show that a quasi-arithmetic mean induced by an additive generator of a strict t-norm (strict t-conorm, respectively) can be represented by the composition of a unary operator (called the tau function) and the strict t-norm (strict t-conorm, respectively), both induced by the same generator function as the quasi arithmetic mean. Here, we will also state a connection between the idempotency of a transformed strict t-norm (strict t-conorm, respectively) and the tau function.

1. Preliminaries

Here, we will give an overview of the concepts and notations that will be utilized later on. We will use the common notation $\mathbb{R}$ for the real line and $\overline{\mathbb{R}}$ for the extended real line, i.e., $\overline{\mathbb{R}} = [-\infty, \infty]$. Since we will operate on $\overline{\mathbb{R}}$, based on [1] and [2], we will adopt the following conventions:

$$\frac{1}{0} = \infty, \ \frac{1}{\infty} = 0, \ e^{-\infty} = 0, \ e^{\infty} = \infty, \ \ln(0) = -\infty, \text{ and } \ln(\infty) = \infty.$$

1.1. Strict triangular norms and strict triangular conorms

The Archimedean triangular norms (t-norms in short) and triangular conorms (t-conorms in short) as well as their strict class both play an important role in continuous-valued logic (for more details, see [1]). These norms are defined as follows (see, e.g., [1,3]).

Definition 1. We say that a continuous t-norm $T : [0,1]^2 \to [0,1]$ (t-conorm $S : [0,1]^2 \to [0,1]$, respectively) is Archimedean, if $T(x,x) < x$ ($S(x,x) > x$, respectively) holds for any $x \in (0,1)$.

Definition 2. We say that a continuous Archimedean t-norm T (t-conorm S, respectively) is a strict t-norm (strict t-conorm, respectively), if $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$ (if $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, respectively).

The strict t-norms and t-conorms can be represented as follows (see [1,4]).

* Corresponding author.
E-mail addresses: dombi@inf.u-szeged.hu (J. Dombi), jonas@gtk.elte.hu (T. Jónás).

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Theorem 1. *A function $T: [0,1]^2 \to [0,1]$ ($S: [0,1]^2 \to [0,1]$, respectively) is a strict t-norm (t-conorm, respectively) if and only if there exists a continuous, strictly decreasing (increasing, respectively) function $t: [0,1] \to [0,\infty]$ ($s: [0,1] \to [0,\infty]$, respectively) with $t(0) = \infty$ and $t(1) = 0$ ($s(0) = 0$ and $s(1) = \infty$, respectively), which is uniquely determined up to a positive constant multiplier, such that for any $x, y \in [0,1]$,*

$$T(x,y) = t^{-1}(t(x) + t(y))$$
$$(S(x,y) = s^{-1}(s(x) + s(y)), \textit{respectively}).$$

In Theorem 1, the function t (s, respectively) is called an additive generator of the strict t-norm T (strict t-conorm S, respectively). We will now use the following class of functions.

Notation 1. Let $\mathcal{G}$ denote the set of all continuous and strictly monotonic functions $g: [0,1] \to [0,\infty]$ for which exactly one of the following two cases holds:

(a) g is strictly decreasing with $g(0) = \infty$ and $g(1) = 0$;
(b) g is strictly increasing with $g(0) = 0$ and $g(1) = \infty$.

Based on Theorem 1 and Notation 1, $g \in \mathcal{G}$ means that g is an additive generator function of either a strict t-norm or a strict t-conorm.

Taking into account the associativity of strict t-norms (strict t-conorms, respectively) (see, e.g., [1,5]), we will simply interpret the n-ary strict t-norm (strict t-conorm, respectively) $o_g: [0,1]^n \to [0,1]$, induced by an additive generator $g \in \mathcal{G}$, as

$$o_g(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$.

1.2. Quasi-arithmetic means

Based on [2], we will use the following definition of a quasi-arithmetic mean on $[0,1]^n$ (see also [6]).

Definition 3. Let $n \in \mathbb{N}$, $n \geq 1$, and let $f: [0,1] \to \overline{\mathbb{R}}$ be a continuous and strictly monotonic function. We say that the function $M_f: [0,1]^n \to [0,1]$ is an n-ary quasi-arithmetic mean on $[0,1]^n$ generated by the function f if for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$, $M_f(\mathbf{x})$ is given by

$$M_f(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right). \tag{2}$$

The function f in (2) is called a generator of M_f. Aczél [5] showed that f is uniquely determined up to a linear transformation (also see [7]).

Noting Definition 3, we readily note that for any $g \in \mathcal{G}$,

$$M_g(\mathbf{x}) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) \tag{3}$$

is an n-ary quasi-arithmetic mean, where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$. In this short communication, we will concentrate on the class of n-ary quasi-arithmetic means induced by all the additive generators of strict t-norms and strict t-conorms (i.e., by functions that are members of $\mathcal{G}$).

Based on [2], we will make use of the following definition of the idempotency of an n-ary operator.

Definition 4. Let $n \in \mathbb{N}$ and $n \geq 1$. We say that an operator $o: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ is idempotent if for any $x \in \overline{\mathbb{R}}$,

$$o(\underbrace{x, x, \dots, x}_{n\text{-times}}) = x.$$

It immediately follows from (2) that a quasi-arithmetic mean is idempotent. In contrast, an n-ary strict t-norm (strict t-conorm, respectively) $o_g: [0,1]^n \to [0,1]$, which is induced by an additive generator $g \in \mathcal{G}$ according to (1), is not idempotent. Namely, if o_g is an n-ary strict t-norm (strict t-conorm, respectively), then for any $x \in (0,1)$,

$$o_g(\underbrace{x, x, \dots, x}_{n\text{-times}}) < x$$

$(o_g(\underbrace{x, x, \ldots, x}_{n\text{-times}}) > x, \text{respectively}).$

Later, we will show how the so-called tau function, which we will present in the next subsection, can be used to transform an n-ary strict t-norm (strict t-conorm, respectively) into a quasi-arithmetic mean.

1.3. The tau function

The tau function, which is a unary modifier operator in continuous-valued logic, was first introduced by Dombi (see [8]). This function is defined as follows.

Definition 5. Let $g \in \mathcal{G}$ and let $\nu, \nu_0 \in (0,1)$. We say that the mapping $\tau_{g,\nu,\nu_0} : [0,1] \to [0,1]$ is a tau function with the parameters ν and ν_0, induced by function g, if τ_{g,ν,ν_0} is given by

$$\tau_{g,\nu,\nu_0}(x) = g^{-1}\left(g(\nu_0)\frac{g(x)}{g(\nu)}\right). \tag{4}$$

Here, the function g is called a generator function of τ_{g,ν,ν_0}.

The following properties of a tau function immediately follow from its definition (for more details, see [8]):

(a) τ_{ν,ν_0} is continuous in $[0,1]$
(b) τ_{ν,ν_0} is strictly increasing in $[0,1]$, $\tau_{\nu,\nu_0}(0) = 0$ and $\tau_{\nu,\nu_0}(1) = 1$
(c) $\tau_{\nu,\nu_0}(\nu) = \nu_0$
(d) For any $x \in (0,1)$,
  (d1) if $\nu = \nu_0$, then $\tau_{\nu,\nu_0}(x) = x$
  (d2) if $\nu < \nu_0$, then $\tau_{\nu,\nu_0}(x) > x$
  (d3) if $\nu > \nu_0$, then $\tau_{\nu,\nu_0}(x) < x$.

2. Representation of quasi-arithmetic means induced by additive generators of strict t-norms and strict t-conorms

Now, we will demonstrate that a quasi-arithmetic mean induced by an additive generator of a strict t-norm (strict t-conorm, respectively) is none other than the composition of a tau function and the strict t-norm (strict t-conorm, respectively) both induced by the same generator function as the quasi arithmetic mean.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $g \in \mathcal{G}$. Suppose that $M_g : [0,1]^n \to [0,1]$ is a quasi-arithmetic mean generated by g according to (3), and $o_g : [0,1]^n \to [0,1]$ is an n-ary strict t-norm or strict t-conorm also generated by g according to (1). Then, for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ and a strictly increasing and continuous function $\tau : [0,1] \to [0,1]$,*

$$M_g(\mathbf{x}) = \tau\left(o_g(\mathbf{x})\right) \tag{5}$$

holds if and only if there exist $\nu, \nu_0 \in (0,1)$ such that

$$\nu_0 = g^{-1}\left(\frac{1}{n}g(\nu)\right) \tag{6}$$

and for any $x \in [0,1]$,

$$\tau(x) = \tau_{g,\nu,\nu_0}(x), \tag{7}$$

where $\tau_{g,\nu,\nu_0} : [0,1] \to [0,1]$ is a tau function with the parameters ν and ν_0, induced by the g function.

Proof. Proof of necessity. Assume that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, (5) holds with a strictly increasing and continuous function $\tau : [0,1] \to [0,1]$. Then, based on (1) and (3), we have

$$g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) = \tau\left(g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right)\right). \tag{8}$$

Noting (1), we also have

$$\sum_{i=1}^{n} g(x_i) = g\left(o_g(\mathbf{x})\right),$$

and so (8) can be written as

$$g^{-1}\left(\frac{1}{n}g\left(o_g(\mathbf{x})\right)\right) = \tau\left(o_g(\mathbf{x})\right).$$

The previous equation means that for any $x \in [0,1]$,

$$\tau(x) = g^{-1}\left(\frac{1}{n}g(x)\right).$$

Since for any $x \in [0,1]$, $g(x) \in [0,\infty]$, there exist $v, v_0 \in (0,1)$ such that $\frac{1}{n} = \frac{g(v_0)}{g(v)}$, or equivalently,

$$v_0 = g^{-1}\left(\frac{1}{n}g(v)\right).$$

Therefore we find that for any $x \in [0,1]$,

$$\tau(x) = g^{-1}\left(\frac{1}{n}g(x)\right) = g^{-1}\left(g(v_0)\frac{g(x)}{g(v)}\right),$$

which means that (7) holds.

Proof of sufficiency. Assume that $v, v_0 \in (0,1)$, (6) holds, and for any $x \in [0,1]$, $\tau(x)$ is given by (7). Noting that (6) is equivalent to $\frac{1}{n} = \frac{g(v_0)}{g(v)}$, using (7), (4) and (1), for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$, we can write

$$\begin{aligned}\tau\left(o_g(\mathbf{x})\right) &= \tau_{g,v,v_0}\left(o_g(\mathbf{x})\right) = g^{-1}\left(g(v_0)\frac{g\left(o_g(\mathbf{x})\right)}{g(v)}\right)\\ &= g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right).\end{aligned}$$

Taking into account (3), the last equation means that (5) holds for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$. □

The following theorem concerns the idempotency of the composition $\tau \circ o_g : [0,1]^n \to [0,1]$, where $\tau : [0,1] \to [0,1]$ is a strictly increasing and continuous function.

Theorem 3. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $g \in \mathcal{G}$. Let $o_g : [0,1]^n \to [0,1]$ be an n-ary strict t-norm or strict t-conorm generated by g according to (1). Then, for a strictly increasing and continuous function $\tau : [0,1] \to [0,1]$, the composition $\tau \circ o_g : [0,1]^n \to [0,1]$ is idempotent if and only if there exist $v, v_0 \in (0,1)$ so that (6) holds, and for any $x \in [0,1]$, (7) holds as well.*

Proof. Proof of necessity. Assume that $\tau \circ o_g : [0,1]^n \to [0,1]$ is idempotent. Taking into account (1), the idempotency of $\tau \circ o_g$ means that for any $x \in [0,1]$,

$$\tau(o_g(\underbrace{x, x, \dots, x}_{n\text{-times}})) = \tau\left(g^{-1}(ng(x))\right) = x. \quad (9)$$

Let $y = g^{-1}(ng(x))$. Then, x can be expressed in terms of y as $x = g^{-1}\left(\frac{1}{n}g(y)\right)$, and so the right hand side equation in (9) can be written as

$$\tau(y) = g^{-1}\left(\frac{1}{n}g(y)\right),$$

where $y \in [0,1]$. Again, since $g(y) \in [0,\infty]$, there exist $v, v_0 \in (0,1)$ such that $\frac{1}{n} = \frac{g(v_0)}{g(v)}$ holds. This implies that (6) holds, and for any $x \in [0,1]$, (7) holds as well.

Proof of sufficiency. Assume that $v, v_0 \in (0,1)$ so that (6) holds, and for any $x \in [0,1]$, $\tau(x)$ is given by (7). Let $M_g : [0,1]^n \to [0,1]$ be a quasi-arithmetic mean generated by g according to (3). Then, based on Theorem 2, we immediately find that for any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0,1]^n$, $M_g(\mathbf{x}) = \tau\left(o_g(\mathbf{x})\right)$ holds. Therefore, since the quasi-arithmetic mean M_g is idempotent, we get that $\tau \circ o_g$ is idempotent as well. □

The following corollary is a direct consequence of Theorems 2 and 3.

Corollary 1. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $g \in \mathcal{G}$. Suppose that $o_g : [0,1]^n \to [0,1]$ is an n-ary strict t-norm or strict t-conorm generated by g according to (1). If $v, v_0 \in (0,1)$ such that*

$$v_0 = g^{-1}\left(\frac{1}{n}g(v)\right),$$

then $\tau_{g,v,v_0} \circ o_g : [0,1]^n \to [0,1]$ is the quasi-arithmetic mean M_g given in (3), which is an idempotent n-ary operator.

Corollary 1 tells us that a tau function with suitably chosen parameter values can be viewed as a transformation that converts an n-ary strict t-norm (strict t-conorm, respectively) into an n-ary averaging operator, i.e., quasi arithmetic mean.

Example 1. Let $g(x) = -\ln(x)$, $x \in [0,1]$. It is well known that g is an additive generator of the product t-norm, which is a strict t-norm. Since $g^{-1}(x) = \mathrm{e}^{-x}$, $x \in [0,\infty]$, the n-ary strict t-norm induced by g is

$$o_g(x_1, x_2, \ldots, x_n) = g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right) = \mathrm{e}^{-\sum_{i=1}^{n}(-\ln(x_i))} = \prod_{i=1}^{n} x_i.$$

The tau function with the parameters $\nu, \nu_0 \in (0,1)$ induced by g is

$$\tau_{g,\nu,\nu_0}(x) = g^{-1}\left(g(\nu_0)\frac{g(x)}{g(\nu)}\right) = x^{\frac{\ln(\nu_0)}{\ln(\nu)}}.$$

In line with Corollary 1, if $\nu_0 = g^{-1}\left(\frac{1}{n}g(\nu)\right)$, i.e., $\frac{\ln(\nu_0)}{\ln(\nu)} = \frac{1}{n}$, then

$$\tau_{g,\nu,\nu_0}\left(o_g(x_1, x_2, \ldots, x_n)\right) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

That is, in this case, $\tau_{g,\nu,\nu_0} \circ o_g$ is the geometric mean operator.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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