Doc. Math. 28 (2023), 11–53
DOI 10.4171/DM/912

Hochschild cohomology of generalised Grassmannians

Pieter Belmans and Maxim Smirnov

Abstract. We compute the Hochschild–Kostant–Rosenberg decomposition of the Hochschild cohomology of generalised Grassmannians, i.e., partial flag varieties associated to maximal parabolic subgroups in a simple algebraic group, in terms of representation-theoretic data. We explain how the decomposition is concentrated in global sections for the (co)minuscule and (co)adjoint generalised Grassmannians, and conjecture that for (almost) all other cases the same vanishing of the higher cohomology does not hold. Our methods give an explicit partial description of the Gerstenhaber algebra structure for the Hochschild cohomology of cominuscule and adjoint generalised Grassmannians. We also consider the case of adjoint partial flag varieties in type A, which are associated to certain submaximal parabolic subgroups.

1. Introduction

For partial flag varieties, or compact homogeneous spaces, it has been an important question to compute topological and geometric invariants of G/P in terms of representation-theoretic data. This has a long and rich history, starting with the works of Borel and Hirzebruch. Examples of these invariants are singular cohomology [5] or quantum cohomology [19], and equivariant variations thereupon [10, 38].

In this paper we consider another important algebro-geometric invariant: *Hochschild cohomology*, denoted $\mathrm{HH}^\bullet(G/P) = \bigoplus_{i\geq 0} \mathrm{HH}^i(G/P)$. It controls the (generalised) deformation theory [34, 35], it is related to Poisson geometry [42], and it comes equipped with a rich algebraic structure. It has not been considered before for partial flag varieties, and the results in this paper suggest many interesting features arising in this setting.

One can explicitly compute $\mathrm{HH}^i(X)$ of a smooth variety X (at least as a vector space, and in characteristic 0), using the *Hochschild–Kostant–Rosenberg decomposition* [15], which expresses it as the direct sum

$$\mathrm{HH}^i(X) \cong \bigoplus_{p+q=i} \mathrm{H}^q\left(X, \bigwedge^p \mathrm{T}_X\right) \tag{1.1}$$

in terms of polyvector fields. Whilst formally similar to the Hodge decomposition of the cohomology of X, the right-hand side is more complicated to compute, as for instance there are no symmetries induced by Serre duality or Hodge symmetry.

2020 Mathematics Subject Classification. Primary 14M17; Secondary 14F10.
Keywords. Hochschild cohomology, partial flag varieties, homogeneous spaces, Hochschild–Kostant–Rosenberg decomposition, Bott vanishing.

In this paper we discuss the case where X is of the form G/P, where G is a simple algebraic group defined over an algebraically closed field of characteristic 0, and P is either

- a *maximal* parabolic subgroup, in which case we call G/P a *generalised Grassmannian*, including when G is of exceptional type;
- the submaximal parabolic subgroup in type A_n corresponding to the *adjoint* case, in which case we have that $G/P = \mathrm{Fl}(1, n, n+1)$.

The starting point for this paper is the question whether these varieties are what we will call *Hochschild global*, i.e., whether

$$\mathrm{H}^q\left(G/P, \bigwedge^p \mathrm{T}_{G/P}\right) = 0 \quad \forall q \geq 1, \tag{1.2}$$

so that the Hochschild–Kostant–Rosenberg decomposition (1.1) is concentrated in global sections. Amongst experts there was the expectation that this would indeed be the case for partial flag varieties.

Upon replacing exterior powers of the tangent bundle by *symmetric* powers of the tangent bundle, it can be shown using Grauert–Riemenschneider vanishing and the Leray spectral sequence (see [7, Section A2]) that

$$\mathrm{H}^q(G/P, \mathrm{Sym}^p \mathrm{T}_{G/P}) = 0 \quad \forall q \geq 1, \tag{1.3}$$

as $\mathrm{Sym}^\bullet \mathrm{T}_{G/P}$ are the functions on the total space of the cotangent bundle. So (1.2) can be seen as an odd version of the vanishing in (1.3).

Instead of Hochschild cohomology, one could also study *Hochschild homology*. Here the Hochschild–Kostant–Rosenberg decomposition takes on the form

$$\mathrm{HH}_i(X) \cong \bigoplus_{p-q=i} \mathrm{H}^q(X, \Omega_X^p),$$

which involves more familiar invariants when working over the complex numbers: the pieces of the Hodge decomposition. The description of these is amenable to topological methods as in [6, Section 24]. For Hochschild cohomology such topological methods are not available and new tools are needed.

Algebraic structures. Hochschild cohomology comes equipped with a rich structure, namely that of a *Gerstenhaber algebra*. This combines a graded-commutative cup product with a graded Lie algebra structure of degree -1, the Gerstenhaber bracket, which are related via the Poisson identity. Two important features of the Gerstenhaber bracket are that

(1) in the setting of G/P the degree-1 component $\mathrm{HH}^1(G/P)$ is a Lie subalgebra given by $\mathrm{Lie}\, G = \mathfrak{g}$ (outside a few exceptional cases, see Remark 3 and Lemma 4), which equips all the $\mathrm{HH}^i(G/P)$ with the structure of a $\mathfrak{g}$-representation;

(2) the self-bracket $[\alpha,\alpha] \in \mathrm{HH}^3(G/P)$ for a class

$$\alpha \in \mathrm{HH}^2(G/P) = \mathrm{H}^0\left(G/P, \bigwedge^2 \mathrm{T}_{G/P}\right)$$

can be identified with the Schouten self-bracket[1], whose vanishing is precisely the condition that a bivector $\alpha \in \mathrm{H}^0(G/P, \bigwedge^2 \mathrm{T}_{G/P})$ gives a Poisson structure.

The first feature gives us a convenient method to describe the vector spaces $\mathrm{HH}^i(G/P)$ as representations of $\mathfrak{g}$. The second feature on the other hand highlights that one of the natural next steps in the description of the Gerstenhaber algebra structure (namely the self-bracket of two classes in degree 2) is very complicated, as this description is only known for the generalised Grassmannians $\mathbb{P}^3$ and Q^3 [33, 42], with a classification in higher dimensions being wide open, see also Remark 20.

Vanishing. The first result we describe is a positive answer to the vanishing question suggested above in an important class of examples. The notions of (co)minuscule and (co)adjoint are recalled in Section 2.2, in particular the following theorem concerns the varieties listed in Tables 2–4.

Theorem A (Vanishing). *Let G/P be either a generalised Grassmannian which is (co)minuscule or (co)adjoint, or an adjoint partial flag variety in type A_n. Then*

$$\mathrm{H}^q\left(G/P, \bigwedge^p \mathrm{T}_{G/P}\right) = 0 \quad \forall q \geq 1.$$

For generalised Grassmannians this vanishing result can in fact be deduced from the vanishing results in [29], but we will give a streamlined proof below. The vanishing in the adjoint case in type A_n is new and a representation-theoretic proof is given. Alternatively the vanishing can be deduced using the description of the adjoint partial flag variety in type A_n as $\mathbb{P}(\mathrm{T}_{\mathbb{P}^n})$.

The complication (outside the cominuscule case) is that $\bigwedge^i \mathrm{T}_{G/P}$ is an equivariant vector bundle, but it is not completely reducible. Therefore one cannot immediately apply the Borel–Weil–Bott theorem. This can be dealt with by using an appropriate filtration on the exterior powers of the tangent bundle so that the associated graded is completely reducible, and one has a spectral sequence (3.3) computing the cohomology we are interested in.

An explicit description. In almost all cases covered by Theorem A we can actually give a description of the Hochschild cohomology $\mathrm{HH}^i(G/P)$ as a representation of the Lie algebra $\mathrm{HH}^1(G/P) \cong \mathfrak{g}$. By Remark 7 we can and will ignore the minuscule case. In the cominuscule case we can state the following theorem giving this description.

[1] This identification needs to use Kontsevich's refined Hochschild–Kostant–Rosenberg isomorphism giving an isomorphism of Gerstenhaber algebras between Hochschild cohomology and the cohomology of exterior powers of the tangent bundle. But by the vanishing $\mathrm{H}^i(G/P, \mathrm{T}_{G/P}) = 0$ for all $i \geq 1$ [8, Theorem VII], so that G/P is (locally) rigid as a variety, we obtain an identification of the two brackets for classes of degree 2. See Section 2.4 for more details.

Theorem B (Cominuscule decomposition). *Let G/P be a cominuscule generalised Grassmannian, where P is associated to the cominuscule weight ω_k. Then*

$$\mathrm{HH}^i(G/P) \cong \mathrm{H}^0\left(G/P, \bigwedge^i \mathrm{T}_{G/P}\right) \cong \bigoplus_{\substack{w \in {}^{\mathfrak{l}}\mathrm{W} \\ \ell(w) = \dim G/P - i}} \mathrm{V}_{\mathfrak{g}}^{w\cdot 0 + \mathrm{i}_{G/P}\,\omega_k} \tag{1.4}$$

as representations of $\mathrm{HH}^1(G/P) \cong \mathfrak{g}$, *where $\mathfrak{l}$ is the Lie algebra of the Levi quotient of P.*

Here $\mathrm{i}_{G/P}$ denotes the *index* of G/P, i.e., the maximal integer r such that we can write the anticanonical line bundle $\omega_{G/P}^{\vee}$ as $\mathcal{L}^{\otimes r}$ for an ample line bundle $\mathcal{L}$, and ${}^{\mathfrak{l}}\mathrm{W}$ are the minimal length coset representatives of the Weyl group $\mathrm{W}_{\mathfrak{l}}$ in $\mathrm{W}_{\mathfrak{g}}$. This decomposition is obtained from Kostant's theorem on the Lie algebra cohomology for the nilradical of a parabolic subalgebra [30, Corollary 8.2].

Outside this particularly nice situation the description of the Hochschild cohomology becomes more complicated, and no existing results can be applied. The adjoint case is closest in complexity to the cominuscule case, in which case we obtain the following description. For notational ease, we will write

$$\mathrm{K}(G, P, i, j) := \bigoplus_{\substack{w \in {}^{\mathfrak{l}}\mathrm{W} \\ \ell(w) = \dim G/P - i}} \mathrm{V}_{\mathfrak{g}}^{w\cdot 0 + (\mathrm{i}_{G/P} + j)\omega_k}$$

as a $\mathfrak{g}$-representation, where the right-hand side describes the result of a suitable modification of Kostant's description of the Lie algebra cohomology of the nilradical of Lie P. For our application we want to restrict this sum to those weights $w \cdot 0 + (\mathrm{i}_{G/P} + j)\omega_k$ which are regular, for which we will use the notation

$$\mathrm{K}(G, P, i, j)^{\mathrm{reg}}. \tag{1.5}$$

Theorem C (Adjoint decomposition). *Let G/P be an adjoint generalised Grassmannian, or the adjoint partial flag variety of type A_n. Then* $\dim G/P = 2r + 1$ *for some r, and*

$$\mathrm{HH}^i(G/P) \cong \mathrm{H}^0\left(G/P, \bigwedge^i \mathrm{T}_{G/P}\right)$$
$$\cong \begin{cases} \left(\bigoplus_{p=0}^{\lfloor \frac{i}{2} \rfloor} \mathrm{K}^{\mathrm{reg}}(G, P, i - 2p, p)\right) \\ \qquad \oplus \left(\bigoplus_{p=0}^{\lfloor \frac{i-1}{2} \rfloor} \mathrm{K}^{\mathrm{reg}}(G, P, i - 2p - 1, p + 1)\right) & i \leq r \\ \left(\bigoplus_{p=0}^{\lfloor \frac{2r-i}{2} \rfloor} \mathrm{K}^{\mathrm{reg}}(G, P, i + 1 + 2p, -p - 1)\right) \\ \qquad \oplus \left(\bigoplus_{p=0}^{\lfloor \frac{2r-i+1}{2} \rfloor} \mathrm{K}^{\mathrm{reg}}(G, P, i + 2p, -p - 2)\right) & i \geq r \end{cases} \tag{1.6}$$

as representations of $\mathrm{HH}^1(G/P) \cong \mathfrak{g}$.

All these results only give a small piece of the whole Gerstenhaber algebra structure, see Remark 20 for more details.

Non-vanishing. In addition to the vanishing results and explicit descriptions obtained above we want to highlight the following surprising phenomenon, showing that the expected vanishing does not always hold, even for maximal parabolic subgroups, contrary to experts' belief.

Proposition D. *For all $n \geq 4$ we have that*

$$\mathrm{H}^1\left(\mathrm{SGr}(3,2n), \bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}\right) \cong \mathrm{V}^{\omega_4}_{\mathfrak{sp}_{2n}},$$

as representations of $\mathrm{HH}^1(\mathrm{SGr}(3,2n)) \cong \mathfrak{sp}_{2n}$.

Here $\mathrm{SGr}(3,2n)$ is the symplectic Grassmannian parametrising 3-dimensional isotropic subspaces of a $2n$-dimensional symplectic vector space, associated to the maximal parabolic subgroup P_3 of a simple group of type C_n.

In particular, not every generalised Grassmannian is Hochschild global in the sense of (1.2). That this can happen for *full* flag varieties became clear after computer calculations by Knutson and Schedler for the flag variety G/B in type A (see [39, Remark 2.2]) and the generalisation of these computer calculations to all Dynkin types (but still for the full flag variety) by the authors. But any systematic description is out of reach in this setting.

Bott vanishing. Bott vanishing is a strong vanishing property for the sheaf cohomology of certain vector bundles on $\mathbb{P}^n$, subsequently generalised to certain other settings. We refer to Section 5.3 for more context. In particular, it is expected to fail for all G/P which are not projective space. Proposition D gives the following corollary.

Corollary E. *Bott vanishing fails for* $\mathrm{SGr}(3,2n)$.

This gives the first instance of the failure of Bott vanishing for generalised Grassmannians in the non-cominuscule case. For generalised Grassmannians this was only known in the cominuscule case [11, Section 4.3] using the method used for Theorem B.

Conjectural non-vanishing. The methods to prove the non-vanishing in Proposition D can be implemented in computer algebra, and computations up to rank 10 for maximal parabolic subgroups show that the vanishing result in Theorem A is in fact (very close to) an if and only if in these cases. Let us phrase this optimistically as the following conjecture, with an important caveat being discussed in Remark 1.

Conjecture F. *Let G/P be a generalised Grassmannian which is* not *(co)minuscule or (co)adjoint. Then*

$$\mathrm{H}^q\left(G/P, \bigwedge^p \mathrm{T}_{G/P}\right) \neq 0$$

for some $p \geq 2$ and $q \geq 1$.

Remark 1. The conjecture is phrased optimistically: there is one family of generalised Grassmannians where our computational methods do not give a definitive answer. These are the orthogonal Grassmannians $\mathrm{OGr}(n-1, 2n+1)$ for $n \geq 4$ associated to a simple algebraic group of type B_n and the maximal parabolic subgroup P_{n-1}. As explained in Section 5.2 our methods are inconclusive because it is possible that all higher cohomology gets cancelled in the spectral sequence we use to analyse the Hochschild–Kostant–Rosenberg decomposition.

For all other cases up to rank 10 (except E_8) the computer calculations precisely tell us that all generalised Grassmannians in Conjecture F are *not* Hochschild global in the sense of (1.2). Hence there is ample computational evidence for the conjecture.

The conjecture is only phrased for generalised Grassmannians. For G of rank up to 3, and also in type A_4, it has been computationally confirmed in [22] that G/P is Hochschild global in the sense of (1.2) for *all* possible parabolic subgroups P. For G/B in type A_4 this was also confirmed in [25, Example 3.3]. This is consistent with the computations due to Knutson–Schedler and ourselves for full flag varieties in arbitrary type, where the non-vanishing in type A starts for rank ≥ 5, and e.g. in other types occurs for G/B in type D_5 or F_4.

Related works. Some related computations appear in [22, 25]. The methods in op. cit. are different from ours, using the Bernstein–Gelfand–Gelfand resolution in relative Lie algebra cohomology to compute multiplicities of representations in the sheaf cohomology of polyvector fields, using [31, Proposition 2.8].

We expect the interaction between the methods from op. cit. and this paper will prove useful in understanding the precise conjecture for generalised Grassmannians, and the general picture for arbitrary partial flag varieties.

In [3] the Hochschild cohomology of Fano 3-folds is computed, also using representation-theoretic methods.

Structure of the paper. We start with a lengthy introduction in Section 2, in order to make the computations accessible to algebraic geometers without a representation-theoretic background. In Section 3 we give a self-contained proof of Theorem A. This result can be deduced from [29], but we will reprove it to set up the notation and machinery for later arguments.

In Section 4 we will then prove Theorems B and C. We will illustrate both descriptions in some examples.

In Section 5 we show that *not* every generalised Grassmannian is Hochschild global in sense of (1.2) by explicitly studying the first example where this is the case. We moreover discuss the phenomenon discussed in Remark 1, and the link with Bott vanishing.

2. Preliminaries

2.1. Setup and notation

For the purposes of our paper it is enough to work with homogenous spaces G/P, where G is assumed to be a connected simply-connected simple algebraic group over $\mathbf{k}$. Most of the time the parabolic subgroup P is assumed to be maximal. Below we recall the relevant notation, mostly following [27, Section II.1].

Roots and coroots. Let G be a connected simply-connected simple algebraic group over $\mathbf{k}$. Let $T \subset G$ be a maximal torus and let $\mathrm{X}(T)$ be its group of characters. The group G acts on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ via the adjoint action and we obtain a decomposition into root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{R}} \mathfrak{g}_\alpha,$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathrm{R} = \mathrm{R}(G) \subset \mathrm{X}(T)$ are the *roots* of G.

Let $\mathrm{Y}(T)$ be the group of cocharacters of T. We denote by

$$\langle -, - \rangle \colon \mathrm{X}(T) \times \mathrm{Y}(T) \to \mathbb{Z} \tag{2.1}$$

the natural *perfect pairing* that gives rise to an isomorphism of abelian groups

$$\mathrm{Y}(T) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{X}(T), \mathbb{Z}\right).$$

For each root $\alpha \in \mathrm{R}$ there is a uniquely defined *coroot* $\alpha^\vee \in \mathrm{Y}(T)$, and the set of roots R together with the map $\alpha \mapsto \alpha^\vee$ defines a root system in $\mathrm{X}(T)_{\mathbb{R}}$ in the sense of [9, Chapter VI, Section 1, no. 1]. For each $\alpha \in \mathrm{R}$ we denote by s_α the corresponding reflection on $\mathrm{X}(T)$

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha,$$

and we extend it to $\mathrm{X}(T)_{\mathbb{R}}$ (resp. $\mathrm{X}(T)_{\mathbb{Q}}$) by extending $\alpha^\vee \in \mathrm{Y}(T) \cong \mathrm{X}(T)^\vee$ to $\mathrm{X}(T)_{\mathbb{R}}$ (resp. $\mathrm{X}(T)_{\mathbb{Q}}$).

The reflections s_α for $\alpha \in \mathrm{R}$ generate the *Weyl group* of G

$$\mathrm{W}_G = \langle s_\alpha \mid \alpha \in \mathrm{R} \rangle \cong \mathrm{N}_G(T)/T.$$

The Weyl group acts linearly on $\mathrm{X}(T)$ and $\mathrm{Y}(T)$ and leaves pairing (2.1) invariant.

Weights and coweights. Let $\mathrm{R}^+ \subset \mathrm{R}$ be a subset of *positive roots* and $\mathrm{S} \subset \mathrm{R}^+$ be the *simple roots*. We denote by $\mathrm{R}^- = -\mathrm{R}^+$ the *negative roots*. We define an order $\leq$ on $\mathrm{X}(T)$ by setting

$$\lambda \leq \mu \iff \mu - \lambda \in \sum_{\alpha \in \mathrm{S}} \mathbb{Z}_{\geq 0}\, \alpha.$$

Since G is semisimple, the simple roots S form a basis of $\mathrm{X}(T)_{\mathbb{Q}}$ and the corresponding simple coroots $\mathrm{S}^\vee = \{\alpha^\vee \mid \alpha \in \mathrm{S}\} \subset \mathrm{Y}(T)$ form a basis of $\mathrm{Y}(T)_{\mathbb{Q}}$.

We define the *fundamental weights* $(\omega_\alpha)_{\alpha\in\mathrm{S}} \in \mathrm{X}(T)_{\mathbb{Q}}$ by

$$\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta} \quad \text{for } \alpha, \beta \in \mathrm{S},$$

and the *fundamental coweights* $(\omega_\alpha^\vee)_{\alpha\in\mathrm{S}} \in \mathrm{Y}(T)_{\mathbb{Q}}$ by

$$\langle \alpha, \omega_\beta^\vee \rangle = \delta_{\alpha,\beta} \quad \text{for } \alpha, \beta \in \mathrm{S}.$$

A priori the fundamental weights ω_α live in $\mathrm{X}(T)_{\mathbb{Q}}$. However, since we assume G to be simply-connected, they live in $\mathrm{X}(T)$ and form a basis of it. Consequently, the simple coroots form the dual basis of $\mathrm{Y}(T)$.

A weight $\lambda = \sum_{\alpha\in\mathrm{S}} \ell_\alpha\omega_\alpha \in \mathrm{X}(T)$ is called *G-dominant* (or simply *dominant*) if

$$\langle \lambda, \alpha^\vee \rangle \geq 0 \quad \text{for all } \alpha \in \mathrm{S} \iff \ell_\alpha \geq 0 \quad \text{for all } \alpha \in \mathrm{S}.$$

Fundamental weights form a cone $\mathrm{X}(T)_G^+ \subset \mathrm{X}(T)$ called the *dominant cone*.

A weight $\lambda = \sum_{\alpha\in\mathrm{S}} \ell_\alpha\omega_\alpha \in \mathrm{X}(T)_G^+$ is called *strictly dominant* if $\ell_\alpha > 0$ for all $\alpha \in \mathrm{S}$.

For $\lambda \in \mathrm{X}(T)$ we denote by $\lambda^\vee$ the unique coweight defined by the identity

$$\langle \lambda, \alpha^\vee \rangle = \langle \alpha, \lambda^\vee \rangle \quad \text{for all } \alpha \in \mathrm{S}.$$

Thus, for $\lambda = \sum_{\alpha\in\mathrm{S}} \ell_\alpha\omega_\alpha$ we have $\lambda^\vee = \sum_{\alpha\in\mathrm{S}} \ell_\alpha\omega_\alpha^\vee$. In particular, we have $(\omega_\alpha)^\vee = \omega_\alpha^\vee$.

We define the weight

$$\rho = \sum_{\alpha\in\mathrm{S}} \omega_\alpha = \frac{1}{2}\sum_{\alpha\in\mathrm{R}^+} \alpha$$

and the *dot-action* of W_G on $\mathrm{X}(T)$ by the formula

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Since G is simple, its root system is irreducible, and therefore there exists (up to a non-zero factor) a unique W_G-invariant scalar product on $\mathrm{X}(T)_{\mathbb{R}}$, (see [9, Chapter VI, Section 1, no. 2])

$$(-,-)\colon \mathrm{X}(T)_{\mathbb{R}} \times \mathrm{X}(T)_{\mathbb{R}} \to \mathbb{R}. \tag{2.2}$$

We choose the standard scaling as in [9]. Now using (2.2) we can identify $\mathrm{X}(T)_{\mathbb{R}}$ and $\mathrm{Y}(T)_{\mathbb{R}}$. Thus, for us both roots and coroots will live in the same space $\mathrm{X}(T)_{\mathbb{R}}$ and we have

$$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha \quad \text{for } \alpha \in \mathrm{R}.$$

Parabolic subgroups. We denote by B^+ and B the Borel subgroups of G corresponding to the positive and negative roots respectively. We have

$$B^+ \cap B = T.$$

We want to stress that B *corresponds to the negative roots*.

For subset $I \subset \mathrm{S}$ one defines the *standard parabolic subgroup* P containing B such that

$$P = LU = L \ltimes U$$

where L is the *Levi factor* of P and U is the unipotent radical of P, and the subset of simple roots of the (reductive) group L is precisely I. The group L is a reductive subgroup of G containing T and its roots with respect to T are

$$\mathrm{R}_L = \mathrm{R} \cap \mathbb{Z}I.$$

We also introduce the notation

$$\mathrm{S}_L = \mathrm{S} \cap \mathrm{R}_L = I \quad \text{and} \quad \mathrm{R}_L^{\pm} = \mathrm{R}_L \cap \mathrm{R}^{\pm}.$$

For the Weyl group of L we have

$$\mathrm{W}_L = \langle s_\alpha \mid \alpha \in \mathrm{R}_L \rangle,$$

and it is generated by the simple reflections s_α with $\alpha \in I$.

Associated Lie algebras. We denote Lie algebras of the aforementioned algebraic groups by

$$\begin{aligned} \mathfrak{g} &= \mathrm{Lie}(G), & \mathfrak{t} &= \mathrm{Lie}(T), & \mathfrak{b} &= \mathrm{Lie}(B), \\ \mathfrak{p} &= \mathrm{Lie}(P), & \mathfrak{l} &= \mathrm{Lie}(L), & \mathfrak{n} &= \mathrm{Lie}(U). \end{aligned}$$

We have the following decompositions

$$\begin{aligned} &\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{R}} \mathfrak{g}_\alpha, \quad \mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{R}^-} \mathfrak{g}_\alpha, \\ &\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}, \\ &\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{R}_L} \mathfrak{g}_\alpha, \quad \mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{R}^- \cup \mathrm{R}_L} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \bigoplus_{\alpha \in \mathrm{R}^- \setminus \mathrm{R}_L^-} \mathfrak{g}_\alpha. \end{aligned}$$

Varieties. In Section 2.2 we will introduce the partial flag varieties of interest, for which we use the following notation:

- Q^n, the n-dimensional smooth quadric hypersurface in $\mathbb{P}^{n+1}$;
- $\mathrm{Gr}(d, n)$, the Grassmannian of d-subspaces in an n-dimensional vector space;
- $\mathrm{OGr}(d, n)$, the *orthogonal Grassmannian* of isotropic d-dimensional subspaces in an n-dimensional vector space equipped with a nondegenerate symmetric bilinear form, which is an isotropic Grassmannian in type B (resp. D) depending on the parity of n;
- $\mathrm{SGr}(d, 2n)$, the *symplectic Grassmannian* of isotropic d-dimensional subspaces in a $2n$-dimensional vector space equipped with a nondegenerate skew-symmetric bilinear form, which is an isotropic Grassmannian in type C.

Representations and equivariant vector bundles. The representation theory of simple Lie algebras and algebraic groups allows us to describe irreducible representations using highest weights; and as the categories of representations are equivalent we will interchangeably use $\mathfrak{g}$ and G. We will denote

- $\mathrm{V}_{\mathfrak{g}}^{\lambda}$ (resp. V_{G}^{λ}), the irreducible $\mathfrak{g}$-representation (resp. G-representation) associated to the $\mathfrak{g}$-dominant highest weight $\lambda \in \mathrm{X}(T)_{\mathfrak{g}}^{+}$;
- $\mathrm{V}_{\mathfrak{l}}^{\lambda}$ (resp. V_{L}^{λ}), the irreducible $\mathfrak{l}$-representation (resp. L-representation) associated to the $\mathfrak{l}$-dominant highest weight $\lambda \in \mathrm{X}(T)_{\mathfrak{l}}^{+}$;
- $\mathcal{E}^{\lambda}$, the G-equivariant vector bundle on G/P associated to $\mathrm{V}_{\mathfrak{l}}^{\lambda}$.

Notation for tables. In some cases we will give a description of the associated graded of $\bigwedge^p \mathrm{T}_{G/P}$ in the sense of Definition 22, see Tables 5–7. Each row is an irreducible summand, and the columns are to be interpreted as:

weight	the weight of the (irreducible) vector bundle, as a coefficient vector for the fundamental weights
rank	the rank of the vector bundle
degree	the degree in which its cohomology lives according to the Borel–Weil–Bott theorem, or empty if the weight is not regular in the setting of Borel–Weil–Bott
representation	if the cohomology is nonzero, the highest weight of the representation obtained from the Borel–Weil–Bott theorem
dimension	the dimension of this representation, if nonzero
sum of roots	the weight of the vector bundle, as a coefficient vector for the simple roots

The coefficient vectors are given with respect to the fundamental weights (resp. the simple roots) instead of an explicit description of the weight, using the labeling of the vertices from Bourbaki [9] and recalled in Table 1.

2.2. Partial flag varieties

Here we fix notation and terminology related to partial flag varieties G/P.

Classification of partial flag varieties. To isolate certain well-behaved families of partial flag varieties, we need to talk about their explicit geometric realisation, although the proofs will not use this. We will use the Bourbaki convention for labelling the simple roots, which is recalled in Table 1.

To a G-dominant weight $\lambda \in \mathrm{X}(T)_{G}^{+}$, we associate the unique closed G-orbit in $\mathbb{P}((\mathrm{V}_{G}^{\lambda})^{\vee})$. This is the orbit of the (line spanned by) the lowest weight vector $v_{-\lambda}$ of weight $-\lambda$ of the representation $(\mathrm{V}_{G}^{\lambda})^{\vee}$ and its stabiliser is the standard parabolic subgroup P associated to the subset $I \subset \mathrm{S}$ defined by

$$I := \{\alpha \in \mathrm{S} \mid (\alpha, \lambda) = 0\} \subset \mathrm{S}.$$

This gives an explicit realisation of G/P.

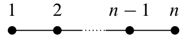
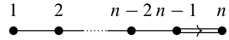
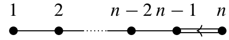
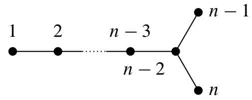
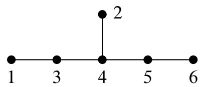
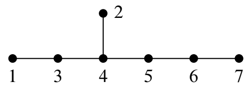
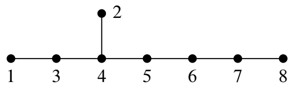
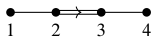
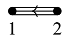
type	labelling
A_n	
B_n	
C_n	
D_n	
E_6	
E_7	
E_8	
F_4	
G_2	

Table 1. Bourbaki labelling for simple roots.

We will specify a partial flag variety by describing the (sum of) simple roots which are *not* included in the parabolic subgroup, e.g. (A_n, α_1) corresponds to $\mathbb{P}^{n+1}$. For a maximal parabolic subgroup there thus is a single simple root. In general we can describe P by crossing out these simple roots in the Dynkin diagram. Hence (A_n, α_1) is described by

 .

When P is a *maximal* parabolic subgroup we will say that the partial flag variety G/P is a *generalised Grassmannian*.

The two following remarks explain why we can *and will* ignore certain descriptions of generalised Grassmannians.

Remark 2. Exceptional isomorphisms of Lie algebras in low rank and symmetries of the Dynkin diagrams account for the following isomorphisms of generalised Grassmannians:

- $(A_n, \alpha_i) = (A_n, \alpha_{n+1-i})$, as $\mathrm{Gr}(i, n+1) \cong \mathrm{Gr}(n-i, n+1)$;
- $(A_3, \alpha_2) = (D_3, \alpha_1)$, which are isomorphic to Q^4;
- $(D_n, \alpha_{n-1}) = (D_n, \alpha_n)$, the $n(n-1)/2$-*dimensional spinor variety*, which is one of the connected components of the space of maximal isotropic subspaces for a nondegenerate symmetric bilinear form in a $2n$-dimensional vector space;

- $(\mathrm{D}_4, \alpha_1) = (\mathrm{D}_4, \alpha_3) = (\mathrm{D}_4, \alpha_4)$, which are isomorphic to Q^6;
- $(\mathrm{E}_6, \alpha_1) = (\mathrm{E}_6, \alpha_6)$, the *Cayley plane*;
- $(\mathrm{E}_6, \alpha_3) = (\mathrm{E}_6, \alpha_5)$.

Remark 3. On the other hand one also has the following exotic isomorphisms which are not related to an exceptional isomorphism of the associated simple Lie algebras or an obvious symmetry of the Dynkin diagram:

(1) $(\mathrm{B}_{n-1}, \alpha_{n-1}) = (\mathrm{D}_n, \alpha_n)$, giving an alternative description of the spinor varieties;

(2) $(\mathrm{C}_n, \alpha_1) = (\mathrm{A}_{2n-1}, \alpha_1)$, isomorphic to $\mathbb{P}^{2n-1}$;

(3) $(\mathrm{G}_2, \alpha_1) = (\mathrm{B}_3, \alpha_1)$, isomorphic to Q^5.

This second class of exotic isomorphisms explains the caveat in the following lemma [18, Section 2].

Lemma 4. *Let G/P be a partial flag variety. Then*

$$\mathrm{H}^0(G/P, \mathrm{T}_{G/P}) \cong \mathfrak{g},$$

unless P is a maximal parabolic subgroup and G/P is of type (B_n, α_n)*,* (C_n, α_1) *or* (G_2, α_1).

In the cases which are ruled out in the statement of the lemma, we notice that we obtain a Lie subalgebra of the Lie algebra associated to the automorphism group of G/P. We will exclude these cases without further mention from our analysis.

Cominuscule and (co)adjoint partial flag varieties. We will now introduce the terminology used to distinguish several special classes of partial flag varieties. These go by different names in the literature, but we will be using the terminology from [17] and mention other terminology as we go along. Here we will use the explicit geometric realisation of G/P as the unique closed orbit in $\mathbb{P}((\mathrm{V}_G^\lambda)^\vee)$.

Definition 5. Let $\lambda \in \mathrm{X}(T)_G^+$ be a dominant weight of G. We will say that λ is

(1) *minuscule* if
$$(\lambda, \alpha^\vee) \leq 1 \quad \forall \alpha \in \mathrm{R}^+$$

(2) *cominuscule* if
$$(\alpha, \lambda^\vee) \leq 1 \quad \forall \alpha \in \mathrm{R}^+ \tag{2.3}$$

(3) *adjoint* if λ is the highest weight of the adjoint representation[2] of G, i.e., $\lambda = \Theta$ is the highest (long) root of G;

(4) *coadjoint* if λ is the highest short root θ.

[2]Since G is assumed to be simple, its adjoint representation is irreducible.

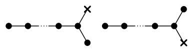
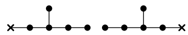
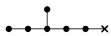
type	variety	diagram	dimension	index	Cartan label
(A_n, α_1)	$\mathbb{P}^n$		n	$n+1$	AIII
(A_n, α_2)	$\mathrm{Gr}(2, n+1)$		$2(n-1)$	$n+1$	AIII
(A_n, α_i)	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$(\mathrm{A}_n, \alpha_{n-1})$	$\mathrm{Gr}(n-2, n+1) \cong \mathrm{Gr}(2, n+1)$		$2(n-1)$	$n+1$	AIII
(A_n, α_n)	$\mathbb{P}^{n,\vee} \cong \mathbb{P}^n$		n	$n+1$	AIII
(B_n, α_1)	Q^{2n-1}		$2n-1$	$2n-1$	BDI
(C_n, α_n)	$\mathrm{SGr}(n, 2n) = \mathrm{LGr}(2n)$		$n(n+1)/2$	$n+1$	CI
(D_n, α_1)	Q^{2n-2}		$2n-2$	$2n-2$	BDI
$(\mathrm{D}_n, \alpha_{n-1}) = (\mathrm{D}_n, \alpha_n)$	$\mathrm{OGr}(n-1, 2n)$		$n(n-1)/2$	$2n-2$	DIII
$(\mathrm{E}_6, \alpha_1) = (\mathrm{E}_6, \alpha_6)$	Cayley plane		16	12	EIII
(E_7, α_7)	Freudenthal variety		27	17	EVII

Table 2. Cominuscule partial flag varieties.

Note that if G is simply-laced, then the notions of minuscule and cominuscule (resp. adjoint and coadjoint) coincide.

Definition 6. Let P be the standard parabolic subgroup of G associated to a weight λ. Then we say that the partial flag variety G/P is *minuscule* (resp. *cominuscule*, *adjoint*, *coadjoint*) if λ is. In such a case we also call the parabolic P *minuscule* (resp. *cominuscule*, *adjoint*, *coadjoint*).

Remark 7. For the purposes of our analysis we can ignore the minuscule case. The only generalised Grassmannians which are minuscule but not cominuscule are associated to (B_n, α_n) and (C_n, α_1) respectively. But by Remark 3 these are isomorphic to the generalised Grassmannians associated to $(\mathrm{D}_{n+1}, \alpha_{n+1})$ and $(\mathrm{A}_{2n}, \alpha_1)$ respectively, which are cominuscule as can be seen in Table 2, and we will use the latter realisations for our analysis.

In Table 2 we have collected the cominuscule generalised Grassmannians, and their relevant properties.

Remark 8. Over the complex numbers cominuscule generalised Grassmannians are also known as *compact hermitian symmetric spaces*, and they are often referred to as such in the literature. We have included the Cartan labelling for them in Table 2.

For each Dynkin type and rank there is a unique *adjoint* partial flag variety. In Table 3 we have collected the adjoint partial flag varieties (excluding type C, see below), and their relevant properties.

Two special cases for us are

- in type A, where it is not a generalised Grassmannian, as the associated parabolic subgroup is submaximal such that $\operatorname{rk} \operatorname{Pic} G/P = 2$: in this case it is isomorphic to $\mathbb{P}(\mathrm{T}_{\mathbb{P}^n})$, the relative Proj of $\operatorname{Sym}^\bullet \mathrm{T}^\vee_{\mathbb{P}^n}$;

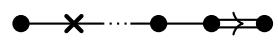
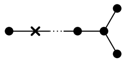
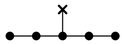
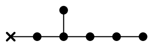
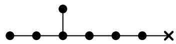
type	variety	diagram	dimension	index
$(\mathrm{A}_n, \alpha_1 + \alpha_n)$	$\mathbb{P}(\mathrm{T}_{\mathbb{P}^n})$		$2n-1$	n
(B_n, α_2)	$\mathrm{OGr}(2, 2n+1)$		$4n-5$	$2n-2$
(D_n, α_2)	$\mathrm{OGr}(2, 2n)$		$4n-7$	$2n-3$
(E_6, α_2)			21	11
(E_7, α_1)			33	17
(E_8, α_8)			57	29
(F_4, α_1)			15	8
(G_2, α_2)	$\mathrm{G}_2\mathrm{Gr}(2, 7)$		5	3

Table 3. Adjoint partial flag varieties.

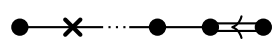
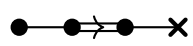
type	variety	diagram	dimension	index
(C_n, α_2)	$\mathrm{SGr}(2, 2n)$		$4n-5$	$n+1$
(F_4, α_4)			15	11

Table 4. Coadjoint but not adjoint partial flag varieties.

- in type C, where the highest root is $2\omega_1$, but by Remark 3 these generalised Grassmannians are isomorphic to $\mathbb{P}^{2n-1}$ (and the adjoint realisation is the second Veronese embedding) and will be omitted from the analysis.

Finally, similar to Remark 7 we need to consider the non-simply-laced case and classify the coadjoint but not adjoint partial flag varieties. These cannot be omitted from the analysis. In Table 4 we have collected the remaining coadjoint partial flag varieties, and their relevant properties.

For more on the geometry of generalised Grassmannians one is referred to [2].

Equivariant vector bundles and Borel–Weil–Bott. For a partial flag variety there exists an equivalence

$$\mathrm{coh}^G G/P \cong \mathrm{rep}\, P \tag{2.4}$$

of monoidal abelian categories between the category of G-equivariant vector bundles on G/P and the category of finite-dimensional representations of P [4, 23]; under this equivalence a G-equivariant vector bundle E is sent to its fiber $E_{[P]}$ at the point $[P] \in G/P$. As P is not reductive, the category $\mathrm{rep}\, P$ is not semisimple and its representation theory is hard to understand. An interesting full subcategory of $\mathrm{rep}\, P$ is given by $\mathrm{rep}^{\mathrm{ss}}\, P$, with objects the completely reducible representations, which is a semisimple category. We have

$$\mathrm{rep}^{\mathrm{ss}}\, P \cong \mathrm{rep}\, L \subseteq \mathrm{rep}\, P \cong \mathrm{coh}^G G/P, \tag{2.5}$$

where L is the Levi factor $L \subset P$ (see Section 2.1).

Given an L-dominant weight $\lambda \in \mathrm{X}(T)_L^+$ we get an irreducible L-representation V_L^λ with the highest weight λ, which we can extend to a representation of P by letting the unipotent radical U act trivially, and hence a G-equivariant vector bundle $\mathcal{E}^\lambda$ on G/P.

Unfortunately, the inclusion in (2.5) is strict, and not all equivariant vector bundles we are interested in arise as representations of L, see Section 3.1. But for those which are associated to completely reducible representations, there is a strong tool to compute their sheaf cohomology: the Borel–Weil–Bott theorem.

Recall that a weight $\mu \in \mathrm{X}(T)_G$ is called *G-regular* (or *regular*), if it does not lie on a wall of a Weyl chamber of G. Equivalently, a weight μ is regular if and only if

$$(\mu, \alpha) \neq 0 \quad \forall \alpha \in \mathrm{R}.$$

Otherwise the weight is called *G-singular* (or *singular*).

Theorem 9 (Borel–Weil–Bott). *Let $\mathcal{E}^\lambda$ be the G-equivariant vector bundle on G/P given by the irreducible L-representation with highest weight $\lambda \in \mathrm{X}(T)_L^+$. Then one of the following holds:*

(1) *if $\lambda + \rho$ is G-singular, then*

$$\mathrm{H}^i(G/P, \mathcal{E}^\lambda) = 0$$

for all i;

(2) *if $\lambda + \rho$ is G-regular, then there exists a unique $w \in \mathrm{W}_G$ such that $w(\lambda + \rho)$ is G-dominant, and then*

$$\mathrm{H}^i(G/P, \mathcal{E}^\lambda) \cong \begin{cases} \mathrm{V}_G^{w(\lambda+\rho)-\rho} & i = \ell(w) \\ 0 & i \neq \ell(w), \end{cases}$$

as G-representations, where $\ell(w)$ denotes the length of the element $w \in \mathrm{W}_G$.

In most cases we cannot apply the Borel–Weil–Bott theorem on the nose to compute the sheaf cohomology of $\bigwedge^p \mathrm{T}_{G/P}$. These are the vector bundles we will be interested in when studying Hochschild cohomology, but in general these equivariant vector bundles are not completely reducible. This fact is the main reason for the existence of this paper.

Remark 10. When describing Hochschild cohomology it is more natural to emphasise the structure as a representation of the Lie algebra, i.e., we will rather write $\mathrm{V}_{\mathfrak{g}}^{w(\lambda+\rho)-\rho}$.

On dominant weights. The following lemma summarises some standard properties.

Lemma 11. *Let G and P be as before.*

(1) *Let λ be an L-dominant weight. If we have $(\lambda, \alpha) \geq 0$ for $\alpha \in \mathrm{S} \setminus \mathrm{S}_L$, then $(\lambda, \alpha) \geq 0$ for any $\alpha \in \mathrm{R}^+$.*

(2) *Let λ be an L-dominant weight. Then*

   (a) *$(\lambda + \rho, \alpha) > 0$ for any $\alpha \in \mathrm{R}_L^+$;*

   (b) *If $(\lambda + \rho, \alpha) \neq 0$ for any $\alpha \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$, then the weight $\lambda + \rho$ is regular.*

(3) *Let λ be a strictly G-dominant weight and $\alpha \in \mathrm{R}$ a root. Then we have*

(a) α*is positive* $\Leftrightarrow (\lambda, \alpha) > 0$;

(b) α*is negative* $\Leftrightarrow (\lambda, \alpha) < 0$.

(4) *For $w \in \mathrm{W}_G$ we have*

$$\ell(w) = \#\mathrm{R}(w),$$

where

$$\mathrm{R}(w) = \{\alpha \in \mathrm{R}^+ \mid w(\alpha) \in \mathrm{R}^-\}.$$

(5) *Let λ be an L-dominant weight, such that $\lambda + \rho$ is regular, and let $w \in \mathrm{W}_G$ be the unique Weyl group element, such that $w(\lambda + \rho)$ is strictly G-dominant. Then we have*

$$\ell(w) = \#\{\alpha \in \mathrm{R}^+ \setminus \mathrm{R}_L^+ \mid (\lambda + \rho, \alpha) < 0\}.$$

Proof. Items (1), (2), and (3) follow immediately from the definitions given Section 2.1. For Item (4) we refer to [26, Lemma 10.3A].

Let us prove Item (5). Let μ be the unique G-dominant weight such that

$$w(\lambda + \rho) = \mu + \rho.$$

By Items (3) and (4) and W_G-invariance of the scalar product we have

$$\mathrm{R}(w) = \{\alpha \in \mathrm{R}^+ \mid (\mu + \rho, w(\alpha)) < 0\} = \{\alpha \in \mathrm{R}^+ \mid (\lambda + \rho, \alpha) < 0\}.$$

Applying Item (2) we get

$$\mathrm{R}(w) = \{\alpha \in \mathrm{R}^+ \setminus \mathrm{R}_L^+ \mid (\lambda + \rho, \alpha^\vee) < 0\}.$$

Now the desired equality follows from Item (4). ∎

2.3. Lie algebra cohomology

In the description of the Hochschild–Kostant–Rosenberg decomposition for cominuscule and adjoint varieties we will need some results on Lie algebra cohomology. Let us briefly introduce the required notions.

Definition 12. Let $\mathfrak{g}$ be a Lie algebra, and V a representation of $\mathfrak{g}$. Then the *ith Lie algebra cohomology of $\mathfrak{g}$ with values in V* is

$$\mathrm{H}^i_{\mathrm{CE}}(\mathfrak{g}, V) := \mathrm{Ext}^i_{\mathrm{U}\mathfrak{g}}(\mathbf{k}, V)$$

where $\mathrm{U}\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

Although confusion between sheaf cohomology and Lie algebra cohomology is unlikely, we will always denote Lie algebra cohomology as $\mathrm{H}^\bullet_{\mathrm{CE}}(\mathfrak{g}, V)$.

To compute it one often uses an explicit projective resolution of the trivial $\mathrm{U}\mathfrak{g}$-module k which gives rise to the *Chevalley–Eilenberg complex* $\mathrm{Hom}_{\mathbf{k}}(\bigwedge^{\bullet}\mathfrak{g}, V)$, with differential

$$\begin{aligned}
&\mathrm{d}(f)(g_1 \wedge \cdots \wedge g_{n+1}) \\
&\quad := \sum_i (-1)^{i+1} g_i f(g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_{n+1}) \\
&\qquad + \sum_{i<j} (-1)^{i+j} f\big([g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_{n+1}\big)
\end{aligned}$$

for $f \in \mathrm{Hom}_{\mathbf{k}}(\bigwedge^n \mathfrak{g}, V)$.

The straightforward observation to make from this definition is that if $\mathfrak{g}$ is abelian and the action of $\mathfrak{g}$ on V is trivial, then the differentials in this complex vanish, and $\mathrm{H}^i_{\mathrm{CE}}(\mathfrak{g}, V) \cong \bigwedge^i \mathfrak{g} \otimes_{\mathbf{k}} V$.

We will compute Lie algebra cohomology for the nilpotent radical of parabolic subalgebras in (semi)simple Lie algebras, and Lie algebras constructed out of it. For this we will use a result of Kostant [30, Corollary 8.2]. The setting we work in is that of a simple Lie algebra $\mathfrak{g}$, with parabolic subalgebra $\mathfrak{p}$ and unipotent radical $\mathfrak{n}$,

$$\mathfrak{n} \subseteq \mathfrak{p} \subseteq \mathfrak{g}.$$

We have the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{l}$ is the Levi subalgebra. If V is a $\mathfrak{p}$-representation, then $\mathrm{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{n}, V)$ has the structure of a $\mathfrak{l}$-representation.

We need one more piece of notation. If I denotes the set of simple roots added to the Borel subalgebra to obtain $\mathfrak{p}$, then we denote

$${}^{\mathfrak{l}}\mathrm{W} := \{w \in \mathrm{W}_{\mathfrak{g}} \mid \forall \alpha \in I : \ell(s_\alpha w) = \ell(w) + 1\} = \{w \in \mathrm{W}_{\mathfrak{g}} \mid w^{-1}\big(\mathrm{X}(T)^+_{\mathfrak{l}}\big) \subseteq \mathrm{X}(T)^+_{\mathfrak{g}}\}$$

the set of minimal length (right) coset representatives of the Weyl group $\mathrm{W}_{\mathfrak{l}}$ in $\mathrm{W}_{\mathfrak{g}}$.

The statement of Kostant's theorem, computing the $\mathfrak{l}$-structure of the Lie algebra cohomology of $\mathfrak{n}$ with values in $V = \mathbf{k}$ the trivial representation – there exists a more general version with coefficients, but we will not need it – reads as follows.

Theorem 13 (Kostant). *There exists an isomorphism*

$$\mathrm{H}^i_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k}) \cong \bigoplus_{\substack{w \in {}^{\mathfrak{l}}\mathrm{W} \\ \ell(w) = i}} \mathrm{V}_{\mathfrak{l}}^{w \cdot 0}$$

of $\mathfrak{l}$-modules.

This result is particularly interesting when $\mathfrak{n}$ has an easy structure. In this paper we consider the cases where $\mathfrak{n}$ is abelian (see Lemma 25 (3)) or Heisenberg (see Lemma 28). This allows us to obtain the descriptions in Sections 4.1 and 4.2.

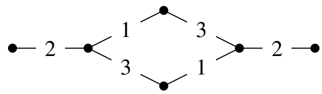


Figure 1. Parabolic Bruhat graph for the (cominuscule) parabolic $\bullet\!-\!\times\!-\!\bullet$.

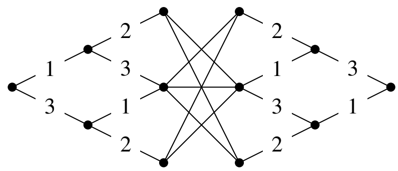


Figure 2. Parabolic Bruhat graph for the (adjoint) parabolic $\times\!-\!\bullet\!-\!\times$.

Example 14. We can conveniently visualise the result of Kostant's theorem, by giving the *parabolic Bruhat graph* of ${}^{\mathrm{I}}\mathrm{W}$. An introductory reference (where it is called the *Hasse diagram*) is [16, Section 3.2]. We will only need that ${}^{\mathrm{I}}\mathrm{W}$ can be interpreted as a subset of $\mathrm{W}_{\mathfrak{g}}$, which has the Bruhat order, which we will write from left to right. We then restrict this Bruhat order to ${}^{\mathrm{I}}\mathrm{W}$ to obtain the parabolic Bruhat graph, and we label an edge by the Weyl group element which sends the source to the target, and this will be a simple reflection.

For example, let us consider the parabolic subalgebra of $\mathfrak{sl}_4$ (of type A_3) corresponding to

$$\bullet\!-\!\times\!-\!\bullet\,.$$

Computing ${}^{\mathrm{I}}\mathrm{W}$ in this case gives rise to the parabolic Bruhat graph in Figure 1. We will revisit this example in Example 36.

We can also consider the parabolic subalgebra of $\mathfrak{sl}_4$ given by

$$\times\!-\!\bullet\!-\!\times\,.$$

This is associated to an *adjoint* parabolic subalgebra. The parabolic Bruhat graph is given in Figure 2. We will revisit this example in Example 41.

2.4. Hochschild cohomology

We now introduce the invariant we are trying to compute in this paper. Originally Hochschild cohomology was introduced for associative algebras, where it governs the deformation theory (as an associative algebra). Later its definition has been generalised to algebraic geometry, and more generally abelian and suitably enhanced triangulated categories.

Definition 15. Let X be a smooth projective variety. Then the *ith Hochschild cohomology* of X is

$$\mathrm{HH}^i(X) := \mathrm{Ext}^i_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

where $\Delta\colon X \hookrightarrow X \times X$ is the diagonal embedding.

To compute Hochschild cohomology one needs a convenient resolution of $\Delta_*\mathcal{O}_X$. It turns out that $\mathbf{L}\Delta^* \circ \Delta_*(\mathcal{O}_X)$ is quasi-isomorphic to $\bigoplus_{i=0}^{\dim X} \Omega_X^i[-i]$. In the affine setting this result is due to Hochschild–Kostant–Rosenberg [24], and various generalisations to the (quasi)projective setting (without an attempt to be exhaustive) are due to e.g. Gerstenhaber–Schack [20], Swan [45], Markarian [36] and Yekutieli [48].

Theorem 16 (Hochschild–Kostant–Rosenberg). *Let X be a smooth projective variety. Then*

$$\mathrm{HH}^i(X) \cong \bigoplus_{p+q=i} \mathrm{H}^q\left(X, \bigwedge^p \mathrm{T}_X\right),$$

as vector spaces.

In the study of partial flag varieties we wish to reduce to the case of G a simple, and not just semisimple, algebraic group. This is done using the following two lemmas. By the Künneth formula and the isomorphism $\mathrm{T}_{X\times Y} \cong \pi_X^*\mathrm{T}_X \oplus \pi_Y^*\mathrm{T}_Y$ we obtain the first lemma.

Lemma 17 (Künneth formula for Hochschild cohomology). *Let X and Y be smooth projective varieties. Then*

$$\begin{aligned}
\mathrm{HH}^i(X\times Y) &\cong \bigoplus_{p+q=i} \mathrm{H}^q\left(X\times Y, \bigwedge^p \mathrm{T}_{X\times Y}\right) \\
&\cong \bigoplus_{p+q=i}\bigoplus_{n+m=p} \mathrm{H}^q\left(X\times Y, \bigwedge^n \mathrm{T}_X \boxtimes \bigwedge^m \mathrm{T}_Y\right) \\
&\cong \bigoplus_{p+q=i}\bigoplus_{n+m=p}\bigoplus_{a+b=q} \mathrm{H}^a\left(X, \bigwedge^n \mathrm{T}_X\right) \otimes_{\mathbf{k}} \mathrm{H}^b\left(Y, \bigwedge^m \mathrm{T}_Y\right)
\end{aligned}$$

If a partial flag variety G/P is associated to a semisimple but not simple algebraic group G, then we always have an isomorphism

$$G/P \cong G_1/P_1 \times \cdots \times G_r/P_r$$

where G_i is simple and P_i is a parabolic subgroup of G_i. Hence in order to answer questions about the (non-)vanishing of components in the Hochschild–Kostant–Rosenberg decomposition, by Lemma 17 it suffices to consider only *simple* algebraic groups G.

Algebraic structure. Hochschild cohomology comes in complete generality equipped with the extra structure of a Gerstenhaber algebra. This is a graded-commutative algebra together with a Lie bracket of degree -1 which are compatible in a way which is not relevant for this paper.

Because the Lie bracket has degree -1, we get that every $\mathrm{HH}^1(X)$ has the structure of a Lie algebra, and every $\mathrm{HH}^i(X)$ is a representation for this Lie algebra.

On the level of polyvector fields we have that $\mathrm{H}^1(X, \mathcal{O}_X) \oplus \mathrm{H}^0(X, \mathrm{T}_X)$ also has the structure of a Lie algebra: $\mathrm{H}^1(X, \mathcal{O}_X)$ is the tangent space at the Picard variety and is an abelian Lie algebra, whilst $\mathrm{H}^0(X, \mathrm{T}_X)$ is the Lie algebra of the automorphism group of X. In the case of $X = G/P$ there is only the contribution of the automorphism group, and by Lemma 4 we have that $\mathrm{H}^0(G/P, \mathrm{T}_{G/P}) \cong \mathfrak{g}$ in all relevant cases.

The Hochschild–Kostant–Rosenberg decomposition from Theorem 16 is only on the level of vector spaces. But by twisting the isomorphism by the square root of the Todd class $\sqrt{\operatorname{td} X} \in \bigoplus_{i=0}^{\dim X} \mathrm{H}^i(X, \Omega_X^i)$ it is possible to upgrade it to an isomorphism of Gerstenhaber algebras [13, 14].

Remark 18. If X is Hochschild global in the sense of (1.2), then the twist by the square root of the Todd class is necessarily trivial, as in each component it is the cup product

$$\mathrm{H}^i(X, \Omega_X^i) \otimes_{\mathbf{k}} \mathrm{H}^q\left(X, \bigwedge^p \mathrm{T}_X\right) \to \mathrm{H}^{q+i}\left(X, \bigwedge^{p-i} \mathrm{T}_X\right),$$

with $(\sqrt{\operatorname{td}_X})_i$, which for $i = 0$ is the identity. In this case the vector space isomorphism is in fact a Gerstenhaber algebra isomorphism, and this paper provides new instances where this is the case.

We obtain the following

Lemma 19. *Let G/P be a partial flag variety. Then each $\mathrm{HH}^i(X)$ has the structure of a $\mathfrak{g}$-representation.*

In the setting of Remark 3 we opt to use the largest possible Lie algebra, to get the most economical description.

Remark 20. What we describe in this paper is only a portion of the full Gerstenhaber structure on $\mathrm{HH}^\bullet(G/P)$, namely the part involving the Gerstenhaber bracket of classes in degrees 1 and i. Other interesting questions, aside from the (non-)vanishing of the components, are e.g.

(1) whether $\mathrm{HH}^\bullet(G/P)$ is generated as an algebra by $\mathrm{HH}^1(G/P)$ (the answer can be checked to be yes for $\mathbb{P}^n$);
(2) what the other Gerstenhaber brackets are, in particular those involving $\mathrm{HH}^2(G/P)$, to get a classification of the Poisson structures on G/P.

As already mentioned in the introduction, the classification of Poisson structures is seemingly a very hard problem, and the answer (in case it is non-trivial) is only known for the generalised Grassmannians $\mathbb{P}^3$ and Q^3 [33, 41]. Observe that on G/P there is always the (non-zero) standard Poisson structure, see e.g. [21].

3. Vanishing for cominuscule and (co)adjoint varieties

In this section we prove Theorem A. It is possible to deduce this result for generalised Grassmannians from the vanishing result [29, Theorem 4.2.3 (i)] due to Konno. In the notation of loc. cit. we need that the difference between the index $k(G/P)$ (denoted $\mathrm{i}_{G/P}$

in this work) and $k'(G/P)$ as defined in equation (4.2.1) of op. cit. is at most 1. One can check that this is the case if and only if G/P is of (co)minuscule or (co)adjoint type, following the description in Section 2.2.

We give an alternative proof, relying more on the geometry of the varieties involved and which also covers the adjoint variety in type A. Along the way we also set up the machinery used in the proofs of Theorems B and C.

3.1. Tangent bundle of a partial flag variety

We will discuss some preliminary facts on the tangent bundle $\mathrm{T}_{G/P}$ of G/P and its exterior powers.

Lemma 21. *Let G and P be as before.*

(1) *The tangent bundle $\mathrm{T}_{G/P}$ is G-equivariant and corresponds via (2.5) to the quotient $\mathfrak{g}/\mathfrak{p}$ endowed with the adjoint action of P.*
(2) *The weights of $\mathfrak{g}/\mathfrak{p}$ are the* non-parabolic positive roots $\mathrm{R}^+ \setminus \mathrm{R}_L^+$. *These are those positive roots of G, whose decomposition in terms of simple roots necessarily involves α_k with a positive coefficient.*
(3) *There is an isomorphism of P-representations $\mathfrak{n}^\vee \cong \mathfrak{g}/\mathfrak{p}$.*
(4) *The exterior powers $\bigwedge^p \mathrm{T}_{G/P}$ are G-equivariant and correspond to the representations $\bigwedge^p \mathfrak{g}/\mathfrak{p}$.*
(5) *Weights of $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ are sums of p distinct non-parabolic positive roots $\mathrm{R}^+ \setminus \mathrm{R}_L^+$. In particular, for any weight β appearing in $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ we have*

$$(\beta, \omega_k^\vee) \geq p.$$

Proof. Items (1), (2), and (3) can be found in [1, Section 3]. Item (4) follows from the equivalence (2.4) and the identification of P-representations with $\mathfrak{p}$-representations. Item (5) follows from Item (2). ∎

A filtration for exterior powers of the tangent bundle. Often the tangent bundle $\mathrm{T}_{G/P}$ and its exterior powers $\bigwedge^p \mathrm{T}_{G/P}$ are not completely reducible, see Lemma 25. Hence, one cannot directly apply Borel–Weil–Bott to compute cohomology of $\bigwedge^p \mathrm{T}_{G/P}$.

However, one can try to bypass this obstacle by considering a filtration of $\bigwedge^p \mathrm{T}_{G/P}$, or equivalently of $\bigwedge^p \mathfrak{g}/\mathfrak{p}$, whose associated graded is completely reducible. One possibility is using a composition series in the setting of the Jordan–Hölder theorem, but following Konno [28, Section 3] we define the following filtration, which is shorter but nevertheless gives a completely reducible associated graded.

Definition 22. Fix $p \geq 0$. We define a filtration

$$\bigwedge^p \mathfrak{g}/\mathfrak{p} = \mathrm{F}^0\left(\bigwedge^p \mathfrak{g}/\mathfrak{p}\right) \supseteq \mathrm{F}^1\left(\bigwedge^p \mathfrak{g}/\mathfrak{p}\right) \supseteq \cdots \tag{3.1}$$

where

$$\mathrm{F}^i\left(\bigwedge^p \mathfrak{g}/\mathfrak{p}\right) := \left\{\text{subspace with weights } \beta \text{ such that } (\beta, \omega_k^\vee) \leq M_p - i\right\} \tag{3.2}$$

with M_p being the maximum of $(\beta, \omega_k^\vee)$ with β ranging over the weights of $\bigwedge^p \mathfrak{g}/\mathfrak{p}$. Using the equivalence (2.5) we have an associated filtration

$$\bigwedge^p \mathrm{T}_{G/P} = \mathrm{F}^0\left(\bigwedge^p \mathrm{T}_{G/P}\right) \supseteq \mathrm{F}^1\left(\bigwedge^p \mathrm{T}_{G/P}\right) \supseteq \cdots$$

of equivariant vector bundles.

This filtration has the desired properties, by the following lemma.

Lemma 23. *The filtration* (3.1) *is a finite decreasing filtration of* $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ *by subrepresentations of* P. *Its associated graded pieces are completely reducible* P*-representations.*

Proof. Since $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ is finite-dimensional, it is clear that the filtration is finite and decreasing.

To prove the complete reducibility of $\mathrm{F}^i / \mathrm{F}^{i+1}$ it is enough to show that $\mathfrak{n}$ acts trivially on $\mathrm{F}^i / \mathrm{F}^{i+1}$. Indeed, in such a case the representation is induced from the Levi subalgebra $\mathfrak{l}$, and is automatically completely reducible.

Recall from Section 2.1 that we have the decomposition

$$\mathfrak{n} = \bigoplus_{\beta \in \mathrm{R}^- \setminus \mathrm{R}_L^-} \mathfrak{g}_\beta.$$

Hence, for any $\mathfrak{g}_\beta$ appearing in this decompositon the root β must have α_k with a negative coefficient in its expression in terms of simple roots.

Let V be a representation of $\mathfrak{p}$, and recall the basic fact that a root space $\mathfrak{g}_\beta$ maps a weight space V_α to the weight space $V_{\alpha+\beta}$, if such a weight space exists, or to zero otherwise. Applying this to our situation, we immediately see that $\mathfrak{n}$ maps F^i to F^{i+1}. Therefore, the action of $\mathfrak{n}$ on $\mathrm{F}^i / \mathrm{F}^{i+1}$ is trivial. ∎

Remark 24. For $p = 1$ this filtration is dual to the lower central series for $\mathfrak{n}$.

Spectral sequence associated to the filtration. We will denote the ith piece of the associated graded as

$$\mathrm{G}^i\left(\bigwedge^p \mathrm{T}_{G/P}\right) := \mathrm{F}^i\left(\bigwedge^p \mathrm{T}_{G/P}\right) / \mathrm{F}^i\left(\bigwedge^p \mathrm{T}_{G/P}\right).$$

We obtain the following spectral sequence for every $p \geq 0$:

$$\mathrm{E}_1^{i,q-i} = \mathrm{H}^q\left(G/P, \mathrm{G}^i\left(\bigwedge^p \mathrm{T}_{G/P}\right)\right) \Rightarrow \mathrm{H}^q\left(G/P, \bigwedge^p \mathrm{T}_{G/P}\right). \tag{3.3}$$

On the E_1-page only cohomology of completely reducible equivariant vector bundles appears, for which one can use the Borel–Weil–Bott theorem. All maps in the spectral sequence are equivariant.

3.2. Vanishing for cominuscule varieties

We have the following characterisation of cominuscule maximal parabolics.

Lemma 25. *Let G and B be as before.*

(1) *P is cominuscule if and only if the filtration (3.1) on $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ is a one-step filtration for all p;*

(2) *If P is cominuscule, then the P-representations $\bigwedge^p \mathfrak{g}/\mathfrak{p}$ are completely reducible for all p;*

(3) *P is cominuscule if and only if the nilradical $\mathfrak{n}$ is an abelian Lie algebra.*

Proof. (1) This follows from (2.3), Lemma 21 (5), and (3.2).

(2) This follows from the first claim and Lemma 23.

(3) This fact is well-known [43, Lemma 2.2]. ■

Let G/P be a cominuscule variety. By Lemma 25 the tangent bundle $\mathrm{T}_{G/P}$ and its exterior powers $\bigwedge^p \mathrm{T}_{G/P}$ are completely reducible G-equivariant vector bundles. Hence, one can use the Borel–Weil–Bott theorem to compute their cohomology by applying it to each irreducible summand $\mathcal{E}^\lambda$ individually.

Proposition 26. *The highest weight λ of an irreducible summand $\mathcal{E}^\lambda$ of $\bigwedge^p \mathrm{T}_{G/P}$ on a cominuscule variety G/P is G-dominant. In particular, $\bigwedge^p \mathrm{T}_{G/P}$ have no higher cohomology.*

Proof. Recall that our maximal parabolic P corresponds to the kth vertex of the Dynkin diagram. Thus, it is enough to show $(\lambda, \alpha_k) \geq 0$. By Lemma 21, any such λ is of the form $\beta_1 + \cdots + \beta_p$ with $\beta_i \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$. Hence, it is enough to show that $(\beta, \alpha_k) \geq 0$ for all $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$, which in turn is equivalent to showing that $(\gamma, \alpha_k) \leq 0$ for all $\gamma \in \mathrm{R}^- \setminus \mathrm{R}_L^-$.

Recall from Section 2.1 that we have

$$\mathfrak{n} = \bigoplus_{\gamma \in \mathrm{R}^- \setminus \mathrm{R}_L^-} \mathfrak{g}_\gamma.$$

Since $\mathfrak{n}$ is abelian by Lemma 25, it follows that for any $\gamma_{1,2} \in \mathrm{R}^- \setminus \mathrm{R}_L^-$ their sum $\gamma_1 + \gamma_2$ is never a root of $\mathfrak{g}$. Therefore, as $-\alpha_k$ is in $\mathrm{R}^- \setminus \mathrm{R}_L^-$, we obtain that for any $\gamma \in \mathrm{R}^- \setminus \mathrm{R}_L^-$ the difference $\gamma - \alpha_k$ is never a root of $\mathfrak{g}$. Hence, we obtain $(\gamma, \alpha_k) \leq 0$ (see [26, Lemma 9.4]). ■

3.3. Vanishing for adjoint varieties

We begin with the description of the nilradical $\mathfrak{n}$ in the adjoint case.

Definition 27. The *rth Heisenberg Lie algebra* $\mathfrak{n}_r$ is the $2r+1$-dimensional Lie algebra defined as a central extension by $\mathbf{k} \cdot e_0$ of a $2r$-dimensional abelian Lie algebra spanned by elements $e_1, \ldots, e_{2r}$ such that $[e_i, e_{i+r}] = -[e_{i+r}, e_i] = e_0$ for all $i = 1, \ldots, r$ with all other brackets of basis vectors being zero.

In particular, the lower central series of the Heisenberg Lie algebra $\mathfrak{n} = \mathfrak{n}_r$ is of the form

$$0 \to [\mathfrak{n}, \mathfrak{n}] \to \mathfrak{n} \to \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}^{\mathrm{ab}} \to 0, \tag{3.4}$$

with $\dim_{\mathbf{k}}[\mathfrak{n}, \mathfrak{n}] = 1$.

We have an ample supply of parabolic subalgebras whose nilradical has this property.

Lemma 28. *Let G/P be an adjoint generalised Grassmannian. Then the nilradical $\mathfrak{n}$ is a Heisenberg Lie algebra.*

Proof. Recall that our maximal parabolic P corresponds to the kth vertex of the Dynkin diagram. From the description of the highest root in [9, Appendix] we obtain that

$$\begin{aligned} &(\alpha, \omega_k^\vee) \le 1 \quad \text{for } \alpha \in \mathrm{R}^+ \setminus \{\Theta\}, \\ &(\Theta, \omega_k^\vee) = 2, \end{aligned} \tag{3.5}$$

where Θ is the highest root. This corresponds to the classification of quasi-cominuscule non-cominuscule weights in [32, Remark 2.3]. From the root decomposition for $\mathfrak{n}$ and (3.5) it follows that $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{g}_{-\Theta} \subset \mathrm{Z}(\mathfrak{n})$. One can conclude as in [16, Section 4.2.1] that the induced pairing is actually non-degenerate. ∎

Thus, for an adjoint generalised Grassmannian G/P we have the following description of the nilradical

$$\mathfrak{n} = \mathfrak{g}_{\gamma_0} \oplus \left(\bigoplus_{i=1}^{r} \mathfrak{g}_{\gamma_i} \right) \oplus \left(\bigoplus_{i=r+1}^{2r} \mathfrak{g}_{\gamma_i} \right), \tag{3.6}$$

where $\gamma_i \in \mathrm{R}^- \setminus \mathrm{R}_L^-$ corresponds to e_i in Definition 27.

Let G/P be an adjoint generalised Grassmannian. In this case neither the tangent bundle $\mathrm{T}_{G/P}$, nor its exterior powers $\bigwedge^p \mathrm{T}_{G/P}$, are completely reducible, and, therefore, we cannot directly apply the Borel–Weil–Bott theorem to prove vanishing and we need to appeal to an appropriate filtration whose associated graded is completely reducible.

Let $\mathcal{E}^\lambda$ be an irreducible direct summand of a graded piece of the filtration with highest weight λ. To show vanishing of higher cohomology of $\bigwedge^p \mathrm{T}_{G/P}$ it is enough to show vanishing of higher cohomology of any such $\mathcal{E}^\lambda$.

Proposition 29. *Let G/P be an adjoint generalised Grassmannian. Any irreducible direct summand $\mathcal{E}^\lambda$ of a graded piece of the filtration on $\bigwedge^p \mathrm{T}_{G/P}$ has no higher cohomology.*

Proof. Recall that our maximal parabolic P corresponds to the kth vertex of the Dynkin diagram. Let λ be the highest weight of such an irreducible direct summand $\mathcal{E}^\lambda$. As $\mathcal{E}^\lambda$ is irreducible, we can use Borel–Weil–Bott to compute its cohomology. Assume that $\mathcal{E}^\lambda$ has non-trivial cohomology, then the weight $\lambda + \rho$ has to be regular. We are going to show that this non-trivial cohomology can only live in degree zero.

By Lemma 11 (5) it is enough to show that $(\lambda + \rho, \alpha_k^\vee) \ge 0$, which in its turn is equivalent to $(\lambda, \alpha_k^\vee) \ge -1$. By Lemma 21 (5), any such λ is of the form $\beta_1 + \cdots + \beta_p$

with $\beta_i \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$. Hence, the statement of the proposition follows from the following simple lemma. ∎

Lemma 30. *If* $\dim_{\mathbf{k}}[\mathfrak{n},\mathfrak{n}]=1$*, then there exists a unique* $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$ *such that* $(\beta,\alpha_k^\vee)<0$*. Namely, we have*

$$\beta = \Theta - \alpha_k \quad \textit{and} \quad (\beta,\alpha_k^\vee) = -1,$$

where Θ *is the highest root.*

Proof. We begin by proving that there exists at most one $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$ with $(\beta,\alpha_k^\vee) < 0$. If $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$ is such an element, then by [26, Lemma 9.4] the sum $\beta + \alpha_k$ is a root (and lies in $\mathrm{R}^+ \setminus \mathrm{R}_L^+$). Since $\beta + \alpha_k$ is a root, the root subspaces $\mathfrak{g}_{-\beta}$ and $\mathfrak{g}_{-\alpha_k}$ have a non-trivial Lie bracket equal to $\mathfrak{g}_{-\beta-\alpha_k}$. Using (3.6), we obtain that $-\beta - \alpha_k = \gamma_i$ for some $i \in \{0,1,\dots,2m\}$. From the explicit description of the Lie bracket of a Heisenberg Lie algebra (see Definition 27) we conclude that $-\beta - \alpha_k = \gamma_0 = -\Theta$, and obtain the desired $\beta = \Theta - \alpha_k$.

Since Θ is the kth fundamental weight and since by definition we have $(\alpha_k,\alpha_k^\vee) = 2$, the equality $(\beta,\alpha_k^\vee) = -1$ follows. ∎

Remark 31. We will now explain how the discussion changes for the adjoint partial flag variety in type A, which is not a generalised Grassmannian.

First we note that Lemma 28 and its proof carry over almost verbatim to this case. The coweight $\omega_k^\vee$ needs to be replaced by the coweight $(\omega_1 + \omega_n)^\vee$ (defined in Section 2.1).

The setup for Proposition 29 carries over as follows: taking any filtration of $\mathrm{T}_{G/P}$ with completely reducible associated graded (e.g. the Jordan–Hölder filtration, or a suitable generalisation of Konno's filtration) we again apply Lemma 11 (5) and need to show

$$(\lambda,\alpha_k^\vee) \geq -1 \quad \text{for } k = 1, n.$$

Hence, the statement of the proposition follows from the analogue of Lemma 30 given below. Thus, we have the desired vanishing.

Lemma 32. *Let* G/P *be the adjoint variety in type A. For* $k = 1, n$ *there exists a unique* $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$ *such that* $(\beta,\alpha_k^\vee) < 0$*. Namely, we have*

$$\beta = \Theta - \alpha_k \quad \textit{and} \quad (\beta,\alpha_k^\vee) = -1,$$

where Θ *is the highest root.*

Proof. As in the proof of Lemma 30 for $k = 1, n$ one shows that an element $\beta \in \mathrm{R}^+ \setminus \mathrm{R}_L^+$ with the property $(\beta,\alpha_k^\vee) < 0$ must be of the form $\beta = \Theta - \alpha_k$. Since $\Theta = \omega_1 + \omega_n$, the equality $(\beta,\alpha_k^\vee) = -1$ also follows. ∎

3.4. Vanishing for coadjoint non-adjoint varieties

Now we are left with just two cases: the symplectic Grassmannians $\mathrm{SGr}(2,2n)$ and the exceptional Grassmannian (F_4,α_4). We need an analogue of Lemma 30 for these cases.

Instead of adapting the argument from the previous section to the dual root system we are going to show this by a direct computation, using the description of roots from [9, Appendix].

Lemma 33. *For the symplectic Grassmannian* $\mathrm{SGr}(2,2n)=(\mathrm{C}_n,\alpha_2)$ *there exist a unique* $\beta\in\mathrm{R}^+\setminus\mathrm{R}_L^+$ *such that* $(\beta,\alpha_2^\vee)<0$*. Namely, we have*

$$\beta=\theta-\alpha_2 \quad \textit{and} \quad (\beta,\alpha_2^\vee)=-1,$$

where θ *is the highest short root.*

Proof. The second coroot is

$$\alpha_2^\vee=e_2-e_3.$$

Computing the pairing $(\alpha,\alpha_2^\vee)$ for all $\alpha\in\mathrm{R}^+$ we see that the only possible negative values of the pairing are -1 and -2.

(1) The value -2 arises only once as $(2e_3,\alpha_2^\vee)=-2$. However, since

$$2e_3=2\alpha_3+2\alpha_4+\cdots+2\alpha_{n-1}+\alpha_n,$$

the root $2e_3$ in not in $\mathrm{R}^+\setminus\mathrm{R}_L^+$.

(2) The value -1 arises multiple times. Namely, the positive root α can be e_1+e_3, e_3+e_j with $j>3$, e_1-e_2 or e_3-e_j with $j>3$. Now one checks easily that in all cases except the first one the roots are not in $\mathrm{R}^+\setminus\mathrm{R}_L^+$. In the first case we have

$$e_1+e_3=\alpha_1+\alpha_2+2\alpha_3+\cdots+2\alpha_{n-1}+\alpha_n=\theta-\alpha_2.$$

This finishes the proof. ∎

Lemma 34. *For the generalised Grassmannian* (F_4,α_4) *there exist a unique* $\beta\in\mathrm{R}^+\setminus\mathrm{R}_L^+$ *such that* $(\beta,\alpha_4^\vee)<0$*. Namely, we have*

$$\beta=\theta-\alpha_4 \quad \textit{and} \quad (\beta,\alpha_4^\vee)=-1,$$

where θ *is the highest short root.*

Proof. The fourth coroot is

$$\alpha_4^\vee=e_1-e_2-e_3-e_4.$$

One checks easily that the positive roots with negative pairing with $\alpha_4^\vee$ are

$$\begin{aligned}&e_i \quad \text{for } 2\le i\le 4\\ &\frac{1}{2}(e_1+e_2+e_3+e_4)\end{aligned}$$

in which case the pairing is -1, and

$$e_i + e_j \quad \text{for } 2 \leq i < j \leq 4,$$

in which case the pairing is -2. Rewriting all of them in terms of simple roots we obtain the list $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_2 + \alpha_3$, α_3, $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$, $\alpha_2 + 2\alpha_3$, $\alpha_1 + \alpha_2 + 2\alpha_3$, $\alpha_1 + 2\alpha_2 + 2\alpha_3$ and we see that the only non-parabolic root is

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) = \theta - \alpha_4.$$

This finishes the proof. ■

4. Description for cominuscule and adjoint varieties

We will now prove Theorems B and C, by explicitly computing the global sections of $\bigwedge^i \mathrm{T}_{G/P}$ for cominuscule and adjoint varieties, as representations of $\mathfrak{g}$. In this way we will have described a part of the Gerstenhaber algebra structure on $\mathrm{HH}^\bullet(G/P)$.

4.1. Hochschild cohomology of cominuscule varieties

As a warmup for the proof in Section 4.2 and to keep the discussion in this paper self-contained, we will give some details on the proof of Theorem B. This result is classical, and follows readily from Kostant's theorem. Recall that the nilpotent radical $\mathfrak{n}$ as a representation of P is associated to the cotangent bundle, with its dual $\mathfrak{g}/\mathfrak{p}$ associated to the tangent bundle.

Proof of Theorem B. Let G/P be a cominuscule variety associated to the fundamental weight ω_k. By Lemma 25 (3) we have that $\mathfrak{n}$ is an abelian Lie algebra. Therefore we get that the differential in the Chevalley–Eilenberg complex vanishes, hence

$$\mathrm{H}^i_{\mathrm{CE}}(\mathfrak{n}, k) \cong \bigwedge^i \mathfrak{n}$$

whilst Kostant's theorem gives

$$\mathrm{H}^i_{\mathrm{CE}}(\mathfrak{n}, k) \cong \bigoplus_{\substack{w \in {}^{\mathfrak{l}}\mathrm{W} \\ \ell(w)=i}} \mathrm{V}_{\mathfrak{l}}^{w \cdot 0}$$

as $\mathfrak{l}$-representations, but also as $\mathfrak{p}$-representations as $\mathfrak{n}$ acts trivially (see Lemma 25 (2)). Therefore under the equivalences from (2.5) we have that

$$\Omega^i_{G/P} \cong \bigoplus_{\substack{w \in {}^{\mathfrak{l}}\mathrm{W} \\ \ell(w)=i}} \mathcal{E}^{w \cdot 0}$$

weight	rank	degree	representation	dimension	sum of roots
$(0,0,0)$	1	0	$(0,0,0)$	1	$(0,0,0)$
$(1,0,1)$	4	0	$(1,0,1)$	15	$(1,1,1)$
$(0,1,2)$	3	0	$(0,1,2)$	45	$(1,2,2)$
$(2,1,0)$	3	0	$(2,1,0)$	45	$(2,2,1)$
$(1,2,1)$	4	0	$(1,2,1)$	175	$(2,3,2)$
$(0,4,0)$	1	0	$(0,4,0)$	105	$(2,4,2)$

Table 5. Associated graded for $\bigwedge^\bullet \mathrm{T}_{\mathrm{Gr}(2,4)}$.

as equivariant vector bundles. Using that

$$\bigwedge^i \mathrm{T}_{G/P} \cong \Omega_{G/P}^{\dim G/P - i} \otimes \omega_{G/P}^\vee,$$

and $\omega_{G/P}^\vee \cong \mathcal{E}^{\mathrm{i}_{G/P}\omega_k}$ where $\mathrm{i}_{G/P}$ is the index of G/P, we apply the Borel–Weil–Bott theorem. As already observed in Proposition 26, the resulting weights for the summands of $\bigwedge^i \mathrm{T}_{G/P}$ are all dominant, thus contribute (non-trivially) to the global sections (and not in higher degree), leading to the decomposition (1.4) in the statement of Theorem B. ∎

So what allowed us to conclude in this case is the fact that $\mathfrak{n}$ is abelian, effectively reducing the computation to Kostant's theorem and applying Borel–Weil–Bott.

Remark 35. Without the twist by $\omega_{G/P}^\vee$ one is actually computing the Hodge numbers of G/P. By Borel–Hirzebruch we know that the non-zero Hodge numbers $\mathrm{h}^{i,i}(G/P)$ are given by the cardinality of the subset of ${}^{\mathrm{I}}\mathrm{W}$ of elements of length i, i.e., if one were to use Borel–Weil–Bott to compute this, all weights are regular but not dominant for $i \geq 1$, and their index is precisely i.

Because this result is standard, we will only give one small example.

Example 36. In type A every Grassmannian is a cominuscule partial flag variety. Let us consider the case $\mathrm{Gr}(2,4)$, which has dimension 4 and index 4 (and it is isomorphic to the quadric Q^4). The parabolic Bruhat graph describing ${}^{\mathrm{I}}\mathrm{W}$ is given in Figure 1, so we just compute the weights $w \cdot 0 + 4\omega_2$ and obtain Table 5.

4.2. Hochschild cohomology of adjoint varieties

In this section we prove Theorem C. By Lemma 28 the nilradical in this case is a Heisenberg Lie algebra, which means that Kostant's theorem doesn't compute the exterior powers of the tangent bundle on the nose. But it is possible to bootstrap from this theorem, as the structure of $\mathfrak{n}$ is still manageable.

One of the ingredients in the proof of Theorem C is the following description of the Betti numbers of $\mathrm{H}^\bullet_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k})$, for which an elementary proof can be found as [44, Theorem 2.2 (i)]. A more conceptual (and lengthier) proof can be found as [12, Corollary 4.4].

Proposition 37 (Santharoubane). *Let $\mathfrak{n}_r$ be the Heisenberg Lie algebra of dimension $2r+1$. Then*

$$\dim_{\mathbf{k}} \mathrm{H}^i_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k}) = \begin{cases} \binom{2r}{i} - \binom{2r}{i-2} & i = 0, \dots, r \\ \binom{2r}{i-2} - \binom{2r}{i} & i = r+1, \dots, 2r+1. \end{cases}$$

The Hochschild–Serre spectral sequence. The Hochschild–Serre spectral sequence associated to the sequence (3.4) is

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, \mathrm{H}^q_{\mathrm{CE}}\left([\mathfrak{n},\mathfrak{n}], \mathbf{k}\right)\right) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k}).$$

In the adjoint case, $[\mathfrak{n},\mathfrak{n}]$ is 1-dimensional, so the sequence is concentrated in 2 rows, and it degenerates at the E_3-page. As will become clear, the spectral sequence is highly non-degenerate on the E_2-page.

Since $\mathrm{H}^0_{\mathrm{CE}}([\mathfrak{n},\mathfrak{n}], \mathbf{k}) \cong \mathbf{k}$ and $\mathrm{H}^1_{\mathrm{CE}}([\mathfrak{n},\mathfrak{n}], \mathbf{k}) \cong [\mathfrak{n},\mathfrak{n}]$ as $\mathfrak{l}$-representations, the E_2-page of the Hochschild–Serre spectral sequence has the form

$$\begin{tikzcd} \mathrm{H}^0_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]\right) \arrow[drr, "\mathrm{d}_2^{0,1}"] & \mathrm{H}^1_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]\right) \arrow[drr, "\mathrm{d}_2^{1,1}"] & \mathrm{H}^2_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]\right) \arrow[drr, "\mathrm{d}_2^{2,1}"] & \mathrm{H}^3_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]\right) & \cdots \\ \mathrm{H}^0_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & \mathrm{H}^1_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & \mathrm{H}^2_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & \mathrm{H}^3_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & \cdots \end{tikzcd} \tag{4.1}$$

with all terms zero outside $\{0,1\} \times \{0,1,\dots,2r\}$. As $\mathfrak{n}^{\mathrm{ab}}$ is abelian, and the action of $\mathfrak{n}^{\mathrm{ab}}$ on both $\mathbf{k}$ and $[\mathfrak{n},\mathfrak{n}]$ is trivial by (3.6), the differential in the Chevalley–Eilenberg complex vanishes, and we have that

$$\dim_{\mathbf{k}} \mathrm{E}_2^{p,q} = \binom{2r}{p} \tag{4.2}$$

for $q = 0, 1$ and $p = 0, \dots, 2r$.

The following lemma is the key result in describing the Hochschild cohomology of partial flag varieties of adjoint type.

Lemma 38. *The differentials $\mathrm{d}_2^{i,1} \colon \mathrm{H}^i_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]) \to \mathrm{H}^{i+2}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k})$ in the Hochschild–Serre spectral sequence (4.1) are*

(1) *injective for $i \leq r-1$;*

(2) *surjective for $i \geq r-1$.*

In particular, the differential $\mathrm{d}_2^{r-1,1}$ is an isomorphism.

Proof. The proof for injectivity is by induction on i. The statement is vacuous for $i = -2, -1$ as the domain is zero. By Proposition 37 we have that

$$\dim_{\mathbf{k}} \mathrm{H}^1_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k}) = \binom{2r}{1} = \dim_{\mathbf{k}} \mathrm{E}_\infty^{0,1} + \dim_{\mathbf{k}} \mathrm{E}_\infty^{1,0}.$$

Because $\mathrm{E}_2^{1,0}$ has no incoming differential and is $\binom{2r}{1}$-dimensional, we see that $\mathrm{d}_2^{0,1}$ must be injective, so that $\mathrm{E}_3^{0,1} = \mathrm{E}_\infty^{0,1} = 0$. Continuing by induction for $i = 2, \dots, r$ we use

Proposition 37, together with (4.2) to conclude that all differentials must be injective so that the appropriate dimension in the abutment is reached.

The proof for surjectivity is by a descending induction on i, and is similar. ■

Hence the entries $E_3^{i,j}$ on the $E_3 = E_\infty$-page look like

$$\begin{matrix} 0 & 0 & 0 & 0 & \cdots \\ H^0_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & H^1_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) & \operatorname{coker} d_2^{0,1} & \operatorname{coker} d_2^{1,1} & \cdots \end{matrix} \tag{4.3}$$

for $i = 0, 1, 2, 3$, resp.

$$\begin{matrix} \cdots & 0 & 0 & \ker d_2^{r,1} & \ker d_2^{r+1,1} & \ker d_2^{r+2,1} & \cdots \\ \cdots & \operatorname{coker} d_2^{r-4,1} & \operatorname{coker} d_2^{r-3,1} & \operatorname{coker} d_2^{r-2,1} & 0 & 0 & \cdots \end{matrix} \tag{4.4}$$

for $i = r-2, r-1, \ldots, r+2$, resp.

$$\begin{matrix} \cdots & \ker d_2^{2r-3,1} & \ker d_2^{2r-2,1} & H^{2r-1}_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n}, \mathfrak{n}]\right) & H^{2r}_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n}, \mathfrak{n}]\right) \\ \cdots & 0 & 0 & 0 & 0 \end{matrix} \tag{4.5}$$

for $i = 2r-3, 2r-2, 2r-1, 2r$.

Using Kostant's theorem (see Theorem 13) we have a description for the Lie algebra cohomology $H^\bullet_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k})$, but there is no immediate link with exterior powers of the (co)tangent bundle anymore. Rather we have the following lemma.

Lemma 39. *Let G/P be an adjoint variety of dimension $2r+1$. There exists a short exact sequence*

$$0 \to \mathcal{E} \to T_{G/P} \to \mathcal{L} \to 0 \tag{4.6}$$

where

- *$\mathcal{L}$ is the line bundle $\mathcal{O}_{G/P}(1)$ (in type A more appropriately written $\mathcal{O}_{G/P}(1,1)$),*
- *$\mathcal{E}$ is the vector bundle of rank $2r$ associated to the dual of the L-representation $\mathfrak{n}^{\mathrm{ab}}$.*

Outside type A we have that $\mathfrak{n}^{\mathrm{ab},\vee}$ is irreducible, in type A is the direct sum of two irreducible representations.

Proof. The sequence is the dual of the short exact sequence of equivariant vector bundles associated to (3.4). As $[\mathfrak{n}, \mathfrak{n}]$ is one-dimensional we have that it is irreducible, and the highest weight of its dual corresponds to the adjoint representation. As the action of $\mathfrak{n}$ on $\mathfrak{n}^{\mathrm{ab}}$ is trivial, we have that it is completely reducible. ■

If we wish to compute the global sections of $\bigwedge^p T_{G/P}$ we are reduced to computing the global sections of the short exact sequence

$$0 \to \bigwedge^p \mathcal{E} \to \bigwedge^p T_{G/P} \to \mathcal{L} \otimes \bigwedge^{p-1} \mathcal{E} \to 0. \tag{4.7}$$

The outer terms are *completely reducible* equivariant vector bundles associated to the duals of the L-representations $\mathrm{H}^{p-1}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}])$ and $\mathrm{H}^{p}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k})$. They can be determined inductively from the Hochschild–Serre spectral sequence and the knowledge of its abutment as follows.

Proof of Theorem C. From the vanishing result in Proposition 29, we obtain that

$$\begin{aligned}\mathrm{HH}^i(G/P) &\cong \mathrm{H}^0\left(G/P, \bigwedge^i \mathrm{T}_{G/P}\right)\\ &\cong \mathrm{H}^0\left(G/P, \bigwedge^i \mathcal{E}\right) \oplus \mathrm{H}^0\left(G/P, \mathcal{L}\otimes \bigwedge^{i-1}\mathcal{E}\right).\end{aligned}$$

Hence it suffices to describe the highest weights that determine the bundles $\bigwedge^i \mathcal{E}$, from which the description for $\mathcal{L}\otimes\bigwedge^{i-1}\mathcal{E}$ follows. Recall that by Borel–Weil–Bott the weights for both are either regular dominant, or singular[3].

To determine $\mathrm{H}^{i}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k})$ (or rather its dual) we use the description of the E_3-pages (4.3), (4.4) and (4.5). For $i = 0, 1$ it is given by Kostant's description of $\mathrm{H}^{i}_{\mathrm{CE}}(\mathfrak{n}, \mathbf{k})$.

For $i = 2, \dots, r$ there is a recursion involving the contributions of $\mathrm{H}^{i-2}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}])$ which are determined by the isomorphism

$$\mathrm{H}^{i-2}_{\mathrm{CE}}\left(\mathfrak{n}^{\mathrm{ab}}, [\mathfrak{n},\mathfrak{n}]\right) \cong \mathrm{H}^{i-2}_{\mathrm{CE}}(\mathfrak{n}^{\mathrm{ab}}, \mathbf{k}) \otimes_{\mathbf{k}} [\mathfrak{n},\mathfrak{n}].$$

The argument is dual for the second half, starting with $i = 2r$ and recursing downwards to $i = r$. There is a shift by an extra copy of $[\mathfrak{n},\mathfrak{n}]^\vee$ originating from the fact that (4.5) has zeroes on the bottom row, so that Kostant's theorem is rather describing the *top row* of the E_3-page.

Now the formula in (1.6) is obtained by keeping track of the recursion with steps of size 2 and the contributions of $\bigwedge^i \mathcal{E}$ and $\mathcal{L}\otimes\bigwedge^{i-1}\mathcal{L}$. ∎

Example 40. The necessity to restrict only to regular weights is obvious already for $\mathrm{T}_{G/P}$. The sequence (4.6) gives rise to the short exact sequence

$$0 \to 0 \to \mathfrak{g} \to \mathfrak{g} \to 0$$

after taking global sections, so there is no contribution from $\mathcal{E}$ in the description (1.6).

We will give two examples in full detail, to illustrate the somewhat involved recursive procedure outlined above in practice.

Example 41. The adjoint partial flag variety in type A_3 is $\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})$, which has dimension 5 and index 3. The parabolic Bruhat graph in Figure 2 can be used in conjunction with Theorem C to determine the Hochschild cohomology.

[3] This observation explains why we have to consider the restricted sum in (1.5) when describing the global sections (see also Example 40).

weight	rank	degree	representation	dimension	sum of roots
$(0,0,0)$	1	0	$(0,0,0)$	1	$(0,0,0)$
$(1,0,1)$	1	0	$(1,0,1)$	15	$(1,1,1)$
$(1,1,-1)$	2				$(1,1,0)$
$(-1,1,1)$	2				$(0,1,1)$
$(0,1,2)$	2	0	$(0,1,2)$	45	$(1,2,2)$
$(2,1,0)$	2	0	$(2,1,0)$	45	$(2,1,0)$
$(-1,0,3)$	1				$(0,1,2)$
$(3,0,-1)$	1				$(2,1,0)$
$(1,0,1)$	1	0	$(1,0,1)$	15	$(1,1,1)$
$(0,2,0)$	3	0	$(0,2,0)$	20	$(1,2,1)$
$(4,0,0)$	1	0	$(4,0,0)$	35	$(3,2,1)$
$(0,0,4)$	1	0	$(0,0,4)$	35	$(1,2,3)$
$(2,0,2)$	1	0	$(2,0,2)$	84	$(2,2,2)$
$(1,2,1)$	3	0	$(1,2,1)$	175	$(2,3,2)$
$(0,1,2)$	2	0	$(0,1,2)$	45	$(1,2,2)$
$(2,1,0)$	2	0	$(2,1,0)$	45	$(2,2,1)$
$(1,1,3)$	2	0	$(1,1,3)$	256	$(2,3,3)$
$(3,1,1)$	2	0	$(3,1,1)$	256	$(3,3,2)$
$(2,0,2)$	1	0	$(2,0,2)$	84	$(2,2,2)$
$(3,0,3)$	1	0	$(3,0,3)$	300	$(3,3,3)$

Table 6. Associated graded for $\bigwedge^\bullet \mathrm{T}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})}$.

In Table 6 the associated graded of $\bigwedge^i \mathrm{T}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})}$ is given, for $i = 0, \ldots, 5$. The decomposition obtained from (4.6) and (4.7) is indicated by the grouping of the terms: we first give $\bigwedge^{i-1} \otimes \mathcal{L}$. Again the need for the restriction to only regular weights in Theorem C is immediate.

For $\mathrm{T}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})}$ we have 3 summands: one coming from $\mathcal{L}$, the other two coming from $\mathcal{E}$. That there are two follows from the fact that there are two Weyl group elements of colength 1 in Figure 2.

For $\bigwedge^2 \mathrm{T}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})}$ there are 6 summands: 2 coming from $\mathcal{E} \otimes \mathcal{L}$, the other 4 coming from $\bigwedge^2 \mathcal{E}$. That there are 4 is part of the recursion: there are three Weyl group elements of colength 2, and one of colength 0.

For $\bigwedge^3 \mathrm{T}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^3})}$ the roles are reversed: 4 summands come from $\bigwedge^2 \mathcal{E}$ which was determined in the previous step, whilst there are 2 summands coming from $\bigwedge^3 \mathcal{E}$. The rest is similar.

Example 42. The adjoint partial flag variety in type B_3 is $\mathrm{OGr}(2,7)$, which has dimension 7 and index 4. The parabolic Bruhat graph in this case is given in Figure 3.

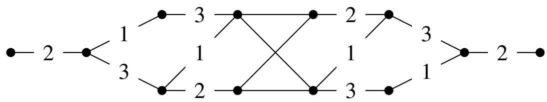


Figure 3. Parabolic Bruhat graph for the (adjoint) parabolic $\bullet\!-\!\times\!\Rightarrow\!\bullet$.

weight	rank	degree	representation	dimension	sum of roots
$(0,0,0)$	1	0	$(0,0,0)$	1	$(0,0,0)$
$(0,1,0)$	1	0	$(0,1,0)$	21	$(1,2,2)$
$(1,-1,2)$	6				$(1,1,2)$
$(1,0,2)$	6	0	$(1,0,2)$	189	$(2,3,4)$
$(0,-1,4)$	5				$(1,2,4)$
$(0,1,0)$	1	0	$(0,1,0)$	21	$(1,2,2)$
$(2,-1,2)$	9				$(2,2,3)$
$(0,0,4)$	5	0	$(0,0,4)$	294	$(2,4,6)$
$(0,2,0)$	1	0	$(0,2,0)$	168	$(2,4,4)$
$(2,0,2)$	9	0	$(2,0,2)$	616	$(3,4,5)$
$(1,0,2)$	6	0	$(1,0,2)$	189	$(2,3,4)$
$(3,0,0)$	4	0	$(3,0,0)$	77	$(3,3,3)$
$(1,-1,4)$	10				$(2,3,5)$
$(1,1,2)$	6	0	$(1,1,2)$	1617	$(3,5,6)$
$(3,1,0)$	4	0	$(3,1,0)$	819	$(4,5,5)$
$(1,0,4)$	10	0	$(1,0,4)$	1386	$(3,5,7)$
$(0,0,4)$	5	0	$(0,0,4)$	294	$(2,4,6)$
$(2,0,2)$	9	0	$(2,0,2)$	616	$(3,4,5)$
$(0,2,0)$	1	0	$(0,2,0)$	168	$(2,4,4)$
$(0,1,4)$	5	0	$(0,1,4)$	2310	$(3,6,8)$
$(2,1,2)$	9	0	$(2,1,2)$	4550	$(4,6,7)$
$(0,3,0)$	1	0	$(0,3,0)$	825	$(3,6,6)$
$(1,1,2)$	6	0	$(1,1,2)$	1617	$(3,5,6)$
$(1,2,2)$	6	0	$(1,2,2)$	7722	$(4,7,8)$
$(0,3,0)$	1	0	$(0,3,0)$	825	$(3,6,6)$
$(0,4,0)$	1	0	$(0,4,0)$	3003	$(4,8,8)$

Table 7. Associated graded for $\bigwedge^\bullet \mathrm{T}_{\mathrm{OGr}(2,7)}$.

In Table 7 the associated graded of $\bigwedge^i \mathrm{T}_{\mathrm{OGr}(2,7)}$ is given, for $i = 0, \ldots, 7$. As before, the parabolic Bruhat graph in Figure 3 can be used in conjunction with Theorem C to determine the Hochschild cohomology. The decomposition obtained from (4.6) and (4.7) is indicated by the grouping of the terms: we first give $\bigwedge^{i-1} \otimes \mathcal{L}$. Again the need for the restriction to only regular weights in Theorem C is immediate.

For $\mathrm{T}_{\mathrm{OGr}(2,7)}$ we have two summands: one coming from $\mathcal{L}$, the other coming from $\mathcal{E}$. That there is only a single summand here is immediately visible from the parabolic Bruhat graph.

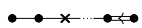
For $\bigwedge^2 \mathrm{T}_{\mathrm{OGr}(2,7)}$ there are three "new" summands coming from $\bigwedge^2 \mathcal{E}$. This is again visible from the parabolic Bruhat graph, where there is one contribution of a line bundle from Kostant's theorem for $i = 7$ and two contributions from Kostant's theorem for $i = 5$, whose weights happen to be singular. The rest of the example proceeds along similar lines.

5. On the (possible) non-vanishing of the higher cohomologies

In this section we discuss what we know for generalised Grassmannians which are *not* covered by Theorem A. The two main results are a proof of Proposition D and an elaboration of the caveat in Remark 1 to Conjecture F. To conclude this section we explain the relationship between our vanishing results and Bott vanishing: our results give new cases in which Bott vanishing fails for generalised Grassmannians.

5.1. Non-vanishing for SGr(3, 2n)

Consider symplectic isotropic Grassmannians $\mathrm{SGr}(3, 2n)$ with $n \geq 4$, which can be realized as the quotient of the symplectic group Sp_{2n} with respect to the maximal parabolic subgroup attached to the third node of the Dynkin diagram C_n, i.e., for

 .

Setup. Let V be a $2n$-dimensional vector space endowed with a symplectic form ω, and let $v_1, \ldots, v_{2n}$ be a basis of V such that $\omega(v_i, v_{2n+1-i}) = 1$ for $1 \leq i \leq n$ and all other pairings between basis elements vanish. For $1 \leq k \leq n$ the symplectic isotropic Grassmannian $\mathrm{SGr}(k, V) = \mathrm{SGr}(k, 2n)$ is the variety parametrising isotropic k-dimensional subspaces in V. As any isotropic subspace is a subspace, we have a natural closed immersion

$$\mathrm{SGr}(k, 2n) \hookrightarrow \mathrm{Gr}(k, V) = \mathrm{Gr}(k, 2n).$$

The symplectic form ω gives rise to a global section s_ω of the vector bundle $\bigwedge^2 \mathcal{U}^\vee$ on $\mathrm{Gr}(k, 2n)$, and the subvariety $\mathrm{SGr}(k, V)$ is the zero locus of s_ω.

One can realise $\mathrm{SGr}(k, 2n)$ and $\mathrm{Gr}(k, 2n)$ as quotients

$$\mathrm{SGr}(k, 2n) = \mathrm{Sp}_{2n} / P_\omega \quad \text{and} \quad \mathrm{Gr}(k, 2n) = \mathrm{GL}_{2n} / P,$$

where we have taken P and P_ω to be the stabilisers of the "standard" isotropic k-dimensional subspace spanned by the basis vectors $v_{2n}, v_{2n-1}, \ldots, v_{2n-k+1}$. Naturally, we have the inclusion of the parabolics $P_\omega \subseteq P$.

Similarly, we have the embedding of the corresponding Levi subgroups

$$L_\omega = \mathrm{Sp}_{2n-2k} \times \mathrm{GL}_k \subset L = \mathrm{GL}_{2n-k} \times \mathrm{GL}_k .$$

In the setup above we have that the maximal torus T is given by the diagonal matrices of the form $(t_1, \dots t_n, t_n^{-1}, \dots, t_1^{-1})$. This way we identify the weight lattice of Sp_{2n} with $\mathbb{Z}^n$ in such a way that the simple roots are

$$\alpha_i = e_i - e_{i+1} \quad \text{for } 1 \leq i \leq n-1, \quad \text{and} \quad \alpha_n = 2e_n,$$

and the fundamental weights are

$$\omega_i = e_1 + \cdots + e_i \quad \text{for } 1 \leq i \leq n,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{Z}^n$.

Lemma 43. *For $n \geq 4$ the associated graded of $\mathrm{T}_{\mathrm{SGr}(3,2n)}$ is given by*

$$\mathrm{gr}(\mathrm{T}_{\mathrm{SGr}(3,2n)}) = \mathcal{E}^{2\omega_1} \oplus \mathcal{E}^{\omega_1 - \omega_3 + \omega_4}.$$

In the special case $n = 3$ the tangent bundle is irreducible and we have $\mathrm{T}_{\mathrm{SGr}(3,6)} = \mathcal{E}^{2\omega_1}$.

Proof. Since the symplectic isotropic Grassmannian $\mathrm{SGr}(3, 2n)$ is a closed subvariety of the ordinary Grassmannian $\mathrm{Gr}(3, 2n)$ cut out by a regular section of $\bigwedge^2 \mathcal{U}^\vee$, we have the short exact sequence

$$0 \to \mathrm{T}_{\mathrm{SGr}(3,2n)} \to i^*\mathrm{T}_{\mathrm{Gr}(3,2n)} \to i^* \bigwedge^2 \mathcal{U}^\vee \to 0 \tag{5.1}$$

of Sp_{2n}-equivariant bundles. By (2.5) it corresponds to a short exact sequence of representations of the parabolic subgroup P of Sp_{2n}. By restricting these representations to the Levi L_ω we will be able to deduce the desired description of the associated graded $\mathrm{gr}(\mathrm{T}_{\mathrm{SGr}(3,2n)})$. That is we need to determine the representations of L_ω corresponding to $i^*\mathrm{T}_{\mathrm{Gr}(3,2n)}$ and $i^* \bigwedge^2 \mathcal{U}^\vee$, and then we can just remove the contribution of the latter bundle from the former.

The tangent bundle to $\mathrm{Gr}(3, 2n)$ is $\mathrm{T}_{\mathrm{Gr}(3,2n)} = \mathcal{Q} \otimes \mathcal{U}^\vee$ where

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathrm{Gr}(3,2n)} \to \mathcal{Q} \to 0$$

is the universal short exact sequence on $\mathrm{Gr}(3, 2n)$. Hence it is the bundle corresponding to the representation

$$V/U \otimes U^\vee,$$

of $L = \mathrm{GL}_{2n-3} \times \mathrm{GL}_3$ where U is the standard subspace described above. Here the representation $U^\vee$ is the dual of the standard representation of GL_3, and the representation V/U is the standard representation of GL_{2n-3}.

Now we need to consider these representations as representations of the Levi quotient $L_\omega = \mathrm{Sp}_{2n-6} \times \mathrm{GL}_3$. Restricting to L_ω we get

$$(W \oplus U^\vee) \otimes U^\vee,$$

where W is the standard representation of Sp_{2n-6}. We can further rewrite it as

$$(W \otimes U^\vee) \oplus \left(\bigwedge^2 U\right)^\vee \oplus (\mathrm{Sym}^2 U)^\vee.$$

In terms of fundamental weights of Sp_{2n} we get

$$\begin{aligned} W \otimes (U)^\vee &\leftrightarrow (1,0,0;1,0,\ldots,0) = \omega_1 - \omega_3 + \omega_4 \\ \left(\bigwedge^2 U\right)^\vee &\leftrightarrow (1,1,0;0,\ldots,0) = \omega_2 \\ (\mathrm{Sym}^2 U)^\vee &\leftrightarrow (2,0,0;0,\ldots,0) = 2\omega_1. \end{aligned}$$

The summand with weight ω_2 gets cancelled in (5.1) and we obtain the claim.

When $n = 3$ it suffices to observe that $W = 0$. ■

Lemma 44. *For $n \geq 5$ the associated graded of $\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}$ is given by*

$$\begin{aligned} \mathrm{gr}\left(\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}\right) = {} & \mathcal{E}^{2\omega_1+\omega_2} \oplus \mathcal{E}^{3\omega_1-\omega_3+\omega_4} \oplus \mathcal{E}^{\omega_1+\omega_2-\omega_3+\omega_4} \\ & \oplus \mathcal{E}^{\omega_2-2\omega_3+2\omega_4} \oplus \mathcal{E}^{2\omega_1-\omega_3+\omega_5} \oplus \mathcal{E}^{2\omega_1}. \end{aligned}$$

For $n = 4$ the summand $\mathcal{E}^{2\omega_1-\omega_3+\omega_5}$ should be omitted. For $n = 3$ we have $\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,6)} = \mathcal{E}^{2\omega_1+\omega_2}$.

Proof. Note that we have

$$\bigwedge^2 (\mathcal{E}^{2\omega_1} \oplus \mathcal{E}^{\omega_1-\omega_3+\omega_4}) = \bigwedge^2 \mathcal{E}^{2\omega_1} \oplus (\mathcal{E}^{2\omega_1} \otimes \mathcal{E}^{\omega_1-\omega_3+\omega_4}) \oplus \bigwedge^2 \mathcal{E}^{\omega_1-\omega_3+\omega_4}.$$

From this we compute that

$$\begin{aligned} \bigwedge^2 \mathcal{E}^{2\omega_1} &\cong \mathcal{E}^{2\omega_1+\omega_2} \\ \mathcal{E}^{2\omega_1} \otimes \mathcal{E}^{\omega_1-\omega_3+\omega_4} &= \mathcal{E}^{3\omega_1-\omega_3+\omega_4} \oplus \mathcal{E}^{\omega_1+\omega_2-\omega_3+\omega_4} \\ \bigwedge^2 \mathcal{E}^{\omega_1-\omega_3+\omega_4} &= \mathcal{E}^{\omega_2-2\omega_3+2\omega_4} \oplus \mathcal{E}^{2\omega_1-\omega_3+\omega_5} \oplus \mathcal{E}^{2\omega_1} \end{aligned}$$

which proves the claim for $n \geq 5$. For $n = 4$ (resp. $n = 3$) we discard the contributions involving ω_5 (resp. ω_4 and ω_5). ■

Proof of Proposition D. Applying Borel–Weil–Bott (complemented with Items (1) and (5) of Lemma 11) we see that the only summands of $\mathrm{gr}(\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)})$ with non-trivial cohomology are

$$\mathrm{H}^\bullet\left(\mathrm{SGr}(3,2n), \mathcal{E}^{2\omega_1+\omega_2}\right) \cong \mathrm{V}^{2\omega_1+\omega_2}_{\mathrm{Sp}_{2n}}[0] \tag{5.2}$$

$$\mathrm{H}^\bullet\left(\mathrm{SGr}(3,2n), \mathcal{E}^{2\omega_1}\right) \cong \mathrm{V}^{2\omega_1}_{\mathrm{Sp}_{2n}}[0] \tag{5.3}$$

$$\mathrm{H}^\bullet\left(\mathrm{SGr}(3,2n), \mathcal{E}^{\omega_2-2\omega_3+2\omega_4}\right) \cong \mathrm{V}^{\omega_4}_{\mathrm{Sp}_{2n}}[-1] \tag{5.4}$$

weight	rank	degree	representation	dimension	sum of roots
$(0,0,-2,3)$	4				$(1,2,3,3)$
$(1,1,-1,1)$	16				$(2,3,3,2)$
$(4,0,0,0)$	15	0	$(4,0,0,0)$	330	$(4,4,4,2)$
$(0,2,0,0)$	6	0	$(0,2,0,0)$	308	$(2,4,4,2)$
$(2,1,0,0)$	15	0	$(2,1,0,0)$	594	$(3,4,4,2)$
$(2,1,-2,2)$	45	1	$(2,0,0,1)$	1155	$(3,4,4,3)$
$(1,0,-1,2)$	9				$(2,3,4,3)$
$(1,2,-1,1)$	30				$(3,5,5,3)$
$(3,1,-1,1)$	48				$(4,5,5,3)$
$(2,0,0,1)$	12	0	$(2,0,0,1)$	1155	$(3,4,5,3)$
$(3,0,1,0)$	10	0	$(3,0,1,0)$	3696	$(4,5,6,3)$
$(0,3,0,0)$	10	0	$(0,3,0,0)$	2184	$(3,6,6,3)$

Table 8. Associated graded for $\bigwedge^p \mathrm{T}_{\mathrm{SGr}(3,8)}$ for $p = 3,4,5,6$.

where $[-i]$ indicates the degree in which the cohomology lives. In particular, we see that in the spectral sequence obtained from the filtration on $\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}$ no cancellation can happen and (5.4) contributes non-trivially to $\mathrm{HH}^3(\mathrm{SGr}(3,2n))$ in the term $\mathrm{H}^1(\mathrm{SGr}(3,2n), \bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)})$ of the Hochschild–Kostant–Rosenberg decomposition. ■

Remark 45. In the special case $n = 3$ we have that SGr(3, 6) is a cominuscule variety, in which case it is covered by Theorems A and B. From Lemma 44 we obtain the isomorphism $\bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,6)} \cong \mathcal{E}^{2\omega_1+\omega_2}$ which by the proof of Proposition D (only) has global sections.

Remark 46. Having shown that $\mathrm{H}^1(\mathrm{SGr}(3,2n), \bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}) \neq 0$ we can wonder what happens for higher exterior powers. Applying this method for higher exterior powers and $n = 4$ reveals that for $\bigwedge^p \mathrm{T}_{\mathrm{SGr}(3,8)}$, with $p = 3,4,5,6$ there are summands in the associated graded which by Borel–Weil–Bott have an H^1. But every representation that arises in this H^1 also appears as the H^0 of a different summand. Hence in the spectral sequence it is possible that these get cancelled, and we cannot conclude whether they are preserved in the abutment.

In Table 8 we have collected the summands and their cohomology as obtained from the Borel–Weil–Bott theorem. The Euler characteristic of the isotypical component associated to the highest weight $2\omega_1 + \omega_4$ is zero, hence it is not clear whether it is being cancelled or not in the spectral sequence. The same is true for $\bigwedge^p \mathrm{T}_{\mathrm{SGr}(3,8)}$ with $p = 4,5,6$: it is not possible from the components in the spectral sequence to deduce (non-)vanishing.

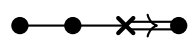
5.2. Potential vanishing for OGr($n-1, 2n+1$)

In this section we discuss the caveat expressed in Remark 1 by explaining the indeterminacy in our methods for the orthogonal Grassmannian $\mathrm{OGr}(n-1, 2n+1)$, in particular

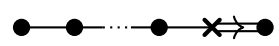
weight	rank	degree	representation	dimension	sum of roots
$(0,1,0,0)$	3	0	$(0,1,0,0)$	36	$(1,2,2,2)$
$(2,0,-1,2)$	18				$(2,2,2,3)$
$(0,1,-2,4)$	15	1	$(0,0,0,2)$	126	$(1,2,2,4)$
$(0,0,0,2)$	3	0	$(0,0,0,2)$	126	$(1,2,3,4)$
$(1,1,-1,2)$	24				$(2,3,3,4)$
$(1,0,1,0)$	3	0	$(1,0,1,0)$	594	$(2,3,4,4)$

Table 9. Associated graded for $\bigwedge^2 \mathrm{T}_{\mathrm{OGr}(3,9)}$.

when $n = 4$. Using the method outlined below we can compute the sheaf cohomology of the associated graded of the equivariant vector bundle $\bigwedge^p \mathrm{T}_{\mathrm{OGr}(3,9)}$ associated to the marked Dynkin diagram

 .

What happens in this case, and likewise for $n = 5, 6, 7, 8$, is similar to the phenomenon described in Remark 46 for $\bigwedge^3 \mathrm{T}_{\mathrm{SGr}(3,8)}$. We expect it is to continue for all $\mathrm{OGr}(n-1, 2n+1)$ with $n \geq 4$, associated to the marked Dynkin diagram

 .

This makes it impossible to conclude anything from the spectral sequence (3.3) without a better understanding of the differentials.

Potential vanishing. We can now elaborate on the "potential vanishing phenomenon" in the smallest case, where all the possibly non-zero differentials are in fact surjections, which means that any contributions to higher cohomology get cancelled. In this case the phenomenon is restricted to degrees 0 and 1. For other generalised Grassmannians (where Conjecture F predicts non-vanishing) there exist examples where the associated graded has cohomology in higher degrees, making the analysis of the spectral sequence harder. The result in Section 5.1 is an instance where the differential is actually zero.

Example 47. Consider $\bigwedge^2 \mathrm{T}_{\mathrm{OGr}(3,9)}$. One can compute that its associated graded has 6 summands. These are collected in Table 9, and one can read off from the table that Konno's filtration is a 2-step filtration in this case. The relevant part of the E_1-page of the spectral sequence has the form

$$\begin{array}{rl} \mathrm{H}^0\left(\mathrm{OGr}(3,9), \mathcal{E}^{2\omega_4}\right) \oplus \mathrm{H}^0\left(\mathrm{OGr}(3,9), \mathcal{E}^{\omega_1+\omega_3}\right) \xrightarrow{\mathrm{d}_1} & \mathrm{H}^1\left(\mathrm{OGr}(3,9), \mathcal{E}^{\omega_2-2\omega_3+\omega_4}\right) \\ & \mathrm{H}^0\left(\mathrm{OGr}(3,9), \mathcal{E}^{\omega_2}\right) \end{array}$$

which after applying Borel–Weil–Bott becomes in terms of $\mathfrak{so}_9$-representations

$$\begin{array}{rl} \mathrm{V}_{\mathfrak{so}_9}^{2\omega_4} \oplus \mathrm{V}_{\mathfrak{so}_9}^{\omega_1+\omega_3} \xrightarrow{\mathrm{d}_1} & \mathrm{V}_{\mathfrak{so}_9}^{2\omega_4} \\ & \mathrm{V}_{\mathfrak{so}_9}^{\omega_2}. \end{array}$$

This method does not tell whether the map d_1 is zero or not. If it is zero, i.e., the spectral sequence degenerates on the E_1-page, then

$$\begin{aligned} H^0\left(\mathrm{OGr}(3,9), \bigwedge^2 T_{\mathrm{OGr}(3,9)}\right) &\cong V^{2\omega_4}_{\mathfrak{so}_9} \oplus V^{\omega_1+\omega_3}_{\mathfrak{so}_9} \oplus V^{\omega_2}_{\mathfrak{so}_9} \\ H^1\left(\mathrm{OGr}(3,9), \bigwedge^2 T_{\mathrm{OGr}(3,9)}\right) &\cong V^{2\omega_4}_{\mathfrak{so}_9} \end{aligned}$$

whereas if it is nonzero the spectral sequence degenerates on the E_2-page after cancelling two representations, and

$$\begin{aligned} H^0\left(\mathrm{OGr}(3,9), \bigwedge^2 T_{\mathrm{OGr}(3,9)}\right) &\cong V^{\omega_1+\omega_3}_{\mathfrak{so}_9} \oplus V^{\omega_2}_{\mathfrak{so}_9} \\ H^1\left(\mathrm{OGr}(3,9), \bigwedge^2 T_{\mathrm{OGr}(3,9)}\right) &\cong 0. \end{aligned} \tag{5.5}$$

Remark 48. We have been informed by Nicolas Hemelsoet that he has used the method from [22] to check that $V^{2\omega_4}_{\mathfrak{so}_9}$ does not appear in $H^0(\mathrm{OGr}(3,9), \bigwedge^2 T_{\mathrm{OGr}(3,9)})$, hence the differential d_1 is in fact nonzero, and we are in situation (5.5).

Therefore, as expressed in Remark 1, it is not clear in the statement of Conjecture F whether to include or exclude the family $\mathrm{OGr}(n-1, 2n+1)$ for $n \geq 4$.

5.3. Remarks on Bott vanishing

Finally we wish to give a brief overview of the relationship of the (non-)vanishing results in this paper and the notion of Bott vanishing. This is the following vanishing property, which is rather strong as the discussion following the definition shows.

Definition 49. A smooth projective variety X satisfies *Bott vanishing* if

$$H^j(X, \Omega^i_X \otimes \mathcal{L}) = 0$$

for all ample line bundles $\mathcal{L}$, all $i \geq 0$ and all $j \geq 1$.

In particular, a smooth projective *Fano* variety (such as G/P) satisfying Bott vanishing immediately satisfies the vanishing property for the Hochschild–Kostant–Rosenberg decomposition we set out to study for generalised Grassmannians.

It is known (due to Bott) that $\mathbb{P}^n$ satisfies Bott vanishing. More generally toric varieties satisfy Bott vanishing, in which case it is called Danilov–Steenbrink–Bott vanishing. Hence Fano toric varieties are automatically Hochschild global in the sense of (1.2), and a combinatorial description of the Hochschild cohomology can be obtained from [37, Theorems 2.14 (2) and 3.6 (2)]. Recently the first non-toric Fano variety, namely $\mathrm{Bl}_4\,\mathbb{P}^2$, satisfying Bott vanishing was found by Totaro [47, Theorem 2.1], and this was generalised by Torres in [46].

On the other hand it is expected (see [11, Remark 2]) that Bott vanishing does *not* hold for any partial flag variety which is not $\mathbb{P}^n$, hence the vanishing result in the cominuscule and (co)adjoint case in Theorem A cannot be the consequence of Bott vanishing for the ample line bundle $\omega_{G/P}^\vee$.

In the case of full flag varieties, the failure of Bott vanishing is shown in [40, Corollary 13]. The situation for generalised Grassmannians is far less well-understood:

- In the cominuscule case explicit examples of the failure of Bott vanishing are given in [11, Section 4.3].
- In the (co)adjoint case the methods used in this paper might be useful in exhibiting examples of the failure of Bott vanishing for generalised Grassmannians, but we leave this for future work. The adjoint case in type A is covered by [11, Section 4.2].
- Outside these cases, a positive answer to Conjecture F would give explicit examples of the failure of Bott vanishing for $\mathcal{L} \cong \omega_{G/P}^\vee$.

Hence Proposition D provides the first example of the failure of Bott vanishing for a generalised Grassmannian which is not cominuscule, which corresponds to Corollary E.

Proof of Corollary E. Consider the (very) ample line bundle $\mathcal{L} := \omega_{\mathrm{SGr}(3,2n)}^\vee$. Then

$$\mathrm{H}^1\left(\mathrm{SGr}(3,2n), \Omega_{\mathrm{SGr}(3,2n)}^{\dim \mathrm{SGr}(3,2n)-2} \otimes \omega_{\mathrm{SGr}(3,2n)}^\vee\right) \cong \mathrm{H}^1\left(\mathrm{SGr}(3,2n), \bigwedge^2 \mathrm{T}_{\mathrm{SGr}(3,2n)}\right) \neq 0$$

by Proposition D. ∎

Acknowledgements. We would like to thank Michel Brion, Friedrich Knop, Alexander Kuznetsov, Catharina Stroppel, and Michel Van den Bergh for interesting discussions.

We want to especially thank Travis Schedler and Allen Knutson for sharing their computations for full flag varieties in type A, which suggested that the situation was even more interesting than a priori expected.

We want to thank Nicolas Hemelsoet for interesting discussions regarding Example 47, and sharing his computations.

Funding. The first author acknowledges the support of the FWO (Research Foundation—Flanders). The second author was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 448537907. We want to thank the Max Planck Institute for Mathematics for the pleasant working conditions during the start of this project, and its high performance computing infrastructure.

References

[1] R. J. Baston and M. G. Eastwood, *The Penrose transform*. Oxford Math. Monogr., Oxford University Press, New York, 1989 Zbl 0726.58004 MR 1038279

[2] P. Belmans, Grassmannian.info: A periodic table of (generalised) Grassmannians. https://grassmannian.info

[3] P. Belmans, E. Fatighenti, and F. Tanturri, Polyvector fields for Fano 3-folds. *Math. Z.* **304** (2023), no. 1, Paper No. 12 Zbl 07680043 MR 4578397

[4] A. I. Bondal and M. M. Kapranov, Homogeneous bundles. In *Helices and vector bundles*, pp. 45–55, London Math. Soc. Lecture Note Ser. 148, Cambridge Univ. Press, Cambridge, 1990 Zbl 0742.14011 MR 1074782

[5] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces. I. *Amer. J. Math.* **80** (1958), 458–538 Zbl 0097.36401 MR 102800

[6] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces. II. *Amer. J. Math.* **81** (1959), 315–382 Zbl 0097.36401 MR 110105

[7] W. Borho and H. Kraft, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen. *Comment. Math. Helv.* **54** (1979), no. 1, 61–104 Zbl 0395.14013 MR 522032

[8] R. Bott, The index theorem for homogeneous differential operators. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pp. 167–186, Princeton University Press, Princeton, NJ, 1965 Zbl 0173.26001 MR 0182022

[9] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles 1337, Hermann, Paris, 1968 Zbl 0186.33001 MR 0240238

[10] M. Brion, Equivariant cohomology and equivariant intersection theory. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, pp. 1–37, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 514, Kluwer Acad. Publ., Dordrecht, 1998 Zbl 0946.14008 MR 1649623

[11] A. Buch, J. F. Thomsen, N. Lauritzen, and V. Mehta, The Frobenius morphism on a toric variety. *Tohoku Math. J. (2)* **49** (1997), no. 3, 355–366 Zbl 0899.14026 MR 1464183

[12] L. Cagliero and P. Tirao, The cohomology of the cotangent bundle of Heisenberg groups. *Adv. Math.* **181** (2004), no. 2, 276–307 Zbl 1037.17020 MR 2026860

[13] D. Calaque, C. A. Rossi, and M. Van den Bergh, Căldăraru's conjecture and Tsygan's formality. *Ann. of Math. (2)* **176** (2012), no. 2, 865–923 Zbl 1252.18035 MR 2950766

[14] D. Calaque and M. Van den Bergh, Hochschild cohomology and Atiyah classes. *Adv. Math.* **224** (2010), no. 5, 1839–1889 Zbl 1197.14017 MR 2646112

[15] A. Căldăraru, The Mukai pairing. II. The Hochschild–Kostant–Rosenberg isomorphism. *Adv. Math.* **194** (2005), no. 1, 34–66 Zbl 1098.14011 MR 2141853

[16] A. Čap and J. Slovák, *Parabolic geometries. I*. Math. Surveys Monogr. 154, American Mathematical Society, Providence, RI, 2009 Zbl 1183.53002 MR 2532439

[17] P. E. Chaput and N. Perrin, On the quantum cohomology of adjoint varieties. *Proc. Lond. Math. Soc. (3)* **103** (2011), no. 2, 294–330 Zbl 1267.14065 MR 2821244

[18] M. Demazure, Automorphismes et déformations des variétés de Borel. *Invent. Math.* **39** (1977), no. 2, 179–186 Zbl 0406.14030 MR 435092

[19] W. Fulton, On the quantum cohomology of homogeneous varieties. In *The legacy of Niels Henrik Abel*, pp. 729–736, Springer, Berlin, 2004 Zbl 1083.14065 MR 2077592

[20] M. Gerstenhaber and S. D. Schack, Algebraic cohomology and deformation theory. In *Deformation theory of algebras and structures and applications (Il Ciocco, 1986)*, pp. 11–264, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 247, Kluwer Acad. Publ., Dordrecht, 1988 Zbl 0676.16022 MR 981619

[21] K. R. Goodearl and M. Yakimov, Poisson structures on affine spaces and flag varieties. II. *Trans. Amer. Math. Soc.* **361** (2009), no. 11, 5753–5780 Zbl 1179.53087 MR 2529913

[22] N. Hemelsoet and R. Voorhaar, A computer algorithm for the BGG resolution. *J. Algebra* **569** (2021), 758–783 Zbl 1471.17037 MR 4187255

[23] L. Hille, Homogeneous vector bundles and Koszul algebras. *Math. Nachr.* **191** (1998), 189–195 Zbl 0957.14035 MR 1621314

[24] G. Hochschild, B. Kostant, and A. Rosenberg, Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.* **102** (1962), 383–408 Zbl 0102.27701 MR 142598

[25] A. Huang, B. Lian, S.-T. Yau, and C. Yu, Jacobian rings for homogenous vector bundles and applications. 2018, arXiv:1801.08261v1

[26] J. E. Humphreys, *Introduction to Lie algebras and representation theory*. Grad. Texts in Math. 9, Springer, New York, 1978 Zbl 0447.17001 MR 499562

[27] J. C. Jantzen, *Representations of algebraic groups*. 2nd edn., Math. Surveys Monogr. 107, American Mathematical Society, Providence, RI, 2003 Zbl 1034.20041 MR 2015057

[28] K. Konno, Infinitesimal Torelli theorem for complete intersections in certain homogeneous Kähler manifolds. *Tohoku Math. J. (2)* **38** (1986), no. 4, 609–624 Zbl 0627.14009 MR 867067

[29] K. Konno, Generic Torelli theorem for hypersurfaces of certain compact homogeneous Kähler manifolds. *Duke Math. J.* **59** (1989), no. 1, 83–160 Zbl 0704.14006 MR 1016881

[30] B. Kostant, Lie algebra cohomology and the generalized Borel–Weil theorem. *Ann. of Math. (2)* **74** (1961), 329–387 Zbl 0134.03501 MR 142696

[31] A. Lachowska and Y. Qi, The center of small quantum groups I: The principal block in type A. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 20, 6349–6405 Zbl 1460.17021 MR 3872326

[32] V. Lakshmibai, C. Musili, and C. S. Seshadri, Geometry of G/P. IV. Standard monomial theory for classical types. *Proc. Indian Acad. Sci. Sect. A Math. Sci.* **88** (1979), no. 4, 279–362 Zbl 0447.14013 MR 553746

[33] F. Loray, J. V. Pereira, and F. Touzet, Foliations with trivial canonical bundle on Fano 3-folds. *Math. Nachr.* **286** (2013), no. 8-9, 921–940 Zbl 1301.37032 MR 3066408

[34] W. Lowen and M. Van den Bergh, Hochschild cohomology of abelian categories and ringed spaces. *Adv. Math.* **198** (2005), no. 1, 172–221 Zbl 1095.13013 MR 2183254

[35] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories. *Trans. Amer. Math. Soc.* **358** (2006), no. 12, 5441–5483 Zbl 1113.13009 MR 2238922

[36] N. Markarian, The Atiyah class, Hochschild cohomology and the Riemann–Roch theorem. *J. Lond. Math. Soc. (2)* **79** (2009), no. 1, 129–143 Zbl 1167.14005 MR 2472137

[37] E. N. Materov, The Bott formula for toric varieties. *Mosc. Math. J.* **2** (2002), no. 1, 161–182, 200 Zbl 1080.14540 MR 1900589

[38] L. C. Mihalcea, On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms. *Duke Math. J.* **140** (2007), no. 2, 321–350 Zbl 1135.14042 MR 2359822

[39] C. Negron and T. Schedler, The Hochschild cohomology ring of a global quotient orbifold. *Adv. Math.* **364** (2020), 106978, 49 Zbl 1435.14005 MR 4057490

[40] K. Paramasamy, Twisted holomorphic forms on generalized flag varieties. *Proc. Indian Acad. Sci. Math. Sci.* **114** (2004), no. 2, 123–140 Zbl 1067.14048 MR 2062394

[41] B. Pym, Quantum deformations of projective three-space. *Adv. Math.* **281** (2015), 1216–1241 Zbl 1338.14007 MR 3366864

[42] B. Pym, Constructions and classifications of projective Poisson varieties. *Lett. Math. Phys.* **108** (2018), no. 3, 573–632 Zbl 1390.53093 MR 3765972

[43] R. Richardson, G. Röhrle, and R. Steinberg, Parabolic subgroups with abelian unipotent radical. *Invent. Math.* **110** (1992), no. 3, 649–671 Zbl 0786.20029 MR 1189494

[44] L. J. Santharoubane, Cohomology of Heisenberg Lie algebras. *Proc. Amer. Math. Soc.* **87** (1983), no. 1, 23–28 Zbl 0509.17006 MR 677223

[45] R. G. Swan, Hochschild cohomology of quasiprojective schemes. *J. Pure Appl. Algebra* **110** (1996), no. 1, 57–80 Zbl 0865.18010 MR 1390671

[46] S. Torres, Bott vanishing using GIT and quantization. 2020, arXiv:2003.10617v1

[47] B. Totaro, Bott vanishing for algebraic surfaces. *Trans. Amer. Math. Soc.* **373** (2020), no. 5, 3609–3626 Zbl 1437.14028 MR 4082249

[48] A. Yekutieli, The continuous Hochschild cochain complex of a scheme. *Canad. J. Math.* **54** (2002), no. 6, 1319–1337 Zbl 1047.16004 MR 1940241

Communicated by Henning Krause

Received 30 August 2021; revised 19 September 2022.

Pieter Belmans
Department of Mathematics, University of Luxembourg, 6, Avenue de la Fonte,
4364 Esch-sur-Alzette, Luxembourg; pieter.belmans@uni.lu

Maxim Smirnov
Instititut für Mathematik, Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany;
maxim.smirnov@math.uni-augsburg.de