# An overlapping local projection stabilization for Galerkin approximations of Stokes and Darcy flow problems

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**Abstract**

An *a priori* analysis for a generalized local projection stabilized finite element approximation of the Stokes, and the Darcy flow equations are presented in this paper. A first-order conforming $\mathbf{P}_1^c$ finite element space is used to approximate both the velocity and pressure. It is shown that the stabilized discrete bilinear form satisfies the inf-sup condition in the generalized local projection norm. Moreover, *a priori* error estimates are established in a mesh-dependent norm as well as in the $\mathrm{L}^2-$norm for the velocity and pressure. The optimal and quasi-optimal convergence properties are derived for the Stokes and the Darcy flow problems. Finally, the derived estimates are numerically validated with appropriate examples.

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**Key words:** Finite element method; Stokes problem; Darcy problems; Generalized local projection stabilization; Inf-sup condition; Error estimates.

**AMS subject classification:** 65N30, 65N15, 65N12, 76M10.

## 1. Introduction

The Stokes problem has considerable practical importance in civil, petroleum, and electrical engineering, such as flow in porous media, heat transfer, semiconductor devices, etc. Several numerical schemes such as conforming and nonconforming finite element methods, finite difference, finite volume methods have been developed for Stokes problem. It is well-known that the choice of equal-order interpolation spaces for the finite element approximations of the pressure and velocity induces spurious oscillations in the numerical solution. Nevertheless, the equal-order interpolation spaces are preferred to avoid mixed finite element spaces and the complexity in implementation. Hence, a stabilization method is used to enhance the stability and accuracy of the standard Galerkin solution with equal-order finite element spaces.

Over the years several stabilization methods such as the streamline diffusion Petrov-Galerkin [43, 44], the residual free bubbles [1, 15, 29], the Galerkin least-squares [1, 11, 35], the edge stabilization [20], the continuous interior penalty [16, 17, 18, 19] and the local projection schemes [9, 24, 30, 31, 41], have been proposed. The relation between the different approaches is also well-understood in most cases. The basic idea of stabilization is to stabilize the Galerkin variational formulation so that the discrete approximation is stable and convergent [6, 7, 17, 21, 22, 40]. Stabilization methods for the Stokes-like operators are well-studied in the literature [3, 30], and a few studies for the Darcy equations have also been presented [6, 7, 39, 40, 41].

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The local projection stabilization (LPS) method has been proposed in [3, 9] for the Stokes problem and subsequently extended to various other classes of problems [4, 10, 30, 31, 34, 38, 41]. The stabilization term in the local projection method is based on a projection of the finite element space that approximates the unknown into a discontinuous space [3, 9]. LPS is very attractive, mainly because of its commutation properties in optimization problems [8] and stabilization properties similar to residual-based approaches [37]. A significant benefit of the local projection method is that the LPS approach uses a symmetric stabilization term and contains fewer stabilization terms than the residual-based stabilization approach. Generalized local projection stabilization (GLPS) is a generalized form of LPS, where the local projection spaces are defined on a patch of cells associated with each node without enriching the finite element space. GLPS was first introduced in [36] for the convection–diffusion problem and later applied to various other classes of problems [5, 26, 32, 33, 38]. Further, unlike LPS, GLPS needs neither a macro grid nor an enrichment of approximation spaces.

In this work, the GLPS conforming finite element scheme for the Stokes and the Darcy flow problems is presented. In particular, the equal order interpolation spaces $\mathbf{P}_1^c/\mathbf{P}_1^c$ are used to approximate the velocity and pressure. Since equal order interpolation does not satisfy the inf-sup stability condition, spurious oscillations in the approximated pressure are unavoidable. GLPS overcomes the space incompatibility issue and suppresses the oscillations. In the stabilized formulation, the pressure stabilization $S_p(\nabla p, \nabla q)$ can rectify the lack of inf-sup stability in finite element spaces. Although approximating the $\mathbf{P}_1^c/\mathbf{P}_1^c$ equal order interpolation spaces, we no longer have mass conservation. This loss can be reduced with the addition of the stabilization term $S_{\mathbf{u}}(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$. The boundary conditions are not used strongly in discrete space; hence, the discrete formulation combines the standard Galerkin formulations, stabilization terms, and weakly imposed boundary conditions.

Further, in [36], it was noted that the stabilization parameter was independent of the diffusion parameter, therefore making the method very well suited for degenerate diffusion problems. The behaviour for the Stokes problem is somewhat different, and the analysis gives different results depending on how the stabilization parameter scales with respect to the mesh-size $h$. Further, an *a priori* error estimates are derived in a mesh-dependent norm as well as in the $\mathrm{L}^2-$norm for the velocity and pressure. The optimal order of convergence is observed for the Stokes equations. For the Darcy equations, the optimality is observed for the divergence of the velocities and suboptimality with the gap of half a power of $h$ for the pressures and velocities in the $\mathrm{L}^2$-norm.

The outline of the article is as follows: In Section 2, the weak formulation of the Stokes problem is introduced. Section 3 is devoted to an overlapping local projection stabilized conforming finite element method. The stability analysis for the Stokes problem with respect to a generalized local projection norm is derived. In section 3.2, an optimal *a priori* error estimates for the Stokes problem with respect to a generalized local projection norm is provided. In Section 4, the above approach is studied for the Darcy flow problem. Taking into account the error estimate of Theorem 3.2, the stabilization parameters are modified. An elementary proof of stability and convergence analysis for the Darcy flow problem is derived. Section 5 presents some numerical experiments that confirm the theoretical analysis.

## 2. The Stokes problem

Consider the following Stokes problem:

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p = \mathbf{f}; \quad \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= 0 \ \text{ on } \partial\Omega.
\end{aligned} \tag{1}
$$

Here, $\Omega$ is an open, bounded subset of $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega$. Here, $\mathbf{u}$ denotes the velocity, $p$ denotes the pressure, $\mathbf{f} \in [\mathrm{L}^2(\Omega)]^2$ is a given data. Throughout this paper, standard notations for Lebesgue and Sobolev spaces are used. The notation $(\cdot,\cdot)$ represents the $\mathrm{L}^2(\Omega)$ inner product; and $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^\infty(\Omega)$ norms are respectively denoted by $\|u\|$ and $\|u\|_\infty$. The standard notation of Sobolev space $\mathrm{H}^{\mathrm{m}}(\Omega)$ for m=1,2 and its norm $\|\cdot\|_{\mathrm{m}}$ are used. The notations $[\mathrm{L}^2(\Omega)]^2$ and $[\mathrm{H}^1(\Omega)]^2$ respectively abbreviate the vector-valued versions of $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$; $\mathrm{H}^1_0(\Omega)$ is a subspace of $\mathrm{H}^1(\Omega)$ with zero trace functions. Now, consider the functional spaces $\mathbf{V} = \{\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^2\}$, and $Q = \mathrm{L}^2_0(\Omega) = \left\{\mathrm{q} \in \mathrm{L}^2(\Omega) | \ \int_\Omega \mathrm{q} \ dx = 0 \ \right\}$.

Further, multiplying the model problem with a test function $(\mathbf{v}, q) \in \mathbf{V} \times Q$ and after integrating over $\Omega$, the weak form of the model problem (1) reads:

Find $(\mathbf{u}, p) \in \mathcal{W} = \mathbf{V} \times Q$ such that

$$
B((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}), \quad \text{for all } (\mathbf{v}, q) \in \mathcal{W}, \tag{2}
$$

where

$$
B((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) - b(p, \mathbf{v}) + b(q, \mathbf{u}),
$$

and $a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})$ and $b(p, \mathbf{v}) = (p, \nabla \cdot \mathbf{v})$.

The existence of a weak solution to this problem follows by applying the Lax–Milgram lemma in the divergence-free subspace of $\mathbf{V}$ and the pressure in $Q$ by the Brezzi condition [14]. We will assume that $\Omega$ is a convex polygonal domain so that the following regularity estimate holds $\|\mathbf{u}\|_2 + \|p\|_1 \lesssim \|\mathbf{f}\|$.

**Remark 2.1.** *The analysis is presented for a two-dimensional case for simplicity. Nevertheless, the study is independent of the dimension, whereas faces instead of edges need to be used to extend to three-dimensions.*

*Finite element formulation*

Let $\mathcal{T}_h$ be a collection of non-overlapping quasi-uniform triangles obtained by decomposition of $\Omega$. Let $h_K = \mathrm{diam(K)}$ for all $K \in \mathcal{T}_h$ and the mesh-size $h = \max_{K \in \mathcal{T}_h} h_K$. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$ be the set of all edges in $\mathcal{T}_h$ where $\mathcal{E}_h^I$ and $\mathcal{E}_h^B$ are the set of all interior and boundary edges respectively and $h_E = \mathrm{diam(E)}$ for all $E \in \mathcal{E}_h$. Let $\mathcal{V}_h := \mathcal{V}_h^I \cup \mathcal{V}_h^B$ be the set of all vertices in $\mathcal{V}_h$ where $\mathcal{V}_h^I$ and $\mathcal{V}_h^B$ are the set of all interior and boundary vertices respectively. For any $a \in \mathcal{V}_h$, $\mathcal{M}_a$ (patch of $a$) denotes the union of all cells that share the vertex $a$. Further, define $h_a = \mathrm{diam}(\mathcal{M}_\mathrm{a})$ for all $a \in \mathcal{V}_h$. Moreover, We use the following norm in the analysis. Let the piecewise constant function $h_\mathcal{T}$ is defined by $h_\mathcal{T}|_K = h_K$ and $s \in \mathbb{R}$ and $\mathrm{m} \geq 0$

$$
\|h_\mathcal{T}^s u\|_{\mathrm{m}} = \left( \sum_{K \in \mathcal{T}_h} h_K^{2s} \|u\|^2_{\mathrm{H}^{\mathrm{m}}(\mathrm{K})} \right)^{\frac{1}{2}} \text{ for all } u \in \ \mathrm{H}^{\mathrm{m}}(\mathcal{T}_\mathrm{h}).
$$

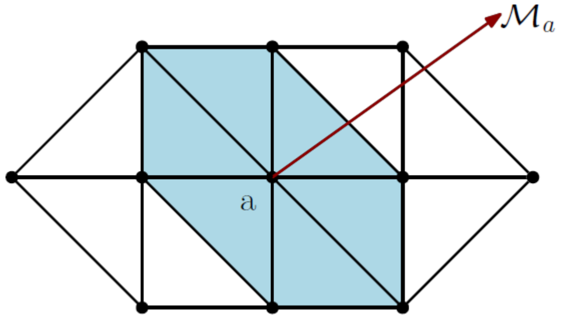



Figure 1: Node patch $\mathcal{M}_a$.

Suppose $I(a)$ denotes the index set for all $K_l$ elements, so that $K_l \subset \mathcal{M}_a$. Then, the local mesh-size associated to $\mathcal{M}_a$ is defined as

$$h_a := \frac{1}{\text{card}(I(a))} \sum_{l \in I(a)} h_l, \quad \text{for each } a \in \mathcal{V}_h,$$

where card$(I(a))$ denotes the number of elements in $\mathcal{M}_a$. Since the mesh $\mathcal{T}_h$ is assumed to be locally quasi-uniform [12], there exists a positive $\zeta \geq 1$ independent of $h$ such that

$$\zeta^{-1} \leq \frac{h_a}{h_l} \leq \zeta \ \text{ for all } l \in I(a).$$

For any $a \in \mathcal{V}_h$, define the fluctuation operator $\kappa_a : \mathrm{L}^2(\mathcal{M}_\mathrm{a}) \to \mathrm{L}^2(\mathcal{M}_\mathrm{a})$ by

$$\kappa_a(v) = v - \frac{1}{|\mathcal{M}_a|} \int_{\mathcal{M}_a} v \ dx.$$

Then,

$$\|\kappa_a\|_{\mathcal{L}(\mathrm{L}^2(\mathcal{M}_\mathrm{a}),\mathrm{L}^2(\mathcal{M}_\mathrm{a}))} \leq C \quad \forall \ a \in \mathcal{V}_h,$$

where $C$ is a constant independent of $h$. For each $a \in \mathcal{V}_h$, let $\beta_a$ be the stabilization parameter. Now, the stabilization term is defined by

$$S_h(u, v) := \sum_{a \in \mathcal{V}_h} \beta_a \int_{\mathcal{M}_a} \kappa_a(u)\kappa_a(v) \ dx.$$

We next define a piecewise polynomial space as

$$\mathbb{P}_k(\mathcal{T}_h) := \left\{ v \in \mathrm{L}^2(\Omega) : \mathrm{v}|_\mathrm{K} \in \mathbb{P}_\mathrm{k}(\mathrm{K}) \quad \forall \mathrm{K} \in \mathcal{T}_\mathrm{h} \right\},$$

where $\mathbb{P}_k(K)$, $k \geq 0$, is the space of polynomials of degree at most $k$ over the element $K$. Further, define a conforming finite element space of piecewise linear

$$\mathbf{P}_1^c(\mathcal{T}_h) := \left\{ v \in \mathrm{H}^1(\Omega) \ : \ \mathrm{v}|_\mathrm{K} \in \mathbb{P}_1(\mathrm{K}) \ \ \forall \ \mathrm{K} \in \mathcal{T}_\mathrm{h} \right\}.$$

Note that throughout this paper, C (sometimes subscripted) denotes a generic positive constant that may depend on the shape-regularity of the triangulation but is independent of the mesh-size. Further, the notation $c \lesssim d$ represents the inequality $c \leq Cd$.

Now recall the following technical results of finite element analysis.

**Lemma 2.1.** Trace inequality *[25]: Suppose $E$ denotes an edge of $K \in \mathcal{T}_h$. For $v_h \in$*

$\mathbb{P}_k(\mathcal{T}_h)$, *there holds*

$$\|v_h\|_{\mathrm{L}^2(\mathrm{E})} \le Ch_K^{-1/2}\|v_h\|_{\mathrm{L}^2(\mathrm{K})}. \tag{3}$$

**Lemma 2.2.** Inverse inequality *[25]: Let $v \in \mathbb{P}_k(\mathcal{T}_h)$, for all $k \ge 0$. Then,*

$$\|\nabla v\|_{\mathrm{L}^2(\mathrm{K})} \le Ch_K^{-1}\|v\|_{\mathrm{L}^2(\mathrm{K})}. \tag{4}$$

**Lemma 2.3.** Poincaré inequality *[13]: For a bounded and connected polygonal domain $\Omega$ and for any $v \in \mathrm{H}^1(\Omega)$,*

$$\left\|v - \frac{1}{|\Omega|}\int_\Omega v\ dx\right\|_{\mathrm{L}^2(\Omega)} \le Ch_\Omega \|\nabla v\|_{\mathrm{L}^2(\Omega)},$$

*where $h_\Omega$ and $|\Omega|$ denote the diameter and measure of domain $\Omega$. In particular, for every vertex $a \in \mathcal{V}_h$ and every function $v \in \mathrm{H}^1(\mathcal{M}_\mathrm{a})$, it holds that*

$$\left\|v - \frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a} v\ dx\right\|_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})} \le Ch_a \|\nabla v\|_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})}, \tag{5}$$

*where the constant $C$ is independent of the mesh-size $h_a$.*

Furthermore, for a locally quasi-uniform and shape-regular triangulation, the $\mathrm{H}^1$-stability, the $\mathrm{L}^2$-stability, and the stability in the weighted the $\mathrm{L}^2$-norm [2] of the $\mathrm{L}^2$-orthogonal projection $I_h : \mathrm{L}^2(\Omega) \to \mathbf{P}_1^\mathrm{c}(\mathcal{T}_\mathrm{h})$ leads to the following approximation properties.

**Lemma 2.4.** $\mathrm{L}^2$*-orthogonal projections: The $\mathrm{L}^2$-projection $I_h : \mathrm{L}^2(\Omega) \to \mathbf{P}_1^\mathrm{c}(\mathcal{T}_\mathrm{h})$ satisfies*

$$\left\|h_\mathcal{T}^{-1}(v - I_h v)\right\| + \|\nabla(v - I_h v)\| \le C\|h_\mathcal{T} v\|_2, \text{ for all } v \in \mathrm{H}^2(\Omega), \tag{6}$$

*For vector-valued functions, $\mathbf{I}_h : [\mathrm{L}^2(\Omega)]^2 \to [\mathbf{P}_1^\mathrm{c}(\mathcal{T}_\mathrm{h})]^2$ satisfies*

$$\left\|h_\mathcal{T}^{-1}(\mathbf{v} - \mathbf{I}_h\mathbf{v})\right\| + \|\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v})\| \le C\|h_\mathcal{T}\mathbf{v}\|_2, \ \forall\ \mathbf{v} \in [\mathrm{H}^2(\Omega)]^2. \tag{7}$$

*Moreover, the trace inequality over each edge implies*

$$\left(\sum_{E\in\mathcal{E}_h}\|\mathbf{v} - \mathbf{I}_h\mathbf{v}\|^2_{\mathrm{L}^2(\mathrm{E})}\right)^{1/2} \le C\left\|h_\mathcal{T}^{3/2}\mathbf{v}\right\|_2 \text{ for all } \ \mathbf{v} \in [\mathrm{H}^2(\Omega)]^2. \tag{8}$$

*The orthogonality relation for all $\mathbf{v}_h \in [\mathbf{P}_1^c(\mathcal{T}_h)]^2$ implies*

$$(\mathbf{v} - \mathbf{I}_h\mathbf{v}, \mathbf{v}_h) = 0. \tag{9}$$

*The following approximation estimates hold for the $L^2$-orthogonal projection operator:*

$$\|\mathbf{I}_h\mathbf{v}\| \le \|\mathbf{v}\|, \quad \left\|h_\mathcal{T}^{-1}\mathbf{I}_h\mathbf{v}\right\| \le C\left\|h_\mathcal{T}^{-1}\mathbf{v}\right\| \quad \textit{and} \quad \|\nabla(\mathbf{I}_h\mathbf{v})\| \le C\|\nabla\mathbf{v}\|. \tag{10}$$

## 3. An overlapping local projection stabilization for Stokes problem

This section describes the conforming finite element method for the Stokes problem (1), where the velocity and the pressure are approximated with the continuous piecewise linear finite element spaces. It is well-known that equal-order interpolation spaces for the pressure and the velocities in the Stokes problem are not inf-sup stable and induce spurious oscillation in the solution. An overlapping local projection stabilization method is introduced to circumvent this stability issue. Further, the Dirichlet boundary condition is not incorporated in the discrete space, and therefore weakly imposed in the discrete formulation using Nitsche's technique [42]. The final formulation combines standard Galerkin formulation, stabilization terms, and weakly imposed Dirichlet boundary condition.

Let $\mathbf{V}_h := [\mathbf{P}_1^c(\mathcal{T}_h)]^2$ and $Q_h := \mathrm{L}_0^2(\Omega)\bigcap \mathbf{P}_1^c(\mathcal{T}_\mathrm{h})$. The stabilised finite element formulation of the Stokes reads:

Find $(\mathbf{u}_h, p_h) \in \mathcal{W}_h = \mathbf{V}_h \times Q_h$ such that

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = L(\mathbf{v}, q), \tag{11}$$

for all $(\mathbf{v}, q) \in \mathcal{W}_h$, where

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = a_h(\mathbf{u}_h, \mathbf{v}) - b_h(p_h, \mathbf{v}) + b_h(\mathbf{u}_h, q) + S_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)), \tag{12}$$

and

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}) :&= (\nabla \mathbf{u}_h, \nabla \mathbf{v}) - \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{v}\; ds - \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{u}_h\; ds \\
&\quad + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E} \mathbf{u}_h \cdot \mathbf{v}\; ds, \\
b_h(p_h, \mathbf{v}) :&= (p_h, \nabla\cdot\mathbf{v}) - \sum_{E\in\mathcal{E}_h^B} \int_E (\mathbf{v}\cdot\mathbf{n}) p_h\; ds, \\
S_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) :&= S_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}) + S_p(p_h, q), \\
S_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}) :&= \sum_{a\in\mathcal{V}_h} \int_{\mathcal{M}_a} \kappa_a(\nabla\cdot\mathbf{u}_h)\kappa_a(\nabla\cdot\mathbf{v})\; dx, \\
S_p(p_h, q) :&= \sum_{a\in\mathcal{V}_h} \mu_a \int_{\mathcal{M}_a} \kappa_a(\nabla p_h)\kappa_a(\nabla q)\; dx, \\
L(\mathbf{v}, q) :&= (\mathbf{f}, \mathbf{v}).
\end{aligned}$$

Note that the parameters $\zeta$ and $\mu_a$ will be chosen later. Further, introduce the generalized local projection norm for $\mathcal{W}_h$ by

$$|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|^2 := \|\nabla\mathbf{u}_h\|^2 + \|p_h\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)). \tag{13}$$

*3.1. The inf-sup condition*

**Theorem 3.1.** *Let $\zeta$ be selected in such a way that $\zeta > \zeta_0 > 0$ with $\zeta_0$ large enough. Assume also that $\mu_a = \mu h_a^2$ for some $\mu > 0$. Then the discrete bilinear form (11) satisfies the following inf-sup condition for some positive constant $\nu$, independent of $h$:*

$$\inf_{(\mathbf{u}_h,p_h)\in\mathcal{W}_h} \sup_{(\mathbf{v}_h,q_h)\in\mathcal{W}_h} \frac{B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|\; |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|} \geq \nu.$$

**Proof.** In order to prove the stability result, it is enough to choose some $(\mathbf{v}_h, q_h) \in \mathcal{W}_h$

for any arbitrary $(\mathbf{u}_h, p_h) \in \mathcal{W}_h$, such that

$$\frac{B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||(\mathbf{v}_h, q_h)|||} \geq \nu \, |||(\mathbf{u}_h, p_h)||| > 0. \tag{14}$$

First, consider the bilinear form in (12) with $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$:

$$B_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) = \|\nabla \mathbf{u}_h\|^2 - 2 \sum_{E \in \mathcal{E}_h^B} \int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{u}_h \; ds + \sum_{E \in \mathcal{E}_h^B} \int_E \frac{\zeta}{h_E} \mathbf{u}_h^2 \; ds \\ + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)). \tag{15}$$

The second term of (15) is handled by using the Cauchy–Schwarz inequality and trace inequality (3),

$$2 \int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{u}_h \; ds \leq 2 \left\| \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \right\|_{\mathrm{L^2(E)}} \|\mathbf{u}_h\|_{\mathrm{L^2(E)}} \leq 2 h_E^{-1/2} \|\nabla \mathbf{u}_h\|_{\mathrm{L^2(K)}} \|\mathbf{u}_h\|_{\mathrm{L^2(E)}} . \tag{16}$$

The sum of all the boundary edges of (16) and using Young's inequality,

$$2 \sum_{E \in \mathcal{E}_h^B} \int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{u}_h \; ds \leq \frac{1}{2} \|\nabla \mathbf{u}_h\|^2 + 2C^2 \sum_{E \in \mathcal{E}_h^B} \int_E \frac{1}{h_E} \mathbf{u}_h^2 \; ds. \tag{17}$$

Substitution of (17) to (15) and the selection of parameter $\zeta > \zeta_0 =: 4C^2$ to obtain

$$\begin{aligned} B_h((\mathbf{u}_h, p_h),&(\mathbf{u}_h, p_h)) \\ &\geq \frac{1}{2} \|\nabla \mathbf{u}_h\|^2 + \frac{\zeta - 2C^2}{\zeta} \sum_{E \in \mathcal{E}_h^B} \int_E \frac{\zeta}{h_E} \mathbf{u}_h^2 \; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \\ &\geq \frac{1}{2} \left( \|\nabla \mathbf{u}_h\|^2 + \sum_{E \in \mathcal{E}_h^B} \int_E \frac{\zeta}{h_E} \mathbf{u}_h^2 \; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \right) . \end{aligned} \tag{18}$$

The selection of the parameter $\zeta$ is given by,

$$\frac{\zeta - 2C^2}{\zeta} \geq \frac{1}{2}.$$

Using the surjectivity of the divergence operator [28], there exists a function $\mathbf{z} \in [\mathrm{H}_0^1(\Omega)]^2$ such that $\nabla \cdot \mathbf{z} = p_h$ and

$$\|\mathbf{z}\|_{1,\Omega} \leq C_1 \|p_h\|. \tag{19}$$

Let $\mathbf{z} \in [\mathrm{H}_0^1(\Omega)]^2$ is defined as in (19). Let $\mathbf{z}_h = \mathbf{I}_h \mathbf{z} \in \mathbf{V}_h$. Then,

$$\|\mathbf{z}_h\|_{1,\Omega} \leq \|\mathbf{z}\|_{1,\Omega} \leq C_1 \|p_h\|. \tag{20}$$

Finally, taking $(\mathbf{v}_h, q_h) = (\mathbf{z}_h, 0)$ in (12),

$$B_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)) = a_h(\mathbf{u}_h, \mathbf{z}_h) - b_h(p_h, \mathbf{z}_h) + S_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)). \tag{21}$$

Let us estimate the above three terms. Consider the first term of (21):

$$a_h(\mathbf{u}_h,\mathbf{z}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{z}_h) - \sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\cdot \mathbf{z}_h\ ds - \sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}}\cdot \mathbf{u}_h\ ds + \sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}\mathbf{u}_h\cdot\mathbf{z}_h\ ds. \tag{22}$$

The first term of (22) is handled by using the Cauchy–Schwarz inequality, (20) and Young's inequality,

$$(\nabla \mathbf{u}_h, \nabla \mathbf{z}_h) \le \|\nabla \mathbf{u}_h\|\,\|\nabla \mathbf{z}_h\| \le C\,\|\nabla \mathbf{u}_h\|\,\|p_h\| \le C\,\|\nabla \mathbf{u}_h\|^2 + \frac{\|p_h\|^2}{10}.$$

Applying the Cauchy–Schwarz inequality, trace inequality (6) and (20),

$$\begin{aligned}\int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\cdot \mathbf{z}_h\ ds = \int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\cdot (\mathbf{z}_h-\mathbf{z})\ ds &\le C\left\|\frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\right\|_{\mathrm{L^2(E)}}\|\mathbf{z}_h-\mathbf{z}\|_{\mathrm{L^2(E)}}\\ &\le C\,\|\nabla \mathbf{u}_h\|_{\mathrm{L^2(K)}}\,\|\nabla \mathbf{z}_h\|_{\mathrm{L^2(K)}}.\end{aligned}$$

and

$$\begin{aligned}\int_E \frac{\zeta}{h_E}\mathbf{u}_h\cdot\mathbf{z}_h\ ds &= \left(\int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\ ds\right)^{\frac{1}{2}}\left(\int_E \frac{\zeta}{h_E}(\mathbf{z}_h-\mathbf{z})^2\ ds\right)^{\frac{1}{2}}\\ &\le C\left(\int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\ ds\right)^{\frac{1}{2}}\|\nabla \mathbf{z}_h\|_{\mathrm{L^2(K)}}.\end{aligned}$$

The sum of all the boundary edges and using (20) and Young's inequality,

$$\begin{aligned}&\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\cdot \mathbf{z}_h\ ds + \sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}\mathbf{u}_h\cdot\mathbf{z}_h\ ds\\ &\qquad\le C\left(\|\nabla \mathbf{u}_h\|^2 + \sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\ ds\right) + \frac{\|p_h\|^2}{10}.\end{aligned}$$

The second term of (22) is handled as:

$$\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}}\cdot \mathbf{u}_h\ ds \le \|\nabla \mathbf{z}_h\|^2 + C\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\ ds \le \frac{\|p_h\|^2}{10} + C\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}\mathbf{u}_h^2\ ds.$$

In the second term of (21), add $0 = (p_h, p_h) - (p_h, -\nabla\cdot\mathbf{z})$ to obtain

$$\begin{aligned}-b_h(p_h,\mathbf{z}_h) &= -(p_h, \nabla\cdot\mathbf{z}_h) + \sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{z}_h\cdot\mathbf{n})p_h\ ds\\ &= \|p_h\|^2 + (p_h, \nabla\cdot(\mathbf{z}-\mathbf{z}_h)) + \sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{z}_h\cdot\mathbf{n})p_h\ ds.\end{aligned} \tag{23}$$

Applying an integration by parts to the second term of (23),

$$(p_h, \nabla\cdot(\mathbf{z}-\mathbf{z}_h)) = -(\nabla p_h, (\mathbf{z}-\mathbf{z}_h)) - \sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{z}_h\cdot\mathbf{n})p_h\ ds.$$

It follows that

$$-b_h(p_h,\mathbf{z}_h) = \|p_h\|^2 - (\nabla p_h, \mathbf{z}-\mathbf{z}_h).$$

Use the canonical nodal basis-function $\phi_a$ at the node $a\in\mathcal{V}_h$ over the mesh $\mathcal{T}_h$. Since, $\sum_{a\in\mathcal{V}_h}\phi_a \equiv 1$,

$$\begin{aligned}(\nabla p_h, \mathbf{z}-\mathbf{z}_h) &= \sum_{K\in\mathcal{T}_h}\int_K \nabla p_h(\mathbf{z}-\mathbf{z}_h)\sum_{a\in\mathcal{V}_h}\phi_a\ dx\\ &= \sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}(\mathbf{z}-\mathbf{z}_h)\ \nabla p_h\phi_a\ dx. \end{aligned}\tag{24}$$

Using the orthogonality property of $\mathrm{L}^2$-projection (9) with the test function $C_a\phi_a\in\mathbf{V}_h$, where $C_a = \frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla p_h\ dx$ and $\|\phi\|_\infty \le 1$,

$$\begin{aligned}(\nabla p_h, \mathbf{z}-\mathbf{z}_h) &= \sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}(\mathbf{z}-\mathbf{z}_h)\left(\nabla p_h - \frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla p_h\ dx\right)\phi_a\ dx\\ &\le \left(\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\mu_a^{-1}(\mathbf{z}-\mathbf{z}_h)^2\ dx\right)^{\frac{1}{2}}\left(\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\mu_a\kappa_a^2(\nabla p_h)\ dx\right)^{\frac{1}{2}}\\ &\le \frac{1}{10}\|p_h\|^2 + CS_p(p_h,p_h). \end{aligned}\tag{25}$$

The last term of (21) is handled by using the Cauchy–Schwarz inequality, boundedness of the local projection operator and (19),

$$\begin{aligned}S_h((\mathbf{u}_h,p_h),(\mathbf{z}_h,0)) &\le [S_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h))]^{\frac{1}{2}}[S_h((\mathbf{z}_h,0),(\mathbf{z}_h,0))]^{\frac{1}{2}}\\ &\le C[S_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h))]^{\frac{1}{2}}\|\nabla\cdot\mathbf{z}_h\|\\ &\le CS_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)) + \frac{1}{10}\|p_h\|^2. \end{aligned}\tag{26}$$

Put together (21) leads to

$$\begin{aligned}&B_h((\mathbf{u}_h,p_h),(\mathbf{z}_h,0))\\ &\quad\ge \frac{1}{2}\|p_h\|^2 - C\left(\|\nabla\mathbf{u}_h\|^2 + \sum_{E\in\mathcal{E}_h^B}\int_E\frac{\zeta}{h_E}\mathbf{u}_h^2\ ds + S_h((\mathbf{u}_h,p_h),(u_h,p_h))\right). \end{aligned}\tag{27}$$

The final selection of $(\mathbf{v}_h,q_h)$ is

$$(\mathbf{v}_h,q_h) = (\mathbf{u}_h,p_h) + \frac{1}{2(C+1)}(\mathbf{z}_h,0).$$

Here $I_h$ is defined in (6). Adding (18), and (27) leads to

$$
\begin{aligned}
&B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \\
&\quad \geq \frac{1}{2}\left( \|\nabla \mathbf{u}_h\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}\mathbf{u}_h^2 \; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \right) + \frac{1}{4(C+1)}\|p_h\|^2 \\
&\quad - \frac{C}{2(C+1)}\left( \|\nabla \mathbf{u}_h\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}\mathbf{u}_h^2 \; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \right) \\
&\quad \geq \frac{1}{2(C+1)}\left( \|\nabla \mathbf{u}_h\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}\mathbf{u}_h^2 \; ds + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \right) + \frac{1}{4(C+1)}\|p_h\|^2 \\
&\quad \geq \frac{1}{2(C+1)} |||(\mathbf{u}_h, p_h)||| . \qquad (28)
\end{aligned}
$$

Applying the triangle inequality

$$
|||(\mathbf{v}_h, q_h)||| \leq |||(\mathbf{u}_h, p_h)||| + \frac{1}{2(C+1)} |||(\mathbf{z}_h, 0)||| \leq \alpha \, |||(\mathbf{u}_h, p_h)||| . \qquad (29)
$$

In the second term of (29), applying (19) and a similar technique as in (26),

$$
|||(\mathbf{z}_h, 0)||| = \|\mathbf{z}_h\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}\mathbf{z}_h^2 \; ds + S_{\mathbf{u}}(\mathbf{z}_h, \mathbf{z}_h) \leq C\|p_h\|^2,
$$

Finally, (28) and (29) lead to (14) and this concludes the claim.

*3.2. A priori error estimates*

We assume $\mathrm{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ regularity for the Stokes problem [30], and derive the following finite element results:

- convergence in triple norm, assuming $p \in \mathrm{H}^1(\Omega)$,
- optimal convergence in the $\mathrm{L}^2$-norm for the velocities.

**Lemma 3.1.** *Assume that $\mu_a = \mu h_a^2$ for some $\mu > 0$. Let $(\mathbf{u}, p) \in [\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}_0^2 \bigcap \mathrm{H}^1(\Omega)$. Then,*

$$
|||(\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p)||| \leq C \left( \|h_{\mathcal{T}}\mathbf{u}\|_2 + \|h_{\mathcal{T}} p\|_1 \right). \qquad (30)
$$

**Proof.** First, consider the terms in $|||\cdot|||$ norm defined in (13),

$$
\begin{aligned}
|||(\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p)|||^2 := \|(\nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u}))\|^2 + \|p - I_h p\|^2 + \sum_{E\in\mathcal{E}_h^B} \int_E \frac{\zeta}{h_E}(\mathbf{u} - \mathbf{I}_h\mathbf{u})^2 \; ds \\
+ S_h((\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p), (\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p)).
\end{aligned}
$$

Using the projection estimates (6)–(7),

$$
\|\nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})\| \leq \|h_{\mathcal{T}}\mathbf{u}\|_2 , \text{ and } \|p - I_h p\| \leq \|h_{\mathcal{T}} p\|_1 .
$$

Using the trace inequality over each edges (8),

$$\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}(\mathbf{u}-\mathbf{I}_h\mathbf{u})^2\ ds \le C\left\|h_\mathcal{T}\mathbf{u}\right\|_2^2.$$

Recall the stabilization term

$$\begin{aligned}S_h((\mathbf{u}-\mathbf{I}_h\mathbf{u},&p-I_hp),(\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp))\\ &=\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\kappa_a^2(\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u}))\ dx+\sum_{a\in\mathcal{V}_h}\mu_a\int_{\mathcal{M}_a}\kappa_a^2(\nabla(p-I_hp))\ dx.\end{aligned} \tag{31}$$

In the first term of (31), using the boundedness of an overlapping local projection operator and (7),

$$\begin{aligned}\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\kappa_a^2(\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u}))\ dx &= \sum_{a\in\mathcal{V}_h}\left\|\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u})-\frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u})\ dx\right\|^2_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})}\\ &\le C\sum_{a\in\mathcal{V}_h}\left\|\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u})\right\|^2_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})}\\ &\le C\left\|h_\mathcal{T}\mathbf{u}\right\|_2^2.\end{aligned} \tag{32}$$

The last term of (31) is handled by following a similar argument as in (32) with $\mu_a=\mu h_a^2$:

$$\sum_{a\in\mathcal{V}_h}\mu_a\int_{\mathcal{M}_a}\kappa_a^2(\nabla(p-I_hp))\ dx\le C\left\|h_\mathcal{T}p\right\|_1^2.$$

The combination of all the above estimates concludes the claim.

**Lemma 3.2.** *Assume that $\mu_a=\mu h_a^2$ for some $\mu>0$. Let $(\mathbf{u},p)\in[\mathrm{H}^2(\Omega)]^2\times\mathrm{L}_0^2\bigcap\mathrm{H}^1(\Omega)$ and for all $(\mathbf{v}_h,q_h)\in\mathcal{W}_h$. Then,*

$$B_h((\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp),(\mathbf{v}_h,q_h))\le C\left(\left\|h_\mathcal{T}\mathbf{u}\right\|_2+\left\|h_\mathcal{T}p\right\|_1\right)\left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|. \tag{33}$$

**Proof.** Consider the bilinear form in (12):

$$\begin{aligned}B_h((\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp),(\mathbf{v}_h,q_h)) = a_h(\mathbf{u}-\mathbf{I}_h\mathbf{u},\mathbf{v}_h)-b_h(p-I_hp,\mathbf{v}_h)+b_h(\mathbf{u}-\mathbf{I}_h\mathbf{u},q_h)\\ +S_h((\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp),(\mathbf{v}_h,q_h)).\end{aligned} \tag{34}$$

Consider the first term of (34):

$$\begin{aligned}a_h(\mathbf{u}-\mathbf{I}_h\mathbf{u},\mathbf{v}_h) = (\nabla(\mathbf{u}-\mathbf{I}_h\mathbf{u}),\nabla\mathbf{v}_h)-\sum_{E\in\mathcal{E}_h^B}\int_E\frac{\partial(\mathbf{u}-\mathbf{I}_h\mathbf{u})}{\partial\mathbf{n}}\cdot\mathbf{v}_h\ ds\\ -\sum_{E\in\mathcal{E}_h^B}\int_E\frac{\partial\mathbf{v}_h}{\partial\mathbf{n}}\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u})\ ds+\sum_{E\in\mathcal{E}_h^B}\int_E\frac{\zeta}{h_E}(\mathbf{u}-\mathbf{I}_h\mathbf{u})\cdot\mathbf{v}_h\ ds.\end{aligned} \tag{35}$$

Applying the Cauchy–Schwarz inequality and $\mathrm{L}^2$-projection property (7),

$$(\nabla(\mathbf{u}-\mathbf{I}_h\mathbf{u}),\nabla\mathbf{v}_h)\le\left\|\nabla(\mathbf{u}-\mathbf{I}_h\mathbf{u})\right\|\left\|\nabla\mathbf{v}_h\right\|\le\left\|h_\mathcal{T}\mathbf{u}\right\|_2\left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|.$$

The second term of (35) is handled by using the Cauchy–Schwarz inequality,

$$\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial(\mathbf{u}-\mathbf{I}_h\mathbf{u})}{\partial\mathbf{n}}\cdot\mathbf{v}_h\ ds \le C\left(\sum_{E\in\mathcal{E}_h^B} h_E\left\|\frac{\partial(\mathbf{u}-\mathbf{I}_h\mathbf{u})}{\partial\mathbf{n}}\right\|^2_{\mathrm{L}^2(\mathrm{E})}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h^B}\frac{\zeta}{h_E}\int_E \mathbf{v}_h^2\ ds\right)^{1/2}.$$

Applying the trace inequality over,

$$\sum_{E\in\mathcal{E}_h^B} h_E\left\|\frac{\partial(\mathbf{u}-\mathbf{I}_h\mathbf{u})}{\partial\mathbf{n}}\right\|^2_{\mathrm{L}^2(\mathrm{E})} \le C\left(\|\mathbf{u}-\mathbf{I}_h\mathbf{u}\|^2_{1,K} + h_K^2\|\mathbf{u}-\mathbf{I}_h\mathbf{u}\|^2_{2,K}\right). \tag{36}$$

Applying an introduction of some nodal interpolation, an inverse inequality and the $\mathrm{H}^1$-stability of $\mathrm{L}^2$-projection in (36),

$$\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial(\mathbf{u}-\mathbf{I}_h\mathbf{u})}{\partial\mathbf{n}}\cdot\mathbf{v}_h\ ds\ \le C\|h_{\mathcal{T}}\mathbf{u}\|_2 \,|||(\mathbf{v}_h,q_h)|||.$$

The third term of (35) is handled by using the Cauchy–Schwarz inequality, trace inequality (3) and (8),

$$\begin{aligned}\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\partial\mathbf{v}_h}{\partial\mathbf{n}}\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u})\ ds\ &\le \left(\sum_{E\in\mathcal{E}_h^B} h_E\left\|\frac{\partial\mathbf{v}_h}{\partial\mathbf{n}}\right\|^2_{\mathrm{L}^2(\mathrm{E})}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h^B}\frac{\zeta}{h_E}\int_E(\mathbf{u}-\mathbf{I}_h\mathbf{u})\ ds\right)^{1/2}\\ &\le C\|h_{\mathcal{T}}\mathbf{u}\|_2\,|||(\mathbf{v}_h,q_h)|||.\end{aligned}$$

The last term of (35) is handled by using the Cauchy–Schwarz inequality and (8),

$$\begin{aligned}\sum_{E\in\mathcal{E}_h^B}\int_E \frac{\zeta}{h_E}(\mathbf{u}-\mathbf{I}_h\mathbf{u})\cdot\mathbf{v}_h\ dsC &\le \left(\sum_{E\in\mathcal{E}_h^B}\frac{1}{h_E}\int_E(\mathbf{u}-\mathbf{I}_h\mathbf{u})^2\ ds\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h^B}\frac{\zeta}{h_E}\int_E \mathbf{v}_h^2\ ds\right)^{1/2}\\ &\le C\|h_{\mathcal{T}}\mathbf{u}\|_2\,|||(\mathbf{v}_h,q_h)|||.\end{aligned}$$

Consider the second term of bilinear form (34):

$$b_h(p-I_hp,\mathbf{v}_h) = (p-I_hp,\nabla\cdot\mathbf{v}_h) - \sum_{E\in\mathcal{E}_h^B}\int_E(\mathbf{v}_h\cdot\mathbf{n})\ (p-I_hp)\ ds. \tag{37}$$

The first term of (37) is handled by using a similar argument as in (24)–(25)

$$
\begin{aligned}
&(p - I_h p, \nabla \cdot \mathbf{v}_h) \\
&\qquad = \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} (p - I_h p) \nabla \cdot \mathbf{v}_h \phi_a \;\; dx \\
&\qquad \leq \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} (p - I_h p)^2 \; dx \right)^{1/2} \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} \left( \nabla \cdot \mathbf{v}_h - \frac{1}{|\mathcal{M}_a|} \int_{\mathcal{M}_a} \nabla \cdot \mathbf{v}_h \; dx \right)^2 \right)^{1/2} \\
&\qquad \leq \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} (p - I_h p)^2 \; dx \right)^{1/2} \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} (\kappa_a (\nabla \cdot \mathbf{v}_h))^2 \; dx \right)^{1/2} \\
&\qquad \leq \| h_{\mathcal{T}} p \|_1 \, ||| (\mathbf{v}_h, q_h) ||| \, .
\end{aligned}
$$

The second term of (37) is handled by using the Cauchy Schwarz inequality and trace inequality over edges,

$$
\begin{aligned}
\sum_{E \in \mathcal{E}_h^B} \int_E (\mathbf{v}_h \cdot \mathbf{n}) \; (p - I_h p) \; ds &\leq C \left( \sum_{E \in \mathcal{E}_h^B} h_E^{-1} \| \mathbf{v}_h \cdot \mathbf{n} \|^2_{\mathrm{L}^2(\mathrm{E})} \right)^{\frac{1}{2}} \left( \sum_{E \in \mathcal{E}_h^B} h_E \| p - I_h p \|^2_{\mathrm{L}^2(\mathrm{E})} \right)^{\frac{1}{2}} \\
&\leq C \, \| h_{\mathcal{T}} p \|_1 \, ||| (\mathbf{v}_h, q_h) ||| \, .
\end{aligned}
$$

Applying an integration by parts in the next term of the bilinear form (34),

$$
\begin{aligned}
&b_h(\mathbf{u} - \mathbf{I}_h \mathbf{u}, q_h) \\
&\qquad = (q_h, \nabla \cdot (\mathbf{u} - \mathbf{I}_h \mathbf{u})) - \sum_{E \in \mathcal{E}_h^B} \int_E ((\mathbf{u} - \mathbf{I}_h \mathbf{u}) \cdot \mathbf{n}) \; q_h \; ds \\
&\qquad = -(\nabla q_h, \mathbf{u} - \mathbf{I}_h \mathbf{u}) + \sum_{E \in \mathcal{E}_h^B} \int_E ((\mathbf{u} - \mathbf{I}_h \mathbf{u}) \cdot \mathbf{n}) \; q_h \; ds - \sum_{E \in \mathcal{E}_h^B} \int_E ((\mathbf{u} - \mathbf{I}_h \mathbf{u}) \cdot \mathbf{n}) q_h \; ds.
\end{aligned} \tag{38}
$$

Applying a similar technique as in (24)–(25), the first term of (38) is estimated as:

$$
(\nabla q_h, \mathbf{u} - \mathbf{I}_h \mathbf{u}) \leq C \, \| h_{\mathcal{T}} \mathbf{u} \|_2 \, ||| (\mathbf{v}_h, q_h) ||| \, .
$$

The last term is estimated in a similar way as in (31),

$$
S_h \left( (\mathbf{u} - \mathbf{I}_h \mathbf{u}, p - I_h p), (\mathbf{v}_h, q_h) \right) \leq \left( \| h_{\mathcal{T}} \mathbf{u} \|_2 + \| h_{\mathcal{T}} p \|_1 \right) ||| (\mathbf{v}_h, q_h) ||| \, .
$$

The combination of the above estimates concludes the claim.

**Lemma 3.3.** *Assume that* $\mu_a = \mu h_a^2$ *for some* $\mu > 0$. *Suppose* $(\mathbf{u}, p) \in [\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}^2_0(\Omega) \cap \mathrm{H}^1(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h$ *are the solutions to (2) and (11) respectively. For any* $(\mathbf{v}_h, q_h) \in \mathcal{W}_h$. *Then,*

$$
B_h((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}_h, q_h)) \leq C \left( \| h_{\mathcal{T}} \mathbf{u} \|_2 + \| h_{\mathcal{T}} p \|_1 \right) ||| (\mathbf{v}_h, q_h) ||| \, .
$$

**Proof.** Using $\mathbf{u} = 0$ over the boundary edges,

$$
B_h((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}_h, q_h)) = S_h((\mathbf{u}, p), (\mathbf{v}_h, q_h)).
$$

$$S_h((\mathbf{u},p),(\mathbf{v}_h,q_h)) = \sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\kappa_a(\nabla\cdot\mathbf{u})\kappa_a(\nabla\cdot\mathbf{v}_h)\;dx + \sum_{a\in\mathcal{V}_h}\mu_a\int_{\mathcal{M}_a}\kappa_a(\nabla p)\cdot\kappa_a(\nabla q_h)\;dx. \tag{39}$$

Using the Cauchy–Schwarz inequality and Poincaré inequality (5) in the first term of (39),

$$\begin{aligned}
&\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\kappa_a(\nabla\cdot\mathbf{u})\kappa_a(\nabla\cdot\mathbf{v}_h)\;dx\\
&\quad\le\left(\sum_{a\in\mathcal{V}_h}\left\|\nabla\cdot\mathbf{u}-\frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla\cdot\mathbf{u}\;dx\right\|^2_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})}\right)^{1/2} S_h^{1/2}((\mathbf{u},p),(\mathbf{v}_h,q_h))\\
&\quad\le\|h_{\mathcal{T}}\mathbf{u}\|_2\,|\!|\!|(\mathbf{v}_h,q_h)|\!|\!|\,.
\end{aligned}$$

In a similar way, the second term is handled as:

$$\sum_{a\in\mathcal{V}_h}\mu_a\int_{\mathcal{M}_a}\kappa_a(\nabla p)\kappa_a(\nabla q_h)\;dx\le\|h_{\mathcal{T}}p\|_1\,|\!|\!|(\mathbf{v}_h,q_h)|\!|\!|\,.$$

The combination of the above estimates concludes the claim.

**Theorem 3.2.** *Suppose* $(\mathbf{u},p)\in[\mathrm{H}^2(\Omega)]^2\times\mathrm{L}^2_0\bigcap\mathrm{H}^1(\Omega)$ *and* $(\mathbf{u}_h,p_h)\in\mathcal{W}_h$ *are the solutions to (2) and (11) respectively. Let* $\mu_a=\mu h_a^2$ *for some* $\mu>0$*. Then it holds that*

$$|\!|\!|(\mathbf{u}-\mathbf{u}_h,p-p_h)|\!|\!|\le C\left(\|h_{\mathcal{T}}\mathbf{u}\|_2+\|h_{\mathcal{T}}p\|_1\right). \tag{40}$$

**Proof.** Using the triangle inequality,

$$|\!|\!|(\mathbf{u}-\mathbf{u}_h,p-p_h)|\!|\!|\le|\!|\!|(\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp)|\!|\!|+|\!|\!|(\mathbf{I}_h\mathbf{u}-\mathbf{u}_h,I_hp-p_h)|\!|\!|\,. \tag{41}$$

The first term of (41) is handled by Lemma 3.1, i.e.,

$$|\!|\!|(\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp)|\!|\!|\le C\left(\|h_{\mathcal{T}}\mathbf{u}\|_2+\|h_{\mathcal{T}}p\|_1\right).$$

The second term of (41) is handled by using Theorem 3.1,

$$\begin{aligned}
|\!|\!|(\mathbf{I}_h\mathbf{u}-\mathbf{u}_h,I_hp-p_h)|\!|\!|&\le 1/\delta\sup_{(\mathbf{v}_h,q_h)\in\mathcal{W}_h}\frac{B_h((\mathbf{I}_h\mathbf{u}-\mathbf{u}_h,I_hp-p_h),(\mathbf{v}_h,q_h))}{|\!|\!|\mathbf{v}_h,q_h|\!|\!|}\\
&\le 1/\delta\sup_{(\mathbf{v}_h,q_h)\in\mathcal{W}_h}\frac{B_h(\mathbf{u}-\mathbf{u}_h,p-p_h),(\mathbf{v}_h,q_h))}{|\!|\!|\mathbf{v}_h,q_h|\!|\!|}\\
&\quad+\sup_{(\mathbf{v}_h,q_h)\in\mathcal{W}_h}\frac{B_h((\mathbf{I}_h\mathbf{u}-\mathbf{u},I_hp-p),(\mathbf{v}_h,q_h))}{|\!|\!|\mathbf{v}_h,q_h|\!|\!|}.
\end{aligned}\tag{42}$$

Finally, the result follows by using Lemma 3.2 and Lemma 3.3 in (42), and this concludes the claim.

We now proceed to derive an $\mathrm{L}^2$-error estimates for the velocities in the case of the Stokes equation. Consider the following dual problem, find $(\phi,\psi)\in\mathcal{W}$ such that

$$a_h(\mathbf{v},\phi)+b_h(q,\phi)-b_h(\psi,\mathbf{v})=(\eta,\mathbf{v})_\Omega,\quad\forall\;(\mathbf{v},q)\in\mathcal{W} \tag{43}$$

and assume that the solution has the additional regularity,

$$\|\phi\|_2 + \|\psi\|_1 \leq C\,\|\eta\|\,, \tag{44}$$

valid if the boundary is sufficiently smooth, [28].

**Theorem 3.3.** *Assume that the solution* $(\mathbf{u}, p)$ *of* (2) *belongs to* $[\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}^2_0 \bigcap \mathrm{H}^1(\Omega)$ *and let* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h$ *be the solution of (11). Assume also that* $\mu_a = \mu h_a^2$ *for some* $\mu > 0$. *Then,*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq C\left(\left\|h_{\mathcal{T}}^2 \mathbf{u}\right\|_2 + \left\|h_{\mathcal{T}}^2 p\right\|_1\right).$$

**Proof.** Choosing $\eta = \mathbf{v} = \mathbf{u} - \mathbf{u}_h, q = p - p_h$ in (43) gives

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_\Omega^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \phi) + b_h(p - p_h, \phi) - b_h(\psi, \mathbf{u} - \mathbf{u}_h) \\
&= \underline{a_h(\mathbf{u} - \mathbf{u}_h, \phi - \mathbf{I}_h\phi) + b_h(p - p_h, \phi - \mathbf{I}_h\phi) - b_h(\psi - I_h\phi, \mathbf{u} - \mathbf{u}_h)} \\
&+ S_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \phi - \mathbf{I}_h\phi) + S_p(p_h, I_h\psi) \\
&= (a) + (b) + (c).
\end{aligned} \tag{45}$$

Let us now estimate these three terms. Consider the first term of (71). Following similar arguments as in Lemma 3.2,

$$(a) \leq C\left(\|\!|\mathbf{u} - \mathbf{u}_h|\!\|\, \|h_{\mathcal{T}}\phi\|_2 + \|p - p_h\|\, \|h_{\mathcal{T}}\phi\|_2 + \|\!|\mathbf{u} - \mathbf{u}_h|\!\|\, \|h_{\mathcal{T}}\psi\|_1\right).$$

Using the Cauchy–Schwarz inequality, and Lemma 3.1,

$$\begin{aligned}
(b) &\leq S_{\mathbf{u}}^{1/2}(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) S_{\mathbf{u}}^{1/2}(\phi - \mathbf{I}_h\phi, \phi - \mathbf{I}_h\phi) \\
&\leq C\,\|\!|\mathbf{u} - \mathbf{u}_h|\!\|\, \|h_{\mathcal{T}}\phi\|_2\,.
\end{aligned}$$

The last term of (71) is handled by using boundedness of the local projection operator, the stability of the projection estimates and $\mu_a = \mu h_a^2$, i.e.,

$$(c) = S_p(p, I_h\psi) - S_p(p - p_h, I_h\psi), \tag{46}$$

$$\begin{aligned}
|S_p(p, I_h\psi)| &\leq \left(\sum_{a \in \mathcal{M}_a} \mu_a \|\kappa_a \nabla p\|^2_{\mathrm{L}^2(\mathcal{M}_a)}\right)^{\frac{1}{2}} S_p(I_h\psi, I_h\psi)^{1/2} \\
&\leq C\,\|h_{\mathcal{T}} p\|_1\, \|h_{\mathcal{T}}\psi\|_1\,.
\end{aligned}$$

$$|S_p(p - p_h, I_h\psi)| \leq C\,\|\!|(\mathbf{u} - \mathbf{u}_h, p - p_h)|\!\|\, \|h_{\mathcal{T}}\psi\|_1\,.$$

The proof concludes by combining the above estimates with the Theorem 3.2, and the assumed regularizing behavior (44).

## 4. An overlapping local projection stabilization for the Darcy flow problem

### *4.1. The Darcy problem*

In this section, we extend the above analysis to the Darcy flow problem. Consider the Darcy flow equations: Find $(\mathbf{u}, p)$ such that

$$\begin{aligned}
\mathbf{u} + \omega\nabla p = \mathbf{f}; \quad \nabla\cdot\mathbf{u} &= g \quad \text{in } \Omega, \\
\mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{47}$$

Here, $\Omega \subset \mathbb{R}^2$ is an open bounded polygonal domain with smooth boundary $\partial\Omega$ and $\mathbf{u}$ is the velocity vector, $p$ is the pressure, $\mathbf{f} \in [\mathrm{L}^2(\Omega)]^2$, $g \in \mathrm{L}_0^2(\Omega)$ are the given data, $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$ and $\omega = \varkappa/\lambda$, $\varkappa > 0$ is the permeability, and $\lambda > 0$ is the viscosity. The divergence constraint implies that the prescribed data must satisfy the condition,

$$\int_\Omega g \; dx = 0.$$

In order to formulate a weak formulation of the Darcy flow equations, consider the following Sobolev spaces,

$$\mathbf{V} := \{\mathbf{v} \in \mathrm{H}(\mathrm{div}, \Omega) | \; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad Q := \mathrm{L}_0^2(\Omega),$$

where $\mathrm{L}^2(\Omega)$ is a space of square-integrable measurable function. Moreover, a weak formulation of the model problem (47) reads as: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) - b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}); \qquad b(\mathbf{u}, q) = (g, q),$$

for all $\mathbf{v} \in \mathbf{V}$ and $q \in Q$, and

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \omega^{-1}(\mathbf{u} \cdot \mathbf{v}) \; dx; \quad b(p, \mathbf{v}) := \int_\Omega p \nabla \cdot \mathbf{v} \; dx.$$

The weak form of the Darcy flow problem can also be defined on $\mathbf{V} \times Q = \mathcal{W}$ and it reads as: Find $(\mathbf{u}, p) \; \in \mathcal{W}$ such that

$$A((\mathbf{u}, p), (\mathbf{v}, q)) = L(\mathbf{v}), \tag{48}$$

for all $(\mathbf{v}, q) \in \mathcal{W}$, where

$$A((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) - b(p, \mathbf{v}) + b(q, \mathbf{u}); \qquad L(\mathbf{v}) := (\mathbf{f}, \mathbf{v}) + (g, q).$$

For sufficiently regular data, the weak formulation (48) is known to possess a unique solution [28].

### *4.2. The inf-sup condition*

This section describes the conforming finite element method for the problem (47), where the velocity and the pressure are approximated with the continuous piecewise linear finite element spaces.

Let $\mathbf{V}_h := [\mathbf{P}_1^c(\mathcal{T}_h)]^2$ and $Q_h := \mathrm{L}_0^2(\Omega) \bigcap \mathbf{P}_1^c(\mathcal{T}_\mathrm{h})$. We propose the following finite element formulation: Find $(\mathbf{u}_h, p_h) \in \mathcal{W}_h = \mathbf{V}_h \times Q_h$ such that

$$A_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = L(\mathbf{v}, q), \tag{49}$$

for all $(\mathbf{v}, q) \in \mathcal{W}_h$, where

$$A_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = a_h(\mathbf{u}_h, \mathbf{v}) - b_h(p_h, \mathbf{v}) + b_h(\mathbf{u}_h, q) + S_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)), \tag{50}$$

and

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}) :&= \sum_{K\in\mathcal{T}_h} \int_K \omega^{-1}(\mathbf{u}_h \cdot \mathbf{v})\ dx,\\
b_h(p_h, \mathbf{v}) :&= (p_h, \nabla\cdot\mathbf{v}) - \sum_{E\in\mathcal{E}_h^B} \int_E (\mathbf{v}\cdot\mathbf{n}) p_h\ ds,\\
S_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) :&= S_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}) + S_p(p_h, q),\\
S_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}) :&= \sum_{a\in\mathcal{V}_h} \beta_a\omega^{-1} \int_{\mathcal{M}_a} \kappa_a(\nabla\cdot\mathbf{u}_h)\kappa_a(\nabla\cdot\mathbf{v})\ dx + \sum_{E\in\mathcal{E}_h^B} \int_E (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})\ ds,\\
S_p(p_h, q) :&= \sum_{a\in\mathcal{V}_h} \beta_a\omega \int_{\mathcal{M}_a} \kappa_a(\nabla p_h)\kappa_a(\nabla q)\ dx,\\
L(\mathbf{v}, q) :&= (\mathbf{f}, \mathbf{v}) + (g, q).
\end{aligned}$$

Note that the stabilization parameters are chosen as $\beta_a = \beta h_a$ for some $\beta > 0$. Further, introduce a generalized local projection norm on $\mathcal{W}_h$ by

$$|||(\mathbf{u}_h, p_h)|||^2 := \|\omega^{-\frac{1}{2}}\mathbf{u}_h\|^2 + \|h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h)\|^2 + \|p_h\|^2 + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)). \tag{51}$$

The main result of this section is the following theorem, which ensures that the discrete bilinear form is well–posed [28].

**Theorem 4.1.** *The discrete bilinear form (49) satisfies the following inf-sup condition for some positive constant $\gamma$, independent of $h$:*

$$\inf_{(\mathbf{u}_h,p_h)\in\mathcal{W}_h} \sup_{(\mathbf{v}_h,q_h)\in\mathcal{W}_h} \frac{A_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||(\mathbf{u}_h, p_h)|||\ |||(\mathbf{v}_h, q_h)|||} \geq \gamma.$$

**Proof.** In order to prove the stability result, it is enough to choose some $(\mathbf{v}_h, q_h) \in \mathcal{W}_h$ for any arbitrary $(\mathbf{u}_h, p_h) \in \mathcal{W}_h$, such that

$$\frac{A_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||(\mathbf{v}_h, q_h)|||} \geq \gamma\ |||(\mathbf{u}_h, p_h)||| > 0. \tag{52}$$

First, consider the bilinear form in (50) with $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$:

$$A_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) = \|\omega^{-\frac{1}{2}}\mathbf{u}_h\|^2 + S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)). \tag{53}$$

Taking $(\mathbf{v}_h, q_h) = (\mathbf{z}_h, 0)$ in (50),

$$A_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)) = a_h(\mathbf{u}_h, \mathbf{z}_h) - b_h(p_h, \mathbf{z}_h) + S_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)). \tag{54}$$

Let us now bound the three contributions. Applying the Cauchy–Schwarz inequality, (19) and Young's inequality,

$$a_h(\mathbf{u}_h, \mathbf{z}_h) \leq \omega^{-1}\|\mathbf{u}_h\|\|\mathbf{z}_h\| \leq \omega^{-1}C_1\|\mathbf{u}_h\|\|p_h\| \leq C\left\|\omega^{-\frac{1}{2}}\mathbf{u}_h\right\|^2 + \frac{1}{8}\|p_h\|^2.$$

The constant $C$ in the above estimates depend on $\omega^{-1}$. In the second term of (54), add

$0 = (p_h, p_h) - (p_h, -\nabla \cdot \mathbf{z})$ to obtain,

$$\begin{aligned} -b_h(p_h, \mathbf{z}_h) &= -(p_h, \nabla \cdot \mathbf{z}_h) + \sum_{E \in \mathcal{E}_h^B} \int_E (\mathbf{z}_h \cdot \mathbf{n}) p_h \; ds \\ &= \|p_h\|^2 + (p_h, \nabla \cdot (\mathbf{z} - \mathbf{z}_h)) + \sum_{E \in \mathcal{E}_h^B} \int_E (\mathbf{z}_h \cdot \mathbf{n}) p_h \; ds. \end{aligned} \tag{55}$$

Applying an integration by parts to the second term of (55),

$$(p_h, \nabla \cdot (\mathbf{z} - \mathbf{z}_h)) = -(\nabla p_h, (\mathbf{z} - \mathbf{z}_h)) + \sum_{E \in \mathcal{E}_h^B} \int_E p_h (\mathbf{z} - \mathbf{z}_h) \cdot \mathbf{n} \; dx.$$

It follows that

$$-b_h(p_h, \mathbf{z}_h) = \|p_h\|^2 - (\nabla p_h, \mathbf{z} - \mathbf{z}_h). \tag{56}$$

Following a similar argument as in (24)–(25), the second term of (56) can be estimated as:

$$\begin{aligned} (\nabla p_h, \mathbf{z} - \mathbf{z}_h) &= \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} (\mathbf{z} - \mathbf{z}_h) \left( \nabla p_h - \frac{1}{|\mathcal{M}_a|} \int_{\mathcal{M}_a} \nabla p_h \; dx \right) \phi_a \; dx \\ &\leq \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} \omega^{-1} \beta_a^{-1} (\mathbf{z} - \mathbf{z}_h)^2 \; dx \right)^{\frac{1}{2}} \left( \sum_{a \in \mathcal{V}_h} \int_{\mathcal{M}_a} \omega \beta_a \kappa_a^2 (\nabla p_h) \; dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{8} \|p_h\|^2 + C S_p(p_h, p_h). \end{aligned} \tag{57}$$

The constant $C$ in the above estimates depend on $\omega^{-1/2}$. The last term of (54) is handled by using the Cauchy–Schwarz inequality, boundedness of the local projection operator and (19),

$$S_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)) \leq \frac{C}{2} S_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) + \frac{1}{4} \|p_h\|^2 .$$

Put together, (54) leads to

$$A_h((\mathbf{u}_h, p_h), (\mathbf{z}_h, 0)) \geq \frac{1}{2} \|p_h\|^2 - C \left( \|\mathbf{u}_h\|^2 + \frac{1}{2} S_h((\mathbf{u}_h, p_h), (u_h, p_h)) \right). \tag{58}$$

Finally, the control of $\left\| h_{\mathcal{T}}^{\frac{1}{2}} (\nabla \cdot \mathbf{u}_h) \right\|^2$ can be obtained by choosing $(\mathbf{v}_h, q_h) = (0, h_{\mathcal{T}}(\nabla \cdot \mathbf{u}_h))$ in (50), that is,

$$A_h((\mathbf{u}_h, p_h), (0, I_h(h_{\mathcal{T}}(\nabla \cdot \mathbf{u}_h)))) = b_h(I_h(h_{\mathcal{T}}(\nabla \cdot \mathbf{u}_h)), \mathbf{u}_h) + S_h((\mathbf{u}_h, p_h), (0, I_h(h_{\mathcal{T}}(\nabla \cdot \mathbf{u}_h)))). \tag{59}$$

By adding and subtracting $\left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2$, the first term of (59) becomes,

$$b_h(I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)),\mathbf{u}_h) = \left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2 + (I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)) - h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h), \nabla\cdot\mathbf{u}_h) - \sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{u}_h\cdot\mathbf{n})\ I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h))\ ds. \qquad (60)$$

The second term of (60) is estimated as:

$$\begin{aligned}
&(I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)) - h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h), \nabla\cdot\mathbf{u}_h)\\
&= \sum_{a\in\mathcal{M}_a}\int_{\mathcal{M}_a} I_h(h_K(\nabla\cdot\mathbf{u}_h)) - h_K(\nabla\cdot\mathbf{u}_h)(\nabla\cdot\mathbf{u}_h)\phi_a\ dx\\
&= \sum_{a\in\mathcal{M}_a}\int_{\mathcal{M}_a} (I_h(h_K(\nabla\cdot\mathbf{u}_h)) - h_K(\nabla\cdot\mathbf{u}_h))\left(\nabla\cdot\mathbf{u}_h - \frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla\cdot\mathbf{u}_h\ dx\right)\phi_a\ dx\\
&\le \left(\sum_{a\in\mathcal{M}_a}\omega\beta_a^{-1}\left\|I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)) - h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)\right\|^2_{\mathrm{L}^2(\mathcal{M}_a)}\right)^{\frac{1}{2}} [S_h((\mathbf{u}_h,0),(\mathbf{u}_h,0))]^{\frac{1}{2}}\\
&\le \frac{1}{6}\left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2 + \frac{C}{2}S_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,q_h)).
\end{aligned}$$

The constant $C$ in the above estimates depend on $\omega^{1/2}$. In the third term of (60), using the Cauchy–Schwarz inequality, trace inequality, stability property of projection operator (10) and Youngs inequality,

$$\begin{aligned}
\sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{u}_h\cdot\mathbf{n})\ I_h(h_K(\nabla\cdot\mathbf{u}_h))\ ds &\le \left(\sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{u}_h\cdot\mathbf{n})^2\ ds\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{E}_h^B}\int_E (I_h(h_K(\nabla\cdot\mathbf{u}_h))^2\ ds\right)^{\frac{1}{2}}\\
&\le \frac{1}{6}\left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2 + \frac{C}{4}S_h((\mathbf{u}_h,0),(\mathbf{u}_h,0)).
\end{aligned}$$

The last term of (59) is handled by using the Cauchy–Schwarz inequality, boundedness of the local projection operator, the stability of the projection operator, inverse inequality and the Young's inequality,

$$\begin{aligned}
&S_h((\mathbf{u}_h,p_h),(0,I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h))))\\
&\qquad \le S_h^{1/2}((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h))S_h^{1/2}((0,I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h))),(0,I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h))))\\
&\qquad \le \frac{1}{6}\left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2 + \frac{C}{4}S_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)).
\end{aligned}$$

Put together, (59) leads to

$$A_h((\mathbf{u}_h,p_h),(0,I_h(h_{\mathcal{T}}(\nabla\cdot\mathbf{u}_h)))) \ge \frac{1}{2}\left\| h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{u}_h) \right\|^2 - \frac{C}{2}S_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)).$$

The selection of $(\mathbf{v}_h, q_h)$ is

$$(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h) + \frac{1}{C+1}(\mathbf{z}_h, 0) + \frac{1}{C+1}(0, I_h(h_{\mathcal{T}}\nabla\cdot\mathbf{u}_h)).$$

Here, $I_h$ is the projection operator. Rest of the proof can be derived following similar steps as in (28)–(29).

### *4.3. A priori error estimates*

Using the $|||\cdot|||$ norm and inf-sup condition described above, we now prove *a priori* error estimates for the discrete solution.

**Lemma 4.1.** *Suppose $\beta_a = \beta h_a$ for some $\beta > 0$. Let $(\mathbf{u}, p) \in [\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}^2_0 \bigcap \mathrm{H}^2(\Omega)$. Then,*

$$|||(\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p)||| \leq C\left(\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2 + \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right). \tag{61}$$

**Proof.** Following a similar argument as in Lemma 3.1, the proof can be derived.

**Lemma 4.2.** *Suppose $\beta_a = \beta h_a$ for some $\beta > 0$. Let $(\mathbf{u}, p) \in [\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}^2_0 \bigcap \mathrm{H}^2(\Omega)$ and for all $(\mathbf{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$. Then,*

$$A_h((\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p), (\mathbf{v}_h, q_h)) \leq C\left(\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2 + \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right)|||(\mathbf{v}_h, q_h)|||. \tag{62}$$

**Proof.** Consider the bilinear form in (50):

$$\begin{aligned} A_h((\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p), (\mathbf{v}_h, q_h)) = a_h(\mathbf{u} - \mathbf{I}_h\mathbf{u}, \mathbf{v}_h) - b_h(p - I_h p, \mathbf{v}_h) + b_h(\mathbf{u} - \mathbf{I}_h\mathbf{u}, q_h) \\ + S_h((\mathbf{u} - \mathbf{I}_h\mathbf{u}, p - I_h p), (\mathbf{v}_h, q_h)). \end{aligned} \tag{63}$$

Applying the Cauchy–Schwarz inequality and $\mathrm{L}^2$-projection property (7),

$$a_h(\mathbf{u} - \mathbf{I}_h\mathbf{u}, \mathbf{v}_h) \leq \left\|\omega^{-\frac{1}{2}}(\mathbf{u} - \mathbf{I}_h\mathbf{u})\right\|\left\|\omega^{-\frac{1}{2}}\mathbf{v}_h\right\| \leq C\left\|h_{\mathcal{T}}^2\mathbf{u}\right\|_2 |||(\mathbf{v}_h, q_h)|||.$$

Note that the constant $C$ in the above estimates depends on $\omega^{-1/2}$. Consider the second term of bilinear form (63):

$$b_h(p - I_h p, \mathbf{v}_h) = (p - I_h p, \nabla\cdot\mathbf{v}_h) - \sum_{E\in\mathcal{E}_h^B}\int_E (\mathbf{v}_h\cdot\mathbf{n})\ (p - I_h p)\ ds. \tag{64}$$

Using the Cauchy–Schwarz inequality and $\mathrm{L}^2$-projection property in the first term of (64),

$$(p - I_h p, \nabla\cdot\mathbf{v}_h) \leq \|p - I_h p\|\,\|\nabla\cdot\mathbf{v}_h\| \leq \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\left\|h_{\mathcal{T}}^{\frac{1}{2}}(\nabla\cdot\mathbf{v}_h)\right\| \leq \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2 |||(\mathbf{v}_h, q_h)|||.$$

The second term of (64) is handled by using the Cauchy–Schwarz inequality and trace inequality over edges,

$$\sum_{E\in\mathcal{E}_h^B}\int_E(\mathbf{v}_h\cdot\mathbf{n})\ (p-I_hp)\ ds \le \left(\sum_{E\in\mathcal{E}_h^B}\|\mathbf{v}_h\cdot\mathbf{n}\|^2_{\mathrm{L}^2(\mathrm{E})}\right)^{\frac{1}{2}}\left(\sum_{E\in\mathcal{E}_h^B}\|p-I_hp\|^2_{\mathrm{L}^2(\mathrm{E})}\right)^{\frac{1}{2}}$$
$$\le \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2 \left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|.$$

Applying an integration by parts in the next term of the bilinear form (63),

$$\begin{aligned}
&b_h(\mathbf{u}{-}\mathbf{I}_h\mathbf{u},q_h)\\
&\quad=(q_h,\nabla\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u}))-\sum_{E\in\mathcal{E}_h^B}\int_E((\mathbf{u}-\mathbf{I}_h\mathbf{u})\cdot\mathbf{n})\ q_h\ ds\\
&\quad=-(\nabla q_h,\mathbf{u}-\mathbf{I}_h\mathbf{u})+\sum_{E\in\mathcal{E}_h^B}\int_E((\mathbf{u}-\mathbf{I}_h\mathbf{u})\cdot\mathbf{n})\ q_h\ ds-\sum_{E\in\mathcal{E}_h^B}\int_E((\mathbf{u}-\mathbf{I}_h\mathbf{u})\cdot\mathbf{n})q_h\ ds.
\end{aligned}\tag{65}$$

Applying a similar technique as in (24), the first term of (65) is estimated as:

$$\begin{aligned}
(\nabla q_h,\mathbf{u}-\mathbf{I}_h\mathbf{u})&=\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}(\mathbf{u}-\mathbf{I}_h\mathbf{u})\left(\nabla q_h-\frac{1}{|\mathcal{M}_a|}\int_{\mathcal{M}_a}\nabla q_h\ dx\right)\phi_a\ dx\\
&\le\left(\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\omega^{-1}\beta_a^{-1}(\mathbf{u}-\mathbf{I}_h\mathbf{u})^2\ dx\right)^{\frac{1}{2}}\left(\sum_{a\in\mathcal{V}_h}\int_{\mathcal{M}_a}\omega\beta_a\kappa_a^2(\nabla q_h)\ dx\right)^{\frac{1}{2}}\\
&\le C\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2\left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|,
\end{aligned}$$

the constant $C$ in the above estimates depends on $\omega^{-1/2}$. The last term is estimated in a similar way as in (31),

$$S_h\left((\mathbf{u}-\mathbf{I}_h\mathbf{u},p-I_hp),(\mathbf{v}_h,q_h)\right)\le C\left(C_1\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2+C_2\left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right)\left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|.$$

The constants $C_1$ and $C_2$ in the above estimates depend on $\omega^{-1/2}$ and $\omega^{1/2}$, respectively. The combination of the above estimates concludes the claim.

**Lemma 4.3.** *Suppose* $(\mathbf{u},p)\in[\mathrm{H}^2(\Omega)]^2\times\mathrm{L}_0^2(\Omega)\cap\mathrm{H}^2(\Omega)$ *and* $(\mathbf{u}_h,p_h)\in\mathcal{W}_h$ *are the solutions to (48) and (49) respectively. For any* $(\mathbf{v}_h,q_h)\in\mathcal{W}_h$*. Then,*

$$A_h((\mathbf{u}-\mathbf{u}_h,p-p_h),(\mathbf{v}_h,q_h))\le C\left(\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2+\left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right)\left|\!\left|\!\left|(\mathbf{v}_h,q_h)\right|\!\right|\!\right|.\tag{66}$$

**Proof.** Following a similar argument as in Lemma 3.3, the proof can be derived.

**Theorem 4.2.** *Suppose* $(\mathbf{u},p)\in[\mathrm{H}^2(\Omega)]^2{\times}\mathrm{L}_0^2\bigcap\mathrm{H}^2(\Omega)$ *and* $(\mathbf{u}_h,p_h)\in\mathcal{W}_h$ *are the solutions to (48) and (49) respectively. Let* $\beta_a=\beta h_a$ *for some* $\beta>0$*. Then it holds that,*

$$\left|\!\left|\!\left|(\mathbf{u}-\mathbf{u}_h,p-p_h)\right|\!\right|\!\right|\le C\left(\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{u}\right\|_2+\left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right).\tag{67}$$

**Proof.** Identical to the proof of Theorem 3.2.

**Corollary 4.1.** *Suppose* $(\mathbf{u}, p) \in [\mathrm{H}^2(\Omega)]^2 \times \mathrm{L}^2_0 \bigcap \mathrm{H}^2(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h$ *are the solutions to (48) and (49) respectively. Let* $\beta_a = \beta$ *for some* $\beta > 0$. *Then it holds that*

$$|\!|\!|(\mathbf{u} - \mathbf{u}_h, p - p_h)|\!|\!| \leq C\left(\|h_{\mathcal{T}}\mathbf{u}\|_2 + \|h_{\mathcal{T}}p\|_2\right). \tag{68}$$

We now proceed to derive an $\mathrm{L}^2$-error estimates for the pressure in the case of the Darcy equation. Consider the following dual problem, for a given $(\phi_v, \phi_p)$, find $(\mathbf{z}_v, z_p) \in \mathcal{W}$ such that

$$a_h(\mathbf{w}_v, \mathbf{z}_v) - b_h(w_p, \mathbf{z}_v) + b_h(\mathbf{w}_v, z_p) = (\phi_p, w_p)_\Omega, \quad \forall\ (\mathbf{w}_v, w_q) \in \mathcal{W} \tag{69}$$

and assume that the solution has the additional regularity, i.e., $(\mathbf{z}_v, z_p) \in [\mathrm{H}^1(\Omega)]^2 \times \mathrm{H}^2(\Omega)$ and it holds the estimate

$$\|\mathbf{z}_v\|_1 + \|z_p\|_2 \leq C\,\|\phi_p\|, \tag{70}$$

valid if the boundary is sufficiently smooth [10].

**Theorem 4.3.** *Assume that the solution* $(\mathbf{v}, p)$ *of (48) belongs to* $[\mathrm{H}^1(\Omega)]^2 \times \mathrm{L}^2_0 \bigcap \mathrm{H}^2(\Omega)$ *and let* $(\mathbf{v}_h, p_h) \in \mathcal{W}_h$ *be the solution of (49). Assume also that* $\beta_a = \beta h_a^2$ *for some* $\beta > 0$. *Then,*

$$\|p - p_h\| \leq C\left(\left\|h_{\mathcal{T}}^{\frac{3}{2}}\mathbf{v}\right\|_1 + \left\|h_{\mathcal{T}}^{\frac{3}{2}}p\right\|_2\right).$$

**Proof.** Choosing, $\phi_p = w_p = p - p_h, \mathbf{w}_v = \mathbf{v} - \mathbf{v}_h$ in (43) gives

$$\begin{aligned}
\|p - p_h\|^2_\Omega &= a_h(\mathbf{v} - \mathbf{v}_h, \mathbf{z}_v) - b_h(p - p_h, \mathbf{z}_v) + b_h(\mathbf{v} - \mathbf{v}_h, z_p)\\
&= \underline{a_h(\mathbf{v} - \mathbf{v}_h, \mathbf{z}_v - \mathbf{I}_h\mathbf{z}_v) + b_h(p - p_h, \mathbf{z}_v - \mathbf{I}_h\mathbf{z}_v) - b_h(\mathbf{v} - \mathbf{v}_h, z_p - I_h z_p)}\\
&+ S_{\mathbf{v}}(\mathbf{v}_h, \mathbf{I}_h\mathbf{z}_v) + S_p(p_h, I_h z_p)\\
&= (a) + (b) + (c).
\end{aligned} \tag{71}$$

Let us now estimate these three terms. Consider the first term of (71). Following similar arguments as in Lemma 4.2,

$$(a) \leq C\left(|\!|\!|\mathbf{v} - \mathbf{v}_h|\!|\!|\,\|h_{\mathcal{T}}\mathbf{z}_v\|_1 + \|p - p_h\|\,\|h_{\mathcal{T}}\mathbf{z}_v\|_1 + |\!|\!|\mathbf{v} - \mathbf{v}_h|\!|\!|\,\|h_{\mathcal{T}}z_p\|_2\right).$$

Consider the second term of (71).

$$(b) \leq S_{\mathbf{v}}(\mathbf{v}, \mathbf{I}_h\mathbf{z}_v) - S_{\mathbf{v}}(\mathbf{v} - \mathbf{v}_h, \mathbf{I}_h\mathbf{z}_v)$$

The first term of above is handled by using boundedness of the local projection operator, the stability of the projection estimates and $\beta_a = \beta h_a^2$, i.e.,

$$\begin{aligned}
|S_{\mathbf{v}}(\mathbf{v}, \mathbf{I}_h\mathbf{z}_v)| &\leq \left(\sum_{a\in\mathcal{M}_a} \beta_a\,\|\kappa_a(\nabla\cdot\mathbf{v})\|^2_{\mathrm{L}^2(\mathcal{M}_a)}\right)^{\frac{1}{2}} S_{\mathbf{v}}(\mathbf{I}_h\mathbf{z}_v, \mathbf{I}_h\mathbf{z}_v)^{1/2}\\
&\leq C\,\|h_{\mathcal{T}}\mathbf{v}\|_1\, S_{\mathbf{v}}(\mathbf{I}_h\mathbf{z}_v, \mathbf{I}_h\mathbf{z}_v)^{1/2}.
\end{aligned} \tag{72}$$

Now,

$$S_{\mathbf{v}}(\mathbf{I}_h\mathbf{z}_v, \mathbf{I}_h\mathbf{z}_v) := \sum_{a\in\mathcal{V}_h} \beta_a \omega^{-1} \int_{\mathcal{M}_a} \kappa_a^2 (\nabla\cdot \mathbf{I}_h\mathbf{z}_v)\ dx + \sum_{E\in\mathcal{E}_h^B} \int_E (\mathbf{I}_h\mathbf{z}_v\cdot\mathbf{n})^2\ ds$$

Consider the second term of above

$$\begin{aligned}\sum_{E\in\mathcal{E}_h^B} \int_E (\mathbf{I}_h\mathbf{z}_v\cdot\mathbf{n})^2\ ds &= \sum_{E\in\mathcal{E}_h^B} \int_E ((\mathbf{z}_v - \mathbf{I}_h\mathbf{z}_v)\cdot\mathbf{n})^2\ ds \\ &\le \left\| h_{\mathcal{T}}^{\frac{1}{2}} \mathbf{z}_v \right\|_1\end{aligned}$$

$$|S_{\mathbf{v}}(\mathbf{v}-\mathbf{v}_h, \mathbf{I}_h\mathbf{z}_v)| \le C\, |||(\mathbf{v}-\mathbf{v}_h, p-p_h)|||\, \left\| h_{\mathcal{T}}^{\frac{1}{2}} \mathbf{z}_v \right\|_1 .$$

The last term of (71) is handled by using boundedness of the local projection operator and the stability of the projection estimates, i.e.,

$$(c) = S_p(p, I_h z_p) - S_p(p-p_h, I_h z_p), \tag{73}$$

$$\begin{aligned}|S_p(p, I_h z_p)| &\le \left( \sum_{a\in\mathcal{M}_a} \beta_a \left\| \kappa_a \nabla p \right\|^2_{\mathrm{L}^2(\mathcal{M}_\mathrm{a})} \right)^{\frac{1}{2}} S_p(I_h z_p, I_h z_p)^{1/2} \\ &\le C \left\| h_{\mathcal{T}} p \right\|_2 \left\| h_{\mathcal{T}} z_p \right\|_2 .\end{aligned}$$

$$|S_p(p-p_h, I_h z_p)| \le C\, |||(\mathbf{v}-\mathbf{v}_h, p-p_h)|||\, \left\| h_{\mathcal{T}} z_p \right\|_2 .$$

The proof concludes by combining the above estimates with the estimate (68), and the assumed regularizing behavior (70).

## 5. Numerical Results

In this section, an array of numerical results is presented to illustrate the derived theoretical estimates. A hierarchy of uniformly-refined triangular meshes having 16, 64, 256, 1024, and 4096 elements is used in all examples. The initial and uniformly-refined mesh is shown in Figure 2.

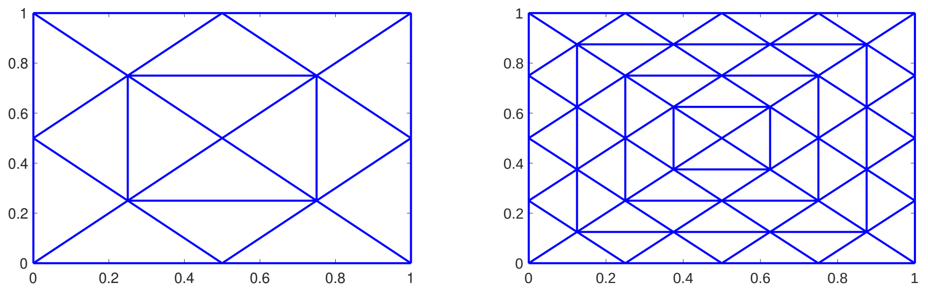



Figure 2: Triangulations used for computations in section 5

### 5.1. *Stokes flow problem*

Consider the model problem (1) in $\Omega = (0,1)^2$ with a given exact solution

$$\mathbf{u}(x,y) = (-\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y),\ \sin(2\pi x)\cos(2\pi y) - \sin(2\pi x))$$
$$p(x,y) = 2\pi(\cos(2\pi y) - \cos(2\pi x)).$$

The stabilization parameters for the discrete variational formulation (12) are chosen as $\mu_a = \mu h_a^2$ with $\mu = 1$ and $\zeta = 2$. The equal-order interpolation spaces $\mathbf{P}_1^c/\mathbf{P}_1^c$ are used to approximate the velocity and pressure approximation. The GLPS formulation (11) suppressed the spurious oscillations in the discrete solutions and succeeded in dealing with the incompatibility of discrete spaces. Figure 3 displays the $\mathbf{P}_1^c/\mathbf{P}_1^c$ stabilized solution for the mesh-size 0.0078. The errors are computed in $\mathrm{L}^2$- norm, $\mathrm{H}^1$-seminorm, and $|||\cdot|||$ stabilized norm. The quantitative and qualitative errors and the order of convergence obtained with $\mathbf{P}_1^c/\mathbf{P}_1^c$ finite element approximations are summarized in Table 1, Table 4 , and in the last plot of Figure 3. Expected convergence rates, *i.e.*, second-order $\mathrm{L}^2$-errors in velocity and pressure and first-order $\mathrm{H}^1-$approximation error in velocity, are obtained.

Table 1: Stokes problem: Errors and convergence orders.

| Mesh-size | $\|\mathbf{u}-\mathbf{u}_h\|$ | Order | $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$ | Order | $\|p-p_h\|$ | Order |
|---|---|---|---|---|---|---|
| 1/16 | 0.2015 | - | 1.9160 | - | 1.5225 | - |
| 1/32 | 0.0403 | 2.3207 | 0.7124 | 1.4273 | 0.3187 | 2.2561 |
| 1/64 | 0.0085 | 2.2394 | 0.3204 | 1.1530 | 0.0514 | 2.6330 |
| 1/128 | 0.0021 | 2.0300 | 0.1578 | 1.0215 | 0.0120 | 2.0936 |

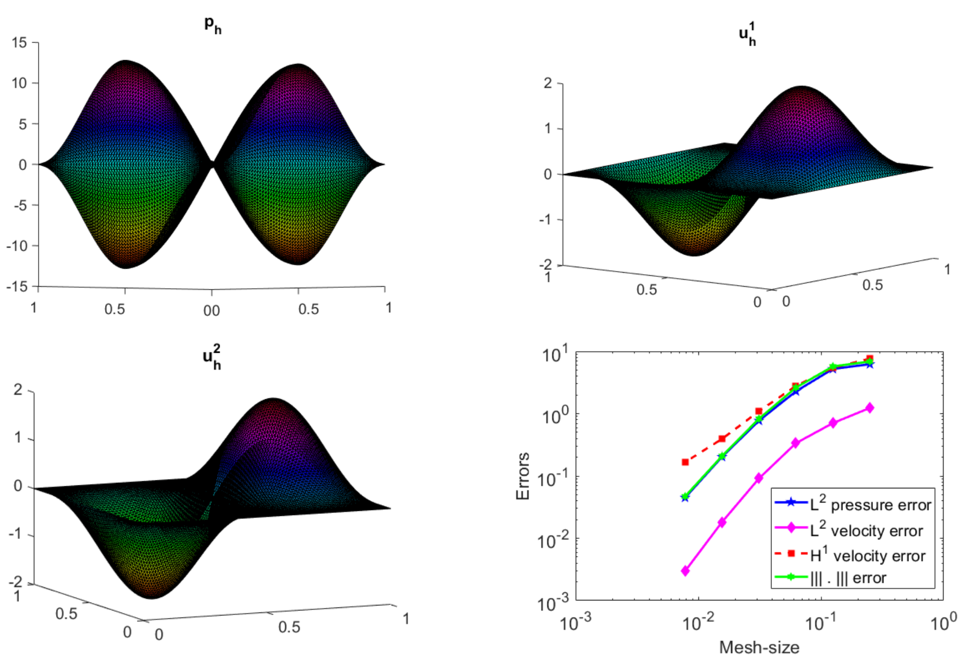



Figure 3: GLPS discrete solution $(\mathbf{u}_h, p_h)$ and convergence plot of the Stokes problem.

*5.2. The Darcy flow problem*

To demonstrate the robustness of the method, we consider the Darcy flow problem as the second numerical test example. Consider the model problem (47) in $\Omega = (0,1)^2$ with a given exact solution:

$$\mathbf{u}(x,y) = (-\pi \sin(2\pi y)\sin^2(\pi x),\ \pi \sin(2\pi x)\sin^2(\pi y)) \quad \text{and} \quad p(x,y) = \sin(2\pi x)\sin(2\pi y),$$

and the stabilization parameters $\beta_a = \beta h_a$, $\beta = 1$. The solution is approximated with the equal-order interpolation spaces $\mathbf{P}_1^c/\mathbf{P}_1^c$ using GLPS finite element formulation (49). Although the velocity and pressure approximation spaces are not *inf-sup* stable for the Darcy problem, the GLP stabilization effectively prevents the spurious oscillations. The effect of parameters $\varkappa$ and $\lambda$ on the rates of convergence are also investigated. Figure 4 shows the $\mathbf{P}_1^c/\mathbf{P}_1^c$ approximation with GLP stabilized finite element solutions for the mesh-size 0.0078 with $\omega = 1$. The computed errors with the $\mathrm{L}^2-$norm and $\mathrm{H}^1-$seminorm are presented in Tables 2 and 3 with ($\omega = 1$) and ($\omega = 0.1$), respectively, whereas Table 4 presents the errors measured in GLP stabilized norm as defined in (51). We can observe a second-order convergence in $\mathrm{L}^2$-norm, the first-order convergence in $\mathrm{H}^1$-seminorm and $\mathcal{O}(h^{3/2})$ convergence in $|\!|\!|\cdot|\!|\!|$. Also, Figure 5 shows the convergence behaviour of $\mathbf{P}_1^c/\mathbf{P}_1^c$ approximation of the Darcy equations with respect to $\mathrm{L}^2$-norm, $\mathrm{H}^1$-seminorm, and the GLP stabilized norm with ($\omega = 1$) and ($\omega = 0.1$) respectively. These numerical results support the estimates derived in the previous section.

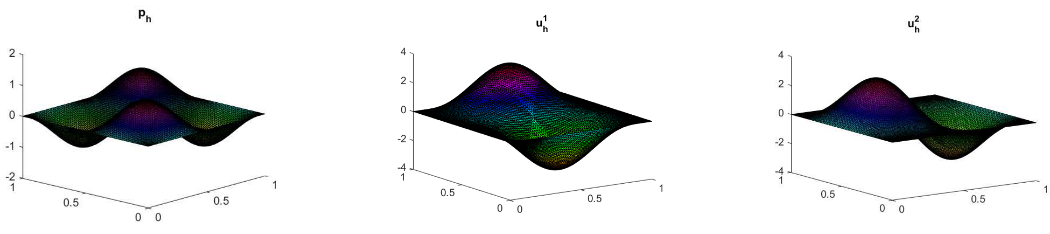


Figure 4: $\mathbf{P}_1^c/\mathbf{P}_1^c$ GLPS discrete solution $(\mathbf{u}_h, p_h)$ with ($\omega = 1,\ \beta = 1$).

Table 2: Darcy problem: Errors and convergence orders with $\omega = 1$.

| Mesh-size | $\|\mathbf{u}-\mathbf{u}_h\|$ | Order | $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$ | Order | $\|p-p_h\|$ | Order |
|---|---|---|---|---|---|---|
| 1/16 | 1.7949 | - | 13.1479 | - | 0.1040 | - |
| 1/32 | 0.5847 | 1.6182 | 5.1579 | 1.3500 | 0.0177 | 2.5549 |
| 1/64 | 0.1395 | 2.0669 | 1.8466 | 1.4819 | 0.0027 | 2.7185 |
| 1/128 | 0.0262 | 2.4128 | 0.5405 | 1.7724 | 0.0005 | 2.5674 |

## 6. Conclusions

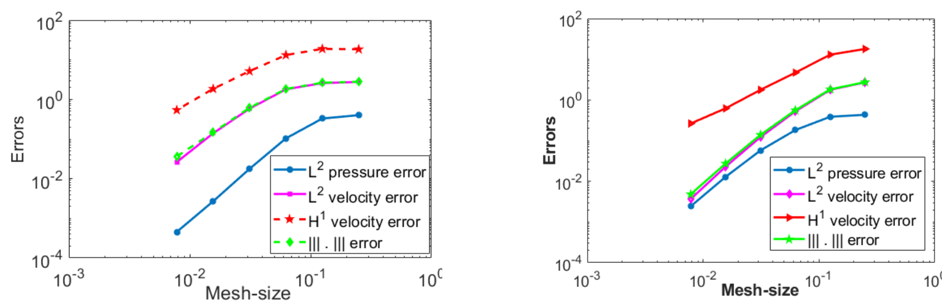
A generalized local projection stabilized (GLPS) conforming finite element scheme for the Stokes and the Darcy flow problems with equal-order interpolation spaces ($\mathbf{P}_1^c/\mathbf{P}_1^c$) is proposed and analyzed in this paper. GLPS allows to use of projection spaces on overlapping sets and avoids the need for a two-level mesh or an enrichment of finite element space. The partition of the unity of the basis functions together with the $\mathrm{L}^2$-orthogonal projection

Table 3: Darcy problem: Errors and convergence orders with $\omega = 0.1$.

| Mesh-size | $\|\mathbf{u} - \mathbf{u}_h\|$ | Order | $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|$ | Order | $\|p - p_h\|$ | Order |
|---|---|---|---|---|---|---|
| 1/16 | 0.5232 | - | 4.7062 | - | 0.1823 | - |
| 1/32 | 0.1223 | 2.0971 | 1.7727 | 1.4086 | 0.0576 | 1.6633 |
| 1/64 | 0.0229 | 2.4166 | 0.6214 | 1.5124 | 0.0129 | 2.1593 |
| 1/128 | 0.0037 | 2.6382 | 0.2639 | 1.2354 | 0.0024 | 2.4026 |

Table 4: Error and convergence orders with respect to $\|\!|\cdot|\!\|$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Darcy flow $\omega = 1$ | Mesh-size $h$ | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
| | $\|\!|\cdot|\!\|$ | 2.7966 | 2.6309 | 1.8345 | 0.6146 | 0.1470 | 0.0363 |
| | Order | - | 0.0881 | 0.5202 | 1.5777 | 2.0639 | 2.0164 |
| Darcy flow $\omega = 0.1$ | Mesh-size $h$ | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
| | $\|\!|\cdot|\!\|$ | 2.7398 | 1.8254 | 0.5594 | 0.1386 | 0.0273 | 0.0048 |
| | Order | - | 0.5859 | 1.7061 | 2.0132 | 2.3420 | 2.5091 |
| Stokes flow | Mesh-size $h$ | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
| | $\|\!|\cdot|\!\|$ | 6.7100 | 5.5089 | 1.6857 | 0.3463 | 0.0649 | 0.0183 |
| | Order | - | 0.2846 | 1.7084 | 2.2834 | 2.4157 | 1.8299 |




(a) (b)

Figure 5: Convergence plots of $\mathbb{P}_1^{nc}/\mathbb{P}_1^{nc}$ approximations with (a) ($\omega = 1,\ \beta = 1$) and (b) ($\omega = 0.1,\ \beta = 1$).

properties is used to derive the stability and convergence estimates. Further, a robust *a priori* error analysis is presented for both problems. An array of numerical experiments is presented to support the derived estimates and to demonstrate the proposed scheme's efficiency in suppressing oscillations without compromising the order of convergence.

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