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Weak precompactness in projective tensor products[☆]

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Abstract

We give a sufficient condition for a pair of Banach spaces (X, Y) to have the following property: whenever $W_1 \subseteq X$ and $W_2 \subseteq Y$ are sets such that $\{x \otimes y : x \in W_1,\ y \in W_2\}$ is weakly precompact in the projective tensor product $X \widehat{\otimes}_\pi Y$, then either W_1 or W_2 is relatively norm compact. For instance, such a property holds for the pair (ℓ_p, ℓ_q) if $1 < p, q < \infty$ satisfy $1/p + 1/q \geq 1$. Other examples are given that allow us to provide alternative proofs to some results on multiplication operators due to Saksman and Tylli. We also revisit, with more direct proofs, some known results about the embeddability of ℓ_1 into $X \widehat{\otimes}_\pi Y$ for arbitrary Banach spaces X and Y, in connection with the compactness of all operators from X to Y^*.

MSC: 46B28; 46B50

Keywords: Projective tensor product; ℓ_1-sequence; Weakly compact set; Weakly precompact set; Coarse p-limited set

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1. Introduction

Let $\mathcal{L}(X)$ denote the Banach space of all (bounded linear) operators on a Banach space X. Given $R, S \in \mathcal{L}(X)$ one can consider the multiplication operator

$$\Phi_{R,S} : \mathcal{L}(X) \to \mathcal{L}(X)$$

defined by

$$\Phi_{R,S}(T) := R \circ T \circ S \quad \text{for all } T \in \mathcal{L}(X).$$

Operator ideal properties of such multiplication operators have been widely studied in the literature. As to weak compactness, it is known that $\Phi_{R,S}$ is weakly compact whenever R is compact and S is weakly compact, or vice versa (see [23, Theorem 2.9]). In the other direction, it is not difficult to check that both R and S are weakly compact whenever they are non-zero and $\Phi_{R,S}$ is weakly compact, but in some cases one can say more. Akemann and Wright (see [1, Proposition 2.3]) proved that for $X = \ell_2$ the weak compactness of $\Phi_{R,S}$ implies that either R or S is compact. Later, Saksman and Tylli (see [23, Propositions 3.2 and 3.8]) showed that this property holds when X is a subspace of ℓ_p for $1 < p < \infty$ or X is the James space. In general, this is not true for arbitrary Banach spaces. See [14,18,19,23–25] for more information on this topic.

The previous circle of ideas is intimately related to weak compactness in projective tensor products. Let X and Y be Banach spaces, let $X\widehat{\otimes}_\pi Y$ be its projective tensor product and let $W_1 \subseteq X$ and $W_2 \subseteq Y$. Then the set

$$W_1 \otimes W_2 := \{x \otimes y : x \in W_1,\ y \in W_2\} \subseteq X\widehat{\otimes}_\pi Y$$

is relatively weakly compact whenever W_1 is relatively norm compact and W_2 is relatively weakly compact, or vice versa. In general, relative weak compactness of both W_1 and W_2 is not sufficient for $W_1 \otimes W_2$ to be relatively weakly compact, neither weakly precompact. Recall that a subset of a Banach space is said to be *weakly precompact* (or *conditionally weakly compact*) if every sequence in it admits a weakly Cauchy subsequence or, equivalently (thanks to Rosenthal's ℓ_1-theorem; see, e.g., [2, Theorem 10.2.1]), if the set is bounded and contains no *ℓ_1-sequence* (that is, a basic sequence which is equivalent to the usual basis of ℓ_1). For instance, if $1 < p, q < \infty$ satisfy $1/p+1/q \geq 1$, then the sequence $(e_n \otimes e'_n)_{n\in\mathbb{N}}$ in $\ell_p\widehat{\otimes}_\pi\ell_q$ is an ℓ_1-sequence, where we denote by $(e_n)_{n\in\mathbb{N}}$ and $(e'_n)_{n\in\mathbb{N}}$ the usual bases of ℓ_p and ℓ_q, respectively (see, e.g., the proof of [5, Proposition 3.6]). The following definition arises naturally:

Definition 1.1. *Let X and Y be Banach spaces. We say that the pair (X, Y) has* property (AW) *if whenever $W_1 \subseteq X$ and $W_2 \subseteq Y$ are sets such that $W_1 \otimes W_2$ is weakly precompact in $X\widehat{\otimes}_\pi Y$, then either W_1 or W_2 is relatively norm compact.*

Clearly, if X and Y are infinite-dimensional Banach spaces such that $X\widehat{\otimes}_\pi Y$ contains no subspace isomorphic to ℓ_1, then the pair (X, Y) fails property (AW) (the unit balls B_X and B_Y fail to be norm compact, while $B_X \otimes B_Y$ is weakly precompact in $X\widehat{\otimes}_\pi Y$). Such an example is given by (ℓ_p, ℓ_q) for $1 < p, q < \infty$ with $1/p + 1/q < 1$, because in this case $\ell_p\widehat{\otimes}_\pi\ell_q$ is reflexive (see, e.g., [22, Corollary 4.24]). In [20, Proposition 3.17] it is shown that the pair (X, Y) has property (AW) whenever X and Y are Banach spaces with unconditional finite-dimensional Schauder decompositions having a disjoint lower p-estimate and a disjoint lower q-estimate, respectively, where $1 < p, q < \infty$ satisfy $1/p + 1/q \geq 1$. In particular, for $1 < p, q < \infty$, the pair (ℓ_p, ℓ_q) has property (AW) if and only if $1/p + 1/q \geq 1$.

The aim of this paper is to go a bit further in the analysis of ℓ_1-sequences in projective tensor products of Banach spaces, property (AW) and its applications to multiplication operators. The paper is organized as follows.

In Section 2 we discuss the relationship between multiplication operators and projective tensor products, specially in connection with weak compactness. Some known results are included for the sake of completeness.

In Section 3 we focus on property (AW). The following property plays an important role in our discussion:

Definition 1.2. *Let X be a Banach space and $1 < p < \infty$. We say that X has* property (R_p) *if for every relatively weakly compact set $A \subseteq X$ which is not relatively norm compact there is an operator $u : X \to \ell_p$ such that $u(A)$ is not relatively norm compact.*

Obviously, ℓ_p has property (R_p) for every $1 < p < \infty$. This property is closely related to the class of coarse p-limited sets introduced in [12] and, in particular, it agrees with the so-called coarse p-Gelfand-Phillips property when $2 \leq p < \infty$ (see Remark 3.3). We prove that the pair (X, Y) has property (AW) whenever X and Y are Banach spaces having properties (R_p) and (R_q), respectively, where $1 < p, q < \infty$ satisfy $1/p + 1/q \geq 1$ (see Theorem 3.4). One of the possible approaches to the previous result sheds some more light on ℓ_1-sequences in this setting: under the same assumptions on X and Y, if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are weakly Cauchy sequences in X and Y, respectively, without norm convergent subsequences, then $(x_n \otimes y_n)_{n\in\mathbb{N}}$ admits an ℓ_1-subsequence in $X\widehat{\otimes}_\pi Y$ (see Theorem 3.8). As an application of Theorem 3.4 and some results of Knaust and Odell [17], we provide new proofs of the aforementioned results on multiplication operators of Saksman and Tylli (see Corollaries 3.16 and 3.17).

In Section 4 we include some complements about ℓ_1-sequences in projective tensor products and we provide more direct proofs of some known results about the embeddability of ℓ_1 into $X\widehat{\otimes}_\pi Y$ for arbitrary Banach spaces X and Y, due to Emmanuele [10] and Xue, Li and Bu [26]. Namely:

(i) If X and Y contain no subspace isomorphic to ℓ_1 and all operators from X to Y^* are compact, then $X\widehat{\otimes}_\pi Y$ contains no subspace isomorphic to ℓ_1, [10, Theorem 3] (see Theorem 4.4).

(ii) If $X\widehat{\otimes}_\pi Y$ contains no subspace isomorphic to ℓ_1 and either X or Y has an unconditional basis, then all operators from X to Y^* are compact, [10, Corollary 6] and [26, Theorem 4] (see Theorem 4.6).

Terminology

We work with real Banach spaces. Let X be a Banach space. The norm of X is denoted by $\|\cdot\|_X$ or simply $\|\cdot\|$. The topological dual of X is denoted by X^*. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by either $x^*(x)$ or $\langle x^*, x\rangle$. By a *subspace* of X we mean a norm closed linear subspace. Given a set $C \subseteq X$, its closed convex hull and its closed linear span (i.e., the subspace of X generated by C) are denoted by $\overline{\text{conv}}(C)$ and $\overline{\text{span}}(C)$, respectively. The closed unit ball of X is $B_X = \{x \in X : \|x\| \leq 1\}$. Given two sets $C_1, C_2 \subseteq X$, its Minkowski sum is $C_1 + C_2 := \{x_1 + x_2 : x_1 \in C_1,\ x_2 \in C_2\}$. By an *operator* we mean a continuous linear map between Banach spaces. Given another Banach space Y, we denote by $\mathcal{L}(X, Y)$ the Banach space of all operators from X to Y, equipped with the operator norm (when $X = Y$ we just write $\mathcal{L}(X)$ instead). As usual, we denote by $T^* \in \mathcal{L}(Y^*, X^*)$ the adjoint of $T \in \mathcal{L}(X, Y)$.

We denote by $\mathcal{B}(X,Y)$ the Banach space of all continuous bilinear functionals on $X\times Y$, with the norm $\|S\|_{\mathcal{B}(X,Y)}=\sup\{|S(x,y)|:\, x\in B_X,\ y\in B_Y\}$. Observe that the spaces $\mathcal{B}(X,Y)$, $\mathcal{L}(X,Y^*)$ and $\mathcal{L}(Y,X^*)$ are isometric in the natural way. Each $S\in\mathcal{B}(X,Y)$ induces a linear functional $\tilde{S}$ in the algebraic tensor product $X\otimes Y$. The *projective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of $X\otimes Y$ when equipped with the norm

$$\|z\|=\sup\{|\tilde{S}(z)|:\, S\in\mathcal{B}(X,Y),\ \|S\|\le 1\},\quad z\in X\otimes Y.$$

Hence, each $S\in\mathcal{B}(X,Y)$ induces an element of $(X\widehat{\otimes}_\pi Y)^*$ (namely, the continuous linear extension of $\tilde{S}$ to $X\widehat{\otimes}_\pi Y$). In fact, this gives an onto isometry between $\mathcal{B}(X,Y)$ and $(X\widehat{\otimes}_\pi Y)^*$ (see, e.g., [22, Section 2.2]). In the sequel we will identify the spaces $(X\widehat{\otimes}_\pi Y)^*$, $\mathcal{B}(X,Y)$, $\mathcal{L}(X,Y^*)$ and $\mathcal{L}(Y,X^*)$ via that isometry.

2. Multiplication operators and tensor products

In this preliminary section we discuss the relationship between multiplication operators and tensor products of operators, in connection with weak compactness. While most of the results are already known, we include their proofs, which can help readers to focus on the subject.

Lemma 2.1. *Let X_1 and X_2 be Banach spaces and let $C_1\subseteq X_1$ and $C_2\subseteq X_2$. The following statements hold:*

(i) If $C_1\otimes C_2$ is relatively weakly compact in $X_1\widehat{\otimes}_\pi X_2$, then both C_1 and C_2 are relatively weakly compact provided that they are not equal to $\{0\}$. The same holds if relative weak compactness is replaced by weak precompactness.

(ii) If C_1 is relatively norm compact and C_2 is relatively weakly compact (resp., weakly precompact), then $C_1\otimes C_2$ is relatively weakly compact (resp., weakly precompact) in $X_1\widehat{\otimes}_\pi X_2$.

Proof. (i) Fix $x_i\in C_i\setminus\{0\}$ for $i\in\{1,2\}$ and consider the isomorphic embeddings

$$\iota_1: X_1\to X_1\widehat{\otimes}_\pi X_2 \quad\text{and}\quad \iota_2: X_2\to X_1\widehat{\otimes}_\pi X_2$$

given by $\iota_1(x):=x\otimes x_2$ for all $x\in X_1$ and $\iota_2(y):=x_1\otimes y$ for all $y\in X_2$. Since both $\iota_1(C_1)$ and $\iota_2(C_2)$ are contained in $C_1\otimes C_2$, the conclusion follows at once.

(ii) Suppose that C_2 is relatively weakly compact. We can assume without loss of generality that $C_1\subseteq B_{X_1}$ and $C_2\subseteq B_{X_2}$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in C_1 and C_2, respectively. By passing to subsequences, we can assume that $(x_n)_{n\in\mathbb{N}}$ is norm convergent to some $x\in X_1$ and that $(y_n)_{n\in\mathbb{N}}$ is weakly convergent to some $y\in X_2$. Given any $T\in\mathcal{L}(X,Y^*)$, for each $n\in\mathbb{N}$ we have

$$\begin{aligned}\big|\langle T,x_n\otimes y_n\rangle-\langle T,x\otimes y\rangle\big| &\le \big|\langle T(x_n)-T(x),y_n\rangle\big|+\big|\langle T(x),y_n-y\rangle\big|\\ &\le \|T\|\|x_n-x\|+\big|\langle T(x),y_n-y\rangle\big|\end{aligned}$$

and so $\langle T,x_n\otimes y_n\rangle\to\langle T,x\otimes y\rangle$ as $n\to\infty$. This shows that $(x_n\otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent to $x\otimes y$ in $X_1\widehat{\otimes}_\pi X_2$. The proof that $C_1\otimes C_2$ is weakly precompact when C_2 is weakly precompact is similar. □

Remark 2.2. In the setting of Lemma 2.1, a similar argument shows that, if either X_1 or X_2 has the Dunford–Pettis and both C_1 and C_2 are relatively weakly compact, then $C_1\otimes C_2$ is relatively weakly compact in $X_1\widehat{\otimes}_\pi X_2$ (which reproves a result of J. Diestel, see [8, Theorem 16]).

Given two operators $T_1 : Y_1 \to X_1$ and $T_2 : Y_2 \to X_2$, where Y_1, Y_2, X_1 and X_2 are Banach spaces, the *projective tensor product* of T_1 and T_2 is the unique operator

$$T_1 \otimes T_2 : Y_1 \widehat{\otimes}_\pi Y_2 \to X_1 \widehat{\otimes}_\pi X_2$$

satisfying

$$(T_1 \otimes T_2)(y_1 \otimes y_2) = T_1(y_1) \otimes T_2(y_2)$$

for every $y_1 \in Y_1$ and for every $y_2 \in Y_2$ (see [22, Proposition 2.3] for details).

Lemma 2.3. *Let Y_1, Y_2, X_1 and X_2 be Banach spaces and let $T_1 : Y_1 \to X_1$ and $T_2 : Y_2 \to X_2$ be operators. Then $T_1 \otimes T_2$ is weakly compact if and only if $T_1(B_{Y_1}) \otimes T_2(B_{Y_2})$ is relatively weakly compact in $X_1 \widehat{\otimes}_\pi X_2$.*

Proof. We have $B_{Y_1 \widehat{\otimes}_\pi Y_2} = \overline{\mathrm{conv}}(B_{Y_1} \otimes B_{Y_2})$ (see, e.g., [22, Proposition 2.2]) and therefore the set $W := T_1(B_{Y_1}) \otimes T_2(B_{Y_2}) = (T_1 \otimes T_2)(B_{Y_1} \otimes B_{Y_2})$ satisfies

$$\overline{\mathrm{conv}}(W) = \overline{(T_1 \otimes T_2)(B_{Y_1 \widehat{\otimes}_\pi Y_2})}.$$

The conclusion now follows from the Krein–Šmulyan theorem asserting that the convex hull of a relatively weakly compact subset of an arbitrary Banach space is relatively weakly compact as well (see, e.g., [9, p. 51, Theorem 11]). □

Proposition 2.4. *Let X, X_1, Y and Y_1 be Banach spaces and let $S : X_1 \to X$ and $R : Y \to Y_1$ be operators. Let us consider the operator*

$$\Phi_{R^{**},S} : \mathcal{L}(X, Y^{**}) \to \mathcal{L}(X_1, Y_1^{**})$$

defined by

$$\Phi_{R^{**},S}(T) := R^{**} \circ T \circ S \quad \text{for all } T \in \mathcal{L}(X, Y^{**}).$$

Then:

(i) $\Phi_{R^{**},S} = (S \otimes R^*)^*$, where as usual we identify $\mathcal{L}(X, Y^{**}) = (X \widehat{\otimes}_\pi Y^*)^*$ and $\mathcal{L}(X_1, Y_1^{**}) = (X_1 \widehat{\otimes}_\pi Y_1^*)^*$.

(ii) $\Phi_{R^{**},S}$ is weakly compact if and only if $S \otimes R^*$ is weakly compact. In this case, the operator

$$\Phi_{R,S} : \mathcal{L}(X, Y) \to \mathcal{L}(X_1, Y_1)$$

defined by

$$\Phi_{R,S}(T) := R \circ T \circ S \quad \text{for all } T \in \mathcal{L}(X, Y)$$

is weakly compact and if, in addition, both S and R are non-zero, then they are weakly compact as well.

Proof. (i) Fix $T \in \mathcal{L}(X, Y^{**})$. Then $(S \otimes R^*)^*(T) = T \circ (S \otimes R^*)$. Given arbitrary $x \in X_1$ and $y^* \in Y_1^*$, we have

$$\begin{aligned} &\big\langle (S \otimes R^*)^*(T), x \otimes y^* \big\rangle = \big(T \circ (S \otimes R^*)\big)(x \otimes y^*) = T\big(S(x) \otimes R^*(y^*)\big) \\ &= \big\langle T(S(x)), R^*(y^*) \big\rangle = \big\langle R^{**}(T(S(x))), y^* \big\rangle = \langle \Phi_{R^{**},S}(T), x \otimes y^* \rangle. \end{aligned}$$

Hence, $(S \otimes R^*)^*(T) = \Phi_{R^{**},S}(T)$.

(ii) The first statement follows from Gantmacher's theorem and (i). Note that the weak compactness of $S \otimes R^*$ is equivalent to the relative weak compactness of the set $S(B_{X_1}) \otimes R^*(B_{Y_1^*})$ in $X_1 \widehat{\otimes}_\pi Y_1^*$ (see Lemma 2.3). Therefore, the second statement is a consequence of Lemma 2.1(i) and the fact that $\Phi_{R,S}$ is the restriction of $\Phi_{R^{**},S}$ to $\mathcal{L}(X,Y)$ as a subspace of $\mathcal{L}(X,Y^{**})$. □

The following result (in slightly less generality) was first proved in [23, Theorem 2.9]. See [18,19,24,25] for other proofs. Our approach is close to that of [19, Proposition 1] and [25, Proposition 2.3(ii)].

Corollary 2.5. Let S and R be as in *Proposition* 2.4. Suppose that either (i) S is compact and R is weakly compact or (ii) S is weakly compact and R is compact. Then $\Phi_{R^{**},S}$ is weakly compact.

Proof. We just prove case (ii) as the other one is similar. Since R is compact, Schauder's theorem ensures that R^* is compact too. Hence, $S(B_{X_1})$ is relatively weakly compact in X and $R^*(B_{Y_1^*})$ is relatively norm compact in Y^*. Then $S(B_{X_1}) \otimes R^*(B_{Y_1^*})$ is relatively weakly compact in $X \widehat{\otimes}_\pi Y^*$ (see Lemma 2.1(ii)) and so Lemma 2.3 applies to deduce that $S \otimes R^*$ is a weakly compact operator. The conclusion now follows from Proposition 2.4(ii). □

Observe that the previous arguments and Remark 2.2 also lead to the next result going back to [19, Proposition 2]:

Remark 2.6. Let S and R be as in Proposition 2.4. Suppose that X or Y^* has the Dunford–Pettis property. If both S and R are weakly compact, then $\Phi_{R^{**},S}$ is weakly compact.

2.1. An observation on the Davis-Figiel-Johnson-Pełczyński factorization

Let us recall the remarkable procedure that Davis, Figiel, Johnson and Pełczyński invented in [7]. Let X be a Banach space and let $W \subseteq X$ be an absolutely convex bounded set. For each $n \in \mathbb{N}$, denote by $|\cdot|_n$ the Minkowski functional of the absolutely convex bounded set $W_n := 2^n W + 2^{-n} B_X \subseteq X$, that is,

$$|x|_n := \inf\{t > 0 : x \in tW_n\} \quad \text{for all } x \in X.$$

Then $X_W := \{x \in X : \sum_{n=1}^\infty |x|_n^2 < \infty\}$ is a linear subspace of X which becomes a Banach space when equipped with the norm

$$\|x\|_{X_W} := \left(\sum_{n=1}^\infty |x|_n^2\right)^{1/2}.$$

The identity map $J_W : X_W \to X$ is an operator and $W \subseteq J_W(B_{X_W})$. Moreover, the space X_W is reflexive if and only if W is relatively weakly compact. The operator J_W will be called the *DFJP operator associated to* W. The reader can find in [3, Section 5.2] the basics on this topic.

The absolutely convex hull (resp., closed absolutely convex hull) of a subset C of a Banach space will be denoted by $\mathrm{aconv}(C)$ (resp., $\overline{\mathrm{aconv}}(C)$).

Proposition 2.7. *Let X_1 and X_2 be Banach spaces. For each $i \in \{1,2\}$, let $C_i \subseteq X_i$ be a bounded set and let $T_i : Y_i \to X_i$ be the DFJP operator associated to $W_i := \mathrm{aconv}(C_i) \subseteq X_i$. Then $T_1 \otimes T_2$ is weakly compact if and only if $C_1 \otimes C_2$ is relatively weakly compact in $X_1 \widehat{\otimes}_\pi X_2$.*

Proof. Since

$$(T_1 \otimes T_2)(B_{Y_1} \otimes B_{Y_2}) = T_1(B_{Y_1}) \otimes T_2(B_{Y_2}) \supseteq W_1 \otimes W_2 \supseteq C_1 \otimes C_2,$$

the set $C_1 \otimes C_2$ is relatively weakly compact in $X_1 \widehat{\otimes}_\pi X_2$ whenever $T_1 \otimes T_2$ is weakly compact.

To prove the converse, let us assume that $C_1 \otimes C_2$ is relatively weakly compact in $X_1 \widehat{\otimes}_\pi X_2$. By the Krein–Šmulyan theorem (see, e.g., [9, p. 51, Theorem 11]), its absolutely convex hull $\mathrm{aconv}(C_1 \otimes C_2)$ is relatively weakly compact in $X_1 \widehat{\otimes}_\pi X_2$ and, hence, the same holds for $W_1 \otimes W_2 \subseteq \mathrm{aconv}(C_1 \otimes C_2)$. The statement is obvious if either W_1 or W_2 equals to $\{0\}$, so we assume that this is not the case. Then W_1 and W_2 are relatively weakly compact (see Lemma 2.1(i)).

Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ large enough such that $2^{-m} \le \varepsilon$ and define $n := 2^m$. Then

$$T_i(B_{Y_i}) \subseteq n\overline{W_i} + \varepsilon B_{X_i} \quad \text{for } i \in \{1, 2\}. \tag{2.1}$$

Indeed, fix $y \in B_{Y_i}$ and take $t > 1 \ge \|y\|_{Y_i}$. By the very definition of the norm of Y_i, we have

$$T_i(y) \in t(2^m W_i + 2^{-m} B_{X_i}) \subseteq t(n\overline{W_i} + \varepsilon B_{X_i}) \subseteq t(n\overline{W_i} + \varepsilon B_{X_i^{**}}) \subseteq X_i^{**}.$$

Since $\overline{W_i}$ is weakly compact in X_i, the set $n\overline{W_i} + \varepsilon B_{X_i^{**}}$ is w^*-closed in X_i^{**} and so, since $t > 1$ is arbitrary, we conclude that

$$T_i(y) \in (n\overline{W_i} + \varepsilon B_{X_i^{**}}) \cap X_i = n\overline{W_i} + \varepsilon B_{X_i},$$

as required. This finishes the proof of inclusion (2.1).

Writing $\rho_i := \|T_i\|^{-1}$, we have

$$H_i := \rho_i T_i(B_{Y_i}) \subseteq \rho_i n\overline{W_i} + \rho_i \varepsilon B_{X_i} \quad \text{for } i \in \{1, 2\}.$$

Since $H_i \subseteq B_{X_i}$, we can apply [6, Lemma 3.10] to deduce that the set

$$U_i := \frac{\rho_i n}{1 + \rho_i \varepsilon} \overline{W_i} \cap B_{X_i}$$

satisfies

$$H_i \subseteq U_i + \frac{2\rho_i \varepsilon}{1 + \rho_i \varepsilon} B_{X_i} \quad \text{for } i \in \{1, 2\}.$$

It follows that

$$T_1(B_{Y_1}) \otimes T_2(B_{Y_2}) \subseteq V + f(\varepsilon) B_{X_1 \widehat{\otimes}_\pi X_2}, \tag{2.2}$$

where

$$V := \rho_1^{-1} U_1 \otimes \rho_2^{-1} U_2 \subseteq X_1 \widehat{\otimes}_\pi X_2$$

and

$$f(\varepsilon) := 2\varepsilon \left(\frac{1}{1 + \rho_1 \varepsilon} + \frac{1}{1 + \rho_2 \varepsilon} + \left(\frac{1}{1 + \rho_1 \varepsilon} \right) \left(\frac{2\varepsilon}{1 + \rho_2 \varepsilon} \right) \right).$$

Writing $\theta_i := n(1 + \rho_i \varepsilon)^{-1}$ for $i \in \{1, 2\}$, we have

$$V \subseteq \theta_1 \overline{W_1} \otimes \theta_2 \overline{W_2} = \theta_1 \theta_2 \big(\overline{W_1} \otimes \overline{W_2}\big) \subseteq \theta_1 \theta_2 \overline{W_1 \otimes W_2} \subseteq \theta_1 \theta_2 \overline{\mathrm{aconv}}(C_1 \otimes C_2),$$

and so (2.2) yields

$$T_1(B_{Y_1}) \otimes T_2(B_{Y_2}) \subseteq \theta_1 \theta_2 \overline{\mathrm{aconv}}(C_1 \otimes C_2) + f(\varepsilon) B_{X_1 \widehat{\otimes}_\pi X_2}.$$

Notice that $\theta_1\theta_2\overline{\mathrm{aconv}}(C_1 \otimes C_2)$ is weakly compact in $X_1\widehat{\otimes}_\pi X_2$ and that $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that $T_1(B_{Y_1}) \otimes T_2(B_{Y_2})$ is relatively weakly compact in $X_1\widehat{\otimes}_\pi X_2$ (see, e.g., [11, Lemma 13.32]). Finally, an appeal to Lemma 2.3 ensures that the operator $T_1 \otimes T_2$ is weakly compact. □

3. Property (AW)

The following notion was introduced in [12]:

Definition 3.1. *Let X be a Banach space and $1 < p < \infty$. A set $A \subseteq X$ is said to be* coarse p-limited *if $T(A)$ is relatively norm compact for every $T \in \mathcal{L}(X, \ell_p)$.*

Remark 3.2. Let X be a Banach space and $1 < p < \infty$. The following statements are equivalent:

(i) X has property (R_p) (see Definition 1.2), i.e., every coarse p-limited relatively weakly compact subset of X is relatively norm compact.
(ii) Every coarse p-limited weakly null sequence in X is norm null.
(iii) Every coarse p-limited weakly precompact subset of X is relatively norm compact.

Proof. The implications (iii)⇒(i)⇒(ii) are obvious. For (ii)⇒(iii), let $A \subseteq X$ be a coarse p-limited weakly precompact subset of X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A. By passing to a subsequence, we can assume that $(x_n)_{n\in\mathbb{N}}$ is weakly Cauchy. We claim that $(x_n)_{n\in\mathbb{N}}$ is norm Cauchy, which is enough to conclude (iii). Indeed, if this is not the case, then we can find $\varepsilon > 0$ and two subsequences $(x_{n_k})_{k\in\mathbb{N}}$ and $(x_{m_k})_{k\in\mathbb{N}}$ such that $\|x_{n_k} - x_{m_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. Note that for every $T \in \mathcal{L}(X, \ell_p)$ the set $T(A)$ is relatively norm compact in ℓ_p and so the same holds for

$$\{T(x_{n_k} - x_{m_k}) : k \in \mathbb{N}\} \subseteq T(A) - T(A).$$

Hence, $(x_{n_k} - x_{m_k})_{k\in\mathbb{N}}$ is a coarse p-limited, weakly null but not norm null sequence, a contradiction. □

Remark 3.3. Let X be a Banach space and $2 \le p < \infty$. Then every coarse p-limited subset of X is weakly precompact (see [12, Proposition 3]). Consequently, X has property (R_p) if and only if it has the *coarse p-Gelfand-Phillips property* of [12], i.e., every coarse p-limited subset of X is relatively norm compact.

The following result provides a sufficient condition on a pair of Banach spaces to have property (AW) (see Definition 1.1):

Theorem 3.4. *Let X and Y be Banach spaces such that X has property (R_p) and Y has property (R_q) for some $1 < p, q < \infty$ satisfying $1/p + 1/q \ge 1$. Then the pair (X, Y) has property (AW).*

First proof of Theorem 3.4. By contradiction, suppose that there exist non relatively norm compact sets $W_1 \subseteq X$ and $W_2 \subseteq Y$ such that $W_1 \otimes W_2$ is weakly precompact in $X\widehat{\otimes}_\pi Y$. Then W_1 and W_2 are weakly precompact (see Lemma 2.1(i)). The assumptions on X and Y together with Remark 3.2 ensure the existence of operators $u : X \to \ell_p$ and $v : Y \to \ell_q$ such that $u(W_1)$ and $v(W_2)$ are not relatively norm compact. As we already mentioned in the introduction, the condition $1/p + 1/q \geq 1$ implies that the pair (ℓ_p, ℓ_q) has property (AW) and so $u(W_1) \otimes u(W_2)$ is not weakly precompact in $\ell_p \widehat{\otimes}_\pi \ell_q$. This is a contradiction, because $u \otimes v : X\widehat{\otimes}_\pi Y \to \ell_p \widehat{\otimes}_\pi \ell_q$ is an operator, $W_1 \otimes W_2$ is weakly precompact in $X\widehat{\otimes}_\pi Y$ and $u(W_1) \otimes u(W_2) = (u \otimes v)(W_1 \otimes W_2)$. □

Theorem 3.8 will provide a different approach to Theorem 3.4. It is convenient to introduce first some terminology. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space is said to be *seminormalized* if it is bounded and $\inf_{n\in\mathbb{N}} \|x_n\| > 0$.

Definition 3.5. *Let X be a Banach space and $1 < p < \infty$. We say that X has* property (P_p) *if every seminormalized weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X admits a basic subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is equivalent to the usual basis of ℓ_p and such that $\overline{\operatorname{span}}(\{x_{n_k} : k \in \mathbb{N}\})$ is complemented in X.*

The following fact is well-known (see, e.g., [2, Proposition 2.1.3]):

Proposition 3.6. *For every $1 < p < \infty$ the space ℓ_p has property (P_p).*

Remark 3.7. Let X be a Banach space and $1 < p < \infty$. If X has property (P_p), then it also has property (R_p).

Proof. It suffices to prove that any coarse p-limited weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X is norm null (see Remark 3.2). By contradiction, suppose this is not the case. Then $(x_n)_{n\in\mathbb{N}}$ admits a seminormalized subsequence and so there is a further subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is a basic sequence equivalent to the usual basis $(e_k)_{k\in\mathbb{N}}$ of ℓ_p and such that $X_0 := \overline{\operatorname{span}}(\{x_{n_k} : k \in \mathbb{N}\})$ is complemented in X. Let $T_0 : X_0 \to \ell_p$ be the isomorphism satisfying $T(x_{n_k}) = e_k$ for all $k \in \mathbb{N}$. Since X_0 is complemented in X, we can extend T_0 to some operator $T : X \to \ell_p$. But $\{T(x_{n_k}) : k \in \mathbb{N}\} = \{e_k : k \in \mathbb{N}\}$ is not relatively norm compact in ℓ_p, which contradicts the fact that $(x_n)_{n\in\mathbb{N}}$ is coarse p-limited. □

Theorem 3.8. *Let X and Y be Banach spaces such that X has property (R_p) and Y has property (R_q) for some $1 < p, q < \infty$ satisfying $1/p + 1/q \geq 1$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be weakly Cauchy sequences in X and Y, respectively, without norm convergent subsequences. Then $(x_n \otimes y_n)_{n\in\mathbb{N}}$ admits an ℓ_1-subsequence in $X\widehat{\otimes}_\pi Y$.*

Proof. The set $\{x_n : n \in \mathbb{N}\}$ is weakly precompact but fails to be norm relatively compact. Since X has property (R_p), there is an operator $u : X \to \ell_p$ such that $\{u(x_n) : n \in \mathbb{N}\}$ is not relatively norm compact in ℓ_p. By passing to a subsequence, we can assume that $(u(x_n))_{n\in\mathbb{N}}$ does not admit norm convergent subsequences. Note that $(u(x_n))_{n\in\mathbb{N}}$ is weakly Cauchy and so weakly convergent to some $z \in \ell_p$ (the space ℓ_p is weakly sequentially complete). Define $z_n := u(x_n) - z$ for all $n \in \mathbb{N}$. Since the weakly null sequence $(z_n)_{n\in\mathbb{N}}$ does not admit norm null subsequences, by passing to a further subsequence we can assume that $(z_n)_{n\in\mathbb{N}}$ is a basic sequence equivalent to the usual basis of ℓ_p and that $\overline{\operatorname{span}}(\{z_n : n \in \mathbb{N}\})$ is complemented in ℓ_p (see Proposition 3.6). In the same way, since the set $\{y_n : n \in \mathbb{N}\}$ is weakly precompact but fails to be norm relatively compact, property (R_q) of Y ensures the existence of an operator $v : Y \to \ell_q$, a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ and $w \in \ell_q$ such that the sequence $(w_k)_{k\in\mathbb{N}}$ defined by $w_k := v(y_{n_k}) - w$ for all $k \in \mathbb{N}$ is a basic sequence equivalent to the usual basis of ℓ_q and that $W := \overline{\operatorname{span}}(\{w_k : k \in \mathbb{N}\})$ is complemented in ℓ_q.

Define $Z := \overline{\operatorname{span}}(\{z_{n_k} : k \in \mathbb{N}\})$. Then $(z_{n_k} \otimes w_k)_{k\in\mathbb{N}}$ is an ℓ_1-sequence in $Z\widehat{\otimes}_\pi W$ (see, e.g., the proof of [5, Proposition 3.6]). Let $\iota_Z : Z \to \ell_p$ and $\iota_W : W \to \ell_q$ be the inclusion operators. Since Z and W are complemented in ℓ_p and ℓ_q, respectively, the operator $\iota_Z \otimes \iota_W : Z\widehat{\otimes}_\pi W \to \ell_p\widehat{\otimes}_\pi \ell_q$ is an isomorphism onto a (complemented) subspace of $\ell_p\widehat{\otimes}_\pi \ell_q$ (see, e.g., [22, Proposition 2.4]). Hence, $(z_{n_k} \otimes w_k)_{k\in\mathbb{N}}$ is also an ℓ_1-sequence in $\ell_p\widehat{\otimes}_\pi \ell_q$. Since

$$h_k := (u \otimes v)(x_{n_k} \otimes y_{n_k}) = u(x_{n_k}) \otimes v(y_{n_k}) = z_{n_k} \otimes w_k + \underbrace{z \otimes w_k + z_{n_k} \otimes w + z \otimes w}_{=:h'_k}$$

for all $k \in \mathbb{N}$ and the sequence $(h'_k)_{k\in\mathbb{N}}$ is weakly convergent (to $z \otimes w$) in $\ell_p\widehat{\otimes}_\pi \ell_q$ (bear in mind that both $(z_{n_k})_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$ are weakly null), we can apply Rosenthal's ℓ_1-theorem (see, e.g., [2, Theorem 10.2.1]) to infer that $(h_k)_{k\in\mathbb{N}}$ admits an ℓ_1-subsequence, say $(h_{k_j})_{j\in\mathbb{N}}$. Since $u \otimes v$ is an operator, it is not difficult to prove that $(x_{n_{k_j}} \otimes y_{n_{k_j}})_{j\in\mathbb{N}}$ is an ℓ_1-sequence in $X\widehat{\otimes}_\pi Y$. This finishes the proof. □

Theorem 3.8 provides an alternative proof of Theorem 3.4, as follows.

Second proof of Theorem 3.4. Let $W_1 \subseteq X$ and $W_2 \subseteq Y$ be sets such that $W_1 \otimes W_2$ is weakly precompact in $X\widehat{\otimes}_\pi Y$. By contradiction, suppose that W_1 and W_2 are not relatively norm compact. Since both W_1 and W_2 are weakly precompact (see Lemma 2.1(i)), there exist weakly Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in W_1 and $(y_n)_{n\in\mathbb{N}}$ in W_2 without norm convergent subsequences. By Theorem 3.8, $(x_n \otimes y_n)_{n\in\mathbb{N}}$ admits an ℓ_1-subsequence in $X\widehat{\otimes}_\pi Y$, which contradicts the weak precompactness of $W_1 \otimes W_2$. □

Let us obtain another consequence of Theorem 3.8 in the context of weakly null sequences in projective tensor products. Observe that the usual basis of ℓ_2 shows that, in general, if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are weakly null sequences in X and Y, respectively, the sequence $(x_n \otimes y_n)_{n\in\mathbb{N}}$ may fail to be weakly null in $X\widehat{\otimes}_\pi Y$. To the best of our knowledge, the following question is open in complete generality.

Question 3.9. *Let X and Y be Banach spaces. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be weakly null sequences in X and Y, respectively, such that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly convergent in $X\widehat{\otimes}_\pi Y$. Is $(x_n \otimes y_n)_{n\in\mathbb{N}}$ weakly null in $X\widehat{\otimes}_\pi Y$?*

Theorem 3.8 allows us to provide the following partial affirmative answer.

Corollary 3.10. Let X and Y be Banach spaces such that X has property (R_p) and Y has property (R_q) for some $1 < p, q < \infty$ satisfying $1/p + 1/q \geq 1$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be weakly null sequences in X and Y, respectively. Then $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly null if (and only if) it is weakly Cauchy in $X\widehat{\otimes}_\pi Y$.

Proof. Suppose that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly Cauchy. By Theorem 3.8, either $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ admits a norm null subsequence. Hence, $(x_n \otimes y_n)_{n\in\mathbb{N}}$ admits a weakly null subsequence (by the proof of Lemma 2.1(ii)). It follows that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly null. □

3.1. Some applications

In this subsection we combine Theorem 3.4 with a deep result of Knaust and Odell [17] (see Theorem 3.13) to get some results on multiplication operators due to Saksman and Tylli [23] (see Corollaries 3.16 and 3.17). We need to introduce some additional terminology.

Definition 3.11. *Let X be a Banach space and $1 < p < \infty$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to have an* upper p-estimate *if there is a constant $c > 0$ such that*

$$\left(\sum_{n=1}^{m} |a_n|^p\right)^{1/p} \geq c \left\| \sum_{n=1}^{m} a_n x_n \right\|$$

for every $m \in \mathbb{N}$ and for all $a_1, \ldots, a_m \in \mathbb{R}$.

Definition 3.12. *Let X be a Banach space and $1 < p < \infty$. We say that X has* property (S_p) *if every seminormalized weakly null sequence in X admits a subsequence having an upper p-estimate.*

The following result can be found in [17, Corollary 2]:

Theorem 3.13 (*Knaust–Odell*)**.** *Let X be a Banach space such that X has property (S_p) and X^* has property $(S_{p'})$, where $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$. The following statements hold:*

(i) If X^* contains no subspace isomorphic to ℓ_1, then X has property (P_p).
(ii) If X contains no subspace isomorphic to ℓ_1, then X^* has property $(P_{p'})$.

Corollary 3.14. Let X be a subspace of ℓ_p, where $1 < p < \infty$. Then:

(i) X has property (P_p).
(ii) X^* has property $(P_{p'})$, where $1 < p' < \infty$ satisfies $1/p + 1/p' = 1$.
(iii) The pair (X, X^*) has property (AW).

Proof. Note that ℓ_p (resp., $\ell_{p'}$) has property (P_p) (resp., $(P_{p'})$), see Proposition 3.6, which in turn implies property (S_p) (resp., $(S_{p'})$). Clearly, property (S_p) is inherited by subspaces, so X has property (S_p). Since quotients of reflexive Banach spaces having property $(S_{p'})$ also have property $(S_{p'})$, it follows that X^* has property $(S_{p'})$. Statements (i) and (ii) now follow at once from Theorem 3.13. Statement (iii) is a consequence of Theorem 3.4 and Remark 3.7. □

Corollary 3.15. Let J be the James space. Then:

(i) J and J^* have property (P_2).
(ii) The pair (J, J^*) has property (AW).

Proof. It is known that both J and J^* have property (S_2), see [4] and [13, Proposition 3.3], respectively. Now, we can argue as in the proof of Corollary 3.14. □

As an application we get the next result (see [23, Proposition 3.2]):

Corollary 3.16 (*Saksman–Tylli*)**.** Let X be a subspace of ℓ_p, where $1 < p < \infty$, and let $R, S \in \mathcal{L}(X)$. If the operator $\Phi_{R,S} : \mathcal{L}(X) \to \mathcal{L}(X)$ is weakly compact, then either R or S is compact.

Proof. The weak compactness of $\Phi_{R,S}$ is equivalent to the weak compactness of $S \otimes R^* \in \mathcal{L}(X\widehat{\otimes}_\pi X^*)$ (see Proposition 2.4), which in turn is equivalent to the relative weak compactness of $S(B_X) \otimes R^*(B_{X^*})$ in $X\widehat{\otimes}_\pi X^*$ (see Lemma 2.3). Since the pair (X, X^*) has property (AW) (see Corollary 3.14(iii)), either $S(B_X)$ or $R^*(B_{X^*})$ is relatively norm compact, that is, either S or R^* is a compact operator. In the second case, Schauder's theorem applies to conclude that R is compact as well. $\square$

Finally, the same argument applies to the following (see [23, Proposition 3.8]):

Corollary 3.17 (*Saksman–Tylli*)**.** Let J be the James space and let $R, S \in \mathcal{L}(J)$. If the operator $\Phi_{R,S} : \mathcal{L}(J) \to \mathcal{L}(J)$ is weakly compact, then either R or S is compact.

4. More on ℓ_1-sequences in projective tensor products

4.1. Basic tensors and unconditional bases

If a sequence in a Banach space fails to be weakly null, then it admits a subsequence which is an ℓ_1^+-*sequence*, in the following sense:

Definition 4.1. *Let X be a Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called an ℓ_1^+-*sequence *if the following equivalent statements hold:*

(i) $0 \notin \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$.
(ii) There is $x^* \in X^*$ such that $x^*(x_n) \geq 1$ for every $n \in \mathbb{N}$.
(iii) There is a constant $C > 0$ such that

$$\left\| \sum_{n=1}^{N} a_n x_n \right\| \geq C \sum_{n=1}^{N} a_n$$

for all $N \in \mathbb{N}$ and for all non-negative real numbers $a_1, \ldots, a_N$.

It is not difficult to check that a bounded unconditional basic sequence is an ℓ_1-sequence if and only if it is an ℓ_1^+-sequence. In the same spirit, we have:

Lemma 4.2. *Let X and Y be Banach spaces such that X has an unconditional basis. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded unconditional basis of X and let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in Y. Then the sequence $(x_n \otimes y_n)_{n\in\mathbb{N}}$ in $X\widehat{\otimes}_\pi Y$ is an ℓ_1-sequence if and only if it is an ℓ_1^+-sequence.*

Proof. Suppose that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is an ℓ_1^+-sequence and fix an operator $T : X \to Y^*$ such that $T(x_n)(y_n) \geq 1$ for all $n \in \mathbb{N}$ (we identify $(X\widehat{\otimes}_\pi Y)^*$ and $\mathcal{L}(X, Y^*)$ as usual). Fix $N \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$. Define an operator $G : X \to Y^*$ by

$$G(x) := \sum_{n=1}^{N} \text{sign}(\lambda_n) x_n^*(x) T(x_n) \quad \text{for all } x \in X,$$

where $(x_n^*)_{n\in\mathbb{N}}$ is the sequence in X^* of biorthogonal functionals associated to the basis $(x_n)_{n\in\mathbb{N}}$. Observe that for every $x \in X$ we have

$$\begin{aligned}\|G(x)\| = \left\| \sum_{n=1}^{N} \text{sign}(\lambda_n) x_n^*(x) T(x_n) \right\| &= \left\| T\left(\sum_{n=1}^{N} \text{sign}(\lambda_n) x_n^*(x) x_n \right) \right\| \\ &\leq \|T\| \left\| \sum_{n=1}^{N} \text{sign}(\lambda_n) x_n^*(x) x_n \right\| \\ &\leq \|T\| K_u \|x\|,\end{aligned}$$

where K_u stands for the unconditional basis constant of $(x_n)_{n\in\mathbb{N}}$. Hence, we have $\|G\| \le \|T\| K_u$ and therefore

$$\begin{aligned}\left\|\sum_{n=1}^{N} \lambda_n x_n \otimes y_n\right\| &\ge \frac{1}{\|T\|K_u}\sum_{n=1}^{N}\lambda_n G(x_n)(y_n)\\ &= \frac{1}{\|T\|K_u}\sum_{n=1}^{N}\lambda_n \mathrm{sign}(\lambda_n) T(x_n)(y_n)\\ &\ge \frac{1}{\|T\|K_u}\sum_{n=1}^{\infty}|\lambda_n|.\end{aligned}$$

This shows that the (bounded) sequence $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is an ℓ_1-sequence. $\square$

The following result provides another partial affirmative answer to Question 3.9. The additional assumption on one of the Banach spaces is weaker than property (P_p) (see Definition 3.5) and holds for c_0, all ℓ_p spaces with $1 \le p < \infty$ (see, e.g., [2, Proposition 2.1.3]) and all $L_p[0,1]$ spaces with $2 < p < \infty$, by a classical result of Kadec and Pełczyński (see [15], Theorem 2 and Corollaries 1 and 4).

Theorem 4.3. *Let X and Y be Banach spaces. Suppose that every seminormalized weakly null sequence in X admits an unconditional basic subsequence whose closed linear span is complemented in X. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in X and let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in Y. If the sequence $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is not weakly null in $X\widehat{\otimes}_\pi Y$, then it admits an ℓ_1-subsequence.*

Proof. Since any non weakly null sequence in a Banach space admits a subsequence which is an ℓ_1^+-sequence, we can assume that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is an ℓ_1^+-sequence. Observe that $(x_n)_{n\in\mathbb{N}}$ cannot be norm null and so it admits a seminormalized subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$. By the assumption on X, we can assume further that $(x_{n_k})_{k\in\mathbb{N}}$ is an unconditional basic sequence and that $X_0 := \overline{\mathrm{span}}(\{x_{n_k} : k \in \mathbb{N}\})$ is complemented in X. Hence, the operator $\iota_{X_0} \otimes \mathrm{id}_Y : X_0\widehat{\otimes}_\pi Y \to X\widehat{\otimes}_\pi Y$ is an isomorphism onto a (complemented) subspace of $X\widehat{\otimes}_\pi Y$, where $\iota_{X_0} : X_0 \to X$ is the inclusion operator and id_Y is the identity operator on Y (see, e.g., [22, Proposition 2.4]). Now, since $(x_{n_k} \otimes y_{n_k})_{k\in\mathbb{N}}$ is also an ℓ_1^+-sequence in $X_0\widehat{\otimes}_\pi Y$, we can apply Lemma 4.2 to conclude that $(x_{n_k} \otimes y_{n_k})_{k\in\mathbb{N}}$ is an ℓ_1-sequence in $X_0\widehat{\otimes}_\pi Y$, and so in $X\widehat{\otimes}_\pi Y$. $\square$

4.2. Embedding ℓ_1 into projective tensor products

Let X and Y be Banach spaces. The subspace of $\mathcal{L}(X, Y^*)$ (resp., $\mathcal{L}(Y, X^*)$) consisting of all compact operators from X to Y^* (resp., from Y to X^*) will be denoted by $\mathcal{K}(X, Y^*)$ (resp., $\mathcal{K}(Y, X^*)$). It is well-known (and not difficult to check) that $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ if and only if $\mathcal{L}(Y, X^*) = \mathcal{K}(Y, X^*)$. The reflexivity of $X\widehat{\otimes}_\pi Y$ is closely related to those equalities. Indeed, if both X and Y are reflexive and $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$, then $X\widehat{\otimes}_\pi Y$ is reflexive; conversely, if $X\widehat{\otimes}_\pi Y$ is reflexive and, in addition, either X or Y has the approximation property, then $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ (see, e.g., [22, Section 4.2]). It is an open problem whether the last statement holds without the approximation property assumption.

As we already mentioned in the introduction, in [10,26] one can find similar results where reflexivity is weakened to "not containing isomorphic copies of ℓ_1". The purpose of this subsection is to provide more direct proofs of those results.

Theorem 4.4. *Let X and Y be Banach spaces such that one of the following conditions holds:*

(i) either X or Y has the Dunford–Pettis property;
(ii) $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$.

Then:

(a) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are weakly Cauchy sequences in X and Y, respectively, then $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly Cauchy in $X\widehat{\otimes}_\pi Y$.
(b) If $W_1 \subseteq X$ and $W_2 \subseteq Y$ are weakly precompact sets, then $W_1 \otimes W_2$ is weakly precompact in $X\widehat{\otimes}_\pi Y$.
(c) If X and Y contain no subspace isomorphic to ℓ_1, then $X\widehat{\otimes}_\pi Y$ contains no subspace isomorphic to ℓ_1.

Proof (*a*)**.** Fix $T \in \mathcal{L}(X, Y^*)$. Note that for every $n, m \in \mathbb{N}$ we have

$$\begin{aligned}&\big|\langle T, x_n \otimes y_n\rangle - \langle T, x_m \otimes y_m\rangle\big| \\ &= \big|\langle T(x_n), y_n\rangle - \langle T(x_m), y_m\rangle\big| \le \big|\langle T(x_n - x_m), y_n\rangle\big| + \big|\langle T(x_m), y_m - y_n\rangle\big|.\end{aligned} \tag{4.1}$$

Suppose that T is compact. Then T is completely continuous, hence we have $\|T(x_n - x_m)\| \to 0$ and so $|\langle T(x_n - x_m), y_n\rangle| \to 0$ as $n, m \to \infty$. In addition, since the set $\{T(x_m) : m \in \mathbb{N}\} \subseteq Y^*$ is relatively norm compact and $y_m - y_n \to 0$ weakly in Y as $n, m \to \infty$, we have $|\langle T(x_m), y_m - y_n\rangle| \to 0$ as $n, m \to \infty$. From (4.1) it follows that $|\langle T, x_n \otimes y_n\rangle - \langle T, x_m \otimes y_m\rangle| \to 0$ as $n, m \to \infty$. This proves that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly Cauchy in $X\widehat{\otimes}_\pi Y$ when $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$.

If Y has the Dunford–Pettis property, then we have $|\langle T(x_n - x_m), y_n\rangle| \to 0$ as $n, m \to \infty$ (because $T(x_n - x_m) \to 0$ weakly in Y^* as $n, m \to \infty$ and $(y_n)_{n\in\mathbb{N}}$ is weakly Cauchy) and $|\langle T(x_m), y_m - y_n\rangle| \to 0$ as $n, m \to \infty$ (because $(T(x_m))_{m\in\mathbb{N}}$ is weakly Cauchy and $y_m - y_n \to 0$ weakly in Y as $n, m \to \infty$). Therefore, from (4.1) we get $|\langle T, x_n \otimes y_n\rangle - \langle T, x_m \otimes y_m\rangle| \to 0$ as $n, m \to \infty$. This proves that $(x_n \otimes y_n)_{n\in\mathbb{N}}$ is weakly Cauchy in $X\widehat{\otimes}_\pi Y$ when Y has the Dunford–Pettis property. By symmetry, the same holds whenever X has the Dunford–Pettis property.

(b) is immediate from (a).

(c) Note that $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y)$ (see, e.g., [22, Proposition 2.2]) and that the closed convex hull of a weakly precompact set in a Banach space is weakly precompact as well, according to a result of Stegall (see [21, Addendum]). The conclusion now follows from (b) and the fact that a Banach space contains no subspace isomorphic to ℓ_1 if and only if its closed unit ball is weakly precompact. □

The following result is implicit in the proof of [16, Theorem 6]. Recall that an *unconditional expansion of the identity* of a Banach space X is a sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(X)$ such that for each $x \in X$ we have $x = \sum_{n\in\mathbb{N}} P_n(x)$, the series being unconditionally convergent in X.

Proposition 4.5. *Let X and Y be Banach spaces and let $(P_n)_{n\in\mathbb{N}}$ be an unconditional expansion of the identity of X (resp., Y^*). If $X\widehat{\otimes}_\pi Y$ contains no complemented subspace isomorphic to ℓ_1, then for each $T \in \mathcal{L}(X, Y^*)$ we have $T = \sum_{n\in\mathbb{N}} T \circ P_n$ (resp., $T = \sum_{n\in\mathbb{N}} P_n \circ T$), the series being unconditionally convergent in $\mathcal{L}(X, Y^*)$.*

Proof. Bearing in mind the identification of $(X\widehat{\otimes}_\pi Y)^*$ and $\mathcal{L}(X, Y^*)$, together with the fact that a Banach space contains no complemented subspace isomorphic to ℓ_1 if and only if its dual contains no subspace isomorphic to c_0 (see, e.g., [3, Theorem 4.68]), the assumption turns out to be equivalent to the fact that $\mathcal{L}(X, Y^*)$ contains no subspace isomorphic to c_0.

Suppose that $(P_n)_{n\in\mathbb{N}}$ is an unconditional expansion of the identity of X (the other case is analogous) and fix $T \in \mathcal{L}(X, Y^*)$. Define $T_J := \sum_{n\in J} T \circ P_n$ for every finite set $J \subseteq \mathbb{N}$. For each $x \in X$ we have $\sup\{\|T_J(x)\|_{Y^*} : J \subseteq \mathbb{N} \text{ finite}\} < \infty$, because the series $\sum_{n\in\mathbb{N}} T(P_n(x))$ is unconditionally convergent in Y^* (with sum $T(x)$). By the uniform boundedness principle,

$$\sup\{\|T_J\|_{\mathcal{L}(X,Y^*)} : J \subseteq \mathbb{N} \text{ finite}\} < \infty.$$

This implies that $\sum_{n\in\mathbb{N}} T \circ P_n$ is a weakly unconditionally Cauchy series in $\mathcal{L}(X, Y^*)$, that is, for every $\varphi \in \mathcal{L}(X, Y^*)^*$ we have $\sum_{n\in\mathbb{N}} |\langle \varphi, T \circ P_n\rangle| < \infty$. Since $\mathcal{L}(X, Y^*)$ contains no subspace isomorphic to c_0, we conclude that the series $\sum_{n\in\mathbb{N}} T \circ P_n$ is unconditionally convergent in $\mathcal{L}(X, Y^*)$ (see, e.g., [3, Theorem 4.49]). Clearly, its sum equals T. $\square$

Clearly, if a Banach space X admits an unconditional basis or just an unconditional FDD (i.e., unconditional finite-dimensional decomposition), then there is an *unconditional finite-dimensional expansion of the identity* of X, that is, an unconditional expansion of the identity consisting of finite rank operators. Of course, this implies that X has the approximation property. As an immediate consequence of Proposition 4.5, we have:

Theorem 4.6. *Let X and Y be Banach spaces such that either X or Y^* admits an unconditional finite-dimensional expansion of the identity. If $X\widehat{\otimes}_\pi Y$ contains no complemented subspace isomorphic to ℓ_1, then $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$.*

The same argument yields the following:

Remark 4.7. Let X and Y be Banach spaces such that either X or Y^* admits an unconditional expansion of the identity consisting of elements of some norm closed operator ideal $\mathcal{A}$. If $X\widehat{\otimes}_\pi Y$ contains no complemented subspace isomorphic to ℓ_1, then all elements of $\mathcal{L}(X, Y^*)$ belong to $\mathcal{A}$.

We finish the paper with a question which is open to the best of our knowledge.

Question 4.8. *Let X and Y be Banach spaces such that $X\widehat{\otimes}_\pi Y$ contains no complemented subspace isomorphic to ℓ_1. Does the equality $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ hold? What if, in addition, either X or Y^* has the approximation property?*

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