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The Evolution from "I think it plus three" Towards "I think it is always plus three." Transition from Arithmetic Generalization to Algebraic Generalization

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Abstract
This paper is part of broader research being conducted in the area of algebraic thinking in primary education. Our general research objective was to identify and describe generalization of a 2nd grade student (aged 7–8). Specifically, we focused on the transition from arithmetic to algebraic generalization. The notion of structure and its continuity in the generalization process are important for this transition. We are presenting a case study with a semi-structured interview where we proposed a task of contextualized generalization involving the function $y=x+3$. Special attention was given to the structures evidenced and the type of generalization expressed by the student in the process. We noted that the student identified the correct structure for the task during the interview and that he evidenced a factual type of algebraic generalization. Due to the student's identification of the appropriate structure and the application of it to other different particular cases, we have observed a transition from arithmetic thinking to algebraic thinking.

Keywords Algebraic thinking · Functional thinking · Generalization · Structure

Introduction

The use of letters is neither a necessary, nor sufficient, condition for algebraic thinking (Radford, 2014). This statement aims to clarify the concept of algebra assumed in our study for primary education. Algebraic thinking does not

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necessarily entail using letters. Between the mathematical contents usually worked on in primary education and the treatment given to algebra in secondary education, there is an abrupt leap (Bednarzet al., 1996; Filloy et al., 2008). The traditional separation between arithmetic and algebra deprives students from powerful ways of thinking about mathematics in primary education which makes it more difficult for them to learn algebra in secondary education (Kieran, 1992). Algebra in school is relegated to symbolic language, removing the sense from the meaning of the symbolic notation used and evidencing the difficulties found by students during the transition from arithmetic to algebra in secondary education (Vergel & Rojas, 2018).

In this regard, *early algebra* emerges as a proposal for curricular change emphasizing the idea of Kaput (2000) on the "algebraization of the curriculum" of mathematics. Early algebra considers the introduction of modes of algebraic thinking from the early education levels to enhance mathematical reasoning and generalization, easing the difficulties students find when approaching algebra in higher grades (Kaput 2000). It is usual for most countries to include elements of this type of thinking in their primary education curricular plans (Morales et al., 2018).

Generalization is an essential process of mathematical reasoning. Considering it in the earliest grades allows, for example, for students to move away from the specificities of arithmetic calculation, based on the observation of structures or behavior patterns, in particular cases given, and the mathematical relations involved (Blanton et al., 2011). Thus, we can begin teaching young students algebra integrating algebraic thinking in school mathematics. The notions of generalization and structure are related since in the process of generalization the structure can be identified from particular cases. Structure is defined as the form in which the regularity between specific values of the variables involved is organized or the manner in which generalization is expressed (Torres et al., 2021).

This article provides elements to delve the category of algebraic thinking proposed by Radford (2010, 2014, 2018) through functional thinking as a subcategory of algebraic thinking. Functional thinking is understood as a type of algebraic thinking where the function is the key mathematical notion. Functional thinking is based on construction, description, representation, and reasoning with and about functions and the elements comprising them (Cañadas & Molina, 2016, p. 212). The identification of structures between variables that covary may lead to generalization processes, arithmetic or algebraic, which can be studied from the characterization suggested by Radford (2008, 2013). Specifically, in this approach, the generalization of structures and the representation of these structures are considered as key elements. Therefore, we assume that functional thinking involves evidencing structures, which can be generalized or not, and which can be expressed through different types of representations (Pinto et al., 2022).

There is growing consensus that algebraic reform requires reconceptualization of the nature of algebra and algebraic thinking, as well as an analysis of when children are capable of reasoning algebraically and when ideas that require algebraic reasoning should be introduced in the curriculum (Carpenter & Levi, 1999). In this sense, we sought to understand the development of algebraic thinking of primary education students towards generalization addressing regularities between the variables evidenced when conducting a generalization task involving a affine function.

There are studies that cover the reform of algebra in the context of primary education mathematics, focusing in particular on the development of algebraic thinking (Carpenter & Levi, 1999). This article presents the results of considering this transition between number and algebraic thinking. We described the process from arithmetic generalization to algebraic generalization in a functional context of school algebra, identifying the notion of structure as a key aspect in the process.

Algebraic Thinking and Functional Thinking

Algebra as a generalization activity has developed as a line of research (e.g. Bell, 1976; Mason & Pimm, 1984). Pattern generalization is considered one of the most significant ways to introduce algebra in the early grades (Radford, 2018; Vergel & Rojas, 2018). Among other aspects, it allows approaching variation situations in the classroom which are necessary to develop algebraic thinking. The approaches based on patterns to introduce algebra in primary education are founded on explorations of visual patterns, which are used to generate generalization expressions. With these patterns, students are required to consider a variation of a set of position-dependent data (that is, as a relation between consecutive terms within the pattern itself). In functional thinking, to get the general expression requires identifying two variables that can also be detected through particular cases. Note in this case, the absence of sequencing in the representation of the variables involved. When referring to generalization expression, both the domain and the route are numerical domains. From this standpoint, Radford (2010) recognized three distinct forms of algebraic thinking depending on the way students communicated their activity during the generalization process. These forms of algebraic thinking were factual thinking, contextual thinking, and symbolic thinking. In factual thinking, students communicate their thoughts through gestures, movement, perceptual activities, and words. At this level of thinking, unknown is implicit, and students work with concrete values, numbers (particular cases) (Radford, 2010). For example, a student indicates with a glance, a finger, pencil movements, or pointing. In studies on the use of verbal language, Mouhayar (2022) highlights the importance of the role of verbal language in the generalization of patterns. Vygotsky recognizes that words are accompanied by gestures, and these allow children to overcome the difficulties caused by verbal communication (Vygotsky, 1978). This author emphasizes the intimate relationship between written signs and gestures. Radford (2005) makes visible in the role of gestures the intentions of communicating some aspect of pattern sequences. According to this author:

> Gestures are indispensable elements as they help students to make their intentions visible, to notice mathematical relationships, and to become aware of conceptual aspects of mathematical objects (p. 3143).

In contextual thinking, gestures and words are replaced by key phrases. In this type of thinking, unknown is explicit, and algebraic formulation is a description of a general term. Finally, symbolic thinking is where key phrases are represented by algebra alphanumeric symbols. At this level, there is a drastic change in the way unknown is referred to (Radford, 2010). These types of algebraic thinking are classified according to how students can communicate. As pointed out by Radford (2010, 2018) and Vergel (2015), this means we need to recognize all those representations such as natural language (oral and written), gestures, and procedures which evidence that students are attempting to build explanations and arguments on general structures and modes of thinking; thus, their lines of argument and explanations are based on particular situations, or concrete actions. It is important to consider the cognitive, physical, and perceptual resources which students mobilize when working with mathematical ideas. These resources include symbolic and oral communications, as well as drawings, gestures, handling of materials, and body movement (Radford et al., 2009). In this sense, Warren et al. (2013), in a study focused on the functional thinking of 5- to 9-year-olds, observed that the use of gestures (by students and interviewers) helped students to look for generalizations and to express these generalities. It also appears that when students became aware of structure, the use of gestures and self-talk tended to decrease.

As mentioned, another way of approaching algebraic thinking is through functional thinking "focused on the relation between two (or more) quantities that vary; specifically, the types of thinking that range from specific relations to relation generalizations" (Kaput, 2008, p. 143). Our study focuses on the introduction of algebra through functions, not with patterns, that is, working with functional thinking as a way of approaching algebra. Within this mode of functional thinking, we are interested in observing generalization as a key element. Within this process, we specifically pay attention to the transition between arithmetic and algebraic generalization. Paying attention to this process will allow us to give more solid arguments to teachers on how to conduct their teaching. In other words, we are providing clues so that we can know when the student is moving from arithmetic generalization to algebraic generalization involving functional thinking. One of the notions involved in functional thinking is generalization of the relations between quantities that covary. Another important part of functional thinking is the expression of these (functional) relations using different representations and applying these expressions to analyze the behavior of a function (Blanton et al., 2011). Research with elementary graders shows that children have a lot more resources to reason about functions than was previously thought. Concerning functional thinking, there is evidence that these students are able to generalize co-varying relationships, identify functional relationships when two variables are involved, represent these relationships in different ways (including with variable notation), and reason with functional relationships to interpret problem situations (Blanton et al., 2015; Cañadas & Molina, 2016). It is not a matter of introducing functions at early education levels in the same way as they are treated in secondary education, but rather of taking advantage of the potential of these mathematical contents to promote skills in children that will be useful for reasoning in general and for mathematics in particular, both at their current school level and future ones (Cañadas & Molina, 2016, p. 8). Functional thinking, in the context of early algebra, also focuses on the relationship between two variables, the study of regularities in concrete or specific cases being essential (Blanton, 2008, p. 30). In general, regularity is that which is repeated. When we observe a regularity, we look for it to be valid for other particular cases and, eventually, for any case within a given situation (Pólya, 1966).

In the area of functional thinking, structure refers to the regularity present in the expression of the relation between the variables of a function. Structure corresponds to the way the elements of a regularity are organized between the variables and the existing relation between said elements (Kieran, 1989). The notion of structure we assume has to do with the terms comprising a functional algebraic expression, with the signs relating them, the order of different operations and the existing relations between the elements. Structure may be evidenced through different representations by students, either when working on particular cases or when generalizing (Pinto & Cañadas, 2017). Some researchers note that before generalizing, we can "see" the structure involved (Mason et al., 2009). Becoming aware of a structure and its stability by working with different particular cases could lead to generalization.

There are publications that are part of research projects in various countries, which emphasize the evidence that children in early grades can think in a rather more sophisticated manner than previously assumed (e.g. Kaput et al., 2008; Pinto & Cañadas, 2017; Radford, 2018; Torres et al., 2018). In particular, research evidences that students' thinking may be truly algebraic even when their production does not include algebra alphanumeric signs (Vergel, 2013).

There are studies that explore generalization of primary school students in functional contexts (e.g. Carraher & Schliemann, 2016; Pinto & Cañadas, 2017). There is numerous research evidencing algebraic and functional thinking of primary school and even preschool children. Specifically, Blanton and Kaput (2004) documented how children aged 6 to 10 can detect addition and multiplication relations among functional relation variables. Blanton et al (2015) evidenced the understanding of letters as variables by students aged 6 to 7; while Vergel (2013) described the variety and blended use of strategies by students of the same age when solving a problem based on a generic example.

Generalization is a key element for algebraic thinking in general and for functional thinking, in particular. We focused on exploring the process of generalization in a student addressing a task in a functional context.

Arithmetic Generalization and **Algebraic Generalization.**

Pólya (1945) took generalization as an inductive empirical activity in which concrete examples are accumulated, and regularity is detected and systematized. Work with particular cases is an essential step towards generalization (Cañadas & Castro, 2007). In Fig. 1, we present a five-step inductive reasoning model of Cañadas and Castro (2007) that emerged as part of research findings that describes work with 359 high school students who performed tasks that involved generalization. The authors emphasized that while the five steps are helpful in helping students progress towards generalization, the end goal may not necessarily be present, appear in the order shown, or have equal weight in inductive reasoning.

Generalization is a key step, while organizing specific cases can be helpful, but is not routinely present.

1. Work on particular cases. Concrete cases or examples initiating the pr
easily observed cases.

2. Identification of regularities. Regularity is that which is common, repeated in different facts or situations and expected to occur again.

3. Formulation of conjectures. A conjecture is a proposition assumed to be true but which has not been subject to examination. Said examination may result
given for which the conjecture is not valid, the latter is rejected. In Popper's terms (1967), the conjecture is refuted.

4. Justification of conjectures. This refers to any reason given to convince of the truth of a statement. We usually distinguish between empirical and deductive justification. Empirical uses examples as a means to convince. The validation of conjectures takes place with new particular cases (different from the previous ones), but not for the general one.

5. Generalization. The conjecture is expressed in such a way that it refers to all the cases of a given class. It implies the extension of reasoning beyond the particular cases studied.

Fig. 1 Inductive reasoning model by Cañadas and Castro (2007)

Another aspect in the generalization process is the work with unknowns which is one of the characteristics of algebra, understood as a method for operating on general forms (Radford, 2011, 2018; Vergel, 2015, Cañadas & Molina 2016). When observations on a given structure (regularities for Cañadas and Castro, 2007) are extended to more cases, generalization is achieved.

As Vergel (2019) points out, work on generalization forces us to specify on at least two types of generalization, which Radford (2008) has categorized as arithmetic generalization and algebraic generalization. For generalization to be algebraic, there must be a deduction [1] of an expression that allows us to calculate the value or image of any term in a sequence. However, in arithmetic generalization, there is no deduction, what there is, is an abduction of the common characteristic that is evidenced in the sequence to be able to move from one term to another (Radford, 2013).

The idea of deduction is key and serves as an operational criterion that allows distinguishing between arithmetic and algebraic thinking (Radford, 2008). Deduction is "all that which is necessarily concluded from other truths known with certainty" (Descartes, 1983, p. 125). For Pappus, "analysis is the movement from that which is given to that which is sought" (Rideout, 2008, p. 62). This is why for Viète, "what is distinctively algebraic (…) is the analytical way in which we think when we think algebraically" (Radford, 2018, p. 6).

As Radford puts it:

> In pattern generalization, an algebraic generalization entails deducing a formula from some terms of a given sequence. That the formula be expressed or not in alphanumeric symbolism is irrelevant. Notice that the fact that the

general term of the sequence be expressed in alphanumeric symbolism does not imply at all that the generalization is the result of thinking algebraically about the sequence 28 (Radford, 2018, p. 9)

Radford (2013, p. 6) put forward that algebraic generalization of patterns considers the following aspects: (a) awareness of a common property noted from work in the phenomenological area of observation of certain particular terms, (b) generalization of said property in the following cases of the sequence and, finally (c) the ability to use that common property in order to obtain a direct expression [2], a formula, that allows calculating the value for any term in the sequence. Within algebraic generalization, there is factual generalization, contextual generalization, and symbolic generalization (Radford, 2018). Factual generalization is the first way of generalizing in which perceptual activities based on students' various means of communication generate a calculation which helps advance abstraction of the particular. This type of algebraic generalization is, therefore, the start of algebraic generalization. It is based on actions conducted upon numbers; actions here consist of words, gestures, and perceptual activities. They are expressed in concrete actions through work upon numbers. It is generalization that allows addressing any particular case, it is the abstraction of concrete actions, that is, it always remains connected to the concrete level (Radford, 2010).

Contextual generalization is the abstraction of a specific action. It differs from factual generalization in that it does not involve dealing with specific numbers. To put it another way, in contextual generalization, indetermination is explicit. In symbolic generalization, it is the representation of sequences with alphanumeric algebraic symbols (Radford, 2010).

Finally, if the confirmed structure can be reaffirmed with undetermined cases or the general one, generalization is obtained. These bases rely on the models by Radford (2013) and the inductive reasoning model by Cañadas and Castro (2007) regarding algebraic generalization. Both consider awareness of a common property based on work with certain particular cases. We assume that generalization is the ability to use this common property to calculate the value of any following term.

Rivera (2017) and Torres et al. (2021) also address the generalization process, including three phases; abduction, induction, and generalization. The abduction phase is where hypotheses are formed that are not confirmed until we have other particular cases in the inductive phase, which is when students have needed to identify a structure in order to be able to continue with the process, as at this age they lack the tools, such as to clearly visualize the quantity, count or draw large quantities. This is where we see the possible confirmation of structures.

The discussion between patterns and functions is not the focus of our work, but we think it is important to emphasize here that in the functions two numerical sets between which there is a covariation relationship become evident.

Now, in this study, we work with tasks that imply functions in which a structure must be identified. In the functions, two numerical sets between which there is a covariation relationship become evident.

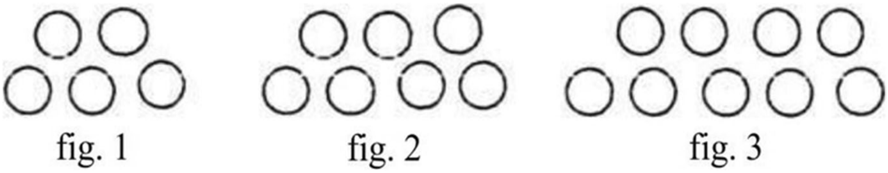

Fig. 2 Pattern task (Vergel, 2015)

Two superheroes, Iron Man and Captain America, have the same birthday.

When Iron Man turned 5, Captain America turned 9.

When Iron Man turned 7, Captain America turned 11…

Fig. 3 Generalization task on the age of superheroes

The idea of pattern is more linked to recurrence rather than to the establishment of a covariance relationship between two quantities (Torres et al., 2021). The study of the pattern and the structure is integrated in a broad range of studies on mathematical development in the early years of learning (Mulligan & Mitchelmore, 2009). For example, in the following task with patterns (see Fig. 1), the regularity can be identified by the recurrence of consecutive terms in a sequence. The position of each element and the sequence in which they appear are a key aspect in this type of tasks, distinguishing them from the generalization tasks we worked with in this study.

We noted the difference with the generalization task based on the age relation of two superheroes given by $y = x + 4$ (Vergel, 2019, Torres et al (2019). Figure 2 shows the introduction to the task (Fig. 3).

Identification of the structure requires a relation between two variables that can also be identified through particular cases. Note in this case the absence of sequencing in the representation of the variables involved. When referring to structure, both the domain and the route are numerical domains. The notions of generalization and structure are related and help characterize students' functional thinking. In general, structure can be identified based on particular cases. Radford (1996) stated that generalization from an educational perspective depends on the mathematical objects being generalized; generalization is not an activity devoid of context. In the case of functional thinking, generalization occurs when establishing and analyzing the relations between variables (Smith, 2003). To encourage generalization, we begin with situations involving particular cases and, observing structures, that is, identifying relations between variables, we aim to attain generalization. Our interest in this study is given by the need to understand what happens between arithmetic and algebraic generalization so that we can promote functional thinking and therefore algebraic thinking in school. Repairing this process will allow teachers to give more solid arguments on how to conduct their teaching. In other words, we are providing clues so that we can know when the student is moving from arithmetic generalization to algebraic generalization.

This study focused on how a student, based on particular cases, can identify structures and then generalize. Our specific objective was to identify and describe the generalization process of a 2nd grade student.

Method

Our study addressed generalization tasks with functions, based on going from work with concrete variable values (particular cases) to work with uncertainties and general cases. We conducted a case study, thus qualitative, exploratory, and descriptive, based on the idea that "it provides a unique example of real people in real situations, enabling readers to understand ideas more clearly than simply by presenting them with abstract theories or principles" (Cohen et al., 2007). It consisted in a semi- structured interview with a 2nd grade student (aged 7–8) from Spain, who had received no prior instruction on functions or generalization in school. The previous knowledge he had about numbers was numbers from 0 to 399, number comparison, and addition and subtraction with borrowing. Therefore, for this case study, we have taken into account the questionnaire and the interview.

First, we applied a questionnaire to the whole class (24 students). Ten days later, we conducted an interview. With the questionnaire, our aim was to explore the generalization process with the function $y=x+3$. From the analysis of the written responses to this questionnaire, we selected 6 students to be interviewed, based on whether or not they had succeeded in generalizing in the questions of the questionnaire. The session was conducted by the researcher-teacher who then conducted the interview. The interview lasted 20 min.

This group of students was an intentional sample, given the availability of the school that authorized us to enter the investigation. As our goal was to describe the process from arithmetic to algebraic generalization, it was essential to select a student who would generalize (Alejandro). To this criterion, we add the participatory attitude of the student to facilitate the conversation in the interview. Both in the questionnaire applied in the session and in the interview, we involved particular cases with close quantities that we increased to approach the work with the indeterminate and guide towards generalization.

We examined how the student related the variables involved and looked at the generalization process, identifying the structures evidenced on the relation between the variables of the function involved. We noted the transition from arithmetic to algebraic generalization.

Data Collection Instrument

We designed a generalization task involving the affine function $y=x+3$ in the context of a machine where you insert balls and more balls come out depending on the function indicated. We applied a questionnaire to a group of students asking various questions about the task. Then, we conducted a semi-structured interview, which was video recorded. We considered questions, which followed the inductive reasoning model by Cañadas and Castro (2007). In Fig. 4, we observe the interview protocol. Some subsequent studies have used this model to design tasks in questionnaires that explore

1. Recall the task of the previous session

    - Do you remember how this machine works?

    – If the answer was yes, we asked, would you give me two examples? What did you do to get that answer?

    – If the answer was no, we reminded them and showed examples and then asked, What is the relation between the balls going in and the ones coming out? Next, we asked again for concrete examples that had emerged.

2. Observe the structure evidenced in the particular cases

    – Particular cases given

    – Particular cases proposed by the student

    – Third-party particular case

3. Unknown cases and the general case

    – We asked, what can we use at the start of the machine to indicate the number of balls going in?

    – Generalization: How can you tell how the machine works?

Fig. 4 Interview protocol

generalization in lower grades (Ayala-Altamirano & Molina, 2020). In this study, we did not focus on analyzing the justification of the conjectures given the young age of the subject for that level of elaboration. When a student expresses a conjecture, we can interpret the structures they evidence. The formulation of conjectures inform of the stability of the structure evidenced, which can then be reformulated with more particular cases before identifying the structure to be generalized. The set of steps described is the basis for the interview conducted in this study. The protocol followed is shown in Fig. 4.

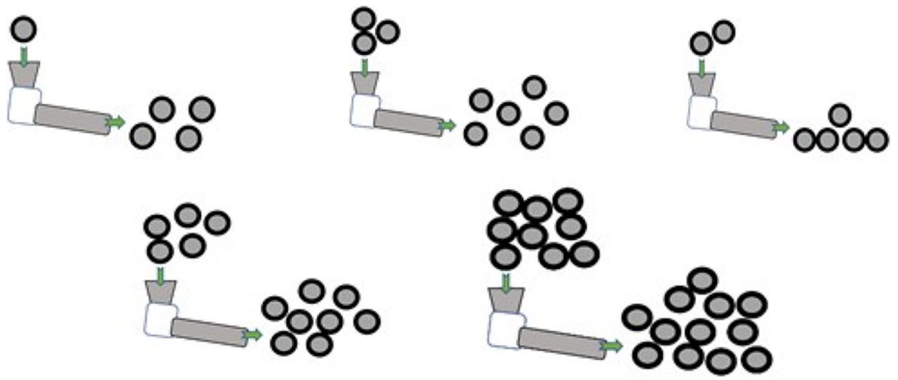

Fig. 5 Particular cases presented (work with concrete variable values)

Figure 5 shows the particular cases we started with (particular cases given) and provided to the student interviewed.

The cases presented were not consecutive to avoid recursiveness in the student's answers. We began with smaller initial quantities which were gradually increased. There were particular cases given by the student as we asked him to give us some quantities of the balls that could be going into the machine. Also, we included the expression "any quantity of balls" or "many balls" as unknown quantities and thus inductively advanced towards algebraic generalization. Following Stacey (1989), the various particular cases presented were different in that with the cases given and the ones proposed by the student; the next term was requested or one that could be obtained by counting, while with others such as the unknown or general cases, it was necessary to know or identify the structure in order to provide an answer.

Data Analysis

We had prepared analysis categories in advance, considering the differences between arithmetic and algebraic generalization. Algebraic generalization addresses the types of algebraic thinking by Radford (2010) and the way students use structures during the generalization process. The reasoning model followed in the generalization process is described with the following elements shown in Fig. 6.

We present a model that is composed of a combination of Radford's theory and the inductive model of Cañadas and Castro (2007). We have conceptualized some terms such as regularities and conjectures by the idea of structure. The notion of structure endows regularities evidenced in a functional context with property.

The categories were formed with the help of Radford's (2010) work but we modified them to give ownership to the functional context in which we studied. We do this by implying a key notion in the generalization process, structure. In addition, we imply the differentiation between arithmetic and algebraic generalization in the overall model. In this model, the student is the subject of the actions. In all reasoning

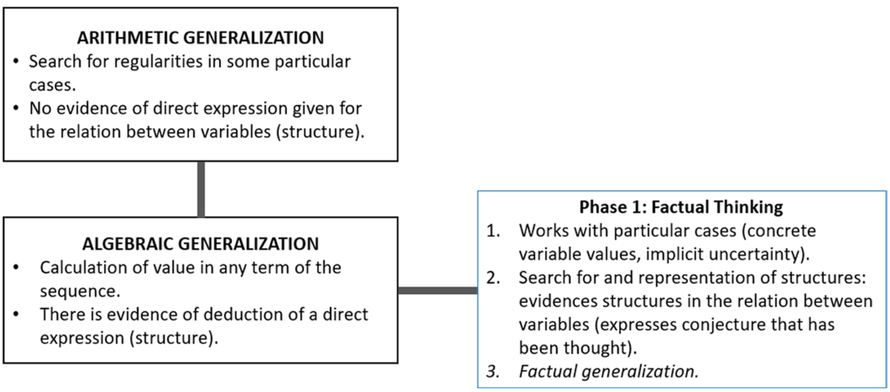

Fig. 6 Generalization process model in a functional context

phases, identifying the structures is key as this allows continuing with the process. We interpreted that a structure was evidenced when the student was able to express a conjecture. The way we recorded what the student said about that being repeated in the particular cases worked on was to identify how he expressed his conjecture. For example, a conjecture could be "plus 3," a verbal representation, (it is what is repeated) but it could be "always add 3" (a more elaborate conjecture). These examples show different levels of thinking, although the structure interpreted is the same: $y = x + 3$. Conjectures can be validated and reformulated in each phase provided since reasoning is dynamic. We express the structure in symbolic form ($y = x + 3$), which is not the representation used by the learner to express his ideas. The symbolic representation of the structure helps us to interpret the identification of variables during the process followed.

The phase where factual thinking takes place (see Fig. 6) is where the possible structures involved in the function are discovered. We identified the structures evidenced by the student both in the particular cases given and in those where the quantities were unknown uncertain. When the student was given new particular cases, he could use "keywords" to describe the general case (Radford, 2010; Vergel & Rojas, 2018). We considered the student had identified a structure when he answered two or more questions following the same regularity or when he generalized. In this case, when we understood that the student had become aware of what was being repeated.

Results and Discussion

We are presenting the results obtained, related to the process followed by the student towards generalization. We focused on language production and the use of signs and appropriation of their meaning during the generalization process (Radford, 2000). We began delving into the student's reasoning based on the work with the particular cases presented in Fig. 5.

The interview extract below shows the moment when Alejandro perceived more balls were coming out than the ones going in.

> Interviewer (I): These are the same examples we used when I presented them in class to all your classmates. What was happening with this machine, Alejandro? What was going into the machine? Student (S): Well, balls.
> I: Balls, ok… and what was coming out?
> S: More balls.
> I: Ok Alejandro, let's look at these machine examples I've brought. If three little balls go in, this is the quantity that came out (they look at the examples on paper).
> S: Six came out (he counts the balls).

There are no indications of a specific structure because he did not quantify the balls that came out versus the ones that went in. The dialogue evidences a fact mediated by the perceptual activity carried out by Alejandro. The interviewer/investigator encouraged the boy to discover structures through work with particular

Fig. 7 Moment when the interviewer begins work with particular cases

cases (concrete variable values) (Fig. 7). The questions asked were important as they moved Alejandro's mathematical activity by starting to identify the variables involved in the functional relation (independent variable: number of balls going in; dependent variable: number of balls coming out). At that moment, the unknown lay on the number level. The student counted the balls going into the machine and the ones coming out to provide the initial answers. The investigator continued asking about other particular cases, as shown in the following fragment of conversation.

> I: For example, in this one (pointing to the machine where five balls are going in), how many are going in?
> S: Six.
> I: Are you sure there are six?
> S: Five, five (counts the balls).
> I: And how many should come out? (Covering the balls exiting the machine)
> S: Eight.
> I: Eight. Why?
> S: Because you have to add three.

This dialogue illustrates that Alejandro identified the regularity represented by three, which corresponds to the number of balls added to the number of balls going into the machine and which, in the end, provided the answer regarding the number of balls coming out.

Through the investigator's eliciting and his strategy to conceal the balls coming out of the machine (Fig. 8), the mathematical relation was identified, which can be symbolically written as $x \rightarrow x+3$: if x number of balls go in, $x+3$ number of balls

Fig. 8 The interviewer concealed the balls coming out of the machine in one of the particular cases

come out. In this case, Alejandro's language used, "Because you have to add three," suggests we are seeing arithmetic generalization. The symbolic expression $x+3$ was not represented by the student. This is the way how we represent the structure evidenced by Alejandro.

At this point of Alejandro's mathematical work, he used abduction (you have to add three) to answer each one of the questions associated to the number of balls going into the machine. There is no evidence of deduction from any direct expression that helps calculate the number of balls coming out of the machine given any number of balls going in. Alejandro's words suggest this is an arithmetic generalization, as his answer was limited to "because you have to add three." In this case, "abduction allows generating a procedure but not a direct expression, in other words, a formula" (Radford, 2013, p. 7).

The investigator continued asking questions about other particular cases such as those shown in the following fragment.

I: How many balls go into this one?
S: Nine (counts the balls).
I: And how many have to come out?
S: 12
I: How do you know it's 12?
S: Because when we did the maths I found a little trick.
I: What was the little trick? Can you tell me?
S: Well, look… to one ball we add three more; to six balls we add another three and so on [with three fingers on his left hand he shows three].

Alejandro emphasized his explanation with gestures to reinforce the truth of what he was saying (Fig. 9). The structure he continued to identify in these particular cases still is, in algebraic terms, $y=x+3$. The student correctly identified the relation between the variables for the particular cases given. In the words, "Well, look, to one ball we add three more; to six balls we add another three and so on," the term "so on" can be considered a time deixis. The student meant it continues "so on," in the same way always. Of course, the adverb always is not explicit in his speech, but it is implicit. Note how in his statement, the modal deixis "so on" suggests recognition and use of what he calls "a little trick," which is no other than a major semiotic resource, which acts as input to make sure those three balls must be added regardless of the particular number of balls going into

Fig. 9 Moment when the student uses gestures to answer

the machine. The adverb "so" evidences one of the generative functions of language, that is, functions that make it possible to describe procedures and actions that can potentially be carried out repeatedly and imagined (Radford, 2003).

In the next interview extract, the interviewer gave the student the chance to validate the prior conjecture on the structure evidenced based on an external example.

I: Then, you have a little trick which you are using for all the questions I'm asking you.
S: Yes.
I: And if I ask you, for example, say any other number, whichever one you want.
S: 19
I: If 19 little balls go in, how many should come out?
S: 22
I: 22. Could you tell me again how you found it?
S: Well, if there is one, four come out, we need to add three more
I: Ok, three more, perfect. There's a kid in class who says that if 25 little balls go in, 27 should come out.
Do you agree with him?
S: No, it's 28.
I: Why 28?
S: Because you have to add three, as I said, and you told me there were 25, there aren't 27

Alejandro validated the conjecture based on the structure evidenced $y = x + 3$. On several occasions, he said you had to "add three." Next, we moved on to the questions the interviewer asked about larger quantities to refer to the number of balls going into the machine.

I: What if, for example, 1 million balls go in.
S: 1 million and three come out.
I: What if I put in 3 million balls.
S: 3 million and 3, you always have to add three.
I: Oh, ok.

Here we observe the transition from the particular cases presented at the beginning to those now being considered. The values covered in this part, one million balls or three million balls, even while concrete, take on an unknown sense for the student interviewed. Here, the student is dealing with quantities that cannot be counted. An analysis of the evidence suggests there has been an evolution. Indeed, the student's production goes from, "Well, look, to one ball we add three, to six we add another three and so on," to "You always have to add three." In this last production, the adverb "always" comes up. Expressions such as these can be seen as "ad hoc language expressions that communicate the idea of the abstraction underlying the generalization of actions" (Radford, 2003, p. 49). These deixes appear to be on the edge between arithmetic thinking and algebraic thinking. However, in this case, we are seeing a generalization of actions that is classified as factual algebraic generalization. That is, a level of thinking where there is evidence of a generalization of actions as an operational outline, connected to the concrete use of number symbols, deixis terms, gestures, and perceptual activity (Radford, 2013; Vergel, 2015).

The analytical nature required for (factual) generalization to be algebraic can be found in the extract where Alejandro states, "You always have to add 3." This analytical nature becomes manifest in the deduction, as movement, which Alejandro expresses in his statement (as opposed to induction), which is based on key information, for example, associated to the questions repeatedly elicited by the investigator and which have enabled moving Alejandro's mathematical activity.

This type of generalization manifests a collective process (investigator and student). Precisely, what Alejandro becomes aware of is this mathematical way of seeing and perceiving a mathematical situation presented in terms of tacit variables and their relation. In other words, we see a covariance that materializes in the specific relation (in symbolic terms: $y=x+3$) between the number of balls going into the machine and the number of balls coming out.

Conclusions

The contribution of this study has been to focus on the transition between arithmetic and algebraic thinking in a concrete approach to algebraic thinking such as functional thinking. Thanks to this we have evidenced characteristics of the different generalizations that we have described in terms of structure and structural awareness, cognitive resources, gestures, and terms that have emerged in the generalization process and in the transition to the algebraic thinking. We will conclude the findings of each of these aspects mentioned.

This study evidences, as anticipated by our predecessors, that a child at an early age can think algebraically (e.g. Kaput et al., 2008; Radford, 2018; Torres et al., 2018; Vergel, 2015). The aim of this research was to identify and describe the generalization process of a 2nd grade student. The analysis of the student's production demonstrates a multimodal way of thinking, as diverse semiotic resources arise triggered by the student in his attempt to generalize (Radford et al., 2009). In this case, Alejandro does not only mobilize cognitive resources, but also perceptual and gestural ones which do not act as peripheral elements in his thinking, but rather are inherent, constituent elements.

Alejandro's work suggests a transition from arithmetic generalization to algebraic generalization with the step from "add 3" to "always plus 3." The notion of structure has been here a key element to detect the transition, as it helped to interpret the relationship between the variables involved in the task and identified by the child ($y=x+3$). The stability of the same structure throughout the interview denotes the creation of awareness by Alejandro of the regularity present among the variables of the function in question.

Radford (2013) gives us an operational criterion for discerning between arithmetic and algebraic generalization and thus between arithmetic and algebraic thinking.

For the generalization to be algebraic, it is necessary that there has been a deduction of an expression that allows us to calculate the value of any term of the sequence.

In the specific case of Alejandro, we have observed that at the beginning, he evidences an arithmetic generalization since there is no indication of deduction when he only expresses that what he has to do to obtain the value is to "add 3" (abduction). Later, as the interview progresses, deduction is observed. Alejandro expresses that what he has to do to obtain the values of the sequence is "always add three" (hypothesis). That is, what is deduced is the expression (not necessarily with alphanumeric signs) that allows to calculate the value of any term of the sequence. This jump shows an arithmetic to algebraic change assuming Radford's criterion. This has meant that abduction no longer appears as a simple possibility or as a plausible hypothesis, but as an assumed principle, that is, a hypothesis to deduce the formula that provides the value of any term. This corresponds to a contribution of the study, the passage from the abductive to the deductive.

The substantial point in Alejandro's production has been the deduction ability manifested in hi answer, "You always have to add 3," a statement that indicates generalization of actions, as an operational outline, which we have classified as factual algebraic generalization. We have denoted this generalization expression symbolically with the structure $y=x+3$.

Keywords such as "always" and "so on" also came up, with which the student indicated that he was focusing on particular cases and thus revealed the acceptance of a sequential continuity. We found a constant in the answers given by Alejandro; he showed stability regarding what was repeated throughout the task with different particular cases, allowing him to arrive at generalization.

With the production provided by Alejandro, we found stability of structure evidenced from the first particular cases worked on, which ensured achievement of generalization. Thus, we verified that the structure is closely related to the generalization process followed, given that identification of the structure in a mathematical situation is key in the generalization process.

Regarding the stability or awareness of the structure, we ascertain that gestures help to emphasize what the student is indicating with his verbal expressions. It would remain for us to do a deeper analysis in the sense of Warren et al. (2013), attending to whether these are reduced once the student creates awareness of the structure involved. In our results, we can intuit that this is indeed the case. The student when generalizing the structure implying the term "always" did not make gestures that complemented what he had expressed.

Finally, we distinguished between arithmetic thinking and algebraic thinking. We evidenced that the transition from arithmetic to algebra occurs through a breakup. This was confirmed in as much as Alejandro got the right structure and deducted thereby other different particular cases. We can recall that for Descartes (1983), deduction is a conclusion based on given facts (truths known with certainty). Herein lies the characteristic of movement in the idea of deduction. In turn, the adjective factual means that the formula variables appear tacitly, that which is unknown (or that which is general) is unnamed, that is, it is not an explicit subject of speech. Another way of saying this is that the formula is expressed through particular instances of the variable (the variable is instanced in specific numbers or "facts") in the form of a concrete rule, for example, "1 million and 3 come out"; "3 million and 3, you always have to add 3."

This research raises an open study question regarding deduction. The deduction could take on different meanings to the extent that it comes from an abduction or a non-abductive assertion. We raise the need for more research to obtain scientific evidence and deeper epistemological reflections. Deduction seems to support algebraic generalizations while inductive ones seem to support arithmetic generalizations. In short, we need to explore more on the transition from arithmetic to algebraic I order to favor students' algebra learning in successive courses.

This research could contribute to teaching by providing information on the guidance that teachers should give in the generalization process of students. We give information about when the student is moving from arithmetic generalization to algebraic generalization and could favor the achievement of generality with effective guidance from the teacher. We are also giving information on the design and creation of specific tasks that could be useful in classrooms for students to achieve generalization. In our methodology, we used the potential of the mathematical object, the function, so that students have developed skills that allow them to think algebraically and reflect on the relationships between.

variables at this educational level.

We recognize as a limitation of the study that the analysis is of a single student. We decided to focus on one case in order to study in more depth aspects that we had not previously considered, such as gestures and perceptual activity. This study is being done with a single student, but it belongs to a broader investigation focused on the study of algebraic thinking from infant age. It can also be seen as a strength, in that the analysis deployed with a real student in a real situation provides the possibility of valuable insight into the student's mathematical thinking.

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