



Simultaneous Approximation via Laplacians on the Unit Ball

Misael E. Marriaga, Teresa E. Pérez and Marlon J. Recarte

Abstract. We study the orthogonal structure on the unit ball \mathbf{B}^d of \mathbb{R}^d with respect to the Sobolev inner products

$$\langle f, g \rangle_{\Delta} = \lambda \mathcal{L}(f, g) + \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx,$$

where $\mathcal{L}(f, g) = \int_{\mathbf{S}^{d-1}} f(\xi)g(\xi) d\sigma(\xi)$ or $\mathcal{L}(f, g) = f(0)g(0)$, $\lambda > 0$, σ denotes the surface measure on the unit sphere \mathbf{S}^{d-1} , and Δ is the usual Laplacian operator. Our main contribution consists in the study of orthogonal polynomials associated with $\langle \cdot, \cdot \rangle_{\Delta}$, giving their explicit expression in terms of the classical orthogonal polynomials on the unit ball, and proving that they satisfy a fourth-order partial differential equation, extending the well-known property for ball polynomials, since they satisfy a second-order PDE. We also study the approximation properties of the Fourier sums with respect to these orthogonal polynomials and, in particular, we estimate the error of simultaneous approximation of a function, its partial derivatives, and its Laplacian in the $L^2(\mathbf{B}^d)$ space.

Mathematics Subject Classification. 33C50, 42C05.

Keywords. Approximation on the ball, inner product via Laplacians, Fourier expansions.

1. Introduction

In [2], Atkinson and Hansen studied the problem of finding the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ with zero boundary

MEM has been supported by the Spanish Agencia Estatal de Investigación under Grant PID2021-122154NB-I00 and by the Comunidad de Madrid multiannual agreement with the Universidad Rey Juan Carlos under the Grant Proyectos I+D para Jóvenes Doctores, Ref. M2731, project NETA-MM. TEP and MJR thank grant A-FQM-246-UGR20 funded by Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía and FEDER, Una manera de Hacer Europa. TEP also thanks IMAG-María de Maeztu Grant CEX2020-001105-M. MJR also thanks Coimbra Group Scholarship Programme for Young Professors and Researchers from Latin American Universities..

conditions on the unit ball \mathbf{B}^d on \mathbb{R}^d , and asked the question of finding an explicit orthogonal basis for the Sobolev inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx, \tag{1.1}$$

where Δ denotes the usual Laplace operator.

Y. Xu answered that question in [11], where he constructed such basis in terms of spherical harmonics and classical Jacobi polynomials of varying parameter. In addition, he studied the orthogonal expansion of a function in that basis, and proved that it can be computed without using the derivatives of the function.

Sobolev orthogonal polynomials, that is, families of polynomials that are orthogonal with respect to inner product involving both values of functions as well as derivative operators, such as partial derivatives, gradients, normal derivatives, Laplacians, and others, have been recently studied. A recent survey on this topic can be found in [7]. A clear range of application of orthogonal polynomials is the field of approximation of functions, with multiple technological applications within the multivariate case. Recently, there has been a renovated interest for approximants based on multivariate Sobolev orthogonal polynomials, showing that it is not necessary to use the derivatives of the function. Examples of this kind of studies can be find in [8, 9, 13], among others.

In this paper, we modify the inner product (1.1) by introducing a term on the spherical border of the ball taking into account the possible information that may be observed from some modification of the nonlinear Poisson equation

$$\begin{aligned} \langle f, g \rangle_{\Delta} &= \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi) \\ &+ \frac{1}{8\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx. \end{aligned}$$

This term is introduced by means of a real positive constant $\lambda > 0$ to modulate the influence of that term in the problem, and the normalization constants are chosen to simplify expressions in the sequel.

In addition, following the ideas in [12], we also modify the inner product (1.1) by introducing a mass point at the origin:

$$\begin{aligned} \langle f, g \rangle_{\Delta}^* &= \frac{\lambda}{\sigma_{d-1}} f(0) g(0) \\ &+ \frac{1}{8\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx. \end{aligned}$$

Our main objective is to study the influence of the additional term in the study of the basis and its impact into the Fourier coefficients and the errors for a given function.

The work is structured in the following way. Section 2 is devoted to establishing the main tools and basic results that we will need for the rest of the paper, including a brief introduction about spherical harmonics, classical orthogonal polynomials in the unit ball of \mathbb{R}^d , Fourier coefficients, among

others facts. Section 3 describes the first considered Sobolev inner product, deducing a Sobolev basis for that inner product, and proving that they satisfy a fourth-order partial differential equation, extending the well known property for ball polynomials since they satisfy a second order PDE. Section 4 is devoted to analyse the Sobolev Fourier orthogonal expansions and approximation, giving explicit bounds for the errors. Finally, in Sect. 5, we study the Sobolev inner product with a mass point at the origin, proving that this second Sobolev inner product and the associated basis can be related with the first one studied.

2. Preliminaries

In this section, we collect the basic background on the classical orthogonal polynomials on the unit ball and orthogonal expansions that will be used throughout this paper.

For $x \in \mathbb{R}^d$, we denote by $\|x\|$ the usual Euclidean norm of x . The unit ball and the unit sphere in \mathbb{R}^d are denoted by $\mathbf{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ and $\mathbf{S}^{d-1} = \{\xi \in \mathbb{R}^d : \|\xi\| = 1\}$, respectively.

For $\mu > -1$, let W_μ be the weight function on the unit ball defined as

$$W_\mu(x) = (1 - \|x\|^2)^\mu, \quad x \in \mathbf{B}^d.$$

This weight function can be used to define the inner product

$$\langle f, g \rangle_\mu = b_\mu \int_{\mathbf{B}^d} f(x)g(x) W_\mu(x) dx, \quad f, g \in L^2(W_\mu; \mathbf{B}^d),$$

where b_μ is the normalization constant, such that $\langle 1, 1 \rangle_\mu = 1$ given by

$$b_\mu = \left(\int_{\mathbf{B}^d} W_\mu(x) dx \right)^{-1} = \frac{\Gamma(\mu + 1 + \frac{d}{2})}{\Gamma(\mu + 1)\pi^{d/2}}.$$

This inner product, in turn, induces the norm $\|\cdot\|_\mu$ defined by

$$\|f\|_\mu = \left(b_\mu \int_{\mathbf{B}^d} f(x)^2 W_\mu(x) dx \right)^{1/2}, \quad f \in L^2(W_\mu; \mathbf{B}^d).$$

For $n \geq 0$, let us denote by Π_n^d the linear space of real polynomials in d variables of total degree less than or equal to n , and by $\Pi^d = \bigcup_{n \geq 0} \Pi_n^d$ the linear space of all real polynomials in d variables.

A polynomial $P \in \Pi_n^d$ is said to be an orthogonal polynomial of (total) degree n if $\langle P, Q \rangle_\mu = 0$ for all $Q \in \Pi_{n-1}^d$. For $n \geq 0$, let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of total degree n . Then, $\dim \mathcal{V}_n^d(W_\mu) = \binom{n+d-1}{n} := r_n^d$.

For $n \geq 0$, let $\{P_\nu^n(x) : 1 \leq \nu \leq r_n^d\}$ be a basis of $\mathcal{V}_n^d(W_\mu)$. Notice that every element of $\mathcal{V}_n^d(W_\mu)$ is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, $\langle P_\nu^n, P_\eta^n \rangle = 0$ whenever $\nu \neq \eta$, we call the basis *mutually orthogonal*. If, in addition, $\langle P_\nu^n, P_\nu^n \rangle = 1$, we say that the basis is *orthonormal*.

2.1. Spherical Harmonics

Let \mathcal{H}_n^d denote the space of harmonic polynomials in d variables of degree n , that is, homogeneous polynomials of degree n satisfying the Laplace equation $\Delta Y = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator. It is well known that

$$a_n^d := \dim \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n}.$$

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}_n^d$, the linear space of spherical harmonics of degree n , then in spherical-polar coordinates $x = r\xi$ where $r > 0$ and $\xi \in \mathbf{S}^{d-1}$, we get

$$Y(x) = r^n Y(\xi),$$

so that Y is uniquely determined by its restriction to the sphere.

We define the linear operator

$$x \cdot \nabla = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i},$$

and, by Euler’s equation for homogeneous polynomials, we deduce that

$$x \cdot \nabla Y(x) = nY(x), \quad \forall Y \in \mathcal{H}_n^d.$$

The differential operators Δ and $x \cdot \nabla$ can be expressed in spherical-polar coordinates as [5]

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_0, \\ x \cdot \nabla &= r \frac{\partial}{\partial r}, \end{aligned} \tag{2.1}$$

where Δ_0 is the spherical part of the Laplacian, called the Laplace–Beltrami operator. The operator Δ_0 has spherical harmonics as eigenfunctions. More precisely, it holds that [5]

$$\Delta_0 Y(\xi) = -n(n+d-2)Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbf{S}^{d-1}. \tag{2.2}$$

We will also need the following family of differential operators, $D_{i,j}$, defined by:

$$D_{i,j} = x_i \partial_j - x_j \partial_i, \quad 1 \leq i < j \leq d.$$

These are angular derivatives, since $D_{i,j} = \partial_{\theta_{i,j}}$ in the polar coordinates of the x_i, x_j -plane, $(x_i, x_j) = r_{i,j} (\cos \theta_{i,j}, \sin \theta_{i,j})$. Furthermore, the angular derivatives $D_{i,j}$ and the Laplace–Beltrami operator Δ_0 are related by

$$\Delta_0 = \sum_{1 \leq i < j \leq d} D_{i,j}^2.$$

Spherical harmonics are orthogonal polynomials on \mathbf{S}^{d-1} with respect to the inner product

$$\langle f, g \rangle_{\mathbf{S}^{d-1}} = \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi) g(\xi) \, d\sigma(\xi),$$

where $d\sigma$ denotes the surface measure and σ_{d-1} denotes the surface area

$$\sigma_{d-1} = \int_{\mathbb{S}^{d-1}} d\sigma(\xi) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

2.2. Mutually Orthogonal Polynomials on the Unit Ball

A mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$ can be given in terms of Jacobi polynomials and spherical harmonics.

For $\alpha, \beta > -1$, the Jacobi polynomial $P_n^{(\alpha, \beta)}(t)$ of degree n is defined as [10]

$$P_n^{(\alpha, \beta)}(t) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (k + \alpha + 1)_{n-k} (k + \alpha + \beta + 1)_k \left(\frac{t-1}{2}\right)^k,$$

where, for $a \in \mathbb{R}, n \geq 0$, $(a)_0 = 1$, and $(a)_n = a(a+1) \cdots (a+n-1)$ denotes the Pochhammer symbol. They are orthogonal with respect to the Jacobi weight function $w_{\alpha, \beta}(t) = (1-t)^\alpha(1+t)^\beta$ on the interval $[-1, 1]$.

The polynomials defined in the following proposition are known in the literature as classical ball polynomials.

Proposition 2.1. [6] For $n \geq 0$ and $0 \leq j \leq \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leq \nu \leq a_{n-2j}^d\}$ denote an orthonormal basis of \mathcal{H}_{n-2j}^d . For $\mu > -1$, define the polynomials

$$P_{j, \nu}^{n, \mu}(x) := P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1) Y_\nu^{n-2j}(x). \tag{2.3}$$

Then, the set $\{P_{j, \nu}^{n, \mu} : 0 \leq j \leq \frac{n}{2}, 1 \leq \nu \leq a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$.

Moreover

$$\langle P_{j, \nu}^{n, \mu}, P_{k, \eta}^{m, \mu} \rangle_\mu = H_{j, n}^\mu \delta_{n, m} \delta_{j, k} \delta_{\nu, \eta},$$

where

$$H_{j, n}^\mu = \frac{(\mu + 1)_j (d/2)_{n-j} (n - j + \mu + d/2)}{j! (\mu + d/2 + 1)_{n-j} (n + \mu + d/2)}. \tag{2.4}$$

The square of the $L^2(w_{\alpha, \beta}, [-1, 1])$ norm of the Jacobi polynomial $P_j^{(\alpha, \beta)}(t)$, given by

$$h_j^{(\alpha, \beta)} := \int_{-1}^1 \left(P_j^{(\alpha, \beta)}(t)\right)^2 w_{\alpha, \beta}(t) dt,$$

is related with $H_{j, n}^\mu$ as follows:

$$H_{j, n}^\mu = \frac{\gamma_{\mu, d}}{2^{n-2j}} h_j^{(\mu, n-2j+\frac{d-2}{2})}, \tag{2.5}$$

where $\gamma_{\mu, d} = \frac{b_\mu \sigma_{d-1}}{2^{\mu+\frac{d}{2}+1}}$.

It is known [6] that the orthogonal polynomials with respect to W_μ are eigenfunctions of the second-order differential operator \mathcal{D}_μ . More precisely, we have

$$\mathcal{D}_\mu P = -(n+d)(n+2\mu-1)P, \quad \forall P \in \mathcal{V}_n^d(W_\mu),$$

where

$$D_\mu := \Delta - \sum_{j=1}^d \frac{\partial}{\partial x_j} x_j \left(2\mu + 1 + \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} \right).$$

The following lemma will be useful in the sequel. For convenience, we define $P_{j,\nu}^{n,\mu}(x) = 0$ if $j < 0$.

Lemma 2.2. [9] *Let $\mu > -1$. Then*

$$\Delta P_{j,\nu}^{n,\mu}(x) = \kappa_{n-j}^\mu P_{j-1,\nu}^{n-2,\mu+2}(x) \quad \text{and} \quad \Delta_0 P_{j,\nu}^{n,\mu}(x) = \varrho_{n-2j} P_{j,\nu}^{n,\mu}(x),$$

where

$$\kappa_n^\mu = 4 \left(n + \mu + \frac{d}{2} \right) \left(n + \frac{d-2}{2} \right) \quad \text{and} \quad \varrho_n = -n(n+d-2).$$

2.3. Fourier Orthogonal Expansion and Approximation

With respect to the basis (2.3), the Fourier orthogonal expansion of $f \in L^2(W_\mu; \mathbf{B}^d)$ is defined by

$$f(x) = \sum_{n=0}^\infty \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \widehat{f}_{j,\nu}^{n,\mu} P_{j,\nu}^{n,\mu}(x) \quad \text{with} \quad \widehat{f}_{j,\nu}^{n,\mu} := \frac{1}{H_{j,n}^\mu} \langle f, P_{j,\nu}^{n,\mu} \rangle_\mu.$$

Since $\|f\|_\mu$ is finite, the Parseval identity holds: for $\mu > -1$

$$\|f\|_\mu^2 = \sum_{n=0}^\infty \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \left| \widehat{f}_{j,\nu}^{n,\mu} \right|^2 H_{j,n}^\mu.$$

Let $\text{proj}_n^\mu : L^2(W_\mu; \mathbf{B}^d) \rightarrow \mathcal{V}_n^d(W_\mu)$ and $S_n^\mu : L^2(W_\mu; \mathbf{B}^d) \rightarrow \Pi_n^d$ denote the projection operator and the partial sum operator, respectively. Then

$$\text{proj}_m^\mu f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{m,\mu} P_{j,\nu}^{m,\mu}(x) \quad \text{and} \quad S_n^\mu f(x) = \sum_{m=0}^n \text{proj}_m^\mu f(x).$$

By definition, $S_n^\mu f = f$ if $f \in \Pi_n^d$. Moreover, for $f \in L^2(W_\mu; \mathbf{B}^d)$, we have $\langle f - S_n^\mu f, Q \rangle_\mu = 0$ for all $Q \in \Pi_n^d$.

We consider the error, $\mathcal{E}_n(f)_\mu$, of best approximation by polynomials in Π_n^d in the space $L^2(W_\mu; \mathbf{B}^d)$, defined by

$$\mathcal{E}_n(f)_\mu = \inf_{p_n \in \Pi_n^d} \|f - p_n\|_\mu,$$

and notice that the infimum is achieved by $S_n^\mu f$.

We define the non-uniform Sobolev space. For $\mathbf{m} \in \mathbb{N}_0^d$, let $|\mathbf{m}| = m_1 + \dots + m_d$ and $\partial^{\mathbf{m}} = \partial_1^{m_1} \dots \partial_d^{m_d}$. For $\mu > -1$ and $s \geq 1$, we denote by $\mathcal{W}_2^s(W_\mu; \mathbf{B}^d)$ the Sobolev space

$$\mathcal{W}_2^s(W_\mu; \mathbf{B}^d) = \{f \in L^2(W_\mu; \mathbf{B}^d); \partial^{\mathbf{m}} f \in L^2(W_{\mu+|\mathbf{m}|}; \mathbf{B}^d), |\mathbf{m}| \leq s, \mathbf{m} \in \mathbb{N}_0^d\}.$$

We say that this Sobolev space is non-uniform, since each derivative of the function belongs to a different L^2 space.

The following estimate was proved in [9]: for $n \geq 2s$ and $f \in \mathcal{W}_2^{2s}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leq \frac{c}{n^{2s}} \left[\mathcal{E}_{n-2s}(\Delta^s f)_{\mu+2s} + \mathcal{E}_n(\Delta_0^s f)_\mu \right], \tag{2.6}$$

and for $n \geq 2s + 1$ and $f \in \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$

$$\mathcal{E}_n(f)_\mu \leq \frac{c}{n^{2s+1}} \left[\sum_{i=1}^d \mathcal{E}_{n-2s-1}(\partial_i \Delta^s f)_{\mu+2s+1} + \sum_{1 \leq i < j \leq d} \mathcal{E}_n(D_{i,j} \Delta_0^s f)_\mu \right].$$

Here and in the sequel, c is a generic constant independent of n and f but may depend on μ and d , and its value may be different from one instance to the next. As pointed out in [9], each term involving Δ and Δ_0 on the right-hand side of the above inequalities is necessary, since the first term deals with the radial component of f and the second one deals with the harmonic component of f defined on the ball.

3. Sobolev Orthogonal Polynomials

This section is devoted to the study of the orthogonal structure on the unit ball with respect to the *Sobolev inner product*

$$\begin{aligned} \langle f, g \rangle_\Delta &= \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi) g(\xi) \, d\sigma(\xi) \\ &+ \frac{1}{8\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] \, dx, \quad \lambda > 0. \end{aligned} \tag{3.1}$$

The normalization constants are chosen to simplify expressions in the sequel. Orthogonal polynomials with respect to inner products involving derivatives are called Sobolev orthogonal polynomials. Let us denote by $\mathcal{V}_n^d(\Delta)$ the space of Sobolev orthogonal polynomials of degree n with respect to (3.1). We point out that when $\lambda = 0$, we recover the inner product studied in [11] up to a normalization constant.

We need the following lemma.

Lemma 3.1. *Let $\beta = n - 2j + \frac{d-2}{2}$ and $Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d$. Then, for any polynomial $q(s)$,*

$$\Delta[(1 - \|x\|^2) q(\|x\|^2) Y_\nu^{n-2j}(x)] = 4(\mathcal{J}_\beta q)(\|x\|^2) Y_\nu^{n-2j}(x), \tag{3.2}$$

where

$$(\mathcal{J}_\beta q)(s) = s(1 - s)q''(s) + (\beta + 1 - (\beta + 3)s)q'(s) - (\beta + 1)q(s).$$

Proof. Using spherical-polar coordinates, we can use (2.1) and (2.2) for the radial and spherical part of Δ , respectively. After a tedious calculation, we get that

$$\begin{aligned} \Delta \left[(1 - \|x\|^2) q(\|x\|^2) Y_\nu^{n-2j}(x) \right] &= \Delta \left[(1 - r^2) q(r^2) r^{n-2j} Y_\nu^{n-2j}(\xi) \right] \\ &= 4 \left[r^2(1 - r^2)q''(r^2) + (\beta + 1 - (\beta + 3)r^2)q'(r^2) - (\beta + 1)q(r^2) \right] Y_\nu^{n-2j}(x). \end{aligned}$$

Setting $s \mapsto r^2$ gives the desired result. □

Inspired by the explicit expression (2.3) for the basis of classical ball polynomials and Theorem 2.4 in [11], we use the univariate Jacobi polynomials and the spherical harmonics to construct the following multivariate polynomials defined on \mathbf{B}^d .

Definition 3.2. For $n \geq 0$ and $0 \leq j \leq \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leq \nu \leq a_{n-2j}^d\}$ denote an orthonormal basis of \mathcal{H}_{n-2j}^d . We define the polynomials

$$\begin{aligned} Q_{0,\nu}^n(x) &:= Y_\nu^n(x), \\ Q_{j,\nu}^n(x) &:= (1 - \|x\|^2) P_{j-1}^{(2,n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1) Y_\nu^{n-2j}(x), \quad 1 \leq j \leq \frac{n}{2}. \end{aligned}$$

It turns out that these polynomials are eigenfunctions of a fourth-order linear partial differential operator.

Proposition 3.3. *The polynomials $Q_{j,\nu}^n$ satisfy*

$$\Delta \left[(1 - \|x\|^2) Q_{j,\nu}^n(x) \right] = c_{j,n} P_{j,\nu}^{n,0}, \quad n \geq 0, \tag{3.3}$$

with

$$c_{j,n} = \begin{cases} -4 \left(n + \frac{d}{2} \right), & j = 0, \\ 4j(j + 1), & 1 \leq j \leq \frac{n}{2}, \end{cases}$$

and

$$(1 - \|x\|^2) \Delta P_{j,\nu}^{n,0}(x) = d_{j,n} Q_{j,\nu}^n(x), \quad n \geq 0, \tag{3.4}$$

with

$$d_{j,n} = \begin{cases} 0, & j = 0, \\ 4 \left(n - j + \frac{d}{2} \right) \left(n - j + \frac{d-2}{2} \right), & 1 \leq j \leq \frac{n}{2}. \end{cases}$$

Proof. For $j = 0$, by (3.2), we have

$$\Delta \left[(1 - \|x\|^2) Q_{0,\nu}^n(x) \right] = \Delta \left[(1 - \|x\|^2) Y_\nu^n(x) \right] = -4 \left(n + \frac{d}{2} \right) Y_\nu^n(x).$$

Now, we deal with $1 \leq j \leq \frac{n}{2}$. The Jacobi polynomials satisfy the following property ([10, p. 71]):

$$(1-s) P_{j-1}^{(2,\beta)}(2s-1) = \frac{1}{2j + \beta + 1} \left[(j + 1) P_{j-1}^{(1,\beta)}(2s-1) - j P_j^{(1,\beta)}(2s-1) \right]. \tag{3.5}$$

Furthermore, the Jacobi polynomials $P_{j-1}^{(1,\beta)}(2s-1)$ satisfy the differential equation

$$s(1-s)y'' + (\beta + 1 - (\beta + 3)s)y' = -(j-1)(j + \beta + 1)y.$$

Using these two facts, we can easily deduce that

$$\begin{aligned}
 & (2j + \beta + 1) \mathcal{J}_\beta \left[(1 - s) P_{j-1}^{(2,\beta)}(2s - 1) \right] \\
 &= (j + 1) \mathcal{J}_\beta P_{j-1}^{(1,\beta)}(2s - 1) - j \mathcal{J}_\beta P_j^{(1,\beta)}(2s - 1) \\
 &= (j + 1) \left[- (j - 1) (j + \beta + 1) - (\beta + 1) \right] P_{j-1}^{(1,\beta)}(2s - 1) \\
 &\quad - j \left[- j (j + \beta + 2) - (\beta + 1) \right] P_j^{(1,\beta)}(2s - 1) \\
 &= -j(j + 1) \left[(j + \beta) P_{j-1}^{(1,\beta)}(2s - 1) - (j + \beta + 1) P_j^{(1,\beta)}(2s - 1) \right].
 \end{aligned}$$

We need yet another formula for Jacobi polynomials ([1, p. 782, (22.7.18)])

$$(2j + \beta + 1) P_j^{(0,\beta)}(2s - 1) = (j + \beta + 1) P_j^{(1,\beta)}(2s - 1) - (j + \beta) P_{j-1}^{(1,\beta)}(2s - 1),$$

which implies immediately that

$$\mathcal{J}_\beta \left[(1 - s) P_{j-1}^{(2,\beta)}(2s - 1) \right] = j(j + 1) P_j^{(0,\beta)}(2s - 1). \tag{3.6}$$

Then, by Eqs. (3.2) and (3.6) with $\beta = n - 2j + \frac{d-2}{2}$,

$$\Delta \left[(1 - \|x\|^2) Q_{j,\nu}^n(x) \right] = 4j(j + 1) P_{j,\nu}^{n,0}(x).$$

This proves (3.3).

Using $\Delta Y_\nu^n(x) = 0$, we get

$$(1 - \|x\|^2) \Delta P_{0,\nu}^{n,0} = 0 = d_{0,n} Q_{0,\nu}^n(x).$$

From Lemma 2.2 and Definition 3.2, we obtain

$$(1 - \|x\|^2) \Delta P_{j,\nu}^{n,0} = d_{j,n} (1 - \|x\|^2) P_{j-1,\nu}^{n-2,2}(x) = d_{j,n} Q_{j,\nu}^n(x),$$

proving (3.4). □

Combining Eqs. (3.3) and (3.4), we get a partial differential equation for the Sobolev orthogonal polynomials.

Corollary 3.4. *The polynomials $Q_{j,\nu}^n$ satisfy*

$$(1 - \|x\|^2) \Delta^2 \left[(1 - \|x\|^2) Q_{j,\nu}^n(x) \right] = \varpi_{n,j} Q_{j,\nu}^n(x), \quad 0 \leq j \leq \frac{n}{2}, \tag{3.7}$$

where

$$\varpi_{n,j} = 16j(j + 1) \left(n - j + \frac{d}{2} \right) \left(n - j + \frac{d - 2}{2} \right).$$

The following relation follows readily from (2.3) and (3.5).

Proposition 3.5. *The polynomials $Q_{j,\nu}^n$ satisfy*

$$\begin{aligned} Q_{0,\nu}^n(x) &= P_{0,\nu}^{n,1}(x), \\ Q_{j,\nu}^n(x) &= \frac{1}{n + \frac{d}{2}} \left[(j + 1) P_{j-1,\nu}^{n-2,1}(x) - j P_{j,\nu}^{n,1}(x) \right], \quad 1 \leq j \leq \frac{n}{2}. \end{aligned} \tag{3.8}$$

In the following proposition, we show that the polynomials in Definition 3.2 constitute a mutually orthogonal basis with respect to the inner product (3.1).

Proposition 3.6. *For $n \geq 0$, $\{Q_{j,\nu}^n : 0 \leq j \leq \frac{n}{2}, 1 \leq \nu \leq a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(\Delta)$. Moreover*

$$\langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta = \tilde{H}_{j,n}^\Delta \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta},$$

where,

$$\tilde{H}_{j,n}^\Delta = \begin{cases} \lambda + n + \frac{d}{2}, & j = 0, \\ \frac{j^2 (j + 1)^2}{2^{n-2j+\frac{d}{2}}} h_j^{(0,n-2j+\frac{d-2}{2})}, & 1 \leq j \leq \frac{n}{2}. \end{cases} \tag{3.9}$$

Proof. If $j = k = 0$, then using the formula

$$\int_{\mathbf{B}^d} f(x) \, dx = \int_0^1 r^{d-1} \int_{\mathbf{S}^{d-1}} f(r \xi) \, d\sigma(\xi) \, dr,$$

and (3.2), we get

$$\langle Q_{0,\nu}^n, Q_{0,\eta}^m \rangle_\Delta = \delta_{n,m} \delta_{\nu,\eta} \left[\lambda + (\beta + 1)^2 \int_0^1 s^\beta \, ds \right] = \delta_{n,m} \delta_{\nu,\eta} [\lambda + \beta + 1],$$

where $\beta = n - 2j + \frac{d-2}{2}$. If $j = 0$ and $k \geq 1$, then

$$\langle Q_{0,\nu}^n, Q_{k,\eta}^m \rangle_\Delta = -(\beta + 1) k (k + 1) \delta_{n,m-2k} \delta_{\nu,\eta} \int_0^1 P_k^{(0,\beta)}(2s - 1) s^\beta \, ds = 0,$$

since the factor $(1 - \|x\|^2)$ vanishes on \mathbf{S}^{d-1} .

For $1 \leq j, k \leq \frac{n}{2}$, applying Green’s identity

$$\int_{\mathbf{B}^d} (u \Delta v - v \Delta u) \, dx = \int_{\mathbf{S}^{d-1}} \left(\frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) \, d\sigma(\xi) \tag{3.10}$$

with $v(x) = (1 - \|x\|^2) Q_{j,\nu}^n(x)$ and $u = (1 - \|x\|^2) Q_{k,\eta}^m(x)$, we get

$$\begin{aligned} \langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta &= \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} Q_{j,\nu}^n(\xi) Q_{k,\eta}^m(\xi) \, d\sigma(\xi) \\ &\quad + \frac{1}{8 \sigma_{d-1}} \int_{\mathbf{B}^d} (1 - \|x\|^2) Q_{k,\eta}^m(x) \Delta^2 [(1 - \|x\|^2) Q_{j,\nu}^n(x)] \, dx. \end{aligned}$$

Then, by (3.7) and Definition 3.2, we can write

$$\begin{aligned} \langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta &= \frac{\varpi_{n,j}}{8 \sigma_{d-1}} \int_{\mathbf{B}^d} P_{j-1,\nu}^{n-2,2}(x) P_{k-1,\eta}^{m-2,2}(x) W_2(x) \, dx \\ &= \frac{\varpi_{n,j}}{8 \sigma_{d-1} b_2} H_{j-1,n-2}^2 \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta}. \end{aligned}$$

Using (2.4), we get

$$\frac{H_{j-1,n-2}^2}{H_{j,n}^0} = \frac{1}{2}j(j+1)\frac{(\frac{d}{2}+1)(\frac{d}{2}+2)}{(n-j+\frac{d}{2})(n-j+\frac{d-2}{2})}.$$

Moreover, using (2.5) and the fact that $b_2 = (\frac{d}{2}+1)(\frac{d}{2}+2)b_0$, we obtain

$$\langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta = \frac{j^2(j+1)^2}{2^{\beta+1}} h_j^{(0,\beta)} \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta}.$$

□

Corollary 3.7. For $n \geq 2$

$$\mathcal{V}_n^d(\Delta) = \mathcal{H}_n^d \oplus (1 - \|x\|^2) \mathcal{V}_{n-2}^d(W_2).$$

Proof. Using the basis (2.3) for $\mathcal{V}_{n-2}^d(W_2)$, it follows that we actually have

$$Q_{j,\nu}^n(x) = (1 - \|x\|^2) F_{j-1,\nu}^{n-2,2}(x),$$

for $j \geq 1$, from which the stated result follows. □

4. Sobolev Fourier Orthogonal Expansions and Approximation

Consider the Sobolev space

$$\mathbb{H}^s(\mathbf{B}^d) = \{f \in C(\mathbf{B}^d); \partial^{\mathbf{m}} f \in L^2(\mathbf{B}^d), |\mathbf{m}| \leq s, \mathbf{m} \in \mathbb{N}_0^d\},$$

where $L^2(\mathbf{B}^d) = L^2(W_0; \mathbf{B}^d)$.

For $f \in \mathbb{H}^2(\mathbf{B}^d)$, let us denote by $\widehat{f}_{j,\nu}^{n,\Delta}$ the Fourier coefficients with respect to the basis of $\mathcal{V}_n^d(\Delta)$ defined in (3.2), that is

$$\widehat{f}_{j,\nu}^{n,\Delta} = \frac{1}{\widetilde{H}_{j,n}^\Delta} \langle f, Q_{j,\nu}^n \rangle_\Delta,$$

where $\widetilde{H}_{j,n}^\Delta$ is given in (3.9).

Let $\text{proj}_m^\Delta : \mathbb{H}^2(\mathbf{B}^d) \rightarrow \mathcal{V}_m^d(\Delta)$ and $S_n^\Delta : \mathbb{H}^2(\mathbf{B}^d) \rightarrow \Pi_n^d$ denote the projection operator and partial sum operators

$$\text{proj}_m^\Delta f(x) = \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{m,\Delta} Q_{j,\nu}^m(x) \quad \text{and} \quad S_n^\Delta f(x) = \sum_{m=0}^n \text{proj}_m^\Delta f(x).$$

We denote by $\|\cdot\|_\Delta$ the norm induced by the inner product (3.1), and by $\mathcal{E}_n(f)_\Delta$ the error of best approximation in $\mathbb{H}^2(\mathbf{B}^d)$ given by

$$\mathcal{E}_n(f)_\Delta = \|f - S_n^\Delta f\|_\Delta.$$

It turns out that the orthogonal expansion can be computed without involving the derivatives of f .

Proposition 4.1. *For $j \geq 1$, let $\beta_j = n - 2j + \frac{d-2}{2}$. Then*

$$\widehat{f}_{j,\nu}^{n,\Delta} = \frac{2j(j+1)}{\sigma_{d-1} \widetilde{H}_{j,n}^\Delta} \left[(\beta_j + j)(\beta_j + j + 1) \int_{\mathbf{B}^d} f(x) Q_{j,\nu}^n(x) dx - \frac{1}{2} \int_{\mathbf{S}^{d-1}} f(\xi) Y_\nu^{n-2j}(\xi) d\sigma(\xi) \right];$$

furthermore, for $j = 0$

$$\widehat{f}_{0,\nu}^{n,\Delta} = \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi) Y_\nu^n(\xi) d\sigma(\xi).$$

Proof. Applying Green’s identity (3.10) with $v(x) = (1 - \|x\|^2) f(x)$ and $u = \Delta[(1 - \|x\|^2) Q_{j,\nu}^n(x)] = 4j(j+1) P_{j,\nu}^{n,0}(x)$, $j \geq 1$, shows

$$\begin{aligned} \widehat{f}_{j,\nu}^{n,\Delta} &= \frac{1}{\sigma_{d-1} \widetilde{H}_{j,n}^\Delta} \left[\lambda \int_{\mathbf{S}^{d-1}} f(\xi) Q_{j,\nu}^n(\xi) d\sigma(\xi) \right. \\ &\quad \left. + \frac{1}{8} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2) f(x)] \Delta[(1 - \|x\|^2) Q_{j,\nu}^n(x)] dx \right] \\ &= \frac{1}{8 \sigma_{d-1} \widetilde{H}_{j,n}^\Delta} \left[\int_{\mathbf{B}^d} (1 - \|x\|^2) f(x) \Delta^2[(1 - \|x\|^2) Q_{j,\nu}^n(x)] dx \right. \\ &\quad \left. - 8j(j+1) \int_{\mathbf{S}^{d-1}} f(\xi) Y_\nu^{n-2j}(\xi) d\sigma(\xi) \right], \end{aligned}$$

where we have used (3.2) and $P_j^{(0,\beta)}(1) = 1$. The stated result for $j \geq 1$ follows from (3.7). The proof of $j = 0$ is similar but easier, in which we need to use $\Delta[(1 - \|x\|^2) Y_\nu^n(x)] = -4(n + \frac{d}{2}) Y_\nu^n(x)$ and $\widetilde{H}_{0,n}^\Delta = \lambda + n + \frac{d}{2}$. \square

We point out that the linear operator \mathcal{D} defined by

$$\mathcal{D}[P(x)] = \Delta[(1 - \|x\|^2) P(x)], \quad P \in \Pi^d,$$

is a bijection on Π^d . Indeed, by Proposition 3.3, $\mathcal{D}[Q_{j,\nu}^n(x)] = c_{j,n} P_{j,\nu}^{n,0}$ with $c_{j,n} \neq 0$. Therefore, for each $n \geq 0$, \mathcal{D} is a one-to-one correspondence between the classical basis for $\mathcal{V}_n^d(W_0)$ and the Sobolev basis for $\mathcal{V}_n^d(\Delta)$ defined in Definition 3.2.

The following results will be used to estimate the error of approximation of the Sobolev orthogonal expansion with respect to the basis in Definition 3.2.

Proposition 4.2. *For $f \in \mathbb{H}^2(\mathbf{B}^d)$ and $m \geq 0$, we have*

$$\mathcal{D} \text{proj}_m^\Delta f(x) \in \mathcal{V}_m^d(W_0).$$

Furthermore, for $m \geq 0$, we have

$$\Delta_0 \text{proj}_m^\Delta f(x) \in \mathcal{V}_m^d(\Delta).$$

Proof. By the definition of $\text{proj}_m^\Delta f(x)$ and (3.3), we have

$$\begin{aligned} \mathcal{D} \text{proj}_m^\Delta f(x) &= \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{m,\Delta} \Delta \left[(1 - \|x\|^2) Q_{j,\nu}^m(x) \right] \\ &= \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a_{m-2j}^d} c_{n,j} \widehat{f}_{j,\nu}^{m,\Delta} P_{j,\nu}^{m,0}(x). \end{aligned}$$

Therefore, $\mathcal{D} \text{proj}_m^\Delta f(x) \in \mathcal{V}_m^d(W_0)$.

The second part of the proposition is a direct consequence of identity (2.2). \square

We use the previous result to show that \mathcal{D} commutes with the partial Fourier sum S_n^Δ .

Proposition 4.3. For $f \in \mathbb{H}^2(\mathbf{B}^d)$

$$\mathcal{D} S_n^\Delta f = S_n^0(\mathcal{D}f) \quad \text{and} \quad \Delta_0 S_n^\Delta f = S_n^\Delta(\Delta_0 f).$$

Proof. By the definition, $f - S_n^\Delta f = \sum_{m=n+1}^{+\infty} \text{proj}_m^\Delta f$. From Proposition 4.2, we get that $\langle \mathcal{D}[f - S_n^\Delta f], P \rangle_0 = 0$ for all $P \in \Pi_n^d$. Consequently, $S_n^0(\mathcal{D}f - \mathcal{D} S_n^\Delta f) = 0$. Since S_n^0 reproduces polynomials of degree at most n , then $S_n^0(\mathcal{D} S_n^\Delta f) = \mathcal{D} S_n^\Delta f$, which implies that

$$0 = S_n^0(\mathcal{D}f - \mathcal{D} S_n^\Delta f) = S_n^0(\mathcal{D}f) - \mathcal{D} S_n^\Delta f,$$

and the commutation relation is proved.

The second part can be established in a similar way taking into account that Δ_0 maps \mathcal{H}_n^d to itself. \square

The relation in the proposition above passes down to the Fourier coefficients.

Proposition 4.4. For $f \in \mathbb{H}^2(\mathbf{B}^d)$

$$\begin{aligned} \widehat{\mathcal{D}f}_{0,\nu}^{n,0} &= -4 \left(n + \frac{d}{2} \right) \widehat{f}_{0,\nu}^{n,\Delta}, \\ \widehat{\mathcal{D}f}_{j,\nu}^{n,0} &= 4j(j+1) \widehat{f}_{j,\nu}^{n,\Delta}, \quad 1 \leq j \leq \frac{n}{2}, \end{aligned}$$

and

$$\widehat{\Delta_0 f}_{j,\nu}^{n,\Delta} = -(n-2j)(n-2j+d-2) \widehat{f}_{j,\nu}^{n,\Delta}, \quad 0 \leq j \leq \frac{n}{2}.$$

Proof. Using the identity $\text{proj}_n^\Delta f = S_n^\Delta f - S_{n-1}^\Delta f$ and Proposition 4.3, we obtain $\mathcal{D} \text{proj}_n^\Delta f = \text{proj}_n^0(\mathcal{D}f)$. Then, the first two identities follow from (3.3). The last one is a direct consequence of (2.2). \square

Theorem 4.5. For $f \in \mathbb{H}^2(\mathbf{B}^d)$

$$\mathcal{E}_n(f)_\Delta = c \mathcal{E}_n(\mathcal{D}f)_0, \quad n \geq 0.$$

Proof. The Parseval identity reads

$$\mathcal{E}_n(f)_\Delta^2 = \|f - S_n^\Delta f\|_\Delta^2 = \sum_{m=n+1}^\infty \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_\nu \left| \widehat{f}_{j,\nu}^{m,\Delta} \right|^2 \widetilde{H}_{j,m}^\Delta = \Sigma_1 + \Sigma_2,$$

where we split the sum as

$$\Sigma_1 = \sum_{m=n+1}^\infty \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \sum_\nu \left| \widehat{f}_{j,\nu}^{m,\Delta} \right|^2 \widetilde{H}_{j,m}^\Delta,$$

$$\Sigma_2 = \sum_{m=n+1}^\infty \sum_\nu \left| \widehat{f}_{0,\nu}^{m,\Delta} \right|^2 \widetilde{H}_{0,m}^\Delta.$$

We estimate Σ_1 first. Using Proposition 4.4, we get

$$\left| \widehat{f}_{j,\nu}^{m,\Delta} \right|^2 = \frac{1}{16j^2(j+1)^2} \left| \widehat{\mathcal{D}f}_{j,\nu}^{m,0} \right|^2.$$

Furthermore

$$\frac{\widetilde{H}_{j,m}^\Delta}{H_{j,m}^0} = \frac{j^2(j+1)^2 h_j^{(0,n-2j+\frac{d-2}{2})}}{2^{n-2j+\frac{d}{2}} b_0 \sigma_{d-1} h_j^{(0,n-2j+\frac{d-2}{2})}} = \frac{2j^2(j+1)^2}{b_0 \sigma_{d-1}}.$$

Consequently, it follows that:

$$\Sigma_1 = \frac{1}{8d} \sum_{m=n+1}^\infty \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \sum_\nu \left| \widehat{\mathcal{D}f}_{j,\nu}^{m,0} \right|^2 H_{j,m}^0.$$

Next, we estimate Σ_2 . Using Proposition 4.4 again, we obtain

$$\left| \widehat{f}_{0,\nu}^{m,\Delta} \right|^2 = \frac{1}{16(m+\frac{d}{2})^2} \left| \widehat{\mathcal{D}f}_{0,\nu}^{m,0} \right|^2.$$

Moreover

$$\frac{\widetilde{H}_{0,m}^\Delta}{H_{0,m}^0} = \frac{(m+\frac{d}{2})(m+\frac{d}{2}+\lambda)}{\frac{d}{2}}.$$

Consequently, it follows that:

$$\Sigma_2 = c \sum_{m=n+1}^\infty \sum_\nu \left| \widehat{\mathcal{D}f}_{j,\nu}^{m,0} \right|^2 H_{j,m}^0.$$

Putting these estimates together completes the proof of the theorem. □

The main result of this section is stated in the following theorem.

Theorem 4.6. For $f \in \mathbb{H}^2(\mathbf{B}^d)$ and $n \geq 1$

$$\left\| \mathcal{D} f - \mathcal{D} S_n^\Delta f \right\|_0 = \mathcal{E}_n(\mathcal{D} f)_0, \tag{4.1}$$

$$\left\| \partial_i \left[(1 - \|x\|^2) (f - S_n^\Delta f) \right] \right\|_0 \leq \frac{c}{n} \mathcal{E}_n(\mathcal{D} f)_0, \quad 1 \leq i \leq d, \tag{4.2}$$

$$\left\| (1 - \|x\|^2) (f - S_n^\Delta f) \right\|_0 \leq \frac{c}{n^2} \mathcal{E}_n(\mathcal{D} f)_0. \tag{4.2}$$

Proof. By Proposition 4.3

$$\| \mathcal{D} f - \mathcal{D} S_n^\Delta f \|_0 = \| \mathcal{D} f - S_n^0(\mathcal{D} f) \|_0 = \mathcal{E}_n(\mathcal{D} f)_0,$$

which proves (4.1).

Now, we deal with (4.2). We use the well-known duality argument, the so called Aubin–Nitsche technique [4]. We use the characterization

$$\left\| (1 - \|x\|^2) (f - S_n^\Delta f) \right\|_0 = \sup_{\|g\|_0 \neq 0} \frac{|\langle g, (1 - \|x\|^2) (f - S_n^\Delta f) \rangle_0|}{\|g\|_0}. \tag{4.3}$$

We introduce the following auxiliary boundary-value problem:

$$\begin{cases} \Delta^2 \varphi_g = g, & \text{in } \mathbf{B}^d, \\ \Delta \varphi_g = 0, & \text{on } \mathbf{S}^{d-1}, \\ \varphi_g = 0. & \text{on } \mathbf{S}^{d-1}. \end{cases} \tag{4.4}$$

Observe that, by Green’s identity, for $h \in \mathbb{H}^2(\mathbf{B}^d)$, we have

$$\begin{aligned} \frac{1}{8 \sigma_{d-1}} \int_{\mathbf{B}^d} \Delta \left[(1 - \|x\|^2) h \right] \Delta \varphi_g \, dx &= \frac{1}{8 \sigma_{d-1}} \int_{\mathbf{B}^d} (1 - \|x\|^2) h \Delta^2 \varphi_g \, dx \\ &= \frac{1}{8 \sigma_{d-1}} \int_{\mathbf{B}^d} (1 - \|x\|^2) h g(x) \, dx \\ &= \frac{1}{8d} \langle g, (1 - \|x\|^2) h \rangle_0, \end{aligned} \tag{4.5}$$

where we have used $b_0 = d/\sigma_{d-1}$. Moreover, since $\varphi_g = 0$ on \mathbf{S}^{d-1} , there is a function $\tilde{\varphi}_g$, such that $\varphi_g = (1 - \|x\|^2) \tilde{\varphi}_g$. If $g = 0$, then $\|\tilde{\varphi}_g\|_\Delta^2 = \langle \tilde{\varphi}_g, \tilde{\varphi}_g \rangle_\Delta = 0$, which implies that $\tilde{\varphi}_g \equiv 0$. This shows that the homogeneous version of (4.4) has a unique solution $\varphi_g = 0$ and, by the linearity of the problem (4.4), we also have that the non-homogeneous problem with $g \in L^2(\mathbf{B}^d)$ has a unique solution.

Using (4.5), we get

$$\langle f - S_n^\Delta f, \tilde{\varphi}_g \rangle_\Delta = \frac{1}{8d} \langle g, (1 - \|x\|^2) (f - S_n^\Delta f) \rangle_0.$$

Since S_n^Δ reproduces polynomials of degree n , it follows that:

$$\langle f - S_n^\Delta f, S_n^\Delta \tilde{\varphi}_g \rangle_\Delta = 0.$$

Consequently

$$\begin{aligned} |\langle g, (1 - \|x\|^2) (f - S_n^\Delta f) \rangle_0| &\leq \langle f - S_n^\Delta f, \tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g \rangle_\Delta \\ &\leq \|f - S_n^\Delta f\|_\Delta \|\tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g\|_\Delta. \end{aligned}$$

Therefore, by (4.3), we have

$$\left\| (1 - \|x\|^2) (f - S_n^\Delta f) \right\|_0 \leq \|f - S_n^\Delta f\|_\Delta \left(\sup_{\|g\|_0 \neq 0} \frac{\|\tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g\|_\Delta}{\|g\|_0} \right).$$

Moreover, by Theorem 4.5 and (2.6)

$$\begin{aligned} \|\tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g\|_\Delta &= c \mathcal{E}_n(\mathcal{D}\tilde{\varphi}_g)_0 = c \mathcal{E}_n(\Delta \varphi_g)_0 \\ &\leq \frac{c}{n^2} \left[\mathcal{E}_{n-2}(\Delta^2 \varphi_g)_2 + \mathcal{E}_n(\Delta_0 \Delta \varphi_g)_0 \right]. \end{aligned} \tag{4.6}$$

Let us bound the term $\mathcal{E}_n(\Delta_0 \Delta \varphi_g)_0$. Since $P_{0,\nu}^{n,0}$ and $P_{\frac{\nu}{2},1}^{n,0}$ are harmonic and radial functions, respectively, then $\Delta_0 \Delta P_{0,\nu}^{n,0} = 0$ and $\Delta_0 \Delta P_{\frac{\nu}{2},1}^{n,0} = 0$. This means that

$$\mathcal{E}_n(\Delta_0 \Delta \varphi_g)_0^2 = \sum_{m=n+1}^\infty \sum_{j=1}^{\lfloor \frac{m-2}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \left| \widehat{\Delta_0 \Delta \varphi_{g,j,\nu}}^{m,0} \right|^2 H_{j,m}^0.$$

We need the following identities (see (3.5) in [9]):

$$\widehat{\Delta_0 \Delta \varphi_{g,j,\nu}}^{m,0} = \lambda_{m-2j} \widehat{\Delta \varphi_{g,j,\nu}}^{m,0} \quad \text{and} \quad \widehat{\Delta^2 \varphi_{g,j-1,\nu}}^{m-2,2} = \kappa_{m-j}^0 \widehat{\Delta \varphi_{g,j,\nu}}^{m,0},$$

for $0 \leq j \leq \frac{m}{2}$ in the first identity and $0 \leq j \leq \frac{m-2}{2}$ in the second identity, where λ_{m-2j} and κ_{m-j}^0 are defined in Lemma 2.2. Using these identities, we get

$$\mathcal{E}_n(\Delta_0 \Delta \varphi_g)_0^2 = \sum_{m=n+1}^\infty \sum_{j=1}^{\lfloor \frac{m-2}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \frac{|\lambda_{m-2j}|^2}{|\kappa_{m-j}^0|^2} \left| \widehat{\Delta^2 \varphi_{g,j-1,\nu}}^{m-2,2} \right|^2 H_{j,m}^0.$$

Moreover

$$\frac{H_{j,m}^0}{H_{j-1,m-2}^2} = \frac{2(m-j + \frac{d-2}{2})(m-j + \frac{d+2}{2})}{(\frac{d+2}{2})j(j+1)}.$$

Therefore, for $1 \leq j \leq \frac{m-2}{2}$

$$\frac{|\lambda_{m-2j}|^2}{|\kappa_{m-j}^0|^2} \frac{H_{j,m}^0}{H_{j-1,m-2}^2} \leq c \frac{m+d}{m + \frac{d-2}{2}}.$$

Consequently

$$\begin{aligned} \mathcal{E}_n(\Delta_0 \Delta \varphi_g)_0^2 &\leq c \sum_{m=n+1}^\infty \frac{m+d}{m + \frac{d-2}{2}} \sum_{j=1}^{\lfloor \frac{m-2}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \left| \widehat{\Delta^2 \varphi_{g,j-1,\nu}}^{m-2,2} \right|^2 H_{j-1,m-2}^2 \\ &\leq c \mathcal{E}_{n-2}(\Delta^2 \varphi_g)_2^2. \end{aligned} \tag{4.7}$$

Putting together (4.6) and (4.7), we get

$$\|\tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g\|_\Delta \leq \frac{c}{n^2} \mathcal{E}_{n-2}(\Delta^2 \varphi_g)_2 \leq \frac{c}{n^2} \|\Delta^2 \varphi_g\|_2 \leq \frac{c}{n^2} \|\Delta^2 \varphi_g\|_0 = \frac{c}{n^2} \|g\|_0,$$

where we have used (4.4) and the fact that $(1 - \|x\|^2)^2 \leq 1$ on \mathbf{B}^d . This implies that

$$\sup_{\|g\|_0 \neq 0} \frac{\|\tilde{\varphi}_g - S_n^\Delta \tilde{\varphi}_g\|_\Delta}{\|g\|_0} \leq \frac{c}{n^2},$$

and, therefore, by Theorem 4.5 again, we have

$$\left\| (1 - \|x\|^2) (f - S_n^\Delta f) \right\|_0 \leq \frac{c}{n^2} \mathcal{E}_n(\mathcal{D}f)_0,$$

which proves (4.2).

The intermediate case follows from the multivariate Landau–Kolmogorov inequality [3]: for $i = 1, 2, \dots, d$

$$\begin{aligned} & \left\| \partial_i \left[(1 - \|x\|^2) (f - S_n^\Delta f) \right] \right\|_0 \\ & \leq c \left\| (1 - \|x\|^2) (f - S_n^\Delta f) \right\|_0^{1/2} \left\| \mathcal{D}f - \mathcal{D}S_n^\Delta f \right\|_0^{1/2}. \end{aligned}$$

□

This kind of result has not been proved yet for the inner product with a mass point at the origin studied in the next section, and will be analysed in the near future.

5. A Sobolev Inner Product with a Mass Point at the Origin

In this section, we consider a Sobolev inner product obtained from $\langle \cdot, \cdot \rangle_\Delta$ by replacing the integral over \mathbf{S}^{d-1} with a mass point at the origin. That is, this new inner product is given by

$$\begin{aligned} \langle f, g \rangle_\Delta^* &= \frac{\lambda^*}{\sigma_{d-1}} f(0) g(0) \\ &+ \frac{1}{8\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx, \quad \lambda^* > 0. \end{aligned} \tag{5.1}$$

As in Definition 3.2, orthogonal polynomials with respect to this inner product can be constructed in a similar way using classical ball polynomials (2.3) and Theorem 2.4 in [11], but, in this case, there is a small modification.

Definition 5.1. For $n \geq 0$ and $0 \leq j \leq \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leq \nu \leq a_{n-2j}^d\}$ denote an orthonormal basis of \mathcal{H}_{n-2j}^d . We define the polynomials

$$\begin{aligned} R_{0,\nu}^n(x) &:= Y_\nu^n(x), \\ R_{j,\nu}^n(x) &:= (-1)^j \|x\|^2 P_{j-1}^{(n-2j+\frac{d-2}{2}, 2)}(1 - 2\|x\|^2) Y_\nu^{n-2j}(x), \quad 1 \leq j \leq \frac{n}{2}. \end{aligned}$$

The following proposition is useful in studying the orthogonality of these polynomials with respect to the inner product (5.1).

Proposition 5.2. *The polynomials $R_{j,\nu}^n$ satisfy*

$$\Delta[(1 - \|x\|^2) R_{j,\nu}^n(x)] = c_{j,n} P_{j,\nu}^{n,0}, \quad n \geq 0, \tag{5.2}$$

where $c_{j,n}$ are constants defined in (3.3).

Proof. For $j = 0$, by (3.2), we have

$$\Delta[(1 - \|x\|^2) R_{0,\nu}^n(x)] = \Delta[(1 - \|x\|^2) Y_\nu^n(x)] = -4 \left(n + \frac{d}{2} \right) Y_\nu^n(x).$$

Now, we deal with $1 \leq j \leq \frac{n}{2}$. The Jacobi polynomials satisfy the following properties ([10, p.71]):

$${}_s P_{j-1}^{(\beta,2)}(1 - 2s) = \frac{1}{2j + \beta + 1} \left[(j + 1) P_{j-1}^{(\beta,1)}(1 - 2s) + j P_j^{(\beta,1)}(1 - 2s) \right],$$

and ([10, p.59]):

$$P_j^{(\beta,1)}(1 - 2s) = (-1)^j P_j^{(1,\beta)}(2s - 1).$$

Therefore

$${}_s P_{j-1}^{(\beta,2)}(1 - 2s) = \frac{(-1)^{j-1}}{2j + \beta + 1} \left[(j + 1) P_{j-1}^{(1,\beta)}(2s - 1) - j P_j^{(1,\beta)}(2s - 1) \right]. \tag{5.3}$$

As in the proof of Proposition 3.3, this implies that

$$\mathcal{J}_\beta \left[{}_s P_{j-1}^{(\beta,2)}(1 - 2s) \right] = (-1)^j j (j + 1) P_j^{(0,\beta)}(2s - 1). \tag{5.4}$$

Then, by (3.2) and (5.4) with $\beta = n - 2j + \frac{d-2}{2}$

$$\Delta \left[(1 - \|x\|^2) R_{j,\nu}^n(x) \right] = 4j(j + 1) P_{j,\nu}^{n,0}(x).$$

This proves (5.2). □

The following relation follows readily from (2.3), (5.3), and (3.8).

Proposition 5.3. *The polynomials $R_{j,\nu}^n$ satisfy*

$$R_{0,\nu}^n(x) = P_{0,\nu}^{n,1}(x),$$

$$R_{j,\nu}^n(x) = \frac{1}{n + \frac{d}{2}} \left[j P_{j,\nu}^{n,1}(x) - (j + 1) P_{j-1,\nu}^{n-2,1}(x) \right], \quad 1 \leq j \leq \frac{n}{2}.$$

In particular, $R_{0,\nu}^n = Q_{0,\nu}^n$ and $R_{j,\nu}^n = -Q_{j,\nu}^n$ for $1 \leq j \leq \frac{n}{2}$.

Let us denote by $\mathcal{V}_n^{d,*}(\Delta)$ the space of Sobolev orthogonal polynomials of degree n with respect to (5.1).

Proposition 5.4. For $n \geq 0$, $\{Q_{j,\nu}^n : 0 \leq j \leq \frac{n}{2}, 1 \leq \nu \leq a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_{n,\nu}^{d,*}(\Delta)$. Moreover

$$\langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_{\Delta}^* = \tilde{H}_{j,n}^* \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta},$$

where,

$$\tilde{H}_{j,n}^* = \begin{cases} \lambda^* \delta_{n,0} + n + \frac{d}{2}, & j = 0, \\ \frac{j^2 (j + 1)^2}{2^{n-2j+\frac{d}{2}}} h_j^{(0,n-2j+\frac{d-2}{2})}, & 1 \leq j \leq \frac{n}{2}. \end{cases}$$

Proof. The proof is similar to that of Proposition 3.6, but the fact that $Q_{j,\nu}^n(0) = -R_{j,\nu}^n(0) = 0$ for $n \geq 0$ must be taken into account. \square

Following the proof of Proposition 4.1 and using Proposition 5.1, we can obtain the Fourier coefficients of a function relative to (5.1) given by

$$\hat{f}_{j,\nu}^{n,*} = \frac{1}{\tilde{H}_{j,n}^*} \langle f, R_{j,\nu}^n \rangle_{\Delta}^*.$$

Proposition 5.5. For $n \geq 1$

$$\hat{f}_{j,\nu}^{n,*} = \hat{f}_{j,\nu}^{n,\Delta}, \quad 0 \leq j \leq \lfloor n/2 \rfloor,$$

and

$$\hat{f}_{0,1}^{0,*} = \frac{1}{\sigma_{d-1} (\lambda^* + \frac{d}{2})} \left[\lambda^* f(0) + \frac{d}{2} \int_{\mathbf{S}^{d-1}} f(\xi) d\sigma(\xi) \right].$$

Proposition 3.6 together with Proposition 5.4 yields the following result.

Proposition 5.6. Define the inner product

$$\begin{aligned} \langle f, g \rangle_{\Delta}^{**} &= \frac{\lambda^*}{\sigma_{d-1}} f(0) g(0) + \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi) \\ &+ \frac{1}{4\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1 - \|x\|^2)f(x)] \Delta[(1 - \|x\|^2)g(x)] dx, \quad \lambda^*, \lambda > 0, \end{aligned} \tag{5.5}$$

and let $\mathcal{V}_n^{d,**}(\Delta)$ be the space of Sobolev orthogonal polynomials of degree n with respect to (5.5). For $n \geq 0$, $\{Q_{j,\nu}^n : 0 \leq j \leq \frac{n}{2}, 1 \leq \nu \leq a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^{d,**}(\Delta)$. Moreover

$$\langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_{\Delta}^{**} = \left(\tilde{H}_{j,n}^{\Delta} + \tilde{H}_{j,n}^* \right) \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta},$$

Conclusions

In this work, we analyse the impact of an additional term in the Sobolev inner product introduced by Y. Xu in [11] used to find the numerical solution of the Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions (Atkinson and Hansen [2]). Each of the two Sobolev inner products considered in this paper includes an additional term that takes into account the values of the functions on the boundary of the unit ball of \mathbb{R}^d (see (3.1)), or the evaluation of the functions at the origin (see (5.1)). These

new Sobolev inner products contemplate the prospect of knowing some additional values of the involved functions and offer the possibility of taking into account more characteristics of the functions that can be approximated. The additional terms are multiplied by a positive real constant λ generalizing the inner product considered by Y. Xu in [11]. Apart from the exhaustive description of the orthogonal structure in both cases, giving explicit expressions for the bases, we provide the Fourier coefficients of the approximations in both cases and related them with the Fourier coefficients for the original case. We must remark that the spherical term in the first Sobolev inner product has influence over the angular part of the orthogonal expansion, that is, the terms associated with the index value $j = 0$ (see (3.9)), and, in the second Sobolev inner product, that is, when the value of the functions at the origin is added, only the first coefficient in the Fourier expansion is affected by the new conditions (Proposition 5.4). Moreover, the error of approximation for the orthogonal expansions with respect to the Sobolev inner products appearing in this work (in particular, the one introduced in [11]) have not been previously studied in the literature.

Acknowledgements

Not applicable.

Author contributions All authors have contributed equally to this article, searching for related results, selecting relevant information, and writing and reviewing the draft.

Funding Funding for open access publishing: Universidad de Granada/CBUA Funding for open access charge: Universidad de Granada.

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare no competing interests.

Ethical Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by

statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York (1972)
- [2] Atkinson, K., Hansen, O.: Solving the nonlinear Poisson equation on the unit disk. *J. Integr. Equ. Appl.* **17**, 223–241 (2005)
- [3] Boyadzhiev, K.N.: A many-variable Landau-Kolmogorov inequality. *Math. Proc. Camb. Philos. Soc.* **101**, 123–129 (1987)
- [4] Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. SIAM Publications, Philadelphia (2002)
- [5] Dai, F., Xu, Y.: Approximation Theory and Harmonic Analysis on Spheres and Balls. Springer Monographs in Mathematics. Springer, New York (2013)
- [6] Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables. Encyclopedia of Mathematics and Its Applications, vol. 155, 2nd edn. Cambridge University Press, Cambridge (2014)
- [7] Marcellán, F., Xu, Y.: On Sobolev orthogonal polynomials. *Expo. Math.* **33**(3), 308–352 (2015)
- [8] Marriaga, M.E., Pérez, T.E., Piñar, M.A., Recarte, M.J.: Approximation via gradients on the ball. The Zernike case. *J. Comput. Appl. Math.* **430**, 115258 (2023)
- [9] Piñar, M.A., Xu, Y.: Best polynomial approximation on the unit ball. *IMA J. Numer. Anal.* **38**(3), 1209–1228 (2018)
- [10] Szegő, G.: Orthogonal Polynomials, American Mathematical Society, Colloquium Publications, vol. 23, 4th ed. Amer. Math. Soc., Providence (1975)
- [11] Xu, Y.: A family of Sobolev orthogonal polynomials on the unit ball. *J. Approx. Theory* **138**(2), 232–241 (2006)
- [12] Xu, Y.: Sobolev orthogonal polynomials defined via gradient on the unit ball. *J. Approx. Theory* **152**, 52–65 (2008)
- [13] Xu, Y.: Approximation and orthogonality in Sobolev spaces on a triangle. *Constr. Approx.* **46**, 349–434 (2017)

Misael E. Marriaga
Departamento de Matemática Aplicada, Ciencia e Ingeniería de Materiales y
Tecnología Electrónica
Universidad Rey Juan Carlos
Madrid
Spain
e-mail: misael.marriaga@urjc.es

Teresa E. Pérez
Instituto de Matemáticas IMAG and Departamento de Matemática Aplicada
Universidad de Granada
Granada
Spain
e-mail: tperez@ugr.es

Marlon J. Recarte
Departamento de Física
Universidad Nacional Autónoma de Honduras en el valle de Sula
San Pedro Sula
Honduras
e-mail: marlon.recarte@unah.edu.hn

Received: January 13, 2023.

Revised: August 23, 2023.

Accepted: September 1, 2023.