Article Identifier
DOI: 10.22032/dbt.58901
URN: urn:nbn:de:gbv:ilm1-2023isc-057:9

60th ILMENAU SCIENTIFIC COLLOQUIUM
Technische Universität Ilmenau, 4 – 8 September 2023
URN: urn:nbn:de:gbv:ilm1-2023isc:1

ON THE OBSERVABILITY OF EMBEDDED POLYNOMIAL DYNAMICAL SYSTEMS

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ABSTRACT

Testing a system for observability is of great practical relevance in technical applications. For linear systems, this problem was solved decades ago. The observability of nonlinear systems can be formally defined, but the actual verification is extremely difficult. For the subclass of poynomial systems, the observability can be decided in a finite number of calculation steps. In this paper, we provide an observability test for embedded polynomial systems. The observability test uses methods of algebraic geometry.

Index Terms— Observability, polynomial systems, embedded systems, algebraic geometry

1. INTRODUCTION

The concept of observability characterizes the possibility of estimating or reconstructing certain system variables from the measurement of other variables. In a narrower sense, it is usually about state observability, i. e., the question whether the state can be determined from the measurement of the output trajectory (and in the case of non-autonomous systems also from the measurement of the input trajectory). The question of observability plays a major role in both the monitoring and the control of technical processes.

In most practical applications the whole state of a dynamical system is not measured directly. Additionally, one wants to reduce the number of sensors required, while the systems state and possibly slowly changing parameters should still be reconstructable from the sensor outputs. If this should be achievable at all, the systems state must be observable. This observability property is important to answer the question of where sensors must be located, or which are redundant [1, 2].

While the observability of a linear time-invariant system can be easily decided by well–known observability criteria [3,4], this becomes more difficult for nonlinear systems. However, many systems are not linear. Thus, a method to decide the observability for nonlinear systems is required. In this contribution, we restrict the system class to those which can be described by polynomial equations. This applies to most mechanical systems, if the systems state is allowed to be embedded into a higher-dimensional space.

Our method allows deciding the observability properties as well as to identify those points where the local observability fails [5]. These points are intrinsic to the system and cannot be compensated by any observer. Furthermore, the algorithm computes the number of output derivatives required. This implies the minimum dimension of the observer state, which may be higher than the dimension of the systems state – a property that does not apply to linear systems.

The paper is structured as follows: In Section 2 we remind the reader of the concepts of local and global observability for nonlinear systems. In order to be able to treat polynomial systems, we provide some basics of algebraic geometry in Section 3. The observability test is described in Section 4. Our method is applied on a mechanical example system in Section 5. Some conclusions are provided in Section 6.

2. NONLINEAR OBSERVABILITY

In this contribution nonlinear systems of the form

$$\dot{x} = f(x), \quad x \in \mathcal{M} \subseteq \mathbb{R}^n \tag{1a}$$
$$y = h(x) \tag{1b}$$

with polynomial vector field f and polynomial scalar fields $h = (h_1, \ldots, h_p)$ are considered. The state space $\mathcal{M}$ is described implicitly by a set of algebraic equations

$$\mathcal{M} = \{x \in \mathbb{R}^n \,|\, g_1(x) = 0, \ldots, g_s(x) = 0\}, \tag{2}$$

which are polynomials, too. Since polynomial functions are analytic, the differential equation (1a) has a locally analytic solution $x(t)$ for each initial value $x_0 \in \mathcal{M}$. Thus, the output trajectory is locally analytic as well. This allows to (locally) expand the output trajectory in a Taylor series

$$y(t) = \sum_{k=0}^{\infty} \tfrac{t^k}{k!} \, \mathrm{L}_f^k h(x(0)), \tag{3}$$

where $\mathrm{L}_f^k h$ denotes the k-th Lie derivative of h along the vector field f. This is the Lie derivative of a scalar field and computed component wise for the output map h and must not be confused with Lie derivative of a vector field [6]. The Lie derivatives can be defined in a recursive way [7, 8]:

$$\mathrm{L}_f^0 h(x) = h(x), \quad \mathrm{L}_f^1 h(x) = \mathrm{L}_f h(x) = \tfrac{\partial h}{\partial x}(x) f(x), \quad \mathrm{L}_f^{k+1} h(x) = \mathrm{L}_f \mathrm{L}_f^k h(x).$$

In the nonlinear case the observability is based on the indistinguishability of system states. Two states $x_1, x_2 \in \mathcal{M}$ of the system (1) are called *indistinguishable* on the interval $[0, t^*]$, if the output trajectories $t \mapsto y(t)$ with initial conditions $x(0) = x_1$ and $x(0) = x_2$, respectively, are equal for all $t \in [0, t^*]$, see [9–11]. Due to the local analyticity of the output trajectories, indistinguishable states evaluate to the same coefficients of the Lie series (3) for the output trajectory. This motivates the definition of the *observability map*

$$q : \mathcal{M} \to \mathbb{R}^{p \times \infty}, \quad x \mapsto q(x) = \big(h(x), \mathrm{L}_f h(x), \mathrm{L}_f^2 h(x), \ldots\big),$$

which maps to the series coefficients. The observability map maps indistinguishable states $x_1, x_2 \in \mathcal{M}$ to the same point: $q(x_1) = q(x_2)$.

Using the notion of indistinguishability, observability is defined as follows [9, 10]: A system is called *globally observable*, if the observability map is injective, i. e.,

$$\forall x, \bar{x} \in \mathcal{M} : q(x) = q(\bar{x}) \implies x = \bar{x}.$$

The system is *locally observable at a point* $x_0 \in \mathcal{M}$, if the observability map is injective in a neighborhood $U_{x_0} \subset \mathcal{M}$ of x_0, i. e.,

$$\forall x, \bar{x} \in U_{x_0} : q(x) = q(\bar{x}) \implies x = \bar{x}.$$

Finally, the system is called *locally observable*, if it is locally observable at every point $x_0 \in \mathcal{M}$.

The injectivity of the observability map is difficult to verify, since its image is of infinite dimension. However, in the case of polynomial systems this problem can be resolved. This, however, requires some algebraic concepts that should be introduced in the following section.

3. POLYNOMIAL IDEALS

The observability criteria are mainly a comparison of algebraic sets, i. e., real sets defined by polynomial equations. Herein, polynomials in the ring $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ are considered. The polynomials, which are equated to zero, can be summarized in a *polynomial ideal*. This is a special subset of the polynomial ring that is closed under addition and multiplication with elements of the ring. Therefore, ideals contains — let alone the trivial ideal that contains only 0 — an infinite number of elements. By Hilbert's basis theorem [12, pp. 76], however, there is finite number of *generator* or *basis* polynomial $g_1, \dots, g_s$ for every ideal $I \subseteq \mathbb{R}[x]$ such that

$$I = \{a_1 g_1 + \cdots + a_s g_s \,|\, a_1, \dots, a_s \in \mathbb{R}[s]\} =: \langle g_1, \dots, g_s \rangle.$$

Such a basis is used to represent polynomial ideals and to carry out computations. There are special bases called *Gröbner bases*. These are useful to decide the ideal membership problem and, thus, enable the comparison of ideals [13].

The common real zero set of the generator polynomials and, therefore, of all polynomials in the ideal, is called the *real variety*. In return, for each variety there is an ideal containing all polynomials that evaluate to zero for each point in the variety. Such an ideal containing all vanishing polynomials is called $real$. There is no bijection between ideals and varieties. However, there is a one-to-one correspondence between real ideals and real varieties. The ideal that contains all polynomials that evaluate to zero at the variety of an ideal I is called the *real radical* of I [14, p. 85] and denoted[1] by $\sqrt{I}$.

Such a variety $V \subseteq \mathbb{R}^n$ can be decomposed into its *irreducible components*: $V = V_1 \cup \cdots \cup V_s$. These are varieties, which cannot be written as a proper union of varieties. Alike, a real ideal can be decomposed into its *prime* components: $I = I_1 \cap \cdots \cap I_s$.

The union of ideals I and J is called the *ideal sum* and denoted by $I + J$. Apart from the different notation the ideal sum is identical to the set theoretic union.

4. ALGEBRAIC OBSERVABILITY TEST

Using these algebraic concepts the observability can finally be tested. This reduced to the comparison of the sets defined by $x = \bar{x}$ and $q(x) = q(\bar{x})$ with $x, \bar{x} \in \mathcal{M}$ for global observability and with $x, \bar{x} \in U_{x_0} \subset \mathcal{M}$ for local observability at a point $x_0 \in \mathcal{M}$. The global case is described in [11]. We refer to [5] for a detailed description of the local case. The general concept is briefly recapped.

Since $\mathcal{M}$ in (2) is variety by definition, so is the set

$$\mathcal{J} = \big\{(x, \bar{x}) \in \mathcal{M}^2 \,|\, x = \bar{x}\big\},$$

whose vanishing ideal will be denoted by J. To show that the set

$$\mathcal{I} = \big\{(x, \bar{x}) \in \mathcal{M}^2 \,|\, q(x) = q(\bar{x})\big\}$$

of indistinguishable points is a variety is less obvious. Consider the defining polynomials, which include Lie derivatives of the output map up to order k, i. e.,

$$I_k = \Big\langle g_1(x), g_1(\bar{x}), \dots, g_s(x), g_s(\bar{x}), h(x) - h(\bar{x}), \mathrm{L}_f h(x) - \mathrm{L}_f h(\bar{x}), \dots, \mathrm{L}_f^k h(x) - \mathrm{L}_f^k h(\bar{x}) \Big\rangle.$$

The variety of this ideal is the set of all initial conditions of two copies of the system, which yield the same output and output derivatives up to order k. Clearly, these ideals form an ascending chain

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots,$$

[1] This notation is usually used for a different radical ideal and the real radical is denoted by $\sqrt[\mathbb{R}]{\cdot}$. Since this is all about real algebra, only the real radical is considered.

which must stabilize after a finite number of ideals by the ascending chain condition [12, p. 80]. The variety of the stabilized ideal I_∞ equals $\mathcal{I}$. In order to detect if the chain has stabilized, the fact $I_k = I_{k+1} \implies I_k = I_{k+2}$ is used. Thus, if two consecutive ideals in the chain are equal, the chain has stabilized. The equality can be detected by comparing the reduced Gröber bases of the ideal. Finally, the system is globally observable, if $I_\infty = J$.

Local observability is not tested point wise. Instead, the variety of all not locally observable points is computed directly. Since the ideals J and I_∞ corresponding to the sets of equal $\mathcal{J}$ and indistinguishable points $\mathcal{I}$, respectively, are known, the equality of their varieties have to be tested locally. The basic idea is that if these varieties are not equal in the neighborhood of a point, the difference set $\mathcal{I} \setminus \mathcal{J}$ is not empty in this neighborhood, i. e., an irreducible component of $\mathcal{I}$ different from $\mathcal{J}$ is contained in the neighborhood. Thus, the not locally observable points are those that are contained in both that variety $\mathcal{J}$ and in an irreducible component of $\mathcal{I}$ different from $\mathcal{J}$ itself.

This can be translated to a corresponding criterion using the ideals: If

$$\sqrt{I_\infty} = J \cap P_1 \cap \cdots \cap P_s$$

is the prime decomposition of $\sqrt{I_\infty}$, the not locally observable points[2] are the variety of

$$J + (P_1 \cap \cdots \cap P_s).$$

5. EXAMPLE

As a simple example consider the physical pendulum depicted in Figure 1. The equations of motion read

$$\begin{aligned} \dot{\varphi} &= \omega \\ \dot{\omega} &= -k \sin \varphi, \end{aligned}$$

where the mass m, inertia J and distance L from the center of mass to the bearing are combined in the quantity $k = \frac{mgL}{J+mL^2}$. In this form the components of the vector field are not polynomials. However, introducing (redundant) coordinates $x_1 = \sin \varphi$, $x_2 = -\cos \varphi$ and $x_3 = \omega$ the differential equations

$$\begin{aligned} \dot{x}_1 &= -x_2 x_3 \\ \dot{x}_2 &= x_1 x_3 \\ \dot{x}_3 &= -k x_1 \end{aligned}$$

result. These redundant coordinates are subject to the algebraic constraint

$$0 = x_1^2 - x_2^2 - 1$$

defining a cylinder embedded in $\mathbb{R}^3$, see Figure 2.

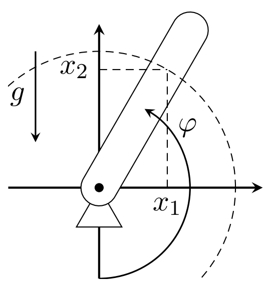

Fig. 1: A physical pendulum described by Cartesian coordinates.

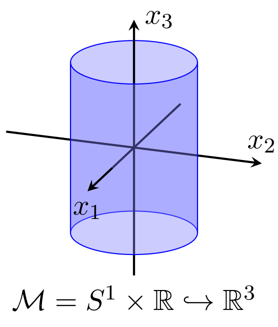

Fig. 2: State space $\mathcal{M}$ of the pendulum embedded in $\mathbb{R}^3$.

[2]To be precise, this variety contains pairs of equal points.

The inertia parameters are assumed to be unknown, so the equation $\dot{k} = 0$ is added to the set of differential equations. Thus, the combined parameter k should be estimated by the observer as well. The motion of the pendulum is measured by an accelerometer attached to the pendulum at distance ℓ from the bearing, which measures the cartesian components of the acceleration, i. e.,

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 & -x_1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} \ell \ddot{x}_1 \\ \ell \ddot{x}_2 + g \end{pmatrix} = \begin{pmatrix} \ell k x_1 - g x_1 \\ g x_2 - \ell \omega^2 \end{pmatrix}.$$

We discuss the case of an unknown distance ℓ first, i. e., the differential equation $\dot{\ell} = 0$ is added to the system. In order to slightly simplify the equations, physical units where $g = 1$ are used in the sequel.

For the observability test the state variables $x = (x_1, x_2, \omega, k, l)$ for the system and $\bar{x} = \left(\bar{x}_1, \bar{x}_2, \bar{\omega}, \bar{k}, \bar{\ell}\right)$ for the copy are used. The combined state space $\mathcal{M}^2$ is described by the implicit equations generated by the ideal

$$I_{-1} = \left\langle x_1^2 + x_2^2 - 1, \bar{x}_1^2 + \bar{x}_2^2 - 1 \right\rangle.$$

The equality of measurements for both system copies leads to the ideal

$$I_0 = \left\langle x_1^2 + x_2^2 - 1, \bar{x}_1^2 + \bar{x}_2^2 - 1, \ell\omega^2 - \bar{\ell}\bar{\omega}^2 - x_2 + \bar{x}_2, \ell k x_1 - \overline{\ell k}\bar{x}_1 - x_1 + \bar{x}_1 \right\rangle.$$

A reduced Gröbner basis with respect to a graded ordering contains additional polynomials:

$$\begin{aligned} I_0 = \langle & x_2^2 k \bar{\omega}^2 \bar{\ell} + x_1 \omega^2 \bar{x}_1 \bar{k} \bar{\ell} - x_2^2 \omega^2 + x_2^3 k - x_1 \omega^2 \bar{x}_1 - x_2^2 k \bar{x}_2 - k \bar{\omega}^2 \bar{\ell} + \omega^2 - x_2 k + k \bar{x}_2, \\ & x_1 k \bar{\omega}^2 \bar{\ell} - \omega^2 \bar{x}_1 \bar{k} \bar{\ell} - x_1 \omega^2 + x_1 x_2 k + \omega^2 \bar{x}_1 - x_1 k \bar{x}_2, \\ & x_2^2 k \ell + x_1 \bar{x}_1 \bar{k} \bar{\ell} - x_2^2 - k\ell - x_1 \bar{x}_1 + 1, \\ & \omega^2 \ell - \bar{\omega}^2 \bar{\ell} - x_2 + \bar{x}_2, \\ & x_1 k \ell - \bar{x}_1 \bar{k} \bar{\ell} - x_1 + \bar{x}_1, \\ & x_1^2 + x_2^2 - 1, \\ & \bar{x}_1^2 + \bar{x}_2^2 - 1 \rangle. \end{aligned}$$

In order to compute the stabilized ideal I_∞, Lie derivatives of all these polynomials are added to the set of generators. After one step, the reduced Gröbner basis of the ideal I_1 contains already 106 polynomials (although the ideal itself can be generated by six polynomials, namely the two copies of the algebraic constraint, the difference of the output maps and their first Lie derivative). However, the Gröbner basis is required in order to decide the ideal membership. After a second iteration the Gröber basis of I_2 contains 751 polynomials. We continue the computation and finally arrive at ideals $I_6 = I_7 = I_\infty$. The reduced Gröbner basis for I_∞ has 151 generators. This ideal is not radical, and the radical $\sqrt{I_\infty}$ has a much simpler representation. Note that we could have tried to compute radicals in between adding Lie derivatives, but this usually get too expensive for the intermediate ideals [15].

The radical $\sqrt{I_\infty}$ can be written as an intersection

$$P_0 \cap P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5$$

of six prime ideals. These components are analysed in order to discuss the system's observability.

The first component is the ideal

$$P_0 = J = \left\langle \bar{x}_1^2 + \bar{x}_2^2 - 1, x_1 - \bar{x}_1, x_2 - \bar{x}_2, \omega - \bar{\omega}, k - \bar{k}, \ell - \bar{\ell} \right\rangle,$$

which describes the set of equal points. This ideal contains all such constructed ideals I_∞. If this was the only component, the system would have been globally observable.

The prime ideals

$$P_1 = \langle x_1, x_2 + 1, \omega, \bar{x}_1, \bar{x}_2 + 1, \bar{\omega}\rangle$$
$$P_2 = \langle x_1, x_2 - 1, \omega, \bar{x}_1, \bar{x}_2 - 1, \bar{\omega}\rangle$$

describes the equilibria of the pendulum. As can be seen, the parameters k and ℓ do not occur in these polynomials, thus, these can be arbitrary. This implies, that an observer cannot estimate these quantities, if the pendulum is in any of these equilibria.

The component

$$P_3 = \left\langle \bar{x}_1^2 + \bar{x}_2^2 - 1, x_1 - \bar{x}_1, x_2 - \bar{x}_2, \omega, k, \bar{\omega}, \bar{k}\right\rangle$$

describes the case $k = 0$, where the pendulums center of mass coincides with the suspension point. If this 'pendulum' is at rest ($\omega = 0$), the position of the accelerometer, i. e., the distance ℓ, cannot be determined. In the case of a known distance, i. e., $\ell = \bar{\ell}$, this prime component would not occur explicitly in the intersection, since then $J \subseteq P_3$. Thus, the ideal P_3 represents a special case, where the quantity ℓ cannot be (locally) observed.

There is another special case due to

$$P_4 = \left\langle \bar{x}_1^2 + \bar{x}_2^2 - 1, \bar{k}\bar{\ell} - 1, x_1 + \bar{x}_1, x_2 - \bar{x}_2, \omega + \bar{\omega}, k - \bar{k}, \ell - \bar{\ell}\right\rangle.$$

The equation $k\ell = 1$ holds e. g. for a mathematical pendulum with the accelerometer placed at the center of mass ($\ell = L$). In general this is a ratio of these parameters, where the accelerometer does not notice an acceleration in one direction, since y_1 is identically zero. Then there is an ambiguity in the direction where the pendulum swings, which can be seen from the mirrored solutions in the system copies exchanging the sign of x_1 and ω.

There is another prime component

$$P_5 = \left\langle \bar{x}_1^2 + \bar{x}_2^2 - 1, \bar{\omega}^2 + 2\bar{x}_2\bar{k}, \bar{k}\bar{\ell} + \tfrac{1}{2}, x_1 - \bar{x}_1, x_2 + \bar{x}_2, \omega + \bar{\omega}, k + \bar{k}, \ell + \bar{\ell}\right\rangle,$$

which comes by a little surprise. Here, the accelerometer is placed at the opposite side with respect to the center of mass such that $kl = -\frac{1}{2}$. There are two different periodic orbits of the pendulum that yield the same output trajectories, but only for special initial conditions. If, however, the sign of k or ℓ is known, this ambiguity can be resolved.

The not locally observable points are computed by first computing the saturation of $\sqrt{I}$ with respect to J. This removed the component $J = P_0$ from the prime decomposition. Computing the ideal sum with J again, leads to the following observation: The equations in the prime component P_5 and J are incompatible such that $P_5 + J = \langle 1\rangle$. This means that P_5 does not contribute the not locally observable points. However, this can still be a concern to an observer.

The ideal $P_4 + J$ is a superset of $(P_1 \cap P_2) + J$, since it has an additional generator $k\ell - 1$. Thus, the ideal $(P_1 \cap P_2 \cap P_3) + J$ remains, which could be written as an intersection the same way. Eliminating the variables $\bar{x}_1, \bar{x}_2, \bar{\omega}, \bar{k}, \bar{\ell}$ for the second system copy finally yields

$$\langle x_1, x_2 + 1, \omega\rangle \cap \langle x_1, x_2 - 1, \omega\rangle \cap \left\langle x_1^2 + x_2^2 - 1, \omega, k\right\rangle,$$

which describes the set of not locally observable points. This is precisely the set of equilibria.

6. CONCLUSION

This example shows that there are quite some ambiguities albeit the simplicity of the considered system. Furthermore, the intermediate expressions can become complicated and sophisticated algebraic methods are required for more complex systems.

The observability map q can also be used to transform the system into its observability canonical form. As could be observed in this example system, higher order derivatives of the output map may be required, compared to the systems state dimension. This leads again to an embedding of the observer state [16]. While this embedding may avoid problematic points for the transformed system, the not locally observable ones are intrinsic.

ACKNOWLEDGEMENT

This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — project number 417698841.

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