Towards a BRST-invariant construction of pure Yang-Mills theory

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Abstract

Yang-Mills theory is a non-Abelian field theory that constitutes the cornerstone of the strong interaction (QCD) and is at the core of the unification of electromagnetic and weak interactions (electroweak). Developing an approach which can monitor and extract its physical properties at all energy scales, while at the same time maintain certain desired symmetries is a task that is as challenging as it is interesting. In this thesis, we concentrate on the sector of QCD that does not include quarks, i.e. pure Yang-Mills, for which we investigate different BRST-invariant realizations in perturbative and non-perturbative settings. Such an analysis yields information regarding long-range physics and the effect of BRST symmetry.

Following the off-shell prescription of the Faddeev-Popov quantization, we set up a gauge-fixing action that is linear in the gauge-fixing condition by choosing instead of the conventional Gaussian, a Fourier weight. This comes at the expense of introducing an external Nakanishi-Lautrup field v. The v field can be considered as a spacetime-dependent set of gauge parameters in the adjoint representation. Then, by employing the background field method we construct an IR regulated background and BRST-invariant action. This is achieved by an appropriate choice of a non-linear gauge-fixing condition that includes regulator-mass parameters.

Following a one-loop perturbative analysis, we find no v-field dependencies for the one-loop effective action which leads to the universal one-loop beta function. The effects of the mass parameters are explored with a phenomenological study of the effective action. Then, we infer that the inclusion of BRST-invariant mass parameters suffices to cure the Nielsen-Olesen instabilities for constant magnetic backgrounds within a certain range of validity. Furthermore, assuming covariantly constant and self-dual backgrounds, we circumvent such tachyonic modes and we observe the presence of a non-trivial minimum which is justified as a manifestation of dynamical breaking of scale symmetry.

An explicit investigation of the v-field dependence as it appears in the one-loop Schwinger functional reveals no additional divergences coming from nonlocal interactions. Furthermore, we classify all possible forms of the associated two-point v-dependent correlator, which vanish in the limit of the Landau gauge.

Finally, we concern ourselves with the extension of our model to mass-dependent renormalization schemes and more precisely within the functional Renormalization Group. For that, we promote the mass parameters to regulators that are introduced in our theory through suitable non-linear gauge-fixing conditions and we construct truncated flow equations compatible with BRST symmetry at any scale. In the absence of a background field, we derive a v-dependent beta function that can be aligned with the universal one-loop beta function by averaging the v field with an appropriate Gaussian distribution. In the presence of a background field, we find a truncated flow equation which is similar to the one found in the literature and reproduces proper universal results without any further restrictions.

Zusammenfassung

Die Yang-Mills-Theorie ist eine nicht-abelsche Feldtheorie, die den Eckpfeiler der Starken Wechselwirkung (QCD) bildet und den Kern der Vereinigung der elektromagnetischen und der schwachen Wechselwirkung (elektroschwach) darstellt. Die Entwicklung eines Ansatzes, mit dem die physikalischen Eigenschaften der Theorie auf allen Energieskalen überwacht und extrahiert werden können, während gleichzeitig bestimmte gewünschte Symmetrien erhalten bleiben, ist eine ebenso anspruchsvolle wie interessante Aufgabe. In dieser Arbeit legen wir den Fokus auf verschiedene BRST-invariante Realisierungen in perturbativen und nicht-perturbativen Einstellungen in dem Sektor der QCD, der keine Quarks enthält, d.h. die reine Yang-Mills-Theorie (pure Yang-Mills theory). Eine solche Analyse liefert Informationen über die Langstreckenphysik und den Effekt der BRST-Symmetrie.

In Anlehnung an die Off-Shell-Vorgehensweise der Faddejew-Popow-Quantisierung richten wir eine Eichfixierungswirkung ein, die linear zur Eichfixierungsbedingung ist, indem wir eine Fourier-Gewichtung anstelle der konventionellen Gauß-Gewichtung wählen. Dies geschieht auf Kosten der Einführung eines externen Nakanishi-Lautrup Feld v. Das v-Feld kann als raumzeitabhängige Menge von Parametern zur Eichfixierung in der adjungierten Darstellung betrachtet werden. Dann konstruieren wir mit Hilfe der Hintergrundfeldmethode eine IR-regulierte, Hintergrund- und BRST-invariante Wirkung. Dies wird durch eine angemessene Wahl einer nichtlinearen Eichfixierungsbedingung erreicht, die Regulatormassenparameter enthält.

Nach einer Störungsrechnung der Ein-Loop-Ordnung finden wir keine v-Feld-Abhängigkeiten für die effektive Ein-Loop-Wirkung, die zur universellen Ein-Loop-Betafunktion führt. Die Auswirkungen der Massenparameter werden mit einer phänomenologischen Studie der effektiven Wirkung untersucht. Daraus folgern wir, dass die Einbeziehung von BRST-invarianten Massenparametern ausreicht, um die Nielsen-Olesen-Instabilitäten für konstante magnetische Hintergründe innerhalb eines bestimmten Gültigkeitsbereichs zu heilen. Unter der Annahme von kovarianzkonstanten und selbstdualen Hintergründen umgehen wir außerdem solche tachyonischen Modi und beobachten das Vorhandensein eines nicht-trivialen Minimums, welche als eine Manifestation der dynamischen Brechung der Skalensymmetrie zu begründen ist.

Eine explizite Untersuchung der Abhängigkeit vom v-Feld, wie sie im einteiligen Schwinger-Funktional auftritt, zeigt keine zusätzlichen Divergenzen aufgrund nicht-lokaler Wechselwirkungen. Außerdem hinaus klassifizieren wir alle möglichen Formen des zugehörigen 2-Punkt-v-abhängigen Korrelators, welche im Limes der Landau-Eichung verschwinden.

Schließlich reformulieren wir unser Modell für massenabhängige Renormierungsschemata und zwar innerhalb der funktionalen Renormierungsgruppe. Dazu machen wir die Massenparameter zu Regulatoren, die in unserer Theorie durch geeignete nichtlineare Eichfixierungsbedingungen eingeführt werden, und konstruieren wir trunkierte Flussgleichungen, die mit der BRST-Symmetrie auf jeder Skala kompatibel sind. In Abwesenheit eines Hintergrundfeldes leiten wir eine v-abhängige Betafunktion ab, die mit der universellen Ein-Loop-Betafunktion in Einklang gebracht werden kann, indem das v-Feld mit einer geeigneten Gauß-Verteilung gemittelt wird. Unter Berücksichtigung eines Hintergrundfeldes finden wir eine trunkierte Flussgleichung, die die richtigen universellen Ergebnisse ohne weitere Einschränkung reproduziert und mit der Literatur übereinstimmt.

Contents

CHAPTER 1

Introduction

The journey for a complete description of the fundamental constituents of Nature has led us to the conclusion that all visible matter is composed of a few basic building blocks, known as the *elementary particles*, governed by the four fundamental forces. These four fundamental forces of Nature are the gravitational, electromagnetic, weak and strong forces. According to our current understanding, the gravitational force emerges from the curvature of spacetime, while the other three forces are mediated by bosonic force-carrier particles between the fundamental fermions that constitute the matter particles and contribute over different energy ranges with varying strength.

The successful integration of the electromagnetic, weak, and strong forces within a unified theoretical framework, known as the *Standard Model (SM) of particles*, represents a significant achievement of theoretical physics in our quest to comprehend the workings of Nature. The development of the SM is predicated on the preservation of certain symmetries, which play a critical role in its construction. As it turns out, the model is described by gauge theories that are invariant under certain local gauge transformations.

The SM enjoys a far-reaching compatibility between theoretical predictions and experimental observations in the subatomic world. Notably, the discovery of the Higgs boson [1, 2] was a major triumph for the SM, as it was conjectured to be responsible for particle mass generation [3–5].

Quantum Chromodynamics (QCD) encodes the physical content of strong interactions in the SM. The force-carrier particles of the strong interactions are called *gluons*, while the matter particles are referred to as *quarks*. Together, they build up hadrons, such as the proton and the neutron.

Several interesting phenomena arise from the study of QCD at different energy scales. At short-range scales, the coupling strength weakens and renders the theory asymptotically free, as was first discovered by Politzer, Gross & Wilczek [6, 7]. However, at lower energies (long-range), perturbation theory exhibits a *Landau pole*, i.e. the coupling strength diverges at a finite energy scale $\Lambda_{\text{QCD}} \sim \mathcal{O}(100)$MeV. Such a breakdown does not indicate a pathological inconsistency of the theory but rather a limitation of the perturbative methodology used to describe the physics at these scales. Consequently in this infrared (IR) energy regime, non-perturbative (in terms of the coupling) methods should be employed. Another interesting aspect which arises in this energy regime of QCD is the origin of mass generation which is associated with both the quark and the gluon sectors.

The origin of the mass generation in the quark sector can be understood by comparing the masses of light hadrons with their current quark masses, acquired through the Higgs mechanism. Given that the current quarks account for only a small fraction of the hadron's total mass, the bulk of the mass is attributed to QCD binding energy obtained from chiral symmetry breaking. Such mass generation has been investigated with various approaches [8–13].

For the gluon sector, this problem is tied to gluon (color) confinement which in turn is related to the emergence of a mass gap in pure Yang-Mills (YM), i.e. QCD without quarks. Such phenomena are less understood and manifest themselves in the correlation functions at the IR energy regime. Therefore, a sophisticated description of pure YM and its building blocks as described by the correlation functions at different energy scales needs to be developed.

For (pure) YM theory, in order to determine finite gauge invariant quantities, the Faddeev-Popov (FP) method is generally employed. Following this procedure, one imposes an appropriate constraint equation which affects the field configuration space, so that as many symmetries of the non-gauge fixed action pass over to the corresponding gauge-fixed action. Additionally, the resulting action should be unitary. The gauge-fixing method is based on the underlying philosophy that one can impose a constraint condition that appropriately selects a unique representative of each gauge-equivalent field configuration, thus fixing the gauge. Such an idealized realization is unfortunately at odds with the current conventional choices.

Gribov was the first to notice that finding a well-behaved constraint equation, also known as gauge-fixing condition, is a highly non-trivial task due to the existence of *Gribov copies*, [14]. These are certain over-counted or under-counted gauge equivalent sections in configuration space. Singer later extended Gribov's work to a larger class of gauge-fixing conditions, [15]. Furthermore, Neuberger analyzed the effect of these Gribov copies on physical observables, particularly on reproducing undetermined $\frac{0}{0}$ results, a problem known as the *Neuberger problem*, [16, 17].

To eliminate the effect of Gribov copies, Gribov proposed a strategy based on exploring the zero modes of the associated Faddeev-Popov (FP) operator[1]. This method involves restricting the analysis to the region where the FP operator is strictly positive-definite, known as the *first Gribov region*. This is equivalent to finding the gauge configurations that minimize the quadratic gauge field functional $\mathrm{tr}_{\mathrm{xcL}}\left(A^2\right)$, as discussed in several works, cf. [21–26]. As a result, the IR gluon propagator is found to be suppressed and no longer possesses a valid Källen-Lehmann spectral representation. Conclusively, it implies the confinement of gluons, affected by the structural properties of the modified theory. An enhanced IR ghost propagator was also deduced, cf. [14]. Zwanziger achieved a successful reformulation of Gribov's domain constraints to a local renormalizable theory, at the expense of introducing additional auxiliary fields, known as the *Gribov-Zwanziger (GZ) model* [27–29]. The gluon and ghost propagators obtained from the GZ action display similar IR behavior as the ones from Gribov's study, cf. [30, 31].

However, it was later shown in [19] that the first Gribov region still contains additional Gribov copies, necessitating a further restriction of the configuration space to only include the set of absolute minima of the associated gauge functional. This restricted space is called the *fundamental modular region (FMR)*. Although analytical calculations within that restricted region are highly

[1] See [18–20] for an explicit construction.

challenging, geometric arguments indicate that the presence of Gribov copies are expected to impact the IR dynamics of non-Abelian field theories [19, 21, 29].

Hence, the presence of Gribov copies can cause inconsistencies in the FP procedure at certain energy scales, when Lorenz-like gauges are considered. In principle, perturbation theory can provide an accurate description of short-range phenomena to a certain extent, due to the irrelevance of Gribov copies. However, perturbation theory beyond a certain validity domain, restricted by the magnitude of the coupling constant, becomes unreliable due to the existence of a Landau pole at finite energy scales, which produces large values of the couplings in the IR regime. Therefore, to study the effects of the IR region, various sophisticated continuous and discrete approaches have been devised in addition to the techniques mentioned previously.

Many studies of the IR properties of non-Abelian field theories focused mainly, but not exclusively cf. [39], on computing the form of the ghost an gluon propagator. This is because the ghost-gluon vertex allows for a determination of the beta function in the Landau gauge. Taylor's non-renormalization theorem for the ghost-gluon vertex function yields a propagator-based definition of the beta function, cf. [37, 38]. However, several computations of associated correlation functions beyond the Landau gauge have been performed, cf. [32–36] for a non-exhaustive literature.

On the discretized front, lattice simulations have been used to derive the IR behavior of relevant correlation functions [40–69]. Lattice computations do not need an implementation of the gauge-fixing condition but the use of gauge fixing allows for a direct comparison with analytic approaches. For gauge-fixed studies on the lattice, the minimum Landau gauge was mainly used, where one chooses an arbitrary minimum of the gauge functional. This corresponds to a random choice of a Gribov copy. As a result, various lattice simulations have inferred a finite ghost dressing function (form factor of the ghost propagator) in the deep IR, along with a saturated and finite IR behavior for the gluon propagator.

On the continuum front, non-perturbative methods such as the Dyson-Schwinger equations (DSE) and the functional Renormalization Group (fRG) contributed to the analytical study of the IR properties of non-Abelian field theories, constructing solutions at different truncation schemes for the behavior of the ghost and gluon propagators [35, 39, 70–99]. The DSE approach treats the correlation functions as self-consistent solutions of a set of integro-differential equations, while the fRG studies the form of the correlation functions by evolving their generating functional with a floating momentum scale, thus potentially exploring the whole energy domain.

Broadly speaking, one can classify the solutions which characterize the long-range behavior of the gluon and ghost propagators into the *scaling* and the *decoupling* solutions. While there have been numerous studies on the effects of these solutions at various dimensions, we will focus on the results in $d = 4$[2]. The scaling solution provides a vanishing saturated gluon propagator and an enhanced ghost propagator in the deep IR, cf. [72–80]. Note that such behavior of correlation functions is consistent with the Kugo-Ojima (K-O) confinement scenario, which is based on global color and Becchi-Rouet-Stora-Tyutin (BRST) invariance, cf. [93, 101, 102]. On the other hand, the decoupling solution predicts a gluon propagator that saturates at a constant value and a finite ghost dressing function. This behavior is similar to the tree-level divergent ghost propagator, in the deep IR [35, 39, 85, 88, 93–99].

[2] See [100] for an analytic study of IR aspects of YM correlation functions at various dimensions in the context of the DSE.

Overall, the results from the continuum approaches seem to be in agreement with the basic features observed in lattice simulations. However, a more detailed quantitative comparison between the lattice predictions and the available non-perturbative solutions indicate a remarkable agreement with the decoupling solution [46].

Furthermore, compatibility between the GZ scenario and lattice simulations was observed, for the IR behavior of gauge theories. This was achieved through a refined version of the GZ model that incorporates dimension-two mass condensates [96, 97, 103–105].

On an alternative continuum front, motivated by the the concept of a massive gauge-fixed YM action, put forth in the Curci-Ferrari model [106], the idea of dynamical mass generation gained traction. Such a realization focuses on the resolution of the Gribov ambiguities and reproduction of discretized observed IR properties of the correlation functions by means of a decoupled gauge field massive sector [89, 107, 108]. However, ad hoc mass deformed actions are incompatible with conventional BRST symmetry (absence of nilpotency), potentially affecting the renormalizability and unitarity of the theory [50, 77, 88, 90, 98, 106, 109–114]. Studying models of massive propagators, by considering such modifications at face value, were found to provide quite accurate phenomenological descriptions of the IR physics in non-Abelian field theories, in accordance with lattice simulations [46, 59, 91, 95, 107, 108, 115–120].

Following the aforementioned philosophy of deforming the gauge-fixed action, a recent study [121] explored a novel approach of including such (non-)perturbative regulator deformation contributions, in a BRST respecting manner. As such, a non-linear and Fourier weighted gauge-fixing condition, with a modified version of the mass sector of the often called Curci-Ferrari-Delbourgo-Jarvis (CFDJ) gauge [106, 117, 122], was utilized to incorporate the regulator dependencies as part of the renormalization procedure. This technique provides different avenues of exploring gauge systems while maintaining the well-established FP procedure. Implementation in the context of the fRG led to a one-loop and BRST-exact flow equation, cf. [121].

The main focus of this work will be around this newly developed framework and our goal will be to examine the properties of such a methodology through the lenses of both perturbative and non-perturbative renormalization schemes.

On the perturbative end, we will recast the aforementioned approach within the framework of the background field method and explore its one-loop perturbative behavior. Its underlying philosophy is based on splitting the gauge field into a fixed background and a fluctuating part. Such formalism introduces a particular class of background covariant gauges, which give rise to correlation functions that ought to respect invariance under a subset of local gauge transformations [123]. The background field method constitutes a powerful tool that greatly facilitates both perturbative [124, 125] and non-perturbative [126–146] calculations implemented at various renormalization and truncations schemes. Such a realization will provide us with a better understanding of how the dynamically generated terms affect different aspects of the theory. In addition, the consideration of the modified mass CFDJ sector will allow us to mimic the decoupling solution and gain further information on its effect.

Note that the background formalism presents a promising avenue for addressing the problem of Gribov ambiguities. Early attempts to utilize the formalism in the GZ framework, motivated by [23], were hampered by a lack of background and BRST invariance [147, 148]. However, recent refinements to the model have resolved these issues by introducing a Stueckelberg-type field, although this has made manipulations of the theory at finite temperatures more challenging,

[149]. Alternatively, a competing formalism was proposed in [150].

On the non-perturbative end, in this work we will study the truncated one-loop flow equation, derived in [121] using conventionally simulated gauges. This will enable us to derive a set of flow equations for the contributing renormalization factors and gain explicit access to the form and behavior of the beta function at varying couplings. By doing so, we will be able to investigate the non-perturbative effects of the dynamically generated terms while maintaining explicit BRST invariance at all energy scales. Such an approach can provide valuable insights into the nature of YM theory and its properties in the deep IR.

The contents of the thesis are organized as follows. Chpts. 2 & 3 are dedicated to setting up a BRST-invariant pure YM model and motivating the need of developing a BRST/gauge-invariant formalism. In Chpt. 4 we perform an explicit study of the model within the background field method (BFM) with the inclusion of an appropriate non-linear gauge-fixing condition. A one-loop perturbative study and phenomenological results of the one-loop Effective Action (EA) are also stated. For Chpt. 5, we turn our attention to the v-dependent part of the Schwinger functional, whose form is investigated after imposing several constraints for the v field. Afterwards, we abandon the perturbative method and in Chpt. 6 we perform a non-perturbative study of the associated flow equation within a certain truncation scheme without any background field and in the presence of it, for which we derive and discuss a Renormalization Group (RG) improved/resummed version of the beta function.

The compilation of this thesis is solely due to the author. However, parts of this work have been developed in collaborations with members of the Theoretical Physical Institute in Jena and the National Institute for Nuclear Physics in Bologna. The results on the one-loop Effective Action and Schwinger functional within a background and BRST-invariant framework, presented in Chpts. 4 & 5 respectively, have been discovered in collaboration with H. Gies & L. Zambelli and published in [151]. The development of a background and BRST-invariant functional renormalization group equation and the complete computation of the beta function for BRST-invariant flows, as displayed in Chpt. 6, is based on completed but so far unpublished material that has been developed in collaboration with S. Asnafi and L. Zambelli [152].

CHAPTER 2

Theoretical Background

In this Chapter, we lay the theoretical foundations for the proper description of non-Abelian gauge theories and introduce the framework that will be adopted in Chpts. 4-6 during the study of pure YM theory. In particular, after formulating, the action for the pure YM theory, within a geometrically focused framework, we explore different ways one can follow in order to properly quantize such a theory. Among these different but ultimately equivalent considerations, we focus our attention on quantization with a constraint equation that includes an isolated stochastic sector. Such a procedure, explored in the literature for the description of field models where the field configurations are constrained by stochastic differential equations, e.g. Langevin equations, provides a valuable input for the implementation of the gauge-fixing condition which can be exploited for gauge theories. From the off-shell quantization procedure of pure YM theory, we observe the emergence of a global supersymmetry called the BRST symmetry. Finally, we discuss the importance of such a symmetry for a well-defined unitary gauge theory via the separation of particle modes to physical and unphysical states with the method of the quartet mechanism from which one is able to deduce, under a certain set of conditions, color confinement. The connection between BRST and gauge symmetry will also be explored.

2.1 Non-abelian gauge theories

We begin by investigating the gauge-invariant action of pure YM theory. To do so, let us construct the underlying symmetric action from first principles by focusing on the geometric interpretation of the building quantities. Note that for our considerations in Miknowski spacetime, the mostly positive metric $g_{\mu\nu} = (-,+,+,+)$ is considered. Let $\mathbb{H}$ be a semi-simple compact Lie group, referred to as the *gauge group* of our system and $\mathfrak{h} = \mathrm{Lie}(\mathbb{H})$ the Lie algebra which spans the infinitesimal group transformations. We want to construct a field theory where the action is invariant under space-dependent group transformations of the fields, called *local gauge transformations*, denoted by $U(x) \in \mathbb{H}$.

Let us introduce the parallel transporter $\mathbf{C}(y, x)$ which is a curve-dependent element of the group representation, called the *comparator* or *Wilson line*, which joins the spacetime point y to

x. Under local gauge transformations the Wilson line changes as,

$$\mathbf{C}'(y,x) = \mathbf{U}(y)\mathbf{C}(y,x)\mathbf{U}^{-1}(x), \tag{2.1}$$

with the boundary behavior of $\mathbf{C}(x,x) = \mathbf{1}$. Consider a differentiable curve connecting these two distinct spacetime points which can be parametrized in terms of $\mathbf{A}_\mu$. Infinitesimally it can be represented as $y_\mu = x_\mu + \mathrm{d}x_\mu$. Performing a Taylor expansion of the Wilson line, this leads to

$$\mathbf{C}(x+\mathrm{d}x, x) = \mathbf{1} + i\bar{g}\mathbf{A}_\mu(x)\mathrm{d}x^\mu, \tag{2.2}$$

where an arbitrary constant $\bar{g}$ has been extracted. Expanding Eq.(2.1) at first order in $\mathrm{d}x_\mu$, one determines how the quantity $\mathbf{A}_\mu(x)$ changes under local gauge transformations,

$$\mathbf{A}'_\mu(x) = \mathbf{U}(x)\mathbf{A}_\mu(x)\mathbf{U}^{-1}(x) - \frac{i}{\bar{g}}\left(\partial_\mu \mathbf{U}(x)\right)\mathbf{U}^{-1}(x), \tag{2.3}$$

As it can be seen in Eq.(2.2), from a geometric point of view, the quantity $\mathbf{A}_\mu(x)$ corresponds to a *connection* which parametrizes an infinitesimal Wilson line. Equivalently, Eq.(2.3) shows that under local gauge transformations, $\mathbf{A}_\mu(x)$ corresponds to a local dynamical field called the *gauge field*.

With the aid of the Wilson line and the connection, we can construct the *covariant derivative*,

$$\mathbf{D}_\mu = \mathbf{1}\partial_\mu + i\bar{g}\mathbf{A}_\mu, \tag{2.4}$$

which changes as a tensor under local gauge transformations of the symmetry group,

$$\mathbf{D}'_\mu = \mathbf{U}(x)\mathbf{D}_\mu\mathbf{U}^{-1}(x). \tag{2.5}$$

From a geometric perspective, the covariant derivative establishes a sensible notion of derivative by comparing neighboring field configurations on an equal footing. In addition, as far as the symmetry group is concerned, the covariant derivative corresponds to a tensorial differential operator.

Taking the commutator of the covariant derivative, we find the *curvature tensor*

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu], \tag{2.6}$$

where under local gauge transformations,

$$\mathbf{F}'_{\mu\nu}(x) = \mathbf{U}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{U}^{-1}(x). \tag{2.7}$$

The curvature tensor can be associated with the parallel transport of an infinitesimal closed Wilson line and is called the *field strength tensor* since it provides the generalization of the electromagnetic tensor of Quantum Electrodynamics (QED) to non-Abelian field theories.
Finally, we can consider infinitesimal local gauge transformations

$$\mathbf{U}(\omega) := \mathbf{U}(x) = \mathbf{1} + i\bar{g}\omega(x) + \mathcal{O}\left(\omega^2\right) \tag{2.8}$$

where the infinitesimal parameter $\omega \in \mathfrak{h}$, under infinitesimal local gauge transformations becomes

$$\omega'(x) = \mathbf{U}(x)\omega(x)\mathbf{U}^{-1}(x). \tag{2.9}$$

The YM action is given by

$$S_{\text{YM}}[\mathbf{A}] = -\frac{1}{4}\int_x \operatorname{tr}\mathbf{F}^2_{\mu\nu}, \tag{2.10}$$

where $\int_x = \int d^dx$. As it can be seen from (2.7), it is invariant under local gauge transformations of the gauge group.

Let $\{\tau^a\}$ be a complete set of generators of $\mathbb{H}$, i.e. a basis of the Lie algebra $\mathfrak{h}$ of the underlying gauge group. In order for $\{\tau^a\}$ to span the Lie algebra of the theory, then the following commutation relations must be satisfied

$$[\tau^a, \tau^b] = if^{abc}\tau^c, \tag{2.11}$$

where f^{abc} correspond to the antisymmetric structure constants and $\operatorname{tr}(\tau^a\tau^b) > 0$. This leads to the following Jacobi identity for the structure constants

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \tag{2.12}$$

The component form of the generators will be determined by the choice of the group representation and equals the dimension of the vector space in which the representation of the group exists whereas the number of the independent generators is associated to the dimension of the vector space spanned by the generators. Thus, the matrix representation of the Lie algebra generators can vary depending on the particular choice of representation. For our purposes, we restrict our attention to Hermitean generators of the *adjoint* group representations

$$(\tau^a_{\text{G}})^{bc} = if^{abc}. \tag{2.13}$$

Note that in the adjoint representation, the following relation holds

$$f^{acd}f^{bcd} = C_2(\text{G})\delta^{ab}, \tag{2.14}$$

where $C_2(\text{G})$ is called the *quadratic Casimir operator*, and is a representation-dependent quantity that commutes with the generators of the Lie algebra.

In order to translate the YM action into a component form, we choose, according to the desired properties, the gauge group $\mathbb{H} = SU(N_c)$ with the Lie algebra $\mathfrak{h} = \mathfrak{su}(N_c)$ and work in the adjoint representation. Then, the fields and differential operators of the action will transform accordingly and the dimension of the Lie algebra denoted by the adjoint indices will be referred to as *colors*. Thus, the gauge field is written as

$$\mathbf{A}_\mu(x) = A^a_\mu(x)\tau^a_{\text{G}}. \tag{2.15}$$

The covariant derivative in the adjoint representation reads as

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + \bar{g} f^{acb} A_\mu^c \tag{2.16}$$

and the components of the field strength tensor $\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}\tau^a_{\mathrm{G}}$ are

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \bar{g} f^{abc} A^b_\mu A^c_\nu. \tag{2.17}$$

It is worth noting that the components of the aforementioned quantities in the adjoint representation exhibit the same transformation rules under local gauge transformations as their corresponding representation-independent counterparts, cf. Eqs.(2.3), (2.5) & (2.7). In addition, in the adjoint representation a finite local transformation takes the form

$$\mathbf{U}(\omega) = \mathrm{e}^{i\bar{g}\omega^a\tau^a_{\mathrm{G}}} = \mathrm{e}^{\bar{g}\omega^a f^{acb}} \tag{2.18}$$

and infinitesimally,

$$\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b. \tag{2.19}$$

Thus, we can now readily find that, the YM action in the adjoint representation takes the form

$$S_{\mathrm{YM}}[A] = -\frac{1}{4}\int_x F^a_{\mu\nu} F^{a\mu\nu}. \tag{2.20}$$

The YM action is still invariant under local gauge transformations. However, due to Eq.(2.18) it gives rise to cubic and quartic gauge-field self interactions. Such a novel characteristic arises due to the non-Abelian character of the gauge group and extends the spectrum of possible interactions with very interesting implications. Finally, note that in the adjoint representation of the $SU(N_c)$ gauge group, $C_2(\mathrm{G}) = N_c$.

2.2 Gauge fixing

In order to implement YM theory for the description of physically interacting systems, the next step is to quantize it. Thus, let us introduce the generating functional of quantum YM theory,

$$\mathcal{Z} = \int_A \mathcal{D}A \, \exp\left[i\left(S_{\mathrm{YM}}[A] + \int_x j^a_\mu A^{a\mu}\right)\right], \tag{2.21}$$

where we have inserted a source j^a_μ for the gluon gauge field A^a_μ to aid in the construction of the building blocks of the theory. A naive attempt to define a sensible quantum field theory from Eq.(2.21) fails due to the fact that the functional integral is ill-defined. The reason is that the functional integral, defined over all possible gauge field configurations, contains a huge redundancy which comes from the physically equivalent field configurations and renders the quantity divergent.

To illustrate this point, let us determine the inverse free gluon propagator operator which corresponds to the quadratic part of the YM action, Eq.(2.20),

$$G_A^{-1} = \left(-\partial^2 \mathbb{1} + \partial \otimes \partial\right)^{-1}. \tag{2.22}$$

The free gluon propagator is singular with vanishing eigenvalues for the longitudinal component of the gauge field. In addition, local gauge invariance, (2.18), yields an infinite number of such equivalent gauge field configurations. A proper definition of the generating functionalrequires to remove these redundancies from the functional integration.

The procedure of removing these gauge equivalent redundant degrees of freedom in a consistent manner is called *gauge fixing*. Even though there are different but equivalent ways on how to perform gauge fixing, the underlying idea is the same. In particular, we wish to impose an appropriate constraint, called the *gauge-fixing condition*, which will restrict the field configuration space over only gauge inequivalent configurations. In other words, we wish to pick one and only one representative out of each set of gauge equivalent configurations, called *gauge orbits*. The task of finding such a gauge-fixing condition turns out, to be highly non-trivial due to the existence of Gribov copies, [14, 15]. As we focus mostly on perturbative applications in this thesis, we will not consider potential modifications of the theory arising from the inclusion of nonlocal terms which account for the existence of Gribov copies and lead to the Gribov-Zwanziger action with implications on the IR behavior of non-Abelian gauge theories cf. [27, 30, 153].

Next, let us review the conventional procedure of gauge fixing. In Subsecs. 2.2.1, 2.2.2 & Sec. 2.3, we further explore the different ways with which one can implement it and the respective advantages of each method. The YM functional integral, Eq.(2.21), is defined over the space of all gauge equivalent field configurations, cf. Eq.(2.3) & (2.18)

$$\mathcal{A} = \left\{ \mathbf{A}_\mu^{\mathbf{U}} := \mathbf{A}'_\mu \middle| \mathbf{U}(\omega) \in SU(N_c) \right\}. \tag{2.23}$$

Then the physically inequivalent configurations will belong to the quotient space of all the configurations over the ones obtained under a gauge transformation, denoted by

$$\mathcal{A} \Big/ SU(N_c) = \{\mathbf{A}_\mu \sim \mathbf{A}_\mu^{\mathbf{U}} | \mathbf{A}_\mu \in \mathcal{A}, \mathbf{U}(\omega) \in SU(N_c)\}$$

Then, we can redefine the path integral in order to extract the contribution of the physically equivalent and inequivalent terms as

$$\int_{\mathcal{A}} \mathcal{D}A \to \int \mathcal{D}\mathbf{U} \int \mathcal{D}\mu[A],$$

where $\mathcal{D}\mathbf{U}$ and $\mathcal{D}\mu[A]$ are measures over gauge equivalent and inequivalent field configurations respectively. Then, we can factor out the contribution of the first part which is equivalent to choosing a representative from each gauge orbit [154]. To perform such a decomposition, we introduce the gauge-fixing condition

$$\mathcal{F}[A] = 0, \tag{2.24}$$

which ideally has a unique solution for each gauge orbit, i.e. $\mathcal{F}[A^{\mathbf{U}}] = 0$. Next, rewriting the partition of unity as the path integral over gauge equivalent configurations of the delta functional times the Jacobian determinant as follows

$$\mathbb{1} = \int \mathcal{D}\mathcal{F}\ \delta[\mathcal{F}] = \int \mathcal{D}\mathbf{U}\ \delta\big[\mathcal{F}[A^{\mathbf{U}}]\big]\ \Delta_{\text{FP}}[A^{\mathbf{U}}],$$

leads to the following decomposition of the functional integration,

$$\int_{\mathcal{A}} \mathcal{D}A = \int \mathcal{D}\mathbf{U} \int_{\mathcal{A}} \mathcal{D}A \; \delta[\mathcal{F}[A]] \; \Delta_{\text{FP}}[A^{\mathbf{U}}] = \int \mathcal{D}\mathbf{U} \int \mathcal{D}\mu[A].$$

Inserting it into Eq.(2.21), leads to

$$\mathcal{Z} = \int_{\mathcal{A}} \mathcal{D}A \; \delta[F[A]] \; \Delta_{\text{FP}}[A] \; e^{iS_{\text{YM}}[A]}, \tag{2.25}$$

where $\int \mathcal{D}\mathbf{U}$ has been neglected as it contributes an overall factor which will not affect the correlation functions. Moreover, the gauge-invariant Jacobian determinant of the transformation, known as the *Faddeev-Popov determinant* is given by

$$\Delta_{\text{FP}}[A] = |\det \mathcal{M}_{\text{FP}}[A]| \, , \text{ where } \mathcal{M}_{\text{FP}}[A] = \left.\frac{\delta \mathcal{F}[A]}{\delta \omega}\right|_{\omega=0} = \left.\frac{\delta \mathcal{F}[A]}{\delta A}\frac{\delta A}{\delta \omega}\right|_{\omega=0}. \tag{2.26}$$

Thus, we obtain the following form of the gauge-fixed YM generating functional,

$$\mathcal{Z} = \int \mathcal{D}A \; \delta\left[\mathcal{F}[A]\right] \; \det\left(\mathcal{M}_{\text{FP}}[A]\right) \exp\left[i\left(S_{\text{YM}}[A] + \int_x j^a_\mu A^{a\mu}\right)\right], \tag{2.27}$$

where functional integration over the space of gauge equivalent field configurations is implied and the modulus signs for the Δ_{FP} have been dropped, which holds in the perturbative domain. The conventional method of dealing with the FP determinant is to replace it with an additional functional integration over some Hermitian auxiliary bosonic Grassmann-valued fields c and $\bar{c}$ called *Faddeev-Popov ghosts* and *antighosts* respectively, as follows [155]

$$\det\left(\mathcal{M}_{\text{FP}}[A]\right) = \int \mathcal{D}c\mathcal{D}\bar{c} \; \exp\left[i\int_{x,y} \bar{c}^a(x)\mathcal{M}^{ab}_{\text{FP}}(x,y)c^b(y)\right]. \tag{2.28}$$

Note that upon substitution of Eq.(2.28) in Eq.(2.27), we observe that implementation of the gauge-fixing procedure results in the insertion of two contributions in the YM generating functional. The Dirac delta functional part is closely associated to the implementation of the gauge-fixing condition, according to the properties discussed before. The FP determinant which gives rise to a deformation of the action originates from the Jacobian of the change of the functional measure over gauge equivalent configurations, cf. Eq.(2.25). However, both additional terms are dependent on the gauge-fixing condition $\mathcal{F}[A]$. Thus, the form of the gauge-fixing condition is expected to affect the generating functional. Moreover, the gauge-fixing procedure lifted the local gauge invariance of our generating functional. In the following we construct concrete expressions for the YM generating functional for an arbitrary gauge-fixing condition, following an on-shell and an off-shell approach. The off-shell approach utilizes the additional freedom of inserting an auxiliary field and encodes information concerning an additional global supersymmetry, called *BRST symmetry*.

2.2.1 Fixing the gauge on-shell

In order to arrive at the on-shell YM generating functional, we consider an extended class of gauge-fixing conditions of the form

$$\mathcal{F}[A] - \mathbf{f}(x) = 0, \tag{2.29}$$

where $\mathbf{f}(x) = f^a(x)(t^a_{\mathrm{G}})^{bc}$ is an arbitrary test function. We rewrite the delta functional as

$$\delta[\mathcal{F}[A]] \to \int \mathcal{D}f \; \delta[\mathcal{F}[A] - \mathbf{f}] \; \exp\left(-\frac{i}{2\xi}\int_x f^a f^a\right) \tag{2.30}$$

with a real constant ξ called the *gauge-fixing parameter*. Note that different values of ξ result in a different implementation of the gauge-fixing condition. Thus, they correspond to different gauges. The most frequently used gauges and the ones that we exclusively consider in this thesis (with some modification) are the *Landau gauge* where $\xi \to 0$ and the *Feynman gauge* where $\xi = 1$. In the Landau gauge, Eq.(2.30) approaches the delta functional up to an overall normalization factor.

Inserting it into the YM generating functional, while integrating over the test function results in an additional contribution to the action. Writing,

$$\mathcal{Z}[j] = \int \mathcal{D}A \; \det\left(\mathcal{M}_{\mathrm{FP}}[A]\right) \; \exp\left[i\left(S_A[A] + \int_x j^a_\mu A^{a\mu}\right)\right], \tag{2.31}$$

the gauge-fixed YM action is given by

$$S_{\mathrm{A}}[A] = S_{\mathrm{YM}}[A] - \frac{1}{2\xi}\int_x \mathcal{F}^a\mathcal{F}^a. \tag{2.32}$$

2.2.2 Fixing the gauge off-shell

The generating functional (2.31), is sometimes referred to as an on-shell formalism in the sense that all fields are dynamical. In this part, we deviate from the aforementioned procedure and construct an off-shell gauge-fixed YM generating functional by means of an auxiliary *Nakanishi-Lautrup (NL)* field $\mathbf{b}(x) = b^a(x)(t^a_{\mathrm{G}})^{bc}$, initially introduced in the context of QED in [156, 157]. The procedure of rewriting the delta functional is similar to Eq.(2.30), with the difference that we introduce the NL as the weight of the delta functional of the extended class of gauge-fixing conditions, i.e.

$$\delta[\mathcal{F}[A]] \to \int \mathcal{D}f\mathcal{D}b \; \exp\left[-\frac{i}{2\xi}\int_x f^a f^a - i\int_x b^a\left(\mathcal{F}^a - f^a\right)\right]. \tag{2.33}$$

Completing the square and ignoring any constant contributions, we find the following YM generating functional

$$\mathcal{Z}[j] = \int \mathcal{D}A\mathcal{D}b \; \det\left(\mathcal{M}_{\mathrm{FP}}[A]\right) \; \exp\left[i\left(S_A[A,b] + \int_x j^a_\mu A^{a\mu}\right)\right], \tag{2.34}$$

where

$$S_A[A,b] = S_{\text{YM}}[A] - \int_x b^a \mathcal{F}^a + \frac{\xi}{2}\int_x b^a b^a. \tag{2.35}$$

Eliminating the contribution of the NL field from Eq.(2.35), by means of its equations of motion yields the on-shell gauge-fixed generating functional, Eq.(2.31). This illustrates the equivalence between the methods used for implementing the corresponding gauge-fixing condition. From Eq.(2.35) one can naturally interpret the NL field in the Landau gauge as a Lagrange multiplier which implements the constraints of the gauge-fixing condition.

Even though the two previously developed ways of finding the gauge-fixed YM generating functional are equivalent on-shell, the off-shell formalism exhibits, due to the presence of the NL field, invariance under a new global supersymmetry, discovered by Benchi, Rouet and Stora [158] and independently by Tyutin [159, 160], called the BRST symmetry[1]. Invariance under BRST transformations can be readily displayed by rewriting the FP determinant in terms of the FP ghosts, cf. Eq.(2.28), thus finding the following off-shell YM action

$$S_A[A,b,c,\bar{c}] = S_{\text{YM}}[A] - \int_x b^a \mathcal{F}^a + \frac{\xi}{2}\int_x b^a b^a + \int_{x,y} \bar{c}^a(x)\mathcal{M}^{ab}_{\text{FP}}(x,y)\bar{c}^b(y). \tag{2.36}$$

Introducing a Grassmann *BRST operator* s which acts on the fields, the BRST transformation reads

$$\begin{aligned} (\mathrm{s}A)^\mu_a &= D^{ab}_\mu c^c, & (\mathrm{s}c)^a &= -\frac{\bar{g}}{2} f^{abc} c^b c^c, \\ (\mathrm{s}\bar{c})^a &= b^a, & (\mathrm{s}b)^a &= 0. \end{aligned} \tag{2.37}$$

One very important property of the BRST operator is its nilpotency, i.e. $\mathrm{s}^2 = 0$. This trait can be used to show the BRST invariance of the gauge-fixed YM action. Using Eq.(2.26) we can rewrite the YM action, Eq.(2.36), as

$$S_A[A,b,c,\bar{c}] = S_{\text{YM}}[A] + \mathrm{s}\Psi[\bar{c},A], \tag{2.38}$$

where $\Psi[\bar{c},A] = \bar{c}^a \mathcal{F}^a[A]$ and spacetime integration is implied. This was first observed in the context of covariant gauges in [161]. Note that for the gauge field, the BRST transformation corresponds to an infinitesimal local gauge transformation, cf. Eq.(2.8), with the infinitesimal parameter replaced by a ghost field. This is justified due to the anticommuting character of the BRST operator. Therefore, the YM part of the gauge-fixed action is BRST invariant. In addition, due to the nilpotency of the BRST operator we can readily deduce that

$$\mathrm{s}S_A[A,b,c,\bar{c}] = 0. \tag{2.39}$$

This shows that the off-shell gauge-fixed YM action is indeed invariant under this residual BRST symmetry generated by the transformations in Eq.(2.37), as long as the BRST operator is nilpotent.

[1] Note that BRST symmetry does not necessarily require the presence of the NL field. But with the NL field, BRST is nilpotent.

2.3 Gauge fixing with a stochastic variable

The gauge-fixed YM action that we have obtained after implementing the gauge-fixing procedure describes a constraint system of gauge field configurations, governed by the dynamics and so on of the underlying pure YM theory that solve the constraint equation (2.24). For such an action, we have observed the natural emergence of a global supersymmetry on the quantum level, the BRST symmetry. The various implications of such a symmetry in the context of non-Abelian field theores will be discussed in Sec. 2.4. In this section, within the framework of gauge theories (which is where conventionally but not solely BRST symmetry is studied), we explore the connection between field theories with field configurations satisfying a constraint equation and BRST symmetry. This alternative and well established viewpoint provides a valuable assistance in the understanding of the gauge-fixing procedure. This section, motivated by [162, 163], is one of the ingredients for the results derived in Sec. 2.2 and its underlying philosophy is an integral part of the framework developed in this thesis.

Note that BRST symmetry arises naturally not only in the framework of gauge theories but also in statistical field models governed by *Langevin equations* [162]. Langevin equations correspond to linear in time stochastic differential equations related to stochastic processes such as diffusion and Brownian motion, [164–166] and have been proposed to describe the dynamics of critical phenomena, cf. [167–173]. By introducing a stochastic field configuration $n(x)$ called the *noise field*, one can impose the corresponding Langevin equation and construct a generating functional for the constraint system. The noise field is conventionally implemented as a Gaussian distribution (Gaussian white noise) and effectively reflects the way by which the Langevin equation (constraint equation) is imposed. Such a system exhibits invariance under an associated BRST symmetry. Given that the solutions of these equations exhibit divergences on a perturbative level, the emergent BRST symmetry of the associated action which arises through the implementation of these constraint equations allows to prove the stability of the Langevin equations under renormalization.

Therefore, it is worth exploring the appearance of BRST symmetry on a more generic setting and more specifically for a system subject to a stochastic constraint equation with a decoupled stochastic sector. By adopting this method for pure YM, we will show how to relate it to the conventional gauge-fixing result derived in Sec. 2.2, while having an additional degree of freedom encoded in the noise field.

We begin our study by introducing a stochastic gauge-fixing condition, with the stochastic variable $\mathbf{n}(x) = n^a (t^a_{\mathrm{G}})^{bc} \in SU(N_c)$, as follows

$$\mathcal{E}[A, n] = \mathcal{F}[A] - \phi(\mathbf{n}) = 0, \tag{2.40}$$

where $\phi(\mathbf{n})$ is an arbitrary function of the stochastic variable or noise and the noise field has a normalized probability distribution, $\mathcal{D}n$. Similarly to Sec. 2.2, taking the variation of Eq.(2.40) with respect to the infinitesimal gauge transformations, Eq.(2.8), results in the Faddeev-Popov matrix, i.e.

$$\mathcal{M}_{\mathrm{FP}}[A] = \left.\frac{\delta \mathcal{E}[A, n]}{\delta \omega}\right|_{\omega=0} = \left.\frac{\delta \mathcal{F}[A]}{\delta A}\frac{\delta A}{\delta \omega}\right|_{\omega=0}. \tag{2.41}$$

Next, we shall derive a generic form for the unity under the generalized constraint (2.40). To do so, let us first derive the formal expression for an arbitrary function $\Phi(A)$ after imposing the gauge-fixing condition, Eq.(2.40). Due to the stochastic character of the gauge-fixing condition, we are ultimately interested in expectation values of functions of gauge fields, i.e.

$$\langle \Phi(A) \rangle_n = \int \mathcal{D}\mathcal{E}\mathcal{D}n \; \delta[\mathcal{E}[A,n]] \; \Phi(A) = \int \mathcal{D}U\mathcal{D}n \; \delta\big[\mathcal{E}[A^{\mathbf{U}},n]\big] \; \mathfrak{J}[A^{\mathbf{U}}] \; \Phi\left(A^{\mathbf{U}}\right), \tag{2.42}$$

where similarly to Sec. 2.2, the Jacobian of the transformation for the stochastic gauge-fixing condition takes the form

$$\mathfrak{J}\big[A^{\mathbf{U}}\big] = \mathcal{N} \det\left(\mathcal{M}_{\text{FP}}[A^{\mathbf{U}}]\right) = \mathcal{N} \int \mathcal{D}c\mathcal{D}\bar{c} \; \exp\left[i \int_{x,y} \bar{c}^a(x) \mathcal{M}^{ab}_{\text{FP}}[A^{\mathbf{U}}] \bar{c}^b(y)\right], \tag{2.43}$$

with $\mathcal{N}$ an overall normalization constant. Next, we rewrite the delta functional in the Fourier representation as

$$\delta[\mathcal{E}[A]] \to \int \mathcal{D}b \; \exp\left[-i \int_x b^a \mathcal{E}^a[A,n]\right]. \tag{2.44}$$

Inserting Eqs.(2.43) and (2.44) in Eq.(2.42), we arrive at

$$\langle \Phi(A) \rangle_n = \mathcal{N} \int \mathcal{D}U\mathcal{D}b\mathcal{D}n \; \det\left(\mathcal{M}_{\text{FP}}[A^{\mathbf{U}}]\right) \; \Phi(A^{\mathbf{U}}) \; \mathrm{e}^{-ib^a \mathcal{E}^a[A^{\mathbf{U}},n]}, \tag{2.45}$$

where integration over spacetime points is implied.

Thus, the partition of unity can be written as

$$\mathbb{1} = \mathcal{N} \int \mathcal{D}U\mathcal{D}b\mathcal{D}n \; \Delta_{\text{FP}}[A^{\mathbf{U}}] \; \mathrm{e}^{-ib^a \mathcal{E}^a[A^{\mathbf{U}},n]}. \tag{2.46}$$

Taking into account that the inverse Jacobian is a gauge invariant quantity and changing $A^{\mathbf{U}} \to A$, we find that up to an irrelevant normalization factor, the gauge-fixed YM generating functional reads

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}n \; \mathrm{e}^{iS_A[A,c,\bar{c},b,n]}, \tag{2.47}$$

with the gauge-fixed action

$$S_A[A,c,\bar{c},b,n] = S_{\text{YM}}[A] - b^a \mathcal{F}^a + S_{\text{noise}}[b,n] + \bar{c}^a \mathcal{M}^{ab}_{\text{FP}} c^b. \tag{2.48}$$

The noise action $S_{\text{noise}}[b,n]$ comes as a result of the stochastic gauge-fixing condition, Eq.(2.40). We can radily integrate out the noise field

$$\mathrm{e}^{iS_{\text{NL}}[b]} = \int \mathcal{D}n \; \mathrm{e}^{iS_{\text{noise}}[b,n]} \tag{2.49}$$

and associate it with the NL auxiliary field that appears during the off-shell quantization, cf. Subsec. 2.2.2. The importance of this method lies in the fact that by inserting the noise field as an extended sector of the gauge-fixing condition, we gain an additional degree of freedom, by properly manipulating the noise field, on the gauge-fixing procedure. In extension, integrating out the noise field gives us direct access to the implementation of the gauge-fixing condition in our theory. Note that as long as we consider a sensible form of the noise action, BRST symmetry of the action is manifest. In the following, in order to make contact with well-known results and also exploit our additional freedom to facilitate further calculations, we choose to integrate out the noise field using *Gaussian* and *Fourier distributions*.

2.3.1 Yang-Mills action with a Gaussian noise distribution

Let us now determine the form of the gauge-fixed YM action by choosing a Gaussian weight for the noise action of the form [121],

$$S_{\text{noise}}[b,n] = -\frac{1}{2\xi} n^a n^a - b^a n^a. \tag{2.50}$$

Averaging over the noise field and associating it with the NL field according to Eq.(2.49) we find that

$$S_{\text{NL}}[b] = \frac{\xi}{2} b^a b^a, \tag{2.51}$$

which reproduces the BRST-invariant off-shell YM action, cf. (2.36).

2.3.2 Yang-Mills action with a Fourier noise distribution

In this case, we choose a Fourier weight for the noise action of the form [121, 151],

$$S_{\text{noise}}[b,n] = n^a (v^a - b^a), \tag{2.52}$$

where $\mathbf{v} = v^a (t^a_{\text{G}})^{bc}$ corresponds to an external NL-type scalar field which extends the space of colored fields of the theory. Integrating over the noise field, according to Eq.(2.49), we find that

$$\mathrm{e}^{iS_{\text{NL}}[b]} = \int \mathcal{D}n \; \mathrm{e}^{-in^a(b^a - v^a)} = \mathcal{N}\delta[b^a - v^a]. \tag{2.53}$$

As a result, the gauge-fixed generating functional becomes

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \; \delta[b^a - v^a] \; \exp\left[i\left(S_{\text{YM}} - b^a\mathcal{F}^a + \bar{c}^a\mathcal{M}^{ab}_{\text{FP}}c^b\right)\right] = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c} \; \mathrm{e}^{iS_A[A,c,\bar{c},v]}, \tag{2.54}$$

with the gauge-fixed action

$$S_A[A,c,\bar{c},v] = S_{\text{YM}}[A] - v^a\mathcal{F}^a + \bar{c}^a\mathcal{M}^{ab}_{\text{FP}}c^b. \tag{2.55}$$

Next, let us address the BRST invariance of the generated action. Firstly, note that due to the presence of the auxiliary v field, the gauge-fixed action, Eq.(2.55), is still off shell. As mentioned in Sec. 2.4, the existence of an auxiliary NL field is essential for a BRST-invariant action. This role is played by the v field. Indeed, the off-shell action is invariant under the following BRST transformations,

$$\begin{aligned} (\mathrm{s}A_\mu)^a &= D^{ab}_\mu c^b, \quad & (\mathrm{s}c)^a &= -\frac{\bar{g}}{2} f^{abc} c^b c^c, \\ (\mathrm{s}\bar{c})^a &= v^a, & (\mathrm{s}v)^a &= 0. \end{aligned} \tag{2.56}$$

This can readily be seen by bringing Eq.(2.55) in the form

$$S_A[A, c, \bar{c}, v] = S_{\text{YM}}[A] + s\Psi[\bar{c}, A], \tag{2.57}$$

which is similar to Eq.(2.38). Thus, taking into account the nilpotency of the BRST operator, we verify the BRST invariance of the action, i.e.

$$\mathrm{s}S_A[A, c, \bar{c}, v] = 0. \tag{2.58}$$

Notice the mutual connection between the different steps followed in this section. In summary, by extending the gauge-fixing condition by a noise sector decoupled from the dynamical fields, we were able to get access to the way with which the gauge-fixing condition is implemented in the action by properly tuning the corresponding form of the noise action and associating it with the NL field, cf. Eq.(2.49). The choice of a Gaussian weight reproduced the conventional off-shell gauge-fixed YM action which upon integration of the NL field will depend quadratically on the gauge-fixing condition, cf. Eq.(2.32). Choosing a Fourier weight for the noise action at the expense of introducing an additional NL-type v field, we obtained a modified version of the gauge-fixed YM action, which upon integration over the b field was linear to the gauge-fixing condition, cf. Eq.(2.55). In addition, we deduced that the off-shell action enjoys invariance under BRST transformations, cf. Eq.(2.56). Given that the upcoming Chapters deal with a non-linear form for the gauge-fixing condition, a linear dependence on the action is a very desirable trait since it does not give rise to complicated interactions that require more computational power. It makes any non-trivial inclusion through the gauge-fixing condition straightforward. Due to this reasoning, the following parts of the thesis rely exclusively on the Fourier weighted gauge-fixed YM action given by Eq.(2.55).

2.4 Importance of BRST symmetry

Even though the invariance of the YM action under the BRST transformations seems to have appeared arbitrarily, BRST symmetry is a fundamental symmetry of gauge theories with tremendous physical implications. Here, we shall present the most important aspects that highlight the physical meaning of a BRST-invariant theory in the context of non-Abelian gauge theories. Note that the results mentioned below were established on covariant (mainly Lorenz) gauges.

2.4.1 Unitarity and the quartet mechanism

The power of the FP method lies in the formal introduction of the FP ghost fields. The reason behind it is that in non-Abelian theories, as opposed to Abelian theories, unitarity is only maintained as a virtue of the inclusion of the FP ghosts. The inconsistency of unitarity for non-Abelian theories when only intermediate contributions of gauge fields are considered, was initially pinpointed by Feynman for the case of a fermion-antifermion scattering to lowest non-trivial order of the coupling constant, [174–176]. Inclusion of ghosts in closed loops gives rise to additional graph contributions essential in order to obtain a unitary S-matrix element as outlined in [177].

The main goal of this subsection is to motivate, from a physical point of view, the fact that in the context of non-Abelian gauge theories, unitarity is closely connected to the construction of a well-defined physical Fock space $\mathcal{H}_{\text{phys}}$, which itself is related to the BRST symmetry. The key for such an identification comes from the analysis of the structure of the state vector space $\mathcal{V}$.

Given that BRST symmetry is a global symmetry, from Noether's theorem, it comes with an associated BRST charge, Q_{B}, which itself generates the BRST transformations[2]. The introduction of Q_{B} facilitates the definition of the physical subspace $\mathcal{V}_{\text{phys}}$. In particular, an elegant *subsidiary condition* was proposed by Kugo and Ojima [178, 179], where

$$Q_{\text{B}} \left|\text{phys}\right\rangle = 0, \qquad \mathcal{V}_{\text{phys}} = \left\{ \left|\phi\right\rangle \,\middle|\, Q_{\text{B}} \left|\phi\right\rangle = 0 \right\}. \tag{2.59}$$

In a covariant formulation of gauge theories, the state vector space $\mathcal{V}$, contains potentially also negative norm states. This translates to the fact that $\mathcal{V}$ is a vector space with an *indefinite metric*. For a meaningful interpretation of the underlying theory, one must properly consider a physical subspace of the state vector space, $\mathcal{V}_{\text{phys}} = \{\left|\text{phys}\right\rangle\}$ that exhibits norm positivity.

To clarify this point, it is worth to make a distinction between *genuine unitarity* of the physical S-matrix with respect to physical states with positive norm and *(pseudo-)unitarity* of the total S-matrix with respect to states of indefinite norm in $\mathcal{V}$. For a physical S-matrix between states in $\mathcal{V}_{\text{phys}}$ that satisfies genuine unitarity, the following conditions should be met:

(i) **(Pseudo-)unitarity of the total S-matrix**. Such a criterion is equivalent to having a Hermitian Lagrangian, which motivates the consideration of the FP ghosts in (4.16), such that $c^\dagger = c$ and $\bar{c}^\dagger = \bar{c}$, cf. [179].

(ii) **Temporal stability of $\mathcal{V}_{\text{phys}}$**. In other words, the physical subspace should not be affected by the total S-matrix. This criterion is automatically satisfied due to the conserved nature of the BRST charge Q_{B}.

(iii) **Positive semi-definiteness of $\mathcal{V}_{\text{phys}}$**. This criterion is translated to

$$\left|\psi\right\rangle \in \mathcal{V}_{\text{phys}} \Rightarrow \left\langle\psi|\psi\right\rangle \geq 0. \tag{2.60}$$

Satisfying this criterion is non-trivial and model dependent. However, such an analysis results in a general norm-cancellation mechanism called the *quartet mechanism*. Kugo and Ojima [101], introduced this mechanism which was a breakthrough in understanding the role of BRST symmetry in gauge theories. The quartet mechanism is a generalization of the Gupta-Bleuler formalism [180, 181] to non-Abelian theories. In order to be able to appreciate this machinery to its full extend, we should fix some relevant terminology.

BRST cohomology

Nilpotency of the BRST operator, $\mathrm{s}^2 = 0$, infers the nilpotency of the BRST charge, $Q_{\text{B}}^2 = 0$. Then, any state can be classified according to the dimension of the representation of Q_{B} to a *BRST singlet* and a *BRST doublet*, cf. [182].

Any state $\left|\theta\right\rangle \in \mathcal{V} : \left|\theta\right\rangle = Q_{\text{B}} \left|\phi\right\rangle \neq 0$ is called *BRST exact*. The two state vectors $\left|\phi\right\rangle$ & $\left|\theta\right\rangle$ are called a *parent* state and a *daughter* state respectively and they constitute a BRST doublet. The parent state $\left|\phi\right\rangle \in \text{Im}(Q_{\text{B}})$.

[2] In canonical quantization, the BRST charge is promoted to an operator by noting that $[Q_{\text{B}}, \Phi] = \mathrm{s}\Phi$, which generates the BRST symmetry [178, 179].

Any state $|\Theta\rangle$ which satisfies the subsidiary condition, Eq.(2.59), is called *BRST closed* with $|\Theta\rangle \in \mathrm{Ker}(Q_\mathrm{B})$. If a BRST closed state is accompanied by a parent state, i.e. $|\Theta\rangle = Q_\mathrm{B}|\phi\rangle$, then it is the daughter state of a BRST doublet whereas otherwise it corresponds to a BRST singlet. BRST-exact states have zero norm and are orthogonal to BRST closed states. Hence, we write $\mathrm{Im}(Q_\mathrm{B}) \equiv \mathcal{V}_0$.

There is an ambiguity in identifying a BRST singlet state and a parent state in a BRST doublet due to the arbitrary addition of daughter states. Such an ambiguity can be alleviated by considering the *BRST cohomology group*, i.e.

$$\mathcal{H}(Q_\mathrm{B}) = \frac{\mathrm{Ker}(Q_\mathrm{B})}{\mathrm{Im}(Q_\mathrm{B})} = \frac{\mathcal{V}_\mathrm{phys}}{\mathcal{V}_0}. \tag{2.61}$$

$\mathcal{H}(Q_\mathrm{B})$ consists of BRST singlet states where their difference is given by a daughter state. In other words, we create equivalence classes of BRST closed states differing only by a BRST-exact one. Such states have the same norm. Thus, all physical contents of the theory should correspond to BRST singlets in $\mathcal{H}(Q_\mathrm{B})$.

Classification of BRST states

With the help of the FP ghost numbers, N_FP, assigned to particle states, we can classify the state space to different classes which encode the physical and unphysical content of the underlying theory. Note that from the study of the BRST cohomology, any state with non-vanishing N_FP requires the existence of an additional FP conjugate state with opposite FP ghost number, $-N_\mathrm{FP}$, [182]. In particular, the BRST singlets can be decomposed into the following classes:

(I) **BRST singlets with $N_\mathrm{FP} = 0$**. Under the assumption of asymptotic fields in the state space, cf. [101], it was deduced that the states of this class constitute the *genuine particle states* of the theory.

(II) **BRST singlets with $N_\mathrm{FP} \neq 0$**. This class can be further divided into the following subclasses:

  (a) **Unpaired singlets**. They correspond to BRST singlets with a FP conjugate state as an unphysical parent state in a BRST doublet.

  (b) **Singlet pairs**. They correspond to FP conjugate pairs in two BRST doublets.

  In view of generic arguments discussed in [182], the BRST singlet states with $N_\mathrm{FP} \neq 0$ do not enter the physical state space and thus this class of states can be neglected.

For the case of BRST doublets, an additional BRST doublet to the original one is required which implies the final category:

(III) **Quartets**. They correspond to FP conjugate pairs of BRST doublets. Quartet states appear in gauge theories [101, 183].

From the aforementioned classification, we can conclude that the relevant states which can potentially affect the physical contents of the theory enter as BRST singlets with $N_\mathrm{FP} = 0$ or as quartets. However, quartets only contribute in zero norm combinations, thus not affecting the properties of $\mathcal{H}(Q_\mathrm{B})$ which is shown with the aid of the quartet mechanism.

Quartet mechanism for off-shell YM theory with unbroken global color symmetry

In order to implement the quartet mechanism, one introduces a projection operator $P^{(n)}$, which projects the particle states onto the sector of state space with n unphysical particles, cf. [178, 179]. Then, it can be deduced that every quartet state enters in the physical theory as zero norm combinations. Through the projection operator, one defines a physical Fock space of genuine states, $\mathcal{H}_{\text{phys}}$, which is isomporphic to the BRST cohomology $\mathcal{H}(Q_{\text{B}})$, i.e.

$$P^{(0)}\mathcal{V} = \mathcal{H}_{\text{phys}} \cong \mathcal{H}(Q_{\text{B}}) = \frac{\mathcal{V}_{\text{phys}}}{\mathcal{V}_0}. \tag{2.62}$$

This ensures a semi-positive norm of genuine physical particles and the condition (iii) is met, which further guarantees, the unitarity of the physical S-matrix. Eq.(2.62) clearly displays that in a gauge theoretic framework, a seemingly arbitrary global symmetry turns out to be an integral part for the establishment of a sensible theory.

An essential property of the construction of the aforementioned mechanism is the nilpotency of the BRST operator. The importance of the nilpotency can be viewed by considering an explicit mass term in pure YM theory. Such a contribution will result in the breaking of this property and a non-unitary S-matrix [109].

Implementing the quartet mechanism in the description of the off-shell pure YM theory with an unbroken global color symmetry[3], while ignoring potential IR divergences, results in the classification of the corresponding field configuration modes [184, 185]. In particular, it turns out that the modes of the transverse gauge field are associated with positive norm BRST singlets which are classified as genuine physical particle states. The unphysical modes of the longitudinal gauge field, the NL field and the FP ghosts constitute the so called *elementary quartet* and as such are confined, maintaining the norm positivity of $\mathcal{H}_{\text{phys}}$. Here, one must highlight the presence of the NL field which is essential for the proper norm cancellation in the quartet mechanism of pure YM.

Note that, as previously mentioned, for the study of the norm positivity of the physical Fock space in pure YM, the presence of IR divergences can affect the unitarity property of the S-matrix. However, in such a case one departs from the perturbative regime and should explore alternative non-perturbative routes, e.g. lattice methods, fRG, etc. Considerations of such generalizations will be addressed on the level of the fRG but their structural effect on the theory in terms of the K-O construction will not be further explored in this thesis.

2.4.2 The Kugo-Ojima color confinement scenario

Based on global color invariance, Kugo & Ojima derived a scenario [101], in view of the conserved color charge, that ensures color confinement in non-Abelian theories. Initially, from the equations of motion for the gauge field, one obtains the following Maxwell-like form [186]

$$\bar{g}\, \mathrm{j}^a_\mu = \partial^\nu F^a_{\mu\nu} + \{Q_{\text{B}}, (D_\mu \bar{c})^a\}\,, \tag{2.63}$$

[3]Invariance under global gauge transformations is to be understood when global color invariance is mentioned.

where j^a_μ is the Noether current of global color symmetry. This current yields the global color charge Q^a, of the global color symmetry, given by

$$\bar{g}\, Q^a = \mathbb{G}^a + \mathbb{N}^a = \int_{\mathbf{x}} \partial_i F^a_{0i} + \int_{\mathbf{x}} \{Q_{\mathrm{B}}, (D_0 \bar{c})^a\}\,, \tag{2.64}$$

with $\int_{\mathbf{x}} = \int d^3x$. Note that $\mathbb{G}^a$ & $\mathbb{N}^a$ are conserved charges of the associated conserved currents that constitute j^a_μ.

Due to the ambiguity of Noether's current, one can add an arbitrary anticommuting term without affecting the dynamics of the theory, thus rewriting the color charge as

$$Q^a = \int_{\mathbf{x}} \left[\mathrm{j}^a_0 - \frac{1}{\bar{g}} \partial_i F^a_{0i}\right] = \frac{1}{\bar{g}} \int_{\mathbf{x}} \{Q_{\mathrm{B}}, (D_0 \bar{c})^a\}, \tag{2.65}$$

which takes a BRST-exact form. Then, any genuine physical state (BRST singlet) when acted by Q^a

$$Q^a\, |\mathrm{phys}\rangle = 0, \tag{2.66}$$

since, according to the quartet mechanism, the genuine physical states obey by definition the subsidiary condition Eq.(2.59). Eq.(2.66) implies that colored elementary particles (which can appear as final states) are confined to the unphysical part of the Hilbert space and all final particle states are *color singlets*. Such a result would generically impose color confinement.

However, the global color charge, defined in Eq.(2.65) is not well-defined, due to non-convergence of the spatial integration. Hence, we turn our attention back to Eq.(2.64) and study each contributing charge individually. Note that $\mathbb{G}^a$ is explicitly related to gluons, whereas $\mathbb{N}^a$ mixes different field configurations.

Study of the $\mathbb{G}^a$ part

If the conserved current, $\partial^\nu F^a_{\mu\nu}$ contains no discrete massless pole, then

$$\mathbb{G}^a = 0, \tag{2.67}$$

as it corresponds to a spatial integration over a total derivative. In addition, Eq.(2.67) is satisfied even for massive genuine gauge fields as the unphysical (longitudinal) configurations drop out for both the massless and massive case due to the antisymmetry of the field strength tensor.

Study of the $\mathbb{N}^a$ part

In order to examine the behavior of the $\mathbb{N}^a$ charge, we introduce the dynamical parameter u^{ab}, called the *K-O function* as the IR limit of the two-point correlator of the composite operator $\mathbb{O}^a(x) = \bar{g} f^{abc} A^b_\mu c^c$,

$$\int_x \mathrm{e}^{ip(x-y)}\, \langle 0|\mathrm{T}\Big[(D_\mu c)^a\,(x)\mathbb{O}^b(y)\Big]|0\rangle = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) u^{ab}(p^2). \tag{2.68}$$

The well-definedness of $\mathbb{N}^a$ is realized from the study of the asymptotic behavior of $(D_\mu \bar{c})^a$

which contains the aforementioned composite operator $\mathbb{O}^a$. Then,

$$\left(D_\mu \bar{c}\right)^a(x) \xrightarrow{x_0 \to \mp\infty} \left(\delta^{ab} + u^{ab}\right) \partial_\mu \bar{\gamma}^b, . \tag{2.69}$$

where $\bar{\gamma}^a(x)$ corresponds to the massless asymptotic field of $\bar{c}^a(x)$. Eq.(2.69) entails that if the following condition (written in matrix notation), called the *color confinement criterion*, is satisfied

$$\mathbf{u}(0) + \mathbb{1} = 0, \tag{2.70}$$

then the conserved $\mathbb{N}^a$ charge generates a well-defined global color charge, which in combination with Eq.(2.67) equals to Eq.(2.65).

Color confinement

We can collectively formulate the previously obtained results from the study of each constituting charge in the following concluding statement. *Color confinement*, is realized as a well-defined color charge Q^a[4], where

$$Q^a = 0, \quad \text{in } \mathcal{H}_{\text{phys}}, \tag{2.71}$$

or equivalently,

$$\langle \phi | Q^a | \psi \rangle = 0, \quad \forall \left|\psi\right\rangle, \left|\phi\right\rangle \in \mathcal{V}_{\text{phys}}. \tag{2.72}$$

Such condition is met if the two following confinement criteria are satisfied:

(i) There is no discrete massless pole in $\partial^\nu F^a_{\mu\nu}$ for massless/massive gauge field configurations.

(ii) The IR pole residue of the composite operator $\mathbb{O}^a(x)$, $\mathbf{u}(0)$, satisfies Eq.(2.70).

In other words, when the K-O color confinement criteria (i) & (ii) are met, then elementary particles correspond to confined colored states and genuine physical particle states appear only as color singlets. The K-O confinement scenario encapsulates both quark and gluon condensate depending on the underlying theory. The K-O confinement criterion, Eq.(2.70), was found to be a necessary condition for the restoration of a broken residual local gauge symmetry [187, 188], thought of as combinations of global gauge transformations with spacetime independent parameters [189, 190].

One important consequence of the K-O confinement scenario, as computed in [191] is an IR enhanced ghost propagator. In particular, the ghost propagator in the Landau gauge can be parametrized as

$$D^{ab}(p^2) = -\delta^{ab} \frac{\mathcal{C}_{\text{c}}(p^2)}{p^2}, \tag{2.73}$$

with

$$\lim_{p^2 \to 0} \mathcal{C}_{\text{c}}(p^2) = \frac{1}{1 + \mathbf{u}(0)}. \tag{2.74}$$

[4] This is equivalent to the statement that global color symmetry is not spontaneously broken.

Finally, following a perturbative study, the stability of the K-O function, Eq.(2.68), was explored in the IR regime which naturally furnished the K-O confinement criterion, Eq.(2.70), cf. [101].

2.4.3 BRST symmetry and gauge invariance

Most interestingly, BRST symmetry inherits the essence of local gauge transformations. In particular, Refs. [192, 193] were motivated by the Maurer-Cartan structure of the ghost BRST transformation, cf. Eq.(2.37) and provided a geometrical interpretation of the FP ghosts as Maurer-Cartan forms that live on the gauge group of infinitesimal gauge transformations. They achieved a restriction of the gauge group over solely infinitesimal local gauge transformations by expanding the finite local gauge transformations, Eq.(2.18) and truncating at order $\mathcal{O}\left(\bar{g}^2\right)$. In such a way, one promotes the approximate relation of the infinitesimal gauge transformations to an exact one which becomes identical to its finite transformation. In addition, with an appropriate identification of the nilpotent BRST operator, they discovered a mutual relation between the nilpotency of the BRST operator, the BRST transformation of the ghost and preservation of closed Lie algebra structure (Jacobi identity in the adjoint representation). Providing two of the aforementioned conditions, the BRST transformations and the corresponding nilpotency follows. Therefore, by identifying the FP ghosts as Maurer-Cartan forms, they concluded that invariance of the theory under BRST transformations entails invariance under infinitesimal gauge transformations.

On the same year, [194] promoted the BRST symmetry as the fundamental symmetry of gauge theories and by constructing BRST invariant actions of physical and ghost fields from first principles, elaborated that they lead to gauge independent theories. Such a procedure, not only facilitates the construction of the theory by providing a consistent underlying symmetry which is not affected by the procedure of gauge fixing, but it can also be related to the conventional FP approach for linear gauges fixing conditions on-shell. This construction serves as an improvement of the conventional FP approach. Moreover, constructing an off-shell BRST invariant action allows for the presence of quartic ghost interactions, as required in [195, 196] but are absent in the FP method.

We see that there is quite a lot of motivation towards adopting the off-shell quantization procedure of YM and maintaining BRST invariance. This argumentation in combination with the study of an extended gauge-fixing condition with a stochastic variable, carried in Sec. 2.3, corresponds to the main motivation of developing our theory in an off-shell framework.

CHAPTER 3

Functional Methods

This chapter will be dedicated to introducing all essential concepts and quantities that will be instrumental later on in the construction and renormalizability of a manifestly BRST-invariant theory. By enforcing BRST symmetry, we will promote invariance under underlying symmetry transformations in correlation functions, albeit at the expense of introducing functional constraint equations. As we will examine both perturbative and non-perturbative renormalization schemes in this thesis, we aim at a transition from the (perturbative) RG to the fRG and provide a brief introduction to the basics of the fRG. We conclude this chapter by investigating the compatibility of the underlying symmetries for mass-dependent renormalization schemes, such as the fRG, encoded in the aforementioned constraint equations, with the associated flow equation.

3.1 Basic functional tools

This section will serve as a short introduction to the definition and form of the generating functionals required to perform perturbative and non-perturbative calculations, adjusted for pure YM theory.

As found in Subsec. 2.3.2, cf. Eq.(2.55), the action functional for pure YM theory, when a Fourier weight is considered, reads

$$S_A[A,v] = S_{\text{YM}}[A] - v^a \mathcal{F}^a. \tag{3.1}$$

From this action, the following path integral is generated

$$Z[j;v] = \int \mathcal{D}A \ \det\left(\mathcal{M}_{\text{FP}}[A]\right) \mathrm{e}^{i\left(S_A[A,v] + j^a_\mu A^{a\mu}\right)}. \tag{3.2}$$

To facilitate further computations, we perform a Wick rotation of the gauge-fixed generating functional, Eq.(3.2). Then, the Euclidean path integral takes the form

$$\mathcal{Z}[j;v] = \int \mathcal{D}A \ \det\left(\mathcal{M}_{\text{FP}}[A]\right) \mathrm{e}^{-\left(S_{\text{E}}[A,v] - j^a_\mu A^a_\mu\right)}, \tag{3.3}$$

with a Euclidean gauge-fixed action

$$S_{\mathrm{E}}[A,v] = S_{\mathrm{YM}}[A] + v^a \mathcal{F}^a \tag{3.4}$$

and a Euclidean YM action

$$S_{\mathrm{YM}} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}. \tag{3.5}$$

Note that we have suppressed the subscript E which highlights the Euclidean character. Given that we mostly work in Euclidean spacetime we do not bother about the position of spacetime indices.

The FP determinant is generated by the following Euclidean integral,

$$\det\left(\mathcal{M}_{\mathrm{FP}}[A]\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\; \mathrm{e}^{-S_{\mathrm{gh}}[A,c,\bar{c}]}, \tag{3.6}$$

with the Euclidean ghost action

$$S_{\mathrm{gh}}[A,c,\bar{c}] = -\bar{c}^a \mathcal{M}^{ab}_{\mathrm{FP}}[A] c^b. \tag{3.7}$$

In order to promote the FP ghosts to elementary fields, we introduce appropriate sources $(\bar{\eta},\eta)$ for $(\mathbf{c},\bar{\mathbf{c}})$ respectively, so that these fields are treated on an equal footing as the gauge field. After this addition of extra sources, we find the generating functional

$$\mathcal{Z}[j,\eta,\bar{\eta};v] = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\; \mathrm{e}^{-(S[A,c,\bar{c},v]+S_{\mathrm{sou}})}, \tag{3.8}$$

where

$$S[A,c,\bar{c},v] = S_{\mathrm{YM}}[A] + v^a\mathcal{F}^a + S_{\mathrm{gh}}[c,\bar{c}], \tag{3.9}$$

$$S_{\mathrm{sou}} = -j^a_\mu A^a_\mu - \bar{\eta}^a c^a - \bar{c}^a \eta^a. \tag{3.10}$$

Eq.(3.8) corresponds to the generating functional of the full correlation functions. The generating functional of the *connected* correlation functions, denoted by $W[j,\eta,\bar{\eta};v]$ and called the *Schwinger functional*, is given by

$$\mathcal{Z}[j,\eta,\bar{\eta};v] = \mathrm{e}^{W[j,\eta,\bar{\eta};v]}. \tag{3.11}$$

The Legendre transform[1] of the Schwinger functional, known as the *effective action (EA)*,

$$\Gamma[A,c,\bar{c};v] = \sup_{j,\eta,\bar{\eta}} \left\{ j^a_\mu A^a_\mu + \bar{\eta}^a c^a + \bar{c}^a\eta^a - W[j,\eta,\bar{\eta};v] \right\}, \tag{3.12}$$

generates the one-particle irreducible (1PI) correlation functions. It is worth noting that at the

[1]The Legendre transform is a mathematical tool used in physics to establish a connection between conjugate quantities. Note that even though the Legendre transform of a convex function is convex, this is not an invertible statement, meaning that a convex Legendre transformed function does not guarantee the convexity of the original function. A prime example is the Schwinger functional which can be non-convex whereas its Legendre transform is a convex functional. Furthermore, it relates functions of velocity, the Lagrangian in classical mechanics and the Helmholtz free energy in statistical mechanics, to functions of momentum, the Hamiltonian and the Gibbs free energy respectively, [124, 197, 198].

supremum, the sources are functionals of the classical fields, which correspond to the expectation values of the associated quantum fields. Furthermore, in a common abuse of notation, we will refer to the classical fields using the same notation as the quantum fields.

Eq.(3.12) entails that the classical fields, conjugate to the sources, satisfy

$$A^a_\mu = \frac{\delta W}{\delta j^a_\mu}, \quad c^a = \frac{\delta W}{\delta \bar{\eta}^a}, \quad \bar{c}^a = -\frac{\delta W}{\delta \eta^a} = W\frac{\overleftarrow{\delta}}{\delta \eta^a}. \tag{3.13}$$

Combining Eq.(3.12) with Eqs.(3.13), one deduces the quantum equations of motion

$$j^a_\mu = \frac{\delta \Gamma}{\delta A^a_\mu}, \quad \eta^a = \frac{\delta \Gamma}{\delta \bar{c}^a}, \quad \bar{\eta}^a = -\frac{\delta \Gamma}{\delta c^a} = \Gamma\frac{\overleftarrow{\delta}}{\delta c^a}. \tag{3.14}$$

3.1.1 Gauge and BRST symmetry generators

Let us now introduce a useful tool that will aid the construction of the functional constraint equations, as discussed in Sec. 3.3. Promoting the FP ghosts to dynamical fields and considering a Fourier noise distribution results in the off-shell generating functional of Eq.(3.8) with an action given by Eq.(3.9). As discussed in Sec. 2.4, the gauge-fixed theory still exhibits manifest global color invariance. Global gauge invariance, i.e. invariance under the transformations with the Lie-valued parameter $\omega = \text{const}$, dictates that the additional fields change homogeneously under local gauge transformations. Combining their change with the change of the gauge field, Eq.(2.3), we find the off-shell extended finite local gauge transformations

$$\begin{aligned} \mathbf{A}^{\mathbf{U}}_\mu &= \mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1} - \frac{i}{\bar{g}}\left(\partial_\mu \mathbf{U}\right)\mathbf{U}^{-1}, \quad & \mathbf{c}^{\mathbf{U}} &= \mathbf{U}\mathbf{c}\mathbf{U}^{-1}, \\ \bar{\mathbf{c}}^{\mathbf{U}} &= \mathbf{U}\bar{\mathbf{c}}\mathbf{U}^{-1}, & \mathbf{v}^{\mathbf{U}} &= \mathbf{U}\mathbf{v}\mathbf{U}^{-1}. \end{aligned} \tag{3.15}$$

Then, one finds the set of off-shell extended infinitesimal local gauge transformations

$$\begin{aligned} \delta_\omega A^a_\mu &= D^{ab}_\mu \omega^b, \qquad & \delta_\omega c^a &= \bar{g} f^{acb}\omega^b c^c, \\ \delta_\omega \bar{c}^a &= \bar{g} f^{acb}\omega^b \bar{c}^c, & \delta_\omega v^a &= \bar{g} f^{acb}\omega^b v^c. \end{aligned} \tag{3.16}$$

Equivalently, Eqs.(3.16) are induced by the off-shell generator of infinitesimal extended local gauge transformations which correspond to the following functional operator

$$\mathcal{G}^a_{\text{local}} = \mathcal{G}^a_{\text{global}} + \mathcal{G}^a_{(\partial)} = D^{ab}_\mu \frac{\delta}{\delta A^b_\mu} + \bar{g} f^{acb}\left(c^c \frac{\delta}{\delta c^b} + \bar{c}^c \frac{\delta}{\delta \bar{c}^b} + v^c \frac{\delta}{\delta v^b}\right). \tag{3.17}$$

Note that the off-shell generator of global color rotations corresponds to a homogeneous change in all fields and reads

$$\mathcal{G}^a_{\text{global}} = \bar{g} f^{acb}\left(A^c_\mu \frac{\delta}{\delta A^b_\mu} + c^c \frac{\delta}{\delta c^b} + \bar{c}^c \frac{\delta}{\delta \bar{c}^b} + v^c \frac{\delta}{\delta v^b}\right) \tag{3.18}$$

and $\mathcal{G}^a_{(\partial)} = \partial_\mu \frac{\delta}{\delta A^a_\mu}$. Indeed, Eq.(3.17) reproduces the infinitesimal off-shell transformations, as it can be seen from

$$\omega^b \mathcal{G}^b_{\text{local}} \Theta^\dagger_i = \delta_\omega \Theta^\dagger_i, \tag{3.19}$$

where a spacetime integration is involved, cf. [199]. Note that we have introduced the following collective representation which includes all fields

$$\Theta_i^\dagger = \left(A_\mu^a, -c^a, \bar{c}^a, v^a\right), \quad \Theta_i = \begin{pmatrix} A_\mu^a \\ c^a \\ -\bar{c}^a \\ v^a \end{pmatrix}. \tag{3.20}$$

Even though at this stage, the gauge-fixed action, Eq.(3.9) and as a consequence the generating functional Eq.(3.8) are not invariant under local gauge transformations, Eq.(3.17) will be of great assistance in the formulation of the functional constraint equation when we restore manifest local gauge invariance on the level of the path integral, see Subsec. 3.3.2.

Finally, we formulate the BRST transformations, Eq.(2.56), with the aid of the BRST generator, which corresponds to a non-linear to the fields functional operator of the form

$$\mathcal{G}_{\text{BRST}} = (D_\mu c)^a \frac{\delta}{\delta A_\mu^a} - \frac{\bar{g}}{2} f^{abc} c^b c^c \frac{\delta}{\delta c^a} + v^a \frac{\delta}{\delta \bar{c}^a} = (\mathrm{s}\Theta)_i \frac{\delta}{\delta \Theta_i^\dagger}. \tag{3.21}$$

Note that the BRST generator is nilpotent, i.e. $\mathcal{G}_{\text{BRST}}^2 = 0$ in agreement with the BRST operator. In contrast to Eq.(3.17), the BRST symmetry is non-linear to the fields and as such when incorporated in the theory can make the renormalization procedure more intricate, a fact that can give rise to nonlocalities. Similar to the previous case, the BRST generator will help to restore manifest BRST invariance for the generating functional using the Zinn-Justin equation, as explained and derived in Subsec. 3.3.3.

3.2 Fundamentals of the background field method

In this section, we briefly go through the main idea behind the well-established, in different contexts, *BFM* [200], applied to the gauge-fixed YM action. Even though the gauge-fixing procedure followed in Sec. 2.2 resulted in an action which is no longer invariant under local gauge transformations, but rather the residual BRST symmetry, the BFM allows to maintain the local symmetry of background gauge transformations in a manifest way. Such a manifest invariance significantly simplifies the renormalization procedure by allowing only contributions that are consistent with the background gauge symmetry and filtering out the rest. More importantly, without any loss of information, such symmetry properties can be associated with the full EA, cf. [123, 201].

In order to transition to the BFM, let us strip off any FP ghost dependence from the generating functional, Eq.(3.8), i.e.

$$\mathcal{Z}[j; v] = \int \mathcal{D}A \, \det\left(\mathcal{M}_{\text{FP}}[A]\right) \mathrm{e}^{-\left(S[A,v] - j_\mu^a A_\mu^a\right)}. \tag{3.22}$$

Then, we decompose the gauge field A_μ^a

$$A_\mu^a \to \bar{A}_\mu^a + a_\mu^a, \tag{3.23}$$

into an auxiliary non-dynamical field $\bar{A}_\mu^a$, called the *background field* and its quantum fluctuations, denoted by a_μ^a. Such a split will affect all contributing quantities of the theory. In particular, the field strength tensor becomes,

$$F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + \left(\bar{D}_\mu a_\nu\right)^a - \left(\bar{D}_\nu a_\mu\right)^a + \bar{g} f^{abc} a^b_\mu a^c_\nu \tag{3.24}$$

with the *background field strength tensor*,

$$\bar{F}^a_{\mu\nu} = \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu + \bar{g} f^{abc} \bar{A}^b_\mu \bar{A}^c_\nu \tag{3.25}$$

and the *background covariant derivative* in the adjoint representation

$$\bar{D}^{ab}_\mu = \partial_\mu \delta^{ab} - \bar{g} f^{abc} \bar{A}^c_\mu. \tag{3.26}$$

The action now becomes

$$S[a, \bar{A}, v] = S_{\text{YM}}[a, \bar{A}] + v^a \mathcal{F}^a[a, \bar{A}], . \tag{3.27}$$

where we allow the gauge-fixing condition $\mathcal{F}^a$ to depend separately on $\bar{A}$ and a.

Next, the *background generating functional* is related to a full generating functional via the relation[2]

$$\bar{\mathcal{Z}}[j, \bar{A}; v] = \mathcal{Z}[j, \bar{A}; v]\, \mathrm{e}^{-j^a_\mu \bar{A}^a_\mu}, \tag{3.28}$$

as obtained by shifting $\bar{A}^a_\mu \to \bar{A}^a_\mu - a^a_\mu$ and keeping a fixed background which entails that the functional integration is carried entirely over the quantum fluctuations, i.e. $\mathcal{D}A \to \mathcal{D}a$. Note that the shift of the background, which in scalar theories connects the conventional and the background functionals, is spoiled in gauge theories due to the field dependence in the gauge-fixing condition. As a result, the full gauge obtained from the shift of the background may take an unusual form, $\mathcal{F}' = \mathcal{F}[\bar{A} - a, \bar{A}]$. Such unconventional gauge-fixing condition leads to the full generating functional $\mathcal{Z}[j, \bar{A}; v]$ which differs in general from the conventional one, cf. Eq.(3.22).

From Eq.(3.28), the *background Schwinger functional* is found to be

$$\bar{W}[j, \bar{A}; v] = W[j, \bar{A}; v] - j^a_\mu \bar{A}^a_\mu, \tag{3.29}$$

where $W[j, \bar{A}; v]$ corresponds to the full Schwinger functional. Taking the Legendre transform of Eq.(3.29), one finds the *background EA*

$$\bar{\Gamma}[a, \bar{A}; v] = \sup_j \left\{ j^a_\mu a^a_\mu - \bar{W}[j, \bar{A}; v] \right\} = \sup_j \left\{ j^a_\mu \left(a^a_\mu + \bar{A}^a_\mu\right) - W[j, \bar{A}; v] \right\}, \tag{3.30}$$

computed at the supremum $j = j[a]$. The *background classical field* a^a_μ and the quantum equation of motion are given by

$$a^a_\mu = \frac{\delta \bar{W}[j, \bar{A}; v]}{\delta j^a_\mu}, \qquad j^a_\mu = \frac{\delta \bar{\Gamma}[a, \bar{A}; v]}{\delta a^a_\mu}. \tag{3.31}$$

[2]In this context, the term full generating functional implies that its associated source couples to the full gauge field as opposed to the background generating functional where the source couples only to the fluctuations.

Taking the functional derivative of Eq.(3.30) with respect to the source, results in

$$a_\mu^a = A_\mu^a - \bar{A}_\mu^a, \tag{3.32}$$

which relates the background with the full classical field. The relation between the background EA and the full EA is obtained by inserting Eq.(3.32) in Eq.(3.30). Setting $A = \bar{A}$, one finds that

$$\bar{\Gamma}[0, A; v] = \Gamma[A; v], . \tag{3.33}$$

where the full EA, on the right side of the relation, depends on the background field both through the gauge-fixing condition and because $A = \bar{A}$, [123, 201–203]. Even though, the shift property is no longer guaranteed, it allows to preserve background invariance which yields the same observables as the usual full gauge, despite the difference in the correlation functions reflecting the choice of the gauge [123, 201].

Hence, the driving force behind adopting the BFM is unveiled when one considers local gauge transformations. Due to Eq.(3.23), there are different ways of writing the local gauge transformations in terms of the decomposed gauge field.

The *quantum gauge transformations* transform only the fluctuation field, e.g.

$$\mathbf{a}_\mu^{\mathbf{U}} = \mathbf{U}\left(\bar{\mathbf{A}}_\mu + \mathbf{a}_\mu\right)\mathbf{U}^{-1} - \frac{i}{g}\left(\partial_\mu \mathbf{U}\right)\mathbf{U}^{-1} - \bar{\mathbf{A}}_\mu, \quad \bar{\mathbf{A}}_\mu^{\mathbf{U}} = \bar{\mathbf{A}}_\mu \tag{3.34}$$

and infinitesimally,

$$\delta_\omega^{\mathrm{Q}} a_\mu^a = \left(D_\mu \omega\right)^a, \quad \delta_\omega^{\mathrm{Q}} \bar{A}_\mu^a = 0. \tag{3.35}$$

The *background gauge transformations* affect both the background and the fluctuation fields, e.g.

$$\mathbf{a}_\mu^{\mathbf{U}} = \mathbf{U}\mathbf{a}_\mu\mathbf{U}^{-1}, \quad \bar{\mathbf{A}}_\mu^{\mathbf{U}} = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1} - \frac{i}{g}\left(\partial_\mu \mathbf{U}\right)\mathbf{U}^{-1}, \tag{3.36}$$

or infinitesimally,

$$\delta_\omega^{\mathrm{B}} a_\mu^a = \left[D_\mu^{ab}(a+\bar{A}) - \bar{D}_\mu^{ab}\right]\omega^b = \bar{g} f^{acb}\omega^b a_\mu^c, \quad \delta_\omega^{\mathrm{B}} \bar{A}_\mu^a = \left(\bar{D}_\mu \omega\right)^a. \tag{3.37}$$

Note that both the quantum and background gauge transformations add up to the local gauge transformations of the full gauge field, cf. Eq.(2.3), as they should. In addition, it is the quantum gauge transformations which must be fixed during the gauge-fixing procedure to make sense of the path integral in perturbation theory and thus their breaking gives rise to the residual global BRST symmetry.

Reinstating the FP ghost contribution, as in Eq.(3.6), results in the ghost action in the BFM

$$S_{\mathrm{gh}}[a, \bar{A}, c, \bar{c}, v] = -\bar{c}^a \mathcal{M}_{\mathrm{FP}}^{ab}[a, \bar{A}, v] c^b. \tag{3.38}$$

At this point one should mention that no FP ghosts or v-field background split took place. This can be justified since these fields appear as the product of the gauge-fixing condition and their decomposition would be irrelevant on the level of the EA in the limit $A = \bar{A}$. Hence, their classical fields and sources are identical to the ones in Eqs.(3.13) & (3.14) respectively.

However, the introduction of a v field through the Fourier weighted gauge-fixing condition in combination with the BFM can provide valuable insights into its properties. The v field has been treated as a background/external field, in the sense that it is not an integration variable as a consequence of the Fourier weight chosen during the gauge-fixing procedure and as such is unaffected by the background decomposition. Even though it differs from the NL b field, they are both products of the gauge fixing and affect the same sectors of the theory. Therefore, it is of interest to investigate whether one can relate these field configurations by exploiting the BFM.

An alternative approach to understanding these fields is to view the v field as the background field of the NL field. This can be illustrated by examining the Fourier weighted NL sector. Manipulating Eq.(2.53), one can find that

$$\int \mathcal{D}b \; \mathrm{e}^{-S_{\mathrm{NL}}[b+v]-b^a\mathcal{F}^a} \to \mathrm{e}^{v^a\mathcal{F}^a}. \tag{3.39}$$

The relation above suggests that the v field can be considered as a background field, while the b field corresponds to the fluctuations of the NL field, i.e. $\mathcal{B}^a = b^a + v^a$. Then, the Fourier noise prescription encoded in Eq.(2.52) leads to the right side of Eq.(3.39) and corresponds to the limit where the fluctuations of the NL field are frozen, leaving only the fixed background v field to contribute.

Such a behavior is attainable in regimes where the NL field can exhibit a preferred non-vanishing expectation value and thus be described to first approximation by the considered limit.

3.2.1 Background gauge and BRST symmetry generators

Similarly to Subsec. 3.1.1, one can extend the background gauge transformations, to include a homogeneous transformation of the remaining fields $(c^a, \bar{c}^a, v^a)$. Thus, we are led to the off-shell extended background gauge field transformations

$$\begin{aligned} &\mathbf{a}_\mu^{\mathbf{U}} = \mathbf{U}\mathbf{a}_\mu\mathbf{U}^{-1}, \quad \bar{\mathbf{A}}_\mu^{\mathbf{U}} = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1} - \frac{i}{\bar{g}}\left(\partial_\mu\mathbf{U}\right)\mathbf{U}^{-1}, \\ &\mathbf{c}^{\mathbf{U}} = \mathbf{U}\mathbf{c}\mathbf{U}^{-1}, \qquad \bar{\mathbf{c}}^{\mathbf{U}} = \mathbf{U}\bar{\mathbf{c}}\mathbf{U}^{-1}, \quad \mathbf{v}^{\mathbf{U}} = \mathbf{U}\mathbf{v}\mathbf{U}^{-1}, \end{aligned} \tag{3.40}$$

or infinitesimally,

$$\begin{aligned} &\delta_\omega^{\mathrm{B}} a_\mu^a = \bar{g} f^{acb}\omega^b a_\mu^c, \quad \delta_\omega^{\mathrm{B}}\bar{A}_\mu^a = \left(\bar{D}_\mu\omega\right)^a, \\ &\delta_\omega^{\mathrm{B}} c^a = \bar{g} f^{acb}\omega^b c^c, \quad \delta_\omega^{\mathrm{B}}\bar{c}^a = \bar{g} f^{acb}\omega^b\bar{c}^c, \quad \delta_\omega^{\mathrm{B}} v^a = \bar{g} f^{acb}\omega^b v^c. \end{aligned} \tag{3.41}$$

Note that the homogeneous change of the remaining fields under the extended background transformations is governed by the condition that background gauge invariance is manifest on the level of the gauge-fixed action, Eq.(3.27). This places a restriction on the class of admissible gauge-fixing conditions.

The extended off-shell background gauge transformations are generated by the background functional operator $\bar{\mathcal{G}}_{\mathrm{B}}^a$, such that

$$\omega^a\bar{\mathcal{G}}_{\mathrm{B}}^a = \omega^a\bar{D}_\mu^{ab}\frac{\delta}{\delta\bar{A}_\mu^b} + \bar{g} f^{acb}\omega^a\left(a_\mu^c\frac{\delta}{\delta a_\mu^b} + c^c\frac{\delta}{\delta c^b} + \bar{c}^c\frac{\delta}{\delta\bar{c}^b} + v^c\frac{\delta}{\delta v^b}\right) = \left(\delta_\omega^{\mathrm{B}}\bar{\Theta}_i\right)\frac{\delta}{\delta\bar{\Theta}_i^\dagger}, \tag{3.42}$$

where $\bar{\Theta}_i$ represents the collective field in the BFM, an extension of Eq.(3.20) which trivially includes the background field.

Moreover within the BFM, the action can be brought into the same form as Eq.(2.57), which implies invariance under the following nilpotent BRST transformations that derive from the quantum gauge transformations

$$\begin{aligned} (\mathrm{s}a_\mu)^a &= (D_\mu c)^a\,, \qquad \left(\mathrm{s}\bar{A}_\mu\right)^a = 0, \\ (\mathrm{s}c)^a &= -\frac{\bar{g}}{2} f^{abc} c^b c^c, \quad (\mathrm{s}\bar{c})^a = v^a, \quad (\mathrm{s}v)^a = 0. \end{aligned} \tag{3.43}$$

The BRST generator of Eq.(3.43) takes the same form as in Eq.(3.21) with the difference that the BRST transformation of the full gauge field is carried by the fluctuating field, i.e.

$$\bar{\mathcal{G}}_{\mathrm{BRST}} = (D_\mu c)^a \frac{\delta}{\delta a_\mu^a} - \frac{\bar{g}}{2} f^{abc} c^b c^c \frac{\delta}{\delta c^a} + v^a \frac{\delta}{\delta \bar{c}^a} = \left(\mathrm{s}\bar{\Theta}\right)_i \frac{\delta}{\delta \bar{\Theta}_i^\dagger}. \tag{3.44}$$

Note that the form of the extended BRST transformations in the BFM, Eq.(3.43), is not unique. In fact, there are alternative extended versions of the BRST transformation for the BFM, under which the background field changes as a BRST closed ghost field, thus extending the color space, cf [204]. Such realizations are motivated from the simplicity of the associated Zinn-Justin equation and have been further implemented in various models in [205–208]. Such potential deformations of the BRST transformations will not be explored in this thesis.

3.3 Functional constraint equations - Ward identities

Constraint equations are a crucial aspect for a theory to be well-defined. As illustrated in Chpt. 2, during quantization of pure YM, a constraint equation at the level of the gauge field configurations, the gauge-fixing condition, was required to obtain a finite generating functional, at the expense of breaking manifest local gauge invariance. However, this gave rise to an additional global supersymmetry of the action, the BRST symmetry, which turned out to be a vital component for many physical aspects of the theory, as discussed in Sec. 2.4.

To compute the building blocks of the theory, such as correlation functions, the EA etc., which were introduced in Sec. 3.1, source terms must be introduced for each associated dynamical field configuration. Generally, we want to investigate under what circumstances a symmetry or previously valid symmetry of the action can be realized at the level of the generating functional and, in turn, for the EA. To achieve this, one imposes manifest invariance under a desired symmetry at the level of the EA, which yields a set of functional constraint equations known as the *Ward Identities (WIs)*. In principle, these WIs correspond to constraint equations of the theory's functionals that, once obeyed, guarantee invariance under the associated symmetry. It should be highlighted that the difference between the constraint equation, introduced in Section 2.2, and that of the functional constraint equations is that the former was imposed at the level of the field configurations, restricting the space of integration, whereas the latter is employed at the level of the functional itself, restricting its admissible form as an effect.

3.3.1 Ward identities from global color invariance

Before we embark on the study of more complicated forms of WIs, it is more instructive to motivate their structural importance with a simpler example within the context of YM theory. In particular, let us consider global color rotations. These transformations can be realized linearly to the fields and are generated by $\mathcal{G}^a_{\text{global}}$, cf. Eq.(3.18). Under global color rotations, the action and the functional measure remain invariant.

Enforcing such a symmetry on the level of the generating functional translates into

$$0 = \frac{\mathcal{G}^a_{\text{global}} \mathcal{Z}[\mathcal{J}; v]}{\mathcal{Z}[\mathcal{J}; v]} = \frac{1}{\mathcal{Z}[\mathcal{J}; v]} \int \mathcal{D}\Phi \; \mathcal{G}^a_{\text{global}} \; \mathrm{e}^{-(S[\Theta] + S_{\text{sou}})}. \tag{3.45}$$

The generator of global color rotations will act on the action as

$$0 = \frac{1}{\mathcal{Z}[\mathcal{J}; v]} \int \mathcal{D}\Phi \left[-\mathcal{G}^a_{\text{global}} S[\Theta] + \mathcal{J}_i^\dagger \left(\mathcal{G}^a_{\text{global}} \Phi_i \right) \right] \mathrm{e}^{-(S[\Theta] + S_{\text{sou}})}, \tag{3.46}$$

where Θ_i is the same as in Eq.(3.19) and we collect the dynamical fields and their associated sources as

$$\begin{aligned} \Phi_i^\dagger &= \left(A^a_\mu, -c^a, \bar{c}^a \right), \quad \Phi_i = \begin{pmatrix} A^a_\mu \\ c^a \\ -\bar{c}^a \end{pmatrix}, \\ \mathcal{J}_i^\dagger &= \left(j^a_\mu, \bar{\eta}^a, \eta^a \right), \qquad \mathcal{J}_i = \begin{pmatrix} j^a_\mu \\ \bar{\eta}^a \\ \eta^a \end{pmatrix}. \end{aligned} \tag{3.47}$$

Eq.(3.47) was chosen such that

$$S_{\text{sou}}[\Phi] = -\mathcal{J}_i^\dagger \Phi_i = -\Phi_i^\dagger \mathcal{J}_i \tag{3.48}$$

and will significantly compactify the upcoming computations. Hence, Eq.(3.46) leads to the equality

$$\langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle = \langle \mathcal{G}^a_{\text{global}} S_{\text{sou}}[\Phi] \rangle \tag{3.49}$$

In order to extract the physical meaning of the constraint, we observe that the first part of Eq.(3.46) corresponds to the expectation value of the variation of the action under global color rotations whereas the second term can be manipulated as follows

$$0 = -\langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle + \frac{1}{\mathcal{Z}[\mathcal{J}; v]} \int \mathcal{D}\Phi \; \mathcal{J}_i^\dagger \left(\mathcal{G}^a_{\text{global}} \frac{\delta}{\delta \mathcal{J}_i^\dagger} \right) \mathrm{e}^{-(S[\Theta] + S_{\text{sou}})} \tag{3.50}$$

Exchanging the functional with the implied spacetime integral of the source action, we arrive at

$$0 = -\langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle + \frac{1}{\mathcal{Z}[\mathcal{J}; v]} \; \mathcal{J}_i^\dagger \left(\mathcal{G}^a_{\text{global}} \frac{\delta \mathcal{Z}[\mathcal{J}; v]}{\delta \mathcal{J}_i^\dagger} \right). \tag{3.51}$$

Eq.(3.51) can be rewritten in terms of the Schwinger functional, (3.11), as

$$0 = -\langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle + \mathrm{e}^{-W[\mathcal{J}; v]} \; \mathcal{J}_i^\dagger \left(\mathcal{G}^a_{\text{global}} \frac{\delta W[\mathcal{J}; v]}{\delta \mathcal{J}_i^\dagger} \right) \mathrm{e}^{W[\mathcal{J}; v]}. \tag{3.52}$$

In the collective field representation, the EA, Eq.(3.12), can be rewritten as

$$\Gamma[\Phi; v] = \sup_{\mathcal{J}_i} \left\{ \mathcal{J}_i^\dagger \Phi_i - W[\mathcal{J}; v] \right\} . \tag{3.53}$$

In addition, the equivalent expressions of Eqs.(3.13) & (3.14) that connect the classical fields and their sources with the Schwinger functional and the EA respectively in the collective field representation reads,

$$\begin{aligned} \Phi_i &= \frac{\delta W[\mathcal{J}; v]}{\delta \mathcal{J}_i^\dagger}, \quad \Phi_i^\dagger = W[\mathcal{J}; v] \frac{\overleftarrow{\delta}}{\delta \mathcal{J}_i}, \\ \mathcal{J}_i &= \frac{\delta \Gamma[\Phi; v]}{\delta \Phi_i^\dagger}, \quad \mathcal{J}_i^\dagger = \Gamma[\Phi; v] \frac{\overleftarrow{\delta}}{\delta \Phi_i}. \end{aligned} \tag{3.54}$$

Performing a Legendre transform at the supremum, cf. (3.53), then $\mathcal{J}_i = \mathcal{J}_i[\Phi]$ which results into a mutual cancellation of the exponential terms of the Schwinger functional. In addition, using Eq.(3.54), we are left with

$$0 = - \langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle + \Gamma[\Phi; v] \frac{\overleftarrow{\delta}}{\delta \Phi_i} \left(\mathcal{G}^a_{\text{global}} \Phi_i \right) , \tag{3.55}$$

where the fields involved in the second term should be realized as the classical fields. Finally, noting that

$$\mathcal{G}^a_{\text{global}} \Gamma[\Phi; v] = \Gamma[\Phi + \mathcal{G}_{\text{global}} \Phi; v] - \Gamma[\Phi; v] = \Gamma[\Phi; v] \frac{\overleftarrow{\delta}}{\delta \Phi_i} \left(\mathcal{G}^a_{\text{global}} \Phi_i \right) ,$$

helps us obtaining

$$\mathcal{G}^a_{\text{global}} \Gamma[\Phi; v] = \langle \mathcal{G}^a_{\text{global}} S[\Theta] \rangle . \tag{3.56}$$

In a broader context, the relationship illustrated by Eq.(3.56) indicates that a linear transformation in the fields that preserves the action corresponds to a symmetry of the EA, provided that the functional measure remains invariant. The above relation corresponds to the *Ward Identity for global color rotations* in YM theory and yields a symmetry of the EA since the off-shell gauge-fixed action is manifestly invariant under the off-shell global gauge transformations, i.e. $\mathcal{G}^a_{\text{global}} S[\Theta_i] = 0$.

However, the WIs associated with local gauge symmetry, which is no longer a manifest symmetry, and with BRST transformations, which are non-linear in the fields, exhibit a more intricate structural form. To clarify the terminology, we shall refer to the WIs of local gauge symmetry as *Ward-Takahashi Identities (WTIs)* and those of BRST symmetry as the *Zinn-Justin equation* or conventionally as the *Slavnov-Taylor Identities (STIs)*, as originally proposed in [37, 209], albeit from a different perspective. The Zinn-Justin equation is also known as the *master equation* since, as argued in Subsec. 2.4.3, it embodies the information about local gauge transformations. Furthermore, when deriving the functional constraint equations, we will not assume any specific form of the gauge-fixing condition, but rather treat it as a generic functional of the relevant field configurations. The investigation of these functional constraints in diverse contexts will be the topic of the forthcoming Subsecs. 3.3.2-3.3.4.

3.3.2 Ward-Takahashi identities

Let us now study the WIs which correspond to the functional constraints of the EA when enforcing local gauge invariance. During the gauge-fixing procedure, local gauge invariance is spoiled on purpose by additional contributions in the action. For clarity, we rewrite Eq.(3.9),

$$S[\Theta] = S_{\text{YM}} + S_{\text{gf}} + S_{\text{gh}}.$$

Given that the YM action, S_{YM} is invariant under local gauge transformations by construction, i.e. $\mathcal{G}^a_{\text{local}} S_{\text{YM}} = 0$, then the Ward-Takahashi Identity reads

$$\mathcal{G}^a_{\text{local}} \Gamma[\Phi; v] = \langle \mathcal{G}^a_{\text{local}} \left(S_{\text{gf}} + S_{\text{gh}} \right) \rangle . \tag{3.57}$$

Recalling that functional derivatives of the EA with respect to the classical fields reproduces the 1-Particle Irreducible (1PI) correlation functions, the solutions of Eq.(3.57) clearly place a constraint on the underlying building blocks of the theory and the admissible form of the EA with vital implications for the renormalization of the theory.

For a Lorenz gauge-fixing condition in the Landau gauge expressed in terms of the bare fields and couplings, Eq.(3.57) connects the 1PI correlation functions at L-loops with up to $(\text{L}+2)$-loop corrections to the connected ghost, gluon propagators and ghost-gluon vertex function, see [199]. In addition, an explicit inclusion of a gluon mass parameter in this gauge is prohibited according to the WTIs.

3.3.3 Zinn-Justin equation

Next, we focus our attention on the derivation of the WIs for BRST symmetry, the so called Zinn-Justin equation or STIs or master equation. BRST transformations are generated by the non-linear generator Eq.(3.21). If we were to follow the same steps as for the global and color gauge transformations, such a non-linearity would propagate into the WIs, e.g.

$$\langle \mathcal{G}_{\text{BRST}} S_{\text{sou}} \rangle = 0. \tag{3.58}$$

Note that the generator of BRST transformations is non-linear to the fields and as such the study of the 1PI correlation functions as solutions of the WIs, Eq.(3.58), ends up being more involved.

Following [210, 211], one obtains a simpler expression for the associated WIs, by introducing appropriate BRST-exact sources of the BRST variations of the fields, extending the source action to

$$S_{\text{sou}} = -\Phi_i^\dagger \mathcal{J}_i + \text{K}^a_\mu \left(\text{s}A\right)^a_\mu + \text{L}^a \left(\text{s}c\right)^a , \tag{3.59}$$

with K^a_μ and L^a being anticommuting and commuting sources respectively. The WIs for the extended action (including the BRST sources) take the expected form, cf. Eq.(3.18)

$$\langle \mathcal{G}_{\text{BRST}} S_{\text{sou}} \rangle = \langle \left(\text{s}\Theta\right)_i \left(\frac{\delta S_{\text{sou}}}{\delta \Theta_i^\dagger} \right) \rangle = 0. \tag{3.60}$$

Inserting Eq.(3.59), we arrive at

$$\langle \mathrm{s}\Phi_i^\dagger \rangle \, \mathcal{J}_i = 0, \tag{3.61}$$

because the contribution of the additional sources vanishes due to their BRST-exact character, i.e.

$$\langle \mathcal{G}_{\mathrm{BRST}} \, \mathrm{K}_\mu^a \, (\mathrm{s}A)_\mu^a \rangle = \langle \mathcal{G}_{\mathrm{BRST}} \, \mathrm{L}^a \, (\mathrm{s}c)^a \rangle = 0.$$

Expanding the collective fields and sources in Eq.(3.61), one finds

$$j_\mu^a \, \langle (\mathrm{s}A)_\mu^a \rangle - \bar{\eta}^a \, \langle (\mathrm{s}c)^a \rangle + \eta^a \, \langle (\mathrm{s}\bar{c})^a \rangle = 0. \tag{3.62}$$

Given the inclusion of the BRST source, one can establish a relation between the expectation values of the BRST transformation, which appear in the WIs and the EA. Then, one finds that

$$\langle (sA)_\mu^a \rangle = \frac{\delta \Gamma}{\delta \mathrm{K}_\mu^a}, \quad \langle (\mathrm{s}c)^a \rangle = \frac{\delta \Gamma}{\delta \mathrm{L}^a}, \quad \langle (\mathrm{s}\bar{c})^a \rangle = v^a. \tag{3.63}$$

Substituting Eq.(3.63) into Eq.(3.62), we get

$$\frac{\delta \Gamma}{\delta A_\mu^a} \frac{\delta \Gamma}{\delta \mathrm{K}_\mu^a} + \frac{\delta \Gamma}{\delta c^a} \frac{\delta \Gamma}{\delta \mathrm{L}^a} + v^a \frac{\delta \Gamma}{\delta \bar{c}^a} = 0. \tag{3.64}$$

The WIs of Eq.(3.64) correspond to the so called *Zinn-Justin* or *master equation* for the gauge-fixed YM action, cf. Eqs.(3.9) & (3.59), i.e.

$$S[A, c, \bar{c}, v, \mathrm{K}, \mathrm{L}] = S_{\mathrm{YM}}[A] + v^a \mathcal{F}^a + S_{\mathrm{gh}}[c, \bar{c}] + S_{\mathrm{sou}}.$$

The Zinn-Justin equation, is satisfied by the total action, thus

$$\frac{\delta S}{\delta A_\mu^a} \frac{\delta S}{\delta \mathrm{K}_\mu^a} + \frac{\delta S}{\delta c^a} \frac{\delta S}{\delta \mathrm{L}^a} + v^a \frac{\delta S}{\delta \bar{c}^a} = 0, \tag{3.65}$$

which encodes the BRST symmetry of the action and entails that $\Gamma[\Phi; v] = S[\Theta, \mathrm{K}, \mathrm{L}]$ is a solution of the Zinn-Justin equation. The significance of Eq.(3.65) as a solution of Eq.(3.64) lies in its predictive power for constructing a renormalizable theory. As elaborated in [163], a construction of a renormalized action from first principles and building on Eq.(3.65)is possible by virtue of power counting at the expense of an emerging BRST-exact term quartic in the FP ghosts. In that case, Eq.(3.65) is stable under renormalization and BRST symmetry is ensured. Hence, we see that the Zinn-Justin equation translates the connection between renormalization and BRST symmetry to a restrictive class of admissible actions.

3.3.4 Ward identities in the background field formalism

During the BFM, we have imposed on the level of the action manifest invariance under BRST and background gauge transformations, cf. Eqs.(3.43) & (3.41), generated by Eqs.(3.44) & (3.42) respectively. Motivated by [212, 213], we derive a compatibility relation between the background gauge and BRST generators which entails a necessary condition for the form of the gauge-fixing condition.

Firstly, let us rewrite the action in the BFM in a BRST closed form,

$$S[a, \bar{A}, c, \bar{c}, v] = S_{\text{YM}}[a, \bar{A}] + \bar{\mathcal{G}}_{\text{BRST}}\Psi[a, \bar{A}, \bar{c}, v], \tag{3.66}$$

with $\Psi[a, \bar{A}, \bar{c}, v] = \bar{c}^a \mathcal{F}^a[a, \bar{A}, v]$.

Manifest background gauge invariance implies that

$$\bar{\mathcal{G}}^a_{\text{B}} S[\bar{\Theta}] = 0. \tag{3.67}$$

Furthermore, manifest BRST invariance of the action translates to the following condition

$$\bar{\mathcal{G}}_{\text{BRST}} S[\bar{\Theta}] = 0. \tag{3.68}$$

One can verify, that the BRST and background gauge transformations generators commute with each other [212],

$$\left[\bar{\mathcal{G}}^a_{\text{B}}, \bar{\mathcal{G}}_{\text{BRST}}\right] = 0. \tag{3.69}$$

Applying Eq.(3.68), while considering Eqs.(3.66) & (6.27), one readily derives that

$$\omega^a \bar{\mathcal{G}}^a_{\text{B}} \Psi = \left(\delta^{\text{B}}_\omega \bar{\Theta}_i\right) \frac{\delta}{\delta \bar{\Theta}^\dagger_i} \Psi = 0. \tag{3.70}$$

Expanding the Ψ functional in Eq.(3.70), then

$$\left(\delta^{\text{B}}_\omega \bar{\Theta}_i\right) \frac{\delta}{\delta \bar{\Theta}^\dagger_i} \mathcal{F}^a = \omega^b \bar{\mathcal{G}}^b_{\text{B}} \mathcal{F}^a = \bar{g} f^{acb} \omega^b \mathcal{F}^c. \tag{3.71}$$

Eq.(3.71) shows that under background gauge transformations, the gauge-fixing condition must change homogeneously in order to maintain manifest background gauge invariance. This condition implies that within the BFM, non-linear gauge-fixing conditions are valid and in accordance with the underlying symmetries, as long as they are constructed from tensorial quantities of the gauge group. Such a remark will be exploited in Chpts. 4 & 5.

Finally, suppressing the FP ghost dependence for a moment, manifest background invariance imposes a functional constraint on the background EA and gives rise to WIs of the form

$$\bar{\mathcal{G}}^a_{\text{B}} \bar{\Gamma}[\bar{A}; v] = 0. \tag{3.72}$$

However, as shown in Eq.(3.33), the background EA can be related to a full EA in the limit of $\bar{A} = A$. In that case, $\bar{\mathcal{G}}^a_{\text{B}} \to \mathcal{G}^a_{\text{local}}$. Thus, Eq.(3.72) reduces to

$$\mathcal{G}^a_{\text{local}} \Gamma[A; v] = 0, \tag{3.73}$$

which implies that the full EA constructed within the formalism of BFM is a gauge invariant quantity and thus consists of only gauge-invariant building blocks, resulting to appropriate physical observables.

Hence, we infer that invariance of the theory under the auxiliary background gauge transformations is associated to the local gauge transformations of the full gauge field, since it leads to a gauge invariant full EA, which can differ from the standard EA but yields the same physical observables. Therefore, it is sensible and many times computationally advantageous to work within the background field formalism without loss of information.

3.4 From RG to fRG

The effective action (EA) is a crucial quantity in the study of a theory as it provides information about the theory's long-range behavior and the realization of symmetries. Physical information, encoded in the EA, can be extracted using perturbative or non-perturbative techniques. To briefly elaborate on this point, let us rewrite the EA of Eq.(3.12), in terms of a functional integral in which, for the moment only, the macroscopic dynamical collective fields are denoted by Φ', and an ultraviolet (UV) cutoff scale is introduced (cf. Subsec. 3.4.2),

$$\mathrm{e}^{-\Gamma[\Phi';v]} = \int_\Lambda \mathcal{D}\Phi\ \mathrm{e}^{-S[\Phi+\Phi',v]+\Phi_i^\dagger \frac{\delta\Gamma}{\delta\Phi_i'^\dagger}}. \tag{3.74}$$

Eq.(3.74) implies that solving a set of first-order non-linear functional differential equations yields the EA. One way to do this is by performing a vertex expansion and determining the expansion coefficients that correspond to the 1PI proper vertices, which results in a set of integro-differential equations known as the *Dyson-Schwinger equations*, cf. [214–217]. Such an implementation of functional methods allows access to the non-perturbative regime of gauge theories at the cost of introducing appropriate truncation schemes to obtain an approximate solution of the corresponding EA.

3.4.1 The procedure of perturbative renormalization

In the context of perturbative treatments, the EA can be expanded to small values of the coupling constant. The resulting perturbative series expansion can be organized in terms of loop contributions, providing a useful approach for analyzing the theory. However, it should be noted that this approach has a limited range of validity. This technique can be used to derive an explicit expression for the EA up to a certain loop order, by writing the right-hand side of Eq.(3.74) as a Gaussian of the dynamical fields and then explicitly performing the functional integral. In this way, the one-loop perturbative EA can be brought to the conventional form of

$$\Gamma_{1\mathrm{L}} = S + \frac{1}{2}\,\mathrm{tr}\ln S^{(2)}, \tag{3.75}$$

as found in Chpt. 4, where $S^{(2)}$ is to be understood as the Hessian of the action in terms of the dynamical fields.

Further computation of associated correlation functions may result in the appearance of divergences arising from loop integrals. Nonlocal interactions can be a source of these emerging divergent contributions. In perturbation theory, these divergences can be handled through the process of *renormalization*. Renormalization is a valuable tool in the study of quantum field theory, allowing for the removal of infinities that arise in loop calculations.

There are various methods that can be used to renormalize a theory and remove its divergences. In Chpt. 4, a two-step procedure will be employed to achieve this, first by regularizing the divergences and then by reparametrizing the field and coupling constant accordingly. This adjustment requires the introduction of renormalization constants which serve as counterterms and connect the initial or *bare* quantities to the reparametrized or *renormalized* ones. By adjusting these constants to counteract loop divergences, the resulting theory will have finite correlation functions. The degree to which the theory can be renormalized is determined by the number of independent divergent loop contributions that must be regularized using the available parameters that can be reparametrized. In Chpts. 4 & 5, the perturbative study of the one-loop EA and 1-Particle Reducible (1PR) correlators for pure YM with nonlocal interactions is explored.

However, it is possible to achieve renormalization of a theory in a more controlled manner. It is important to note that there is not a unique method of renormalization and that observables must ultimately be independent of the chosen renormalization scheme. As a result, during renormalization, a mass scale k can be introduced into the theory which makes the couplings scale-dependent. By appropriately varying the renormalization mass scale, new couplings are induced at the newly varied scale. Repeating this procedure results in the *RG*, that generates a set of scale-dependent couplings which in turn are associated with the correlation functions at that scale. This approach provides an algorithmic method for dealing with divergences by consistently screening the energy scale at which they appear, but their effect remains integrated in the scale dependence of the effective couplings.

This scale dependence is described in terms of a *beta function* for the corresponding coupling g_i, defined as

$$\beta_{g_i} = k\frac{\mathrm{d}g_i}{\mathrm{d}k}. \tag{3.76}$$

The RG machinery offers valuable insights into the effects of microscopic degrees of freedom by studying the macroscopic aspects of a determined Effective Field Theory (EFT). This approach establishes a useful interplay between energy scales and renormalization, revealing information of the "full" theory from the structural study of an EFT confined within a well-defined scale domain. Ultimately, the RG provides a powerful tool for analyzing the behavior of a theory and extracting valuable information regarding its underlying structure. Therefore, renormalization plays an essential role in quantum field theory, enabling precise predictions and facilitating a better understanding of the behavior of quantum systems at small and large scales.

3.4.2 Functional renormalization group

Other successful non-perturbative versions for obtaining approximate solutions of the EA by solving a set of exact functional RG equations have been devised throughout the years [218–226]. In this work, we shall concern ourselves with the *functional renormalization group (fRG)* approach [227–231] which has been successfully implemented in the investigation of different physical models, cf. [232–251] for a non-exhaustive list and cf. [71, 136, 199, 244, 246, 252–254] for a sample of reviews on the topic.

The fRG is a non-perturbative approach to solve Eq.(3.75) by implementing the Wilsonian RG formulation [219–221, 255–259]. According to Wilson's idea, one can construct an EFT of low energy modes (long distance) by integrating over high energy ones (short distance). Such a procedure is based on the philosophy that the contribution of long range fluctuations, also called *slow modes*, is not affected by the short distance fluctuations or *fast modes* and as such they can

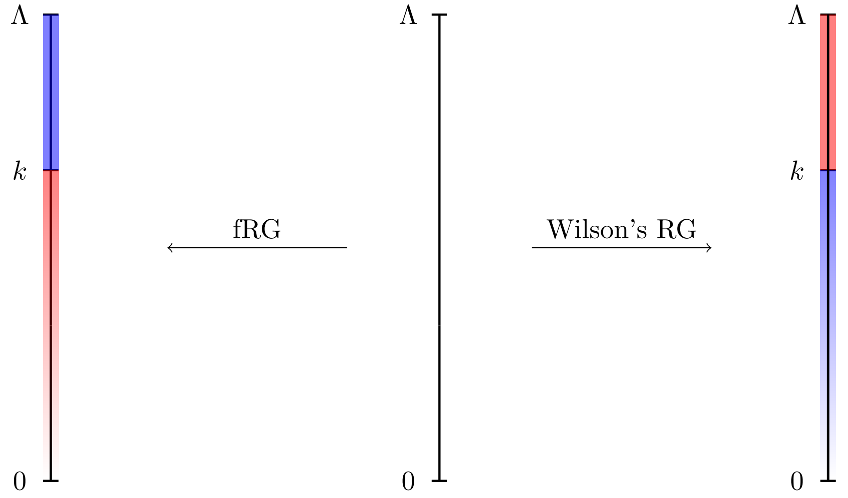

Figure 3.1: Graphical depiction of the possible energy scales of a regularized theory, as indicated by the cutoff Λ and integrating out modes by following Wilson's RG and fRG approaches. The central image shows the energy scales of the theory, with the cutoff Λ at the top. To the right and left of it, the underlying philosophy of integrating out modes by following Wilson's RG approach and fRG is shown in an intuitive manner. The blue shaded region corresponds to the energy domain of modes that contribute in the corresponding flows whereas the red shaded region represents the energy domain of modes that are integrated in the microscopical flow of Wilson's approach or that will be integrated in the macroscopic flow of the fRG. The role of the RG scale k, can also be clearly observed, where in Wilson's RG corresponds to a UV cutoff whereas in the fRG to an IR cutoff.

be *integrated out*.

Wilson's approach to the RG involves starting with a QFT that has been regularized up to a cutoff scale Λ. This cutoff scale is related to relevant microscopic scales, such as an inverse lattice spacing in a discrete setting and governs the microscopic dynamics. Consequently, the functional measure is restricted to an integration over fields with momenta up to the cutoff scale, i.e. $\int \mathcal{D}\Phi \to \int_\Lambda \mathcal{D}\Phi$. By employing the RG procedure, one can reformulate the underlying theory up to a different energy scale k, potentially experimentally accessible, by integrating out the fast modes and studying the effects of such a scale reduction on the microscopic dynamics.

The RG is generated by employing infinitesimally, coarse graining and successively a scale transformation in an iterative fashion. Coarse graining is responsible for modifying the theory's scale down to the desired scale k, providing an EFT for the slow modes with momenta $p < k$, above which all quantum fluctuations are integrated out. Computationally, this is achieved by splitting the slow from the fast varying modes and then integrating over the fast modes on the partition function. This results in a modified action. In such a way, there might not be an explicit dependence of the partition functional on the integrated out fast modes, but their information is encoded in the modified action.

However, as noted in Subsec. 3.4.1, each modified action can be associated with a modified correlation function at each scale. In order to extract meaningful information, the correlation functions from different energy scales need to be comparable, which can be performed by rescaling the momenta and field configurations to match those of the previous iterations. This is accomplished by applying an infinitesimal scale transformation on top of the coarse graining already applied. Repetition of the aforementioned procedure will result in a history of modified actions, generated at each coarse graining step, that interpolate between the cutoff scale Λ and the desired scale k. Such a dataset of modified action traces a trajectory in the space of actions, called the *RG trajectory*.

In the spirit of Wilson's idea, the fRG adopts an "inverse" logic and by employing functional methods manipulates the RG on a macroscopic scale, in contrast to the previous microscopic level. The main feature of the fRG is that instead of computing the history of modified actions of the slow modes, one derives a one-parameter family of scale-dependent EA, called *Effective Average Action (EAA)* of the fast modes, generated by the RG machinery. This approach shifts the focus of the RG procedure from the microscopic action to the macroscopic EAA, thereby enabling the exploration of the system's properties at different energy regimes by appropriately adjusting the scale k. Notably, the scale k no longer corresponds to a UV cutoff for the contributing slow modes, but instead to an IR cutoff regulated by the fast modes. Consequently, the IR limit corresponds to $k \to 0$, whereas the UV limit of the regulated theory corresponds to $k \to \Lambda$. When the UV regulator Λ is removed, i.e., $\Lambda \to \infty$, we gain access to the deep UV domain. A graphical representation of the two differing methods is depicted in Fig. 3.1.

Thus, according to the aforementioned structure, the EAA must interpolate between two fixed asymptotic behaviors. The microscopic bare action at the UV cutoff $k \to \Lambda$, since at that limit there are no integrated out fast modes and the full quantum EA of the theory at the IR limit $k \to 0$,

$$\lim_{k\to\Lambda} \Gamma_k[\Phi; v] \sim S[\Phi; v], \quad \lim_{k\to 0} \Gamma_k[\Phi; v] = \Gamma[\Phi; v]. \tag{3.77}$$

In order to build the EAA, due to the underlying principle, one should decouple the slow and fast varying modes of the theory on the level of the generating functional. This can be achieved by associating a large mass to the slow modes, by explicitly implementing an *IR regulator* $\Delta S_k[\Phi]$ in the generating functional. Thus, one finds an IR regulated, scale-dependent generating functional

$$\mathcal{Z}_k[\mathcal{J}; v] = \int_\Lambda \mathcal{D}\Phi \; \mathrm{e}^{-S[\Theta] - \Delta S_k[\Phi] + \mathcal{J}_i^\dagger \Phi_i} \equiv \mathrm{e}^{W_k[\mathcal{J}, v]}, \tag{3.78}$$

where in general, the IR regulator has a momentum dependent mass form

$$\Delta S_k[\Phi] = \frac{1}{2}\int_p A_\mu^a(-p)\; (R_k^{\mathrm{gl}})_{\mu\nu}(p)\; A_\mu^a(p) + \int_p \bar{c}^a(-p)\; (R_k^{\mathrm{gh}})(p)\; c^a(p), \tag{3.79}$$

where for each associate sector, appropriate gluon and ghost regulators were introduced. Next, we can define the *Effective Average Action (EAA)* as a modified Legendre transform

$$\begin{aligned} \Gamma_k[\Phi; v] &= \sup_{\mathcal{J}} \Big\{ \mathcal{J}_i^\dagger \Phi_i - W_k[\mathcal{J}; v] \Big\} - \Delta S_k[\Phi] \\ &= \tilde{\Gamma}[\Phi; v] - \Delta S_k[\Phi]. \end{aligned} \tag{3.80}$$

Furthermore, at $\mathcal{J} = \mathcal{J}_{\sup}$, we find the scale-independent macroscopic fields and scale-dependent quantum equations of motion, (fRG extension of Eqs.(3.54)),

$$\begin{aligned} \Phi_i &= \frac{\delta W[\mathcal{J}; v]}{\delta \mathcal{J}_i^\dagger}, & \Phi_i^\dagger &= W[\mathcal{J}; v] \frac{\overleftarrow{\delta}}{\delta \mathcal{J}_i}, \\ \mathcal{J}_i &= \frac{\delta \Gamma[\Phi; v]}{\delta \Phi_i^\dagger} + \frac{\delta \Delta S_k[\Phi]}{\delta \Phi_i^\dagger}, & \mathcal{J}_i^\dagger &= \Gamma[\Phi; v] \frac{\overleftarrow{\delta}}{\delta \Phi_i} + \Delta S_k[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi_i}. \end{aligned} \tag{3.81}$$

At this point, we can study the expected behavior of the *regulator function* $R_k(p)$ of Eq.(3.79). In particular, it must satisfy the following conditions:

- At $k \to 0$, when all quantum fluctuations are integrated out, it should vanish in order to ensure that the generating functional coincides with the scale-independent one i.e.

$$\lim_{k^2/p^2 \to 0} R_k(p) = 0 \Leftrightarrow \lim_{k \to 0} \mathcal{Z}_k[\mathcal{J}, v] = \mathcal{Z}[\mathcal{J}, v]. \tag{3.82}$$

- At $k \to \Lambda$, when no quantum fluctuations are integrated out, then, in order to reproduce the proper behavior of the EAA, cf. Eq.(3.77), the regulator function must diverge,

$$\lim_{k^2 \to \Lambda} R_k(p) \to \infty \Leftrightarrow \lim_{k^2 \to \Lambda} \Gamma[\Phi] = S[\Phi] + \text{const.}. \tag{3.83}$$

  Note that these requirements can be justified by exploiting the convex property of the EAA through a saddle point expansion of the corresponding action and regulator. At one-loop order and taking into account the scale-dependent quantum equations of motion in Eq.(3.81), any first functional derivative contribution will vanish and only the Hessians of the action and the regulators will survive in the functional integral. However, imposing the condition Eq.(3.83) will result in a Gaussian integral in terms of the field-independent Hessian of the regulator, i.e. $\left(\Delta S_k^{(2)}\right)_{\Phi\Phi} \sim R_k$. Performing the remaining Gaussian functional integral explicitly will result in a field-independent constant contribution which can be filtered out, thus reproducing the desired behavior of Eq.(3.77). Similar behavior will be exhibited if one considers the deep UV region, $k^2 \to \Lambda \to \infty$.

- At $0 < k < \Lambda$, then the fast modes should be unaffected by the regulator whereas the slow modes are regularized in a mass-like fashion, $m^2 \sim k^2$,

$$\lim_{p^2/k^2 \to 0} R_k(p) > 0. \tag{3.84}$$

The flow equation of the EAA with respect to the RG scale k is described by the *Wetterich equation*, cf. [231, 260–262],

$$\partial_t \Gamma_k = \frac{1}{2} \operatorname{tr} \left[(\partial_t R_k) \left(\Gamma_k^{(2)} + R_k \right)^{-1} \right], \tag{3.85}$$

where $t = \log k$, is called the *RG "time"*, $\left(\Gamma_k^{(2)}\right)_{\Phi_i \Phi_j} = \frac{\delta}{\delta \Phi_i^\dagger} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Phi_j}$ represents the Hessian of the EAA in terms of all dynamical macroscopic fields and tr accounts for all relevant field dependencies and contributes an additional minus in fermionic sectors.

Long-range physics should be independent of the choice of the regularization scheme and in extend from the choice of the regulator. This can be achieved by finding an exact solution for the associated flow equation, subject to an appropriate adjustment of the initialization of the action, which is a very involved task.

In practice, we employ certain approximations/truncations of the EAA which in turn can spoil this expected behavior. Therefore, it is important to select an optimized regulator that minimizes the effect of this breaking and yields the appropriate physical observables. Despite the aforementioned restrictions there is still a large class of possible regulators. For convenience, let us reparametrize the regulator function as

$$R(p^2) = p^2 r(p^2) \tag{3.86}$$

where $r(p^2)$ is called *regulator shape function*. In this work we shall exclusively choose the *Litim regulator* shape function for concrete computations, [263], which reads

$$r(p^2) = \left(\frac{k^2}{p^2} - 1\right) \theta(k^2 - p^2). \tag{3.87}$$

Due to the presence of the step function $\theta(k^2 - p^2)$, the behavior of the shape function is not smooth for varying RG scales. It should thus be used with care during the derivative expansion [264].

3.5 Towards BRST/gauge invariant fRG flows

In Sec. 3.3, we discussed the introduction of functional constraint equations or WIs as means of monitoring global and local symmetries on the level of the building blocks of non-Abelian field theories. Among the different symmetries involved, we highlighted the importance of BRST symmetry and its close relation to gauge invariance, cf. Sec. 2.4. On that note, we incorporated the information of BRST symmetry on the level of the correlation functions spanning across all loop orders, as a functional constraint equation called the master equation, cf. Subsec. 3.3.3, at the expense of introducing additional BRST sources in the generating functional, Eq.(3.59). Furthermore, we argued that such a master equation provides a valuable tool to extract information for various aspects of the theory, among which is the stability under renormalization. Thus, it is natural to examine the possible modification of certain WIs within the context of mass-dependent regularization schemes and more precisely the fRG, cf. Sec. 3.4.

At the setting of the fRG, the presence of the regulator in the EAA, Eq.(3.79), affects the underlying symmetry invariance of the theory and in extent results in a modification of the associated functional constraints, leading to the so called *modified Ward Identities (mWIs)*. Note that at the physical limit of $k \to 0$, the mWIs must agree with the WIs of the scale-independent theory, cf. [265–269].

Formulating the mWIs of the explicitly broken local gauge symmetry, i.e. the WTIs, in the fRG setup, one finds [199]

$$\mathcal{W}_k = \mathcal{G}^a_{\text{local}} \Gamma_k[\Phi; v] + \mathcal{G}^a_{\text{local}} \Delta S_k[\Phi] - \langle \mathcal{G}^a_{\text{local}} (S_{\text{gf}} + S_{\text{gh}} + \Delta S_k)[\Theta] \rangle = 0, \tag{3.88}$$

which correspond to the *modified Ward-Takahashi Identities (mWTIs)*. Comparing Eq.(3.57) to Eq.(3.88), we observe that the explicit breaking of local gauge invariance is enhanced by the presence of the regulator contributions.

Due to the close interconnectivity between BRST and gauge symmetry, cf. Subsec. 2.4.3, in order to gain insight into the gauge invariance of the theory one usually studies the implications of the WIs of the BRST symmetry, i.e. the master equation or STIs, cf. Subsec. 3.3.3. The elegance and practical accessibility of the master equation, Eq.(3.64), lies in the systematic way with which one can construct iteratively perturbative correlation functions by employing algebraic cohomological methods. Furthermore, its resolution does not require the computation of loop terms. In the context of the fRG, the corresponding mWIs of BRST symmetry read

$$\mathcal{S}_k = \langle \mathcal{G}_{\rm BRST} S_{\rm sou} \rangle + \mathcal{G}_{\rm BRST} \Delta S_k[\Phi] - \langle \mathcal{G}_{\rm BRST} \Delta S_k[\Phi] \rangle = 0, \tag{3.89}$$

called the *modified Slavnov-Taylor Identities (mSTIs)*. The mSTIs encode the gauge invariance of correlation functions generated by the EAA for fRG flows. Moreover, Eq.(3.89) reproduces the STIs, Eq.(3.58), at the physical limit $k \to 0$. Maintaining this limit is important to guarantee physical gauge invariance. Since $R_k \to 0$ eliminates the regulator-dependent contributions, we rediscover the standard STIs in the physical limit of $k \to 0$. Furthermore, in [261], the mSTIs were rephrased in terms of a regulator-dependent extension of the master equation Eq.(3.64), called the *modified master equation (mME)*. For further details see reviews [136, 199, 253, 254, 270].

Owing to the quadratic structure of the regulator Eq.(3.79), the regulator-dependent terms introduce one-loop contributions which make the resolution of the mSTIs more involved [271, 272]. Furthermore, due to the presence of the regulator, manifest BRST invariance is lost during the functional RG flow at finite scales k. However, the structural form of the mSTIs signifies the *soft breaking* of BRST invariance and as an extension of gauge invariance, since at high energies the operators associated to the regulator are suppressed, cf. [254] and references therein.

The flow of the mWTIs can be compactly written as [199]

$$\partial_t \mathcal{W}_k = -\frac{1}{2} G_k^{AB} \partial_t R_k^{AC} G_k^{CD} \frac{\delta}{\delta \Phi_D^\dagger} \mathcal{W}_k \frac{\overleftarrow{\delta}}{\delta \Phi_B}, \tag{3.90}$$

where $G_k^{AB} = \left(\frac{1}{\Gamma_k^{(2)} + R_k} \right)_{AB}$ corresponds to the AB component of the fully dressed propagator of the theory. Note that in order to avoid confusion with the notation of the collective fields and the RG scale, we denote the collective indices as $\{A, B, C, \ldots\}$.

Eq.(3.90) dictates how the mWTIs and in extension the functional constraints between correlation functions change at different RG scales. Most importantly, Eq.(3.90) entails that the resolution of the mWTIs, i.e. $\mathcal{W}_k = 0$, is a fixed point of the flow, i.e. $\partial_t \mathcal{W}_k = 0$. This ensures that if the mWTIs are respected at some scale k, at which a Γ_k is generated, then they are also met at a different scale k' with $\Gamma_{k'}$, [199].

Broadly speaking, the flow of the mWTIs or mSTIs monitors potential violations of gauge invariance in the case of mWTIs, or BRST symmetry (and in extension gauge symmetry) in the case of the mSTIs, in mass-dependent regularization schemes such as the fRG. They describe the limit where the symmetry is softly broken due to the inclusion of regulators and is reinstated at the physical limit $k \to 0$. However, such a description is unambiguously valid when one can derive the corresponding histories of generated EAAs that exactly solve the flow equation. In the case of gauge theories, the flow equation is solved approximately, which implies that the EAA is subject to the chosen truncation scheme. To that extent, the confidence level of the flow of the mWIs is directly related to the selected approximation scheme. Hence, potential violations of the underlying symmetries, as encoded by the flow of the mWIs, can arise from relevant or irrelevant RG operators. Thus for practical applications where truncation schemes are inevitable, the mWIs control the corresponding truncation by providing self-consistency and stability checks [271, 272].

Even though irrelevant symmetry violating RG operators do not affect the flow at the physical

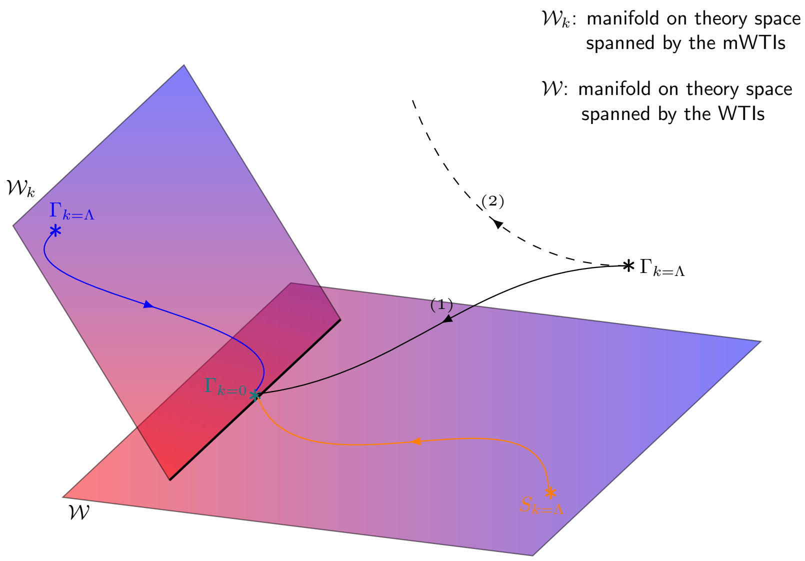

Figure 3.2: Graphical depiction of gauge invariant flows in theory space realized by the associated constraint equations and truncation schemes [199, 274]. In the above:
*: Generates a history of gauge invariant action functionals as determined by the WTIs.
*: Generates a history of gauge invariant EAAs as determined from the flow of the mWTIs and agrees with the WTIs in the physical limit of $k \to 0$ where the symmetry is softly broken due to the regulator insertion. However, it requires an EAA that solves the fRG equation exactly, which constitutes a strong constraint.
*: Belongs neither on the WTIs nor the mWTIs spanned manifolds and its flow depends on the chosen truncation scheme, thus potentially diverging the flow from the physical limit. Resolution of the truncated flow equation in accordance to the scale-dependent mWTIs dictates the class of admissible operators.
(1): Resolution of the flow equation within a truncation where the deviation from the $\mathcal{W}_k$ subspace is given only by RG-irrelevant operators, thus reproducing the correct physical limit.
(2): Resolution of the flow equation within a truncation where the deviation from the $\mathcal{W}_k$ subspace is given also by RG-relevant operators, which can result to a divergent trajectory and should be avoided.

limit, thus resulting in a resolution of the underlying functional constraint equations, violations induced by relevant RG operators as part of the truncation can potentially affect the behavior in the physical limit. Consequently, we require computational tools that will alleviate this ambiguity. A naive classification of the operators with power counting methods in the fRG setup falls short. As it turns out, resolution of the associated mWIs works hand in hand with that of the flow equation, such that seemingly relevant RG operators are suppressed at $k \to 0$, [87, 273].

In other words, the mWIs assist during the RG flow, by regulating the considered approximation according to the symmetries, in a controlled manner, realizing possible classes of admissible operators that become RG irrelevant in the physical limit. This procedure can be generalized to the mSTIs. A graphical depiction of the aforementioned procedure, for the case of the flow of the mWTIs can be found in Fig. 3.2.

Furthermore, consistency of a truncation in terms of the mWIs is linked to the IR behavior of correlation functions [87, 90, 93], which governs the physical properties of the system, cf. Chpt. 1. Therefore, the development of a procedure that applies the fRG flows coupled to the resolution of the underlying mWIs beyond perturbation theory, is a key ingredient in the understanding of gauge theories and there are several proposals for such a procedure for truncated fRG flows [79, 93, 271, 275].

A convenient option to devise non-perturbative truncations in gauge theories relies on the BFM with the construction of background gauge invariant flows. Such an approach, not only preserves gauge invariance encoded in mWTIs/mSTIs, but also includes constraint equations which monitor the gauge/background independence of correlation functions. These constraint equations are known as *modified Nielsen Identities (mNIs)*, which, in the absence of regulators, reduce to the *Nielsen Identities (NIs)* and on-shell incorporate the background independence of proper vertices, $\frac{\delta\Gamma}{\delta\bar{A}} = 0$, [136, 254]. In practice, they have been treated on an approximate level [132–135, 143, 145, 146].

In another approach, gauge-invariant functional flows that are stable under renormalization were constructed, at the expense of introducing composite operators that lead to regulator-dependent BRST transformations for the renormalized fields. Such a construction infers a modified version of the associated symmetry constraint equations [270, 276–278]. Applications of the compatibility of the derived constraint equations and their functional flows have been studied in the context of QED [279, 280] and perturbative YM theory [281].

Alternatively, the authors in [282, 283] constructed manifestly gauge-invariant exact RG flows, without the use of gauge fixing, for $SU(N)$ pure YM theory. This was achieved by embedding the theory into a supergauge group using a higher derivatives regularization scheme and a manifest spontaneous breaking of supergauge invariance $SU(N|N) \to SU(N) \times SU(N)$. Following this gauge-invariant approach, several loop computations for various theoretic frameworks were carried out, including that of the one-loop and two-loop YM beta functions [253, 284–289].

Due to the conceptual and computational accessibility of the gauge-fixing procedure, it remains advantageous to explore the behavior of non-Abelian field theories at different energy regimes by employing the FP quantization. Built on this reasoning, the authors of [121] derived BRST-invariant flows through an fRG equation, compatible with the extended STIs in a suitable truncation scheme.

A novel feature of this method is that the BRST-breaking regulators are incorporated as part of the regularization procedure by imposing a non-trivial gauge-fixing condition. As discussed in [121] & Sec. 2.3, such a modification comes at the expense of extending the color space with the inclusion of a NL-type v field which is expected to appear in the generated loop graphs. Furthermore, the truncation scheme, due to the inclusion of extra BRST sources, allows for additional regulator-dependent vertex interactions. Therefore, it is of interest to examine whether one can reproduce the universal one-loop beta function from the associated flow equation, considering the additional degrees of freedom which come from the v field. A preliminary study of the gluon anomalous dimension compared quite favorably with existing literature results for an appropriate choice of the weight of the v field, [121].

The computation of the one-loop beta function in a manifestly BRST-invariant manner is the subject of study in Chpt. 6.

CHAPTER 4

Background and BRST-Invariant One-Loop EA for Non-Linear Gauge Fixing

This chapter focuses on the perturbative analysis of our model within the BFM. The goal is to construct a modified background invariant non-linear gauge-fixing condition with a decoupled mass-regulator sector. The Fourier weighted gauge-fixing procedure used in Subsec. 2.3.2 introduces a v-field dependence in the bare action, which affects the classical equations of motion. This, in turn, constrains the admissible forms of the v field. However, the gauge-fixing condition introduces nonlocal terms, which may cause additional divergences. To address this, we study the perturbative stability of our model obtaining explicit expressions for the one-loop perturbative EA and Schwinger functional in the BFM. The focus will be on the presence of the v field in the structure of these quantities and the impact of the regulator parameters. This analysis reveals a v-independent EA and a v-dependent Schwinger functional, which will be further studied in Chpt. 5. Next, from the renormalization of the one-loop EA we determine the associated beta function under various regularization schemes. Finally, a phenomenological study of the potential-like quantity in terms of the action density provides insight into the role of the regulator parameters that govern the IR dynamics of the model.

4.1 Background-invariant non-linear gauge-fixing condition

In order to proceed with further calculations in our framework, let us turn our attention to determining a concrete form of the gauge-fixing condition, in accordance with the desired symmetries. For that, we impose the BFM. As mentioned in Chpt. 1 and introduced in Sec. 3.2, the BFM constitutes a valuable tool that can greatly facilitate (non-)perturbative calculations due to manifest background gauge invariance, which can be associated to a gauge invariant EA, cf. Subsec. 3.3.4. Furthermore, in Subsec. 3.3.4, we deduced that manifest BRST and background gauge invariance of the action restricts the class of gauge-fixing conditions to those that change in a tensorial manner which, in turn, dictates the admissible construction terms, cf. Eq.(3.71). Building on this principle, one can generically choose a set of Lorenz-like linear covariant gauges [123, 201]. However, we deviate from this conventional choice. Following [121], we choose a non-linear background gauge-fixing condition of the form with a decoupled modified mass CFDJ sector

$$\mathcal{F}^a[a, \bar{A}, v] = a^b_\mu \mathrm{Q}^{abc}_{\mu\nu} a^c_\nu + \mathfrak{L}^{ab}_\mu a^b_\mu, \tag{4.1}$$

where

$$\begin{aligned} \mathrm{Q}^{abc}_{\mu\nu} &= \frac{v^a}{2|v|^2}\left[\bar{m}^2 \delta^{bc}_{\mu\nu} - \frac{1}{\xi}\left(\bar{D}_\mu \bar{D}_\nu\right)^{bc}\right], \\ \mathfrak{L}^{ab}_\mu &= \left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)\bar{D}^{ab}_\mu . \end{aligned} \tag{4.2}$$

Here, we have used the condensed notation,

$$\bar{D}^{ab}_\mu - \bar{m}^2_{\text{gh}}\left(\frac{1}{\bar{D}^2}\bar{D}_\mu\right)^{ab} = \left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)\bar{D}^{ab}_\mu .$$

The choice of the non-linear gauge-fixing condition, may appear arbitrary at first sight but it becomes clear when writing down the resulting gauge-fixed action,

$$\begin{aligned} S_{\text{gf}}[a, \bar{A}, v] &= S_{\text{Lorenz}}[a, \bar{A}] + S_{\text{mCFDJ}}[a, \bar{A}, v] \\ &= -\frac{1}{2\xi} a^a_\mu \left(\bar{D}_\mu \bar{D}_\nu\right)^{ab} a^b_\nu + \frac{\bar{m}^2}{2} a^a_\mu a^a_\mu + v^a\left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)\bar{D}^{ab}_\mu a^b_\mu . \end{aligned} \tag{4.3}$$

Note that the quadratic contribution reproduces the conventional background gauge term S_{Lorenz}, whereas the combination of the linear and quadratic parts gives rise to the mass-regulator parts for the dynamical fields. In particular, S_{mCFDJ} corresponds to a modified version of the CFDJ gauge [106, 117, 122], adjusted for the Fourier weight distribution that was chosen during the FP procedure. Here, let us highlight once more that, even though the inclusion of a ghost mass regulator seems to be at odds with associated considerations, such an IR contributing effect will be responsible for potential regularization of emerging divergences in mass-dependent renormalization schemes such as the fRG. It will also provide the means of simulating on a perturbative level the effects of the well-established, on a non-perturbative level, decoupling solution, cf. Sec. 4.5. In such a way, it creates a bridge between perturbative and non-perturbative treatments, while still maintaining the FP procedure and respecting the desired symmetries. Such a modifications of the gauge-fixing condition comes at the cost of the inclusion of the NL-type v field and a nonlocal action. Both points require further investigation, which will be carried out in the following.

For the ghost sector, the FP determinant can be cast into a nonlocal ghost action, according to Eq.(3.6), which reads

$$\begin{aligned} S_{\text{gh}}[a, \bar{A}, c, \bar{c}, v] = \bar{c}^a \Bigg\{ \frac{v^a}{2|v|^2}\left[2\bar{m}^2 a^b_\mu - \frac{1}{\xi}\left(\bar{D}_\mu \bar{D}_\nu\right)^{bc} a^c_\nu - \frac{1}{\xi} a^c_\nu \left(\bar{D}_\nu \bar{D}_\mu\right)^{cb}\right] \\ + \left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)\bar{D}^{ab}_\mu \Bigg\} \left(D_\mu c\right)^b . \end{aligned} \tag{4.4}$$

It is a straightforward task to prove the invariance of the generated action under background gauge transformations, Eqs.(3.40) & (3.41). In particular, noting that the background covariant derivatives change homogeneously under background gauge transformations, Eq.(4.1) implies that the gauge-fixing condition changes homogeneously as well. Thus, as discussed in Subsec. 3.3.4, this is a sufficient condition to ensure invariance of the generated ghost and gauge-fixing actions under the extended background gauge transformations. Furthermore, the fact that the modified contributions were introduced through the gauge-fixing sector entails that the action can still be organized in the form of Eq.(3.66) which implies manifest BRST invariance.

4.2 Equations of motion in the background formalism

As the v field enters the action, it is expected to affect the associated background field equations of motion. Firstly, using Eq.(3.24), one writes the YM action, Eq.(3.5), in the background field formalism as

$$\begin{aligned} S_{\text{YM}}[a, \bar{A}] =& \frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}^a_{\mu\nu} + \bar{F}^a_{\mu\nu}\left(\bar{D}_\mu a_\nu\right)^a - \frac{1}{2}\left(\bar{D}_\mu a_\nu\right)^a\left(\bar{D}_\nu a_\mu\right)^a \\ &+ \frac{1}{2}\left(\bar{D}_\mu a_\nu\right)^a\left(\bar{D}_\mu a_\nu\right)^a + \frac{\bar{g}}{2} f^{abc}\bar{F}^a_{\mu\nu}a^b_\mu a^c_\nu + \mathcal{O}(a^3). \end{aligned} \tag{4.5}$$

We introduce the following shorthand notation for the vector boson's action, i.e. the total action at vanishing ghost fields,

$$S_{\text{v}}[a, \bar{A}, v] = S_{\text{YM}}[a, \bar{A}] + S_{\text{gf}}[a, \bar{A}, v]. \tag{4.6}$$

Inserting, Eqs.(4.4) & (4.5) in Eq.(4.6), one obtains the relation

$$\begin{aligned} S_{\text{v}}[a, \bar{A}, v] =& \frac{1}{4}\bar{F}^a_{\mu\nu} + \left[\bar{F}^a_{\mu\nu} + v^a\left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)\right]\left(\bar{D}_\mu a_\mu\right)^a \\ &+ \frac{1}{2}a^a_\mu\left\{\left[-\left(\bar{D}^2\right) + \bar{m}^2\delta^{ab}\right]\delta_{\mu\nu} + \left(1 + \frac{1}{\xi}\right)\left(\bar{D}_\mu\bar{D}_\nu\right)^{ab}\right\}a^b_\nu \\ &+ \bar{g}f^{abc}a^b_\mu a^c_\nu\bar{F}^a_{\mu\nu} + \mathcal{O}(a^3). \end{aligned} \tag{4.7}$$

For the classical equations of motion in the BFM, we assume a vanishing classical configuration of the FP ghosts, which eliminates the S_{gh} contribution from the equations of motion and set the quantum field fluctuations to zero. Thus, the classical equations of motion are obtained from the following relation

$$\left.\frac{\delta S_{\text{v}}[a, \bar{A}, v]}{\delta a^a_\nu}\right|_{a\to 0} = 0. \tag{4.8}$$

Taking Eq.(4.7) into account, the classical equations of motion in the BFM for the non-linear gauge-fixing condition Eq.(4.1) are written as

$$\bar{D}^{ab}_\mu F^b_{\mu\nu} = -J^a_\nu[\bar{A}, v], \tag{4.9}$$

where the information associated with the non-conventional gauge-fixing procedure is encoded in an external current

$$J^a_\mu[\bar{A}, v] = \bar{D}^{ab}_\mu\left(1 + \frac{\bar{m}^2_{\text{gh}}}{-\bar{D}^2}\right)v^b. \tag{4.10}$$

In other words, such a term expresses the deviations from the classical equations of motion, i.e. $DF = 0$, as a result of the novel linear consideration in the gauge-fixing condition. However, background covariant current conservation places a constraint on the admissible class of v-field configurations, as they must satisfy a massive Klein-Gordon equation governed by the ghost regulator mass, i.e.

$$\bar{D}_{\mu}^{ab} J_{\mu}^{b} = 0 \Leftrightarrow \left(\bar{D}^2 - \bar{m}_{\text{gh}}^2\right) v^a = 0. \tag{4.11}$$

Thus, in order for the background equations of motion Eq.(4.9) to be consistent, the v field is restricted within a class of solutions dictated by Eq.(4.11). Such a realization will be exploited during the forthcoming Chpt. 5 and corresponds to the background field generalization of the massive Klein-Gordon equation found in [121].

4.3 One-loop Schwinger functional and effective action

The goal of this section is to investigate the explicit form the one-loop EA and Schwinger functional. The study of such quantities is of vital importance since they provide a linkage between the non-linear character of the gauge-fixing condition, the mass-regulator parameters and the correlation functions generated by the modified generating functionals. Such a modification can potentially affect the stability of the theory under renormalization with the emergence of additional divergences. In the following, we will drop the bar of the background field, i.e $\bar{A} \to A$ and denote the quantum fluctuations as a' in order to distinguish them from the classical fields, for clarity's sake.

In order to carry out a one-loop perturbative study of our model, we expand the corresponding action Eq.(4.7) to second order around small quantum field fluctuations,

$$S_{\text{v}}[a', A, v] \simeq S_{\text{v}}[A, v] + \frac{\delta S_{\text{v}}[A, v]}{\delta {a'_\mu}^a} {a'_\mu}^a + \frac{1}{2} {a'_\mu}^a M_{\mu\nu}^{ab}[A, v] {a'_\nu}^b. \tag{4.12}$$

Inserting the one-loop expansion in the background Schwinger functional, Eqs.(3.28) & (3.29), one obtains the integral equation

$$\mathrm{e}^{W_{1\text{L}}[j,A;v]} = \mathrm{e}^{-S_{\text{v}}[A,v]}\, \Delta_{\text{FP}}[A, v] \int \mathcal{D}a'\, \mathrm{e}^{-\frac{1}{2}{a'_\mu}^a M_{\mu\nu}^{ab} {a'_\nu}^b + \frac{\delta S_{\text{v}}}{\delta {a'_\mu}^a} {a'_\mu}^a}. \tag{4.13}$$

Completing the square in the exponent of the integral equation reproduces a Gaussian integral which upon computation leads to the Schwinger functional

$$-W_{1\text{L}}[j, A; v] = S_{\text{v}}[A; v] - \ln \Delta_{\text{FP}}[A] + \frac{1}{2} \ln \det M[A] - W_{\text{source}}[j, A; v], \tag{4.14}$$

where

$$S_{\text{v}}[A] = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \tag{4.15}$$

$$\Delta_{\text{FP}}[A] = \det\left[\left(1 + \frac{\bar{m}_{\text{gh}}^2}{-D^2}\right)\left(D^2\right)^{ab}\right], \tag{4.16}$$

$$M_{\mu\nu}^{ab}[A; \xi] = \bar{m}^2 \delta_{\mu\nu}^{ab} + 2\bar{g} f^{abc} F_{\mu\nu}^c - \left(D^2\right)^{ab} \delta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \left(D_\mu D_\nu\right)^{ab}, \tag{4.17}$$

$$W_{\text{source}}[j, A; v] = \frac{1}{2} \left(\mathcal{K}_\mu^a[A, v] + j_\mu^a\right) \left(M^{-1}\right)_{\mu\nu}^{ab} [A] \left(\mathcal{K}_\nu^b[A, v] + j_\nu^b\right). \tag{4.18}$$

Here, we denote the Hessian of the gluon action as $M = \frac{\delta^2 S_{\text{v}}}{\delta a' \delta a'}$, which in turn represents the inverse gluon propagator in the background field formalism. We have also introduced

$$\mathcal{K}^a_\mu[A, v] = -\frac{\delta S_{\text{v}}[A, v]}{\delta {a'_\mu}^a} = (D_\nu F_{\nu\mu})^a + J^a_\mu[A, v] \tag{4.19}$$

The effect of the choice of the non-linear gauge-fixing condition can be observed both in the form of the FP determinant, Eq.(4.16), as well as that of the gluon fluctuation operator, (4.17). In particular, both quantities are shifted by a corresponding mass regulator as compared to their respected form in the linear gauge, cf. [202, 290, 291]. The result for the Schwinger functional however is also interesting due to the appearance of the novel W_{source} contribution which is a direct consequence of the particular non-linear choice of the gauge-fixing condition, Eq.(4.1) and carries all the v-field dependence. It explicitly enters from the linear part $\frac{\delta S_{\text{v}}}{\delta {a'_\mu}^a}$ in the expansion of the bare action. In addition, during the computation of the integral equation, we consider that the gluonic fluctuation operator M is invertible, with the computation of the form of the inverse to be of main interest in the upcoming sections.

Next, let us determine the form of the one-loop EA. Taking the exponential of Eq.(3.30), one finds

$$\mathrm{e}^{-\Gamma[a,A;v]} = \int \mathcal{D}a' \, \mathrm{e}^{-S_{\text{v}}[a',A,v] + \frac{\delta\Gamma[a,A;v]}{\delta a^a_\mu} {a'}^a_\mu - \frac{\delta\Gamma[a,A;v]}{\delta a^a_\mu} a^a_\mu} \, \Delta_{\text{FP}}[a', A, v]. \tag{4.20}$$

By performing the shift $a' \to a' + a$, Eq.(4.20) becomes

$$\mathrm{e}^{-\Gamma[a,A;v]} = \int \mathcal{D}a' \, \mathrm{e}^{-S_{\text{v}}[a'+a,A,v] + \frac{\delta\Gamma[a,A;v]}{\delta a^a_\mu} {a'}^a_\mu} \Delta_{\text{FP}}[a' + a, A, v]. \tag{4.21}$$

Next, we perform an expansion to Gaussian order, similar to Eq.(4.12). Then, the constituents of Eq.(4.21) take the form

$$\begin{aligned} S_{\text{v}}[a' + a, A, c] &= S_{\text{v}}[a, A, v] + \frac{\delta S_{\text{v}}[a, A, v]}{\delta {a'}^a_\mu} {a'}^a_\mu + \frac{1}{2} {a'}^a_\mu \frac{\delta^2 S_{\text{v}}[a, A]}{\delta {a'}^a_\mu \delta {a'}^b_\nu} {a'}^b_\nu + \mathcal{O}({a'}^3), \\ \Delta_{\text{FP}}[a' + a, A, v] &= \Delta_{\text{FP}}[a, A, v] + \mathcal{O}(a'). \end{aligned}$$

This expansion corresponds to one-loop order. The EA in the background field formalism is related to the full EA according to Eq.(3.33). Then, we obtain the integral equation for the full one-loop EA in the limit $a \to 0$

$$\mathrm{e}^{-\Gamma_{1\text{L}}[0,A;v]} = e^{-S_{\text{v}}[A,v]} \, \Delta_{\text{FP}}[A, v] \int \mathcal{D}a' \, \mathrm{e}^{\left(\mathcal{K}^a_\mu[A,v] + \frac{\delta\Gamma[0,A;v]}{\delta a^a_\mu}\right) {a'_\mu}^a} \, \mathrm{e}^{-\frac{1}{2} {a'_\mu}^a M^{ab}_{\mu\nu}[A;\xi] {a'_\nu}^b} \tag{4.22}$$

The term $\frac{\delta\Gamma[0,A;v]}{\delta a^a_\mu}$ requires a more detailed study. In general, one can decompose the EA into the bare action plus some loop corrections as follows

$$\Gamma[a, A; v] = S_{\text{v}}[a, A; v] + \Delta\Gamma_{\text{v}}[a, A; v]. \tag{4.23}$$

Taking the functional derivative in the limit where $a \to 0$ results in

$$\frac{\delta\Gamma[A;v]}{\delta a^a_\mu} = -\mathcal{K}^a_\mu[A;v] + \frac{\delta\Delta\Gamma_{\text{V}}[A;v]}{\delta a^a_\mu}. \tag{4.24}$$

Inserting Eq.(4.24) into Eq.(4.22), we get

$$\mathrm{e}^{-\Gamma_{1\text{L}}[A]} = \mathrm{e}^{-S_{\text{V}}[A]}\ \Delta_{\text{FP}}[A] \int \mathcal{D}a'\ \mathrm{e}^{-\frac{1}{2}{a'_\mu}^a M^{ab}_{\mu\nu} {a'_\nu}^b + \frac{\delta\Delta\Gamma_{\text{V}}[A]}{\delta a^a_\mu} {a'_\mu}^a}. \tag{4.25}$$

The linear part which is proportional to the external current gets canceled by the first functional derivative of the EA. Performing the Gaussian integral by completing the square results in the following form of the one-loop EA

$$\Gamma_{1\text{L}}[A] = S_{\text{V}}[A] - \ln\Delta_{\text{FP}}[A] + \frac{1}{2}\ln\det M[A] + \frac{1}{2}\frac{\delta\Delta\Gamma_{\text{V}}[A]}{\delta a^a_\mu}\left(M^{-1}\right)^{ab}_{\mu\nu}\frac{\delta\Delta\Gamma_{\text{V}}[A]}{\delta a^b_\nu}. \tag{4.26}$$

The effect of Eq.(4.24) appears as the last term in the EA. However, such a term is a higher-loop contribution and thus it does not contribute to one-loop order.

Thus, the one-loop EA for the non-linear gauge-fixing condition takes the form

$$\Gamma_{1\text{L}}[A] = S_{\text{V}}[A] - \ln\Delta_{\text{FP}}[A] + \frac{1}{2}\ln\det M[A]. \tag{4.27}$$

One can notice that the quantum corrections which arise on the right side of Eq.(4.27) correspond to the expected ones from the ghost loop and gluon loop. The kinetic operators appearing in these loops are, for the present gauge fixing choice, the usual background covariant Lorentz-DeWitt operators, augmented by two standard mass terms featuring regulator-mass parameters $\bar{m}^2$ and $\bar{m}^2_{\text{gh}}$ for the gluons and ghosts respectively. In order to further evaluate these quantities, conventional heat-kernel techniques can be employed. This is explored in the Sec. 4.4. Notice that, as seen from Eqs.(4.16) & (4.17), the fact that the mass matrices are proportional to the identity will simplify our computation. Furthermore, even though the one-loop EA came out to be v-field independent, we found that the one-loop Schwinger functional indeed exhibits a v-field dependence, Eq.(4.14), due to the presence of W_{source} which is closely related to the background equations of motion. From a structural perspective, the EA and the Schwinger functional represent generating functionals of the 1PI and 1PR correlators. Thus, the W_{source} contribution has a structure of a 1PR correlator with an internal gluon propagator. In the background field formalism, the difference between the Schwinger functional and the EA on the level of the 1PR correlators can be constructed also at higher-loop orders [292–296].

4.4 Stability under renormalization to one-loop order

Despite using a non-linear gauge-fixing condition, we have derived a one-loop EA that takes on a standard form, comprising the bare action, the ghost and gluon sectors. Importantly, this process ensures manifest background gauge and BRST invariance of the theory. However, this has come at a cost of introducing an auxiliary v field and modifying the form of the quantities involved in the EA, namely the FP determinant and the gluon fluctuation operator. While the dependence on the auxiliary v field drops out at one-loop order, the form of these quantities is modified by a shift of the corresponding mass regulators, as shown in Eqs.(4.16) & (4.17) respectively. Consequently, a natural question arises as to whether such novel contributions have an effect on the stability of the renormalization procedure.

To address this question, we employ perturbative renormalization, within various renormalization schemes, outlined in Subsec. 3.4.1, such that we determine the RG flow of the coupling constant encoded within the one-loop beta function Eq.(3.76). This incorporates information about the change of our model across different scales. To facilitate comparison with existing literature, we will focus on covariantly constant background fields in this section, which by definition obey

$$(D_\mu F_{\nu\rho})^a = 0. \tag{4.28}$$

Consequently, under this assumption and by choosing the field strength tensor to be solely dependent on the constant magnetic field, cf. Eq.(A.4), the bare action can be written as

$$S_{\mathrm{v}}[A] = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} = \frac{1}{2} B^2. \tag{4.29}$$

To accommodate further manipulations, we briefly introduce the eigenvalues ν_ℓ of the color matrix $\hat{n}^a \left(\tau^a_{\mathrm{G}}\right)^{bc}$, with $\ell = 1, \ldots, N_c^2 - 1$ and $\bar{B}_\ell = \bar{g} B \nu_\ell$. For further clarification and a discussion on the implications of covariantly constant backgrounds see App. A and references therein.

Finally, let us rewrite the one-loop EA of Eq.(4.27) as

$$\Gamma_{1\mathrm{L}}[A] = S_{\mathrm{v}}[A] + \Delta\Gamma_{\mathrm{gh}}[A] + \Delta\Gamma_{\mathrm{gf}}[A]. \tag{4.30}$$

4.4.1 Study of the ghost sector

Let us begin by examining the ghost sector. Write Eq.(4.16) as,

$$\log \Delta_{\mathrm{FP}} = \log \det \left(D^2 - \bar{m}^2_{\mathrm{gh}} \right) = \mathrm{tr}_{\mathrm{xcL}} \left[\log \left(D^2 - \bar{m}^2_{\mathrm{gh}} \right) \right], \tag{4.31}$$

where the relation $\log(\det A) = \mathrm{tr}(\log A)$ for a matrix A was used. To simplify our notation, we have introduced the functional trace, which depending on its corresponding indices can denote, integration over spacetime points (x), summation over all color (c) or Lorentz (L) indices.

Multiplying both sides of Eq.(4.31) with $\log(\det(-1))$, results in

$$\log(-\Delta_{\mathrm{FP}}) = \mathrm{tr}_{\mathrm{xcL}} \left[\log \left(-D^2 + \bar{m}^2_{\mathrm{gh}} \right) \right]. \tag{4.32}$$

Subtracting the divergent vacuum contribution of the functional supertrace as an overall constant leads to

$$\Delta\Gamma_{\mathrm{gh}}[A] = -\,\mathrm{tr}_{\mathrm{xcL}} \left[\log \left(\frac{-D^2 + \bar{m}^2_{\mathrm{gh}}}{-\partial^2 + \bar{m}^2_{\mathrm{gh}}} \right) \right]. \tag{4.33}$$

One can further simplify the functional trace by introducing the proper-time representation of the logarithm of a fraction, see Eq.(B.26) and [297, 298]. Then, Eq.(4.33) becomes

$$\Delta\Gamma_{\text{gh}}[A] = \int_0^\infty \frac{\mathrm{d}s}{s}\,\mathrm{tr}_{\text{x}}\left[\mathrm{e}^{-s\left(-\partial^2+\bar{m}_{\text{gh}}^2\right)}\right] - \int_0^\infty \frac{\mathrm{d}s}{s}\,\mathrm{tr}_{\text{xcL}}\left[\mathrm{e}^{-s\left(-D^2+\bar{m}_{\text{gh}}^2\right)}\right]. \tag{4.34}$$

Employing conventional heat-kernel techniques, discussed in App. B, by using Eq.(B.15) for each functional trace and then substituting Eqs.(B.19) & (B.17) for the first and second functional traces respectively, one finds in d-dimensions

$$\Delta\Gamma_{\text{gh}} = -\frac{\Omega_d}{(4\pi)^{d/2}}\int_0^\infty \frac{\mathrm{d}s}{s^{1+d/2}}\,\mathrm{e}^{-\bar{m}_{\text{gh}}^2 s}\sum_{\ell=1}^{N^2-1}\left[1-\frac{sB_\ell}{\sinh(sB_\ell)}\right], \tag{4.35}$$

where Ω_d is the d-dimensional spacetime volume. The integral is divergent at the lower bound of the proper-time parameter, thus resulting in the emergence of a UV divergence.

4.4.2 Study of the gluon sector

We continue with the contribution of the gluon sector. First, we rewrite the gluon fluctuation operator, Eq.(4.17), in terms of the transversal kinetic operator. For convenience, we perform the computation in the Feynman gauge $\xi = 1$, which eliminates the dependence on the longitudinal kinetic operator and reduces its form to

$$M_{\mu\nu}^{ab}[A;\xi=1] = \bar{m}^2\delta_{\mu\nu}^{ab} + (\mathfrak{D}_{\text{T}})_{\mu\nu}^{ab}, \tag{4.36}$$

where $(\mathfrak{D}_{\text{T}})_{\mu\nu}^{ab} = -\delta_{\mu\nu}\left(D^2\right)^{ab} + 2igF_{\mu\nu}^{ab}$ denotes the spin-1 Laplacian. Subtracting the divergent vacuum contribution from the gluon loop, we obtain

$$\Delta\Gamma_{\text{gl}}[A] = \frac{1}{2}\,\mathrm{tr}\left[\ln\left(\frac{\bar{m}^2+\mathfrak{D}_{\text{T}}[A]}{\bar{m}^2+\mathfrak{D}_{\text{T}}[0]}\right)\right]. \tag{4.37}$$

Following similar steps as in the preceding case, while tracing over the spectrum of the transversal kinetic operator, Eq.(B.18), one finds in d dimensions:

$$\Delta\Gamma_{\text{gl}} = \frac{\Omega_d\, d}{2(4\pi)^{d/2}}\int_0^\infty \frac{\mathrm{d}s}{s^{1+d/2}}\,\mathrm{e}^{-\bar{m}^2 s}\sum_{\ell=1}^{N^2-1}\left[1-\frac{sB_\ell}{\sinh(sB_\ell)}-\frac{4sB_\ell}{d}\sinh(sB_\ell)\right]. \tag{4.38}$$

Similarly to the ghost sector, the integral expression exhibits a UV divergence.

4.4.3 One-loop running coupling

Substituting Eqs.(4.29), (4.35) & (4.38) into Eq.(4.30),

$$\begin{aligned}\Gamma_{1\text{L}} = \frac{1}{2}\Omega_d B^2 + \frac{\Omega_d}{(4\pi)^{d/2}}\sum_{\ell=1}^{N^2-1}\int_0^\infty \frac{\mathrm{d}s}{s^{1+d/2}}\Bigg[\,&\mathrm{e}^{-\bar{m}_{\text{gh}}^2 s}\left(1-\frac{sB_\ell}{\sinh(sB_\ell)}\right)\\ &-\frac{d}{2}\,\mathrm{e}^{-\bar{m}^2 s}\left(1-\frac{sB_\ell}{\sinh(sB_\ell)}-\frac{4sB_\ell}{d}\sinh(sB_\ell)\right)\Bigg].\end{aligned} \tag{4.39}$$

In order to address the UV divergences which arise in the ghost and gluon sectors, we reparametrize our field and coupling parameters by introducing a renormalized coupling g_{R} and renormalized field parameters. Note that B_R can be associated to the gauge field under the assumption of covariantly constant backgrounds Eq.(A.4). Manifest background gauge invariance implies that the wave-function renormalization of the background field and the renormalization constant of the coupling, denoted by Z_F and $Z_{\bar{g}}$, are related with each other as, cf. App. C and [123, 201]

$$Z_{\bar{g}} = \sqrt{Z_\mathrm{F}}. \tag{4.40}$$

Therefore, the bare and renormalized quantities in our framework are associated with each other

$$g_\mathrm{R} = \sqrt{Z_\mathrm{F}}\bar{g}, \quad B_\mathrm{R} = \frac{B}{\sqrt{Z_\mathrm{F}}}. \tag{4.41}$$

In terms of the reparametrized quantities, we can rewrite the one-loop EA as

$$\begin{aligned}\Gamma_{1\mathrm{L}} &= \frac{\Omega_d}{2}B_\mathrm{R}^2 + \Delta\Gamma_\mathrm{gl} + \Delta\Gamma_\mathrm{gh} - \frac{\Omega_d}{2}(1 - Z_\mathrm{F})B_\mathrm{R}^2 \\ &= \frac{\Omega_d}{2}B_\mathrm{R}^2 + \Delta\Gamma_\mathrm{gl,R} + \Delta\Gamma_\mathrm{gh,R}.\end{aligned} \tag{4.42}$$

We have chosen Z_F so that the UV log-divergences from the bare loop contributions $\Delta\Gamma_\mathrm{gl/gh}$ are canceled and render finite renormalized quantities $\Delta\Gamma_\mathrm{gl/gh,\,R}$. It is worth noting that in $d = 4$, only terms quadratic to the renormalized constant field will diverge at the lower bound of s due to the form of the integrals in Eq.(4.39). Consequently, by performing a weak field expansion $B_\mathrm{R} \to 0$, we can identify the divergent contributions in the ghost and gluon sectors.[1] As it turns out,

$$Z_\mathrm{F} = 1 + \frac{N_c\, g_\mathrm{R}^2}{6(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}s}{s^{-1+d/2}} \left[(24 - d)\, \mathrm{e}^{-\bar{m}^2 s} + 2\mathrm{e}^{-\bar{m}_\mathrm{gh}^2 s}\right], \tag{4.43}$$

where we have used Eq.(A.5).

After isolating the divergent contributions by means of a redefinition of the associated terms, the next step is to regularize them. This procedure serves a two-fold purpose for studying the stability of the theory under renormalization. Firstly, it provides an algorithmic asset for extracting an analytic expression of divergent contributions which can be further manipulated during renormalization. Regularization introduces a notion of a mass scale which is then used to study the underlying theory at different energy regimes. This step enables us to derive the one-loop beta function, which characterizes the running of the coupling constant at one-loop level. Although different regularization schemes may not be expected to yield the same one-loop beta function, the one-loop beta function of Yang-Mills in $d = 4$ is known to be universal and hence renormalization scheme independent under certain assumptions [124]. Therefore, it is instructive to compute the one-loop beta function by choosing different regularization schemes and compare the results. In our study, we will compute the one-loop beta function using both dimensional regularization with the $\overline{\mathrm{MS}}$ scheme and also employ a sharp UV proper-time cutoff. In the following we shall concern ourselves not only with different regularization schemes but also with different renormalization scales.

[1] The form of divergence is obtained in but not restricted to the weak field expansion $B_\mathrm{R} \to 0$ but it is valid for arbitrary magnetic fields.

4.4.3.1 Dimensional regularization and $\overline{\text{MS}}$

In this case, we express the divergent integral Eq.(4.43) in terms of a dimensionless coupling by introducing an arbitrary mass scale μ and subsequently analytically continue to $d = 4$. This will allow us to regularize the divergent quantity through dimensional manipulations. For the first step, we define the dimensionless renormalized coupling as

$$g^2 = \mu^{4-d} g_{\text{R}}^2. \tag{4.44}$$

Choosing $d = 4 - 2\epsilon$, then

$$Z_{\text{F}} = 1 + \frac{N_c\, g^2 \mu^{2\epsilon}}{3\,(4\pi)^{2-\epsilon}} \int_0^\infty \frac{ds}{s^{1-\epsilon}} \Big[(10+\epsilon)\, e^{-\bar{m}^2 s} + e^{-\bar{m}_{\text{gh}}^2 s}\Big]. \tag{4.45}$$

Expanding around $\epsilon \to 0$ within the $\overline{\text{MS}}$ scheme, while considering Eq.(B.29) results in

$$Z_{\text{F}} = 1 - \frac{22 N g^2}{6\,(4\pi)^2} \left[-\frac{1}{\epsilon} + \frac{11}{10} \ln \frac{\bar{m}^2}{\mu^2} + \frac{1}{11} \ln \frac{\bar{m}_{\text{gh}}^2}{\mu^2} - \ln 4\pi + \gamma_{\text{E}} - \frac{1}{11} \right], \tag{4.46}$$

where γ_{E} corresponds to the Euler–Mascheroni constant. It is worth mentioning that only logarithmic divergences arose from the gluon and ghost sector as can be seen from Eq.(4.46) (due to the $\frac{1}{\epsilon}$ form).

The divergent part which appears in Eq.(4.46) is of the form discussed in App. C, cf. Eq.(C.6). Therefore, the one-loop beta function, taking into account Eq.(C.8) reads

$$\beta_{g^2} = \mu \frac{\mathrm{d}g^2}{\mathrm{d}\mu} = -\frac{22 N_c}{3\,(4\pi)^2} g^4, \tag{4.47}$$

which corresponds to the universal one-loop beta function for Yang-Mills theory in $d = 4$ [124].

4.4.3.2 Sharp UV proper-time cutoff

Next, let us derive the one-loop beta function by regularizing the divergent integral in Eq.(4.43). For this, we impose a sharp proper-time cutoff at the (divergent) lower bound of the integral (UV region). Then in $d = 4$, $g_{\text{R}} = g$ and the computation of the divergent integral is possible. The wave-function renormalization becomes

$$Z_{\text{F}} = 1 + \frac{N_c g^2}{6\,(4\pi)^2} \int_{\frac{1}{\Lambda^2}}^\infty \frac{\mathrm{d}s}{s} \Big(20\, \mathrm{e}^{-\bar{m}^2 s} + 2\, \mathrm{e}^{-\bar{m}_{\text{gh}}^2 s}\Big), \tag{4.48}$$

with Λ corresponding to a UV regulator. The corresponding integral was computed in Eq.(B.35). Then, the wave-function renormalization becomes

$$Z_{\text{F}} = 1 + \frac{N_c\, g^2}{3\,(4\pi)^2} \left[10\, \Gamma\!\left(0, \frac{\bar{m}^2}{\Lambda^2}\right) + \Gamma\!\left(0, \frac{\bar{m}_{\text{gh}}^2}{\Lambda^2}\right) \right]. \tag{4.49}$$

Identifying the sharp UV proper-time cutoff Λ as the renormalization scale, the one-loop beta function can readily be deduced

$$\beta_{g^2} = -\Lambda \frac{\mathrm{d}g^2}{\mathrm{d}\Lambda} = -\frac{N_c\, g^4}{3\,(4\pi)^2} \left(20\, \mathrm{e}^{-\frac{\bar{m}^2}{\Lambda^2}} + 2\, \mathrm{e}^{-\frac{\bar{m}^2_{\mathrm{gh}}}{\Lambda^2}} \right). \tag{4.50}$$

During the previous calculation we have kept Λ finite in order to obtain a generic expression for the one-loop beta function. However, setting $\Lambda \to \infty$ or $\bar{m}^2, \bar{m}^2_{\mathrm{gh}} \to 0$ one reproduces the universal one-loop result for the beta function, cf. Eq.(4.47).

Finally, it is worth exploring the form of the one-loop beta function following the same regularization procedure but considering a different RG scale. For convenience and independently of the scheme, we choose a common (renormalization) scale k for the ghost and gluon masses,

$$k^2 = \bar{m}^2 = \bar{m}^2_{\mathrm{gh}}. \tag{4.51}$$

In $d = 4$, the wave-function renormalization becomes

$$Z_{\mathrm{F}} = 1 + \frac{22 N_c\, g^2}{6\,(4\pi)^2}\, \Gamma\!\left(0, \frac{k^2}{\Lambda^2}\right). \tag{4.52}$$

Considering a finite Λ, the one-loop beta function takes the form

$$\beta_{g^2} = k \frac{\mathrm{d}g^2}{\mathrm{d}k} = -\frac{22 N_c\, g^4}{3\,(4\pi)^2}\, \mathrm{e}^{-\frac{k^2}{\Lambda^2}}. \tag{4.53}$$

Once more, Eq.(4.53) agrees with the universal one-loop beta function result of Eq.(4.47) in the limit of $\Lambda \to \infty$ or $k \to 0$.

In summary, the results of various one-loop beta functions computed either in different regularization schemes or at different renormalization scales show an agreement in the appropriate limits. This finding is consistent with the expected universality of the YM one-loop beta function in $d = 4$. Additionally, a mass-dependent scheme with a finite Λ contribution, results in a beta function with a threshold behavior when the mass scale surpasses the UV regulator. As the mass scale dominates over the UV regulator, integrated out considerations become incompatible, leading to a beta function that approaches zero, indicating mode decoupling.

4.5 Phenomenological study of the action density

Let us now turn our focus towards the implications of the inclusion of mass parameters within our theoretical framework from a phenomenological standpoint. Specifically, we will investigate the behavior of the *action density*, which is the finite part of the one-loop EA, Eq.(4.42), for various values of the coupling. To accomplish this, we will utilize different assumptions for the background field, enabling us to explore a wider range of coupling constant validity. A discussion on the applied set of assumptions for the background field can be found in App. A and [146, 291]. As the one-loop EA we have derived for pure YM only differs from the conventional one-loop EA due to the presence of mass parameters, any changes in its form will be a direct consequence of the mass deformation. Thus, the study of the action density can provide insights into how the mass parameters affect the underlying YM theory.

4.5.1 Action density for constant magnetic backgrounds

Let us begin our study by considering the same set of assumptions that were introduced in Sec. 4.4, i.e. covariantly constant backgrounds and a constant magnetic field, cf. Eqs.(4.28) & (A.4) respectively. Then, the renormalized one-loop action density, after subtraction of the counterterms from the ghost and gluon loops takes the following form

$$\frac{\Gamma_{1\mathrm{L}}}{\Omega_4} = \frac{1}{2}B_{\mathrm{R}}^2 + \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\mathrm{e}^{-k^2 s}\sum_{\ell=1}^{N^2-1}\left[1 + \frac{11(s\bar{B}_\ell)^2}{6} - \frac{s\bar{B}_\ell}{\sinh\left(s\bar{B}_\ell\right)} - 2s\bar{B}_\ell\sinh\left(s\bar{B}_\ell\right)\right]. \tag{4.54}$$

The first term in the action density corresponds to the classical contribution whereas the second one encodes the one-loop quantum corrections. The latter consists of four terms in the proper-time integral of Eq.(4.54). The first two terms correspond to the contribution of the vacuum subtraction and the inclusion of the counterterms, which render the one-loop action density finite. The third term arises from the functional trace of the heat kernel of the covariant Laplacian of gluons and ghosts, cf. Eq.(B.17), whereas the last term comes from the functional trace of the gluonic heat kernel of the transversal kinetic operator, $\mathfrak{D}_{\mathrm{T}}$ in $d = 4$, cf. Eq.(B.18). Such functional trace computations involve tracing over the spectrum of the corresponding operators, which under the assumption of covariantly constant magnetic background can be found in [143] (see App. B within). A particularity comes from the spectrum of $\mathfrak{D}_{\mathrm{T}}$ which contains negative modes. Such "tachyonic" field fluctuations are called *Nielsen-Olesen unstable modes* and tend to destabilize the Savvidy QCD vacuum [299]. Several attempts have been made to address this instability, leading to modified QCD vacuum models ("Spaghetti vacuum") [300–304] or formal mathematical techniques [305, 306].

In general, the existence of such modes spoils perturbative calculations [307–310]. However, in our formalism, the inclusion of symmetry respecting regulator parameters provides a natural tool to handle these gluonic modes, for $k^2 \geq \bar{B}_\ell$. This implies that a sufficiently large BRST invariant mass scale k^2, suppresses the divergent contribution of the unstable modes, allowing us to investigate the behavior of the action density, Eq.(4.54), within the corresponding range of validity. Fig. 4.1 displays such a study of the action density in terms of the dimensionless parameter $\zeta = \frac{\bar{B}_\ell}{k^2}$ for various values of the coupling constant. Note that the computation of the action density was restricted within the range of validity, $k^2 \geq \bar{B}_\ell$ as in further regions the unstable gluonic modes dominate over the mass scale leading to an IR divergent result. Furthermore, we restricted our computation of the action density to the $SU(2)$ gauge group, which produces color eigenvalues $\nu_\ell = -1, 0, 1$.

In Fig. 4.1, we observe that for small values of the coupling constant, the classical contribution dominates over the quantum corrections in the action density. As we approach the boundary $\zeta = \frac{g_{\mathrm{R}}B_{\mathrm{R}}}{k^2} = 1$, the quadratic increase begins to stabilize due to the effect of unstable modes, as seen in the upper part of the line with $g_{\mathrm{R}} = 3.5$ or $\alpha = \frac{g_{\mathrm{R}}^2}{4\pi} = 0.98$. Increasing the coupling constant to higher values results in a reversal, where the quantum corrections become increasingly dominant over the classical contribution. At a critical value of $g_{\text{crit}} = 7.94$ or $\alpha_{\text{crit}} = 5.02$, the quantum corrections become of the same order as the classical contribution. An ad hoc further increase in the coupling results in an overtaking of the classical by the quantum corrections, as seen in the line with $g_{\mathrm{R}} = 12$ or $\alpha = 11.46$. As a preliminary conclusion, we can deduce that our system exhibits a tendency of deviating from the conventional minimum, signaling the existence of a gluon condensate, even though our considerations for the coupling constant lie outside the regime of perturbation theory.

The aforementioned consideration of the action density at varying values of the couplings, despite the presence of unstable modes, is a novel aspect of our one-loop model that originates from the inclusion of mass-regulator parameters. In particular, our modified gauge-fixing procedure allows us to rigorously control the instabilities, thus extending the validity domain of the one-loop EA, without the need for an analytic continuation of the gluon determinant. The latter is required in a conventional one-loop treatment. In such cases, the corresponding counterterm contribution $\frac{11}{3}(sB)^2$ dominates in the proper-time integral, thus imposing a $B^2 \ln \frac{B^2}{k^2}$ behavior which also displays a non-trivial minimum [311].

The expression given in Eq.(4.54) represents the one-loop EA for the background field, which depends on the renormalized coupling, field strength and an IR regulator parameter k introduced through the gauge-fixing condition and regularization of loop integrals. It is expected that this dependence on k should be removable, as gauge-fixing parameters should not affect BRST-invariant quantities and regularization scales should not affect observables. This expectation is indeed realized through a systematic cancellation of the gauge-parameter dependence in any renormalization scheme by redefining the essential coupling constants of the gauge theory, as has long been established [312–315]. Similarly, the independence of observables from the renormalization scale is guaranteed by the transformation of coupling constants under RG evolution.

Therefore, our study highlights the significance of incorporating suitable mass parameters in pure YM theory, which screen the unstable modes, within a certain validity domain and provide a consistent way to handle such instabilities for a wide range of coupling values. This approach entails valuable phenomenological conclusions for non-perturbative phenomena (e.g. gluon condensate). The absence of a non-trivial minimum of the action density, which would more conclusively indicate the existence of a gluon condensate, can be attributed to two factors. Either a different set of assumptions that makes better use of the perturbative form of the action density is needed to obtain more definitive results, or higher-order loop contributions could modify the action density and drive the system to a finite non-trivial minimum. The former reasoning will be explored in the next section where we will consider the behavior of the action density under a different set of assumptions.

4.5.2 Action density for self-dual backgrounds

Considering a constant magnetic field led to a finite action density within a limited range of validity which was controlled by the BRST invariant mass parameters. Beyond this range, the Nielsen-Olesen unstable modes spoils the behavior of the action density which made it difficult to draw conclusions. Therefore, it turns out to be advantageous to choose a different set of assumptions in order to extend the domain of study of the action density and possibly reach phenomenological conclusions.

Therefore, we shall use a known choice for the background field which is stable, i.e. covariantly constant and self-dual backgrounds [316–319]. The basic considerations for self-dual backgrounds is briefly addressed in Appendix A. By analyzing the spectrum of $\mathfrak{D}_{\mathrm{T}}$, cf. Eq.(A.12), we no longer encounter negative modes, which previously led to instabilities. However we do find

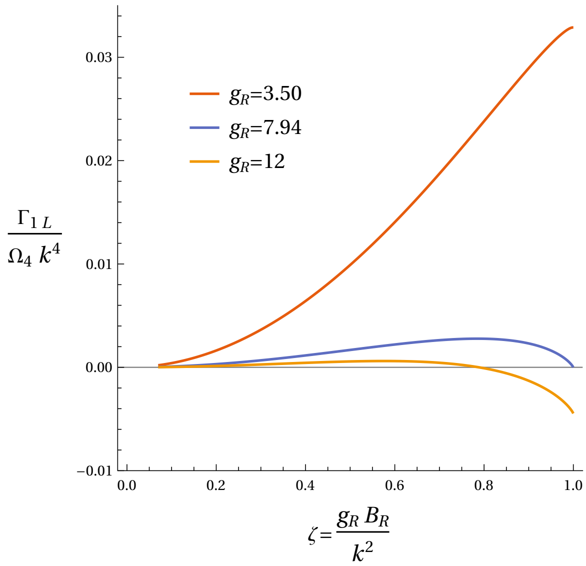

Figure 4.1: Numerical result for the dimensionless action density, as determined in (4.54), in terms of the dimensionless parameter $\zeta = \frac{g_R B_R}{k^2} \leqslant 1$, for various values of the renormalized coupling constant. Increasing the renormalized coupling g_R results in an increasing but finite contribution of the quantum part of the one-loop EA eventually dominating the classical one.

zero modes which at $d = 4$ have a multiplicity of 2. These modes, called *chromons*, [318, 319], can significantly contribute and their effect on the action density requires a careful treatment.

First, let us rewrite the bare action as follows

$$S_V[A] = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} = f^2 \tag{4.55}$$

Choosing a self-dual field strength tensor, see Eq.(A.9), the functional traces in the one-loop EA, Eq.(4.27), can be written in terms of $f_\ell = \bar{g} f \nu_\ell$.

As an additional consideration, we choose a vanishing ghost mass $\bar{m}_{gh} = 0$ and a finite gluon mass $\bar{m} > 0$ parameters. Such a choice of the mass parameters is motivated from non-perturbative studies and is constructed so that it mimics on a perturbative level the so called *decoupling solution*, which corresponds to a non-perturbative solution of the DSE that describes the IR properties of the fully dressed propagators of gauge theories [35, 39, 46, 66–70, 88, 94, 96–99, 320].

Hence, similarly to the case of a constant magnetic field, we encounter the heat kernel of the operators $-D^2, \mathfrak{D}_T$ for self-dual backgrounds. Their form is determined in App. B, cf. Eqs.(B.20) & (B.21), where one can notice the explicit contribution of the zero modes. Renormalizing the divergences in the ghost and gluon sectors, we relate the bare with the renormalized quantities as follows

$$g_R = \sqrt{Z_F}\bar{g}, \qquad f_R = \frac{f}{\sqrt{Z_F}}. \tag{4.56}$$

Subtracting the divergences appropriately in $d = 4$ and abbreviating $\bar{f}_\ell = g_R f_R \nu_\ell$, we find the finite action density

$$\frac{\Gamma_{1\mathrm{L}}}{\Omega_4} = f_{\mathrm{R}}^2 + \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\sum_{\ell=1}^{N^2-1}\left\{ \mathrm{e}^{-\bar{m}^2 s}\left[2 + \frac{11(s\bar{f}_\ell)^2}{3} - \frac{2(s\bar{f}_\ell)^2}{\sinh^2\left(s\bar{f}_\ell\right)} - 4(s\bar{f}_\ell)^2\right] \right. \\ \left. - \left(1 - \frac{(\bar{f}_\ell s)^2}{\sinh^2(\bar{f}_\ell s)}\right)\right\}. \tag{4.57}$$

Notice that the counterterms have been subtracted entirely from the gluon loop, resulting in a well-defined proper-time integral. This implies that the one-loop quantum corrections in the action density are both UV and IR convergent. By contrast, subtracting the divergences separately from the ghost and gluon loops would artificially induce an IR divergence in the ghost term, rendering the action density IR divergent. This would become visible at the upper bound of the proper-time integral. This unregularized divergence is an artifact of the specific choice of mass regulators used to simulate the decoupling solution, rather than a natural occurrence within the ghost sector and as such it can be circumvented with an appropriate placement of the counterterms. The current prescription ensures a pure UV subtraction scheme, as required from the ghost and gluon loops.

An interesting aspect of the action density comes from the last term of the first line in the gluon loop, i.e. $-4\left(s\bar{f}_\ell\right)^2$, which arises from the zero mode contribution. This term has a significant impact on the behavior of the action density. In the large field limit, it decreases as $-f^2 \ln\frac{f_{\mathrm{R}}^2}{\bar{m}^4}$ which dominates the quadratic increase of the classical contribution, i.e. f_{R}^2. This spoils the large field behavior of the action density which becomes unbounded from below. Hence, an inherent IR attribute of our theory, such as the zero modes ends up affecting the large field behavior. Such an effect illustrates the interplay between UV and IR properties, discussed within the framework of QED in [321]. However, one can regulate the large field effect of the zero modes in a consistent manner, by an ad hoc inclusion of an IR suppressing regulator for the corresponding term, for instance

$$-4\left(s\bar{f}_\ell\right)^2 \to -4\left(s\bar{f}_\ell\right)^2 \exp\left(\frac{s^2}{L^4}\right), \tag{4.58}$$

where $L \gg \frac{1}{\bar{m}}$. After addressing the seeming inconsistency of the unboundedness from below of the action density in the large field limit, it becomes possible to focus on the well-behaved regime of the theory in the small field domain.

In Fig. 4.2, we present a study of the behavior of the action density for $SU(2)$ by fixing different values of the renormalized coupling constant in gluon mass units, as we did for the constant magnetic field case, cf. Fig. 4.1. The results show that for small values of the renormalized coupling, the classical contribution dominates over the one-loop corrections in the action density, as seen in the upper line with $g_{\mathrm{R}} = 4$ i.e. $\alpha_{\mathrm{R}} = 1.3$. This behavior persists up to the critical coupling $g_{\mathrm{R,crit}} \simeq 5$, i.e. $\alpha_{\mathrm{R,crit}} \simeq 2$, above which the quantum corrections drive the system to a non-trivial minimum for small values of the field, as seen in the middle line with $g_{\mathrm{R}} = 7.5$ i.e. $\alpha_{\mathrm{R}} = 4.5$. Increasing the renormalized coupling even further drives this minimum to larger values of the field, as seen in the lowest line with $g_{\mathrm{R}} = 8$ i.e. $\alpha_{\mathrm{R}} = 5.1$. Such a behavior implies the existence of a phase transition to a *gluon condensation* phase.

It remains to demonstrate the order of the phase transition. To this end, in Fig. 4.3 we il-

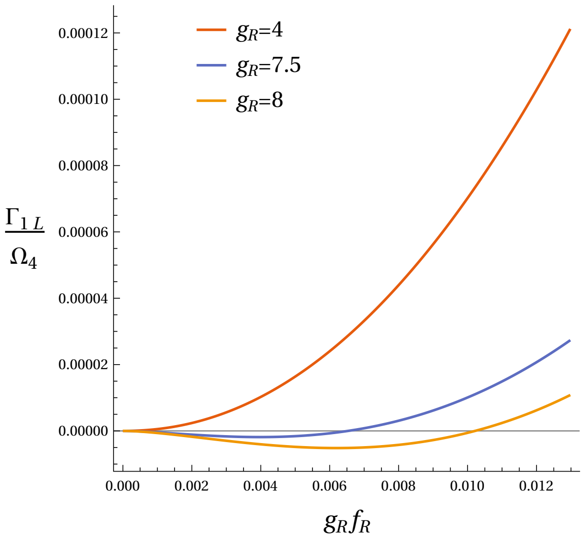

Figure 4.2: Action density for a self-dual background field, Eq.(4.57), in terms of the parameter $g_\mathrm{R} B_\mathrm{R}$. All quantities are plotted in units of the gluon mass which we choose as $\bar{m}^2 = 1$ here. An increase of the renormalized coupling beyond a critical coupling $g_{\mathrm{R,crit}} \simeq 5$ results in the appearance of a non-trivial minimum which can be taken as an indication for a gluon condensate.

lustrate the location of the minimum of the action density, as determined from Eq.(4.57), as a function of the renormalized coupling. The graph shows the appearance of a non-trivial minimum above the critical value of the coupling $\alpha_{\mathrm{R,crit}} \simeq 2$ which initially displays a quadratic increase, followed by a linear increase as the coupling is increased further. This structural behavior suggests a second order phase transition to a gluon condensate phase.

Our perturbative study of pure YM theory reveals that the action density for self-dual background displays a non-trivial minimum as the coupling strength increases. Such novel behavior is a direct consequence of the inclusion of mass-regulator parameters and is qualitatively consistent with non-perturbative fRG studies, although performed for the so-called scaling solution [146]. Therefore, a sensible incorporation of these parameters allows for the appearance of previously observed non-perturbative phenomena, such as the gluon condensate, within a perturbative framework. However, it should be noted that our perturbative study extends to couplings outside the regime of perturbation theory. Despite this, the agreement of our results with the literature within the perturbative range, as well as the behavior observed outside that domain, provides an additional validation of our phenomenological results.

The consistent qualitative behavior of our model seems to be at odds with quantitative estimates and warrants further investigation. In our model, the value of the condensate depends on the choice of an appropriate IR scale, which is determined by the energy range of interest and governed by the gluon mass parameter and an additional input for the IR properties of the coupling. For instance, selecting an IR bound at a typical hadronic scale of $\bar{m} = 1$ GeV and an observed IR coupling range of $\alpha_\mathrm{R} \in [2, 8]$ yields a condensate of the order of $f_{\mathrm{R,min}} \bar{m}^2 \simeq 0.21$ GeV^2 in our conventions, which deviates from phenomenological estimates [322]. Such a quantitative discrepancy can be attributed to the limitations of the one-loop expansion. Thus, our one-loop consideration still leaves open the question of how higher loops would modify the observed behavior.

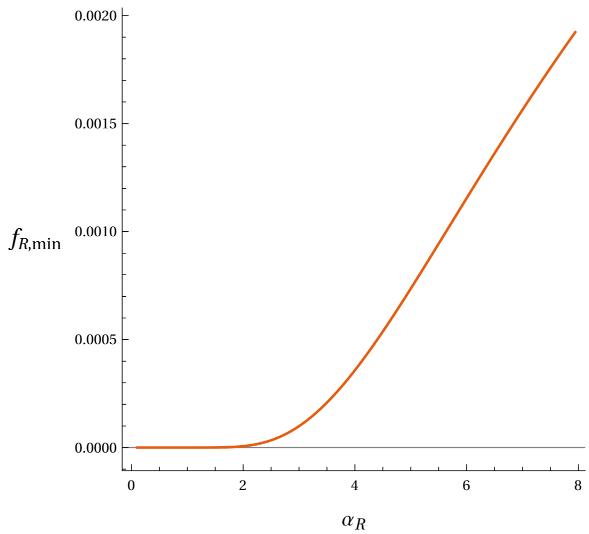

Figure 4.3: Gluon condensate $f_{\mathrm{R,min}}$, in units of the gluon mass, corresponding to the non-trivial minimum of the action density as a function of the renormalized coupling $\alpha_{\mathrm{R}} = \frac{g_{\mathrm{R}}^2}{4\pi}$. Our result shows a continuous increase of the condensate beyond a critical coupling $\alpha_{\mathrm{R}} > \alpha_{\mathrm{R,crit}} \simeq 2$.

CHAPTER 5

Schwinger Functional and the Two-Point Correlation Function

In this chapter, we investigate the impact of the v-field-dependent source term in the Schwinger functional, Eq.(4.14), which is of nonlocal nature and hence requires an in-depth analysis. To this end, we deduce a vanishing contribution in the large gluonic mass expansion (LGME) limit, for covariantly constant background Eq.(4.28) and covariantly conserved external current Eq.(4.11). Next relaxing these constraints, we use heat-kernel techniques and treat the v field as a disorder field, which yields a v-independent result free of divergences and consistent with the conventional theory in Landau gauge. Furthermore, we examine the effects of this term on the construction of the building blocks of the theory, e.g. two-point correlators. To that extent, we explore the contribution of the v field on the level of a two-point correlator at various settings and comment on its asymptotic behavior. We conclude the chapter by organizing all adopted approaches and the associated form of the two-point correlator in a tabular form.

5.1 Source term in the LGME limit for covariantly constant backgrounds

In our previous work, we investigated the effect of a non-linear gauge-fixing condition, which resulted, among others, in the emergence of a source term with a non-conventional form, as presented in Sec. 4.3 and expressed by Eq.(4.18). This source term, beyond the structural difference, introduced a dependence on the v field, which is a consequence of the chosen gauge-fixing procedure and should not affect observables. However, to fully understand the implications of this term, it is crucial to study its form and behavior. We will focus on examining the case of vanishing external sources ($j = 0$), which simplifies Eq.(4.18) to

$$W_{\text{source}}[A; v] = \frac{1}{2}\mathcal{K}^a_\mu[A, v] \left(M^{-1}\right)^{ab}_{\mu\nu} [A]\mathcal{K}^b_\mu[A, v], \tag{5.1}$$

where $\mathcal{K} = DF + J$ is given by Eq.(4.19).

The study of Eq.(5.1) requires to determine the inverse gluonic fluctuation operator, M^{-1}. We begin by calculating such a contribution in the approximate case of the LGME limit. To do so, we rewrite the gluonic fluctuation operator, given by Eq.(4.17), as

$$M^{ab}_{\mu\nu}[A;\xi] = \bar{m}^2 \delta^{ab}_{\mu\nu} + \mathcal{Q}^{ab}_{\mu\nu}[A;\xi] = \bar{m}^2 \left(\delta^{ab}_{\mu\nu} + \frac{1}{\bar{m}^2} \mathcal{Q}^{ab}_{\mu\nu}[A;\xi] \right), \tag{5.2}$$

where

$$\mathcal{Q}^{ab}_{\mu\nu}[A;\xi] = -2\bar{g} f^{abc} F^c_{\mu\nu} - \left(D^2\right)^{ab} \delta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) D^{ac}_{\mu} D^{cb}_{\nu}. \tag{5.3}$$

Schematically M and M^{-1} can be written as

$$M = \bar{m}^2 \left[\mathbb{1} + \bar{m}^{-2}\mathcal{Q}\right], \quad M^{-1} = \bar{m}^{-2} \left[\frac{1}{1 + \bar{m}^{-2}\mathcal{Q}}\right].$$

Considering a large gluonic mass, we expand these formulas in powers of the operator $\mathcal{Q}$ and obtain the Neumann series

$$\left(M^{-1}\right)^{ab}_{\mu\nu} = -\sum_{n=0}^{\infty} \left[\left(\frac{i}{\bar{m}}\right)^{2n+2} \left(\mathcal{Q}^n\right)^{ab}_{\mu\nu} \right], \tag{5.4}$$

where for the product of the $\mathcal{Q}$ operators, the following condensed notation was adopted

$$\left(\mathcal{Q}^n\right)^{ab}_{\mu\nu} = \underbrace{\mathcal{Q}^{ac}_{\mu\rho} \mathcal{Q}^{cd}_{\rho\sigma} \cdots \mathcal{Q}^{er}_{\kappa\tau} \mathcal{Q}^{rb}_{\tau\nu}}_{\text{n times}}.$$

Up to this point, we have not considered any restrictions on the form of the background and v field. One can immediately deduce a vanishing contribution of this term in the case where the background equations of motion are satisfied, Eq.(4.9), since in that case the current $J[A, v]$ vanishes as well. However, our goal is to display the form of this novel term as accurately and generically as possible. To that extent, certain soft constraints will be placed on the background and v field.

In the LGME limit, it turns out to be appropriate to choose covariantly constant backgrounds and current conservation of the source, cf. Eqs.(4.28) & (4.11) respectively. For these choices of field configurations, the source contribution, takes the form

$$W_{\text{source}}[A;v] = \frac{1}{2} J^a_{\mu}[A,v] \left(M^{-1}\right)^{ab}_{\mu\nu} [A] J^b_{\nu}[A,v]. \tag{5.5}$$

For the study of the inverse gluonic fluctuation operator to all orders, we shall rewrite it in terms of the longitudinal $\mathfrak{D}_{\mathrm{L}}$ and transversal $\mathfrak{D}_{\mathrm{T}}$ kinetic operators

$$(\mathfrak{D}_{\mathrm{T}})^{ab}_{\mu\nu} = -\,\delta_{\mu\nu} \left(D^2\right)^{ab} - 2i\bar{g}F^{ab}_{\mu\nu}, \tag{5.6}$$

$$(\mathfrak{D}_{\mathrm{L}})^{ab}_{\mu\nu} = -\,D^{ac}_{\mu} D^{cb}_{\nu}. \tag{5.7}$$

These are further discussed in App. D and are motivated by [132, 133, 323]. The operator $\mathcal{Q}_{\mu\nu}$ splits into the sum

$$\mathcal{Q}^{ab}_{\mu\nu}[A] = (\mathfrak{D}_{\mathrm{T}})^{ab}_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) (\mathfrak{D}_{\mathrm{L}})^{ab}_{\mu\nu}. \tag{5.8}$$

Some useful properties of the kinetic operators that will be used in the following are

$$[\mathfrak{D}_{\mathrm{T}}, \mathfrak{D}_{\mathrm{L}}]^{ab}_{\mu\nu} = 0, \tag{5.9}$$

$$(\mathfrak{D}_{\mathrm{L}})^{ab}_{\mu\nu}\, J^b_\nu = 0, \quad J^a_\mu\, (\mathfrak{D}_{\mathrm{L}})^{ab}_{\mu\nu} = 0. \tag{5.10}$$

Eq.(5.9) is proven in App. D, and is a consequence of the assumption of covariantly constant background field. Moreover, Eqs.(5.10) are a direct consequence of the definition of the longitudinal kinetic operator given by Eq.(5.7) and of current conservation, Eq.(4.11).

The source contribution to the Schwinger functional, Eq.(5.5), in the LGME limit takes the following form

$$W_{\text{source}}[A; v] = -\frac{1}{2} J^a_\mu \sum_{n=1}^{\infty} \left[\left(\frac{i}{\bar{m}}\right)^{2n+2} (\mathfrak{D}^n_{\mathrm{T}})^{ab}_{\mu\nu} \right] J^b_\nu, \tag{5.11}$$

which implies that there is no contribution from the longitudinal kinetic operator and hence no dependence on the gauge parameter. Thus, all possible nonvanishing contributions will come from the transversal kinetic operator $\mathfrak{D}_{\mathrm{T}}$. This result holds true to all orders of the expansion.

It remains to be determined whether this term provides a nonvanishing contribution to the one-loop Schwinger functional. To that extent, we must examine how the transversal kinetic operator acts in the expansion, Eq.(5.11). This is done by employing the commutation relation of the covariant Laplacian with the covariant derivative for covariantly constant background fields

$$\left[D^2, D_\mu\right]^{ab} = 2i\bar{g}\, (F_{\alpha\mu} D_\alpha)^{ab}. \tag{5.12}$$

Then,

$$J^a_\mu \left(M^{-1}\right)^{ab}_{\mu\nu} J^b_\nu = -\sum_{n=1}^{\infty} \left(\frac{i}{\bar{m}}\right)^{2n+2} J^a_\mu \left(\mathfrak{D}^{n-1}_{\mathrm{T}}\right)^{ac}_{\mu\rho} (\mathfrak{D}_{\mathrm{T}})^{cb}_{\rho\nu} J^b_\nu.$$

Considering, Eqs.(5.6), (5.12) & Eq.(4.10),

$$\begin{aligned} J^a_\mu \left(M^{-1}\right)^{ab}_{\mu\nu} J^b_\nu = \sum_{n=1}^{\infty} \left(\frac{i}{\bar{m}}\right)^{2n+2} & \left[J^a_\mu \left(\mathfrak{D}^{k-1}_{\mathrm{T}}\right)^{ac}_{\mu\nu} D^{cd}_\nu D^{db}_\alpha J^b_\alpha \right. \\ & - 2i\bar{g} J^a_\mu \left(\mathfrak{D}^{k-1}_{\mathrm{T}}\right)^{ac}_{\mu\rho} F^{cb}_{\rho\nu} J^b_\nu \\ & \left. + 2i\bar{g} J^a_\mu \left(\mathfrak{D}^{k-1}_{\mathrm{T}}\right)^{ac}_{\mu\rho} F^{cb}_{\rho\nu} J^b_\nu \right]. \end{aligned}$$

However, from current conservation follows that the first term is zero and the other two exactly cancel each other. Hence we conclude that

$$W_{\text{source}}[A] \mid_{DF=0, D\cdot J=0} = 0, \tag{5.13}$$

which is valid to all orders in the LGME limit. This leads to a one-loop Schwinger functional of the form

$$-W_{1\mathrm{L}}[A] \Big|_{DF=0, D\cdot J=0} = S_{\mathrm{v}}[A] - \ln \Delta_{\mathrm{FP}}[A] + \frac{1}{2} \ln \det M[A], \tag{5.14}$$

where one can notice the absence of any v-field contribution. Therefore, for covariantly constant background fields and covariantly conserved external current, the one-loop Schwinger functional has a conventional form which is exhausted by a ghost loop and by a gluon loop.

5.2 Schwinger functional with a disorder v field

In our investigation of pure YM theory, we have succeeded in developing a formulation that is both background and BRST invariant. However, achieving this required implementing a Fourier weight during the gauge-fixing procedure, see Subsec. 2.3.2, leading to a v-dependent generating functional, Eq.(3.8). The dependence of the v field is inherited by the FP determinant and subsequently affects the ghost action through the gauge-fixing procedure, see Eq.(4.4).

At one-loop, this v dependence drops out of the FP determinant, Eq.(4.16) and provides a v-independent one-loop EA which exhibits a regulator deformed structure, see Secs. 4.3-4.5. However, the v field still plays a role in the Schwinger functional through a novel source contribution, Eq.(4.18), potentially affecting the 1PR correlators. Therefore, it is of interest to study the impact of the v-dependent terms on the correlation functions and their effects, e.g. potential emergence of additional divergences.

One approach to do so is by directly computing the contribution of these terms to the 1PR correlation functions, which will be the subject of Secs. 5.3 & 5.4. Another method is to integrate out the external v field. Although the v field itself is not expected to affect the observables, the terms arising from it may have an impact. Therefore, averaging over the v field is a great way to eliminate the contribution of an external field while studying the results it entails. Either method of study requires a thorough understanding of how to handle the v field in the Schwinger functional Eq.(4.14).

In Sec. 5.1 we proved that such a v-field dependence drops out on the level of the Schwinger functional when certain consistency and/or on-shell conditions are satisfied, cf. Eq.(5.13). However, it is beneficial to investigate the structural form of the Schwinger functional from a broader perspective, i.e. when no such conditions are imposed. To achieve this, we treat the v field as an external stochastic field, similar to a disorder field in a statistic field theory.

In statistical physics, the disorder field can be treated through either the *annealed* or the *quenched average disorder* methods. These two methods differ in the way they average over the disorder field. The difference between these two methods can be understood from the two following relations of the disordered-independent Schwinger functional

$$\langle W_{1\mathrm{L}}\rangle_{\mathrm{a}}[j,A] = \mathcal{N}\log\left[\int \mathcal{D}v\;\mathrm{d}\rho(v)\;\mathcal{Z}_{1\mathrm{L}}[j,A;v]\right] = \mathcal{N}\log\langle \mathcal{Z}_{1\mathrm{L}}[j,A;v]\rangle, \tag{5.15}$$

for an annealed average, whereas for a quenched average,

$$\langle W_{1\mathrm{L}}\rangle_{\mathrm{q}}[j,A] = \mathcal{N}\int \mathcal{D}v\;\mathrm{d}\rho(v)\;\log\left(\mathcal{Z}_{1\mathrm{L}}[j,A;v]\right) = \mathcal{N}\left\langle \log \mathcal{Z}_{1\mathrm{L}}[j,A;v]\right\rangle. \tag{5.16}$$

Note that $\mathrm{d}\rho(v)$ denotes the probability distribution of the disorder field which in our case will be considered to have a Gaussian form. Eqs.(5.15) & (5.16) correspond to different ways of averaging over the disorder field and typically result in different disorder-independent correlation functions.

In Eq.(5.15) the disorder field and quantum fluctuations are treated on the same footing, making the disorder yet another degree of freedom. Alternatively, Eq.(5.16) first integrates over the quantum fluctuations, i.e. quantum degrees of freedom to obtain the disorder-dependent Schwinger functional and subsequently averages over the disorder field with an appropriate distribution. In such a way, the disorder degrees of freedom are "frozen".

To study the Schwinger functional, we will use the quenched average method to integrate out the disorder v field. This choice is motivated by the fact that in our case, the v field, as the product of the gauge-fixing condition, should not be viewed as an extension of our theory but rather as a removable external contribution and as such should be integrated out independently of the field configurations. Additionally, the quenched average method is appropriate for statistical field theoretic systems with disorder contributions that exhibit self-averaging properties [324]. *Self-averaging* quantities are statistically stable and their disorder-independent value can be obtained by averaging over the disorder [325]. The Schwinger functional is an example of a self-averaging quantity, while the partition function is generally not self-averaging. Therefore, using the quenched average method is expected to yield consistent disorder-independent observables.

Let us begin our study, by performing the quenched Gaussian average over the disorder v field of the Schwinger functional, then

$$\langle W_{1\mathrm{L}} \rangle [A] = \mathcal{N} \int \mathcal{D}v \; \mathrm{e}^{-S_{\mathrm{NL}}[v]} \; W_{1\mathrm{L}}[A;v], \tag{5.17}$$

$$S_{\mathrm{NL}}[v] = \int \mathrm{d}^d x \, \frac{v^2}{2\alpha}, \tag{5.18}$$

where $\mathcal{N}$ is a normalization constant determined by the constant condition $\langle 1 \rangle = 1$. Note that we dropped the subscript q from the average. Since only the source contribution in Eq.(4.14) will be affected by the averaging, it is useful to denote

$$\langle W_{\text{source}} \rangle [A] = -\frac{\mathcal{N}}{2} \int \mathcal{D}v \; \mathrm{e}^{-S_{\mathrm{NL}}[v]} \; W_{\text{source}}[A;v]. \tag{5.19}$$

In the following, we perform an explicit study of the source contribution.

5.2.1 Functional traces of nonlocal operators

As mentioned before, the study of the source contribution Eq.(5.19) is of particular interest. It corresponds to a new term which arises due to the choice of the non-linear gauge-fixing condition, Eq.(4.1), in order to maintain BRST and background invariance. Inserting the definition of the current J^a_μ Eq.(4.10) one obtains four terms of the following form up to a normalization constant

$$
\begin{aligned}
\langle W_{\text{source}}\rangle [A] = -\left(\frac{1}{2}\right)\int \mathcal{D}v\, \mathrm{e}^{-\int \frac{v^2}{2\alpha}} \Big[& (D_\mu v)^a \left(M^{-1}\right)^{ab}_{\mu\nu} (D_\nu v)^b && \text{(i)} \\
& - \bar{m}^2_{\text{gh}} (D_\mu v)^a \left(M^{-1}\right)^{ab}_{\mu\nu} \left(D_\nu \left(\frac{1}{D^2}\right) v\right)^b && \text{(ii)} \\
& - \bar{m}^2_{\text{gh}} \left(D_\mu \left(\frac{1}{D^2}\right) v\right)^a \left(M^{-1}\right)^{ab}_{\mu\nu} (D_\mu v)^b && \text{(iii)} \\
& + \bar{m}^4_{\text{gh}} \left(D_\mu \left(\frac{1}{D^2}\right) v\right)^a \left(M^{-1}\right)^{ab}_{\mu\nu} \left(D_\nu \left(\frac{1}{D^2}\right) v\right)^b \Big]. && \text{(iv)}
\end{aligned}
\tag{5.20}
$$

Performing the Gaussian functional integral and recombining the remaining terms to functional traces, the four previously determined terms become

$$\text{(i)} = \left(\frac{\alpha}{2}\right) \mathrm{tr}_{\text{xc}} \left[D_\mu M^{-1}_{\mu\nu} D_\nu\right], \tag{5.21a}$$

$$\text{(ii)} = \text{(iii)} = -\left(\frac{\alpha \bar{m}^2_{\text{gh}}}{2}\right) \mathrm{tr}_{\text{xc}} \left[\frac{1}{D^2} D_\mu M^{-1}_{\mu\nu} D_\nu\right], \tag{5.21b}$$

$$\text{(iv)} = \left(\frac{\alpha \bar{m}^4_{\text{gh}}}{2}\right) \mathrm{tr}_{\text{xc}} \left[\frac{1}{D^2}\frac{1}{D^2} D_\mu M^{-1}_{\mu\nu} D_\nu\right], \tag{5.21c}$$

where the cyclic property of the trace has been taken into account. In addition, for the last three functional traces the following identity of the inverse Laplacian was employed $\left(\frac{1}{D^2}\right)^{ab}_{xy} = \left(\frac{1}{D^2}\right)^{ba}_{yx}$. Finally, one can notice that (ii) & (iii) are equal to each other.

The aforementioned functional traces can be further simplified to the following form

$$\text{(i)} = \left(\frac{\alpha}{2}\right) \mathrm{tr}_{\text{xc}} \left[\frac{1}{\bar{m}^2 - \left(\frac{1}{\xi}\right) D^2} D^2\right], \tag{5.22a}$$

$$\text{(ii)} = \text{(iii)} = -\left(\frac{\alpha \bar{m}^2_{\text{gh}}}{2}\right) \mathrm{tr}_{\text{xc}} \left[\frac{1}{D^2}\frac{1}{\bar{m}^2 - \left(\frac{1}{\xi}\right) D^2} D^2\right], \tag{5.22b}$$

$$\text{(iv)} = \left(\frac{\alpha \bar{m}^4_{\text{gh}}}{2}\right) \mathrm{tr}_{\text{xc}} \left[\frac{1}{D^2}\frac{1}{D^2}\frac{1}{\bar{m}^2 - \left(\frac{1}{\xi}\right) D^2} D^2\right]. \tag{5.22c}$$

In App. E, an explicit derivation of the functional traces Eqs.(5.22a)-(5.22c) from Eqs.(5.21a)-(5.21c) can be found. The assumptions made in this derivation include the existence of an inverse gluonic matrix M^{-1} and the condition of a covariantly constant background, as given in Eq.(4.28). It should be emphasized that the assumption of a covariantly constant background plays a crucial role in simplifying the four functional traces Eq.(5.21a)-(5.21c). Specifically, this condition allows for a cancellation of terms, which greatly facilitates the implementation of diagonal heat-kernel techniques in studying the functional trace behavior. Furthermore, there is a convergence limit in the Landau gauge, $\xi \to 0$, where all functional traces vanish.

Finally, the functional traces have an unconventional structure as they involve different powers of the nonlocal operator $\frac{1}{D^2}$. This raises the question of whether such a novel term introduces additional divergences that could affect the 1PR correlators of the theory through the Schwinger functional. To address this question, we employ heat-kernel techniques to regularize each trace.

Study of the (i) functional trace

In this section, we adopt the conventional approach to handle functional traces of functions of the Laplacian. We achieve this by Laplace transforming each function in the functional trace, resulting in an integration over the Laplace transformed function and the heat kernel of the Laplacian. The integration variable corresponds to the proper-time or Schwinger parameter. Once we substitute the expression for the heat kernel of the Laplacian, as given in Eq.(B.17), the computation reduces to a proper-time integral. For a more comprehensive analysis, see App. B.2.

Starting from the first functional trace, Eq.(5.22a), we write symbolically the auxiliary function in the trace as

$$h(x) = -\frac{\xi x}{x + \xi \bar{m}^2}, \tag{5.23}$$

which makes the trace

$$\mathrm{tr}_{\mathrm{xc}}\left[h(x)\right]\big|_{x=-D^2} = \mathrm{tr}_{\mathrm{xc}}\left[\frac{\xi x}{x + \xi \bar{m}^2}\right]\bigg|_{x=-D^2}. \tag{5.24}$$

Similarly to the ghost and gluon loops, cf. Sec. 4.4, we subtract the infinite vacuum field contribution. Then, by considering the Laplace transformation of the function, Eq.(B.22) and the corresponding heat-kernel relations, Eqs.(B.17) & (B.19), we obtain

$$\mathrm{tr}_{\mathrm{xc}}\left[h\left(-D^2\right) - h\left(-\partial^2\right)\right] = -\frac{\Omega_4}{(4\pi)^2}\sum_{\ell=1}^{N^2-1}\int_0^\infty \frac{ds}{s^2}\left[(\xi\bar{m})^2\,\mathrm{e}^{-\xi\bar{m}^2 s} - 2\xi\delta(s)\right]\left[\frac{sB_\ell}{\sinh sB_\ell} - 1\right]. \tag{5.25}$$

At the lower boundary of the proper-time parameter, $\delta(s)$ is to be understood as contributing half of its weight, cf. Eq.(B.2). Considering a weak field expansion, see Eq.(5.34), then according to Eq.(B.9) we have

$$\mathrm{tr}_{\mathrm{xc}}\left[h\left(-D^2\right) - h\left(-\partial^2\right)\right] = \frac{\Omega_4}{(4\pi)^2}\sum_{\ell=1}^{N^2-1}\int_0^\infty \frac{\mathrm{d}s}{s^2}\left[(\xi\bar{m})^2\,\mathrm{e}^{-\xi\bar{m}^2 s} - 2\xi\,\delta(s)\right]\frac{(sB_\ell)^2}{6}\quad . \tag{5.26}$$

Computing the convergent integrals leads to

$$\frac{1}{\Omega_4}(\mathrm{i})_{\mathrm{vs}} = 0 + \mathcal{O}\left(\frac{B^4}{\bar{m}^8}\right), \tag{5.27}$$

where the index vs denotes the subtraction of the infinite vacuum field contribution.

Study of the (ii), (iii) functional traces

For the second and third functional traces, we abbreviate the function inside the trace as

$$g(x) = \frac{\xi}{x + \xi \bar{m}^2} \tag{5.28}$$

Following similar steps to the first functional trace, while considering Eq.(B.23) for the Laplace transformation of the function, we obtain

$$\mathrm{tr}_{\mathrm{xc}} \left[g\left(-D^2\right) - g\left(-\partial^2\right) \right] = \Omega_4 \frac{\xi}{(4\pi)^2} \sum_{\ell=1}^{N_c^2-1} \int_0^\infty \frac{\mathrm{d}s}{s^2} \mathrm{e}^{-\xi \bar{m}^2 s} \left(\frac{sB_\ell}{\sinh sB_\ell} - 1 \right). \tag{5.29}$$

Performing a weak field expansion to second order, see Eq.(5.34), we are left only with a convergent integral of the form

$$\frac{1}{\Omega_4} (\mathrm{ii})_{\mathrm{vs}} = \frac{1}{\Omega_4} (\mathrm{iii})_{\mathrm{vs}} = \left(\frac{1}{4\pi} \right)^2 \left(\frac{\bar{m}_{\mathrm{gh}}}{\bar{m}} \right)^2 \left(\frac{N_c \alpha}{12} \right) (\bar{g}B)^2 + \mathcal{O}\left(\frac{B^4}{\bar{m}^8} \right), \tag{5.30}$$

where we have used Eqs.(A.5) & (A.6).

Study of the (iv) functional trace

For the fourth functional trace, Eq.(5.22c), the situation is a bit more involved. In particular, the function under consideration reads

$$f(x) = -\frac{\xi}{x\left(x + \xi \bar{m}^2\right)}, \tag{5.31}$$

which leads to the following vacuum subtracted functional trace

$$\mathrm{tr}_{\mathrm{xc}} \left[f\left(-D^2\right) - f\left(-\partial^2\right) \right] = -\frac{\Omega_4}{(4\pi \bar{m})^2} \sum_{\ell=1}^{N^2-1} |B_\ell| \int_0^\infty \frac{\mathrm{d}x}{x^2} \left[1 - \mathrm{e}^{-\frac{\xi \bar{m}^2}{|B_\ell|} x} \right] \left[\frac{x}{\sinh x} - 1 \right], \tag{5.32}$$

where $x = s|B_\ell|$. A linear dependence in the constant magnetic field seems to exist. A naive weak field expansion of the form $|B_\ell| \to 0$ would not take into account all contributing terms and would result in possible fictitious divergences. In addition, due to the variable change, it would also create problems with the upper boundary of integration since it would correspond to an undetermined value. Hence, one needs to use a different method to treat the integral. This can be done by splitting the full integral into its divergent and convergent parts. Then, we can regularize its divergent parts, whereas its convergent part can be trivially calculated. Only after the divergent integrals have been regularized and brought to an appropriate form, we impose a weak field expansion.

For the regularization of the divergent integrals, we introduce a convergence enforcing term $\lim_{\epsilon \to 0} x^\epsilon$. We have the freedom to introduce such a term, without the appearance of an extra mass scale, since by dimensional analysis one can see that x is dimensionless. In particular, in $d = 4$ we find that $[A_\mu] = 1$ which means that $[F_{\mu\nu}] = 2 = [B]$ and $[\bar{g}] = 0$. This implies that $[\beta] = \left[\frac{\xi \bar{m}^2}{B_\ell} \right] = 0$ and subsequently $[x] = 0$. The explicit form of the divergent integrals, treated with the ϵ technique is given in Eqs.(B.30) & (B.31).

Computing the full integral, the divergences of the contributing integrals exactly cancel with each other, cf. App. B see Eq.(B.34). The vacuum subtracted fourth functional trace then takes the form

$$\frac{1}{\Omega_4}(\mathrm{iv})_{\mathrm{vs}} = -\left(\frac{1}{4\pi}\right)^2\left(\frac{\bar{m}_{\mathrm{gh}}}{\bar{m}}\right)^4\left(\frac{\alpha}{2}\right)\left[\bar{m}^2\ln\frac{1}{2}\sum_{\ell=1}^{\infty}|B_\ell| + \frac{N}{6\xi}\left(\bar{g}B\right)^2\right] + \mathcal{O}\left(\frac{B^4}{\bar{m}^8}\right). \tag{5.33}$$

Note that for the computation of the aforementioned relation we have assumed a *weak-field expansion*, via the following condition

$$\frac{|B_\ell|}{\xi\bar{m}^2} \ll 1. \tag{5.34}$$

Consequently, Eq.(5.34) is to be understood as the range of validity for the weak field expansion.

In support of the analytic result of the fourth functional trace, we have performed a numerical computation of the same integral as in Eq.(5.32). According to the weak-field expansion, we compute the integral within the following range of values

$$\frac{|B_\ell|}{\xi\bar{m}^2} \in \left[10^{-6}, 0.3\right]. \tag{5.35}$$

Considering a linear line fitting of the numerical values of the integral in terms of our range of values, we find

$$\mathrm{I}\left(\frac{|B_\ell|}{\xi\bar{m}^2}\right) = -0.693 + 0.166\,\frac{|B_\ell|}{\xi\bar{m}^2}. \tag{5.36}$$

Inserting the numerical result for the proper-time integral of the functional trace, Eq.(5.36), in the vacuum subtracted expression, Eq.(5.22c),

$$\frac{1}{\Omega_4}(\mathrm{iv})_{\mathrm{num}} = -\left(\frac{1}{4\pi}\right)^2\left(\frac{\bar{m}_{\mathrm{gh}}}{\bar{m}}\right)^4\left(\frac{\alpha}{2}\right)\left[-0.693\,\bar{m}^2\sum_{\ell=1}^{\infty}|B_\ell| + \frac{0.166N}{\xi}\left(\bar{g}B\right)^2\right] + \mathcal{O}\left(\frac{B^4}{\bar{m}^8}\right). \tag{5.37}$$

At the outset, we observe that both the numeric and analytic approaches yield similar structural forms, as they both exhibit contributions from linear and quadratic magnetic fields. Specifically, a comparison of the prefactors obtained from the analytic calculation in Eq.(5.33) with those from the numeric calculation in Eq.(5.37) confirms this agreement up to the third decimal point, thereby validating the analytic result. Notably, the analytic method provides a solution to a question that arises during the numerical integration. To be more precise, the range of our parameter does not start from zero but close to it, cf. Eq.(5.36). Analytically, this is due the variable change which results in an undetermined value of the upper boundary of the integral. Numerically, this reasoning is no longer valid since we have considered well-behaved integration limits. Therefore it most probably corresponds to the inability of Mathematica to perform the numerical integration at that point. This raises the question of whether this is a shortcoming of the method of numerical integration that was considered or an inherent feature of the integral, possibly resulting in divergences at $|B_\ell|/\left(\xi\bar{m}^2\right) \to 0$. Here, the analytic computation is decisive, demonstrating that such possible divergences cancel out. Thus, the analytic method complements the numeric method, with the numeric result offering justification for the validity of the analytic result! We proceed with the elegant analytic result.

5.2.2 Study the form of $\langle W_{\text{source}}\rangle [A]$

Collecting the results deduced for the functional traces in the small magnetic field expansion, e.g. Eqs.(5.27), (5.30) & (5.33) and inserting them into the source contribution to the Schwinger functional, Eq.(5.20), we determine

$$\langle W_{\text{source}}\rangle [A] = \frac{\Omega_4}{(4\pi)^2}\left(\frac{\bar{m}_{\text{gh}}}{\bar{m}}\right)^2\left(\frac{\alpha}{2}\right)\left[-\bar{m}^2\ln\frac{1}{2}\sum_{l=1}^{N^2-1}|B_\ell| + \frac{\bar{g}^2 N_c}{6}\left(1-\frac{\bar{m}_{\text{gh}}^2}{\xi\bar{m}^2}\right)B^2\right]. \tag{5.38}$$

An important finding is that the contribution from the disordered source, which arises from the dependence on the v field, itself stemming from the non-linear gauge fixing condition, produces a finite term in the relevant part of the disordered one-loop Schwinger functional,

$$-\langle W_{1\text{L}}\rangle [A] = S_{\text{v}}[A] - \ln\Delta_{\text{FP}}[A] + \frac{1}{2}\ln\det M[A] - \langle W_{\text{source}}\rangle [A]. \tag{5.39}$$

It is in order to comment on the results that we derived. Starting off from the functional traces, Eqs.(5.22a)-(5.22c), there exists two significant convergence limits. In particular, in the Landau gauge, $\xi\to 0$ or at vanishing disorder parameter $\alpha\to 0$, all functional traces vanish and give a zero contribution of $\langle W_{\text{source}}\rangle [A]$. However, such a convergence limit seems to be lost on the level of the disordered one-loop Schwinger functional, Eq.(5.39). In that case, the convergence limit $\alpha\to 0$ is still valid, however in the Landau gauge we no longer obtain a vanishing result. The assumption behind this illusive inconsistency is the range of validity of the weak-field expansion. Performing such an expansion implies that we restrict our quantities within the corresponding range denoted by Eq.(5.34). However, in that range, we can no longer consider the Landau gauge, $\xi\to 0$. Therefore, the inconsistent dependence of $\langle W_{\text{source}}\rangle$ on the gauge parameter ξ between Eqs.(5.20) & (5.39), is merely an artifact of the weak-field expansion we chose to perform for the computation of the functional traces rather than an inconsistency of our calculational procedure.

Lastly, as it can be seen from Eq.(5.39), considering $\bar{m}_{\text{gh}}^2 = \xi\bar{m}^2$ eliminates the quadratic magnetic field dependence from the source contribution. In that case, we obtain

$$\begin{aligned}\langle W_{\text{source}}\rangle [A] &= \frac{\Omega_4}{(4\pi)^2}\left(\frac{\bar{m}_{\text{gh}}}{\bar{m}}\right)^2\left(\frac{\xi\alpha}{2}\right)\bar{m}^2\ln 2\sum_{l=1}^{N^2-1}|B_\ell| \\ &= \frac{\Omega_4}{(4\pi)^2}\left(\frac{\bar{m}_{\text{gh}}}{\bar{m}}\right)^2\left(\frac{\alpha}{2}\right)\bar{m}_{\text{gh}}^2\ln 2\sum_{l=1}^{N^2-1}|B_\ell|\end{aligned} \tag{5.40}$$

5.3 Two-point correlation function of $W_{\text{source}}[A; v]$

In Sec. 5.2, we discussed an important aspect of our formalism which pertains to the v field and its potential contribution to the building blocks of the theory, when considered as a disorder field and the quenched average method is applied. Although not expected to contribute at the level of observables, it is nevertheless worthwhile to study its effects on the correlation functions while keeping an explicit v dependence. Thus in this section and the following, we focus on the study of a two-point correlator that stems from the source term of the Schwinger functional, W_{source}, Eq.(5.1). To this end, as long as it is computationally feasible, we shall maintain an explicit v dependence. By employing different assumptions for the v field, we shall determine its impact on the following two-point correlator

$$W_v^{(2)} = \left.\frac{\delta^2 W_{\text{source}}[A;v]}{\delta A \delta A}\right|_{A\to 0}, \tag{5.41}$$

in the limit of vanishing backgrounds $A \to 0$. Note that up to this point, no assumption for the form of the v field has been made, only on the vanishing form of the background.

Computing the second functional derivative of Eq.(5.1), we find that

$$\left(W_v^{(2)}\right)^{cd}_{\alpha\beta} = \left.\frac{\delta \mathcal{K}^a_\mu}{\delta A^c_\alpha} \left(M^{-1}\right)^{ab}_{\mu\nu} \frac{\delta \mathcal{K}^b_\nu}{\delta A^d_\beta}\right|_{A\to 0} + \left.\frac{\delta \mathcal{K}^a_\mu}{\delta A^d_\beta} \left(M^{-1}\right)^{ab}_{\mu\nu} \frac{\delta \mathcal{K}^b_\nu}{\delta A^c_\alpha}\right|_{A\to 0}. \tag{5.42}$$

For the derivation of the Eq.(5.42), we have used that

$$\mathcal{K}^a_\mu\big|_{A\to 0} = J^a_\mu\big|_{A\to 0} = 0, \tag{5.43}$$

which makes use of current conservation, Eq.(4.11). For a generic proof of this result, see Eqs.(F.9) & (F.11) in App. F.

Finally, the two missing pieces in the computation of Eq.(5.42) are the inverse gluonic operator $M^{-1}\big|_{A\to 0}$ and $\frac{\delta J}{\delta A}\big|_{A\to 0}$. Schematically, the inverse gluonic operator for vanishing fields in our gauge reads

$$M^{-1}\big|_{A\to 0} = \left(\frac{1}{\bar{m}^2 - \partial^2}\right)\Pi_{\text{T}} + \left(\frac{1}{\bar{m}^2 - \frac{\partial^2}{\xi}}\right)\Pi_{\text{L}}, \tag{5.44}$$

where we have introduced the longitudinal and transversal projectors

$$(\Pi_{\text{L}})^{ab}_{\mu\nu} = \left(\frac{\partial_\mu \partial_\nu}{\partial^2}\right)\delta^{ab}, \quad (\Pi_{\text{T}})^{ab}_{\mu\nu} = \delta_{\mu\nu}\delta^{ab} - (\Pi_{\text{L}})^{ab}_{\mu\nu}. \tag{5.45}$$

The second important piece of the puzzle has been computed in Eq.(F.11). Then, collecting all aforementioned results, we can determine the form of the two-point correlator by implementing different or no extra constraints on it.

5.3.1 v-dependent two-point correlator

A straightforward insertion of the results mentioned before in Eq.(5.42) leads to the following two-point correlator

$$\begin{aligned}\left(W_v^{(2)}\right)^{cd}_{\alpha\beta}(p_1,p_2) =& \delta^{cd}\frac{p_1^2}{\bar{m}^2 + p_1^2}\left(p_1^2\delta_{\alpha\beta} - p_{1\alpha}p_{1\beta}\right)\delta_{p_1,-p_2} \\ &+ \bar{g}^2 f^{abc} f^{aed} \int_q \frac{(2q+p_1)_\alpha}{\bar{m}^2 + \frac{q^2}{\xi}} \frac{(2q-p_2)_\beta}{q^2} v^e_{-q-p_1} v^b_{q-p_2}.\end{aligned} \tag{5.46}$$

The term in the first line, which is diagonal in momentum space and v-independent comes from the $\left.\frac{\delta^2 (DF) M^{-1}(DF)}{\delta A \delta A}\right|_{A\to 0}$ term. The v-dependent contribution comes solely from the $\left.\frac{\delta^2 J M^{-1} J}{\delta A \delta A}\right|_{A\to 0}$ part of the two-point correlator. Finally, the $\left.\frac{\delta^2 J M^{-1} (DF)}{\delta A \delta A}\right|_{A\to 0}$ vanishes. Thus, Eq.(5.46) corresponds to the two-point correlator of the Schwinger functional at vanishing background and sources when the auxiliary v field is restricted to obey the massive Klein-Gordon equation, as implied by current conservation, Eq.(4.11).

5.3.2 v-independent two-point correlator

From the form of the two-point correlator, Eq.(5.46), we see that there is indeed a contribution coming from the v field that enters at the level of the correlation function. It is interesting to study the effect that the v-dependent term has on the two-point correlator by averaging over the auxiliary v field. However, unlike in the previous case, cf. Sec. 5.2, the v field is constrained to obey the massive Klein-Gordon equation as a result of current conservation. This will be implemented as an additional constraint by means of a Lagrange multiplier λ^a on top of the Gaussian average of the v field.

Another important distinction to the previous case is that the averaging takes place on the level of the correlation function, instead of that of the Schwinger functional as before. The averaged two-point correlator is

$$\langle W_v^{(2)}\rangle_{\alpha\beta}^{cd}(p_1,p_2) = \mathcal{N}\int \mathcal{D}\lambda^a\mathcal{D}v^a \mathrm{e}^{i\lambda^a\left(-\partial^2+\bar{m}_{\mathrm{gh}}^2\right)-\frac{v^2}{2\alpha}}\, W_v(p_1,p_2).. \tag{5.47}$$

Substituting the v-dependent two-point correlator, Eq.(5.46), in the above quantity there is a constant and a quadratic contribution of the v field to the Gaussian integral coming from the first and second term in the two-point correlator respectively. These Gaussian integrals have been computed in App. G, cf. Eqs.(G.8) & (G.10) for the contribution of each term. Thus, the complicated quadratic v-field contribution drops out, leaving

$$\langle W_v^{(2)}\rangle_{\alpha\beta}^{cd}(p_1,p_2) = \delta^{cd}\frac{p_1^4}{\bar{m}^2+p_1^2}\left(\delta_{\alpha\beta}-\frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right)\delta_{p_1,-p_2}. \tag{5.48}$$

Notably, the asymptotic behavior of the propagator-like quantity $\langle W_v^{(2)}\rangle^{-1}$ is of interest. In particular, at large momenta it exhibits a usual decay as $\langle W_v^{(2)}\rangle^{-1} \sim \frac{1}{p_1^2}$, indicating a tree-like behavior of the S-matrix in the perturbative domain. At the other end of the spectrum at small momenta, we observe an enhanced decay of $\langle W_v^{(2)}\rangle^{-1} \sim \frac{1}{p_1^4}$, reminiscent of IR slavery that agrees with non-perturbative realizations of the underlying regime, i.e. mass gap of the background field excitations.

5.3.3 Two-point correlator for a localized v field

Finally, let us study the form of the two-point correlator by choosing the following constant localized form for the v field

$$v_q^a = \sqrt{(2\pi)^d}\,v^a\delta(q). \tag{5.49}$$

In that case, the class of possible v fields is restricted by current conservation Eq.(4.11), which at vanishing background becomes a massive Klein-Gordon equation. Then a constant solution for the external v field is admissible only at vanishing ghost mass $\bar{m}_{\text{gh}} = 0$:

$$\partial^2 v^a = 0.$$

Incidentally, this scenario is in line with the decoupling solution used in Sec. 4.5. Performing the computation for a vanishing $\bar{m}_{\text{gh}}$, leads to

$$\begin{aligned}\left(W_v^{(2)}\right)^{cd}_{\alpha\beta}(p) =& \bar{g}^2 f^{abc} f^{aed} v^b v^e \delta_{p_1,-p_2} \left\{ \left[\delta_{\alpha\beta} - \frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right] \frac{1}{\bar{m}^2 + p_1^2} + \frac{p_{1\alpha}p_{1\beta}}{p_1^2} \frac{1}{\bar{m}^2 + \frac{p_1^2}{\xi}} \right\} \\ &+ \delta^{cd} \frac{p_1^4}{\bar{m}^2 + p_1^2} \left(\delta_{\alpha\beta} - \frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right) \delta_{p_1,-p_2}.\end{aligned} \tag{5.50}$$

Note that for a constant v field the two-point correlator turns out to be diagonal in momentum space.

In summary, the two-point correlators obtained with an explicit v dependence, given by Eqs.(5.46) & (5.50) for a general gauge-fixing parameter, display both longitudinal and transversal contributions. However, in the Landau gauge, $\xi \to 0$, the longitudinal part decouples leaving only a purely transversal contribution. Remarkably, in the Landau gauge, the two-point correlator Eq.(5.46) becomes v independent and equal to the two-point correlator with an annealed disorder, Eq.(5.48), whereas the two-point correlator for a constant v field, Eq.(5.50), retains its v-field dependence.

5.4 Two-point correlation function of $W_{\textbf{source}}[A, v[A]]$

In our study, we initially considered a v field without any constraints up to the level of the Schwinger functional, where it entered through the source term contribution W_{source}, as given in Eq.(4.18). In order to gain a deeper understanding of the form and impact of this term on our theory, we subsequently imposed the condition of current conservation at various stages, as described in Secs. 5.1 & 5.3. This restriction limited the admissible v fields to those that satisfy the massive Klein-Gordon equation. In this section we will consider current conservation from the start, which entails a background-dependent $v[A]$ field. Using the same approach as before, we shall study the new form of the two-point correlator $W_v^{(2)}$ at vanishing background which is expected to differ from the one we determined in the preceding section.

A background dependent $v[A]$ field gives rise to an external current of the form

$$J^a_\mu[A, v[A]] = D^{ab}_\mu \left[\left(1 + \frac{\bar{m}^2_{\text{gh}}}{-D^2}\right) v[A]\right]^b \tag{5.51}$$

The functional derivative of the external current provides an additional contribution to the result of the previous section as found in Eq.(F.11),

$$\frac{\delta J[A, v[A]]}{\delta A} = \frac{\delta J[A, v]}{\delta A}\Big|_{v=v[A]} + N. \tag{5.52}$$

The second term comes due to the background dependence of the $v[A]$ field. At vanishing background N becomes

$$\left(N^{ac}_{\mu\alpha}\right)_{xy}\Big|_{A\to 0} = \partial^x_\mu \left(\frac{\delta \mathrm{v}^a_x}{\delta A^c_{\alpha y}}\right) - \bar{m}^2_{\text{gh}} \int_w \left(\frac{1}{\partial^2}\right)_{xw} \partial^w_\mu \left(\frac{\delta \mathrm{v}^a_w}{\delta A^c_{\alpha y}}\right), \tag{5.53}$$

where we have denoted

$$\mathrm{v}^a = v^a[A \to 0], \quad \frac{\delta \mathrm{v}^a}{\delta A^c_\alpha} = \frac{\delta v^a}{\delta A^c_\alpha}\Big|_{A\to 0}. \tag{5.54}$$

In order to compute the novel term N and hence the extra contributions to the two-point correlator, we first need to specify the form of $\frac{\delta \mathrm{v}}{\delta A}$. To do so, we make use of the current conservation condition,

$$(D^2 - \bar{m}^2_{\text{gh}}) v[A] = 0. \tag{5.55}$$

Functionally differentiating this relation with respect to A and then setting $A \to 0$, we reach to the following equation

$$\int_z \frac{\delta \left(D^2\right)^{ab}_{xz}}{\delta A^c_{\alpha y}}\Bigg|_{A\to 0} \mathrm{v}^b_z = (\partial^2_x - \bar{m}^2_{\text{gh}}) \left(\frac{\delta \mathrm{v}^a_x}{\delta A^c_{\alpha y}}\right). \tag{5.56}$$

Multiplying both sides with the inverse of the kernel of the massive Klein-Gordon operator results to the desired equation. Performing a Fourier transformation, we find that it reads in momentum space

$$\frac{\delta \mathrm{v}^a_q}{\delta A^c_{\alpha p}} = i\bar{g} f^{abc} \left(\frac{1}{\bar{m}^2_{\text{gh}} + q^2}\right) (2q - p)_\alpha \, \mathrm{v}^b_{q-p}. \tag{5.57}$$

Inserting it into Eq.(5.53) and then into Eq.(5.52) we determine the functional derivative of the external current at vanishing background for a background-dependent $v[A]$ field,

$$\frac{\delta J^a_{\mu q}}{\delta A^c_{\alpha p}}\Bigg|_{A\to 0} = \bar{g} f^{abc} \frac{q_\mu \left(2q - p\right)_\alpha}{q^2} \left[1 - \frac{p^2 - 2p\cdot q}{\bar{m}^2_{\text{gh}} + q^2}\right] \mathrm{v}^b_{q-p}. \tag{5.58}$$

5.4.1 $v[A]$-dependent two-point correlator

Having computed all the essential elements, we insert them into the two-point correlator, which is given by the same relation as before, Eq.(5.42). After a straightforward computation we find that

$$\begin{aligned}\left(W_v^{(2)}\right)^{cd}_{\alpha\beta}(p_1,p_2) =&\delta^{cd}\frac{p_1^2}{\bar{m}^2+p_1^2}\left(p_1^2\delta_{\alpha\beta}-p_{1\alpha}p_{1\beta}\right)\delta_{p_1,-p_2}\\ &+\bar{g}^2 f^{abc}f^{aed}\int_q\left(\frac{1}{\bar{m}^2+\frac{q^2}{\xi}}\right)\left(\frac{1}{q^2}\right)(2q+p_1)_\alpha(2q-p_2)_\beta\\ &\times\left[\left(\frac{1}{\bar{m}^2_{\text{gh}}+q^2}\right)^2\left(p_1^2+2p_1\cdot q\right)\left(p_2^2-2p_2\cdot q\right)\right.\\ &\left.-\frac{(p_2^2-2p_2\cdot q)}{\bar{m}^2_{\text{gh}}+q^2}+\frac{(p_1^2+2p_1\cdot q)}{\bar{m}^2_{\text{gh}}+q^2}+1\right]\mathrm{v}^b_{q-p_2}\mathrm{v}^e_{-q-p_1}.\end{aligned} \tag{5.59}$$

The background-dependence resulted in the appearance of extra terms in the two-point function in relation to the one we found in the preceding section, cf. Eq.(5.46), as a consequence of the consistency condition. These are the terms which depend on the $\bar{m}_{\text{gh}}$ in the square brackets. Therefore in the limit of large ghost mass, $\bar{m}_{\text{gh}} \to \infty$, we recover the background independent result. Finally, let us remark that the structure of the two-point correlator is the same as the one in the background independent case.

5.4.2 $v[A]$-independent two-point correlator

Similarly to the previous section, let us average over the $v[A]$ field in the two-point correlator, Eq.(5.59), under the constraint of current conservation. The structure of the two-point correlator, computed in this section and the one found in Eq.(5.46) have a similar structural form. For the computation we will require the same Gaussian integrals as before, i.e. Eqs.(G.8) & (G.10). Substituting them we find

$$\langle W_v^{(2)}\rangle^{cd}_{\alpha\beta}(p_1,p_2) = \delta^{cd}\frac{p_1^4}{\bar{m}^2+p_1^2}\left(\delta_{\alpha\beta}-\frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right)\delta_{p_1,-p_2}. \tag{5.60}$$

Thus, after averaging over the $v[A]$ field on the level of the two-point function, one discovers the same result for the two-point function regardless of the assumption of a background-dependent or background-independent v field.

5.4.3 Two-point correlator for a localized $v[A]$ field

Let us conclude this section with a computation of the two-point correlator for a constant $v[A]$ field solution, cf. Eq.(5.49). In that case, we find that

$$\begin{aligned}\left(W_v^{(2)}\right)^{cd}_{\alpha\beta}(p) =&\bar{g}^2 f^{abc}f^{aed}v^b v^e\delta_{p_1,-p_2}\left\{\left[\delta_{\alpha\beta}-\frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right]\frac{1}{\bar{m}^2+p_1^2}+\frac{p_{1\alpha}p_{1\beta}}{p_1^2}\frac{2}{\bar{m}^2+\frac{p_1^2}{\xi}}\right\}\\ &+\ \delta^{cd}\frac{p_1^4}{\bar{m}^2+p_1^2}\left(\delta_{\alpha\beta}-\frac{p_{1\alpha}p_{1\beta}}{p_1^2}\right)\delta_{p_1,-p_2},\end{aligned} \tag{5.61}$$

which up to a constant factor of 2 in the longitudinal part, agrees with the background-independent result, Eq.(5.50). In the Landau gauge these results agree in full since in that case the longitudinal contribution decouples.

Different Approaches			
Section	5.1	5.3	5.4
Background (A)	$D_\mu F_{\mu\nu} = 0$	$A \to 0$	$A \to 0$
v-field constraint	$\left(D^2 - \bar{m}^2_{\text{gh}}\right) v = 0$	$\left(\partial^2 - \bar{m}^2_{\text{gh}}\right) v = 0$	$\left(D^2 - \bar{m}^2_{\text{gh}}\right) v[A] = 0$
$M^{-1}[A]$	Eq.(5.4)	Eq.(5.44)	Eq.(5.44)
$W_{\text{source}}[A, v]$	0	0	0
$W_v^{(2)}(p_1, p_2)$	–	Eq.(5.46)	Eq.(5.59)
$\langle W_v^{(2)} \rangle (p_1, p_2)$	–	Eq.(5.48)	Eq.(5.60)
$W_v^{(2)}(p)$	–	Eq.(5.50)	Eq.(5.61)

Table 5.1: Summary of the approaches adopted in the evaluation of the new source-like contributions to the background connected correlation functions, associated to the special gauge-fixing sector constructed in Sec. 4.1. These contributions depend on two external fields: the background gluon field A, and the Nakanishi-Lautrup field v, which have been chosen as specified in the second and third rows respectively. Notice that the $v[A]$ of the third column differs from the v of the second column, in that we consider the v field as a functional of A *before* taking functional derivatives w.r.t. an arbitrary A. The main actor in the evaluation of the source-like contributions is the gluon propagator M^{-1}, for which we refer to the corresponding explicit expression. Under each assumption considered, we report the contributions to the zero-point function $W_{\text{source}}[A]$ and to the two-point function $W_{\text{v}}^{(2)}(p_1, p_2)$. Special forms of the latter have been obtained, either by averaging over the v field with a constrained Gaussian distribution, or by assuming v to be constant. The corresponding results are respectively recalled in the last two rows. The empty entries in the lower-left corner correspond to computations of the two-point correlator in non-vanishing backgrounds and are left for future studies.

In Tab. 5.1 we have summarized all different approaches and assumptions that were considered in this chapter for the study of the novel source term in the Schwinger functional W_{source} with their corresponding sections.

CHAPTER 6

fRG Study of BRST-Invariant Flows

As motivated in Sec. 3.5, this chapter extends the work of [121] to the computation of the one-loop beta function and the model developed within the BFM in Chpts. 4 & 5 to a non-perturbative setting. Consequently, in the forthcoming parts, initially we shall abandon the BFM and analyze the implications of the associated fRG equation and then continue this non-perturbative study within the BFM. In both cases the generated flows will be compatible with BRST symmetry.

6.1 Non-linear gauge fixing in the fRG

In order to transition the non-linear gauge-fixing condition to the fRG setup, it is natural to promote the mass parameters, initially introduced in Eq.(4.2) in the BFM, to IR regulator kernels each associated with the fields to be regularized. In particular,

$$\begin{aligned} \bar{m}^2\delta_{\mu\nu} &\to R_{\mu\nu}(\partial), \\ \bar{m}^2_{\text{gh}} &\to R_{\text{gh}}(\partial) = (-\partial^2)r_{\text{gh}}(-\partial^2). \end{aligned} \tag{6.1}$$

Next, we consider the following non-linear gauge-fixing condition

$$\mathcal{F}^a[A,v] = A^b_\mu \mathrm{Q}^{abc}_{\mu\nu} A^c_\nu + \mathcal{L}^{ab}_\mu A^b_\mu, \tag{6.2}$$

where,

$$\begin{aligned} \mathrm{Q}^{abc}_{\mu\nu} &= \frac{v^a}{2|v|^2}\mathrm{Q}_{\mu\nu}\delta^{bc}, \quad \mathrm{Q}_{\mu\nu} = R_{\mu\nu}(\partial) - \frac{1}{\xi}\partial_\mu\partial_\nu, \\ \mathcal{L}^{ab}_\mu &= \left(1 + r_{\text{gh}}(-\partial^2)\right)\partial_\mu\delta^{ab}. \end{aligned} \tag{6.3}$$

Given that $R_{\mu\nu}$ is a symmetric tensor and an even differential operator, a possible choice of its form reads [121]

$$R^{\mu\nu}(\partial) = R_{\text{L}}(-\partial^2)\Pi^{\mu\nu}_{\text{L}} + R_{\text{T}}(-\partial^2)\Pi^{\mu\nu}_{\text{T}}. \tag{6.4}$$

Note that $R_{\text{gh,T,L}}$ are the associated IR regulators that appear in the flow equation, exhibiting a behavior in momentum space as described in Subsec. 3.4.2.

The gauge-fixing action for the non-linear gauge-fixing condition Eq.(6.2) is given by

$$S_{\text{gf}}[A,v] = \frac{1}{2} A_\mu^a \mathrm{Q}_{\mu\nu} A_\mu^a + v^a \left(1 + r_{\text{gh}}(-\partial^2)\right) \partial_\mu A_\mu^a, \tag{6.5}$$

whereas the associated ghost action reads

$$S_{\text{gh}}[A,c,\bar{c},v] = -\,\bar{c}^a \left(1 + r_{\text{gh}}(-\partial^2)\right) (\partial_\mu D_\mu c)^a - \frac{v^a}{2|v|^2} \bar{c}^a \left[\left(\mathrm{Q}_{\mu\nu} A_\nu^b \right) (D_\mu c)^b + A_\mu^b \mathrm{Q}_{\mu\nu} \left(D_\nu c\right)^b \right]. \tag{6.6}$$

Next, we extend even further the source action, cf. Eq.(3.59), by introducing further BRST sources coupled to appropriate composite operators [121], denoted as

$$\begin{aligned} \Omega_\mu^a &= \frac{v^b}{|v|^2} \bar{c}^b A_\mu^a, \\ \mathcal{A}_\mu^a &= \frac{v^b}{|v|^2} \bar{c}^b \left(D_\mu c\right)^a. \end{aligned} \tag{6.7}$$

Then, the source action takes the form

$$\begin{aligned} S_{\text{sou}} &= \mathcal{J}_i^\dagger \Phi_i + S_{\text{so}}^{\text{BRST}} \\ &= \mathcal{J}_i^\dagger \Phi_i + \mathrm{K}_\mu^a \left(\mathrm{s}A\right)_\mu^a + \mathrm{L}^a \left(\mathrm{s}c\right)^a + \mathrm{I}_\mu^a \Omega_\mu^a - \mathrm{M}_\mu^a \mathcal{A}_\mu^a. \end{aligned} \tag{6.8}$$

Note that $\{\mathrm{I}_\mu^a, \mathrm{K}_\mu^a\}$ correspond to anticommuting sources whereas $\{\mathrm{L}^a, \mathrm{M}_\mu^a\}$ to commuting ones. Furthermore, the BRST variation of the composite operators infers that they are both nilpotent and the difference $(\mathrm{s}\Omega)_\mu^a = A_\mu^a - \mathcal{A}_\mu^a$ is BRST exact and as such, $\{A_\mu^a, \mathcal{A}_\mu^a\}$ belong in the same cohomology class. For compactness and following a logic similar to Eq.(3.47), we collect all BRST sources using the following collective notation

$$\mathcal{I}_i^\dagger = \left(\mathrm{K}_\mu^a, \mathrm{L}^a, \mathrm{M}_\mu^a, \mathrm{I}_\mu^a\right), \quad \mathcal{I}_i = \begin{pmatrix} \mathrm{K}_\mu^a \\ \mathrm{L}^a \\ \mathrm{M}_\mu^a \\ \mathrm{I}_\mu^a \end{pmatrix}. \tag{6.9}$$

Then, in the collective field representation we obtain the following generating functional

$$\mathcal{Z}[\mathcal{J},\mathcal{I};v] = \mathrm{e}^{W[\mathcal{J},\mathcal{I};v]} = \int \mathcal{D}\Phi \; \mathrm{e}^{S[\Theta] - S_{\text{sou}}[\mathcal{I},v]}, \tag{6.10}$$

where, $S[\Theta]$ corresponds to the BRST-invariant nonlocal action which contains the gauge-fixing and the ghost actions Eqs.(6.5) & (6.6), i.e.

$$S[\Theta] = S_{\text{YM}}[A] + S_{\text{gf}}[A,v] + S_{\text{gh}}[A,c,\bar{c},v]. \tag{6.11}$$

As it was anticipated in our previous study within the BFM, we inferred a nonlocal but BRST-invariant action. This result highlights the trade-off between manifest BRST invariance and non-locality on the level of the action. In the fRG setup, the inclusion of regulators through the gauge-fixing condition can result in non-trivial regulator-dependent interactions, e.g. in the ghost-gluon vertex and as such their effect merits further investigation.

In our approach, a unique feature is the introduction of additional BRST sources of composite operators. This is motivated by our requirement to establish BRST invariance on the level of the EAA and serves a dual purpose. Firstly, their inclusion in the generating functional is essential for deriving a one-loop flow equation (Wetterich-like) [136]. Secondly, the extra sources aid in monitoring BRST symmetry on the level of the associated EAA, as encoded by the master equation, by bringing it in a compact form. Then, with an appropriate truncation of the EAA one can verify the compatibility between the flow and the master equation, thereby ensuring BRST-invariant flows, cf. Sec. 3.5.

6.2 BRST-invariant flow equation

Next, we sketch the derivation of the flow equation which arises within our framework, see [121] for a more detailed derivation. We denote by $\tilde{\Gamma}$, the Legendre transform of the regularized Schwinger functional Eq.(6.10), which we call *Legendre EA*

$$\tilde{\Gamma}[\Phi, \mathcal{I}; v] = \sup_{\mathcal{J}_i} \Big\{ \mathcal{J}_i^{\dagger} \Phi_i - W[\mathcal{J}, \mathcal{I}; v] \Big\}, \tag{6.12}$$

as opposed to Γ which represents the EAA and is obtained after subtracting the regulator dependence in accordance with Eq.(3.80), i.e.

$$\Gamma[\Phi, \mathcal{I}; v] = \tilde{\Gamma}[\Phi, \mathcal{I}; v] - \Delta S[\Phi; v]. \tag{6.13}$$

Furthermore, ΔS incorporates both the gluon and ghost regulator dependencies as they contribute to the gauge-fixing and ghost sectors,

$$\Delta S[\Phi; v] = \Delta S_{\text{gf}}[A; v] + \Delta S_{\text{gh}}[\Phi; v], \tag{6.14}$$

where

$$\begin{aligned} \Delta S_{\text{gf}}[A; v] &= \frac{1}{2} A_\mu^a R_{\mu\nu} A_\nu^a + v^a r_{\text{gh}}(-\partial^2) \partial_\mu A_\mu^a, \\ \Delta S_{\text{gh}}[\Phi; v] &= -\bar{c}^a r_{\text{gh}}(-\partial^2) \left(\partial_\mu D_\mu c\right)^a - \frac{v^a}{2|v|^2} \bar{c}^a \left[\left(R_{\mu\nu} A_\nu^b \right) \left(D_\mu c\right)^b + A_\mu^b \left(R_{\mu\nu} D_\nu c\right)^b \right]. \end{aligned} \tag{6.15}$$

Making use of the functional relations for standard sources and fields, Eq.(3.54), one can determine useful relations between the Hessians of the Schwinger functional and Legendre EA, abbreviated with the following shorthand notation

$$\begin{aligned} W^{(2)}_{\mathcal{J}_i \mathcal{J}_j} &= \frac{\delta}{\delta \mathcal{J}_i^{\dagger}} W[\mathcal{J}, \mathcal{I}; v] \frac{\overleftarrow{\delta}}{\delta \mathcal{J}_j}, \\ \tilde{\Gamma}^{(2)}_{\Phi_i \Phi_j} &= \frac{\delta}{\delta \Phi_i^{\dagger}} \tilde{\Gamma}[\Phi, \mathcal{I}; v] \frac{\overleftarrow{\delta}}{\delta \Phi_j}. \end{aligned} \tag{6.16}$$

Therefore, $\{W^{(2)}, \tilde{\Gamma}^{(2)}\}$ can be thought of as matrix operators with entries classified by the associated functional differentiation. Hence, one can easily derive a relation between their diagonal elements. In particular,

$$\begin{aligned} W^{(2)}_{\mathcal{J}_i\mathcal{J}_j} &= \left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i\Phi_j}, \\ W^{(2)}_{\mathcal{I}_i\mathcal{I}_j} &= -\tilde{\Gamma}^{(2)}_{\mathcal{I}_i\mathcal{I}_j}. \end{aligned} \tag{6.17}$$

For the non-diagonal elements of the Hessians, the following slightly non-tivial relations hold,

$$\begin{aligned} W^{(2)}_{\mathcal{J}_i\mathcal{I}_j} &= -\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i\Phi_k}\tilde{\Gamma}^{(2)}_{\Phi_k\mathcal{I}_j}, \\ W^{(2)}_{\mathcal{I}_i\mathcal{J}_j} &= -\tilde{\Gamma}^{(2)}_{\mathcal{I}_i\Phi_k}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_k\Phi_j}. \end{aligned} \tag{6.18}$$

Having associated all relevant Hessian components, the form of the flow equation can be determined after some algebraic manipulations, by considering the scale derivative of the generating functional, Eq.(6.10) and translating the result in terms of the Schwinger functional. Then, one finds

$$\begin{aligned} \partial_t W =& \frac{1}{2}\left(\partial_t R_{\mu\nu}\delta^{ab}\right)\left[\frac{\delta W}{\delta \mathrm{M}^a_\mu}\left(W\frac{\overleftarrow{\delta}}{\delta j^b_\nu}\right) + W^{(2)}_{\mathrm{I}^a_\mu \mathrm{K}^b_\nu} + \frac{\delta W}{\delta \mathrm{I}^a_\mu}\left(W\frac{\overleftarrow{\delta}}{\delta \mathrm{K}^b_\nu}\right) + W^{(2)}_{\mathrm{I}^a_\mu \mathrm{K}^b_\nu} - \frac{\delta W}{\delta j^a_\mu}\left(W\frac{\overleftarrow{\delta}}{\delta j^b_\nu}\right) - W^{(2)}_{j^a_\mu j^b_\nu}\right] \\ &+ \left(\partial_\mu r_{\mathrm{gh}}\partial_\mu\delta^{ab}\right)\left[v^b\frac{\delta W}{\delta j^a_\mu} - \frac{\delta W}{\delta \mathrm{K}^a_\mu}\left(W\frac{\overleftarrow{\delta}}{\delta \eta^b}\right) - W^{(2)}_{\mathrm{K}^a_\mu \eta^b}\right]. \end{aligned} \tag{6.19}$$

Relating the Schwinger functional to the Legendre EA, Eq.(6.12), its first functional derivative to the associated dynamical macroscopic fields, Eq.(3.54) and its Hessian to that of the Legendre EA, Eqs.(6.17) & (6.18), one arrives at the flow equation of the Legendre EA

$$\begin{aligned} \partial_t\tilde{\Gamma} =& \frac{1}{2}\left(\partial_t R_{\mu\nu}\delta^{ab}\right)\left[\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu A^b_\nu} + \tilde{\Gamma}^{(2)}_{\mathrm{M}^a_\mu\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i A^b_\nu} + \tilde{\Gamma}^{(2)}_{\mathrm{K}^b_\nu \mathrm{I}^a_\mu} + \frac{\delta\tilde{\Gamma}}{\delta \mathrm{M}^a_\mu}A^b_\nu - \left(\frac{\delta}{\delta \mathrm{K}^b_\nu}\tilde{\Gamma}\right)\left(\tilde{\Gamma}\frac{\overleftarrow{\delta}}{\delta \mathrm{I}^a_\mu}\right)\right] \\ &+ \left(\partial_\mu r_{\mathrm{gh}}\partial_\mu\delta^{ab}\right)\left[\tilde{\Gamma}^{(2)}_{\mathrm{K}^a_\mu\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i(-\bar{c}^b)} - \bar{c}^b\frac{\delta\tilde{\Gamma}}{\delta \mathrm{K}^a_\mu}\right] + \partial_t\Delta S_{\mathrm{gf}}. \end{aligned} \tag{6.20}$$

Note that according to the condensed notation, cf. App. F, differential operators acting on δ^{ab} imply differentiation with respect to x_a. Eq.(6.20) can be rewritten in terms of the EAA, Eq.(6.13) and yields a Wetterich-like flow equation [121]. The last term denotes the scale derivative of the regulator dependent part in the gauge-fixing sector, given in Eq.(6.15) and reads as

$$\partial_t\Delta S_{\mathrm{gf}} = \frac{1}{2}\left(\partial_t R_{\mu\nu}\delta^{ab}\right)A^a_\mu A^b_\nu + \left(\partial_t r_{\mathrm{gh}}\partial_\mu\delta^{ab}\right)v^b A^a_\mu. \tag{6.21}$$

To further computational manipulations, we require a suitable class of truncations for the Legendre EA that reproduces flows of the associated functional, compatible with the BRST symmetry. As determined in [121], such a behavior for the Legendre EA can be realized in a truncation scheme where the BRST sources, collectively denoted by $\mathcal{I}$ and found in Eq.(6.8), contribute in a linear fashion similar to the bare action, i.e.

$$\tilde{\Gamma}_k[\Phi,\mathcal{I};v] = \tilde{\Gamma}[\Phi;v] + S^{\mathrm{BRST}}_{\mathrm{sou}}[\Phi,\mathcal{I};v]. \tag{6.22}$$

Then, Eq.(6.22) can straightforwardly be translated in the language of the truncated EAA, which takes the form

$$\Gamma_k[\Phi,\mathcal{I};v] = \Gamma[\Phi;v] + S^{\text{BRST}}_{\text{sou}}[\Phi,\mathcal{I};v]. \tag{6.23}$$

For this family of truncations, the flow equation for the BRST source-independent part of the truncated EAA simplifies to

$$\begin{aligned}\partial_t\Gamma[\Phi;v] =& \frac{1}{2}\left(\partial_t R_{\mu\nu}\delta^{ab}\right)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu A^b_\nu} + \frac{1}{2}\left(\partial_t R_{\mu\nu}\delta^{ab}\right)\tilde{\Gamma}^{(2)}_{\mathrm{M}^a_\mu\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i A^b_\nu} \\ &+ \left(\partial_t r_{\text{gh}}\partial_\mu\delta^{ab}\right)\tilde{\Gamma}^{(2)}_{\mathrm{K}^a_\mu\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i(-\bar{c}^b)}.\end{aligned} \tag{6.24}$$

Let us turn our attention to the compatibility of the aforementioned truncated flow equation with the (modified) master equation of the theory. Due to the inclusion of the additional BRST sources of composite operators, BRST symmetry is encoded in a scale-dependent constraint equation of the following form

$$\mathcal{S}[W] = \left(j^a_\mu\frac{\delta}{\delta\mathrm{K}^a_\mu} + \bar{\eta}^a\frac{\delta}{\delta\mathrm{L}^a} + v^a - \mathrm{M}^a_\mu\frac{\delta}{\delta\mathrm{K}^a_\mu} + \mathrm{I}^a_\mu\frac{\delta}{\delta j^a_\mu} - \mathrm{I}^a_\mu\frac{\delta}{\delta\mathrm{M}^a_\mu}\right)W = \tilde{\mathcal{G}}_{\text{BRST}}W = 0. \tag{6.25}$$

The compatibility between the flow equation and the (modified) master equation, was proven in [121] and translates on the level of the generating functional as

$$\partial_t\mathcal{S}[W] = \mathcal{G}_t\mathcal{S}[W], \tag{6.26}$$

where $\mathcal{G}_t$ corresponds to the generator of the RG transformations of the Schwinger functional, Eq.(6.19). One can verify the compatibility relation, Eq.(6.26), by noting that the generator of the RG transformations and the BRST generator, commute with each other, i.e.

$$[\tilde{\mathcal{G}}_{\text{BRST}},\mathcal{G}_t] = 0. \tag{6.27}$$

Thus, the compatibility relation describes the flow of the associated master equation and incorporates information both for the flow equation and the underlying BRST symmetry. In particular, it denotes that the (modified) master equation $\mathcal{S}[W] = 0$ is a fixed point of its flow equation and as such it is satisfied at all scales by a Schwinger functional that solves the associated flow equation, cf. Eq.(6.19), subject to the requirement that at an initial scale $k = \Lambda$ the master equation is met. Such compatibility statement can be straightforwardly generalized to the Legendre EA (and in extend to all other associated quantities), in the (modified) master equation,

$$\mathcal{S}[\tilde{\Gamma}] = \frac{\delta\tilde{\Gamma}}{\delta A^a_\mu}\frac{\delta\tilde{\Gamma}}{\delta\mathrm{K}^a_\mu} + \frac{\delta\tilde{\Gamma}}{\delta c^a}\frac{\delta\tilde{\Gamma}}{\delta\mathrm{L}^a} + v^a\frac{\delta\tilde{\Gamma}}{\delta\bar{c}^a} + \mathrm{M}^a_\mu\frac{\delta\tilde{\Gamma}}{\delta\mathrm{K}^a_\mu} + A^a_\mu\mathrm{I}^a_\mu + \frac{\delta\tilde{\Gamma}}{\delta\mathrm{M}^a_\mu}\mathrm{I}^a_\mu = 0. \tag{6.28}$$

In this section, we have introduced various quantities that are related to the EA. To summarize, in Eq.(6.12), we defined the Legendre transform of the Schwinger functional, which we called Legendre EA and denoted by $\tilde{\Gamma}$. The Legendre EA was associated with the EAA, denoted by Γ in Eq.(6.13). Note that due to the gauge-fixing procedure, scale dependence, introduced via the regulators, enters from the non-linear gauge-fixing condition, Eq.(6.2). This implies that both the Legendre EA as well as the EAA are scale-dependent quantities and as such one can determine a flow equation for each one, cf. Eq.(6.20) for the flow equation of the Legendre EA. Furthermore, we introduced truncated versions of both quantities as realized in Eqs.(6.22) & (6.23) and denoted by $\tilde{\Gamma}_k$ & Γ_k for the truncated Legendre EA and the truncated EAA respectively, which have corresponding truncated flow equations, cf. Eq.(6.24) for the flow equation of the truncated EAA. This truncated flow equation for the EAA in combination with the vertex functions, dictate the contributing terms in the associated flows of the coupling at vanishing macroscopical fields.

6.3 Renormalized flow and simulated conventional gauges

Having chosen an appropriate truncation scheme, Eq.(6.22), for the Legendre EA and in extension for the EAA, Eq.(6.23), it is left to specify the form of their BRST-independent truncated parts. We use the ansatz that they take a form identical to the bare action. Thus, the BRST source-independent part of the truncated Legendre EA reads

$$\tilde{\Gamma}[\Phi; v] = Z_{\mathrm{T}}\, S_{\mathrm{YM}}[A] + \tilde{\Gamma}_{\mathrm{gf}}[A; v] + \tilde{\Gamma}_{\mathrm{gh}}[\Phi; v], \tag{6.29}$$

where, from Eqs.(6.5) & (6.6)

$$\begin{aligned} \tilde{\Gamma}_{\mathrm{gf}}[A; v] &= \frac{1}{2} A^a_\mu \mathrm{Q}_{\mu\nu} A^a_\nu + v^a Z_{\mathrm{gh}} \left(1 + r_{\mathrm{gh}}(-\partial^2)\right) \partial_\mu A^a_\mu, \\ \tilde{\Gamma}_{\mathrm{gh}}[\Phi; v] &= -Z_{\mathrm{gh}} \bar{c}^a \left(1 + r_{\mathrm{gh}}(-\partial^2)\right) (\partial_\mu D_\mu c)^a - \frac{v^a}{2|v|^2} \bar{c}^a \left[\left(\mathrm{Q}_{\mu\nu} A^b_\nu \right) (D_\mu c)^b + A^b_\mu \mathrm{Q}_{\mu\nu} (D_\nu c)^b \right]. \end{aligned} \tag{6.30}$$

Note that wave-function renormalization constants have been appropriately included. To further the renormalization procedure and reproduce the renormalized BRST source-independent truncated Legendre EA (and EAA) with a form similar to the renormalized bare action, it is necessary for the differential operator $\mathrm{Q}_{\mu\nu}$ to depend on the longitudinal and transversal wave-function renormalization constants. This can be achieved by performing a rescaling,

$$\begin{aligned} R_{\mathrm{T}} &\to Z_{\mathrm{T}} R_{\mathrm{T}}, \quad \xi \to \frac{\xi}{Z_{\mathrm{L}}}, \\ R_{\mathrm{L}} &\to \frac{Z_{\mathrm{L}}}{\xi} R_{\mathrm{L}}, \quad r_{\mathrm{gh}} \to Z_{\mathrm{gh}} r_{\mathrm{gh}}. \end{aligned} \tag{6.31}$$

Then, the BRST source-independent part of the truncated EAA

$$\Gamma[\Phi; v] = Z_{\mathrm{T}}\, S_{\mathrm{YM}}[A] + \Gamma_{\mathrm{gf}}[A; v] + \Gamma_{\mathrm{gh}}[\Phi; v], \tag{6.32}$$

consists of

$$\begin{aligned} \Gamma_{\mathrm{gf}}[A; v] &= \frac{Z_{\mathrm{L}}}{2\xi} \left(\partial_\mu A^a_\mu\right)^2 + Z_{\mathrm{gh}} v^a \partial_\mu A^a_\mu, \\ \Gamma_{\mathrm{gh}}[\Phi; v] &= -Z_{\mathrm{gh}} \bar{c}^a (\partial_\mu D_\mu c)^a + \frac{Z_{\mathrm{L}} v^a}{2\xi |v|^2} \bar{c}^a \left[\left(\partial_\mu \partial_\nu A^b_\nu \right) (D_\mu c)^b - A^b_\mu \partial_\mu \partial_\nu (D_\nu c)^b \right]. \end{aligned} \tag{6.33}$$

As it is convenient to proceed with the calculations within a certain gauge by specifying the gauge-fixing parameter ξ, an identification of the corresponding gauge from the gauge-fixing parameter is not guaranteed due to the non-linear gauge-fixing condition, Eq.(6.2). In the following, we present a prescription that allows us to identify the commonly used gauges, i.e. Feynman and

Landau gauge, within our framework.

One way to identify the Feynman or Landau gauge in the non-linear gauge-fixing condition is to specify the gauge-fixing parameter ξ that yields the expected form of the associated gluon propagator. In the Feynman and Landau gauge within the Lorenz gauge-fixing condition, the gluon propagator has certain properties that allow for a different approach. For instance, in the Landau gauge, the gluon propagator is purely transversal. Thus, instead of specifying a value for ξ, we can characterize gauges based on the form of the gluon propagator. This leads to a constraint relation which unambiguously identifies the gauge within the non-linear gauge-fixing condition, but can involve a larger number of parameters. Therefore, the key ingredient in this construction is to note the properties of the gluon propagator at the underlying gauges.

Hence, from Eq.(6.29) we find that in momentum space, the kernels of the transversal and longitudinal gluon propagators correspond to

$$G_{\mathrm{T}}(p) = \frac{1}{Z_{\mathrm{T}}\left(R_{\mathrm{T}}(p) + p^2\right)}, \quad G_{\mathrm{L}}(p) = \frac{1}{\frac{Z_{\mathrm{L}}}{\xi}\left(R_{\mathrm{L}}(p) + p^2\right)}. \tag{6.34}$$

Landau gauge

In the Landau gauge, the gluon propagator is purely transversal, which according to Eq.(6.34) corresponds to the choice

$$\xi = 0, \quad R_{\mathrm{T}}(p^2) = R_{\mathrm{L}}(p^2). \tag{6.35}$$

Therefore, we find that we can reproduce the Landau gauge of the Lorenz gauge-fixing condition within our non-linear gauge-fixing condition by choosing appropriate constraint relations.

Feynman gauge

In the Feynman gauge, the longitudinal and transversal parts of the gluon propagator are equal. According to Eq.(6.34), this holds true for

$$Z_{\mathrm{T}} = \frac{Z_{\mathrm{L}}}{\xi} = Z, \quad R_{\mathrm{T}}(p^2) = R_{\mathrm{L}}(p^2), \tag{6.36}$$

Similarly, Eq.(6.36) corresponds to the generic constraint which allows us to simulate the conventional Feynman gauge of the Lorenz gauge-fixing condition within our non-linear gauge-fixing condition. Furthermore, for $\xi = 1$ we obtain the special solution which was used in [121]. We proceed by employing the generic Feynman gauge constraint relations.

We reparametrize the gluon and ghost regulator according to Eq.(3.86)

$$R(p^2) = p^2 r(p^2), \quad r_{\mathrm{gh}}(p^2) = r(p^2), \tag{6.37}$$

for which we choose the Litim regulator shape function, Eq.(3.87). Finally, in $d = 4$ for the Feynman gauge we define the dimensionless renormalized couplings and fields as

$$g^2 = \frac{\bar{g}^2}{Z}, \quad \tilde{v}^a = \frac{Z_{\mathrm{gh}}\bar{g}}{Zk^2}v^a. \tag{6.38}$$

6.4 Beta function from BRST-invariant flows

We have reached a stage in our analysis where we can implement our chosen truncation scheme in the flow equation and extract the flow of $\{Z_{\mathrm{T}}, Z_{\mathrm{gh}}, g\}$ by studying different vertex interactions. These flows are generated from one-loop diagrams with regulator-dependent vertices that may receive contributions from the v field, as prescribed by the truncated fRG equation, Eq.(6.20). The momentum space representation of the truncated fRG equation as well as all relevant n-point vertex functions are detailed in App. H.

Our goal is to obtain the beta function, generated by BRST-invariant flows, in the presence of regulator-dependent vertex interactions and a fixed external/background v field. Furthermore, we aim to examine the impact of the v field on the renormalizability of the theory and whether universal results can be reproduced by properly treating it. Such an investigation can facilitate more involved v-field calculations and provide insights on the consistency of the theory.

In the course of our calculations, we must handle external and loop momenta contributions carefully. To do so in accordance with the considered truncation scheme, we assume that the external momenta are small compared to loop momenta [124] and collect from the non-vanishing contributions terms of the same order as those on the left side of the associated flow equation. This momentum expansion is also expected to affect the regulator-dependent terms. Note that such novel features (regulator-dependent vertices) serve as a check on whether the Litim regulator is a valid choice for the description of our model. With these considerations in mind, we proceed to study the one-loop generated contributions to the gluon and ghost anomalous dimensions, as well as the ghost-gluon vertex.

In order to provide a more holistic and intuitive depiction of the contributing terms in the associated flow equations, we introduce a Feynman diagrammatic representation within our framework. This representation allows us to easily identify contributions from all relevant v-field sectors, including the v-independent, v-dependent, and v^2-dependent sectors. An overview of all non-vanishing diagrams that contribute in the associated flow equations, are depicted in Fig. 6.1.

In particular, Fig. 6.1 illustrates the non-vanishing contributions of the transversal wave-function renormalization Z_{T}, which determines the transversal gluon anomalous dimensions η_{T}, the ghost renormalization Z_{gh}, which determines the ghost anomalous dimension η_{gh}, and the $\bar{c}cA$-vertex. The BRST source and regulator-dependent character of the truncated EAA introduces novel vertices, which can be found in App. H.2 and are diagrammatically represented according to the following notation. Regulator dependencies can appear both in the gluon and ghost internal lines as $\{\partial_t R_{\mu\nu}, \partial_t r_{\mathrm{gh}}\}$ respectively, as well as on vertices and are displayed with a crossed circle. Vertices denoted with an empty circle or a full square are obtained by acting with the $\{\mathrm{M}, \mathrm{K}\}$ BRST sources respectively. Full dots represent conventional 3 & 4-gluon vertices. Finally, the v field has been treated as a fixed background/external field and as such it can contribute to the form of vertices. Therefore, vertices with an anchored field indicate the presence of a v field. The symmetry factor which appears in front of some graphs corresponds to an exchange of the external gluons legs.

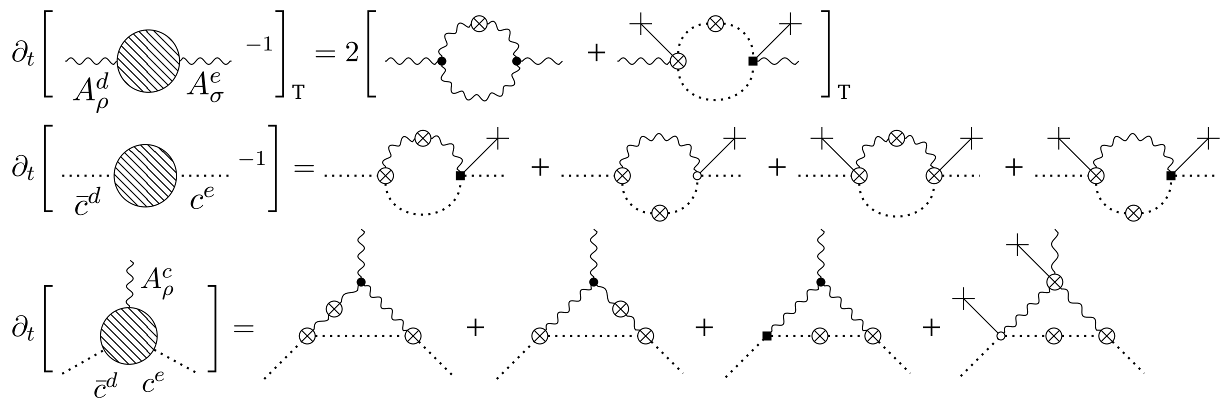

Figure 6.1: Diagrammatic representation of all non-vanishing contributions which arise from the truncated flow equation, Eq.(6.24).

6.4.1 Gluon anomalous dimension

In the study of the wave-function renormalization, we observe that only a subset of graphs are non-vanishing. Several generated contributions either cancel, vanish after projecting onto external momenta or identically drop as a result of the considered truncation. All vanishing diagrammatic contributions can be found in Fig. 6.2. Then, one arrives at the following flow equation

$$(\partial_t Z_{\mathrm{T}})\,\delta^{ab} = Z_{\mathrm{T}} g^2 2\mathrm{v}_4 \left[\frac{1}{6}\left(19 - 5\eta_{\mathrm{T}}\right) N_c \delta^{ab} + \frac{\tilde{v}^a \tilde{v}^b}{4|\tilde{v}|^4}\frac{1}{3}\left(5 - \frac{\eta_{\mathrm{gh}}}{2}\right)\right], \tag{6.39}$$

where we have used the abbreviation $\mathrm{v}_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)$. The color index notation on both sides indicates that also the right side still has to be projected onto the terms proportional to δ^{ab}. Outside of the v contribution, notice the explicit dependence of the right side on the anomalous dimensions

$$\eta_{\mathrm{T}} = -\partial_t \ln Z_{\mathrm{T}}, \quad \eta_{\mathrm{gh}} = -\partial_t \ln Z_{\mathrm{gh}}, \tag{6.40}$$

which originate from a resummation of a large class of diagrams, manifesting the "RG improvement". Ignoring any resummation contributions, our result agrees with the perturbative one obtained in [121].

6.4.2 Ghost anomalous dimension

Following the same line of reasoning, we arrive at the flow equation for Z_{gh}

$$\left(\partial_t Z_{\mathrm{gh}}\right)\delta^{ab} = Z_{\mathrm{gh}} g^2 2\mathrm{v}_4 \left[f^{abc}\frac{\tilde{v}^c}{2|v|^2}\frac{3}{4} + \frac{\tilde{v}^a\tilde{v}^b}{4|v|^4}\frac{1}{8}\left(32 - \frac{109}{30}\eta_{\mathrm{T}} + \frac{19}{10}\eta_{\mathrm{gh}}\right)\right]. \tag{6.41}$$

The vanishing contributions of the flow of Z_{gh} are represented in Fig. 6.3. It is worth mentioning that due to the antisymmetry of the structure constants and the cancellation that occurs between different diagrams, all v-dependent graphs with a fixed v anchored to the external antighosts vanish. Furthermore, all v-independent contributions drop out within our truncation.

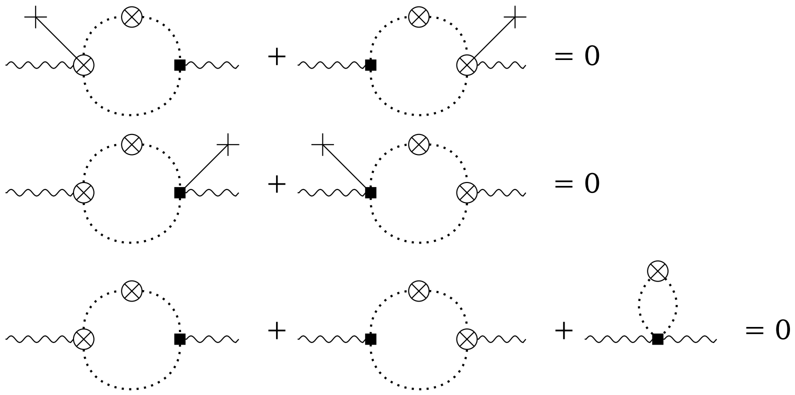

Figure 6.2: Vanishing contributions from the gluon anomalous dimension.

6.4.3 Ghost-gluon vertex

The study of the ghost-gluon vertex turns out to be more intricate and a careful treatment is required in order to extract the associated flow equation. The relevant graphs that contribute in the flow equation are determined by taking appropriate functional derivatives of the truncated flow equation. In the case of the ghost-gluon vertex this is achieved by computing

$$\frac{\delta}{\delta A^c_\rho(q_3)}\frac{\delta}{\delta \bar{c}^d(q_1)}\partial_t\Gamma\frac{\overleftarrow{\delta}}{\delta c^e(q_2)}\Bigg|_{\Phi\to 0} = \frac{\delta}{\delta A^c_\rho(q_3)}\frac{\delta}{\delta \bar{c}^d(q_1)}\,[\text{R.H.S. of Eq.(H.1)}]\,\frac{\overleftarrow{\delta}}{\delta c^e(q_2)}\Bigg|_{\Phi\to 0}\,. \tag{6.42}$$

Due to energy-momentum conservation, we find an overall $\delta(q_1+q_2+q_3)$, which imposes the condition $q_1+q_2+q_3=0$. This constraint fixes only one external momentum in terms of the other two, leading to an ambiguity when expanding in external momenta. Since we have to expand up to q_1^3, then we are left with two possible choices for the remaining momenta. One is to keep q_2 fixed and expand around $q_1=-q_3=q\to 0$, or fix q_3 and expand around $q_1=-q_2=q\to 0$. The latter leads to divergent graphs that do not cancel whereas the former results in convergent and vanishing graphs Fig. 6.4. Note that such issues do not arise in the case of the gluon and ghost anomalous dimensions since in that case energy conservation fixes the external momenta uniquely as $q_1=-q_2=q$. This serves as the motivation for adopting the scheme where we fix the ghost external momentum.

Thus from the ghost-gluon vertex, we arrive at the following flow equation

$$f^{abc}\partial_t\Big(gZ_{\text{gh}}\sqrt{Z_{\text{T}}}\Big) = \frac{g^3}{2}Z_{\text{gh}}\sqrt{Z_{\text{T}}}2\text{v}_4\left[f^{abc}N_c\left(2-\frac{5\eta_{\text{T}}}{8}\right)-f^{adc}\frac{\tilde{v}^b\tilde{v}^d}{4|v|^4}\left(1-\frac{\eta_{\text{T}}}{6}\right)\right]. \tag{6.43}$$

Notice that the flow equation exhibits non-vanishing contributions up to $\mathcal{O}(v^2)$ order. However, similarly to the BFM case such novel contributions do not give rise to any divergences within a suitable scheme.

Prior to deriving the one-loop beta function it is of essence to test the validity of the derived flow equations, Eqs.(6.41)-(6.43), by rediscovering established results in the perturbative limit.

6.4.4 Rediscovering perturbative results

The confidence in the aforementioned flow equations within our considered truncated framework can be put to the test by reproducing the associated perturbative results. To achieve such a

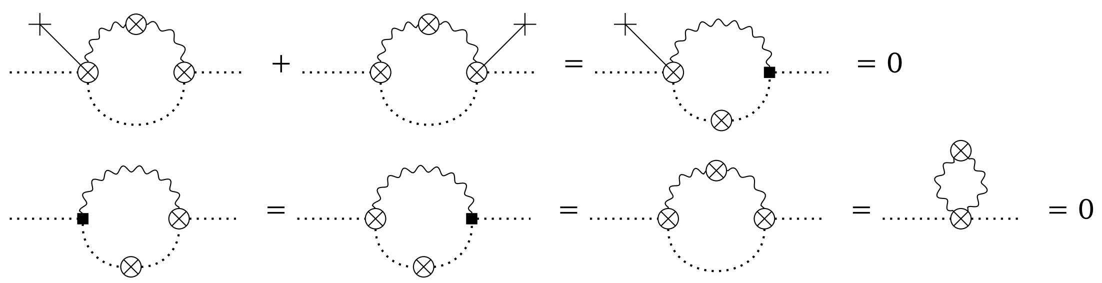

Figure 6.3: Vanishing contributions from the ghost anomalous dimension.

comparison, we start from a modified version of the truncated EAA and repeat the same procedure as before. In particular, the form of the truncated EAA is determined as follows. Firstly we strip-off the regulator dependence from the vertex functions, next we neglect any generated RG-resummation terms from the flow equations and finally we replace the v-field contribution appropriately.

In order to obtain regulator-independent vertices, it is sufficient to set in Eq.(6.3)

$$\mathrm{Q}_{\mu\nu}(p^2)\big|_{R=0} = Z p_\mu p_\nu. \tag{6.44}$$

The effect of such a replacement in the ghost-gluon vertex is depicted in Fig. 6.5. Furthermore, neglecting any RG resummation terms is equivalent to setting $\eta_{\mathrm{T}} = \eta_{\mathrm{gh}} = 0$ on the right side of the generated flows when such terms appear. Tuning the v field appropriately is an important step in this comparison, since it appears in the flow equations as the product of the gauge-fixing procedure. For that, we replace $-\tilde{v}^a\tilde{v}^b/\left(4|\tilde{v}|^4\right)$ by $\delta^{ab}g^2N_c$. Such a procedure, reproduces the perturbative ghost-loop contribution in the Feynman gauge [121, 124].

Note that Eq.(6.44), directly affects the form of the truncated EAA. This, in turn, generates a different class of diagrams when inserted into the fRG equation. Therefore, a naive implementation of the aforementioned perturbative conditions on the already derived flow equations will not accurately reproduce the perturbative limit. To achieve this, one needs to first input the perturbative conditions and subsequently compute the associated flows.

At the perturbative limit, the flow of Z_{gh} generates a single one-loop diagram, cf. Fig. 6.5. Then, one finds the perturbative ghost anomalous dimension

$$\eta^0_{\mathrm{gh}} = \frac{N_c}{(4\pi)^2}g^2 + \mathcal{O}(g^4), \tag{6.45}$$

which agrees with current literature [124].

6.4.5 Beta function

Our final task is to determine the YM beta function within our BRST-invariant framework. Combining all flows Eqs.(6.41)-(6.43), one finds that

$$\begin{aligned}\beta(g)\delta^{ab} = -g^3 2\mathrm{v}_4 \Bigg[&\frac{1}{12}\left(7 - \frac{5}{4}\eta_{\mathrm{T}}\right)N_c\delta^{ab} + \frac{\tilde{v}^a\tilde{v}^b}{4|\tilde{v}|^4}\frac{1}{6}\left(29 + \frac{37}{40}\eta_{\mathrm{gh}} - \frac{109}{40}\eta_{\mathrm{T}}\right)\\ &+ \frac{\tilde{v}^d\tilde{v}^e}{4|\tilde{v}|^4}\frac{f^{acd}f^{bce}}{2N_c}\left(1 - \frac{\eta_{\mathrm{T}}}{6}\right)\Bigg].\end{aligned} \tag{6.46}$$

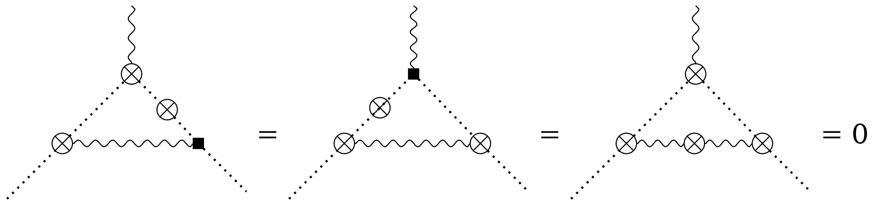

Figure 6.4: Depiction of vanishing graphs when q_2 is kept fixed that diverge in the case of a fixed q_3 from the ghost-gluon vertex.

Note that terms of the order $\mathcal{O}(v)$ in the ghost-gluon vertex were neglected, due to computational complexity and since in our scheme such contributions will drop out after averaging over the v field.

As the v field corresponds to an external/background field, it can be treated appropriately. Thus, we choose to integrate out the v field with a Gaussian weight of an adjusted width of the following form

$$\frac{\tilde{v}^a \tilde{v}^b}{4|\tilde{v}|^4} \to \frac{37}{64} N_c \delta^{ab}. \tag{6.47}$$

Interpreting the v field as a quenched disorder field, Eq.(6.47) fixes the disorder amplitude. This leads to the beta function

$$\beta(g) = -\frac{g^3 N_c}{16\pi^2} \left(\frac{11}{3} - \frac{6373}{15360} \eta_{\mathrm{T}} + \frac{37}{1536} \eta_{\mathrm{gh}} \right). \tag{6.48}$$

Combining the flow equations in Eqs.(6.39) & (6.41), one can find an analytic expression for the resummation contributions in the one-loop beta function in different powers of the coupling, cf. Eq.(I.1). Expanding these expressions to lowest order and inserting the result into Eq.(6.48), one arrives at an estimate for the two-loop beta function

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} N_c + \frac{11}{500} N_c^2 g^2 \right) + \mathcal{O}(g^7), \tag{6.49}$$

which to lowest order in the coupling reproduces the universal one-loop beta function.

Note that in order to arrive at Eq.(6.48), we had to adjust the v-field contribution by appropriately choosing the width of a Gaussian distribution. Tuning the width of the Gaussian weight corresponds to an additional degree of freedom. Different widths are expected to affect differently the underlying flow equations and correspondingly the beta function. However, universal results, such as the one-loop YM beta function in $d = 4$ alleviate this ambiguity and fix this seemingly extra degree of freedom in a unique manner.

One would naively anticipate the substitution performed for the v field in Subsec. 6.4.4 to persist in the derivation of Eq.(6.48). Such a reasoning would not take into account that we ad hoc eliminated the regulator dependence from the vertices in the perturbative limit. Such a consideration, reproduces the same diagrammatic contributions as in the perturbative case, but alters the form of the truncated EAA. In this modified truncation scheme, the specific substitution of the v field, as discussed in Subsec. 6.4.4 is sufficient to reproduce the universal one-loop beta function. However, in the case where the truncated EAA is given by Eq.(6.32), the flow equation gives rise to an extended class of diagrams which differ from those at the perturbative limit exactly due to the presence of regulator-dependent vertices. In that case, selecting a Gaussian weight

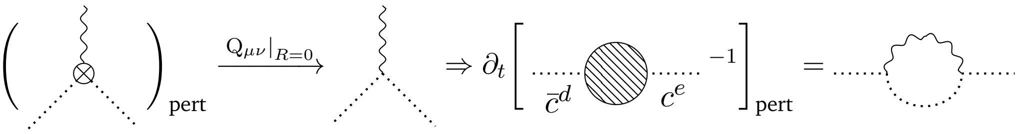

Figure 6.5: Graphical depiction of the vertex and diagrammatic modifications for the ghost anomalous dimension.

with a width dictated by Eq.(6.47) reproduces the universal one-loop beta function.

In other words, our formalism seems to provide an extra degree of freedom with the inclusion of a novel v field. Such a condition is constrained and uniquely determined by associated universal results, unless there are additional divergences. Therefore, we can draw a correspondence between changes in such a condition and novel contributions within our formalism.

At two-loop level, even though the beta function Eq.(6.49) is in qualitative agreement with literature results obtained both perturbatively and within mass-dependent renormalization schemes, the quantitative comparison is not favorable [143]. To improve the results of the associated flow equation, two approaches can be considered. Firstly, an appropriate modification of the selected truncation scheme can be made to include a larger class of contributions, thereby potentially enhancing the accuracy at higher-loop orders. Secondly, an alternative framework for our non-perturbative model, such as the BFM, can be adopted. The former approach extends the admissible diagrammatic contributions of the previously established flow equation Eq.(6.20), while the latter can alter the form of the flow equation and is explored in Sec. 6.5.

6.5 Background & BRST-invariant RG flows

We close this Chapter by deriving an explicit BRST-invariant flow equation in the BFM. To achieve this, we apply the BFM as introduced in Chpt. 4, within the previously developed non-perturbative setup, for which we derive the associated fRG equation. Given that the forthcoming construction is quite close to the one followed in Secs. 4.1, 6.1 & 6.2, to avoid repetition, we refer the reader to the relevant relations and only highlight any minor modifications as necessary.

We commence our background-field analysis by constructing a background-invariant gauge-fixing condition. Motivated by Sec. 4.1, this can be achieved by replacing the mass parameters in Eqs.(4.1) & (4.2) with regulators, as in Eq.(6.1), that now depend on the background covariant derivatives/Laplacian. Then, the background-invariant gauge-fixing condition corresponds to

$$\mathcal{F}^a[a, \bar{A}, v] = a^b_\mu \bar{\mathrm{Q}}^{abc}_{\mu\nu} a^c_\nu + \bar{\mathcal{L}}^{ab}_\mu a^b_\mu, \tag{6.50}$$

where,

$$\begin{aligned} \bar{\mathrm{Q}}^{abc}_{\mu\nu} &= \frac{v^a}{2|v|^2}\bar{\mathrm{Q}}^{bd}_{\mu\nu}\delta^{dc}, \quad \bar{\mathrm{Q}}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu}(-\bar{D}^2) - \frac{1}{\xi}\left(\bar{D}_\mu \bar{D}_\nu\right)^{ab}, \\ \bar{\mathcal{L}}^{ab}_\mu &= \left(1 + r_{\mathrm{gh}}(-\bar{D}^2)\right)^{ac} \bar{D}^{cb}_\mu . \end{aligned} \tag{6.51}$$

Note that due to background invariance, the regulators can be represented as non-diagonal matrices in color space.

The background gauge-fixing action is given by

$$S_{\mathrm{gf}}[a, \bar{A}, v] = \frac{1}{2} a^a_\mu \bar{\mathrm{Q}}^{ab}_{\mu\nu} a^b_\mu + v^a \left(1 + r_{\mathrm{gh}}(-\bar{D}^2)\right)^{ab} (\bar{D}_\mu a_\mu)^b, \tag{6.52}$$

whereas the background ghost action reads

$$S_{\text{gh}}[a,\bar{A},c,\bar{c},v] = -\,\bar{c}^a\left(1+r_{\text{gh}}(-\bar{D}^2)\right)^{ab}\left(\bar{D}_\mu D_\mu c\right)^b - \frac{v^a}{2|v|^2}\bar{c}^a\left[\left(\bar{\mathrm{Q}}_{\mu\nu}a_\nu\right)^b\left(D_\mu c\right)^b + a_\mu^b\left(\bar{\mathrm{Q}}_{\mu\nu}D_\nu c\right)^b\right]. \tag{6.53}$$

As expected, the background gauge fixing and ghost actions are associated with the previously determined Eqs.(6.5) & (6.6) respectively by replacing the full gauge field with the field fluctuations, $A_\mu^a \to a_\mu^a$ and the partial derivative with the background covariant derivative, $\partial_\mu \to \bar{D}_\mu^{ab}$. This will be the recipe that shall connect most background quantities with the ones without background.

Further, in order to ensure the compatibility of the generated background flow equation with the BRST symmetry as in Eq.(6.26), we introduce additional BRST sources of composite operators of the form

$$\begin{aligned}\Omega_\mu^a &= \frac{v^b}{|v|^2}\bar{c}^b a_\mu^a,\\ \mathcal{A}_\mu^a &= \frac{v^b}{|v|^2}\bar{c}^b\left(D_\mu c\right)^a.\end{aligned} \tag{6.54}$$

Similarly to Eq.(6.7), we find that the composite operators are nilpotent with $(\mathrm{s}\Omega)_\mu^a = a_\mu^a - \mathcal{A}_\mu^a$, which implies that $\{a_\mu^a, \mathcal{A}_\mu^a\}$ belong in the same cohomological class.

Introducing the collective field notation for the BRST sources as in Eq.(6.9), while replacing into Eq.(3.47) the entry for full gauge field with its field fluctuations, we arrive at the background Legendre EA and EAA that are of the same form as in Eqs.(6.12) & (6.13) respectively. The gauge fixing and ghost sectors of the regulators, in the presence of a background field take the form

$$\begin{aligned}\Delta S_{\text{gf}}[a,\bar{A};v] &= \frac{1}{2}a_\mu^a R_{\mu\nu}^{ab}a_\nu^b + v^a r_{\text{gh}}^{ab}(-\bar{D}^2)\left(D_\mu a_\mu\right)^b,\\ \Delta S_{\text{gh}}[\Phi,\bar{A};v] &= -\bar{c}^a r_{\text{gh}}^{ab}(-\bar{D}^2)\left(\bar{D}_\mu D_\mu c\right)^b - \frac{v^a}{2|v|^2}\bar{c}^a\left[\left(R_{\mu\nu}a_\nu\right)^b\left(D_\mu c\right)^b + a_\mu^b\left(R_{\mu\nu}D_\nu c\right)^b\right].\end{aligned} \tag{6.55}$$

Now we are in a position to derive the background flow equation for the Legendre EA

$$\begin{aligned}\partial_t\tilde{\Gamma} =& \frac{1}{2}\left(\partial_t R_{\mu\nu}^{ab}\right)\left[\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{a_\mu^a a_\nu^b} + \tilde{\Gamma}^{(2)}_{\mathrm{M}_\mu^a\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i a_\nu^b} + \tilde{\Gamma}^{(2)}_{\mathrm{K}_\nu^b\mathrm{I}_\mu^a} + \frac{\delta\tilde{\Gamma}}{\delta\mathrm{M}_\mu^a}a_\nu^b - \left(\frac{\delta}{\delta\mathrm{K}_\nu^b}\tilde{\Gamma}\right)\left(\tilde{\Gamma}\frac{\overleftarrow{\delta}}{\delta\mathrm{I}_\mu^a}\right)\right]\\ &+ \left(\partial_t r_{\text{gh}}\bar{D}_\mu\right)^{ab}\left[\tilde{\Gamma}^{(2)}_{\mathrm{K}_\mu^b\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i(-\bar{c}^a)} - \bar{c}^a\frac{\delta\tilde{\Gamma}}{\delta\mathrm{K}_\mu^b}\right] + \partial_t\Delta S_{\text{gf}},\end{aligned} \tag{6.56}$$

where

$$\partial_t\Delta S_{\text{gf}} = \frac{1}{2}\left(\partial_t R_{\mu\nu}^{ab}\right)a_\mu^a a_\nu^b + \left(\partial_t r_{\text{gh}}\bar{D}_\mu\right)^{ab}v^b a_\mu^a. \tag{6.57}$$

Choosing a truncation linear to the BRST sources for the background Legendre EA and EAA, as in Eqs.(6.22) & (6.23) respectively, we arrive at the truncated background flow equation for the BRST source-independent part of the EAA,

$$\begin{aligned}\partial_t\Gamma[\Phi,\bar{A};v] =& \frac{1}{2}\left(\partial_t R_{\mu\nu}^{ab}\right)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{a_\mu^a a_\nu^b} + \frac{1}{2}\left(\partial_t R_{\mu\nu}^{ab}\right)\tilde{\Gamma}^{(2)}_{\mathrm{M}_\mu^a\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i a_\nu^b}\\ &+ \left(\partial_t r_{\text{gh}}\bar{D}_\mu\right)^{ab}\tilde{\Gamma}^{(2)}_{\mathrm{K}_\mu^b\Phi_i}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\Phi_i(-\bar{c}^a)}.\end{aligned} \tag{6.58}$$

Similarly to Eq.(6.29), we consider the BRST source-independent part of the background truncated Legendre EA and EAA to be of the form of the bare background action with gauge fixing and ghost parts according to Eqs.(6.52) & (6.53) respectively. Thus, we have

$$\begin{aligned}\tilde{\Gamma}_{\text{gf}}[a,\bar{A};v] &= \frac{1}{2}a^a_\mu \bar{\mathrm{Q}}^{ab}_{\mu\nu} a^b_\nu + v^a Z_{\text{gh}}\left(1+r_{\text{gh}}(-\bar{D}^2)\right)^{ab}\left(\bar{D}_\mu a_\mu\right)^b,\\ \tilde{\Gamma}_{\text{gh}}[\Phi,\bar{A};v] &= -Z_{\text{gh}}\bar{c}^a\left(1+r_{\text{gh}}(-\bar{D}^2)\right)^{ab}\left(\bar{D}_\mu D_\mu c\right)^b - \frac{v^a}{2|v|^2}\bar{c}^a\left[\left(\bar{\mathrm{Q}}_{\mu\nu}a_\nu\right)^b\left(D_\mu c\right)^b + a^b_\mu\left(\bar{\mathrm{Q}}_{\mu\nu}D_\nu c\right)^b\right].\end{aligned} \tag{6.59}$$

However, we are interested in the form of the flow equation in the limit where $\bar{A}=A$ and $\Phi_i\to 0$ after taking the associated functional derivatives. Furthermore, due to the linear dependence of the chosen truncation on the BRST sources, the non-vanishing vertex interactions that appear in Eq.(6.58) and involve BRST sources, have the same field dependence as the one presented in App. H, with the distinction of replacing $A^a_\mu \to a^a_\mu$ when needed. This implies that $\tilde{\Gamma}^{(2)}_{\mathrm{M}\Phi}\Big|_{\Phi\to 0}=0$ which eliminates the second term from the first line in the truncated background flow equation. Then, we find that

$$\partial_t\Gamma[A] = \frac{1}{2}\left(\partial_t R^{ab}_{\mu\nu}\right)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu A^b_\nu} + \left(\partial_t r_{\text{gh}} D^{ab}_\mu\right)\tilde{\Gamma}^{(2)}_{\mathrm{K}^b_\mu c^c}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{c^c(-\bar{c}^a)}. \tag{6.60}$$

Recalling that $\tilde{\Gamma}^{(2)}_{\mathrm{K}^b_\mu c^c} = D^{bc}_\mu$, Eq.(6.60) becomes

$$\partial_t\Gamma[A] = \frac{1}{2}\left(\partial_t R^{ab}_{\mu\nu}\right)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu A^b_\nu} - \left(\partial_t R^{ab}_{\text{gh}}\right)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\bar{c}^a c^b}, \tag{6.61}$$

where we have used that $R_{\text{gh}}(-D^2) = (-D^2)r_{\text{gh}}(-D^2)$. This reduced flow equation displays an identical form as the conventional background flow equation for pure YM theory found in the literature [199]. Furthermore, it does not exhibit any v-field dependence, in agreement with the background one-loop perturbative study that was carried in Chpt. 4.

In the derivation of Eq.(6.61), other than the linear dependence of the truncated Legendre EA on the BRST sources, cf. Eq.(6.23), we did not make use of the ansatz Eq.(6.59). We can further exploit Eq.(6.59) in order to align our result with the one found in the literature. In particular, performing the rescaling

$$R^{ab}_{\mu\nu}\to Z R^{ab}_{\mu\nu},\quad r^{ab}_{\text{gh}}(-\bar{D}^2)\to Z_{\text{gh}}\, r^{ab}_{\text{gh}}(-\bar{D}^2),\quad \xi\to\frac{\xi}{Z}, \tag{6.62}$$

one finds inverse regularized gluon and ghost propagators of the form

$$G^{-1}_{AA} = Z\,(\mathfrak{D}_{\mathrm{T}}+R),\quad G^{-1}_{\bar{c}c} = Z_{\text{gh}}\left(-D^2+R_{\text{gh}}\right). \tag{6.63}$$

Upon insertion into Eq.(6.61) and further specification of the gluon regulator as $R=\mathfrak{D}_{\mathrm{T}}\, r(\mathfrak{D}_{\mathrm{T}})$ and $Z_{\text{gh}}=1$, one reproduces the same background flow equation which has been established to reproduce the universal one-loop beta function and is even in line, within a small margin of error, with the associated two-loop result [199].

CHAPTER 7

Outlook and Optimization

In this thesis, we have dealt with BRST-invariant representations of pure quantum YM theory. The preservation of BRST symmetry is integral for the construction of non-Abelian field theories, as it has implications for the IR regime of the theory which is governed by non-perturbative effects. Consequently, through the lens of BRST invariance, our work has aimed to serve the purpose of both analyzing the stability of the model under perturbative and non-perturbative renormalization schemes and also performing a phenomenological study at different energy scales.

For our investigation, we have adopted the off-shell FP procedure as outlined in [121], which enabled us to derive a gauge-fixed bare action that exhibited a linear dependence on the gauge-fixing condition, at the expense of introducing an additional background/external v field. This was achieved during the gauge-fixing procedure, after choosing a Fourier weight for the noise action instead of the conventionally used Gaussian weight. In that way, we were able to study the stability and properties of the model, by introducing appropriate mass-regulator parameters in a BRST-invariant manner, thus bridging the chasm between perturbation theory and non-perturbative aspects of non-Abelian field theories, without deviating from the computationally accessible FP procedure.

Our perturbative study was carried out in the BFM, which greatly facilitates perturbative calculations. We then examined a modified version of the non-linear gauge-fixing condition proposed in [121] that includes gluon and ghost mass-regulator parameters and reduces to the conventional Landau-DeWitt gauge in their absence. From the background equations of motion we found that the v-field dependence acts as an external current and its conservation places a constraint on the class of admissible forms of the v field. Using perturbative functional analysis, we determined that the one-loop 1PI EA still comprehends the standard contribution of ghost and gluon loops, augmented however by the associated mass-regulator parameters, while the structural form of the Schwinger functional is v dependent as observed in an isolated source sector, W_{source}.

We proceeded to investigate the stability of our model under renormalization, where we deduced the universal one-loop beta function by employing dimensional regularization in the $\overline{\text{MS}}$ scheme. Additionally, we applied a proper-time regularization scheme to obtain an analytic form for the renormalized one-loop EA and the one-loop beta function. Our analysis revealed a threshold behavior for the beta function, which nevertheless reproduced the universal result in the deep UV limit and/or at vanishing mass parameters, as it should.

Based on the renormalized one-loop EA, we discovered that by introducing sufficiently large mass parameters, the Nielsen-Olesen instabilities arising from covariantly constant pseudo-Abelian magnetic backgrounds can be mitigated within a specific range of applicability. Additionally, it was found that quantum corrections progressively overtake classical contributions and dominate above a critical value of the coupling, indicating the tendency towards a non-trivial minimum. However, beyond this range of validity, the unstable gluonic modes destabilize our system thus rendering any concrete results inconclusive.

To avoid instabilities caused by these "tachyonic" modes, we utilized covariantly constant and self-dual backgrounds, which is a stable approximation for the background field. Within this approach, we incorporated non-perturbative information for the gluon and ghost propagators, allowing us to mimic the decoupling solution. As it turned out, this extended the domain of validity up to certain large field values, above which zero gluonic modes prevailed. However by introducing a suitable IR regulator, we demonstrated that the one-loop EA, supports a gluon condensate beyond a critical coupling. This result compares quite favorably with non-perturbative studies [146, 326] and is in agreement with the indications observed from the previous set of approximations.

Such computations provide valuable insights into the interpretation of the mass-regulator parameters. Given that these parameters enter through the gauge-fixing sector, they correspond to BRST-exact deformations of the classical action. Hence, one might expect the associated scale symmetry breaking to remain confined within the unphysical BRST-exact sector. However, as elaborated from the study of the one-loop EA, the dynamical breaking of scale symmetry is driven by quantum corrections which are structurally affected by the mass parameters. Therefore, we infer that the tree-level scale symmetry breaking that occurs in the BRST-exact sector due to the inclusion of the mass parameters manifests itself in the quantum corrections and as such propagates by radiative corrections to the physical sector of theory space. We can further draw a correspondence between our perturbative treatment of the non-linear gauge fixing condition and mass-dependent renormalization schemes. For the latter, such phenomena emerge as part of the inherent renormalization scale at the cost of BRST invariance, while in our case such a floating scale is replaced by BRST-invariant mass parameters in a BRST-respecting manner.

To our knowledge such a novel perturbative technique to investigate pure YM theory in a BRST-respecting manner has not been carried out in the literature and as such a higher-loop study that analyzes the structural form of the EA provides an intriguing optimization avenue for future research.

Having addressed the v-field independence of the one-loop EA and elaborated the role of the mass parameters in the BFM, we focused on the v-dependent part of the Schwinger functional in different settings. Firstly, we derived an analytic expression at the LGME limit which vanishes by assuming covariantly constant backgrounds and color current conservation, thus reproducing the conventional one-loop Schwinger functional.

Relaxing the aforementioned constraints, we followed a different approach where the associated background two-point correlator at vanishing backgrounds was computed by imposing color conservation at different stages of the calculation. In one approach, this constraint was imposed after varying the Schwinger functional with respect to the background field. In another, color conservation was assumed from the start, which entailed a background-dependent $v[A]$ field. In both cases, we deduced v-dependent correlation functions that were classified to appropriate longitudinal and transversal parts after averaging over the v-field contribution with a suitable Lagrange multiplier to account for current conservation or by considering a constant (i.e. homogeneous) v field as in [121].

Alternatively, we explored the potential emergence of divergences in the Schwinger functional due to the novel v-field contribution by treating it as a quenched disorder field. This allowed us to employ conventional heat-kernel techniques and investigate in detail the structural form of W_{source} to which we inferred no additional divergences. Our analysis supported the previously found finite form of the corresponding two-point correlator. Thus, our findings for W_{source} further substantiated our conclusion that at one-loop order no major interpretational novelties appear.

We then departed from the BFM framework and delved into the non-perturbative extension of the pure YM theory which has been developed in the literature. The transition from our previously discussed model to a non-perturbative study occurred by a mere replacement of the regulator-mass parameters with associated scale-dependent regulators which translates to the appearance of regulator-dependent vertices in our theory. In particular, the construction of a BRST-invariant functional RG equation was discussed, at the expense of introducing additional BRST sources. Choosing an appropriate truncation linear to the BRST sources, we extended the work of [121].

Deviating from the conventional approach in the literature, we generalized the conditions for the most usual gauges, i.e. Feynman and Landau gauge, within our truncation scheme. For the purposes of this thesis, we considered the Feynman gauge. As an internal consistency check, we introduced perturbative conditions by dropping the regulator dependencies on the vertices and ignoring higher-loop resummation terms. Then, the perturbative ghost anomalous dimension was rediscovered through an appropriate treatment of the v field. In this case, the v field was viewed as an auxiliary degree of freedom that upon integration reproduces expected or universal results. Even though such an interpretation seems abstract, further confidence on our results is gained by noting that following the same treatment for the v field within the perturbative limit reproduces the one-loop ghost loop and ghost anomalous dimension found in literature [121, 124]. Even though it has not been explicitly computed yet, we anticipate that a similar calculation for the ghost-gluon vertex in the perturbative limit will yield conventional results, thus after combination with the ones found before they shall reproduce the universal one-loop beta function in a perturbative setting.

After finding an analytic expression for the flow of the ghost, gluon renormalization factors and the ghost-gluon vertex from the truncated BRST-invariant flow equation, we adopted a suitable diagrammatic representation of the underlying interactions that includes the v-field dependence. Further algebraic manipulations resulted in a v-dependent beta function, which was brought into the universal one-loop beta function by averaging over the v field with a Gaussian weight of an adjusted width.

Further interesting conclusions can be inferred from the form of the beta function. In particular, the beta function depends on both the ghost and gluon anomalous dimensions which enter with an opposite sign. These resummation contributions combine in such a manner that they provide a two-loop beta function with the correct sign. In addition, the form of these RG-improved terms feature a beta function which is free from singularities. This is an improvement compared to earlier fRG studies where a pole necessarily results from RG resummation terms in low-order truncations [131].

In the last part of the thesis we introduced the BFM into our non-perturbative setup. By constructing a background-invariant non-linear gauge fixing condition, we derived the associated truncated background flow equation. From a structural perspective, the truncated background flow equation was found to be v independent with a conventional form. Such a flow equation leads to interesting conclusions. Initially, it extends the class of generated graphs due to resummation contributions. This implies that even beyond one loop and up to the extend specified by the background flow equation, our model remains v independent with a form that is found to reproduce universal results [199]. Furthermore, phenomena that were observed in our perturbative study but could not be verified with certainty due to the limitations of perturbation theory, are expected to manifest in our non-perturbative model.

Moving forward, a detailed analysis of our model and the study of emergent phenomenological properties would be a natural continuation of our work. Additionally, the coupling of our YM theory to matter presents an exciting opportunity for further investigation and study.

A critical view on our non-perturbative studies gives rise to the following question. How is it that the truncated flow equation in the absence of a background field exhibits an additional free parameter, as displayed with the appearance of the v field, whereas such a dependence does not arise in the truncated background flow equation?

Although we currently lack a definitive answer, one potential explanation for this discrepancy can be traced to the introduction of the new BRST sources. Recall that in order to derive a flow equation that is compatible with BRST symmetry, we introduced two additional sources $\{\Omega^a_\mu, \mathcal{A}^a_\mu\}$. In the absence of a background field, we found that $\Omega^a_\mu \propto A^a_\mu$. However, in the presence of a background field, $\Omega^a_\mu \propto a^a_\mu$, i.e. it is related to the field fluctuations. This modification induces a difference in the source action, which due to the selected truncation scheme, is reflected in the associated flow equation thus rendering their direct treatment on an equal footing inconsistent.

In that case, a possible resolution could be to introduce $\Omega^a_\mu - \bar{\Omega}^a_\mu \propto A^a_\mu$ instead of $\Omega^a_\mu \propto a^a_\mu$ in the source action and then compute the corresponding background flow equation. Preliminary calculations indicate that such an inclusion induces an explicit v-field dependent deformation of the gluon and ghost sectors in the background flow equation, which is in line with the observed behavior of the flow equation in the absence of a background field. However, further computations must be performed in order to deduce more conclusive results. Such an extension corresponds to another promising avenue for future research.

APPENDIX A

Covariantly Constant Backgrounds

Here, we mention the implications of considering covariantly constant backgrounds. This was an essential assumption in the computation of various quantities in the main body. We start by imposing the covariantly constant condition which reads

$$(D_\mu F_{\nu\rho})^a = 0. \tag{A.1}$$

If such a condition holds true, then the gauge field can be written as

$$A_\mu = n^a \left(t^a\right)_{bc} \left[-\frac{1}{2}\mathsf{F}_{\mu\nu} x_\nu - \frac{i}{g}\left(\partial_\mu U\right) U^{-1}\right], \tag{A.2}$$

which demonstrates that the gauge field is Abelian for covariantly constant fields, up to a pure gauge term. Therefore, we can always consider an Abelian gauge field by performing an appropriate gauge transformation which will eliminate the pure gauge contribution, [291]. The gauge field and subsequently the field strength tensor take the form

$$A^a_\mu = \hat{n}^a \mathsf{A}_\mu \Rightarrow F^a_{\mu\nu} = \hat{n}^a \mathsf{F}_{\mu\nu}, \tag{A.3}$$

where A_μ & $\mathsf{F}_{\mu\nu}$ correspond to the Abelian gauge field and field strength tensor.

At this level, we still have the freedom to consider an explicit form for the Abelian field strength tensor. In order to compare with current literature results, we choose to work with a field strength tensor proportional to a constant magnetic field and with a self-dual one, cf. Subsecs. 4.5.1 & 4.5.2. In the following we explore the implications of each set of assumptions. Within the BFM, such a specification on the form of the field strength facilitates perturbative calculations since only terms proportional to $F^2 = F^a_{\mu\nu} F^a_{\mu\nu}$ are relevant due to background gauge invariance.

A.1 Constant magnetic field

For our purposes it is convenient to choose

$$\mathsf{F}_{\mu\nu} = \partial_\mu \mathsf{A}_\nu - \partial_\nu \mathsf{A}_\mu = B\epsilon^{\perp}_{\mu\nu} = \text{constant}. \tag{A.4}$$

The gauge field then represents a constant magnetic field in the direction of the constant unit vector $\hat{n}^a$ in color space, $\hat{n}^a\hat{n}^a = 1$. Furthermore, $\epsilon^{\perp}_{\mu\nu}$ corresponds to the spatial directions affected by the constant magnetic field, i.e. $\mathsf{F}_{12} = -\mathsf{F}_{21} = B$ for a magnetic field pointing into the 3-direction.

Then, the Euclidean bare action, Eq.(4.15), can be brought into the form

$$S_{\text{v}}[A] = \frac{1}{2}B^2.$$

Due to this pseudo-Abelian field consideration, all color dependence is expected to come in the form of $\hat{n}^a\,(t^a)_{bc} = -i\hat{n}^a f^{abc}$, for $SU(N)$ in the adjoint representation. Thus, we consider ν_ℓ with $\ell = 1, ..., N^2-1$ to be the eigenvalues of the structure constants. For the contribution to the constant magnetic field, the following identity shall be used

$$\sum_{\ell=1}^{N^2-1} \nu_\ell^2 = N_c. \tag{A.5}$$

Finally, in terms of the eigenvalues ν_ℓ let us denote

$$B_\ell = gB\nu_\ell. \tag{A.6}$$

Such a diagonalization in color space enables us to compute the spectrum of a series of operators which constitutes an essential tool in the computation of functional traces using the heat-kernel technique, cf. App. B.2. During our study such functional traces appear in the the one-loop EA, Eq.(4.27). In particular, the spectrum of the ghost and gluon operators, cf. Eqs.(4.16) & (4.36), need to be specified. In $d = 4$, the relevant spectra required for their specification read [143],

$$\text{Spec}\left\{-D^2\right\} = q^2 + (2n+1)\,B_\ell, \quad n = 0, 1, ..., \tag{A.7}$$

$$\text{Spec}\left\{\mathfrak{D}_{\text{T}}\right\} = \begin{cases} q^2 + (2n+1)\,B_\ell, & \text{multiplicity } 2 \\ q^2 + (2n+3)\,B_\ell, & \text{multiplicity } 1 \\ q^2 + (2n-1)\,B_\ell, & \text{multiplicity } 1, \end{cases} \tag{A.8}$$

where q_μ corresponds to a 2-dimensional Fourier momentum in the spacetime directions not affected by the magnetic field and n is the quantum number which labels the Landau levels. Notice the existence of negative modes in the spectrum of the transversal kinetic operator, $\mathfrak{D}_{\text{T}}$, for small enough momenta for $n = 0$. These correspond to the Nielsen-Olesen modes, which are discussed in Sec. 4.5.

A.2 Self-dual field strength tensor

Another assumption for the field strength which we employed in the main text corresponds to a self-dual field strength tensor, $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$. This results in a covariantly constant self-dual background which is free from the unstable Nielsen-Olesen instabilities.

In order to reproduce self-duality of the field strength tensor, we choose the components of the Abelian field strength

$$\mathsf{F}_{\mu\nu} = \begin{cases} \mathsf{F}_{12} = \mathsf{F}_{34} = f = \text{constant} \\ \mathsf{F}_{ij} = 0, \quad \text{otherwise}, \end{cases} \tag{A.9}$$

where $f^2 = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$. In that case, it is straightforward to show that we obtain a self-dual field strength tensor, i.e. $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$, with $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma}$ representing the dual field strength.

Similarly to the constant magnetic field case, we diagonalize the color matrix contribution $-i\hat{n}^a f^{abc}$ in the adjoint representation and express it in terms of its eigenvalues ν_ℓ. This entails that the spectra of the operators, $-D^2, \mathfrak{D}_\mathrm{T}$, are in terms of

$$f_\ell = g f \nu_\ell. \tag{A.10}$$

Computing the spectra of the relevant operators in $d = 4$ gives [146]

$$\text{Spec}\left\{-D^2\right\} = 2\left(n+m+1\right) f_\ell, \quad n, m = 0, 1, \ldots \tag{A.11}$$

$$\text{Spec}\left\{\mathfrak{D}_\mathrm{T}\right\} = \begin{cases} 2\left(n+m+2\right) f_\ell, & \text{multiplicity } 2 \\ 2\left(n+m\right) f_\ell, & \text{multiplicity } 2. \end{cases} \tag{A.12}$$

Note that the spectrum of the transversal kinetic operator, $\mathfrak{D}_\mathrm{T}$, does not exhibit negative modes but it contains zero modes for $n = m = 0$. When heat-kernel techniques are implemented, the contribution of these zero modes requires an independent and careful treatment, as followed in Subsec. 4.5.2.

APPENDIX B

Computation of Functional Traces

Computation of functional traces using heat-kernel techniques has been extensively used in QFT for perturbative and non-perturbative calculations. As a result, in several Sections of the main text, such functional traces appear. The goal of this Appendix is to provide a step by step treatment of dealing with such quantities and to highlight the computational power of the heat kernel in performing analytical calculations. Even though the integrals to be discussed correspond to the ones that appear in our theory, the different steps provided can be generalized to more generic forms of the heat-kernel operators.

B.1 Elementary relations

Euler-Mascheroni Constant:

$$\gamma_{\mathrm{E}} = 0.57721\ldots . \tag{B.1}$$

Step function:

$$\theta(-z) = \int_0^{\infty} \mathrm{d}z\ \delta(z) = \begin{cases} 1 & \text{for } z < 0 \\ \frac{1}{2} & \text{for } z = 0 \\ 0 & \text{for } z > 0 \end{cases} . \tag{B.2}$$

ψ-function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = (\ln \Gamma(z))' . \tag{B.3}$$

ϵ-expansion:

$$\alpha^{-\epsilon} = 1 - \epsilon \ln \alpha + \frac{1}{2} (\epsilon \ln \alpha)^2 + \mathcal{O}\left(\epsilon^3\right) \tag{B.4}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \frac{1}{2}\left(\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{6}\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) \tag{B.5}$$

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + (\gamma_{\mathrm{E}} - 1) + \left(-\frac{1}{2}\gamma_{\mathrm{E}}^2 - \frac{\pi^2}{12} - 1 + \gamma_{\mathrm{E}}\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) \tag{B.6}$$

$$\Gamma(\epsilon - n) = \frac{(-1)^n}{n!}\left\{\frac{1}{\epsilon} + \psi\left((n+1) + \frac{1}{2}\left[\frac{\pi^2}{3} + (\psi(x+1))^2 - \psi'(n+1)\right]\epsilon + \mathcal{O}\left(\epsilon^2\right)\right\} \tag{B.7}$$

$$\zeta\left(\epsilon, \alpha\right) = \frac{1}{2} - \alpha + \epsilon\zeta'\left(0, \alpha\right) + \mathcal{O}\left(\epsilon^2\right). \tag{B.8}$$

Asymptotic behavior for $|z| \to 0$:

$$\frac{z}{\sinh z} = 1 - \frac{z^2}{6} + \mathcal{O}\left(z^4\right) \tag{B.9}$$

$$z \sinh z = z^2 + \mathcal{O}\left(z^4\right). \tag{B.10}$$

Asymptotic behavior for $|z| \to \infty$:

$$\ln \Gamma(z) \sim \left(z + \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{B.11}$$

$$\ln\left(z+1\right) = \ln z + \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{B.12}$$

$$\Gamma(0, z) = -\ln z - \gamma_{\mathrm{E}} - \sum_{k=1}^{\infty}\frac{(-1)^k}{k\,(k!)}z^k \tag{B.13}$$

$$\zeta'(0, z) = \ln\Gamma(z) + \zeta'(0) \overset{\text{(B.11)}}{=} \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right). \tag{B.14}$$

B.2 Functional traces with heat-kernel techniques

During the study of the one-loop EA and Schwinger functional, we are required to calculate different functional traces, see Secs. 4.4 & 5.2. The power of the heat-kernel formalism comes from relating the functional trace of the heat kernel with functional traces that are a function of the covariant Laplacian. This is achieved by computing the Laplace transform of the so called Schwinger or proper-time parameter s. The relation follows as

$$\mathrm{tr}\left[i(x)\right]\big|_{x=-D^2} = \int_0^\infty \mathrm{d}s\, \tilde{i}(s)\, \mathrm{tr}\left[\mathrm{e}^{-s\left(-D^2\right)}\right], \tag{B.15}$$

where the the trace of the covariant Laplacian which appears on the right side of the equation is a well-known quantity and

$$i(x) = \mathfrak{L}\left\{\tilde{i}(s)\right\} = \int_0^\infty \mathrm{d}s\, \tilde{i}(s)\, \mathrm{e}^{-sx}, \tag{B.16}$$

is the Laplace transformation of $i(x)$.

Next, our task is to compute the trace of the heat kernel. Thankfully, this has already been worked out extensively in the literature, thus we only need to find the appropriate formula of the trace of the heat kernel that is in accordance with the assumptions for the field configurations.

Trace of the heat kernel for a constant magnetic field

In our case, we have considered covariantly constant backgrounds, see App. A for more details. By tracing over the spectrum, the trace of the heat kernel has been worked out to be,

$$\mathrm{tr}_{\mathrm{xcL}}\left[\mathrm{e}^{-s(-D^2)}\right] = \Omega_d \sum_{\ell=1}^{N^2-1} \frac{2s^{-\frac{d}{2}}}{2\,(4\pi)^{\frac{d}{2}}} \frac{sB_\ell}{\sinh sB_\ell}, \tag{B.17}$$

where $\Omega_d = \int d^dx$ is the d-dimensional spacetime volume and $B_\ell = gB\nu_\ell$, [143].

For the gluon sector, in Sec. 4.4, we need to compute the functional trace of the logarithm of the gluonic fluctuation operator which is a function of the transversal kinetic operator Eq.(5.6), cf. Eq.(4.37). This entails, according to Eq.(B.17), the computation of the functional trace of the aforementioned operator. However this has also been worked out in [143], by tracing over the spectrum, yielding

$$\mathrm{tr}_{\mathrm{xcL}}\left[\mathrm{e}^{-s\mathfrak{D}_{\mathrm{T}}}\right] = \Omega_d \sum_{\ell=1}^{N^2-1} \frac{2s^{-\frac{d}{2}}}{2\,(4\pi)^{\frac{d}{2}}} \left(d\frac{sB_\ell}{\sinh sB_\ell} + 4sB_\ell \sinh sB_\ell\right). \tag{B.18}$$

In addition, we need to subtract from the functional traces the divergent zero-field contribution. This is achieved by subtracting the same functional trace at vanishing field configurations. In our case, it gives rise to the heat kernel of the d'Alembert operator, which can readily be determined to be

$$\mathrm{tr}_{\mathrm{x}}\left[\mathrm{e}^{-s(-\partial^2)}\right] = \Omega_d \frac{2s^{-\frac{d}{2}}}{2\,(4\pi)^{\frac{d}{2}}}. \tag{B.19}$$

Another advantage of using the heat-kernel technique is that the function, we wish to trace, can also contain nonlocal operators as long as they are functions of the covariant Laplacian, e.g. $\frac{1}{D^2}, \frac{1}{D^4}, \dots$. In that way, one can study the structural form of such terms in a straightforward manner. Such a study of the functional traces of terms with nonlocal contribution appears during our study of the one-loop Schwinger functional for a quenched field, cf. Eqs.(5.22a)-(5.22c). In the next part we shall focus our attention in the computation of the Fourier transformation of these novel nonlocal functional traces.

Trace of the heat kernel for a self-dual background

Considering the alternative choice of a self-dual background, then the trace of the heat kernel of the covariant Laplacian operator $-D^2$ equals to

$$\mathrm{tr}_{\mathrm{xc}}\left[\mathrm{e}^{-s(-D^2)}\right] = \frac{\Omega_d}{(4\pi)^{\frac{d}{2}}} \sum_{\ell=1}^{N^2-1} \left(\frac{f_\ell}{\sinh sf_\ell}\right)^{\frac{d}{2}}, \tag{B.20}$$

where $f_\ell = gf\nu_\ell$.

Incidentally, the trace of the heat-kernel of the $\mathfrak{D}_{\mathrm{T}}$ operator can be expressed in terms of the trace of the heat kernel of $-D^2$ and the contribution of the zero modes of its spectrum, see

Eq.(A.12), i.e.

$$\mathrm{tr}_{\mathrm{xcL}}\left[\mathrm{e}^{-s\mathfrak{D}_{\mathrm{T}}}\right] = \frac{d\,\Omega_d}{(4\pi)^{\frac{d}{2}}}\sum_{\ell=1}^{N^2-1}\left[\left(\frac{f_\ell}{\sinh s f_\ell}\right)^{\frac{d}{2}} + 2^{\frac{d}{2}-1} f_\ell^{\frac{d}{2}}\right]. \tag{B.21}$$

Laplace transformation

The task of this section is to determine the inverse Laplace transform of the functions inside the following functional traces:

$$\mathrm{tr}_{\mathrm{xc}}\left[\frac{1}{\bar{m}^2 - \frac{1}{\xi}D^2}D^2\right], \quad \mathrm{tr}_{\mathrm{xc}}\left[\frac{1}{D^2}\frac{1}{\bar{m}^2 - \frac{1}{\xi}D^2}D^2\right],$$
$$\mathrm{tr}_{\mathrm{xc}}\left[\frac{1}{D^2}\frac{1}{\bar{m}^2 - \frac{1}{\xi}D^2}D^2\right], \quad \mathrm{tr}_{\mathrm{xc}}\left[\frac{1}{D^2}\frac{1}{D^2}\frac{1}{\bar{m}^2 - \frac{1}{\xi}D^2}D^2\right].$$

The inverse Laplace transform of the functions inside the trace reads

$$h(x) = \frac{x}{x+\alpha} \Leftrightarrow \mathfrak{L}^{-1}\{h(x)\} = \tilde{h}(s) = -\alpha\,\mathrm{e}^{-\alpha s} + \delta(s), \tag{B.22}$$
$$g(x) = \frac{1}{x+\alpha} \Leftrightarrow \mathfrak{L}^{-1}\{g(x)\} = \tilde{g}(s) = \mathrm{e}^{-\alpha s}, \tag{B.23}$$
$$f(x) = \frac{1}{x(x+\alpha)} \Leftrightarrow \mathfrak{L}^{-1}\{f(x)\} = \tilde{f}(s) = \frac{1}{\alpha}\left[1 - \mathrm{e}^{-\alpha s}\right], \tag{B.24}$$

where $\mathfrak{L}\{...\}$ denotes the Laplace transformation. The validity of the previously determined quantities can be verified by substituting the result we obtained in the Laplace transformation Eq.(B.16). Then, it reproduces the initial function.

Therefore, once the Fourier transformation of the underlying function is determined, given that the trace of the heat kernel is a well-understood quantity, one can complete the computation by calculating the integral which is the goal of the following part. Furthermore, we have restricted our attention to the study of the Fourier transformation of the functions included in the aforementioned functional traces since in our study, they are the only non-trivial functions which require the computation of the inverse Laplace transform.

However as it can be seen in Sec. 4.4, one is required to compute the functional trace of the logarithm of an operator. To do so, one needs to follow a slightly different strategy in order to bring the functional trace to a form of a proper-time integral.

Functional trace of the logarithm of an operator

The main goal of this part is to bring the functional trace of the logarithm of an operator to a form where one can apply the heat-kernel tools in order to simplify the resulted calculations. Such a procedure is facilitated significantly by considering the proper-time representation of the logarithm. This will change the functional trace of the logarithm to a functional trace of an exponential. Then, employing the relations for the heat kernel will bring the functional trace of the logarithm to a mere computation of an integral over the proper-time parameter. However, the computation of the functional trace of the exponential is not a trivial task given that its form depends of the function on the logarithm. Thankfully, the traces of the heat kernel that appear in the main text, cf. Sec. 4.4, are well understood quantities, see Eqs.(B.17)-(B.19).

Let us summarize the most basic identities of the logarithm in the proper-time representation,

$$\log x = \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} - \frac{\mu^{2\epsilon}}{\epsilon\, \Gamma(\epsilon)} \int_0^\infty ds\ s^{\epsilon - 1} \mathrm{e}^{-sx} \right], \tag{B.25}$$

$$\operatorname{tr} \left[\log \left(\frac{x}{y} \right) \right] = - \int_0^\infty \frac{ds}{s} \left[\operatorname{tr} \mathrm{e}^{-sx} - \operatorname{tr} \mathrm{e}^{-sy} \right], \tag{B.26}$$

where due to the implementation of the ϵ-technique, a mass scale $\mu^{2\epsilon}$ has to be introduced in order to render $\left[\mu^{2\epsilon} s\right] = 0$. In addition, Eq.(B.26) follows straightforwardly from Eq.(B.25).

B.3 Proper-time integrals

After the implementation of heat-kernel techniques, the functional trace has been reduced to the computation of a proper-time integral. In general such a computation is non-trivial since the computation of the integral depends on the form of the function in the functional trace and divergences can occur. In order to regularize the divergent proper-time integrals we shall consider the ϵ technique by introducing an appropriate factor. The underlying idea is based on the dimensional regularization scheme where an analytic continuation of the divergent quantity gives rise to the ϵ factor that assists to isolate the corresponding divergences. Another technique to regularize an infinite integral comes by by introducing a sharp regulator, which we will call Λ, on the divergent boundary of integration, e.g. $\int_0^\infty A(s) \to \int_{1/\Lambda^2}^\infty A(s) < \infty$. Both regularization schemes are used in various Sections of the main text, thus we will summarize all the results for the integrals which appear.

General integrals

Stating relevant integrals from [327],

$$\int_0^\infty \frac{\mathrm{d}s}{s^{1-\nu}} \mathrm{e}^{-\mu s} = \frac{\Gamma(\nu)}{\mu^\nu} \qquad \left[\mathrm{Re}\mu > 0, \mathrm{Re}\nu > 0\right], \tag{B.27}$$

$$\int_0^\infty \frac{\mathrm{d}s}{s^{1-\mu}} \frac{\mathrm{e}^{-\alpha s}}{\sinh s} = 2^{1-\mu} \Gamma(\mu)\, \zeta\left[\mu, \frac{1}{2}(\alpha + 1)\right] \qquad \left[\mathrm{Re}\mu > 1, \mathrm{Re}\alpha > -1\right], \tag{B.28}$$

we present the general form of the integrals which appear in the main text.

Computation of divergent and convergent integrals

We provide the computation of the divergent integrals that appear in Secs. 4.4 & 5.2 as part of the corresponding regularization schemes that were chosen.

Divergent ϵ-Integrals $(\alpha > 0)$:

$$\int_0^\infty \frac{\mathrm{d}s}{s^{1-\epsilon}}\ \mathrm{e}^{-\alpha s} = \alpha^{-\epsilon} \Gamma(\epsilon) \overset{(B.5)}{=} \frac{1}{\epsilon} - \ln \alpha - \gamma_{\mathrm{E}}, \tag{B.29}$$

$$\int_0^\infty \frac{\mathrm{d}s}{s^{2-\epsilon}}\ \mathrm{e}^{-\alpha s} = \alpha^{1-\epsilon} \Gamma(1-\epsilon) \overset{(B.6)}{=} -\frac{\alpha}{\epsilon} + \alpha \ln \alpha + \alpha\left(\gamma_{\mathrm{E}} - 1\right), \tag{B.30}$$

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}s}{s^{1-\epsilon}}\frac{\mathrm{e}^{-\alpha s}}{\sinh s} &= 2^{1-\epsilon}\Gamma(\epsilon)\,\zeta\left[\epsilon, \frac{1}{2}(\alpha+1)\right] \\ &\overset{(B.5),\,(B.8)}{=} -\frac{\alpha}{\epsilon} - (\alpha+1) + 2\zeta'\left[0, \frac{1}{2}(\alpha+1)\right] + \alpha\gamma_{\mathrm{E}} + \alpha\ln 2 \\ &\overset{(B.14),\,(B.12)}{=} -\frac{\alpha}{\epsilon} + \frac{1}{6\alpha} + \alpha\ln\alpha + \alpha\left(\gamma_{\mathrm{E}} - 1\right),\end{aligned} \tag{B.31}$$

where in the last equality an expansion for $|\alpha| \to \infty$ was considered.

Convergent Integrals ($\alpha > 0$):

$$\int_0^\infty \frac{\mathrm{d}s}{s^2}\Big(1 - \frac{s}{\sinh s}\Big)\,\mathrm{e}^{-\alpha s} \overset{(B.30),\,(B.31)}{=} \frac{1}{6\alpha}, \tag{B.32}$$

$$\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\Big(\frac{s}{\sinh s} - 1\Big) = \ln\frac{1}{2}, \tag{B.33}$$

$$\int_0^\infty \frac{\mathrm{d}s}{\mathrm{s}^2}\,\left(1 - \mathrm{e}^{-\alpha s}\right)\Big(\frac{s}{\sinh s} - 1\Big) \overset{(B.32),\,(B.33)}{=} \ln\frac{1}{2} + \frac{1}{6\alpha}. \tag{B.34}$$

Divergent Integrals with a Cutoff Regulator Λ:

$$\int_{\frac{1}{\Lambda^2}}^\infty \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-\alpha^2 s} = \Gamma\left(0, \frac{\alpha^2}{\Lambda^2}\right) \overset{(B.13)}{=} -\ln\frac{\alpha^2}{\Lambda^2} - \gamma_{\mathrm{E}} + \mathcal{O}\left(\frac{\alpha^2}{\Lambda^2}\right), \tag{B.35}$$

where $\mathcal{O}\left(\frac{\alpha^2}{\Lambda^2}\right) = -\sum_{k=1}^\infty \frac{(-1)^k}{k(k!)}\left(\frac{\alpha}{\Lambda}\right)^{2k}$ signifies all higher-order contributions.

APPENDIX C

One-Loop Beta Function in the Background Field Formalism

This appendix is dedicated to the derivation of the generic form of the one-loop beta function by considering covariantly constant backgrounds. The discussion will provide essential results of the counterterm renormalization procedure in the BFM.

Let us begin by relating the renormalized with the bare quantities. Due to the covariantly constant background assumption, there is a proportionality of the background gauge field A_μ with the magnetic field B, cf. Eqs.(A.2) & (A.4). Then, the bare and renormalized quantities would correspond to

$$B_{\mathrm{R}} = \frac{B}{\sqrt{Z_{\mathrm{F}}}}, \quad g_{\mathrm{R}} = Z_{\bar{g}} \bar{g}, \tag{C.1}$$

where $Z_{\mathrm{F}}, Z_{\bar{g}}$ correspond to the wave-function and coupling constant renormalization constants respectively. In addition, $B_{\mathrm{R}}, g_{\mathrm{R}}$ corresponds to the renormalized field and coupling constant whereas $B, \bar{g}$ represent the bare field and coupling constant.

Manifest background gauge invariance on the level of the EA, implies that gauge covariant quantities must remain RG invariant. Such a quantity corresponds to the covariant derivative which entails the product $\bar{g}A_\mu$. For covariantly constant backgrounds this product can be identified by $\bar{g}B$ which must also remain RG invariant. Such a condition holds true if

$$Z_{\bar{g}} = \sqrt{Z_{\mathrm{F}}}. \tag{C.2}$$

Then Eq.(C.1) gives Eqs.(4.41) & (4.56) for covariantly constant background and for self-dual background.

In d-dimensions, the dimensionless coupling is related to the bare coupling via the wave-function renormalization as

$$g^2 = Z_{\mathrm{F}} \mu^{d-4} \bar{g}^2. \tag{C.3}$$

Then, the beta function can readily be deduced to be equal to

$$\beta_{g^2} = \mu \frac{dg^2}{d\mu} = \mu^{d-4} \left[\mu \frac{dZ_{\mathrm{F}}}{d\mu} - 2\left(d-4\right) Z_{\mathrm{F}} \right] \bar{g}^2. \tag{C.4}$$

Choosing $d = 4 - 2\epsilon$ and taking Eq.(C.3) into account, the preceding equation takes the form

$$\beta_{g^2} = -2\epsilon g^2 + \frac{d \ln Z_{\mathrm{F}}}{dg^2} \beta_{g^2} g^2. \tag{C.5}$$

We can further determine Eq.(C.5) by specifying the form of the wave-function renormalization Z_{F}. In particular, during our study of divergent contributions which arise in the one-loop EA from the gluon and ghost sectors following dimensional regularization, we employed the $\overline{\mathrm{MS}}$ scheme, cf. Subsec. 4.4.3. In this renormalization scheme, we absorb both the infinities and a universal constant which appear in the EA by introducing counterterms as parts of the infinite renormalization constants. Doing so then in general the wave-function renormalization, has the form

$$Z_{\mathrm{F}} = 1 + \sum_{n=1}^{\infty} \left[\frac{Z_{\mathrm{F}}^{(n)}}{\epsilon^n} + C^{(n)} \right], \tag{C.6}$$

where n represents the generated number of loops and $C^{(n)}$ the universal constant which appears at n loops. However, given that the beta function must be finite, whereas Z_{F} contains a sum over loops that runs over arbitrary powers of $\frac{1}{\epsilon}$, then the only possible finite combination comes when the linear to ϵ part of the β_{g^2} is multiplied with the one-loop expansion of Z_{F}. In that case, one finds that

$$\beta_{g^2} = -2\epsilon g^2 - 2g^4 \left[\frac{dZ_{\mathrm{F}}^{(1)}}{dg^2} + \epsilon \frac{dC^{(1)}}{dg^2} \right]. \tag{C.7}$$

For $\epsilon \to 0$, then

$$\beta_{g^2} = -2g^4 \frac{dZ_{\mathrm{F}}^{(1)}}{dg^2}, \tag{C.8}$$

which corresponds to the one-loop beta function in the BFM. Here one should highlight the fact that the form of the beta function would have been the same had one chosen to employ the MS renormalization scheme since the universal constant $C^{(n)}$ drops out on the level of the one-loop beta function.

APPENDIX D

Longitudinal $\mathfrak{D}_\mathsf{L}$ and Transversal $\mathfrak{D}_\mathsf{T}$ Kinetic Operators

Motivated by [132, 133, 323] and due to the form of the $\mathcal{Q}$ operator that determines the inverse gluonic fluctuation operator, it is essential to introduce and discuss some properties of the following longitudinal and transversal operator matrices

$$(\mathfrak{D}_\mathrm{T})_{\mu\nu} = -\,\delta_{\mu\nu} D^2 + 2igF_{\mu\nu}, \tag{D.1}$$

$$(\mathfrak{D}_\mathrm{L})_{\mu\nu} = -\,D_\mu D_\nu. \tag{D.2}$$

These operators will help to parametrize the inverse gluonic fluctuation operator and determine a systematic study of its form to all orders in the LGME limit. However, it is of vital importance to discuss some important properties of these operators that will apply in our study as well. The goal of this Appendix is to determine the commutation relations of the longitudinal and transversal operators for covariantly constant backgrounds. Consequently, the operators in the adjoint representation take the form

$$(\mathfrak{D}_\mathrm{T})^{ab}_{\mu\nu} = -\,\delta_{\mu\nu} \left(D^2\right)^{ab} + 2igF^{ab}_{\mu\nu}, \tag{D.3}$$

$$(\mathfrak{D}_\mathrm{L})^{ab}_{\mu\nu} = -\,D^{ac}_\mu D^{cb}_\nu. \tag{D.4}$$

For our subsequent study, we require the following relations

$$\left[D^2, D_\mu\right]^{ab} = -2igF^{ac}_{\alpha\mu} D^{cb}_\alpha,$$
$$\left[D_\mu, F_{\mu\nu}\right]^{ab} = 0.$$

Then, the commutator of the transversal and longitudinal operators takes the form

$$\begin{aligned}\left[\mathfrak{D}_\mathrm{T}, \mathfrak{D}_\mathrm{L}\right]^{ab}_{\mu\nu} &= (\mathfrak{D}_\mathrm{T})^{ac}_{\mu\rho} (\mathfrak{D}_\mathrm{L})^{cb}_{\rho\nu} - (\mathfrak{D}_\mathrm{L})^{ac}_{\mu\rho} (\mathfrak{D}_\mathrm{T})^{cb}_{\rho\nu} \\ &= \left(D^2\right)^{ac} D^{cd}_\mu D^{db}_\nu - D^{ad}_\mu D^{dc}_\nu \left(D^2\right)^{cb} - 2ig\left[F^{ac}_{\mu\rho} D^{cd}_\rho D^{db}_\nu - D^{ad}_\mu F^{dc}_{\rho\nu} D^{cb}_\rho\right].\end{aligned}$$

However, from the commutator of the Laplacian with the covariant derivative, we obtain that

$$\begin{aligned}\left(D^2 D_\mu\right)^{ad} &= \left(D_\mu D^2\right)^{ad} - 2igF^{ac}_{\alpha\mu}D^{cd}_{\alpha},\\ \left(D_\nu D^2\right)^{db} &= \left(D^2 D_\nu\right)^{db} + 2igF^{dc}_{\alpha\nu}D^{cd}_{\alpha},\end{aligned}$$

where inserting in the commutator we arrive at

$$[\mathfrak{D}_{\mathrm{T}}, \mathfrak{D}_{\mathrm{L}}]^{ab}_{\mu\nu} = 0. \tag{D.5}$$

APPENDIX E

Simplification of Functional Traces

This Appendix is dedicated to the derivation of the functional traces Eqs.(5.22a)-(5.22c). As mentioned in the main text, for the following only the assumption of covariantly constant backgrounds will be considered. We shall make use of the following relations

$$[D_\mu, F_{\mu\nu}]^{ab}_{xy} = 0, \tag{E.1}$$

$$\left[D^2, D_\nu\right]^{ab}_{xy} = -2ig\left(F_{\mu\nu}D_\mu\right)^{ab}_{xy}. \tag{E.2}$$

Furthermore, the definition of the gluonic operator will be used so it is instructive to display its form here explicitly,

$$\left(M^{ab}_{\mu\nu}\right)_{xy} = \bar{m}^2\delta^{ab}_{\mu\nu}\delta_{xy} + 2ig\left(F^{ab}_{\mu\nu}\right)_x \delta_{xy} - \left(D^2\right)^{ab}_{xy}\delta_{\mu\nu} + \left(1-\frac{1}{\xi}\right)\left(D_\mu D_\nu\right)^{ab}_{xy}, \tag{E.3}$$

where we have denoted $F^{ab}_{\mu\nu} = -if^{abc}F^c_{\mu\nu}$ and $\delta(x-y) = \delta_{xy}$.

Taking into account that $MM^{-1} = 1$ and reinserting all contributing indices, then

$$\begin{aligned}\delta^{ac}_{\mu\rho}\delta_{xy} &= \int_z \left(M^{ab}_{\mu\nu}\right)_{xz}\left(M^{-1}_{\nu\rho}\right)^{bc}_{zy} \\ &= \int_z \left[\bar{m}^2\delta^{ab}_{\mu\nu}\delta_{xz} - \left(D^2\right)^{ab}_{xz}\delta_{\mu\nu} + 2ig\left(F^{ab}_{\mu\nu}\right)_x\delta_{xz} + \left(1-\frac{1}{\xi}\right)\left(D_\mu D_\nu\right)^{ab}_{xz}\right]\left(M^{-1}_{\nu\rho}\right)^{bc}_{zy}.\end{aligned} \tag{E.4}$$

Next, we multiply Eq.(E.3) by $\int_x \left(D^{da}_\mu\right)_{wx}$ from the left (the direction is irrelevant due to the form of the covariant derivative but for completeness it shall be mentioned), then

$$\begin{aligned}\int_{x,z}\Big[&\bar{m}^2\left(D^{db}_\nu\right)_{wx}\delta_{xz} - \left(D^{da}_\nu\right)_{wx}\left(D^2\right)^{ab}_{xz} + 2ig\left(D^{da}_\nu\right)_{wx}\left(F^{ab}_{\mu\nu}\right)_{xz} \\ &+ \left(1-\frac{1}{\xi}\right)\left(D^{da}_\mu\right)_{wx}\left(D_\mu D_\nu\right)^{ab}_{xz}\Big]\left(M^{-1}_{\nu\rho}\right)^{bc}_{zy} = \left(D^{ab}_\rho\right)_{wy}.\end{aligned} \tag{E.5}$$

Combining the color indices, performing the delta integration and employing Eq.(E.2) for the second term and Eq.(E.1) for the third term of Eq.(E.5), we find that

$$\left[\bar{m}^2\delta^{ac} - \left(\frac{1}{\xi}\right)\left(D^2\right)^{ac}_{x}\right]\left(D_\nu M^{-1}_{\nu\rho}\right)^{cb}_{xy} = \left(D^{ab}_{\rho}\right)_{xy}, \tag{E.6}$$

where we have changed the spacetime index not affected by integration from $w \to x$. In addition, one should notice the difference between covariant derivatives with one and two spacetime indices. In particular, covariant derivative with one spacetime index as the one that appears inside the square brackets on the left side of Eq.(E.6) denotes an operator that acts on every quantity to its right with the same spacetime dependence. To be more precise, $\left(D^{ab}_{\mu}\right)_x = \delta^{ab}\partial^x_\mu + gf^{abc}A^c_{\mu x}$, whereas the one in the covariant derivative with two spacetime indices indicates the existence of a delta function in their definition is implied and corresponds to the one used extensively in the main text. Therefore, one should be careful on how the covariant derivatives are moved in an equation since the ones with two spacetime indices can be freely moved whereas the ones with one index can be moved only through partial integration. Finally, for convenience, let us denote the operator in the square brackets of Eq.(E.6) as

$$O^{ab}_x = \bar{m}^2\delta^{ab} - \left(\frac{1}{\xi}\right)\left(D^2\right)^{ab}_x, \tag{E.7}$$

which makes Eq.(E.6),

$$O^{ac}_x\left(D_\nu M^{-1}_{\nu\rho}\right)^{cb}_{xy} = \left(D^{ab}_{\rho}\right)_{xy}. \tag{E.8}$$

Next, let us multiply Eq.(E.8) by $\int_y \left(D^{be}_\rho\right)_{yz}$ from the right. Then,

$$O^{ac}_x\left(D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{cb}_{xy} = \left(D^2\right)_{xy}, \tag{E.9}$$

where we have renamed some Lorentz indices and renamed $z \to y$.
Finally, we multiply by $\left(O^{-1}_x\right)^{ea}$ from the left, then

$$\left(D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left(O^{-1}D^2\right)^{ab}_x\delta_{xy}. \tag{E.10}$$

Considering Eqs.(E.7), then Eq.(E.10) takes the form

$$\left(D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left[\frac{1}{\bar{m}^2 - \left(\frac{1}{\xi}\right)D^2_x}D^2_x\right]^{ab}\delta_{xy}. \tag{E.11}$$

Inserting the identity

$$\int_x\left(D^2\frac{1}{D^2}\right)^{ca}_{zx} = \int_x\delta^{ca}\delta_{zx}, \tag{E.12}$$

in both sides of Eq.(E.10), we arrive at

$$\left(D^2_z\right)^{cd}\left(\frac{1}{D^2}D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{bd}_{zy} = \left(O^{-1}D^2\right)^{cb}_z\delta_{zy}. \tag{E.13}$$

Multiplying by $\left(\frac{1}{D^2}\right)_z^{ac}$ from the left, we have

$$\left(\frac{1}{D^2}D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left(\frac{1}{D^2}O^{-1}D^2\right)^{ab}_{x}\delta_{xy}, \tag{E.14}$$

or by inserting Eq.(E.7),

$$\left(\frac{1}{D^2}D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left[\left(\frac{1}{D^2_x}\right)\frac{1}{\bar{m}^2-\left(\frac{1}{\xi}\right)D^2_x}D^2_x\right]^{ab}\delta_{xy}. \tag{E.15}$$

Similarly, one can readily deduce that

$$\left(\frac{1}{D^2}\frac{1}{D^2}D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left(\frac{1}{D^2}\frac{1}{D^2}O^{-1}D^2\right)^{ab}_{x}\delta_{xy}. \tag{E.16}$$

or by inserting Eq.(E.7),

$$\left(\frac{1}{D^2}\frac{1}{D^2}D_\mu M^{-1}_{\mu\nu}D_\nu\right)^{ab}_{xy} = \left[\left(\frac{1}{D^2_x}\right)\left(\frac{1}{D^2_x}\right)\frac{1}{\bar{m}^2-\left(\frac{1}{\xi}\right)D^2_x}D^2_x\right]^{ab}\delta_{xy} \tag{E.17}$$

Taking the trace of Eqs.(E.11), (E.15) & (E.17), we deduce the functional traces which appear in Sec. 5.2.

APPENDIX F

Useful Relations for the Background Two-Point Correlator

In this Appendix we will summarize all useful results in position and momentum space, used in Secs. 5.3 & 5.4 that lead to the two-point correlator. However, before doing so, it turns out to be more trustworthy for the validity of the final results if we were to reinsert the explicit spacetime dependence of the underlying quantities. This will will also facilitate the computation of certain inverse operators. Therefore, we shall deviate from the condensed notation which was mainly employed in the main body. By condensed notation, one means that the color indices represent spacetime indices as well. For instance, the covariant derivative in the condensed notation is written as

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + \bar{g} f^{acb} A_\mu^c,$$

Reinstating the explicit spacetime dependence

$$D_\mu^{ab}(x, y) = \left(\delta^{ab} \partial_{x^\mu} + \bar{g} f^{acb} A_\mu^c(x)\right) \delta(x - y).$$

Let us begin by establishing our Fourier conventions

$$\begin{gathered} \Phi_x^i = \int_p e^{-ipx} \Phi_p^i, \quad \Phi_p^i = \int_x e^{ipx} \Phi_x^i, \\ \delta_{xy} = \int_p e^{-ip(x-y)}, \end{gathered} \tag{F.1}$$

where

$$\int_p = \int \frac{d^d p}{(2\pi)^d}, \quad \int_x = \int d^d x.$$

In this slightly more compact convention, we can write the covariant derivative as

$$\left(D_\mu^{ab}\right)_{xy} = \left(\delta^{ab} \partial_\mu^x + \bar{g} f^{acb} A_{\mu x}^c\right) \delta_{xy}.$$

Having set our conventions, in the following, we provide all intermediate steps for the computation of the two-point correlator. For the Laplacian in position space,

$$\left(D^2\right)^{ab}_{xy} = \left[\delta^{ab}\partial^2_x + 2\bar{g}f^{acb}A^x_{\mu x}\partial^x_\mu + \bar{g}f^{acb}\left(\partial^x_\mu A^c_{\mu x}\right) + \bar{g}^2 f^{acd}f^{cbe}A^d_{\mu x}A^e_{\mu x}\right]\delta_{xy}, \tag{F.2}$$

whereas its functional derivative at vanishing background reads

$$\left.\frac{\delta\left(D^2\right)^{ab}_{xy}}{\delta A^c_{\alpha z}}\right|_{A\to 0} = 2\bar{g}f^{acb}\delta_{xz}\left(\partial^x_\alpha\delta_{xy}\right) + \bar{g}f^{acb}\left(\partial^x_\alpha\delta_{xz}\right)\delta_{xy}. \tag{F.3}$$

The inverse Laplacian at vanishing background,

$$\left.\left(\frac{1}{D^2}\right)^{ab}_{xy}\right|_{A\to 0} = \delta^{ab}K(x-y), \tag{F.4}$$

where $K(x-y)$ corresponds to the kernel of the massless propagator in position space and obeys $\partial^2_x K(x-y) = \delta_{xy}$.

For the functional derivative of the inverse Laplacian operator at vanishing background, we make use of the following relation for the functional derivative of the inverse of a generic operator,

$$\frac{\delta\left(\Theta^{-1}\right)^{ab}_{\mu\nu}}{\delta A^c_\rho} = -\left(\Theta^{-1}\right)^{al}_{\mu\lambda}\left(\frac{\delta\Theta^{lk}_{\lambda\kappa}}{\delta A^c_\rho}\right)\left(\Theta^{-1}\right)^{kb}_{\kappa\nu}. \tag{F.5}$$

Then, one can deduce the derivative of the inverse Laplacian operator at vanishing background which in momentum space reads

$$\frac{\delta}{\delta A^c_{\alpha p}}\left(\frac{1}{D^2}\right)^{ab}_{q_1 q_2} = -i\bar{g}f^{acb}\frac{(q_1+q_2)_\alpha}{q_1^2 q_2^2}\delta_{q_1 q_2 p} \tag{F.6}$$

where we have abbreviated $\delta(q_1 - q_2 - p_1) = \delta_{q_1 q_2 p_1}$.

External current conservation, Eq.(4.11), at vanishing backgrounds constraints the external v field to obey a massive Klein-Gordon equation

$$\left.\left(D^{ab}_\mu\right)_{xy} J^b_\mu\right|_{A\to 0} = 0 \Rightarrow \left(\partial^2_x - \bar{m}^2_{\text{gh}}\right)v^a_x = 0, \tag{F.7}$$

which in momentum space takes the form

$$\left(q^2 + \bar{m}^2_{\text{gh}}\right)v^a_q = 0. \tag{F.8}$$

The external current can also be computed in momentum space. However, its form depends on whether external current conservation holds true or not. In particular

$$\left.J^a_{\mu q}\right|_{A\to 0} = \begin{cases} -iq_\mu\left(1+\frac{\bar{m}^2_{\text{gh}}}{q^2}\right)v^a(q), & D\cdot J|_{A\to 0}\neq 0 \\ 0, & D\cdot J|_{A\to 0} = 0 \end{cases} \tag{F.9}$$

Finally, collecting all pieces one can determine the functional derivative of the external current at vanishing background. Therefore, when Eq.(F.8) is not imposed as a constraint, then

$$\left.\frac{\delta J^a_{\mu q}}{\delta A^c_{\alpha p}}\right|_{A\to 0} = -\bar{g} f^{abc}\left[\delta_{\mu\alpha} + \bar{m}^2_{\text{gh}}\frac{\delta_{\mu\alpha}}{\left(q-p\right)^2} + \bar{m}^2_{\text{gh}}\frac{q_\mu\left(2q-p\right)_\alpha}{q^2\left(q-p\right)^2}\right] v^b_{q-p}, \tag{F.10}$$

whereas imposing the constraint from current conservation, leads to

$$\left.\frac{\delta J^a_{\mu q}}{\delta A^c_{\alpha p}}\right|_{A\to 0} = \bar{g} f^{abc} q_\mu\left(2q-p\right)_\alpha\left(\frac{1}{q^2}\right) v^b_{q-p}. \tag{F.11}$$

APPENDIX G

Elementary Gaussian Functional Integrals

In this Appendix, we shall mention some elementary results from Gaussian integrals which are used in Secs. 5.2-5.4, during the procedure of averaging over the external/disorder v field by introducing a Gaussian weight and additional constraints via Laplace multipliers. Note that the condensed notation has been adopted where repeated color indices are extended to represent integration over the corresponding spacetime points as well.

G.1 Standard Gaussian functional integrals

Normalization condition:

$$\langle 1 \rangle = \mathcal{N} \int \mathcal{D}v \; \mathrm{e}^{-\frac{v^2}{2\alpha}} = \mathcal{N} \int \mathcal{D}v \; \mathrm{e}^{-\frac{1}{2\alpha} v^a \mathrm{K}^{ab} v^b} = \mathcal{N} \, (\det \mathrm{K})^{-\frac{1}{2}} \stackrel{!}{=} 1. \tag{G.1}$$

Normalization constant:

$$\mathcal{N} = \det \left(\frac{\mathrm{K}}{\alpha} \right)^{\frac{1}{2}}. \tag{G.2}$$

Single v-field contribution:

$$\langle v^a(x) \rangle = \mathcal{N} \int \mathcal{D}v \; v^a(x) \; \mathrm{e}^{-\frac{v^2}{2\alpha}} = \mathcal{N} \int \mathcal{D}v \; v^a(x) \; \mathrm{e}^{-\frac{1}{2\alpha} v^a \mathrm{K}^{ab} v^b} = 0. \tag{G.3}$$

Quadratic v-field contribution:

$$\langle v^a(x) v^b(y) \rangle = \mathcal{N} \int \mathcal{D}v \; v^a(x) v^b(y) \; \mathrm{e}^{-\frac{v^2}{2\alpha}} = \mathcal{N} \int \mathcal{D}v \; v^a(x) v^b(y) \; \mathrm{e}^{-\frac{1}{2\alpha} v^a \mathrm{K}^{ab} v^b} = \delta^{ab} \delta(x-y). \tag{G.4}$$

Fourier Transformation of K to momentum space:

$$K^{ab}(x,y) = \delta^{ab} \delta(x-y) \longrightarrow K^{ab}(p,q) = \delta^{ab} \delta(p+q). \tag{G.5}$$

In Sec. 5.2, the Gaussian functional integral in Eq.(G.4) appears.

G.2 Gaussian functional integrals with a constraint

Next, we study the Gaussian functional integrals that contain an extra constraint. The computation of these kinds of Gaussian integrals is mainly based on completing the square of the integrated quantities.

v-field constraint:

$$\Delta_{\text{FP}}^{ab} = \left(-\partial^2 + \bar{m}_{\text{gh}}^2\right)\delta^{ab}. \tag{G.6}$$

Normalization condition:

$$\begin{aligned}\langle 1\rangle &= \mathcal{N}\int \mathcal{D}\lambda\mathcal{D}v\ \mathrm{e}^{i\lambda^a\Delta_{\text{FP}}^{ab}v^a - \frac{v^2}{2\alpha}} = \mathcal{N}\int \mathcal{D}\lambda\mathcal{D}v\ \mathrm{e}^{-\frac{1}{2\alpha}(v^a - i\alpha\Delta_{\text{FP}}\lambda^a)^2}\mathrm{e}^{-\frac{\alpha}{2}\lambda^a\left(\Delta_{\text{FP}}^2\right)^{ab}\lambda^b}\\ &= \mathcal{N}\int \mathcal{D}\lambda\ \mathrm{e}^{-\frac{\alpha}{2}\lambda^a\left(\Delta_{\text{FP}}^2\right)^{ab}\lambda^b}\int \mathcal{D}v\ \mathrm{e}^{-\frac{1}{2\alpha}v^a\mathrm{K}^{ab}v^b}\\ &= \mathcal{N}\det\left(\Delta_{\text{FP}}^2\mathrm{K}\right)^{-\frac{1}{2}} \overset{!}{=} 1.\end{aligned} \tag{G.7}$$

Normalization constant:

$$\mathcal{N} = \det\left(\Delta_{\text{FP}}^2\mathrm{K}\right)^{\frac{1}{2}}. \tag{G.8}$$

Single v-field contribution with a constraint:

$$\begin{aligned}\langle v^a(x)\rangle &= \mathcal{N}\int \mathcal{D}\lambda\mathcal{D}v\ v^a(x)\ \mathrm{e}^{i\lambda^a\Delta_{\text{FP}}^{ab}v^a - \frac{v^2}{2\alpha}}\\ &= \mathcal{N}\int \mathcal{D}\lambda\ \mathrm{e}^{-\frac{\alpha}{2}\lambda^a\left(\Delta_{\text{FP}}^2\right)^{ab}\lambda^b}\int \mathcal{D}v\ v^a(x)\ \mathrm{e}^{-\frac{1}{2\alpha}v^a\mathrm{K}^{ab}v^b}\\ &= 0.\end{aligned} \tag{G.9}$$

Quadratic v-field contribution with a constraint:

$$\langle v^a(x)v^b(y)\rangle = \mathcal{N}\int \mathcal{D}\lambda\mathcal{D}v\ v^a(x)v^b(y)\ \mathrm{e}^{i\lambda^a\Delta_{\text{FP}}^{ab}v^a - \frac{v^2}{2\alpha}} = 0, \tag{G.10}$$

where for the derivation of Eq.(G.10) we completed the square in the exponent and introduced vanishing sources that couple to the Lagrange multiplier λ^a. We make use of this result in the Secs. 5.3 & 5.4.

APPENDIX H

Flow Equation and Vertex Interactions in Momentum Space

In this Appendix, we summarize all relevant results that contribute in the derivation of the associated flow equations which constitute the one-loop beta function. Furthermore, we provide the expanded version of the truncated flow equation in momentum space, upon which the computations for the derivation of the one-loop beta function are built.

H.1 Flow equation in momentum space

The flow equation in momentum space according to the class of truncations which couple the effective average action linearly to the BRST sources cf. Eq.(6.23), reads

$$
\begin{aligned}
\partial_t \Gamma =& \frac{1}{2}\int_{p_1} \partial_t R^{\mu\nu}(p_1)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu(p_1)A^a_\nu(-p_1)} \\
&- i\int_{p_1,p_2,p_3} \partial_t\big(Z_{\rm gh} r_{\rm gh}(p_2^2)\big)p_2^\mu \left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^b_\nu(p_3)\bar{c}^a(-p_1)} \tilde{\Gamma}^{(2)}_{A^b_\nu(-p_3)K^a_\mu(-p_2)}\delta(p_1+p_2) \\
&- i\int_{p_1,p_2,p_3} \partial_t\big(Z_{\rm gh} r_{\rm gh}(p_2^2)\big)p_2^\mu \left(\tilde{\Gamma}^{(2)}\right)^{-1}_{c^b(p_3)\bar{c}^a(-p_1)} \tilde{\Gamma}^{(2)}_{c^b(-p_3)K^a_\mu(-p_2)}\delta(p_1+p_2) \\
&- i\int_{p_1,p_2,p_3} \partial_t\big(Z_{\rm gh} r_{\rm gh}(p_2^2)\big)p_2^\mu \left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\bar{c}^b(p_3)\bar{c}^a(-p_1)} \tilde{\Gamma}^{(2)}_{\bar{c}^b(-p_3)K^a_\mu(-p_2)}\delta(p_1+p_2) \\
&+ \frac{1}{2}\int_{p_1} \partial_t R^{\mu\nu}(p_1)\left[\tilde{\Gamma}^{(2)}_{I^b_\mu(p_1)K^b_\nu(-p_2)} + \int_{p_2}\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^c_\rho(p_2)A^b_\nu(-p_1)} \tilde{\Gamma}^{(2)}_{A^c_\rho(-p_2)M^b_\mu(p_1)}\right] \\
&+ \frac{1}{2}\int_{p_1,p_2} \partial_t R^{\mu\nu}(p_1)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{c^c(p_2)A^b_\nu(-p_1)} \tilde{\Gamma}^{(2)}_{c^c(-p_2)M^b_\mu(p_1)} \\
&+ \frac{1}{2}\int_{p_1,p_2} \partial_t R^{\mu\nu}(p_1)\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\bar{c}^c(p_2)A^b_\nu(-p_1)} \tilde{\Gamma}^{(2)}_{\bar{c}^c(-p_2)M^b_\mu(p_1)}.
\end{aligned}
\tag{H.1}
$$

H.2 Propagators and vertex interactions

As one can observe from the right side of the expanded flow equation, Eq.(H.1), several regulator-dependent and independent interactions can arise as a result of the truncation scheme. Let us summarize the form of all contributing interactions in our theory.

To that extent, we begin by rewriting the relevant quantities from which one can read off all associated interactions, which correspond to the chosen form of the truncation of the Legendre EA Eq.(6.22) as well as its building blocks, Eqs.(6.29) & (6.30)

$$\tilde{\Gamma}_k[\Phi, \mathcal{I}; v] = Z_{\mathrm{T}}\, S_{\mathrm{YM}}[A] + \tilde{\Gamma}_{\mathrm{gf}}[A; v] + \tilde{\Gamma}_{\mathrm{gh}}[\Phi; v] + S^{\mathrm{BRST}}_{\mathrm{sou}}[\Phi, \mathcal{I}; v], \tag{H.2}$$

$$\tilde{\Gamma}_{\mathrm{gf}}[A; v] = \frac{1}{2} A^a_\mu \mathrm{Q}_{\mu\nu} A^a_\nu + v^a Z_{\mathrm{gh}} \left(1 + r_{\mathrm{gh}}(-\partial^2)\right) \partial_\mu A^a_\mu,$$

$$\tilde{\Gamma}_{\mathrm{gh}}[\Phi; v] = -Z_{\mathrm{gh}} \left(1 + r_{\mathrm{gh}}(-\partial^2)\right) \bar{c}^a \left(\partial_\mu D_\mu c\right)^a - \frac{v^a}{2|v|^2} \bar{c}^a \left[\left(\mathrm{Q}_{\mu\nu} A^b_\nu\right) (D_\mu c)^b + A^b_\mu \mathrm{Q}_{\mu\nu} \left(D_\nu c\right)^b\right]. \tag{6.30}$$

Then, the following n-point vertex functions contribute in the associated flow equations

$$\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{\bar{c}^a(p_1)c^b(p_2)} = \frac{\delta^{ab}\delta(p_1+p_2)}{Z_{\mathrm{gh}}\left(p_1^2 + R(p_1)\right)} = \delta^{ab}\delta(p_1+p_2) G_{\mathrm{gh}}(p_1)$$

$$\left(\tilde{\Gamma}^{(2)}\right)^{-1}_{A^a_\mu(p_1)A^b_\nu(p_2)} = \frac{\delta^{ab}\delta_{\mu\nu}\delta(p_1+p_2)}{Z\left(p_1^2 + R(p_1)\right)} = \delta^{ab}\delta_{\mu\nu}\delta(p_1+p_2) G(p_1)$$

$$\tilde{\Gamma}^{(2)}_{c^b(-p_3)\mathrm{K}^a_\mu(-p_2)} = i p_{3\mu}\, \delta^{ab}\delta(p_2+p_3) + \bar{g} f^{abc} \int_q A^c_\mu(q)\delta(q - p_2 - p_3)$$

$$\tilde{\Gamma}^{(3)}_{\bar{c}^a(p_1)A^b_\mu(p_2)c^c(p_3)} = i\bar{g} f^{abc} Z_{\mathrm{gh}} p_{1\mu} \left(1 + r_{\mathrm{gh}}(p_1)\right) \delta(p_1+p_2+p_3) - i\frac{v^a}{2|v|^2}\delta^{bc}\left[\mathrm{Q}_{\mu\nu}(p_2) + \mathrm{Q}_{\mu\nu}(p_3)\right] p_{3\nu}\, \delta(p_1+p_2+p_3)$$

$$\tilde{\Gamma}^{(3)}_{c^b(-p_3)A^d_\rho(q_1)\mathrm{K}^a_\mu(-p_2)} = \bar{g} f^{abd}\delta_{\mu\rho}\delta(q_1 - p_2 - p_3)$$

$$\tilde{\Gamma}^{(3)}_{A^a_\mu(p_1)A^b_\nu(p_2)A^c_\rho(p_3)} = i Z_{\mathrm{T}} \bar{g} f^{abc}\left[\delta_{\mu\nu}\left(p_1 - p_2\right)_\rho + \delta_{\nu\rho}\left(p_2 - p_3\right)_\mu + \delta_{\mu\rho}\left(p_3 - p_1\right)_\nu\right]\delta(p_1+p_2+p_3)$$

$$\tilde{\Gamma}^{(3)}_{\bar{c}^a(p_1)\mathrm{M}^b_\mu(p_2)c^c(p_3)} = -i\frac{v^a}{|v|^2}\delta^{bc} p_{3\mu}\delta(p_1+p_2+p_3)$$

$$\tilde{\Gamma}^{(3)}_{A^a_\mu(p_1)\mathrm{K}^b_\nu(p_2)c^c(p_3)} = \bar{g} f^{abc}\delta_{\mu\nu}\delta(p_1+p_2+p_3)$$

$$\tilde{\Gamma}^{(4)}_{\bar{c}^a(p_1)\mathrm{M}^b_\mu(p_2)c^c(p_3)A^d_\nu(p_4)} = \bar{g}\frac{v^a}{|v|^2} f^{bcd}\delta_{\mu\nu}\delta(p_1+p_2+p_3+p_4)$$

$$\tilde{\Gamma}^{(4)}_{A^a_\mu(p_1)A^b_\nu(p_2)A^c_\rho(p_3)A^d_\sigma(p_4)} = Z_{\mathrm{T}}\bar{g}^2\Big[f^{eab}f^{ecd}\left(\delta_{\sigma\nu}\delta_{\rho\mu} - \delta_{\sigma\mu}\delta_{\nu\rho}\right) + f^{ead}f^{ecb}\left(\delta_{\sigma\nu}\delta_{\rho\mu} - \delta_{\mu\nu}\delta_{\rho\sigma}\right) + f^{eac}f^{edb}\left(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma}\right)\Big]\delta(p_1+p_2+p_3+p_4)$$

$$\tilde{\Gamma}^{(4)}_{\bar{c}^a(p_1)A^b_\mu(p_2)A^c_\nu(p_3)c^d(p_4)} = 0$$

APPENDIX I

Resummation Terms

In this Appendix we present the analytic expressions for the gluon and ghost anomalous dimensions. To do so, we rewrite the left sides of Eqs.(6.39) & (6.41) in terms of the gluon and ghost anomalous dimensions respectively, Eq.(6.40). Then, we obtain a system of two equations with two unknowns. Upon solving this system, one finds that

$$
\begin{aligned}
\eta_{\mathrm{T}} &= -\, g^2 2\mathrm{v}_4 N_c \frac{793}{192} \frac{1 + g^2 2\mathrm{v}_4 N_c \frac{4107}{812032} \frac{32 + \frac{86437}{5760} \frac{g^2 2\mathrm{v}_4 N_c}{1+\frac{5N_c}{6} g^2 2\mathrm{v}_4}}{1+\frac{703}{5120} g^2 2\mathrm{v}_4 N_c + \frac{149221}{1966080} \frac{g^4 (2\mathrm{v}_4)^2 N_c^2}{1+\frac{5N_c}{6} g^2 2\mathrm{v}_4}}}{1 + \frac{5N_c}{6} g^2 2\mathrm{v}_4}, \\
\eta_{\mathrm{gh}} &= -\, g^2 2\mathrm{v}_4 N_c \frac{37}{16} \frac{1 + g^2 2\mathrm{v}_4 N_c \frac{86437}{184320} \frac{1}{1+\frac{5N_c}{6} g^2 2\mathrm{v}_4}}{1 + \frac{703}{5120} g^2 2\mathrm{v}_4 N_c + \frac{149221}{1966080} g^4 (2\mathrm{v}_4)^2 N_c^2 \frac{1}{1+\frac{5N_c}{6} g^2 2\mathrm{v}_4}}.
\end{aligned}
\tag{I.1}
$$

With the help of Eq.(I.1), we find an estimate for the two-loop beta function Eq.(6.49).

Acronyms

1PI 1-Particle Irreducible

1PR 1-Particle Reducible

BFM background field method

BRST Becchi-Rouet-Stora-Tyutin

CFDJ Curci-Ferrari-Delbourgo-Jarvis

DSE Dyson-Schwinger equations

EA Effective Action

EAA Effective Average Action

EFT Effective Field Theory

FMR fundamental modular region

FP Faddeev-Popov

fRG functional Renormalization Group

GZ Gribov-Zwanziger

IR infrared

K-O Kugo-Ojima

LGME large gluonic mass expansion

mME modified master equation

mNIs modified Nielsen Identities

mSTIs modified Slavnov-Taylor Identities

mWIs modified Ward Identities

mWTIs modified Ward-Takahashi Identities

NIs Nielsen Identities

NL Nakanishi-Lautrup

QCD Quantum Chromodynamics

QED Quantum Electrodynamics

RG Renormalization Group

SM Standard Model

STIs Slavnov-Taylor Identities

UV ultraviolet

WIs Ward Identities

WTIs Ward-Takahashi Identities

YM Yang-Mills

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Η Ιθάκη σ’ έδωσε τ’ ωραίο ταξίδι.
Χωρίς αυτήν δεν θα ‘βγαινες στον δρόμο.
Άλλα δεν έχει να σε δώσει πια.
Κι αν πτωχική την βρεις, η Ιθάκη δεν σε γέλασε.
Έτσι σοφός που έγινες, με τόση πείρα,
ήδη θα το κατάλαβες οι Ιθάκες τι σημαίνουν.
[Κωνσταντίνος Καβάφης - Ιθάκη]

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