Lengths of divisible codes with restricted column multiplicities

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Abstract

We determine the minimum possible column multiplicity of even, doubly-, and triply-even codes given their length. This refines a classification result for the possible lengths of q^r-divisible codes over $\mathbb{F}_q$. We also give a few computational results for field sizes $q > 2$. Non-existence results of divisible codes with restricted column multiplicities for a given length have applications e.g. in Galois geometry and can be used for upper bounds on the maximum cardinality of subspace codes.

Keywords: Divisible codes, linear codes, Galois geometry
Mathematics Subject Classification: 51E23 (05B40)

1 Introduction

Adding a parity check bit to a binary linear code yields an even code, i.e., all codewords have an even weight. Doubly-even binary linear codes, where the weights of the codewords are multiples of four, have been the subject of extensive research for decades, see e.g. [DFJ+11]. Also linear codes where all occurring weights are divisible by eight, so-called triply-even codes, have been studied in the literature, see e.g. [BM12, MN19, Rod22]. The mentioned classes of linear codes are special cases of so-called Δ-divisible codes where the weights of the codewords all are divisible by some integer $\Delta > 1$. They have e.g. applications for the maximum size of partial k-spreads, i.e., sets of pairwise disjoint k-dimensional subspaces, see e.g. [HKK18]. More concretely, the non-existence of q^{k-1}-divisible codes over $\mathbb{F}_q$ of a certain length implies an upper bound on the cardinality of partial k-spreads. In [KK20] the possible lengths of q^r-divisible codes over $\mathbb{F}_q$ have been completely characterized. However, on the constructive side some of these codes require a relatively large column multiplicity. In some applications upper bounds on the maximum possible column multiplicity are known. E.g. in the situation of partial k-spreads the codes have a maximum column multiplicity of one, i.e., the codes have to be projective. This special case has received quite some attention, see e.g. [HKKW19, Kur20]. Here we ask more generally for the minimum possible column multiplicity of a Δ-divisible code over $\mathbb{F}_q$ having length n. Those results imply classification results for the possible lengths of Δ-divisible codes over $\mathbb{F}_q$ given any upper bound γ on the allowed column multiplicity, refining the results from [KK20]. A general parametric solution to this problem seems very unlikely, so that we solve the first few smallest cases in this paper. Our utilized arguments will mostly be of geometric nature so that we will use the geometric reformulation of linear codes as multisets of points in projective spaces. Non-existence results for divisible codes of a certain length have applications for covering and packing problems in Galois geometry, see e.g. [Etz14, EKOÖ20]. They can also be used to improve upon the so-called Johnson bound on the size of constant-dimension codes, see [KK20], as well as more general mixed dimension subspaces codes, see [HKK19b].

The remaining part of this paper is structured as follows. In Section 2 we introduce the necessary preliminaries and state our main result as Theorem 13, i.e., for the binary case we completely determine the minimum possible column multiplicity of Δ-divisible codes of length n for each $\Delta \in \{2, 4, 8\}$. Classification results for even and doubly-even binary codes are obtained in Section 3 and used to conclude results for triply-even binary codes in Section 4. We draw a brief conclusion in Section 5.

All of the used arguments are completely theoretical and do not rely on any computer calculations. As a verification and continuation we present computational results in Section B in the appendix. As a small justification why some of our arguments are quite lengthy we also give some information on the combinatorial richness of a special case in Section A.

2 Preliminaries

For a prime power q let $\mathbb{F}_q$ be the finite field with q elements. Let $V \simeq \mathbb{F}_q^v$ be a v-dimensional vector space over $\mathbb{F}_q$ and $\operatorname{PG}(v-1,q)$ the projective space associated to it. By a k-space of $\operatorname{PG}(v-1,q)$ we mean a k-dimensional linear subspace of V, also using the terms points, lines, planes, and hyperplanes for 1-, 2-, 3-spaces, and $(v-1)$-spaces, respectively. We define a multiset $\mathcal{M}$ of points via its point multiplicities $\mathcal{M}(P) \in \mathbb{N}$ for each point P. We allow the addition, subtraction, and scaling with rational factors of multisets of points componentwise as long as the resulting point multiplicities are all natural integers. For an arbitrary subspace K in $\operatorname{PG}(v-1,q)$ we define $\mathcal{M}(K) := \sum_{P \le K} \mathcal{M}(P)$, where we write $A \le B$ if A is a subspace of B and the summation is over all points P. With this, we define the cardinality or size of $\mathcal{M}$ as $\#\mathcal{M} := \mathcal{M}(V)$, i.e., as the sum over all point multiplicities $\mathcal{M}(P)$. A multiset $\mathcal{M}$ of points in $\operatorname{PG}(v-1,q)$ is called spanning if $\langle P : \mathcal{M}(P) \ge 1\rangle_{\mathbb{F}_q} = \mathbb{F}_q^v$. The maximum occurring point multiplicity of $\mathcal{M}$ is denoted by $\gamma_1(\mathcal{M})$, or just γ_1 whenever $\mathcal{M}$ is clear from the context. More generally, for each $1 \le i \le v$ we denote by γ_i, more precisely $\gamma_i(\mathcal{M})$, the maximum of $\mathcal{M}(K)$ where K runs over all i-spaces. E.g., $\gamma_v = \#\mathcal{M}$.

To each multiset $\mathcal{M}$ of n points in $\operatorname{PG}(v-1,q)$ we can assign a q-ary linear code $C(\mathcal{M})$ defined by a generator matrix whose n columns consist of representatives of the n points of $\mathcal{M}$. It is well-known, see e.g. [DS98], that this relation between $C(\mathcal{M})$ and $\mathcal{M}$ associating a full-length linear $[n,v]_q$ code with a multiset $\mathcal{M}$ of n points in $\operatorname{PG}(v-1,q)$ induces a one-to-one correspondence between classes of (semi-)linearly equivalent spanning multisets of points and classes of (semi-)linearly equivalent full-length linear codes. The maximum point multiplicity γ_1 of a multiset $\mathcal{M}$ of points is the same as the maximum column multiplicity of the corresponding linear code C (given an arbitrary generator matrix). So, C is projective iff $\mathcal{M}$ is indeed a set, i.e., $\mathcal{M}(P) \le 1$ for all points P.

A linear code C is said to be Δ-divisible, where $\Delta \in \mathbb{N}_{\ge 1}$, if all of its weights are divisible by Δ. A multiset $\mathcal{M}$ of points is called Δ-divisible iff its corresponding linear code $C(\mathcal{M})$ is Δ-divisible. More directly, a multiset $\mathcal{M}$ of points is Δ-divisible iff we have $\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{\Delta}$ for all hyperplanes H. As for binary linear codes we speak of even, doubly-even, and triply-even multisets of points in $\operatorname{PG}(v-1,2)$ when they are 2-, 4-, and 8-divisible, respectively.

If $\mathcal{M}$ is a multiset of points in $\operatorname{PG}(v-1,q)$ and K some subset of all points (usually a subspace in $\operatorname{PG}(v-1,q)$), then $\mathcal{M}|_K$ denotes the restriction of $\mathcal{M}$ to K, i.e., $\mathcal{M}|_K(P) = \mathcal{M}(P)$ for all $P \le K$ and $\mathcal{M}|_K(P) = 0$ otherwise. If K is a hyperplane then the restricted multiset $\mathcal{M}|_K$ inherits divisibility with a smaller divisibility constant, see e.g. [HKK18, Lemma 7].

Lemma 1. *Let $\mathcal{M}$ be a Δ-divisible multiset of points in $\operatorname{PG}(v-1,q)$. If q divides Δ, then $\mathcal{M}|_H$ is (Δ/q)-divisible for each hyperplane H.*

Of course we can apply the lemma recursively so that $\mathcal{M}|_S$ is $\left(\Delta/q^i\right)$-divisible for each subspace S of codimension i, i.e., dimension $v-i$, if Δ is divisible by q^i.

For an arbitrary subspace K we denote by χ_K its characteristic function, i.e., $\chi_K(P) = 1$ iff $P \le K$ and $\chi_K(P) = 0$ otherwise. The support $\operatorname{supp}(\mathcal{M})$ of a multiset of points $\mathcal{M}$ is the set of points that have non-zero multiplicity.

For a given multiset $\mathcal{M}$ of points in $\operatorname{PG}(v-1,q)$ we denote by a_i the number of hyperplanes H such that $\mathcal{M}(H) = i$. If $\mathcal{M}$ is spanning, then we have $a_{\#\mathcal{M}} = 0$. We say that $\mathcal{M}$ has dimension k if its span is a k-dimensional subspace K. By considering $\mathcal{M}$ restricted to $K \cong \operatorname{PG}(k-1,q)$ we can always assume that $\mathcal{M}$ is spanning if we choose a suitable integer for k. For the ease of notation we assume that $\mathcal{M}$ is spanning in $\mathrm{PG}(k-1,q)$ in the following. Counting the number of hyperplanes in $\mathrm{PG}(k-1,q)$ gives

$$\sum_i a_i = \frac{q^k-1}{q-1} \tag{1}$$

and counting the number of pairs of points and hyperplanes gives

$$\sum_i i a_i = \#\mathcal{M} \cdot \frac{q^{k-1}-1}{q-1}. \tag{2}$$

By λ_i we denote the number of points P such that $\mathcal{M}(P)=i$, so that

$$\sum_i \lambda_i = \frac{q^k-1}{q-1} \tag{3}$$

and

$$\sum_i i\lambda_i = \#\mathcal{M}. \tag{4}$$

Double-counting the incidences between pairs of elements in $\mathcal{M}$ and hyperplanes gives

$$\sum_i \binom{i}{2} a_i = \binom{\#\mathcal{M}}{2} \cdot \frac{q^{k-2}-1}{q-1} + q^{k-2} \cdot \sum_i \binom{i}{2}\lambda_i. \tag{5}$$

We call the equations (1)-(5) the *standard equations* for multisets of points. If $\mathcal{M}$ is a set of points, then Equation (5) simplifies to

$$\sum_i \binom{i}{2} a_i = \binom{\#\mathcal{M}}{2} \cdot \frac{q^{k-2}-1}{q-1}$$

and is complemented to the standard equations (for sets of points) by equations (1) and (2). We call the vector $(a_i)_{i\in\mathbb{N}}$ the *spectrum* of $\mathcal{M}$. As an abbreviation we set $[k]_q := \left(q^k-1\right)/(q-1)$ for all $k \in \mathbb{N}$.

If all hyperplanes have the same multiplicity, then there is a well known classification of the corresponding multisets of points. In order to keep the paper self-contained we give a direct proof.

Lemma 2. *Let $\mathcal{M}$ be a spanning multiset of points of cardinality n in $\mathrm{PG}(k-1,q)$ such that every hyperplane H has multiplicity $\mathcal{M}(H)=s$. Then, we have $\mathcal{M}(P)=t$ for every point P, where $t=n/[k]_q$. If $k\ge 2$, then we additionally have $s=t[k-1]_q$.*

Proof. If $k=1$, then we can choose $t=n$. The unique point P then satisfies $\mathcal{M}(P)=\#\mathcal{M}=n=t$. Now assume $k\ge 2$. Equation (1) gives $a_s=[k]_q$, so that Equation (2) yields $s[k]_q=n[k-1]_q$. If $k=2$, then we have $\mathcal{M}(P)=s=t[k-1]_q=t$ for every point P since every hyperplane is a point for $k=2$. For $k\ge 3$ double-counting the points of $\mathcal{M}$ via the hyperplanes that contain P gives $\mathcal{M}(P)=t$. □

Proposition 3. *Let $0\le l\le r$ be integers and $\mathcal{M}$ be a q^r-divisible multiset of points in $\mathrm{PG}(v-1,q)$ of cardinality $n=q^l\cdot[r+1-l]_q$. Then, there exists a $(r+1-l)$-space K such that $\mathcal{M}=q^l\cdot\chi_K$.*

Proof. If $l=r$, then $\mathcal{M}$ is q^r-divisible with cardinality q^r, so that $\mathcal{M}$ corresponds to a q^r-fold point. If $l<r$, then we have $q^r<n<2q^r$ and all hyperplanes that do not contain all points of $\mathcal{M}$ have the same multiplicity so that we can apply Lemma 2. □

Corollary 4. *Let $\mathcal{M}$ be a Δ-divisible multiset of points in $\mathrm{PG}(v-1,2)$ of cardinality n.*

- *If $\Delta=2$ and $n=2$, then $\mathcal{M}$ is the characteristic function of a double point.*
- *If $\Delta=2$ and $n=3$, then $\mathcal{M}$ is the characteristic function of a line.*

- *If $\Delta = 4$ and $n = 4$, then $\mathcal{M}$ is the characteristic function of a 4-fold point.*
- *If $\Delta = 4$ and $n = 6$, then $\mathcal{M}$ is the characteristic function of a double line.*
- *If $\Delta = 4$ and $n = 7$, then $\mathcal{M}$ is the characteristic function of a plane.*
- *If $\Delta = 8$ and $n = 8$, then $\mathcal{M}$ is the characteristic function of an 8-fold point.*
- *If $\Delta = 8$ and $n = 12$, then $\mathcal{M}$ is the characteristic function of an 4-fold line.*
- *If $\Delta = 8$ and $n = 14$, then $\mathcal{M}$ is the characteristic function of a double plane.*
- *If $\Delta = 8$ and $n = 15$, then $\mathcal{M}$ is the characteristic function of a solid.*

If $\mathcal{M}$ is a multiset of points and Q a point in $\operatorname{PG}(v-1,q)$, where $v \ge 2$, then we can construct a multiset $\mathcal{M}_Q$ in $\operatorname{PG}(v-2,q)$ by *projection* trough Q, that is the multiset image under the map $P \mapsto \langle P, Q\rangle/Q$ setting $\mathcal{M}_Q(L/Q) = \mathcal{M}(L) - \mathcal{M}(Q)$ for every line $L \ge P$ in $\operatorname{PG}(v-1,q)$. We directly verify the following properties:

Lemma 5. *Let $\mathcal{M}$ be a spanning Δ-divisible multiset of points in $\operatorname{PG}(k-1,q)$, where $k \ge 2$, and let $\mathcal{M}_Q$ arise from $\mathcal{M}$ by projection through a point Q. Then we have $\#\mathcal{M}_Q = \#\mathcal{M} - \mathcal{M}(Q)$, $\mathcal{M}_Q$ is Δ-divisible, the span of $\mathcal{M}_Q$ has dimension $k-1$, and $\gamma_1(\mathcal{M}_Q) = \mathcal{M}(L) - \mathcal{M}(Q)$, where L is a line containing Q and maximizing $\mathcal{M}(L)$.*

In the binary case also 2^r-divisible multisets of points of cardinality 2^{r+1} can be characterized easily for each $r \in \mathbb{N}$. We first state an auxiliary result that is also used later on.

Lemma 6. *Let $\mathcal{M}$ be a spanning multiset of points in $\operatorname{PG}(k-1,q)$ with cardinality n, $1 \le l \le k-2$, and K be an arbitrary l-dimensional subspace. If all hyperplanes containing K have cardinality s, then $\mathcal{M}(K) = s - \frac{n-s}{q-1} + \frac{n-s}{q^{k-l-1}(q-1)} > s - \frac{n-s}{q-1}$. If all hyperplanes containing K have cardinality at least s, then $\mathcal{M}(K) \ge s - \frac{n-s}{q-1} + \frac{n-s}{q^{k-l-1}(q-1)} > s - \frac{n-s}{q-1}$.*

Proof. Counting points via the hyperplanes containing K yields

$$[k-l]_q \cdot (s - \mathcal{M}(K)) = [k-l-1]_q \cdot (n - \mathcal{M}(K))$$

in the first case. Solving for $\mathcal{M}(K)$ yields

$$\mathcal{M}(K) = \frac{[k-l]_q \cdot s - [k-l-1]_q \cdot n}{q^{k-l-1}} = s - \frac{n-s}{q-1} + \frac{n-s}{q^{k-l-1}(q-1)} > s - \frac{n-s}{q-1}.$$

In the second case the same reasoning yields

$$\mathcal{M}(K) \ge s - \frac{n-s}{q-1} + \frac{n-s}{q^{k-l-1}(q-1)} > s - \frac{n-s}{q-1}.$$

□

Proposition 7. *Let $r \ge 1$ be an integer and $\mathcal{M}$ be a 2^r-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 2^{r+1}. Then, either $\mathcal{M} = 2^{r+1} \cdot \chi_P$ for a point P or there exist subspaces K and $E \le K$ with $r+1 \ge \dim(E) = \dim(K) - 1 \ge 1$ such that $\mathcal{M} = 2^{r+1-\dim(E)} \cdot \chi_{K \setminus E} = 2^{r+1-\dim(E)} \cdot \chi_K - 2^{r+1-\dim(E)} \cdot \chi_E$.*

Proof. Let k be the dimension of the span of $\mathcal{M}$. If $k = 1$, then there clearly exists a point P with $\mathcal{M} = 2^{r+1} \cdot \chi_P$. For $k \ge 2$ the standard equations yield $a_0 = 1$ and $a_{2^r} = 2^k - 2$. We choose E as the unique hyperplane with multiplicity zero (and K as the entire ambient space). If P is a point with $\mathcal{M}(P) > 0$, then Lemma 6 yields $\mathcal{M}(P) = 2^{r-k+2}$. Since there are exactly 2^{k-1} points outside of E, all points P outside of E have multiplicity $2^{r-k+2} = 2^{r+1-\dim(E)}$. □

In other words, the corresponding multisets of points are suitable multiples of affine spaces or are given by 2^{r+1}-fold points, which might be considered as a degenerated case. We remark that the corresponding situation for $q > 2$ is more complicated, see papers on the so-called cylinder conjecture [DBDMS19, KM21].

Definition 8. *By $\Gamma_q(\Delta, n)$ we denote the minimum of $\gamma_1(\mathcal{M})$ over all Δ-divisible multisets of points $\mathcal{M}$ in $\mathrm{PG}(v-1,q)$ with cardinality n, where v is sufficiently large. If no such multiset of points exist we set $\Gamma_q(\Delta, n) = \infty$.*

In [War81, Theorem 1] it was shown that each $(p^e d)$-divisible code over a finite field with characteristic p, where $\gcd(p,d) = 1$, is a d-fold repetition of a p^e-divisible code. So it suffices to determine $\Gamma_q(\Delta, n)$ for the cases when Δ has no non-trivial factor that is coprime to q. In [KK20, Theorem 1] the possible (effective) lengths of q^r-divisible codes over $\mathbb{F}_q$ were completely characterized for all $r \in \mathbb{N}$. In order to state the result we need a bit more notation. For each $r \in \mathbb{N}$ and each integer $0 \le i \le r$ we define $s_q(r,i) := q^i \cdot [r-i+1]_q$. Note that the number $s_q(r,i)$ is divisible by q^i but not by q^{i+1}. This allows us to create kind of a positional system upon the sequence of base numbers $S_q(r) := \big(s_q(r,0), s_q(r,1), \ldots, s_q(r,r)\big)$. With this, each integer n has a unique $S_q(r)$-*adic expansion*

$$n = \sum_{i=0}^{r} e_i s_q(r,i)$$

with $e_0, \ldots, e_{r-1} \in \{0, \ldots, q-1\}$ and *leading coefficient* $e_r \in \mathbb{Z}$. Rewritten to our geometrical setting [KK20, Theorem 1] says:

Theorem 9. *For $n \in \mathbb{Z}$ and $r \in \mathbb{N}$ the following are equivalent:*

(i) For sufficiently large v there exists a q^r-divisible multiset of points of cardinality n in $\mathrm{PG}(v-1,q)$.

(ii) The leading coefficient e_r of the $S_q(r)$-adic expansion of n is non-negative.

As an example we consider the $S_2(2)$-adic expansion of 9: $1 \cdot 7 + 1 \cdot 6 - 1 \cdot 4$. So, there is no 4-divisible multiset of points with cardinality 9 in $\mathrm{PG}(v-1,2)$, where the dimension v of the ambient space is arbitrary. A direct implication of Theorem 9 is:

Proposition 10. *We have $\Gamma_2(2,1) = \infty$, $\Gamma_2(4,n) = \infty$ for $n \in \{1,2,3,5,9\}$, and $\Gamma_2(8,n) = \infty$ for $n \in \{1,2,3,4,5,6,7,9,10,11,13,17,18,19,21,25,33\}$.*

For $\Delta \in \{2,4,8\}$ the possible cardinalities of Δ-divisible sets of points over $\mathbb{F}_2$ are completely determined, see e.g. [HKK18] and [HKKW19, Theorem 2], where it was shown that no binary 8-divisible projective linear code of effective length 59 exists. In our context this implies:

Proposition 11. *We have $\Gamma_2(2,n) = 1$ iff $n \ge 3$, $\Gamma_2(4,n) = 1$ iff $n \in \{7,8\}$ or $n \ge 14$, and $\Gamma_2(8,n) = 1$ iff $n \in \{15,16,30,31,32,45,46,47,48,49,50,51\}$ or $n \ge 60$.*

We can use the corresponding examples to construct multisets of points of larger divisibility or larger cardinality. If $\mathcal{M}$ is a Δ-divisible multiset of points in $\mathrm{PG}(v-1,q)$, then $q \cdot \mathcal{M}$ is a $q\Delta$-divisible multiset of points in $\mathrm{PG}(v-1,q)$ with cardinality $q \cdot \#\mathcal{M}$ and maximum point multiplicity $q \cdot \gamma_1(\mathcal{M})$. Using the decomposition $\mathbb{F}_q^{v_1} \oplus \mathbb{F}_q^{v_2} \cong \mathbb{F}_q^{v_1+v_2}$ we can also combine two Δ-divisible multisets of points $\mathcal{M}_i$ in $\mathrm{PG}(v_1-1,q)$, where $i = 1,2$, to a Δ-divisible multiset of points in $\mathrm{PG}(v_1+v_2-1,q)$ with cardinality $\#\mathcal{M}_1 + \#\mathcal{M}_2$ and maximum point multiplicity $\max\{\gamma_1(\mathcal{M}_1), \gamma_1(\mathcal{M}_2)\}$. Applied recursively we obtain:

Proposition 12.

(i) We have $\Gamma_2(2,2) = 2$, $\Gamma_2(4,n) = 2$ for $n \in \{6,10,12,13\}$, and $\Gamma_2(8,n) = 2$ for $n \in \{14,28,29,34,36,38,40,42,43,44,52,53,54,55,56,57,58,59\}$.

(ii) We have $\Gamma_2(4,n) \le 4$ for $n \in \{4,11\}$ and $\Gamma_2(8,n) \le 4$ for $n \in \{12,20,24,26,27,35,39,41\}$.

(iii) We have $\Gamma_2(8,n) \le 8$ for $n \in \{8,22,23,37\}$.

The main goal of the remaining part of this paper is to show that the upper bounds in (ii) and (iii) are indeed sharp, see Theorem 13. We remark that the constructions used in [KK20] imply $\Gamma_q(q^r,n)\le r$ whenever $\Gamma_q(q^r,n)\neq\infty$ and $r\in\mathbb{N}$.

Theorem 13.

- *We have* $\Gamma_2(2,1)=\infty$, $\Gamma_2(4,n)=\infty$ *for* $n\in\{1,2,3,5,9\}$, *and* $\Gamma_2(8,n)=\infty$ *for* $n\in\{1,2,3,4,5,6,7,9,10,11,13,17,18,19,21,25,33\}$.
- *We have* $\Gamma_2(2,n)=1$ *iff* $n\ge 3$, $\Gamma_2(4,n)=1$ *iff* $n\in\{7,8\}$ *or* $n\ge 14$, *and* $\Gamma_2(8,n)=1$ *iff* $n\in\{15,16,30,31,32,45,46,47,48,49,50,51\}$ *or* $n\ge 60$.
- *We have* $\Gamma_2(2,2)=2$, $\Gamma_2(4,n)=2$ *for* $n\in\{6,10,12,13\}$, *and* $\Gamma_2(8,n)=2$ *for* $n\in\{14,28,29,34,36,38,40,42,43,44,52,53,54,55,56,57,58,59\}$.
- *We have* $\Gamma_2(4,n)=4$ *for* $n\in\{4,11\}$ *and* $\Gamma_2(8,n)=4$ *for* $n\in\{12,20,24,26,27,35,39,41\}$.
- *We have* $\Gamma_2(8,n)=8$ *for* $n\in\{8,22,23,37\}$.

Another tool that we can use in the task of proving Theorem 13 is the classification of Δ-divisible codes spanned by codewords of weight Δ [KKar]. An exemplary implication is:

Lemma 14. *Let $\mathcal{C}$ be a binary code with non-zero weights in $\{8,16,24\}$ that is spanned by codewords of weight 8. Then, we have*

$$A_8\in\{0,1,2,3,4,6,7,8,9,10,11,13,14,15,16,17,18,21,22,25,29,30,31,33,37,45\}$$

for the number A_8 of the number of codewords of weight 8 in C (including the case that C is empty).

Proof. We apply the classification of [KKar]: The possible weight enumerators of the indecomposable subcodes are given by $1+1x^8$, $1+3x^8$, $1+7x^8$, $1+15x^8$, $1+6x^8+1x^{16}$, $1+10x^8+5x^{16}$, $1+14x^8+1x^{16}$, $1+30x^8+1x^{16}$, $1+15x^8+15x^{16}+1x^{24}$, and $1+21x^8+35x^{16}+7x^{24}$. Combining two such blocks with maximum weight 8 gives the further possibilities $1+2x^8+1x^{16}$, $1+4x^8+3x^{16}$, $1+8x^8+7x^{16}$, $1+2x^8+1x^{16}$, $1+16x^8+15x^{16}$, $1+2x^8+1x^{16}$, $1+6x^8+9x^{16}$, $1+10x^8+21x^{16}$, $1+18x^8+45x^{16}$, $1+14x^8+49x^{16}$, $1+22x^8+105x^{16}$, and $1+30x^8+225x^{16}$. Combining these or a block with maximum weight 16 with a block with maximum weight 16 further possibilities $1+7x^8+7x^{16}+1x^{24}$, $1+9x^8+19x^{16}+3x^{24}$, $1+13x^8+43x^{16}+7x^{24}$, $1+21x^8+91x^{16}+15x^{24}$, $1+11x^8+15x^{16}+5x^{24}$, $1+7x^8+7x^{16}+1x^{24}$, $1+13x^8+35x^{16}+15x^{24}$, $1+17x^8+75x^{16}+35x^{24}$, $1+25x^8+155x^{16}+75x^{24}$, $1+15x^8+15x^{16}+1x^{24}$, $1+17x^8+43x^{16}+3x^{24}$, $1+21x^8+99x^{16}+7x^{24}$, $1+29x^8+211x^{16}+15x^{24}$, $1+31x^8+31x^{16}+1x^{24}$, $1+33x^8+91x^{16}+3x^{24}$, $1+37x^8+211x^{16}+7x^{24}$, and $1+45x^8+451x^{16}+15x^{24}$. □

3 Classification results for even and doubly-even multisets of points

A few classification results are already stated in Corollary 4. Here the characterized multisets of points are given by $\lambda\cdot\chi_K$ for some subspace K. For $n\ge 3$ another construction of a spanning even set of n points in $\mathrm{PG}(n-2,2)$ is given by a so-called *projective base* B_n of size n, i.e., a set of n points such that each $n-1$ points span an $(n-1)$-space.

Proposition 15. *Let $\mathcal{M}$ be a 2-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 5. Then either $\mathcal{M}=\chi_L+2\cdot\chi_P$, where L is a line and P a point, or $\mathcal{M}$ is the characteristic function of a projective base B_5 of size 5.*

Proof. If there exists a point P with $\mathcal{M}(P)\ge 2$, then $\mathcal{M}-2\cdot\chi_P$ is also 2-divisible with cardinality 3, so that we can apply Proposition 3. Thus, we can assume $\gamma_1(\mathcal{M})=1$ in the following so that the standard equations yield $k=4$ for the dimension of the span of $\mathcal{M}$. Since $\gamma_1(\mathcal{M})=1$ no three points form a line L (since otherwise $\mathcal{M}-\chi_L$ would be the characteristic function of a double point). Finally, 2-divisibility implies that no four points can span a plane, which is a hyperplane in our situation. □

Proposition 16. *Let $\mathcal{M}$ be a 2-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 4*. Then either* $\mathcal{M} = 2\cdot\chi_{P_1} + 2\cdot\chi_{P_2}$ *for two points* P_1, P_2 *(that may also be equal) or there exists a plane* E *and line* $L \le E$ *with* $\mathcal{M} = \chi_E - \chi_L = \chi_{E\backslash L}$.

Proof. This is a special case of Proposition 7. □

Proposition 17. *Let $\mathcal{M}$ be a 2-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 6*. Then either $\mathcal{M}$ contains a point of multiplicity at least two, $\mathcal{M}$ is the characteristic function of a projective base B_6 of size* 6 *or* $\mathcal{M} = \chi_{L_1} + \chi_{L_2}$ *for two (disjoint) lines* L_1, L_2.

Proof. If there is no point of multiplicity at least two, then we have $\gamma_1(\mathcal{M}) = 1$, which we assume in the following. If $\mathcal{M}$ contains three points forming a line L, then $\mathcal{M} - \chi_L$ is also 2-divisible of cardinality 3, so that we can apply Proposition 3. In the remaining part we can assume that each three points span a plane and denote the dimension of the span of $\mathcal{M}$ by k. Since each line contains at most two points of $\mathcal{M}$, we have $k \ge 4$. If $k = 4$, then there has to be a plane E containing four points. Since no three points of $\mathcal{M}|_E$ form a line, there exists a line $L \le E$ such that $\mathcal{M}|_E = \chi_E - \chi_L = \chi_{E\backslash L}$. However, then $\mathcal{M}|_E$ is 2-divisible, c.f. Proposition 16, and $\mathcal{M} - \mathcal{M}|_E = 2\cdot\chi_P$ for a suitable point P. Thus, it remains to consider the case $k = 5$. Here 2-divisibility implies $\mathcal{M}(E) \le 3$ for every plane E since otherwise $\mathcal{M}(\langle E, P\rangle) = 6$ for each point $P \notin E$ with $\mathcal{M}(P) = 1$, which contradicts $k = 5$. Moreover, no five points can span a solid, which is a hyperplane for $k = 5$. □

We remark that in the case where $\mathcal{M}$ contains a point P of multiplicity at least two we can apply Proposition 16 to $\mathcal{M} - 2\cdot\chi_P$.

Proposition 18. *Let $\mathcal{M}$ be a 2-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 7*. Then either $\mathcal{M}$ contains a point of multiplicity at least two, there exists a line L such that $\mathcal{M} - \chi_L \ge 0$, or $\mathcal{M}$ is the characteristic function of a projective base B_7 of size* 7.

Proof. W.l.o.g. we assume $\gamma_1(\mathcal{M}) = 1$ and $\mathcal{M}(L) \le 2$ for every line L. If there would be a plane E with $\mathcal{M}(E) \ge 4$, then there would be a line $L \le E$ such that $\mathcal{M}|_E = \chi_{E\backslash L}$. However, in this case $\mathcal{M}|_E$ is 2-divisible and $\mathcal{M} - \mathcal{M}|_E$ would be 2-divisible with cardinality 3, which is possible for the characteristic function of a line only. Thus, we assume $\mathcal{M}(E) \le 3$ for every plane and that each three points of $\mathcal{M}$ span a plane. So, we have $k \ge 5$ for the dimension of the span of $\mathcal{M}$ since the standard equations do not have a solution with $a_5 = 0$ for $k = 4$. For $k = 5$ there would be a solid S with $\mathcal{M}(S) = 5$. Noting that $\mathcal{M}(L) \le 2$ for each line $L \le S$ and $\mathcal{M}(E) \le 3$ for each plane $E \le S$ we conclude that $\mathcal{M}|_S$ would be a projective base of size 5, so that $\mathcal{M} - \mathcal{M}|_S = 2\cdot\chi_P$ for a suitable point P, which contradicts our assumption. Thus, it remains to consider the case $k = 6$. Here we have $\mathcal{M}(S) \le 4$ for each solid since otherwise $\mathcal{M}(\langle S, P\rangle) = 7$ for each point $P \notin S$ with $\mathcal{M}(P) = 1$. From 2-divisibility we conclude $\mathcal{M}(H) \le 5$ for each hyperplane H, so that $\mathcal{M}$ has to be the characteristic function of a projective base of size 7. □

In the case where $\mathcal{M}$ contains a point P of multiplicity at least two we can apply Proposition 15 to $\mathcal{M} - 2\cdot\chi_P$. If a line L with $\mathcal{M} - \chi_L \ge 0$ exists, then we can apply Proposition 16 to $\mathcal{M} - \chi_L$. We remark that for each dimension $3 \le k \le 7$ of the span of $\mathcal{M}$ there exists an up to symmetry unique example, if we assume $\gamma_1(\mathcal{M}) = 1$. For even sets of points over $\mathbb{F}_2$ of cardinality $n \ge 8$ the classification gets more involved, see [HHK+17] for computational results.[1]

Let $\mathcal{M}$ be a doubly-even multiset of points over $\mathbb{F}_2$. Cardinalities $n \in \{4, 6, 7\}$ are characterized in Corollary 4 and cardinality $n = 8$ is characterized in Proposition 7. Due to Proposition 10 we have $\#\mathcal{M} \ge 10$ for all other feasible cases.

Proposition 19. *Let $\mathcal{M}$ be a 4-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 10*. Then there either exists a point P with $\mathcal{M}(P) \ge 4$ or $\mathcal{M} = 2\cdot\chi_{B_5}$ where B_5 denotes a projective base of size* 5.

[1] Note that adding a parity check bit to an arbitrary binary linear code yields a 2-divisible linear code whose effective length is increased by one, so that the classification of even sets of points over $\mathbb{F}_2$ is equivalent to the classification of sets of points over $\mathbb{F}_2$ (in some sense).

Proof. W.l.o.g. we assume $\gamma_1(\mathcal{M}) \le 3$ and that $\mathcal{M}$ is spanning with dimension k. From the standard equations we compute $a_2 = 2^{k-2}+1$ and $a_6 = 3 \cdot 2^{k-2} - 2$, so that $\lambda_2 \ge 1$ since each hyperplane with multiplicity 2 is 2-divisible, see Lemma 1, and so contains a double point P_2. If P_3 is a point with multiplicity 3, then each hyperplane H containing P_3 has multiplicity $\mathcal{M}(H) = 6$, so that Lemma 6 yields $k = 4$ via $\mathcal{M}(P_3) = 2+2^{4-k}$. For the line L spanned by P_2 and P_3 we have $\mathcal{M}(H) = 6$ for all hyperplanes H containing L, so that Lemma 6 yields $\mathcal{M}(L) = 4 < \mathcal{M}(P_2) + \mathcal{M}(P_3)$ – contradiction. Thus, we have $\gamma_1(\mathcal{M}) = 2$ and $k \ge 3$ since $(2^2-1)\cdot 2 = 6 < 10$. Moreover, $k \ne 3$ since $\mathcal{M}'$ defined by $\mathcal{M}'(P) = 2-\mathcal{M}(P)$ is also 4-divisible with cardinality $2 \cdot \left(2^k-1\right) - 10 = 4$, which is impossible.

Let L be a line with $\mathcal{M}(L) \ge 3$, which clearly exists due to $\lambda_2 \ge 1$ and $\lambda_1 + \lambda_2 \ge 2$. Each hyperplane H containing L has multiplicity $\mathcal{M}(H) = 6$, so that Lemma 6 yields $\mathcal{M}(L) = 4$ and $k = 4$. With this, solving the standard equations gives $\lambda_1 = 0$ and $\lambda_2 = 5$. Thus, $\frac{1}{2} \cdot \mathcal{M}$ is 2-divisible with cardinality 5 and we can apply Proposition 15. □

Using Corollary 4 and Proposition 19 we conclude:

Corollary 20. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 10. Then, we have $\mathcal{M}(P) \in \{0,2,4,6\}$ for every point P.*

Lemma 21. *Let $0 \le l < r$ be integers and $\mathcal{M}$ be a spanning q^r-divisible multiset of points in $\mathrm{PG}(k-1,q)$ of cardinality $n = q^l \cdot \frac{q^{r+1-l}-1}{q-1} + q^r$. Then $\gamma_1(\mathcal{M}) = q^l$ or $\gamma_1(\mathcal{M}) = q^r - \frac{q^l - q^{2+r-k}}{q-1}$.*

Proof. The possible hyperplane multiplicities are $m_1 := q^l \cdot \frac{q^{r+1-l}-1}{q-1} = n - q^r$ and $m_2 := q^l \cdot \frac{q^{r-l}-1}{q-1} = n - 2q^r$. Due to Lemma 2 both multiplicities indeed occur. If H is a hyperplane with multiplicity $\mathcal{M}(H) = m_2 = q^l \cdot \frac{q^{r-l}-1}{q-1}$, then Proposition 3 implies the existence of an $(r-l)$-dimensional subspace S' in H with $\mathcal{M}|_H = q^l \cdot \chi_{S'}$. Thus we have $\gamma_1(\mathcal{M}) \ge q^l$. If P is a point with multiplicity $\mathcal{M}(P) > q^l$, then each hyperplane through P has cardinality m_1. Lemma 6 yields

$$\mathcal{M}(P) = n - \frac{q^r}{q^{k-2}} \cdot [k-1]_q = q^r - \frac{q^l - q^{2+r-k}}{q-1}.$$

□

Proposition 22. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 11. Then, $\mathcal{M} = \chi_E + 4 \cdot \chi_P$, where E is a plane and P a point.*

Proof. Since no 4-divisible set of 11 points in $\mathrm{PG}(v-1,2)$ exists, Lemma 21 implies $\gamma_1(\mathcal{M}) = 3+2^{2+r-k} \ge 4$, where k is the dimension of the span of $\mathcal{M}$ and $r = 2$. Reducing the multiplicity of a point P with maximum multiplicity by four gives a 4-divisible multiset of points with cardinality 7, which is the characteristic function of a plane by Proposition 3. □

Proposition 23. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 12. Then either there exists a point with multiplicity at least four or all points have even multiplicity.*

Proof. W.l.o.g. we assume $\gamma_1(\mathcal{M}) \le 3$. Assume that P is a point with multiplicity $\mathcal{M}(P) = 3$. If H is a hyperplane with $P \le H$ and $\mathcal{M}(H) = 4$, then we can apply Proposition 16 to conclude a contradiction since $\mathcal{M}|_H$ is 2-divisible, see Lemma 1. Thus, all hyperplanes containing P have multiplicity 8 and Lemma 6 yields a contradiction. So, we conclude $\gamma_1(\mathcal{M}) = 2$ from Proposition 11.

Denoting the dimension of the span of $\mathcal{M}$ by k we conclude $k \ge 3$ from $2 \cdot \left(2^k-1\right) \ge 11$. Moreover, $k \ne 3$ since $\mathcal{M}'$ defined by $\mathcal{M}'(P) = 2 - \mathcal{M}(P)$ is also 4-divisible with cardinality $2 \cdot 7 - 12 = 2$ otherwise, which is clearly impossible. Now assume that P_1 is a point with multiplicity 1 and P_2 a point with multiplicity 2. Consider the line L spanned by P_1 and P_2. Now observe that $\mathcal{M}(H) \ne 4$ for each hyperplane H containing L since $\mathcal{M}|_H$ is 2-divisible and Proposition 16 would yield a contradiction otherwise. Since we have $\lambda_2 \ge 1$, this implies $\lambda_1 = 0$. □

So, we can read off the explicit classification from Proposition 7 or Proposition 17.

Proposition 24. *Let $\mathcal{M}$ be a 4-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 13*. Then, either $\mathcal{M}=\chi_E+2\cdot\chi_L$, where E is a plane and L a line, or there exists a projective base B_5 of size 5 and a point C outside of the span of B_5 such that $\mathcal{M}(C)=3$, $\mathcal{M}(Q)=1$ if there exists a point P in B_5 such that $Q\in\langle P,C\rangle$, $Q\neq C$, and $\mathcal{M}(Q)=0$ otherwise.*

Proof. Since no 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 9 exists, we have $\gamma_1(\mathcal{M})<4$ and $k\ge 3$ for the dimension k of the span of $\mathcal{M}$. Due to Corollary 4 it suffices to show the existence of a plane E with $\mathcal{M}\ge\chi_E$. If $k=3$, then we consider $\mathcal{M}'$ defined by $\mathcal{M}'(P)=3-\mathcal{M}(P)$. With this, $\mathcal{M}'$ is also 4-divisible, has cardinality $3\cdot\left(2^k-1\right)-\#\mathcal{M}=8$ and maximum point multiplicity at most 3. From Proposition 7 we conclude $\gamma_1(\mathcal{M}')\le 2$, so that we can choose E as the ambient space. In the remaining part we have $k\ge 4$. Since there is no 2-divisible multiset of cardinality 1, we can use the standard equations to compute $a_5=5\cdot 2^{k-3}+1$, $a_9=3\cdot 2^{k-3}-2$, and $\lambda_2=1-3\lambda_3+2^{6-k}$.

Assume that P_1,P_2 are two different points with $\mathcal{M}(P_1),\mathcal{M}(P_2)\ge 2$. Let L be the line spanned by P_1,P_2 and H be an arbitrary hyperplane containing L. Since $\mathcal{M}|_H$ is 2-divisible and contains both P_1 and P_2, Proposition 15 yields $\mathcal{M}(H)=9$. Applying Lemma 6 yields $\mathcal{M}(L)=5+2^{5-k}$. If $k=5$, then $\lambda_2=3-3\lambda_3$ and the assumption $\lambda_2+\lambda_3\ge 2$ implies $\lambda_3=0$ and $\lambda_2=3$, so that $\mathcal{M}\ge 2\cdot\chi_L$ (using $\mathcal{M}(L)=6$). If $k=4$, then we have $\mathcal{M}(L)=7$ and $\lambda_3\le 1$ also implies $\mathcal{M}\ge 2\cdot\chi_L$. Since $2\cdot\chi_L$ is 4-divisible $\mathcal{M}-2\cdot\chi_L$ is also 4-divisible with cardinality 7, Corollary 4 implies the existence of a plane E with $\mathcal{M}=2\cdot\chi_L+\chi_E$.

If $\lambda_2+\lambda_3\le 1$, then $\lambda_2=1-3\lambda_3+2^{6-k}$ implies $\lambda_2=0$, $\lambda_3=1$, and $k=5$. If P is a point with multiplicity 1, then all hyperplanes containing the line L spanned by P and the unique point of multiplicity 3 have multiplicity 9, so that Lemma 6 yields $\mathcal{M}(L)=5$. In other words there exist five pairwise different lines $L_1,\dots,L_5$ that all contain the unique point of multiplicity 3, denoted by P_3, such that $\mathcal{M}=-2\chi_{P_3}+\sum_{i=1}^5\chi_{L_i}$. If S is a solid, then S can contain at most three of the lines L_i (using $\mathcal{M}(S)\le 9$) and intersects the others in a point. Thus, factoring out P_3 from the L_i yields a projective base of size 5. □

Lemma 25. *Let $\mathcal{M}$ be a 4-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with $\#\mathcal{M}=15$ and $2\le\gamma_1(\mathcal{M})\le 3$ that does not contain a plane in its support. Then, we have $k\ge 5$ for the dimension k of the span of $\mathcal{M}$ and there does not exist a line L' with $\mathcal{M}(P)\ge 2$ for all points P in L'. Moreover, we have*

$$a_7 = 7\cdot 2^{k-3}+1-2a_3, \qquad (6)$$

$$a_{11} = 2^{k-3}-2+a_3,\ \text{and} \qquad (7)$$

$$a_3 = (4+\lambda_2+3\lambda_3)\cdot 2^{k-6}-1 \qquad (8)$$

for the spectrum of $\mathcal{M}$ and the multiplicity $\mathcal{M}(S)$ of each subspace S of codimension 2 *is odd.*

Proof. Since $\gamma_1(\mathcal{M})\le 3$, we have $k\ge 3$. If $k=3$, then we consider $\mathcal{M}'$ defined by $\mathcal{M}'(P)=3-\mathcal{M}(P)$. With this, $\mathcal{M}'$ is 4-divisible with cardinality 6, to that Corollary 4 implies $\gamma_1(\mathcal{M}')\le 2$. Thus, we can choose E as the ambient space and have $\mathcal{M}\ge\chi_E$. In the remaining part we have $k\ge 4$. The stated equations for the spectrum can be directly concluded from the standard equations. If $k=4$, then $\lambda_2+\lambda_3\ge 1$ implies $a_3\ge 1$. Let H be a hyperplane with multiplicity 3 and L be a line such that $\mathcal{M}|_H=\chi_L$. For the two other hyperplanes H', H'' that contain L w.l.o.g. we can assume $\mathcal{M}(H')=7$ and $\mathcal{M}(H'')=11$. Since $\mathcal{M}|_{H'}$ is not the characteristic function of a plane there exists a point $P\le H'$ with $\mathcal{M}(P)\ge 2$ so that $\mathcal{M}|_{H'}-\chi_L-2\cdot\chi_P$ is a double point and we have $\lambda_2\ge 2$. Thus, $a_3\in\mathbb{N}$ implies $\lambda_2=4$, $\lambda_3=0$, and $\lambda_1=7$. However, there cannot be seven points of multiplicity 1 and two points of multiplicity 2 in H''.

If L' is a line with $\mathcal{M}(P)\ge 2$ for all points P in L', then $\mathcal{M}-2\cdot\chi_{L'}$ would be 4-divisible with cardinality 9, which is impossible. If S is a subspace of codimension 2, then denote the three hyperplanes containing S by H_1,H_2,H_3, so that $\#\mathcal{M}+2\cdot\mathcal{M}(S)=\mathcal{M}(H_1)+\mathcal{M}(H_2)+\mathcal{M}(H_3)\equiv 1 \pmod 4$ yielding $\mathcal{M}(S)\equiv 1 \pmod 2$. □

Lemma 26. *Let $\mathcal{M}$ be a 4-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with $\#\mathcal{M}=15$ and $2\le\gamma_1(\mathcal{M})\le 3$ that does not contain a plane in its support. There exist a_3 lines L_i sharing a common point B such that*

$\mathcal{M}(P)=1$ *for all points* P *contained in one of the lines* L_i *and* $\lambda_1 \ge 2a_3+1$*. Moreover, we have* $k=5$ *for the dimension* k *of the span of* $\mathcal{M}$*.*

Proof. Let H be a hyperplane of multiplicity 3 and L be a line with $\mathcal{M}|_H = \chi_L$. If H' is another hyperplane with multiplicity 3 and L' a line with $\mathcal{M}|_{H'} = \chi_{L'}$, then we have $L \neq L'$ since otherwise $\mathcal{M}(H\cap H')=3$ and the third hyperplane containing $H\cap H'$ would have multiplicity $15=\#\mathcal{M}$. So, there exist a_3 lines L_i such that $\mathcal{M}(P)=1$ for all points P contained in one of the lines L_i. Moreover any two such lines L_i intersect in exactly a point.

So, if $a_3 \le 2$, then there exist a_3 lines L_i sharing a common point B such that $\mathcal{M}(P)=1$ for all points P contained in one of the lines L_i and $\lambda_1 \ge 2a_3+1$.

If there exist three of the lines L_i with pairwise different intersection points, then they span a plane E with six points of multiplicity 1 noting that the 7th point P has multiplicity 0 since $\mathcal{M}$ does not contain a plane in its support. Since the multiplicity of every subspace of codimension 2 is odd, we have $k\ge 6$, see Lemma 25. With this Equation (8) yields $a_3 \ge 4$. The fourth line L_i also has to be completely contained in E, which is impossible.

Thus, in general all lines L_i intersect in a common point B and we conclude $\lambda_1 \ge 2a_3+1$. Due to Lemma 25 it remains to show $k\le 5$. From $\lambda_2+\lambda_3\ge 1$ we conclude $\lambda_1 \le 13$ and $a_3\le 6$, so that Equation (8) implies $k\le 6$. If $k=6$, then Equation (8) gives $a_3\ge 4$. Let S be a solid spanned by three of the lines L_i. If S also contains a fourth line L_i, then we have $\mathcal{M}(S)\ge 9$ and all three hyperplanes containing S have multiplicity 11 and indeed $\mathcal{M}(S)=9$. Let H be one of these hyperplanes that contains a point Q with multiplicity at least 2, so that indeed $\mathcal{M}(Q)=2$. W.l.o.g. we assume that the four lines in S are labeled $L_1,\dots,L_4$. Note that $\mathcal{M}|_H$, $2\cdot\chi_Q$, χ_{L_1}, and $\chi_{(L_2\cup L_3)\backslash B}$ are 2-divisible so that $\mathcal{M}|_H - 2\cdot\chi_Q - \chi_{L_1} - \chi_{(L_2\cup L_3)\backslash B} = \chi_{L_4\backslash B}$ is also 2-divisible, which is a contradiction. Thus, any four lines L_i span a hyperplane H with multiplicity 11. Pick one of these and a solid S that intersects these four L_i in exactly point B, so that $\mathcal{M}(S)\in\{1,3\}$. If $\mathcal{M}(S)=1$, then the two other hyperplanes that contain S have multiplicity 3, which would imply $\gamma_1(\mathcal{M})=1<2$. If $\mathcal{M}(S)=3$, then one of the other two hyperplanes containing S has multiplicity 3 and one has multiplicity 7, which implies $a_3\ge 5$. With this, Equation (8) gives $a_3=5$, $\lambda_1=11$, $\lambda_2=2$, and $\lambda_3=0$. Let $\mathcal{H}$ denote the 16 hyperplanes not containing B. Since $5\cdot 1+2\cdot 2=9<11$ we have $\mathcal{M}(H)\in\{3,7\}$ for all $H\in\mathcal{H}$, so that $8\cdot 14=\sum_{H\in\mathcal{H}}(\mathcal{M}(H)-1)\ge 6\cdot 16$ – contradiction.

Thus, we have $k\le 5$ and Lemma 25 gives $k=5$. □

Proposition 27. *Let* $\mathcal{M}$ *be a* 4*-divisible multiset of points in* $\operatorname{PG}(v-1,2)$ *with cardinality* 15*. Then either there exists a point of multiplicity at least* 4*,* $\gamma_1(\mathcal{M})=1$*, or there exists a plane* E *with* $\mathcal{M}\ge\chi_E$*.*

Proof. W.l.o.g. we assume $2\le\gamma_1(\mathcal{M})\le 3$ and that $\mathcal{M}$ does not contain a plane in its support. Lemma 26 states $k=5$ for the dimension of the span of $\mathcal{M}$, so that Equation (8) yields $a_3\ge 2$. Using the notation from Lemma 26 we consider the plane $E:=\langle L_1,L_2\rangle$ with $\mathcal{M}(E)\ge 5$. Due to Lemma 25 $\mathcal{M}(E)$ is odd. Since $\mathcal{M}$ does not contain a plane in its support, we have $\mathcal{M}(E)\in\{5,7\}$. If $\mathcal{M}(E)=7$, then there exists a point $P\le E$ with multiplicity $\mathcal{M}(P)=2$. However, E is contained in a hyperplane H of multiplicity 7 and $\mathcal{M}|_H - 2\cdot\chi_P - \chi_{L_1} = \chi_{L_2\backslash B}$ is 2-divisible – contradiction. Thus, $\mathcal{M}(E)=5$ and for the three hyperplanes H_1,H_2,H_3 containing E we can assume w.l.o.g. $\mathcal{M}(H_1)=7$, $\mathcal{M}(H_2)=7$, and $\mathcal{M}(H_3)=11$. Since $\mathcal{M}|_{H_i}-\chi_{L_1}$ is 2-divisible for $i=1,2$ we have $\gamma_1(H_i)=1$. Now let $P\le H_3$ be a point of multiplicity at least 2, so that $\mathcal{M}|_{H_3}-2\cdot\chi_P-\chi_{L_1}$ is 2-divisible with cardinality 6, so that Proposition 17 implies $\lambda_2+\lambda_3\ge 2$. With this we conclude $\lambda_1=11$, $\lambda_2=2$, $\lambda_3=0$, and $(a_3,a_7,a_{11})=(2,25,4)$. For two hyperplanes H, H' of multiplicity 11 let E' denote their intersection. Counting points gives $\mathcal{M}(E')\ge 7$. Since $\mathcal{M}$ does not contain a plane in its support, we have $\mathcal{M}(E')=7$, so that the third hyperplane H'' containing E' has multiplicity 7 and contains a double point P. So, $\mathcal{M}|_{H''}-2\cdot\chi_P$ is 2-divisible with cardinality 5 and dimension at most 3. Using Proposition 15 we conclude that E' contains both points of multiplicity 2 and three points of multiplicity 1 that form a line. Let L' be the line spanned by the two double points and Q be the third point on the line, so that $\mathcal{M}(Q)=1$ and $\mathcal{M}(L')=5$. Each of $\binom{4}{2}=6$ pairs of hyperplanes of multiplicity 11 yields a different plane $E'\ge L'$, so that $\lambda_1\ge 1+6\cdot 2=13$, which is a contradiction. □

We remark that if a 4-divisible multiset of points $\mathcal{M}$ in $\operatorname{PG}(v-1,2)$ with cardinality 15 contains a point P with multiplicity at least 4, then $\mathcal{M}-4\cdot\chi_P$ is also 4-divisible with cardinality 11, so that Proposition 22 implies the existence of a plane in the support of $\mathcal{M}$. For $\gamma_1(\mathcal{M}) = 1$ the possibilities have been classified in [HKK19a]. Except for a single case all point sets also contain a plane in its support. We summarize the result in:

Corollary 28. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 15. Then either there exists a plane E and a 4-divisible multiset of points $\mathcal{M}'$ in $\operatorname{PG}(v-1,2)$ with cardinality 8 such that $\mathcal{M} = \chi_E + \mathcal{M}'$ or there exists a projective base B_7 of seven points and a point P outside of $\langle B_7\rangle$ such that $\mathcal{M}(Q) = 1$ iff there is a point P' in B_7 with $Q \le \langle P', P\rangle$ and $\mathcal{M}(Q) = 0$ otherwise.*

Lemma 29. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 16 and $\gamma_1(\mathcal{M}) \le 3$. Then, we have $\lambda_3 \le 4$ and $\lambda_2 + 3\lambda_3 \le 12$. If $\lambda_3 < 4$, then $\lambda_3 \le 2$.*

Proof. W.l.o.g. we assume $\gamma_1(\mathcal{M}) = 3$. Note that $\lambda_2 \le \frac{16-3\lambda_3}{2}$ implies $\lambda_2 + 3\lambda_3 \le 8 + \frac{3\lambda_3}{2} \le 12.5$ for $\lambda_3 \le 3$. So, let P_1, P_2, P_3 be three arbitrary different points with multiplicity at least 2. If they form a line L, then $\mathcal{M} - 2\cdot\chi_L$ is 4-divisible with cardinality 10, so that Corollary 20 yields a contradiction. Thus, any three points with multiplicity at least two span a plane E. If E contains a fourth point of multiplicity at least 2, then they form an affine plane A, so that we can apply Proposition 7 to conclude the statement. Since there is no 4-divisible multiset of points with cardinality $[3]_2 \cdot 3 - \#\mathcal{M} = 7\cdot 3 - 16 = 5$, we have $k \ge 4$ for the dimension of the span of $\mathcal{M}$. For the case $k = 4$ consider a plane E spanned by three points of multiplicity 3. Since $\mathcal{M}(E) \ge 9$, we have $\mathcal{M}(E) = 12$. However, the three different lines spanned by the three pairs of the considered three points of multiplicity 3 have even cardinality and the third point has multiplicity at most 1, so that E contains three points with multiplicity 0 that form a line L'. Since the fourth point in $E\backslash L'$ has multiplicity at most 1, we obtain the contradiction $\mathcal{M}(E) \le 10$. Thus, we can assume $k \ge 5$ in the following.

Let E be a plane spanned by three points of multiplicity 3. Since $\mathcal{M}(E) \ge 9$ we have $\mathcal{M}(H) = 12$ for every hyperplane H containing E, so that Lemma 6 yields $\mathcal{M}(E) = 8 + 2^{6-k}$. An arbitrary line L that is contained in hyperplanes of multiplicity 12 only has multiplicity $\mathcal{M}(L) = 8 + 2^{5-k}$, which is impossible. Now assume $k = 5$ for a moment, so that $\mathcal{M}(E) = 10$ and E contains an additional point Q with multiplicity 1. If there would be a line $L \le E$ with multiplicity 7, then for a hyperplane $H \ge L$ with multiplicity 8 we would have that $\mathcal{M}|_H - \chi_L$ is 2-divisible with cardinality 5 containing two double points – contradiction. Thus, the four points with non-zero multiplicity in E form an affine plane, i.e., $\mathcal{M}|_E$ is 2-divisible. So, for each of the three hyperplanes H containing E we have $\mathcal{M}(H) = 12$ and $\mathcal{M}|_H - \mathcal{M}|_E$ is 2-divisible with cardinality 2, i.e., a double point. Thus, we conclude $\lambda_3 = 3$, $\lambda_2 = 3$, and $\lambda_1 = 1$. However, for a line L spanned by two points P', P'' of multiplicity 3 we have $\mathcal{M}(L) = 6$ there exists a hyperplane $H \ge L$ with multiplicity $\mathcal{M}(H) = 8$. Thus, $\mathcal{M}|_H - 2\cdot\chi_{P'} - 2\cdot\chi_{P''}$ is 2-divisible with cardinality 4 containing at least two points of multiplicity 2, so that Proposition 7 yields the existence of two points with multiplicity 1 outside of L. This contradicts $\lambda_1 = 1$ and it remains to consider the case $k = 6$. Here the plane E spanned by three points of multiplicity 3 has multiplicity 9 and any solid containing S has multiplicity 10. Thus, we have $\lambda_3 = 3$, $\lambda_2 = 0$, and $\lambda_1 = 7$. Now consider a line L spanned by two points of multiplicity 3, so that $\mathcal{M}(L) = 6$, and let $H \ge L$ be a hyperplane with multiplicity 8, so that Proposition 7 yields that $\mathcal{M}|_H$ spans a plane E' and the four points of non-zero multiplicity form an affine plane. Let S be the solid spanned by E' and the third point of multiplicity 3, so that $\mathcal{M}(S) \ge 8 + 3 = 11$, which is impossible for $k = 6$. □

Lemma 30. *Let $\mathcal{M}$ be a 4-divisible spanning multiset of points in $\operatorname{PG}(k-1,2)$ with cardinality 16, $\gamma_1(\mathcal{M}) = 3$, and $\lambda_2 \ge 2$. Then, we have $(\lambda_1, \lambda_2, \lambda_3) \in \{(7,3,1), (6,2,2), (9,2,1)\}$.*

Proof. Let P_1, P_2, P_3 be three arbitrary different points with multiplicity at least 2. If they form a line L, then $\mathcal{M} - 2\cdot\chi_L$ is 4-divisible with cardinality 10, so that Corollary 20 yields a contradiction. Thus, any three points with multiplicity at least two span a plane E. If E contains a fourth point of multiplicity at least 2, then they form an affine plane A, so that we can apply Proposition 7 to conclude the statement. Assume $\mathcal{M}(P_1) = 3$, $\mathcal{M}(P_2) = \mathcal{M}(P_3) = 2$, and that each plane contains at most three points of multiplicity at least 2 in the following. Since there is no 4-divisible multiset of cardinality 9, E contains at least one point with multiplicity 0, so that $7 \le \mathcal{M}(E) \le 10$. Since $\mathcal{M}|_H$ is 2-divisible for any hyperplane H, we have $\mathcal{M}(H) = 12$ if $H \ge E$, so that Lemma 6 yields $\mathcal{M}(E) = 8 + 2^{6-k}$, which implies $\mathcal{M}(E) \in \{9, 10\}$

and $k \in \{5,6\}$. Since E contains only three points of multiplicity at least 2, we have $a_0 = 0$. With this, we conclude $\lambda_2 + 3\lambda_3 = 2^{6-k} \cdot (3 + a_{12}) - 8$. Let Q be an arbitrary point outside of E and S be the solid spanned by Q and E. If $k = 5$, then $\mathcal{M}(S) = 12$ and $\mathcal{M}(E) = 10$ implies $\mathcal{M}(Q) \le 2$, so that $\lambda_3 = 1$. Using $\lambda_2 + 3\lambda_3 = 2a_{12} - 2$ we conclude $\lambda_2 \equiv 1 \pmod 2$, so that $\lambda_2 \ge 3$. Now consider a plane E' spanned by three points of multiplicity 2. Since E' does not contain a fourth point of multiplicity at least 2, we conclude that every hyperplane $H' \ge E'$ has multiplicity 12. As before, we can conclude $\mathcal{M}(E') = 10$, which then implies $\mathcal{M}(P) \ge 1$ for all $P \le E$ – contradiction. It remains to consider the case $k = 6$ where $\mathcal{M}(E) = 9$ and $\mathcal{M}(S) = 10$ for every solid $S \ge E$, so that $\lambda_2 + \lambda_3 = 3$. □

Lemma 31. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 17 and $2 \le \gamma_1(\mathcal{M}) \le 3$. Then, we have*

$$\begin{aligned} a_5 &= 2^{k-3} + 2 + a_{13}, \\ a_9 &= 7 \cdot 2^{k-3} - 3 - 2a_{13}, \\ \lambda_2 + 3\lambda_3 &= -5 + 2^{6-k} \cdot (3 + a_{13}), \end{aligned}$$

and $k \ge 4$. Moreover, each three points of multiplicity at least 2 span a plane and each four points of multiplicity at least 2 span a solid.

Proof. Since no 2-divisible multiset of points of cardinality 1 exists, the multiplicities of the hyperplanes are contained in $\{5, 9, 13\}$. From the standard equations we compute the stated equations. Clearly we have $k \ge 3$. If $k = 3$, then $\mathcal{M}'$ defined by $\mathcal{M}'(P) = 3 - \mathcal{M}(P)$ would be 4-divisible with $\#\mathcal{M}' = 4$ and $\gamma_1(\mathcal{M}') \le 3$ – contradiction. If L' is a line with $\mathcal{M} \ge 2 \cdot \chi_{L'}$, then $\mathcal{M} - 2 \cdot \chi_{L'}$ would be 4-divisible with cardinality 11, so that Proposition 22 implies $\gamma_1(\mathcal{M}) \ge 4$ – contradiction. In other words each three points of multiplicity at least 2 span a plane. If E is a plane and $L \le E$ a line with $\mathcal{M} \ge 2 \cdot \chi_{E \setminus L}$, then $\mathcal{M} - 2 \cdot \chi_{E \setminus L}$ would be 4-divisible with cardinality 9 – contradiction. Thus, any four points of multiplicity at least 2 span a solid. □

Lemma 32. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 17 and $2 \le \gamma_1(\mathcal{M}) \le 3$. Then there exists a point P with $\mathcal{M}(P) = 2$, $k \ge 6$, or $\lambda_3 = 1$.*

Proof. Assuming $\lambda_2 = 0$ the standard equations yield

$$\begin{aligned} a_5 &= 2^{k-3} + 2 + a_{13}, \\ a_9 &= 7 \cdot 2^{k-3} - 3 - 2a_{13}, \text{ and} \\ 3\lambda_3 &= (3 + a_{13}) \cdot 2^{6-k} - 5, \end{aligned}$$

where k denotes the dimension of the span of $\mathcal{M}$. If $k \le 3$, then $\lambda_3 \le 5$ implies $a_{13} < 0$ – contradiction. If $k = 4$, then $\lambda_3 \equiv 1 \pmod 4$, so that we can assume $\lambda_3 = 5$. With this we have $a_5 = 6$, $a_9 = 7$, $a_{13} = 2$, and $\lambda_1 = 2$. Let L be the line spanned by the two points of multiplicity one. From Proposition 15 we conclude that every hyperplane of multiplicity 5 has to contain L. However, L is contained in three hyperplanes only – contradiction.

Finally, assume $k = 5$ and $\lambda_3 \ge 2$. If $\lambda_3 = 2$, then $3\lambda_3 = (3 + a_{13}) \cdot 2^{6-k} - 5$ would imply that a_{13} is fractional. So, let P_1, P_2, P_3 be different points with multiplicity 3. They cannot form a line L since $\mathcal{M} - 2 \cdot \chi_L$ would be 4-divisible with cardinality 11 but does not contain a point of multiplicity at least 4, which contradicts Proposition 22. So, let E be the plane spanned by P_1, P_2, and P_3. Clearly every hyperplane H containing E has multiplicity at least 9. However, multiplicity 9 is impossible, since otherwise $\mathcal{M}|_H - \sum_{i=1}^{3} 2 \cdot \chi_{P_i}$ would be 2-divisible of cardinality 3, so that Corollary 4 yields that P_1, P_2, P_3 form a line – contradiction. Thus, we have $\mathcal{M}(E) = 11$, which implies $\lambda_3 = 3$, $\lambda_1 = 8$, $a_{13} = 4$, $a_5 = 10$, and $a_9 = 17$. Now let $L \le E$ be a line with multiplicity 7, so that $\mathcal{M}|_L$ is 2-divisible and all 7 hyperplanes containing L have multiplicity 13 – contradiction. □

Lemma 33. *Let $\mathcal{M}$ be a 4-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 17 and $2 \le \gamma_1(\mathcal{M}) \le 3$. If there exists a line L consisting of three points P_i with $\mathcal{M}(P_i) = i$ for $1 \le i \le 3$, then we have $k = 5$,*

$\lambda_1 = 9$, $\lambda_2 = 1$, $\lambda_3 = 2$, $a_{13} = 3$, $a_9 = 19$, *and* $a_5 = 9$. *Up to symmetry a unique representation of* $\mathcal{M}$ *is given by the columns of*

$$\begin{pmatrix} 111 & 1 & 111 & 1 & 00 & 0 & 00 & 00 & 00 \\ 000 & 1 & 111 & 0 & 11 & 0 & 00 & 00 & 11 \\ 000 & 0 & 111 & 1 & 00 & 1 & 01 & 01 & 01 \\ 000 & 0 & 000 & 0 & 00 & 0 & 11 & 00 & 11 \\ 000 & 0 & 000 & 0 & 00 & 0 & 00 & 11 & 11 \end{pmatrix}.$$

Proof. Since $\mathcal{M}(L) = 6 \not\equiv \#\mathcal{M} \pmod 2$, Lemma 1 implies $k \geq 5$. Assume that Q is another point with $\mathcal{M}(Q) \geq 2$ and consider the plane E spanned by L and Q. If $H \geq E$ is a hyperplane with multiplicity 9, then $\mathcal{M}' := \mathcal{M}|_H - \chi_L - 2 \cdot \chi_{P_3} - 2 \cdot \chi_Q$ would be 2-divisible with cardinality 2, so that $\mathcal{M}' = 2 \cdot \chi_P$ for some point P. However, we have $\mathcal{M}'(P_2) = 1$ – contradiction. Thus, every hyperplane $H \geq E$ has multiplicity 13, so that Lemma 6 yields $\mathcal{M}(E) = 9 + 2^{6-k}$.

If $k = 5$, then $\mathcal{M}(E) = 11$ and the other six planes E' containing L have multiplicity 7, so that all points of multiplicity at least 2 are contained in E. Lemma 31 then yields $\lambda_2 + \lambda_3 = 3$, so that $\lambda_2 + 3\lambda_3 = -5 + 2^{6-k} \cdot (3 + a_{13})$ and $a_{13} \geq 3$ implies $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_1 = 9$, and $a_{13} = 3$. Using the equations in Lemma 31 we then compute $a_9 = 19$ and $a_5 = 9$.

If $k = 6$, then $\mathcal{M}(E) = 10$ and each solid $S \geq E$ has multiplicity 11. Thus, as before, we conclude that all points with multiplicity at least 2 are contained in E and $\lambda_2 + \lambda_3 = 3$. Let R_1, R_2, R_3 be pairwise different points such that $\mathcal{M}|_E - \chi_L - 2 \cdot \chi_{P_3} - 2 \cdot \chi_Q = \sum_{i=1}^{3} \chi_{R_i}$. For any two points Z_1, Z_2 of multiplicity 1 outside of E there exists a third point Z_3 (depending on Z_1 and Z_2), so that $\mathcal{M}|_H = \mathcal{M}|_E + \sum_{i=1}^{3} \chi_{Z_i}$ for the hyperplane $H = \langle E, Z_1, Z_2 \rangle$ and $\mathcal{M}' := \sum_{i=1}^{3} \chi_{R_i} + \sum_{i=1}^{3} \chi_{Z_i}$ is 2-divisible consisting of six different points of multiplicity 1. Now we apply Proposition 17. If $\mathcal{M}'$ is the sum of the characteristic functions of two disjoint lines, then one of the two lines has to be contained in E, i.e., the R_i form a line. In that case, also the Z_i have to form a line for each choice of a pair $\{Z_1, Z_2\}$, so that the seven points of multiplicity 1 outside of E form a disjoint plane E'. However, then we can apply Proposition 19 to $\mathcal{M} - \chi_{E'} = \mathcal{M}|_E$ to obtain a contradiction. Thus, $\mathcal{M}'$ is a projective base of size 6 in all cases. This is impossible as it can be seen using coordinate representations: W.l.o.g. we assume $R_i = e_i$ for $1 \leq i \leq 3$, where e_j denotes the jth unit vector. Since $\mathcal{M}$ is spanning, we assume w.l.o.g. that also e_i are points of multiplicity 1 for $4 \leq i \leq 6$. Choosing $4 \leq i < j \leq 6$ the points e_1, e_2, e_3, e_i, e_j are completed by $e_1 + e_2 + e_3 + e_i + e_j$ to a projective base of size 6. However, $e_1, e_2, e_3, e_1 + e_2 + e_3 + e_i + e_j, e_1 + e_2 + e_3 + e_i + e_h$ are completed by $e_j + e_h$ to a projective base of size 6 for each $\{i, j, h\} = \{4, 5, 6\}$, which gives more than seven points of multiplicity 1 outside of E – contradiction.

It remains to consider the case $\lambda_3 = \lambda_2 = 1$ and $\lambda_1 = 12$. The projection $\mathcal{M}_{P_2}$ of $\mathcal{M}$ through P_2 is 4-divisible with cardinality 15 and a unique point P' of cardinality 4 (arising from P_1 and P_3). However, Proposition 22 yields a contradiction for $\mathcal{M}_{P_2} - 4 \cdot \chi_{P'}$.

For the classification of the case $k = 5$, $\lambda_1 = 9$, $\lambda_2 = 1$, $\lambda_3 = 2$, $a_{13} = 3$, $a_9 = 19$, and $a_5 = 9$, we consider a projection $\mathcal{M}_{P_2}$ of $\mathcal{M}$ through the unique point P_2 of multiplicity 2. Note that the line L is mapped to a point $\overline{Q}$ with multiplicity $\mathcal{M}_{P_2}(\overline{Q}) = \mathcal{M}(P_3) + \mathcal{M}(P_1) = 4$, so that $\mathcal{M}_{P_2} - 4 \cdot \chi_{\overline{Q}}$ is 4-divisible with cardinality 11. From Proposition 22 we conclude the existence of a point $\overline{Q}'$ and a plane $\overline{E}'$ such that $\mathcal{M}_{P_2} = 4 \cdot \chi_{\overline{Q}} + 4 \cdot \chi_{\overline{Q}'} + \chi_{\overline{E}'}$. The preimage of $\overline{Q}'$ has to be a line L' consisting of P_2, the second point of multiplicity 3, that we denote by P_3', and a point P_1' of multiplicity 1. Since P_2 and P_3' are contained in E, also P_1' is contained in E. Let $\tilde{L} := \langle P_1, P_1' \rangle$, N be the third point in $\tilde{L}$, and A be the seventh point in E, i.e., the set of points in E is given by $\{P_1, P_2, P_3, P_1', P_3', A, N\}$. Since $\mathcal{M}(E) = 11$ we have $\mathcal{M}(A) + \mathcal{M}(N) = 1$. Each of the three hyperplanes H_1, H_2, H_3 containing E has multiplicity 13 and consists of two points of multiplicity 1 outside of E. By L_i we denote the line spanned by those two points, where $1 \leq i \leq 3$. Since $k = 5$, the line L_i meets E in a point Z_i. We will now determine Z_i and show $Z_1 = Z_2 = Z_3$. Let $Z_i \leq \hat{L} \leq E$ be an arbitrary line and $\hat{E} = \left\langle \hat{L}, L_i \right\rangle$ be a plane depending on the choice of $\hat{L}$. Note that $\mathcal{M}(\hat{E}) = \mathcal{M}(\hat{L}) + 2$ and $\mathcal{M}(\hat{E}) \equiv \#\mathcal{M} \equiv 1 \pmod 2$ due to Lemma 1. Thus Z_i cannot be contained in L or L', which implies $Z_i \in \{A, N\}$. Noting that $N \leq \langle P_3, P_3' \rangle$ we then conclude $Z_i = A$ for all $1 \leq i \leq 3$, which then implies

$\mathcal{M}(A)=1$ and $\mathcal{M}(N)=0$. For a parameterization of $\mathcal{M}$ we denote the ith vector by e_i and choose w.l.o.g. $A=e_3$, $P_2=e_2$, $P_3=e_1$, so that $P_3'=e_1+e_2+e_3$, $P_1=e_1+e_3$, $P_1'=e_1+e_2$, and $N=e_2+e_3$. W.l.o.g. we choose $L_1=\langle e_3,e_4\rangle$ and $L_2=\langle e_3,e_5\rangle$. From 4-divisibility we then conclude $L_3=\langle e_3,e_2+e_4+e_5\rangle$, which can be seen e.g. by looking at the first 15 columns of the stated matrix and using the fact that the number of ones in each row has to be divisible by 4. □

Lemma 34. *Let $\mathcal{M}$ be a spanning 4-divisible multiset of points in $\operatorname{PG}(k-1,2)$ with cardinality 17, $\lambda_2=0$, and $\gamma_1(\mathcal{M})=3$. Then, we have $\lambda_3\in\{1,2\}$ and a line spanned by two points of multiplicity 3 has multiplicity 6. If $\lambda_3=2$, then $k\ge 6$.*

Proof. Assume that L is a line consisting of two points of multiplicity 3 and one point of multiplicity 1. Noting that $\mathcal{M}|_L$ is 2-divisible and $\lambda_2=0$ we conclude $\mathcal{M}(H)=13$ for each hyperplane $H\ge L$. Since $\mathcal{M}(L)=7$, we have $k\ge 4$ and can use Lemma 6 to conclude $\mathcal{M}(L)>9$ – contradiction.

Due to Lemma 32 it suffices to consider the case $\lambda_3\ge 3$. W.l.o.g. we assume that the points with the coordinates e_1, e_2, and e_3 have multiplicity 3, so that the points in $\langle e_1+e_2,e_1+e_3\rangle$ have multiplicity zero. Set $E:=\langle e_1,e_2,e_3\rangle$ and $Q:=\langle e_1+e_2+e_3\rangle$. Since no 4-divisible multiset of points of cardinality 9 exists, we have $\mathcal{M}(Q)\in\{0,1\}$ so that $\mathcal{M}(E)\in\{9,10\}$. Note that $\mathcal{M}|_E$ is not 2-divisible if $\mathcal{M}(E)=9$, so that all hyperplanes $H\ge E$ have multiplicity 13 and Lemma 6 yields $\mathcal{M}(E)=9+2^{6-k}$, so that $k=6$ and $\mathcal{M}(E)=10$, i.e., $\mathcal{M}(Q)=1$. So, $\mathcal{M}|_E$ is 2-divisible and each hyperplane $H\ge E$ has multiplicity 13 and is given by $\mathcal{M}|_H-\mathcal{M}|_E=\chi_{L_H}$ for some line L_H. For an arbitrary point $P\notin E$ with non-zero multiplicity consider the solid $S=\langle E,P\rangle$ and the three hyperplanes H_1,H_2,H_3 containing S. Since $\mathcal{M}(H_1)=\mathcal{M}(H_2)=\mathcal{M}(H_3)=13$, we have $\mathcal{M}(S)=11$ and $\mathcal{M}=\mathcal{M}|_E+\sum_{i=1}^{3}\chi_{L_{H_i}}-2\chi_P$. Using the same argument for a different point $P\neq Q\notin E$ with non-zero multiplicity yields a contradiction. □

4 The minimum possible point multiplicity for triply-even multisets of points

Let $\mathcal{M}$ be an 8-divisible multiset of points in $\operatorname{PG}(v-1,2)$. From Corollary 4 we know $\gamma_1(\mathcal{M})=8$ if $\#\mathcal{M}=8$ and $\gamma_1(\mathcal{M})=4$ if $\#\mathcal{M}=12$.

Lemma 35. *Let $\mathcal{M}$ be an 8-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 22. Then, we have $\gamma_1(\mathcal{M})\ge 8$.*

Proof. The possible hyperplane multiplicities are given by 14 and 6. If H is a hyperplane with $\mathcal{M}(H)=6$, then Corollary 4 yields $\mathcal{M}|_H=2\cdot\chi_L$ for some line L. If P is a point with multiplicity $\mathcal{M}(P)>2$, then all hyperplanes containing P have multiplicity 14 and Lemma 6 yields $\mathcal{M}(P)=6+2^{5-k}$, so that it suffices to consider the case $\mathcal{M}(P)=7$. Since $\mathcal{M}|_H-4\cdot\chi_P$ is 4-divisible with cardinality 10 and a point of multiplicity 3, Corollary 20 gives a contradiction.

It remains to consider the case $\gamma_1(\mathcal{M})=1$. Using the standard equations we compute $\lambda_2=19+2^{8-k}$ for the dimension k of the span of $\mathcal{M}$, which clearly contradicts $\#\mathcal{M}=22$. □

Lemma 36. *Let $\mathcal{M}$ be an 8-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 23. Then, we have $\gamma_1(\mathcal{M})\ge 8$.*

Proof. The possible hyperplane multiplicities are given by 15 and 7. If H is a hyperplane with $\mathcal{M}(H)=7$, then Corollary 4 yields $\mathcal{M}|_H=\chi_E$ for some plane E. If P is a point with multiplicity $\mathcal{M}(P)>1$, then all hyperplanes containing P have multiplicity 15 and Lemma 6 yields $\mathcal{M}(P)>7$. However, not all hyperplanes can have multiplicity 15. □

Lemma 37. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 37 *and* $\gamma_1(\mathcal{M}) \le 7$. *Then, we have*

$$\begin{aligned} a_{13} &= 5\cdot 2^{k-4}+2+a_{29}, \\ a_{21} &= 11\cdot 2^{k-4}-3-2a_{29}, \text{ and} \\ \sum_{i=2}^{7}\binom{i}{2}\lambda_i &= 9+2^{8-k}\cdot(3+a_{29}) \end{aligned} \tag{9}$$

for the spectrum of $\mathcal{M}$, where k is the dimension of the span of $\mathcal{M}$. There do not exist a solid S', a plane E', or a line L' such that $\mathcal{M}\ge\chi_{S'}$, $\mathcal{M}\ge 2\cdot\chi_{E'}$, or $\mathcal{M}\ge 4\cdot\chi_{L'}$, respectively. Moreover, we have $\gamma_1(\mathcal{M})\le 3$, each point of multiplicity at most 3 is contained in a hyperplane with multiplicity 13, and $k\ge 6$.

Proof. Since there is no 4-divisible multiset of cardinality 5, the hyperplane multiplicities are given by 13, 21, and 29. With this, the stated equations follow from the standard equations. If a subspace S', E', or L', as specified in the statement would exist, then $\mathcal{M}-\chi_{S'}$, $\mathcal{M}-2\cdot\chi_{E'}$, or $\mathcal{M}-4\cdot\chi_{L'}$ would be an 8-divisible multiset of cardinality 22, 23, or 25, which is impossible due to Lemma 35, Lemma 36, and Proposition 10.

Assume that P is a point with $\mathcal{M}(P)\ge 4$. Proposition 24 implies $\mathcal{M}(H)\in\{21,29\}$ for each hyperplane H containing P, so that Lemma 6 yields $\mathcal{M}(P)>5$, i.e., $\lambda_4=\lambda_5=0$ and each point with multiplicity at most 3 indeed has to be contained in a hyperplane of multiplicity 13. Double counting the points in the hyperplanes containing P yields that P is contained in $2^{k-5}-2$ hyperplanes of multiplicity 29 if $\mathcal{M}(P)=6$ and $2^{k-4}-2$ hyperplanes of multiplicity 29 if $\mathcal{M}(P)=7$. Thus, we have $k\ge 6$ if $\lambda_6\ge 1$ and $k\ge 5$ if $\lambda_7\ge 1$.

First we will show $\lambda_6+\lambda_7\le 1$ and $k\ge 5$. Assume that L is a line spanned by two points with multiplicity at least 6 and denote the third point of L by P. From Proposition 24 we conclude that each hyperplane H with multiplicity 13 meets L exactly in P, so that $a_{13}\le[k-1]_2-[k-2]_2=2^{k-2}<5\cdot 2^{k-4}+2+a_{29}$ – contradiction. Thus, we have $\lambda_6+\lambda_7\in\{0,1\}$. From $a_{13}\in\mathbb{N}$ we conclude $k\ge 4$. If $k=4$, then Equation (9) implies $\lambda_6+\lambda_7\ge 1$, which is possible for $k\ge 5$ only.

Assume that P is a point of multiplicity at least 6. Further assume the existence of a line L not containing P but whose three points all have multiplicity at least 2. By E we denote the plane spanned by L and P, so that $\mathcal{M}(H)\ge 21$ for every hyperplane H containing E, see Proposition 24. If $\mathcal{M}(H)=21$, then $\mathcal{M}|_H-2\cdot\chi_L-4\cdot\chi_P$ is 4-divisible with cardinality 11, so that Proposition 22 implies the existence of a point Q with multiplicity 4 in $\mathcal{M}|_H-2\cdot\chi_L-4\cdot\chi_P$. Since $\gamma_1(\mathcal{M})<8$ and $\lambda_4+\lambda_5+\lambda_6+\lambda_7\le 1$ this is impossible. Thus, all $2^{k-3}-1$ hyperplanes containing E have multiplicity 29. Now let $\tilde{L}\le E$ be a line with $P\le\tilde{L}$, so that we have $\mathcal{M}(H)\in\{21,29\}$ for every hyperplane containing $\tilde{L}$. Denoting the number of those hyperplanes with multiplicity 29 by x and double counting points gives

$$\left(2^{k-3}-1\right)\cdot(37-\mathcal{M}(\tilde{L}))=\left(2^{k-2}-1\right)\cdot(21-\mathcal{M}(\tilde{L}))+8x,$$

so that $\mathcal{M}(\tilde{L})=5+2^{7-k}+2^{6-k}x$. Using $x\ge 2^{k-3}-1$ we conclude $\mathcal{M}(\tilde{L})\ge 13+2^{6-k}$. However, $\mathcal{M}(\tilde{L})\le 7+2\cdot 3=13$ – contradiction.

So, if P is a point with multiplicity at least 6, then any line L that does not contain P contains a point of multiplicity at most 1. With this, Proposition 24 yields that any hyperplane H with multiplicity 13 contains of a unique point of multiplicity 3 and 10 points of multiplicity 1. Especially, we have $\mathcal{M}(P')\ne 2$ for every point P' and $k\ge 6$. If $k\ge 7$, then let H_1 be a hyperplane with multiplicity 13 and $K\le H_1$ a $(k-2)$-dimensional subspace with multiplicity $\mathcal{M}(K)=13$. With this let H_2 and H_3 be the two other hyperplanes containing K. W.l.o.g. we assume $\mathcal{M}(H_2)=21$ and $\mathcal{M}(H_3)=29$. Clearly $P\not\le H_1$ and since $\mathcal{M}|_{H_2}-\mathcal{M}|_K-4\cdot\chi_P$ would be 4-divisible with cardinality 4, P is also not contained in H_2. However, $\mathcal{M}|_{H_3}-\mathcal{M}|_K-4\cdot\chi_P$ is 4-divisible with cardinality 12. By construction, except at most one point, the point multiplicities are contained in $\{0,1,3\}$, which contradicts Proposition 23. It remains to consider the case $k=6$. If P_6 is a point of multiplicity 6, then it is contained in $2^{k-5}-2=0$ hyperplanes of multiplicity 29, so that the line L spanned by P_6 and a point of multiplicity 3 is contained in hyperplanes of multiplicity 21 only. Lemma 6 then yields the contradiction $\mathcal{M}(L)=7<6+3$. So, let P_7 be a point of multiplicity 7, so that $3\lambda_3=4a_{29}$, which implies $a_{29}\ge 3$ and $\lambda_3\ge 4$. Now let L be a line spanned by two points of multiplicity

3 such that P_7 is not contained in L and E be the plane spanned by L and P_7. If H would be a hyperplane with multiplicity 21, then we can apply Lemma 32 to $\mathcal{M}|_H - 4 \cdot \chi_{P_7}$ – contradiction. So, Lemma 6 yields $\mathcal{M}(E) = 23$ and $\mathcal{M}(S) = 25$ for each solid containing E. Moreover, we have $a_{29} \ge 7$, so that $a_{29} \ge 9$ and $\lambda_3 \ge 12$, which is impossible.

Thus, we finally conclude $\gamma_1(\mathcal{M}) \le 3$.

If $k = 5$, then the existence of a hyperplane of multiplicity 13 implies $\lambda_1 \ge 4$, see Proposition 24, so that $\lambda_2 + 3\lambda_3 \le 33$. With this, Equation (9) yields $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 11$, and $a_{29} = 0$. However, the four points of multiplicity 1 span a unique plane, which contradicts $a_{13} \ge 12 > 3$.

□

Lemma 38. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 37*. Then, we have* $\gamma_1(\mathcal{M}) \ge 8$.

Proof. Assume that $\mathcal{M}$ is an 8-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 37, $\gamma_1(\mathcal{M}) \le 7$, and minimum possible dimension k of its span. Lemma 37 yields $\gamma_1(\mathcal{M}) \le 3$ and $k \ge 6$, so that there clearly exists a point Q with multiplicity zero. Using Lemma 5 we conclude that the projection $\mathcal{M}_Q$ through Q is an 8-divisible multiset of points with cardinality 37, $\gamma_1(\mathcal{M}_Q) \le 2 \cdot \gamma_1(\mathcal{M}) \le 6$, and dimension $k-1$ of its span, which contradicts the minimality of k. □

Lemma 39. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 20*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. The possible hyperplane multiplicities are given by 12 and 4. If the latter occurs, then there is a point of multiplicity 4, see Corollary 4. Otherwise we can apply Lemma 2 to obtain a contradiction. □

Lemma 40. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 24*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. W.l.o.g. we assume $2 \le \gamma_1(\mathcal{M}) \le 3$. The possible hyperplane multiplicities are given by 0, 8, and 16. Let P be a point with multiplicity $\mathcal{M}(P) = 3$. Due to Proposition 7 we have $\mathcal{M}(H) = 16$ for every hyperplane H containing P, so that Lemma 6 yields $\mathcal{M}(P) \ge 9$ – contradiction. Thus, we have $\gamma_1(\mathcal{M}) = 2$ and denote the dimension of the span of $\mathcal{M}$ by k. Using the standard equations we compute

$$\begin{aligned} a_0 &= a_{16} - 2^{k-1} + 2, \\ a_8 &= -2a_{16} + 3 \cdot 2^{k-1} - 3, \text{ and} \\ \lambda_2 &= -108 + (a_{16} + 3) \cdot 2^{8-k}, \end{aligned}$$

so that

$$a_{16} \ge 2^{k-1} - 2 \quad \text{and} \quad \lambda_2 \ge 20 + 2^{8-k} > 20,$$

which contradicts $\lambda_2 \le 24/2 = 12$. □

Lemma 41. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 26*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. W.l.o.g. we assume $2 \le \gamma_1(\mathcal{M}) \le 3$. The possible hyperplane multiplicities are given by 18 and 10. If there exists a point P with multiplicity $\mathcal{M}(P) = 3$, then Corollary 20 implies $\mathcal{M}(H) = 18$ for all hyperplanes H that contain P. With this Lemma 6 yields $\mathcal{M}(P) \ge 11$ – contradiction. Thus, we have $\gamma_1(\mathcal{M}) = 2$ and the standard equations yield $\lambda_2 = 17 + 2^{8-k}$ for the dimension k of the span of $\mathcal{M}$, which clearly contradicts $\#\mathcal{M} = 26$. □

Lemma 42. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 27*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. The possible hyperplane multiplicities are given by 19 and 11. If the latter occurs, then there is a point of multiplicity at least 4, see Proposition 22. Otherwise we can apply Lemma 2 to obtain a contradiction. □

Lemma 43. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 35*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. The possible hyperplane multiplicities are given by 11, 19, and 27. However, if there exists a hyperplane H with $\mathcal{M}(H) = 11$, then Proposition 22 yields the existence of a point with multiplicity at least 4. Otherwise Lemma 6 gives $\mathcal{M}(P) > 3$ for every point P. □

Lemma 44. *Let $\mathcal{M}$ be an 8-divisible multiset of points in* $\mathrm{PG}(v-1,2)$ *with cardinality* 39*. Then, we have* $\gamma_1(\mathcal{M}) \ge 4$.

Proof. Assume that $\mathcal{M}$ is an 8-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 39, $\gamma_1(\mathcal{M}) \le 3$, and minimum possible dimension k of its span. If Q would be a point with multiplicity 2, then the projection $\mathcal{M}_Q$ through Q would be 8-divisible with cardinality 37 and $\gamma_1(\mathcal{M}_Q) \le 6$, so that Lemma 38 yields a contradiction. Thus, we have $\lambda_2 = 0$ and $\gamma_1(\mathcal{M}) = 3$, see Proposition 11. Since $\gamma_1(\mathcal{M}) \le 3$ and $\#\mathcal{M} = 39$, we clearly have $k \ge 4$. If $k = 4$, then the multiset of points $\mathcal{M}'$ defined by $\mathcal{M}'(P) = 3 - \mathcal{M}(P)$ is 8-divisible with cardinality 6 – contradiction. Thus, we have $k \ge 5$. Since each hyperplane H with multiplicity 7 is given by $\mathcal{M}|_H = \chi_E$ for some plane E, see Corollary 4, each hyperplanes H' that contains a point of multiplicity 3 has multiplicity $\mathcal{M}(H') \ge 15$. Lemma 6 yields that each point of multiplicity 3 is contained in a hyperplane H of multiplicity 15. Using $\gamma_1(\mathcal{M}) \le 3$ and $\lambda_2 = 0$, Corollary 28 yields $\mathcal{M}|_H = 3 \cdot \chi_E - 2 \cdot \chi_L$ to a plane E and a line $L \le E$. So, $\lambda_3 \ge 1$ implies $\lambda_3 \ge 4$.

Let H be hyperplane with multiplicity 15 and E a plane, $L \le E$ a line such that $\mathcal{M}|_H = 3 \cdot \chi_E - 2 \cdot \chi_L$. Choose a subspace $K \le H$ of codimension 2 intersecting E in a line of multiplicity 7, i.e., containing two points of multiplicity 3 and one point of multiplicity 1. We denote the other two hyperplanes containing K by H' and H''. W.l.o.g. we assume $\mathcal{M}(H') = 15$ and $\mathcal{M}(H'') = 23$. So, let $E' \le H'$ a plane and $L' \le E'$ a line such that $\mathcal{M}|_{H'} = 3 \cdot \chi_{E'} - 2 \cdot \chi_{L'}$. We conclude $\lambda_3 \ge 6$, $\lambda_1 \ge 5$ and denote the solid spanned by E and E' by S. By construction we have $\mathcal{M}(S) \ge 23$. However, we have $\mathcal{M}(H) \ne 23$ for any hyperplane $H \ge S$ since $\mathcal{M}|_H - 3 \cdot \chi_E + 2 \cdot \chi_L$ would be 4-divisible with cardinality 8 containing two points of multiplicity 3, which contradicts Proposition 7. Thus, we have $\mathcal{M}(H) = 31$ for every hyperplane H that contains S, so that Lemma 6 yields $\mathcal{M}(S) = 23 + 2^{8-k}$. If $k = 5$, then $\mathcal{M}(S) = 31$. However, since no three points of multiplicity 3 form a line, S can contain at most 8 points of multiplicity 3 and seven points of multiplicity 1, which yields $\mathcal{M} \ge \chi_S$ and $\mathcal{M} - \chi_S$ is 8-divisible of cardinality 31, so that Lemma 40 yields a contradiction.

Let H be an arbitrary but fixed hyperplane of multiplicity 15 and E be the corresponding plane that contains the non-zero points of $\mathcal{M}|_H$. Now consider subspace $K \le H$ of codimension 2. In $2^{k-4} - 1$ cases $E \le K$ and the other two hyperplanes containing K have multiplicities 23 and 31. In $6 \cdot 2^{k-4}$ cases K intersects E in a line of multiplicity 7 and the other two hyperplanes containing K have multiplicities 15 and 23. In 2^{k-4} cases K intersects E and a line of multiplicity 3 and there are two cases for the other two hyperplanes H', H'' containing K. Either $\{\mathcal{M}(H'), \mathcal{M}(H'')\} = \{7, 23\}$ or $\mathcal{M}(H') = \mathcal{M}(H'') = 15$. Denote the number of occurrence of the first case by x, so that $x \in \mathbb{N}$ with $x \le 2^{k-4}$. With this, we compute

$$\begin{aligned} a_7 &= x, \\ a_{15} &= 1 + 8 \cdot 2^{k-4} - 2x, \\ a_{23} &= 7 \cdot 2^{k-4} - 1 + x, \text{ and} \\ a_{31} &= 2^{k-4} - 1, \end{aligned}$$

so that $2^{9-k} + 28 + 2^{8-k}x = 3\lambda_3$, which implies $\lambda_3 \ge 10$ and $\lambda_1 \le 9$. However, in E there are three points of multiplicity 1 and there exists a hyperplane $H \ge E$ with $\mathcal{M}(H) = 23$, so that Proposition 7, $\gamma_1(\mathcal{M}) \le 3$, and $\lambda_2 = 0$ imply $\lambda_1 \ge 3 + 8 = 11 > 9$ – contradiction. □

Lemma 45. *Let $\mathcal{M}$ be a spanning 8-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 41 and $1 \le \gamma_1(\mathcal{M}) \le 3$. Then, we have*

$$\begin{aligned} a_{17} &= 9 \cdot 2^{k-4} + 2 + a_{33}, \\ a_{25} &= 7 \cdot 2^{k-4} - 3 - 2a_{33}, \\ \lambda_2 + 3\lambda_3 &= 11 + 2^{8-k} \cdot (3 + a_{33}), \end{aligned}$$

and $k \ge 5$. Moreover, for each line L there exists a hyperplane $H \ge L$ with $\mathcal{M}(H) = 17$ and a point $P \le L$ with $\mathcal{M}(P) \le 1$.

Proof. Since no 4-divisible multiset of points of cardinality 9 exists, the possible multiplicities of the hyperplanes are given by 17, 25, and 33. Using the standard equations we compute the stated equations. Since $\gamma_1(\mathcal{M}) \le 3$, we clearly have $k \ge 4$. If $k = 4$, then the multiset of points $\mathcal{M}'$ defined by $\mathcal{M}'(P) = 3 - \mathcal{M}(P)$ is 8-divisible with cardinality 4, which is impossible.

Let L be an arbitrary line. If L is not contained in a hyperplane of multiplicity 17, then Lemma 6 yields $\mathcal{M}(L) > 25 - 16 = 9$. However, $\gamma_1(\mathcal{M}) \le 3$ implies $\mathcal{M}(L) \le 9$ – contradiction. So, let $H \ge L$ be a hyperplane with $\mathcal{M}(H) = 17$. If $\mathcal{M}(P) \ge 2$ for all points $P \le L$, then $\mathcal{M}|_H - 2 \cdot \chi_L$ is 4-divisible with cardinality 11, so that Proposition 22 contradicts $\gamma_1(\mathcal{M}) \le 3$. □

Lemma 46. *Let $\mathcal{M}$ be a spanning 8-divisible multiset of points in $\operatorname{PG}(v-1,2)$ with cardinality 41 and $1 \le \gamma_1(\mathcal{M}) \le 3$. Then, there does not exist a hyperplane H such that $\mathcal{M}|_H$ is given as specified in Lemma 33.*

Proof. Assume that $\mathcal{M}$ is a spanning 8-divisible multiset of points in $\operatorname{PG}(k-1,2)$ with cardinality 41 and $1 \le \gamma_1(\mathcal{M}) \le 3$, such that $\mathcal{M}|_{H_1}$ is as specified in Lemma 33 for some hyperplane H_1. Since $\mathcal{M}|_{H_1}$ spans a 5-dimensional subspace we have $k \ge 6$ for the dimension of the point set spanned by $\mathcal{M}$. Let $E := \langle e_7, \dots, e_k \rangle$, where we also allow E to be an empty space for $k = 6$. W.l.o.g. we also assume coordinates as in Lemma 33, so that especially $H_1 = \langle e_1, \dots, e_5, E \rangle$. Consider the subspace $S := \langle e_1, e_2, e_4, e_5, E \rangle \le H_1$ with $\mathcal{M}(S) = 9$, so that we have $\mathcal{M}(H_2) = 17$ and $\mathcal{M}(H_3) = 25$ for the two other hyperplanes containing S. Note that the line $L := \langle e_1, e_2 \rangle$ is contained in $H_2 \ge S$ and consists of a point of multiplicity i for all $1 \le i \le 3$, so that we can apply Lemma 33 to $\mathcal{M}|_{H_2}$. W.l.o.g. we assume that the second point of multiplicity 3 in H_2 (that is not contained in S) has coordinates e_6. With this, the point set $\mathcal{M}|_{H_1 \cup H_2}$ is given by the columns of

$$\begin{pmatrix} 1111111100000000000010111 \\ 0001111011000001100011110 \\ 0000111100101010100000000 \\ 0000000000011001100000101 \\ 0000000000000111100000011 \\ 0000000000000000011111111 \end{pmatrix},$$

where the entries in the rows 7 to k are all 0 and not displayed. This multiset of points consists of 14 points of multiplicity 1, a unique point of multiplicity 2, and three points of multiplicity 3.

There exist at least

$$\Omega := 2^k - 1 - (\lambda_1 + \lambda_2 + \lambda_3) - \binom{\lambda_2 + \lambda_3}{2} - \lambda_3 \cdot \lambda_1$$

points Q of multiplicity 0 such that every line that contains Q has multiplicity at most 3. Assume $k \ge 8$. If $\lambda_3 \ge 3 + 4 = 7$, then we have $(\lambda_1, \lambda_2, \lambda_3) \in \{(18,1,7), (16,2,7), (14,3,7), (15,1,8)\}$, where $\Omega \ge 38 > 0$. If $\lambda_3 < 7$, then $\lambda_2 + \lambda_3 \le 12$ implies $\Omega \ge 255 - 30 - 66 - 126 = 43 > 0$. So, such a point Q exists and the projection $\mathcal{M}_Q$ of $\mathcal{M}$ through Q is 8-divisible with cardinality 41 and $\gamma_1(\mathcal{M}_P) \le 3$, see Lemma 5. W.l.o.g. we can assume that k is minimal, so that it suffices to consider the cases $k \in \{6, 7\}$ in the following.

If $k = 6$, then Lemma 45 yields $\lambda_2 + 3\lambda_3 = 23 + 4a_{33}$. Since the points in $H_1 \cup H_2$ contribute $1 \cdot 1 + 3 \cdot 3 = 10$ to $\lambda_2 + 3\lambda_3$ and the points outside of $H_1 \cup H_2$ can contribute at most 16, we have $a_{33} = 0$ and $\lambda_2 \equiv 2 \pmod 3$, which implies $\lambda_2 = 2$, $\lambda_3 = 7$, and $\lambda_1 = 16$. Let P_2' denote the second point of multiplicity 2, which lies outside of S. For the plane $E' := \langle P_2', L \rangle$ we know that all hyperplanes H that contain E' have multiplicity

25, so that Lemma 6 implies $\mathcal{M}(E')=13$ and $\mathcal{M}(S')=17$ for every solid $S'\ge E'$. From Lemma 45 we know that the points with multiplicity at least 2 are contained in an affine plane, so that E' consists of two points of multiplicity 2, two points of multiplicity 3, and three points of multiplicity 1 forming a line. Starting from $L'':=\langle e_2, e_1+e_2+e_3\rangle$ we can construct $E'':=\langle P_2', L''\rangle$ and deduce the same structure information for E''. Since both E' and E'' contain the two points of multiplicity 2, their span $S':=\langle E', E''\rangle$ is a solid with multiplicity at least $\mathcal{M}(E')+2\cdot 3\ge 19>17$ – contradiction.

If $k=7$, then Lemma 45 yields $\lambda_2+3\lambda_3=17+2a_{33}$. We choose $K:=\langle e_1,\dots,e_r\rangle$ and $H_1:=\langle K,e_7\rangle$, so that $\mathcal{M}(K)=\mathcal{M}(H_1)=17$ and $\{\mathcal{M}(H_2),\mathcal{M}(H_3)\}=\{25,33\}$ for the other two hyperplanes that contain K. Let $H_2:=\langle K,e_6\rangle$, so that $e_6\in H_2\backslash K$ is a point of multiplicity 3 and Proposition 7 implies $\mathcal{M}(H_2)=33$. Thus, $\mathcal{M}':=\mathcal{M}|_{H_2}-\mathcal{M}|_K$ is 4-divisible with cardinality 16 and contains at least one point of multiplicity 3 as well as five points of multiplicity 1. Since there can be at most $\lfloor 8/3\rfloor=2$ more points of multiplicity 3, Lemma 29 implies that $\mathcal{M}'$ contains at most two points of multiplicity 3 in total. Since $\mathcal{M}|_{H_3}-\mathcal{M}|_K$ is 4-divisible with cardinality 8, Proposition 7 implies $3\le\lambda_3\le 4$. Combining this with $\lambda_2+3\lambda_3=17+2a_{33}$ and $a_{33}\ge 1$ we conclude $\lambda_2\ge 7$. Using Proposition 7 again we conclude $1\le\lambda_3(\mathcal{M}')$ and $\lambda_2(\mathcal{M}')\ge 2$, so that we can apply Lemma 30. If $\lambda_3=3$, then $\lambda_2\le 1+3+4=8$, which contradicts $\lambda_2+3\lambda_3\ge 19$. Thus, we have $\lambda_3=4$ and the upper bound $\lambda_2\le 1+2+4$ implies $a_{33}=1$, $\lambda_2=7$, and $\lambda_1=15$ using $\lambda_2+3\lambda_3=17+2a_{33}$ and $\lambda_1+2\lambda_2+3\lambda_3=41$. Moreover, the six points of multiplicity 1 and the two points of multiplicity 3 outside of H_1 form an affine solid. However, we have

$$\langle e_6, e_2+e_6, e_1+e_2+e_6, e_1+e_2+e_4+e_6, e_1+e_2+e_5+e_6, e_1+e_4+e_5+e_6\rangle=\langle e_1,e_2,e_4,e_5,e_6\rangle,$$

which is a contradiction. □

Lemma 47. *Let $\mathcal{M}$ be an 8-divisible multiset of points in $\mathrm{PG}(v-1,2)$ with cardinality 41. Then, we have $\gamma_1(\mathcal{M})\ge 4$.*

Proof. Assume that $\mathcal{M}$ is a spanning 8-divisible multiset of points in $\mathrm{PG}(k-1,2)$ with cardinality 41, $\gamma_1(\mathcal{M})\le 3$, and minimum possible dimension k of its span. Proposition 11 yields $\gamma_1(\mathcal{M})\ge 2$.

Assume that P_2 is a point with multiplicity 2. If $L\ge P_2$ is a line that also contains a point of multiplicity 3 and a point of multiplicity 1, then Lemma 45 implies the existence of a hyperplane $H\ge L$ with $\mathcal{M}(H)=17$ and Lemma 33 yields a description of $\mathcal{M}|_H$. However, Lemma 46 gives a contradiction. Using Lemma 45 again we conclude $\mathcal{M}(L')\le 5$ for each line $L'\ge P_2$. So, for the projection $\mathcal{M}_{P_2}$ of $\mathcal{M}$ through P_2 we have $\gamma_1(\mathcal{M}_{P_2})\le 3$. However, $\mathcal{M}_{P_2}$ is 8-divisible with cardinality 39, which contradicts Lemma 44. Thus, we have $\lambda_2=0$ and $\gamma_1(\mathcal{M})=3$.

Lemma 45 gives $3\lambda_3=11+2^{8-k}\cdot(3+a_{33})$, so that $\lambda_3\ge 4$. Let L be a line spanned by two points of multiplicity 3. Lemma 45 yields the existence of a hyperplane $H\ge L$ with $\mathcal{M}(H)=17$, so that Lemma 34 gives $\mathcal{M}(L)=6$, $\lambda_3(\mathcal{M}|_H)=2$, $\lambda_1(\mathcal{M}|_H)=11$, and $k\ge 7$ (for the dimension of $\mathcal{M}$). Since $\lambda_1\ge 11$, we have $\lambda_3\le 10$. So, $4\le\lambda_3\le 10$ implies $\lambda_1+\lambda_3\le 33$, $\binom{\lambda_3}{2}\le 45$, and $\lambda_1\cdot\lambda_3\le 140$. Since $33+45+140<2^8-1$ for $k\ge 8$, there exists a point Q of multiplicity zero such that every line $L'\ge Q$ has multiplicity at most 3. With this, the projection $\mathcal{M}_Q$ of $\mathcal{M}$ through Q is 8-divisible with cardinality 8 and $\gamma_1(\mathcal{M}_Q)=3$, see Lemma 5. Due to the assumed minimality of k we have $k=7$.

Using $k=7$, the equation $3\lambda_3=11+2^{8-k}\cdot(3+a_{33})=17+2a_{33}$ implies $\lambda_3\in\{7,9\}$ and $a_{33}\in\{2,5\}$. Due to Lemma 31 the hyperplane H with multiplicity 17 contains a 5-dimensional subspace K with multiplicity $\mathcal{M}(K)=5$. Since also the two other hyperplanes that contain K then also have multiplicity 17, Lemma 34 yields $\lambda_3\le 3\cdot 2=6$ – contradiction. □

5 Conclusion

We have determined the minimum possible column multiplicities for Δ-divisible binary linear codes for each given length n and all $\Delta\in\{2,4,8\}$. This refines a comprehensive characterization result on the possible length of q^r-divisible linear codes over $\mathbb{F}_q$ from [KK20]. The motivation for this refinement is that in some applications upper bounds on the allowed maximum column multiplicity are given. We mainly use geometric methods to obtain computer-free proofs. While the stated result can also be obtained by an exhaustive computer enumeration, the question arises whether the theoretical tools can be strengthened and approaches be simplified in order to obtain results for wider ranges of parameters. As outlined in Section B in the appendix, currently we cannot go much further even using extensive computer enumerations. Interestingly enough $\Gamma_2(2^r, n)$ is always a power of 2, if finite at all, for $r \in \{1, 2, 3\}$. Our, rather sparse, numerical data might suggest the conjecture that $\Gamma_q(q^r, n)$ always has to be a power of the characteristic p of the underlying field $\mathbb{F}_q$. However, such a strong statement is wrong in general. To this end, note that applying the construction of [LR19, Theorem 10] to the smallest non-trivial blocking set in $\operatorname{PG}(2, p)$, i.e.,the projective triangle, yields a p-divisible multiset $\mathcal{M}$ of points with cardinality $\#\mathcal{M} = \frac{p^2+1}{2} + 2p$ and maximum point multiplicity $\gamma_1(\mathcal{M}) = \frac{p+3}{2}$ for each odd prime p. So, for $p \geq 5$ we have $\Gamma_p(p, \#\mathcal{M}) < p$ while [HKK18, Theorem 11] yields $\Gamma_p(p, \#\mathcal{M}) > 1$.

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A Combinatorial data of the possible 4-divisible multisets of points with cardinality 17

In our discussion of 8-divisible multisets of points of cardinality 41, 4-divisible multisets of points of cardinality 17 played an important role. To this end we have stated several auxiliary results for the latter.

In order to demonstrate the combinatorial richness we have listed key parameters of these objects in Table 1. This data has been obtained by an exhaustive computer enumeration using the software package `LinCode` [BBK21]. The three cases of maximum point multiplicity 1 have also been computationally classified in [HHK+17].

B Computational results

One alternative way to prove Theorem 13 is to use the fact that for every field size q and every divisibility constant $\Delta \in \mathbb{N}$ there exists an integer $N(q,\Delta)$ such that for all $n \ge N(q,\Delta)$ there exists a projective Δ-divisible linear code over $\mathbb{F}_q$ with length n. So, for each pair q, Δ a complete determination of the function $\Gamma_q(\Delta,\cdot)$ amounts to a finite computation. So, we may simply enumerated all Δ-divisible codes over $\mathbb{F}_q$ with length strictly smaller than $N(q,\Delta)$ and determine the corresponding column multiplicities. To this end we have used the software package `LinCode` [BBK21] and list the corresponding enumeration results of semi-linearly non-equivalent linear codes per length and dimension in the subsequent tables. Here a blank entry means that no such code exists.

For 16-divisible binary linear codes we have only partial results. We remark that the smallest attained dimension for a given length can be explained by a statement similar to Lemma 1. Lengths that do not occur at all are explained by Theorem 9. The complete classification of the possible lengths of projective 16-divisible binary linear codes is still an open problem, see e.g. [HKK18, HKKW19]. The same is true for the projective q^2-divisible linear codes over $\mathbb{F}_q$ when $q \ge 3$ and the projective q-divisible linear codes over $\mathbb{F}_q$ when $q \ge 5$.

For ternary linear codes we can state $\Gamma_3(3,n)=1$ iff $n=4$ or $n \ge 8$. Moreover, we have $\Gamma_3(3,n)=3$ iff $n \in \{3,6,7\}$ and $\Gamma_3(3,n)=\infty$ iff $n \in \{1,2,5\}$. For quaternary linear codes we can state $\Gamma_4(4,n)=1$ iff $n \in \{5,10,15,16,17\}$ or $n \ge 20$. Moreover, we have $\Gamma_4(4,n)=2$ iff $n \in \{12,14,18,19\}$, $\Gamma_4(4,n)=4$ iff

n	k	Δ	γ_1	λ_1	λ_2	λ_3	spectrum
17	3	4	7	4	0	2	$(a_5, a_9, a_{13}) = (4, 2, 1)$
17	3	4	5	3	0	3	$(a_5, a_9, a_{13}) = (3, 4, 0)$
17	4	4	7	6	2	0	$(a_5, a_9, a_{13}) = (8, 3, 4)$
17	4	4	6	6	1	1	$(a_5, a_9, a_{13}) = (7, 5, 3)$
17	4	4	5	5	2	1	$(a_5, a_9, a_{13}) = (6, 7, 2)$
17	4	4	4	6	2	1	$(a_5, a_9, a_{13}) = (5, 9, 1)$
17	4	4	4	4	0	3	$(a_5, a_9, a_{13}) = (6, 7, 2)$
17	4	4	3	4	2	3	$(a_5, a_9, a_{13}) = (5, 9, 1)$
17	4	4	3	6	4	1	$(a_5, a_9, a_{13}) = (4, 11, 0)$
17	5	4	7	10	0	0	$(a_5, a_9, a_{13}) = (16, 5, 10)$
17	5	4	6	7	2	0	$(a_5, a_9, a_{13}) = (14, 9, 8)$
17	5	4	5	9	0	1	$(a_5, a_9, a_{13}) = (12, 13, 6)$
17	5	4	5	6	3	0	$(a_5, a_9, a_{13}) = (12, 13, 6)$
17	5	4	4	7	3	0	$(a_5, a_9, a_{13}) = (10, 17, 4)$
17	5	4	4	10	0	1	$(a_5, a_9, a_{13}) = (10, 17, 4)$
17	5	4	4	6	2	1	$(a_5, a_9, a_{13}) = (11, 15, 5)$
17	5	4	3	5	3	2	$(a_5, a_9, a_{13}) = (10, 17, 4)$
17	5	4	3	9	1	2	$(a_5, a_9, a_{13}) = (9, 19, 3)$
17	5	4	3	10	2	1	$(a_5, a_9, a_{13}) = (8, 21, 2)$
17	5	4	3	6	4	1	$(a_5, a_9, a_{13}) = (9, 19, 3)$
17	5	4	2	7	5	0	$(a_5, a_9, a_{13}) = (8, 21, 2)$
17	5	4	2	11	3	0	$(a_5, a_9, a_{13}) = (7, 23, 1)$
17	5	4	2	15	1	0	$(a_5, a_9, a_{13}) = (6, 25, 0)$
17	6	4	4	10	0	1	$(a_5, a_9, a_{13}) = (21, 31, 11)$
17	6	4	4	7	3	0	$(a_5, a_9, a_{13}) = (21, 31, 11)$
17	6	4	3	11	0	2	$(a_5, a_9, a_{13}) = (18, 37, 8)$
17	6	4	3	10	2	1	$(a_5, a_9, a_{13}) = (17, 39, 7)$
17	6	4	3	6	4	1	$(a_5, a_9, a_{13}) = (19, 35, 9)$
17	6	4	3	12	1	1	$(a_5, a_9, a_{13}) = (16, 41, 6)$
17	6	4	2	7	5	0	$(a_5, a_9, a_{13}) = (17, 39, 7)$
17	6	4	2	11	3	0	$(a_5, a_9, a_{13}) = (15, 43, 5)$
17	6	4	2	13	2	0	$(a_5, a_9, a_{13}) = (14, 45, 4)$
17	6	4	2	15	1	0	$(a_5, a_9, a_{13}) = (13, 47, 3)$
17	6	4	1	17	0	0	$(a_5, a_9, a_{13}) = (12, 49, 2)$
17	7	4	2	11	3	0	$(a_5, a_9, a_{13}) = (31, 83, 13)$
17	7	4	2	7	5	0	$(a_5, a_9, a_{13}) = (35, 75, 17)$
17	7	4	2	13	2	0	$(a_5, a_9, a_{13}) = (29, 87, 11)$
17	7	4	2	15	1	0	$(a_5, a_9, a_{13}) = (27, 91, 9)$
17	7	4	1	17	0	0	$(a_5, a_9, a_{13}) = (25, 95, 7)$
17	8	4	1	17	0	0	$(a_5, a_9, a_{13}) = (51, 187, 17)$

Table 1: Combinatorial data of 4-divisible multisets of points of cardinality 17.

n / k	1	2	3	4	5	6	7	8	9
2	1								
3		1							
4	1	1	1						
5		1	1	1					
6	1	2	3	2	1				
7		2	4	4	2	1			
8	1	3	8	10	7	3	1		
9		3	9	18	16	9	3	1	
10	1	4	17	37	46	30	13	4	1

Table 2: Number of even codes per dimension k and effective length n.

n / k	1	2	3	4	5	6	7	8	9
4	1								
6		1							
7			1						
8	1	1	1	1					
10		1	1	1					
11			1	1					
12	1	2	3	4	2				
13			1	1	2				
14		2	4	6	5	4			
15			3	6	6	4	2		
16	1	3	8	18	21	15	7	2	
17			2	7	14	11	5	1	
18		3	9	27	44	45	21	6	
19			6	22	52	62	40	10	
20	1	4	17	64	149	212	156	65	10

Table 3: Number of doubly-even codes per dimension k and effective length n.

n / k	1	2	3	4	5	6	7	8	9	10	11
8	1										
12		1									
14			1								
15				1							
16	1	1	1	1	1						
20		1	1	1							
22			1	1							
23				1	1						
24	1	2	3	4	4	1					
26			1	1	2						
27				1	1	1					
28		2	4	6	7	6	1				
29				1	1	2	1				
30			3	6	8	7	6	2			
31				4	8	8	6	4	1		
32	1	3	8	18	32	34	24	13	5	1	
34			2	7	14	11	5	1			
35				3	7	7	3	1			
36		3	9	27	54	65	36	11	1		
37				2	5	8	5	1			
38			6	22	57	79	61	21	2		
39				10	36	57	49	30	10	1	
40	1	4	17	64	194	347	323	187	59	11	1
41				2	12	29	26	12	3		

Table 4: Number of triply-even codes per dimension k and effective length n.

n / k	1	2	3	4	5	6	7	8	9	10	11	12
16	1											
24		1										
28			1									
30				1								
31					1							
32	1	1	1	1	1	1						
40		1	1	1								
44			1	1								
46				1	1							
47					1	1						
48	1	2	3	4	4	3	1					
52			1	1	2							
54				1	1	1						
55					1	1	1					
56		2	4	6	7	8	3	1				
58				1	1	2	1					
59					1	1	1	1				
60			3	6	8	9	8	4	1			
61					1	1	2	1	1			
62				4	8	10	9	8	4	2		
63					5	10	10	8	6	3	1	
64	1	3	8	18	32	48	48	35	21	11	4	1
68			2	7	14	11	5	1				
70				3	7	7	3	1				
71					3	7	7	3	1			
72		3	9	27	54	75	56	26	6	1		
74				2	5	8	5	1				
75					2	5	5	4	1			
76			6	22	59	86	75	34	9	1		
77					2	5	8	6	4	1		
78				10	36	64	66	52	28	11	2	
79					14	47	71	63	44	23	8	1

Table 5: Number of 16-divisible codes per dimension k and effective length n.

$n \in \{4, 8, 9, 13\}$, and $\Gamma_4(4, n) = \infty$ iff $n \in \{1, 2, 3, 6, 7, 11\}$. For $n \in \{6, 7, 9\}$ there exist projective $[n, 3]_4$ two-weight codes that are 2-divisible. They belong to the families $TF1$, $TF2$, $RT1$, and $RT2$, see [CK86]. Since $19 = 14 + 5$ this gives constructions for all cases with $\Gamma_4(4, n) = 2$.

n / k	1	2	3
3	1		
4		1	
6	1	1	
7		1	1

Table 6: Number of 3-divisible ternary linear codes per dimension k and effective length n.

n / k	1	2	3	4	5	6	7
4	1						
5		1					
8	1	1					
9		1	1				
10		1	1	1			
12	1	2	2				
13		2	3	1			
14		1	5	3	1		
15		1	3	6	2	1	
16	1	4	9	7	2		
17		3	12	9	2		
18		2	18	25	8	1	
19		1	14	42	25	6	1

Table 7: Number of 4-divisible quaternary linear codes per dimension k and effective length n.