A CONTRIBUTION TO THE FOUNDATIONS OF THE THEORY OF QUASIFIBRATION

by

PETER JOSEPH WITBOOI

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Supervisor: Professor K. A. Hardie

Co-supervisor: (Assoc.) Professor P. Cherenack

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ABSTRACT

The concept of quasifibration was introduced by Dold and Thom in their work on infinite symmetric product spaces. Among other things, they prove a theorem [DT; Satz 2.2], which has since been applied widely in the literature.

This thesis presents a study of the notion of n-equivalences and related types of maps. The first of our two main goals is to prove a result, Theorem **5.1**, which generalizes a fundamental theorem of Dold and Thom on globalization of quasifibrations. Secondly we show that by means of adjunction or clutching constructions, this theorem enables us to retrieve the famous results [J_2; Theorem 1.2 and Theorem 1.3] of James in his work on suspension of spheres. The results of James appear in the thesis as Theorem **13.8**.

For some of the applications we require a generalized version of n-equivalence. This generalization entails replacing, in the definition of n-equivalence, the isomorphisms by isomorphisms modulo a suitable Serre class [Se] of abelian groups. Although quasifibrations are often applied in the generalized context in the literature, there is a lack of a formal theorem covering such applications. We fill this gap by proving a generalized version, Theorem **8.5**, of a result [$\mathbf{M}_2$; Theorem 1.2] of May, on n–equivalences and cotriads.

In the process of pursuing the theorems of James, we discover new results as well as new alternative proofs of well-known results. The most prominent example of the latter type of work, is the proof of the homotopy excision due to Blakers and Massey [$\mathbf{BM}_1$]. The homotopy excision theorem appears as Theorem **7.1**. Among the original results, the most prominent ones are the Theorems **11.1** and Theorem **12.1**. The latter result describes how, under suitable conditions, a relative homeomorphism can be extended to give isomorphisms of the relevant homotopy groups in dimensions beyond those of Theorem **7.1**. Theorem 11.1 gives sufficient conditions for a map of pairs of finite dimensional CW–complexes to be a n-equivalence. ∎

CONTENTS.

0. GENERAL BACKGROUND AND OUTLINE

The concept of quasifibration was invented by Dold and Thom [DT]. May [$\mathbf{M_2}$] approached quasifibrations from a new angle, making use of n-equivalences. This dissertation presents a study of the notion of n-equivalences and related types of maps. The first of our two main goals is to prove a result, Theorem **5.1**, which generalizes the fundamental theorem [DT; Satz 2.2] by Dold and Thom on globalization of quasifibrations. Secondly we show that by means of adjunction or clutching constructions, this theorem enables us to retrieve the famous results of James [$\mathbf{J_2}$; Theorem 1.2 and Theorem 1.3] in his work on suspension of spheres. The results of James appear in the thesis as Theorem **13.8**.

For some of the applications we need a generalized version of n-equivalence. This generalization entails replacing, in the definition of n-equivalence, the isomorphisms by isomorphisms modulo a suitable Serre class [Se] of abelian groups. For the sake of having the thesis self-contained, we include a formal discussion of localization of 1-connected spaces and Serre classes of abelian groups. This summarizes the scope of the thesis. More detail on the content of the thesis will be given after we have sketched a historical perspective on quasifibrations.

The homotopy lifting property is defined in Section 1 of this thesis. A map that has the homotopy lifting property is called a fibration. It is known that for a product of topological spaces, the projection map onto one of the factors is a fibration. The simple manner in which a homotopy can be lifted over a projection map, resulted in such maps being called trivial fibrations. The concept of locally trivial fibration (see Section 2) has its roots in the work of Whitney [**Wy**]. Steenrod [$\mathrm{Sd_1}$] proved that if the base of a Whitney fibre space is compact, then the projection is a fibration. This implies that any locally trivial fibration is a *Serre fibration*, that is to say, the map has the homotopy lifting property with respect to compact polyhedra.

Using the homotopy lifting property, it can be shown that for a Serre fibration $p : E \longrightarrow B$ and a point $b \in B$ with $F = p^{-1}(b) \neq \phi$, the induced function on homotopy sets :

$$p_* : \pi_n(E,F) \longrightarrow \pi_n(B,b) \qquad (1)$$

is bijective for each $n \in \mathbb{N}$. In order to pursue this property, the concept of ***quasifibration*** was introduced by Dold and Thom in their pioneering work [**DT**]. In [**DT**], quasifibrations are used extensively to study ***infinite symmetric product spaces*** and related constructions. The authors prove that the inclusion of a space into its infinite symmetric product space, realizes the Hurewicz homomorphism. In particular, the singular integral homology of the space is naturally isomorphic to the homotopy of the infinite symmetric product of the space. A key technical result in [**DT**] is *Satz* 2.2 which gives sufficient conditions for a map which is locally a quasifibration, to be in fact a quasifibration.

This globalization theorem was applied for identification of quasifibrations in numerous papers and in a variety of fields. We can mention among others, the fields of *equivariant topology* for example [**Wa**] and [**CW**], and *shape theory* for example [**Ed**] and [**Fe**]. In a number of cases, the adjunction theorem [**Ha**; Theorem 0.2] due to Hardie is used. When appropriate, this adjunction theorem is simpler to apply than than the original theorem and lemmas of [**DT**]. In what follows, we look at some of the applications which are more relevant to this thesis. The papers cited are only examples and the list is certainly not complete.

Almost immediately after the invention of the notion of quasifibration, Dold and Lashof [**DL**] reported their construction of a ***universal quasifibration*** for an associative H-space. The universal quasifibration is comparable with the universal principal bundle for a group [**Sd$_2$**]. Stasheff gave further generalizations of this universal bundle type of construction. In [**Sf$_1$**] it is shown that for every finite CW-complex F, there exists a fibration that classifies all quasifibrations which have quasifibres of the homotopy type of F and base a CW-complex.

Quasifibration theory was used further in the study of ***associativity in H-spaces***, by among others, Hubbuck and Mimura [**HM**] and Stasheff [$\mathbf{Sf_2}$]. The closer an H-space is to being associative, the further one can carry on with the iterative construction towards the Dold-Lashof universal quasifibration. In these papers, the spaces are generally considered to be *localized* with respect to a specified subset of prime numbers. This idea is also common in many of the other applications we are going to mention.

In the study of ***geometric realizations*** of semisimplicial sets, in particular, the nerve of a category, the theory was also valuable. As examples here we have among others, publications by Anderson [**An**], May [$\mathbf{M_1}$], McCord [**Mc**], Meyer [**Me**] and Quillen [$\mathbf{Q_1}$ and $\mathbf{Q_2}$]. McCord shows in particular that every *finite space* is weakly equivalent to a compact polyhedron. It is interesting to note how Quillen applies the theory of quasifibrations to problems of abstract algebra. The book [**Ss**] includes a brief treatment of the basics of quasifibrations, to enlighten the applications to *algebraic K-theory* as in the work [$\mathbf{Q_1}$] of Quillen. In a recent paper, Weidner and Welker [**WW**] did work on abstract *group theory* related to that in [$\mathbf{Q_2}$]. In both of the latter two papers, quasifibration methods were used.

The techniques of quasifibrations have also impacted on ***homotopy theory***. Baues [Ba] obtains a result of the nature of the Hilton-Milnor theorem. In [$\mathrm{G_2}$] Gray establishes the weak homotopy equivalence between the loop space of the suspension of a space and the James reduced product by these means. This result is originally due to James [$\mathrm{J_1}$] by different methods. In [$\mathrm{G_3}$], Gray investigates for topological group structure on the homotopy fibres of iterated suspension maps. Recently, Wong [**Wong**] has also used quasifibrations to study the homotopy fibres of these maps. Hardie and Porter [**HP**] obtain by means of adjunction of quasifibrations, maps due to James [$\mathrm{J_2}$] of reduced product space skeleta. This construction will be pursued by similar methods in this thesis. In this dissertation we further extend this list of applications, viewed in the setting of n-equivalences.

We give a description of the sequencing of the topics and the contents of chapters. The essence of the thesis is contained in the first three chapters, or more precisely, Chapters II and III. Chapters IV and V contain detailed expositions of work of a more elementary level, required for Chapters II and III.

In Chapter I we continue by discussing the essential preliminaries, such as double mapping cylinder, mapping path fibration and many more. For motivational purposes we include a treatment of locally trivial fibrations. The definitions of the concepts of n-equivalence and quasifibration follow. A number of examples are included, some of which will be revisited at a later stage in the thesis.

In the first part of Chapter II we treat the new globalization theorem for n-equivalences, Theorem **5.1**, and deduce the one by Dold and Thom for quasifibrations. This is followed by some adjunction theorems. Among the latter, at least the first one, Theorem **6.1**, is a new contribution. Furthermore we prove the homotopy excision theorem of Blakers and Massey [**BM$_1$**] by an original method, using adjunction of n-equivalences. This result is labeled Theorem **7.1**.

For most of the further applications, we require a generalized version of n-equivalence. This generalization involves the concept of Serre classes of abelian groups. In the first part of Chapter III we introduce the essentialities regarding this generalization. We also discuss more preliminaries required for further applications. Theorem **11.1** gives sufficient conditions for a map of pairs of finite CW-complexes, to be a weak equivalence. This is also a new contribution to our subject, and has applications comparable to those of the Serre spectral sequence of a fibration. The applications in the latter part of Chapter III to maps involving spheres, include new proofs of famous results as well as results that seem to be new. Among the original results, the most prominent ones are the theorems **11.1** and **12.1**. The latter result describes how, under suitable conditions, a relative homeomorphism can be extended to give isomorphisms of the relevant homotopy groups in dimensions beyond those of Theorem **7.1**. Theorem **11.1** gives sufficient conditions for a map of pairs of finite dimensional CW–complexes to be a n-equivalence. The known theorems for which we obtain new proofs include the results [$\mathbf{J}_2$; 1.2 and 1.3] of James, Theorem **13.8** mentioned in the introductory paragraph. (With a little more effort, besides using adjunction, we retrieve the generalization [T; Theorem 2.11] by Toda of James's results. The latter result is presented as Theorem **A.1** in the appendix.) The main objectives of this dissertation can be considered to have been fulfilled at this stage.

The foundations of n-equivalences is treated in Chapter IV. For completeness we include a discussion of the five-lemma. We treat in detail the action of the group $\pi_1(A)$ on the set $\pi_1(X,A)$. This action does not seem to have been studied in detail before, and certainly has not been given its rightful place in the study of maps of pairs. Hereafter we study maps of pairs and triples. Furthermore we present a detailed proof of a key result, Lemma **5.5**, required for the proof of Theorem **5.1**. Lemma **5.5** is modeled partly on [**DT**; *Hilfssatz* 2.6] in the presentation of Dold and Thom, and partly on [$\mathbf{M}_2$; Lemma 3.3] of in May's treatment of n-equivalences (which is attributed to Sugawara [**Sa**]). Logically, this chapter precedes Chapter II, but since the proof of Lemma **5.5** is lengthy and the other material of Chapter IV is fairly elementary, it has been shifted away from the main development of the dissertation.

Chapter V is included for the sake of making the thesis self-contained as was mentioned formerly. Where references are not provided, the theory is regarded as well-known. We give an account of the relevant facts regarding Serre classes of abelian groups. This is followed by a discussion of localization, with respect to a given set of primes, of abelian groups and of 1-connected spaces. Finally we generalize the concept of n-equivalence, and prove adjunction theorems for such maps. A summary of this work appears in Section **8** for application in Chapter III.

A characteristic feature of this thesis is the geometric methods. The description by Toda [T; Section 5] of the attaching maps of the cells of S^k_∞ is crucial for the geometric proof of Theorem **12.1** (an intermediate step towards James's results [J_2; 1.2 and 1.3]). Nowhere in the thesis, except in the proof of Theorem **A.1**, due to Toda [T; 2.11], do we make use of the cohomology or homology of specific spaces. Also in the proof of Theorem **A.1**, we do make use of cohomology and of Wang's cohomology sequence. Nevertheless, the Wang sequence, and consequently the cohomology ring of ΩS^n, is obtainable by elementary means as we shall point out. Further, the thesis abounds with elementary counter-examples supporting the results and pin-pointing difficulties.

The end of the proof of a result is denoted by the symbol ▌. The symbol ▪ is used when it is considered necessary to clearly mark the end of any other type of discussion, for instance a remark or a definition. We label the ideas or paragraphs in bold print, for example, **5.1** refers to the first (citable) item in Section **5**. The chapters only serve to group together the sections belonging to the same theme. Wherever practical, the main result of a given section appears as $*.1$.

Chapter I : INTRODUCTION AND MOTIVATION

In the first section we discuss preliminaries and establish notation and conventions to be used throughout the dissertation.

The constructions with locally trivial fibrations in Section **2** serves as motivation for the study of quasifibrations and n-equivalences.

In Section **3** we introduce the concepts mentioned in its title. Numerous examples are given in order to illustrate the concepts and to support the results. Some of these are revisited in later chapters.

Section 1. Preliminaries and notation

Section 2. Adjunction of locally trivial fibrations

Section **3**. n-Equivalence and quasifibration.

1. PRELIMINARIES AND NOTATION.

We devote this section to the discussion of preliminaries such as the homotopy lifting and extension properties and elementary facts and terminology about the category Top^2 . The notation for *double mapping cylinder, push-out* and *mapping path fibration* established here, will be used throughout the thesis. The notion of mapping cylinder was invented by J. H. C. Whitehead [W_1] and pursued by Fox [Fo]. The adjunction space concept is also attributed to J. H. C. Whitehead [W_2]. The mapping path fibration construction is due to Cartan and Serre [CS]. Steenrod's NDR-pairs [Sd_3] provide us with an elegant approach to the homotopy extension problem. For many of the concepts in this section though, the history is hard to trace and we shall not aspire to provide a complete set of references.

1.1 The categories Top and Top^2

The term *map* means continuous function between topological spaces. A neighbourhood of a point in a (topological) space will be assumed to be *open*. The category of spaces and maps is denoted by Top . When working with a pair of spaces (X,A), it is assumed that $A \neq \phi$. At times, with due notice, we shall use the same symbol to denote the category of pointed spaces and base point-preserving maps. There is no chance of ambiguity since in a given section we will work consistently in only one of these categories. Top^2 is the category of which the objects are the morphisms of Top , and a morphism in the category Top^2 from the object $\alpha : A \longrightarrow A'$ to the object $\beta : B \longrightarrow B'$ is a pair (μ,μ') of maps $\mu : A \longrightarrow B$ and $\mu' : A' \longrightarrow B'$ such that $\beta \circ \mu = \mu' \circ \alpha$.

1.2 Definition

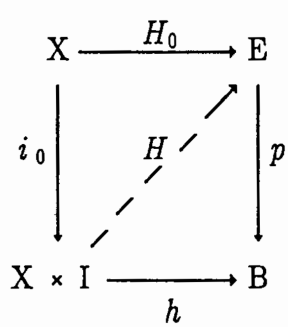
A ***homotopy*** from a map $h_0 : X \longrightarrow B$ to a map $h_1 : X \longrightarrow B$ is a map $h : X \times I \longrightarrow B$ such that $h(x,0) = h_0(x)$ and $h(x,1) = h_1(x)$ for every $x \in X$. If such a homotopy exists, then we say that h_0 is ***homotopic*** to h_1.

A ***homotopy equivalence*** is a map $h : X \longrightarrow B$ such that there exist a map $g : B \longrightarrow X$, called a homotopy inverse of h, with $g \circ h$ homotopic to the identity map of X and $h \circ g$ homotopic to the identity map of B.

1.3 Definition

Let $i_0 : X \longrightarrow X \times I$ be the obvious homeomorphism of X onto the subspace $X \times \{0\}$ of $X \times I$. A map $p : E \longrightarrow B$ is said to have the ***homotopy lifting property*** with respect to the space X if, for maps $h : X \times I \longrightarrow B$ and $H_0 : X \longrightarrow E$ such that $h \circ i_0 = p \circ H_0$, there exists $H : X \times I \longrightarrow E$ such that $p \circ H = h$ and $H \circ i_0 = H_0$. That is to say, given a commutative square such as the one formed by the unbroken arrows in diagram A, there exists an arrow H such that the diagram is commutative.

– A –



1.4 Definition. A map which has the homotopy lifting with respect to all spaces is called a ***fibration***, or more precisely, a ***Hurewicz fibration.***

1.5 Examples

(a) Obviously every homeomorphism is a fibration because given the commutative square of unbroken arrows in diagram **A**, and assuming that p is a homeomorphism, we choose $H = p^{-1} \circ h$.

(b) Let $E = F \times B$ for some space F, and let $p : F \times B \longrightarrow B$ be the projection. Then, given the square of unbroken arrows in diagram **A** above, we define $H : X \times I \longrightarrow E$ by the formula:

$$H(x,t) = (\, q \circ H_0(x),\, h(x,t)\,),$$

where $q : F \times B \longrightarrow F$ is the projection map. Then by [**Ms**; Theorem 7.4 on p109], H is a continuous function. This map H makes diagram **A** commutative. It can thus be seen that p is a fibration.

(c) Every map $f : E \longrightarrow B$ can be decomposed $f = f^{\wedge} \circ f^{-}$ as a homotopy equivalence followed by a fibration. This construction, see **1.6** below, is known as the Cartan-Serre construction [CS]. The fibration $f^{\wedge}$ is called the mapping path fibration of f.

1.6 *Mapping path fibration*

For any map $f : E \longrightarrow B$, we define $f^{\wedge} : E^{\wedge} \longrightarrow B$ as follows. Let $P(B)$ denote the space of all paths $\lambda : [0,1] \longrightarrow B$ in B, with the compact-open topology. Let $E^{\wedge} = \{\, (x,\lambda) \in E \times P(B) : f(x) = \lambda(0)\, \}$ and let $f^{\wedge}(x,\lambda) = \lambda(1)$.

When there is a chance of confusion we write $E^{\wedge}(f)$ instead of $E^{\wedge}$. For any $x \in E$, denote by $\overline{x}$ the stationary path in B at $f(x)$. Then we have a map $f^{-} : E \longrightarrow E^{\wedge}$ given by the formula $f^{-}(x) = (x,\overline{x})$.

1.7 Proposition. With notation as in **1.6** we have the following:

(a) $f^{\wedge} \circ f^{\sim} = f$

(b) $f^{\wedge}$ is a fibration

(c) $f^{\sim}$ is a homotopy equivalence and $f^{\sim}(E)$ is a deformation retract of $E^{\wedge}$.

The proof is routine and we omit it. It is discussed in for example [**Wd**]. ∎

1.8 Remark

For maps $f_i : E_i \longrightarrow B_i$, $i = 1;\ 2$, and a <u>Top</u>^{2}-morphism $(g,h) : f_1 \longrightarrow f_2$, we define a function $E^{\wedge}(g,h) : E^{\wedge}(f_1) \longrightarrow E^{\wedge}(f_2)$ by mapping an element (x,λ) of $E^{\wedge}(f_1)$ onto the element $(g(x), h \circ \lambda)$ in $E^{\wedge}(f_2)$. Then we have a functor $E^{\wedge}$: <u>Top</u>$^2 \longrightarrow$ <u>Top</u>^{2}.

Taking $\eta(f)$ to be the pair $(f^{\sim}, 1)$, with 1 denoting the identity map of the space B, we find that η is a natural transformation from the identity functor on <u>Top</u>^{2}, to $E^{\wedge}$. ∎

The following definition is due to Steenrod [**Sd$_3$**]. We do not restrict ourselves to compactly generated Hausdorff spaces.

1.9 Definition

Given a space X and subspace A, the pair (X,A) is called an ***NDR-pair*** if there exists maps $u : X \longrightarrow I$ and $h : X \times I \longrightarrow X$ such that :

$A = u^{-1}(0)$,

$h(x,0) = x$ for all $x \in X$,

$h(x,t) = x$ for all $(x,t) \in A \times I$, and

$h(x,1) \in A$ whenever $u(x) < 1$.

A ***cofibration*** is an embedding $i : A \longrightarrow X$ such that $(X, i(A))$ is an NDR-pair. ∎

The term cofibration used to be defined in terms of homotopy extension, but in [**Sd$_3$**] it is shown that (at least for compactly generated Hausdorff spaces) the given definition is equivalent to the original one.

1.10 Definition.

The ***mapping cylinder*** $Z(f)$ of the map $f: E \longrightarrow B$ is the space obtained from $E \times I + B$ by making the identifications $(x,1) = f(x)$. For a map $f_0 : E_0 \longrightarrow B_0$ and a $\underline{Top}^2$-morphism $(g,h) : f \longrightarrow f_0$ we define a function $Z(g,h) : Z(f) \longrightarrow Z(f_0)$ to be the one induced from the map :

$$g \times 1 + h \ : \ E \times I + B \longrightarrow E_0 \times I + B_0 .$$

Then $Z(g,h)$ is continuous and so we have a functor $Z : \underline{Top}^2 \longrightarrow \underline{Top}$. There is a projection $Z(f) \longrightarrow B$, determined by the obvious map $E \times I + B \longrightarrow B$. This is a natural transformation from Z to the codomain functor $\underline{Top}^2 \longrightarrow \underline{Top}$. Similarly there is a natural inclusion $E \longrightarrow Z(f)$. This inclusion is a cofibration. Note that the projection has a right inverse and is a homotopy equivalence as well as a cofibration.

1.11 *Push-out*

We explain the push-out, $B = G(h_1, h_2)$ of a cotriad (1) below, in the category $\underline{Top}$.

(1) $$B_1 \xleftarrow{h_1} B_0 \xrightarrow{h_2} B_2$$

A relation R_0 is defined on $B_1 + B_0 + B_2$ by the rule given in (2) below. Let R be the equivalence relation generated by R_0.

(2) $$(b,b') \in R_0 \Leftrightarrow b \in B_0 \text{ and } b' \in \{h_1(b), h_2(b)\}.$$

Let $B = (B_1 + B_2)/R$, and for each $j = 1; 2$, let $\bar{h}_j : B_j \longrightarrow B$ be the restriction of the quotient map $\theta : (B_1 + B_0 + B_2) \longrightarrow B$. Then diagram **B** is a push-out square.

– B –

$$\begin{array}{ccc} B_0 & \xrightarrow{\;h_2\;} & B_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle \bar{h}_1} \\ B_1 & \xrightarrow{\;\bar{h}_2\;} & B \end{array}$$

For a commutative diagram such as **C**, there is an obvious induced function of push-out spaces, $f\colon G(g_1, g_2) \to G(h_1, h_2)$, and this function is continuous by the properties of quotient spaces.

- C –

$$\begin{array}{ccccc} E_1 & \xleftarrow{\;g_1\;} & E_0 & \xrightarrow{\;g_2\;} & E_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} \\ B_1 & \longleftarrow & B_0 & \longrightarrow & B_2 \end{array}$$

– D –

$$\begin{array}{ccc} f_0 & \xrightarrow{\;(g_2, h_2)\;} & f_2 \\ {\scriptstyle (g_1,h_1)}\downarrow & & \downarrow{\scriptstyle (\bar{g}_1, \bar{h}_1)} \\ f_1 & \xrightarrow{\;(\bar{g}_2, \bar{h}_2)\;} & f \end{array}$$

The map f above is such that the square of diagram **D** is a push-out in <u>Top</u>^{2}. During the push-out construction as in diagram **B**, the map h_1 will often be required to be a cofibration. In this case, the space obtained is called an *adjunction space* [**W**$_2$].

1.12 *Double mapping cylinder*

Consider again the cotriad $B_1 \xleftarrow{\;h_1\;} B_0 \xrightarrow{\;h_2\;} B_2$ in the category <u>Top</u>. The double mapping cylinder $D(h_1,h_2)$ of the cotriad is the quotient space obtained from the disjoint union $B^S = B_1 + B_0 \times I + B_2$, by making the identifications as shown in (3) below, and let $\eta : B^S \longrightarrow D(h_1,h_2)$ be the quotient map.

(3) $$(x,0) = h_1(x) \text{ and } (x,1) = h_2(x).$$

This quotient map embeds the subspaces B_1, B_2 and $B_0 \times (0,1)$ in $D(h_1,h_2)$. In what follows we abbreviate $D(h_1,h_2)$ to B' and $D(g_1,g_2)$ to E'. For the commutative diagram **C**, there is a map $f' : E' \longrightarrow B'$.

1.13 Remarks

In diagram **E** we have quotient maps η and θ, and q is the projection $B_0 \times I \longrightarrow B_0$.

$$
-\mathbf{E}- \qquad
\begin{array}{ccc}
B_1 + B_0 \times I + B_2 & \xrightarrow{1+q+1} & B_1 + B_0 + B_2 \\
\downarrow \eta & & \downarrow \theta \\
B' & \dashrightarrow & B
\end{array}
$$

There exists a natural map $\beta : B' \longrightarrow B$, the broken arrow in diagram **E**, such that the square is commutative.

1.14 Proposition

The map $\beta : B' \longrightarrow B$ of **1.13** is a homotopy equivalence if h_1 is a cofibration.

An even stronger result is proved in the book [Br; 7.5.4 p275] of Brown. ■

1.15 Definition

Let α, β, μ and μ' be as in **1.1** above. Then (μ,μ') is said to be :

(a) a ***homeomorphism of fibres*** if for each $a \in A'$, the induced map $\alpha^{-1}(a) \longrightarrow \beta^{-1}(\mu'(a))$ is a homeomorphism.

(b) a ***weak equivalence of fibres*** if for every $a \in A'$ the induced map $\alpha^{-1}(a) \longrightarrow \beta^{-1}(\mu'(a))$ induces bijections of homotopy sets in all dimensions.

(c) a ***weak equivalence of homotopy fibres*** if $E\hat{}(\mu,\mu')$ is a weak homotopy equivalence of fibres. $E\hat{}$ is as in 1.8. ■

2. ADJUNCTION OF LOCALLY TRIVIAL FIBRATIONS

Adjunction of maps is the most prominent of the techniques in this thesis. In a nut shell, an adjunction is a push-out in the category $\underline{\text{Top}}^2$, as explained in the latter part of 1.11. The idea is to input three maps of a given type in a suitable manner, and obtain a similar map in the push-out. Theorem 2.4 in this section gives sufficient conditions on a $\underline{\text{Top}}^2$–cotriad (of locally trivial fibrations) to ensure that the push-out is a locally trivial fibration. As an illustration, we shall view the well known Hopf fibrations in the light of this theorem. The construction described in 2.4 is analogous to that of Dold and Lashof [**DL**] and Hardie [**Ha**] for quasifibrations.

The concept of locally trivial fibration is due to Whitney [**Wy**]. A locally trivial fibration over a paracompact space has been proved by Huebsch [**Hh**] and Hurewicz [**Hz**] to have the homotopy lifting property, i.e. to be a Hurewicz fibration. A simple proof of the fact that every locally trivial fibration has the homotopy lifting property with respect to compact Hausdorff spaces can be found in [G_1]. This implies that the condition ($_1$) in Section 0 is in fact satisfied, and is the motivation for investigating the property more generally, as was done originally by Dold and Thom [DT].

2.1 Definition

Let F be any topological space. A map $p : E \longrightarrow B$ is a ***locally trivial fibration*** with fibre F, if there is an open cover $\mathcal{U}$ of B satisfying the following condition. For every $U \in \mathcal{U}$ there exists a homeomorphism $\varphi : U \times F \longrightarrow p^{-1}(U)$ such that $p\,\varphi(x,a) = x$ for every $(x,a) \in U \times F$.

For a map $p : E \longrightarrow B$, and a subset U of B, U is said to be ***p-projected*** if there exists a space G and a homeomorphism $\varphi : U \times G \longrightarrow p^{-1}(U)$ such that for every $(x,g) \in U \times G$, $p\,\varphi(x,g) = x$. We refer to φ as a ***trivialization*** of p over U. ■

We immediately note that if p is a locally trivial fibration with fibre F, then for every p-projected subset U of B, the space G must be homeomorphic to F.

2.2 Examples. **(a)** Let E be a topological group, and F a topologically closed normal subgroup of E. Then the canonical epimorphism $p : E \longrightarrow E/F$ is a locally trivial fibration with fibre F, provided that p has a local cross-section.

(b) Every covering projection over a connected base space is a locally trivial fibration.

2.3 Proposition

The diagram A below, of maps of topological spaces, is considered to be commutative.

– A –

$$\begin{array}{ccc} E_0 & \xrightarrow{\ g_0\ } & E \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ B_0 & \xrightarrow[\ g\ ]{} & B \end{array}$$

(a) If B is p-projected and the square is a pull-back, then B_0 is p_0-projected.

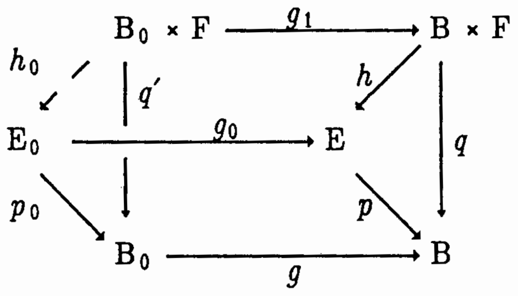
(b) If p is a locally trivial fibration with fibre F and the square is a pull-back, then p_0 is a locally trivial fibration with fibre F.

(c) Suppose that p_0 and p are locally trivial fibrations with fibre F, for some locally compact Hausdorff space F. Suppose further that the group $h(F)$ of self–homeomorphisms of F is a topological group when equipped with the compact-open topology.

If the $\underline{\text{Top}}^2$-morphism $(g_0,g) : p_0 \longrightarrow p$ is a homeomorphism of fibres (see 1.15), then the square is a pull-back.

Proof **(a)** If B is p-projected, then as in diagram B, the projection map q factorizes through p. In diagram B the projection q' is the pull-back of q over g. By pull-back properties, we obtain the map h_0. This map h_0 is a homeomorphism since h is one and the two squares between which they operate are pull-backs. Thus our claim follows.

– B –



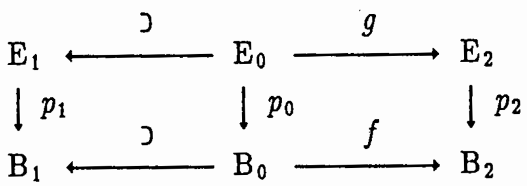
(b) For every p-projected open subset U of B, the set $U_0 = g^{-1}(U)$ is obviously open, but is also p_0-projected by **(a)**. Thus (b) follows.

(c) A proof is given in the book [Po] of Porter. ▮

For a compact Hausdorff space F, h(F) is known [**Sd**$_2$; p20] to be a topological group.

When formulating the adjunction theorem, 2.4 below, we refer to the following diagram, C.

– C –



2.4 Theorem

Suppose that $p_1 : E_1 \longrightarrow B_1$ is a locally trivial fibration with fibre F, and for some $B_0 \subset B_1$, $p_0 : E_0 \longrightarrow B_0$ is the pull-back of p_1 over the inclusion $B_0 \subset B_1$. Further, let us suppose that the following conditions hold.

(1) (E_1,E_0) and (B_1,B_0) are NDR-pairs represented by maps $u : E_1 \longrightarrow I$, $h : E_1 \times I \longrightarrow E_1$ and, respectively, $v : B_1 \longrightarrow I$, $k : B_1 \times I \longrightarrow B_1$ as in **1.9**.

(2) $E_0 = p_1^{-1}(B_0)$, $v \circ p_1 = u$ and $p_1 h(x,1) = k(p_1(x),1)$ for each $x \in E_0$.

(3) For every $x \in v^{-1}[0,1)$, the map θ defined by the formula below, is a homeomorphism.

$$\theta : p_1^{-1}(x) \longrightarrow p_1^{-1} k(x,1), \quad \theta(e) = h(e,1).$$

(4) F is a locally compact Hausdorff space for which h(F) is a topological group.

Suppose further that $p_2 : E_2 \longrightarrow B_2$ is a locally trivial fibration with fibre F and that the Top²-morphism (g,f) from p_0 to p_2 is a homeomorphism of fibres (see **1.15**).

Then the push-out $p : E \longrightarrow B$ of diagram C is a locally trivial fibration with fibre F.

Proof Let z be any point of B. We shall show that z has a p-projected neighbourhood W in B. Let $\eta : B_1 + B_0 + B_2 \longrightarrow B$ be the quotient map.

Suppose that $z \in B \setminus \eta(B_2)$. Let $y \in B_1 \setminus B_0$ be such that $\eta(y) = z$. B_0 is closed in B_1 due to the cofibration, and so $B_1 \setminus B_0$ is open in B_1. Thus $B_1 \setminus B_0$ contains a p_1-projected neighbourhood Z of y. Since the quotient maps embeds $E_1 \setminus E_0$ into E and $B_1 \setminus B_0$ into B, the image of Z in B is an open p-projected subset of B and is our choice for W.

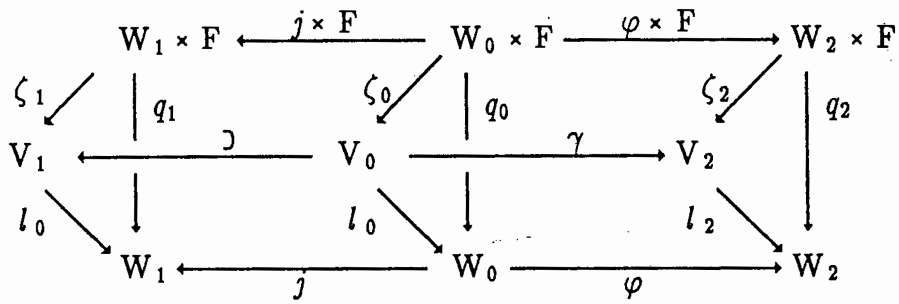
We now show that for an arbitrary point $z \in \eta(B_2)$, z has a p-projected (open) neighbourhood W. We shall find open subsets $W_i \subset B_i$ such that $W_1 + W_0 + W_2$ is saturated with respect to η, and $\eta(W_1 + W_0 + W_2)$ is p-projected. Then we can choose W to be the set $\eta(W_1 + W_0 + W_2)$.

Let $y \in B_2$ be any point such that $z = \eta(y)$ and let W_2 be a p_2-projected neighbourhood of y. We choose W_0 to be the open subset $W_0 = f^{-1}(W_2)$ of B_0. Let W_1 be the set $W = \{x \in v^{-1}[0,1) : k(x,1) \in W_0\}$. Then $W_1 \cap B_0 = W_0$ since $k(x,1) = x$ for every $x \in B_0$. Thus W_1 is open in B_1 and $W_1 + W_0 + W_2$ is saturated with respect to η. There is a map $s : W_1 \longrightarrow W_0$ defined by the formula $s(x) = k(x,1)$. In fact, $W_0 \subset W_1$ and s is a retraction. Let $V_i = p_i^{-1}(W_i)$. Then $V_0 = g^{-1}(V_2)$ and $V_1 = \{x \in u^{-1}[0,1) : h(x,1) \in V_0\}$. Also, $V_0 \subset V_1$ and there is a retraction $r : V_1 \longrightarrow V_0$ defined by the formula $r(x) = h(x,1)$. In diagram **D** the vertical arrows are pull-backs of the maps p_i (by **2.3(b)**, such maps are locally trivial fibrations), and r and s are induced by g and f respectively. The squares are commutative. In the first square, commutativity is due to condition **2.4($_2$)**.

– D –

$$\begin{array}{ccc} V_1 & \xrightarrow{\ r\ } & V_0 \\ \downarrow l_1 & & \downarrow l_0 \\ W_1 & \xrightarrow{\ s\ } & W_0 \end{array} \qquad \begin{array}{ccc} V_0 & \xrightarrow{\ \gamma\ } & V_2 \\ \downarrow l_0 & & \downarrow l_2 \\ W_0 & \xrightarrow{\ \varphi\ } & W_2 \end{array}$$

Each <u>Top</u>^{2}-morphism (r,s) and (γ,φ) is a homeomorphism of fibres, in the case of (r,s) this is due to **2.4($_3$)**. Hence by **2.3(c)**, each of these squares is a pull-back square. Thus for a trivialization ζ_2 of p_2 over W_2, we obtain as in **2.3(a)**, a trivialization ζ_0 of p_0 over W_0 and a trivialization ζ_1 of p_1 over W_1. Since r and s are retractions, in view of **2.4($_2$)** we obtain the commutative diagram **E**. The maps $q_i : W_i \times F \longrightarrow W_i$ are projections and j is an inclusion. The space F is locally compact Hausdorff and thus by [AP; Problem 25 on p330], whenever $\lambda : A \longrightarrow B$ is a quotient map, then the map of product spaces $\lambda \times 1 : A \times F \longrightarrow B \times F$ is a quotient map (1 is the identity map of F).

– E – 

Thus the push-out of the q_i-cotriad is precisely the projection map $q : W \times F \longrightarrow W$. The push-out ζ of the cotriad formed by the maps ζ_i, is a trivialization of p over W, so W is p-projected. This completes the proof. ∎

Let us demonstrate this process by analysing the construction of the Hopf fibrations as in [**Wd**]. We denote the (unreduced) cone of a space X by CX. CX is the quotient space obtained from $I \times X$ by identifying the points $(0,x)$. The image of a point (t,x) under this quotient map is denoted by $t \wedge x$. We regard X as a subspace of CX via the embedding that maps a point x of X to the point $1 \wedge x$, and then the pair (CX,X) has a natural (although not unique) representation as an NDR-pair. The suspension ΣX of X is the quotient space of CX obtained by collapsing the subspace X. For a point $t \wedge x$ of CX, we ambiguously denote its image in ΣX by the same symbol. The join $X * Y$ of spaces X and Y, is the subspace $CX \times Y \cup X \times CY$ of $CX \times CY$.

2.5 Illustration: The Hopf fibrations

We show that for each $n = 1; 3; 7$, there is a map $p : S^{2n+1} \longrightarrow S^{n+1}$, which is a locally trivial fibration with fibre S^n. For each of the given values of n, the sphere S^n admits a continuous multiplication with a unit element such that each element has a unique inverse. Furthermore, inversion is a continuous map, and the following condition is satisfied:

(1) For every $x,y \in S^n$, $(xy)y^{-1} = x$.

The Hopf construction on this multiplication $S^n \times S^n \longrightarrow S^n$, see [**Wd**; p502], yields a map $p : S^n * S^n \longrightarrow \Sigma S^n$, given by the formulae:

$$p\,(t \wedge x, y) = \tfrac{t}{2} \wedge xy, \quad \text{and} \quad p\,(x, t \wedge y) = \tfrac{1+t}{2} \wedge xy.$$

Note that $S^n * S^n$ and (resp.) ΣS^n are homeomorphic to S^{2n+1} and, respectively, S^{n+1}. We shall show that the map p is a locally trivial fibration with fibre S^n. Let us denote by T_-, T_0 and T_+ respectively, the following subspaces of ΣS^n :

$$\{t \wedge x : t \leq \tfrac{1}{2} \text{ and } x \in S^n\}, \ \{t \wedge x : t = \tfrac{1}{2} \text{ and } x \in S^n\} \text{ and } \{t \wedge x : t \geq \tfrac{1}{2} \text{ and } x \in S^n\}.$$

Then p can be seen to be the push-out of the $\underline{\text{Top}}^2$-cotriad in diagram **F**. Each vertical arrow is a pull-back of p.

$$\text{– } \mathbf{F} \text{ –} \qquad \begin{array}{ccccc} CS^n \times S^n & \xleftarrow{\supset} & S^n \times S^n & \longrightarrow & S^n \times CS^n \\ {\scriptstyle p_-}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_+} \\ T_- & \xleftarrow{\supset} & T_0 & \longrightarrow & T_+ \end{array}$$

Each of the vertical arrows in diagram **F** is a locally trivial fibration with fibre S^n. We prove this for p_-. For the other arrows the argument is similar.

Let $h : T_- \times S^n \longrightarrow CS^n \times S^n$ be defined by the formula $h\,(t \wedge x,\ y) = (2t \wedge xy,\ y^{-1})$. Then $p_- \circ h$ coincides with the projection $T_- \times S^n \longrightarrow T_-$ and h is a homeomorphism. The property $(_1)$ of the multiplication is important in this regard. So T_- is in fact p-projected. The pairs (T_-, T_0) and $(CS^n \times S^n, S^n \times S^n)$ can be represented as NDR-pairs in an obvious manner, determined by the NDR-representation of a pair (CX,X), even in such a way as to comply to conditions $(_2)$ and $(_3)$ of Theorem 2.4. Furthermore, the fibres of our maps are homeomorphic to S^n which is a compact Hausdorff space. Thus by Theorem 2.4, p is a locally trivial fibration with fibre S^n. ■

In [**Wd**; Example 1, p503] it is shown that these maps p are (up to homotopy) the Hopf fibrations.

Note that the case n = 1 in **2.5** is an instance of Example **2.2(b)**. For n = 1; 2, an alternative approach is possible, see for example [G_1; Example 4, p78]. The case n = 7 is more complicated since the space lack group structure. In fact, S^7 does not admit even a homotopy associative multiplication [**Sf**$_2$].

3. n-EQUIVALENCE AND QUASIFIBRATION

The concept of quasifibration was invented by Dold and Thom [DT] while working on infinite symmetric product spaces. May [M_2] gives a new perspective on quasifibrations with his notion of n-equivalences. In this section we introduce these concepts. Many of the examples quoted here will be treated more extensively in further chapters. Our in-depth study of the elementary properties of these maps appears in Section **15** and Section **16**.

Spaces will be assumed to be free, not pointed. Base points are chosen explicitly whenever we require them.

3.1 Definition

Let $f\colon X \longrightarrow Y$ be a map of topological spaces. Then f is a 0-***equivalence*** if it induces a surjection of the sets of path components, $\pi_0(X) \longrightarrow \pi_0(Y)$.

For an integer $n \geq 1$, f is said to be a n-***equivalence*** if $f_* : \pi_0(X) \longrightarrow \pi_0(Y)$ is bijective, and for every $x \in X$, $f_* : \pi_r(X,x) \longrightarrow \pi_r(Y,fx)$ is bijective for every $0 < r < n$ and surjective for $r = n$.

The map f is said to be a ***weak equivalence*** if it is a n-equivalence for every $n \geq 0$.

This definition is as in Gray [G_1], and Spanier [Sr]. May [M_2] relativized these notions as follows.

3.2 Definition

A map $f\colon (X,A) \longrightarrow (Y,B)$ of pairs is said to be a 0-***equivalence*** if the condition $(_2)$ below holds. For a positive integer n, $f\colon (X,A) \longrightarrow (Y,B)$ is said to be a n-***equivalence*** if conditions $(_1)$ and $(_2)$ hold.

($_1$) For every $a \in A$ and $b = f(a)$, the function $f_*: \pi_r(X,A,a) \longrightarrow \pi_r(Y,B,b)$ is bijective for $1 \leq r < n$ and surjective for $r = n$.

($_2$) $f_*^{-1}\, \mathrm{Im}\, [\pi_0(B) \longrightarrow \pi_0(Y)] = \mathrm{Im}\, [\pi_0(A) \longrightarrow \pi_0(X)]$.

If for every positive integer n, f is a n-equivalence, then f is said to be a ***weak equivalence***.

3.3 Remarks

(a) If in **3.2** we have $A = f^{-1}(B)$, then condition **3.2(₂)** holds. In fact, the condition is equivalent to the requirement that the following diagram is a weak pull-back square in the category of sets.

$$\begin{array}{ccc} \pi_0(A) & \longrightarrow & \pi_0(X) \\ \downarrow & & \downarrow \\ \pi_0(B) & \longrightarrow & \pi_0(Y) \end{array}$$

(b) The composition of two n-equivalences is again a n-equivalence.

(c) A map $f: (X,A) \longrightarrow (Y,B)$ is a n-equivalence if and only if for every path component Z of Y such that $B \cap Z \neq \phi$, with $Z' = f^{-1}(Z)$, the pull-back $(Z', A \cap Z') \longrightarrow (Z, B \cap Z)$ of f is a n-equivalence.

The proof of the following proposition is obtained by juggling path components and making use of **3.3(c)**.

3.4 Proposition

A map $f: (X,A) \longrightarrow (Y,B)$ is a n-equivalence if and only if whenever Z is a union of path components of Y with $B \cap Z \neq \phi$, then the pull-back $(Z', A \cap Z') \longrightarrow (Z, B \cap Z)$ of f, with $Z' = f^{-1}(Z)$, is a n-equivalence. ■

3.5 Examples

(a) For any Hurewicz fibration $f: X \longrightarrow Y$ and any subset B of Y with $f^{-1}(B) \neq \phi$, the induced map $(X, f^{-1}B) \longrightarrow (Y,B)$ is a weak equivalence. This follows by homotopy lifting, and by **3.3(a)**.

(b) The projection of a wedge of spheres $(S^n \vee S^m, S^m) \longrightarrow (S^n,*)$ onto the first summand, is a $(n+m-1)$-equivalence. This can be proved using the Künneth formula and the Whitehead theorem. However, we prove it in Chapter II by adjunction methods.

(c) For a n-connected space X, i.e. a path connected space X with $\pi_r(X) = 0$ whenever $1 \leq r \leq n$, the suspension map $X \longrightarrow \Omega\Sigma X$ is a $(2n+1)$-equivalence. This is an easy consequence of the homotopy excision theorem [**BM**$_1$]. In Chapter II we give a proof of the homotopy excision theorem using our adjunction theory. In fact it is this result, Theorem **7.1** that implies **3.5**(b) above.

(d) For any pointed space X, we shall denote the James reduced product space [**J**$_1$] by X_∞. This is the free topological monoid generated by X, with the base point $*$ as unit element. By X_r we denote the subspace of words of length at most r. Let n be any odd positive integer, let $f: (S^n_\infty, S^n) \longrightarrow (S^{2n}_\infty, *)$, be any map between reduced products as in [**J**$_1$] of spheres, which extends the following map, f_1 :

$$S_2 \longrightarrow S_2 / S \equiv T \subset T_\infty .$$

The map f_1 is the composition of a pinching map, a homeomorphism and an inclusion. James [**J**$_2$] proved that the map of pairs f is a weak equivalence. An alternative, entirely geometric argument will be given in Chapter III (Theorem **13.8**). ∎

The next definition is as in the paper [$\mathbf{M_2}$] of May. The concept has its origin in [**DT**] by Dold and Thom.

3.6 Definition. For topological spaces X and Y, a map $q: X \longrightarrow Y$ is called a ***quasi-fibration*** if it is surjective and for every $y \in Y$, the map $(X, q^{-1} y) \longrightarrow (Y,y)$ of pairs induced by q, is a weak equivalence.

3.7 Examples

(a) Hurewicz fibrations, locally trivial fibrations and Serre fibrations which are surjective, are quasifibrations.

(b) In the fundamental paper [DT], Dold and Thom constructs several quasifibrations. For a cofibration $i: A \longrightarrow X$, application of the infinite symmetric product functor to the quotient map $X \longrightarrow X/A$ yields a quasifibration $SP(X) \longrightarrow SP(X/A)$ with quasifibres weakly equivalent to $SP(A)$.

(c) Dold and Lashof [DL] shows how to construct, for an associative H-space G, a quasifibration $E \longrightarrow B$, with E weakly contractible and the quasifibres weakly equivalent to G.

(d) Gray [G_2] constructs a quasifibration $(X,A)_\infty \longrightarrow X/A$ with quasifibres weakly homotopy equivalent to A_∞. Here, the space $(X,A)_\infty$ is the subspace of all words in X_∞ of which all letters except perhaps the first, belong to the subspace A. He applies this result to recover the weak equivalence $X_\infty \longrightarrow \Omega\Sigma X$ originally proved by James [J_1]. ■

3.8 Remark. An important difference between Hurewicz fibrations and quasifibrations is that quasifibration is not preserved under pull-back. An example demonstrating this fact can be found in the paper [MP] by Morgan and Piccininni.

Chapter II : GLOBALIZATION OF n-EQUIVALENCES

We present a fundamental theorem, Theorem **5.1**, on n-equivalences. This result generalizes the Dold-Thom theorem on quasifibrations, and the theorem on n-equivalences due to May. From **5.1** we deduce several other globalization and adjunction theorems. We obtain a new proof of the homotopy excision theorem of Blakers and Massey.

We make use of some results which appear in Chapter IV. These are fairly elementary, related to the five-lemma. The detailed proof of a key lemma for the proof of **5.1**, is also deferred to Chapter IV, due to its length.

4. LOCAL n-EQUIVALENCE

The concept of local n-equivalence is introduced in this section. It is in this setting that we shall prove the fundamental globalization theorem for quasifibrations of Dold and Thom [DT] in Section 5. Spaces will not be considered to have fixed base points. Whenever required, base points will be specified explicitly.

Definition **4.1(a)** is the concept of *ausgezeichnete Menge* in [DT]. We generalize the concept in **4.1(b)**.

4.1 Definition

Let $p : E \longrightarrow B$ be a map of topological spaces, and for a subspace U of B, let us denote $p^{-1}(U)$ by U′.

(a) A subset U of B is said to be ***distinguished*** with respect to p if the map $U' \longrightarrow U$ induced by p, is a quasifibration.

(b) Suppose that $V \subset U \subset B$. Then for a non-negative integer m, the pair (U,V) is said to be m-***distinguished*** with respect to p if the map $(U',V') \longrightarrow (U,V)$ induced by p, is a m-equivalence. The pair (U,V) is said to be ***distinguished*** with respect to p if it is m-distinguished for all $m \geq 0$.

4.2 Definition

Let $p : E \longrightarrow B$ be a map and m a non-negative integer. Suppose there is an open cover $\mathcal{U}$ of B satisfying the following conditions for arbitrary $U, V \in \mathcal{U}$:

($_1$) Whenever $V \subset U$, then the pair (U,V) is m-distinguished with respect to p.

($_2$) Whenever $x \in U \cap V$, there exists $W \in \mathcal{U}$ such that $x \in W$ and $W \subset U \cap V$.

Then the pair $(p,\mathcal{U})$ is said to be a ***local*** m-***equivalence***. The pair $(p,\mathcal{U})$ is said to be a ***local weak equivalence*** if for every positive integer k it is a local k-equivalence. ∎

We fix the following notation throughout this section, and in Section **5**.

4.3 Notation

If A is a topological manifold with boundary, we denote its boundary by ∂A. We fix an integer $n \geq 2$ and then define the following sets :

I is the unit interval $[0,1]$;

K is the unit n-cube I^n, and then we have the following subsets of K :

$I_1 = I^{n-1} \times 1$ and J is the closure of $\partial K \setminus I_1$.

For paths μ and ν (maps $I \longrightarrow X$) in a space X with $\mu(1) = \nu(0)$ we define the path $\mu + \nu$ in X by the formula :

$$(\mu+\nu)(t) = \begin{cases} \mu(2t) & 0 \leq t \leq \frac{1}{2} \\ \nu(2t-1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

The path μ^- is defined by the formula $\mu^-(t) = \mu(1-t)$. ∎

4.4 Proposition

Let J be the space as defined in **4.3**, corresponding to $n = 2$. Let $q : (X,Y) \longrightarrow (U,V)$ be a 1-equivalence, let $e \in X$ be any point, and let λ and d be paths in U with $\lambda(0) = d(0)$, $q(e) = d(1)$ and $\lambda(1) \in V$. Then there is a path ζ in X with $\zeta(0) = e$ and $\zeta(1) \in Y$, such that the map $h : J \longrightarrow U$ defined below, can be extended to a map

$$H : (I \times I, 1 \times I) \longrightarrow (U,V) .$$

$$h(s,t) = \begin{cases} q \circ \zeta(s) & t = 1 \\ d(t) & s = 0 \\ \lambda(s) & t = 0. \end{cases}$$

Proof Let $X \xrightarrow{q^{\sim}} U^{\wedge} \xrightarrow{q^{\wedge}} U$ be the mapping path fibration factorization of q. We shall regard $q^{\sim}$ as an inclusion map. Let κ be the path in U defined by $\kappa = d^{-} + \lambda$. By the homotopy lifting property there is a path μ in $U^{\wedge}$ with $\mu(0) = e$ such that $q^{\wedge} \circ \mu = \kappa$. By **16.5,** $q^{\sim}$ gives a surjective function, $\pi_0(Y) \longrightarrow \pi_0(V^{\wedge})$ where $V^{\wedge} = q^{\wedge -1}(V)$, since $q : (X,Y) \longrightarrow (U,V)$ is a 1-equivalence. Thus we can find a path ν in $V^{\wedge}$ from the point $\mu(1)$ to a point in Y. There is a natural retraction $r : U^{\wedge} \longrightarrow X$. We choose $\zeta = r \circ (\mu + \nu)$, which is a path in X with $\zeta(0) = e$ and $\zeta(1) \in Y$. Note that $q \circ \zeta(0) = q \circ r \circ \mu(0) = q \circ \mu(0) = q(e)$, thus $d(1) = q \circ \zeta(0)$, and therefore there is a path $\lambda^{-} + d + (q \circ \zeta)$ in U.

The proof is completed by showing that $\lambda^{-} + d + (q \circ \zeta)$ is path homotopic to ($\simeq_p$) a path in V. To this end we note that $q \circ r$ is homotopic to $q^{\wedge}$ (rel. X). Since the end-points of $\mu + \nu$ is in X, it therefore follows that

$$q \circ \zeta = q \circ r \circ (\mu + \nu) \simeq_p q^{\wedge} \circ (\mu + \nu).$$

Finally, we have the following routine computation:

$$\begin{aligned} \lambda^{-} + d + (q \circ \zeta) &\simeq_p \lambda^{-} + d + [q^{\wedge} \circ (\mu + \nu)] \\ &\simeq_p \lambda^{-} + d + (q^{\wedge} \circ \mu) + (q^{\wedge} \circ \nu) \\ &\simeq_p \lambda^{-} + d + \kappa + (q^{\wedge} \circ \nu) \\ &\simeq_p q^{\wedge} \circ \nu , \end{aligned}$$

and the assertion can be seen to follow. ∎

4.5 Lemma

If $(p,\mathcal{U})$ is a local 1-equivalence, then for every $C \in \mathcal{U}$, the pair (B,C) is 1-distinguished.

Proof (B,C) is 1-distinguished if and only if for any map $g : (I,\partial I,0) \longrightarrow (B,C,p(e))$, there exists a map $F : (I,\partial I,0) \longrightarrow (E,p^{-1}(C),e)$ and a homotopy D_t from g to $p \circ F$, fixing 0 on $p(e)$ and sending 1 into C. We note that we work with complete inverse images and thus by **3.3(a)**, the path component condition **3.2($_2$)** is satisfied.

Let g be as above. The open cover $\{g^{-1}(U) : U \in \mathcal{U}\}$ of the compact metric space I has a Lebesgue number. Thus there exists an integer k, such that for every $r \in \mathbb{N}$ with $r \leq k$, the interval $[\frac{r-1}{k}, \frac{r}{k}]$ is mapped by g into one of the members of $\mathcal{U}$. For each such $1 \leq r \leq k$, one can choose $U_r \in \mathcal{U}$ such that $g[\frac{r-1}{k}, \frac{r}{k}] \subset U_r$. Hereafter we choose, for each positive integer $r < k$, a set $V_r \in \mathcal{U}$ with $g(\frac{r}{k}) \in V_r \subset U_r \cap U_{r+1}$, and a subset V_k of $U_k \cap C$ such that $g(1) \in V_k$ and $V_k \in \mathcal{U}$. We can make our choice such that $U_1 = C$ by choosing k sufficiently big.

Let $X_r = p^{-1}(U_r)$ and $Y_r = p^{-1}(V_r)$. We shall inductively apply **4.4** to the map $(X_r,Y_r) \longrightarrow (U_r,V_r)$. For $r = 1$ we choose, as in **4.4**, λ_1 to be the path in U_1 defined by the formula $\lambda_1(t) = g(\frac{t}{k})$ and $d_1(t) = p(e)$ the constant path in U_1. Then there exists, as in **4.4**, a path ζ_1 in X_1 and a map $H_1 : I \times I \longrightarrow U_1$.

We repeat this process of constructing similar ζ_r's and H_r's for $r = 2; 3; \ldots ; k$. Having obtained the maps for a given stage $r-1$, we choose $d_r(t) = H_{r-1}(t,1)$, $\lambda_r(t) = g(\frac{t+r-1}{k})$ and $e_r = \delta_{r-1}(1)$.

The ζ_r's and respectively the H_r's can be merged together to form the required path F and the homotopy D_t. More precisely, we obtain them by the formulae :

$$F(s) = \zeta_r(ks-r+1) \text{ for } \tfrac{r-1}{k} \leq s \leq \tfrac{r}{k},$$

$$D_t(s) = H_r(ks-r+1, t) \text{ for } \tfrac{r-1}{k} \leq s \leq \tfrac{r}{k}. \qquad \blacksquare$$

5. THE GLOBALIZATION THEOREMS

The globalization theorems for n-equivalences, 5.1, and quasifibrations, 5.2, can now be proved. Similar globalizations have been studied by Huebsch [**Hh**] and Hurewicz [**Hz**] for locally trivial fibrations, and by Dyer and Eilenberg [**DE**] for fibrations. As presented here for n-equivalences, Theorem 5.1 is the most general result in this context. It is this one that we actually prove, and thereafter we derive the theorem [**DT**; Satz 2.2] of Dold and Thom. May [**M$_2$**] gave a similar treatment of the Dold-Thom theorem through n-equivalences. The proof of Theorem 5.1 entails the solution of some homotopy lifting problem, see 5.5, that is solvable locally. We use induction, the first step of which is covered by 4.5.

Base points for spaces will be specified whenever they are required. The main theorems are formulated now. Their proofs will be given at the end of the section.

5.1 Theorem

Let $p : E \longrightarrow B$ be a map and $\mathcal{U}$ an open cover of B. If $(p,\mathcal{U})$ is a local n-equivalence, then for every $C \in \mathcal{U}$, (B,C) is a n-distinguished pair.

5.2 Theorem [Satz 2.2 of DT]

Let $p : E \longrightarrow B$ be a map and $\mathcal{U}$ an open cover of B, of subsets which are distinguished with respect to p. Suppose further that for every $U,V \in \mathcal{U}$, and arbitrary x in $U \cap V$, there exists $W \in \mathcal{U}$ such that $x \in W$ and $W \subset U \cap V$. Then p is a quasifibration.

5.3 An indexing of a finite collection of subsets of a set

Let $\mathcal{U}$ be any finite non-empty collection of subsets of a given set X. Let $\mathcal{V}$ be the collection of all non-empty subsets of X which are intersections of subcollections of $\mathcal{U}$.

Then $\mathcal{V}$ is finite and we can index the members of $\mathcal{V}$: $V_1, V_2, V_3, \dots, V_q$ in such a way that for $1 \leq i < j \leq q$ we have either $V_i \cap V_j = V_r$ for some $r \leq i$ or $V_i \cap V_j = \phi$.

5.4 Remark

Suppose that $(p,\mathcal{U})$ is a local n-equivalence. Let B_0 be a subspace of B which is the intersection of finitely many members of $\mathcal{U}$. Let $p_0 : E_0 \longrightarrow B_0$ be the pull-back of p over the inclusion map $B_0 \subset B$, and let $\mathcal{V}$ be the subclass of $\mathcal{U}$ consisting of all those members lying entirely inside B_0. It follows from the definition that $(p_0,\mathcal{V})$ is a local n-equivalence. ■

The notation of **4.3** will be used in the remainder of the section. Lemma **5.5** on homotopy lifting is required for the proof of the fundamental globalization theorem, **5.1**. Its proof appears in Section **17**.

5.5 Lemma

Let $p : (X,U) \longrightarrow (Y,V)$ be a map of pairs. Then the following three conditions are equivalent (for $n \geq 2$) :

(a) Given maps $f: (J,\partial J) \longrightarrow (X,U)$ and $g : (K,I_1) \longrightarrow (Y,V)$ together with a homotopy $d_t : J \longrightarrow Y$ from $p \circ f$ to the restriction $g|J$ of g to J, such that $d_t(\partial J) \subset V$ for all $t \in I$, there exists an extension $F: (K,I_1) \longrightarrow (X,U)$ of f, and a homotopy $D_t : K \longrightarrow Y$ from $p \circ F$ to g, extending d_t such that $D_t(I_1) \subset V$ for all $t \in I$.

(b) Given maps $\varphi : (J, \partial J) \longrightarrow (X,U)$ and $\gamma : (K,I_1) \longrightarrow (Y,V)$ with $p \circ \varphi = \gamma|J$, then there exists an extension $\Phi : (K,I_1) \longrightarrow (X,U)$ of φ, and a homotopy $\Delta_t : K \longrightarrow Y$ from $p \circ \Phi$ to γ, such that Δ_t is stationary on J and $\Delta_t(I_1) \subset V$ for all $t \in I$.

(c) For every $e \in U$ and $b = p(e)$, the function $p_*: \pi_r(X,U,e) \longrightarrow \pi_r(Y,V,b)$ is injective for $r = n-1$ and surjective for $r = n$. ■

5.6 *Proof of Theorem* **5.1**

The proof is by induction. We inductively prove the statements $\mathcal{S}_q$, $q \geq 0$.

$\mathcal{S}_q$: If $(p,\mathcal{U})$ is a local q-equivalence, then for every $C \in \mathcal{U}$, (B,C) is a q-distinguished pair.

Since we work with complete inverse images, $\mathcal{S}_0$ follows by **3.3(a)**. The statement $\mathcal{S}_1$ is exactly the result **4.5**. Now assume that for some integer n bigger than 1, $\mathcal{S}_q$ is true for all $q \in \{1, 2, 3, \dots, n-1\}$, and suppose that $(p,\mathcal{U})$ is a local n-equivalence and $C \in \mathcal{U}$. We shall prove $\mathcal{S}_n$ by showing that, for $C' = p^{-1}(C)$, the map $(E,C') \longrightarrow (B,C)$ is a n-equivalence. We already know this map of pairs to be a (n–1)-equivalence by the induction assumption, and it suffices to show that:

$(_0)$ $p_* : \pi_r(E,C',e) \longrightarrow \pi_r(B,C,b)$ is injective for $r = n-1$ and surjective for $r = n$,

for every $e \in C'$ and $b \in p(e)$.

This statement $(_0)$ is in the same form as condition **5.5(c)**. We prove the equivalent condition **5.5(b)**, or more precisely, the version of **5.5(b)** in terms of our symbols.

So suppose that we have maps $f: (J,\partial J) \longrightarrow (E,C')$ and $g: (K,I_1) \longrightarrow (B,C)$, and that $d_t : J \longrightarrow B$ is a stationary homotopy from $p \circ f$ to $g|J$. We shall show how f and d_t can be extended systematically over the subsets T_u of K,

$$T_u = J \cup (I^{n-1} \times [0, \tfrac{u}{k}]),$$

by induction on u, $u = 1; 2; \dots ; k$. For a given r, $2 \leq r \leq k$, the obtained extensions of f and d_t over T_{r-1} are further extended over T_r.

The compact subset $g(\mathrm{K})$ of B can be covered by a finite subcollection $\mathcal{U}_1$ of $\mathcal{U}$. We can assume that $\mathrm{C} \in \mathcal{U}_1$. The open cover $\{g^{-1}(\mathrm{U}) : \mathrm{U} \in \mathcal{U}_1\}$ of the compact metric space K has a Lebesgue number. Therefore there exists an integer k such that for every subcube φ of K with side lengths not exceeding $\frac{1}{k}$, φ lies entirely in one of the sets in $\{g^{-1}(\mathrm{U}) : \mathrm{U} \in \mathcal{U}_1\}$. Let $\mathcal{V}$ be the collection of all non-empty proper subsets of B which can be obtained as intersections of subcollections of $\mathcal{U}_1$. Then $\mathcal{V}$ is finite and $\mathcal{U}_1 \subset \mathcal{V}$. Label the members of $\mathcal{V}$ as in 5.3, $\mathcal{V} = \{\mathrm{V}_1, \mathrm{V}_2, \dots, \mathrm{V}_d\}$.

In order to extend f and d_t we shall make use of the facts $(_1)$ and $(_2)$ which shall be proved further on.

$(_1)$ For any $\mathrm{Y}, \mathrm{V} \in \mathcal{V}$ with $\mathrm{V} \subset \mathrm{Y}$, (Y,V) is $(n-1)$-distinguished.

$(_2)$ For any $\mathrm{Y} \in \mathcal{U}$ and $\mathrm{V} \in \mathcal{V}$ with $\mathrm{V} \subset \mathrm{Y}$, (Y,V) is n-distinguished.

For the subset $\mathrm{Q} = \{0; \frac{1}{k}; \frac{2}{k}; \frac{3}{k}; \dots ; 1\}$ of I, let Q^{n-1} be the product of $n-1$ copies of Q. Let Φ the set of all subcubes of I^{n-1} with vertices in Q^{n-1} and volume in $\mathbb{R}^{n-1}$ equal to $(\frac{1}{k})^{n-1}$. Let Φ_r be the set of all r-faces, $r = 0; 1; \dots ; n-1$, of members of Φ. For $\varphi \in \Phi_r$, we say that φ is of type (i,j) if i is the least among the integers q for which $g(\varphi \times \frac{1}{k}) \subset \mathrm{V}_q$, and j is the least integer r such that $g(\varphi \times [0, \frac{1}{k}]) \subset \mathrm{V}_r$.

We now construct the extensions of f and d_t over T_1. Put

$$\Gamma_r = \mathrm{J} \cup (\Phi_r \times [0, \tfrac{1}{k}]) \subset \mathrm{T}_1 .$$

Applying $(_1)$ above and 4.4, we can extend f and d_t over Γ_0, obtaining F and D_t partially, such that for every type (i,j) face $\varphi \in \Phi_0$, the following conditions are satisfied.

$$(3) \qquad F(\varphi \times \tfrac{1}{k}) \subset p^{-1}(V_i) \text{ and } D_t(\varphi \times \tfrac{1}{k}) \subset V_i,$$

$$F(\varphi \times [0, \tfrac{1}{k}]) \subset p^{-1}(V_j) \text{ and } D_t(\varphi \times [0, \tfrac{1}{k}]) \subset V_j.$$

We proceed with this construction over Γ_1, and then over $\Gamma_2, \ldots, \Gamma_{n-2}$ using the fact (1) above and the implication **(c) ⇒ (a)** of Lemma **5.5** (taking $V_i = V$ and $V_j = Y$). At each of these stages, we do it for all $\varphi \in \Phi_r$, in such a way that if φ is of type (i,j), then F and D_t satisfy the conditions **(3)** above.

In order to do the extensions over Γ_{n-1}, the observation **(2)** above is important, and we proceed as follows. Let $\varphi \in \Phi_{n-1}$ be a face of type (i,j). At this stage we no longer require the minimality of V_j. Instead we replace V_j (if necessary) by a set $Y \in \mathcal{U}_1$, in order that by **(2)**, (Y, V_i) is n-distinguished. By **5.5** [**(c) ⇒ (a)**], we can thus extend F and D_t over $\varphi \times [0, \frac{1}{k}]$ in such a way that :

$$F(\varphi \times \tfrac{1}{k}) \subset p^{-1}(V_i) \text{ and } D_t(\varphi \times \tfrac{1}{k}) \subset V_i,$$

$$F(\varphi \times [0, \tfrac{1}{k}]) \subset p^{-1}(Y) \text{ and } D_t(\varphi \times [0, \tfrac{1}{k}]) \subset Y.$$

For the purposes of Lemma **5.5** we use $V_i = V$ and $Y = Y$. Note that we do not quite use the fact that d_t is stationary. The important fact is that for a type (i,j) face φ, d_t keeps $\varphi \times [0, \frac{1}{k}]$ inside V_j and $\varphi \times \frac{1}{k}$ inside V_i. The extension of d_t does the same all along in the process. Thus our construction can be repeated over $T_2, T_3, \ldots, T_k$ and it yields the required extensions. The proof of **5.1** is complete except for the proofs of **(1)** and **(2)** which we supply now.

Proof of (2): Let $V \in \mathcal{V}$ and $Y \in \mathcal{U}$ with $V \subset Y$. We construct a subset Z (not necessarily in $\mathcal{U}$) of V such that (V,Z) is (n–1)-distinguished, (Y,Z) is n-distinguished, and the inclusion $Z \subset V$ is a bijection of path components. Then by **16.8** applied to the triple (Y,V,Z), our assertion $(_2)$ will follow, i.e (Y,V) is n-distinguished. So we proceed to construct the set Z.

Let P be a subset of V containing precisely one element from every path component of V. For a subset A of Y and $c \in P$, we use the symbol A_c to denote the path component of A which contains the point c.

Now choose an arbitrary $c \in P$. Since V is a finite intersection of members of $\mathcal{U}$, by **4.2**$(_2)$, there exists $W \in \mathcal{U}$ such that $c \in W \subset V$ (and we make a fixed choice of such a subset W). By **5.4** and $\mathcal{S}_{n-1}$, the pair (V,W) is (n–1)-distinguished. Since $W,Y \in \mathcal{U}$ the pair (Y,W) is n-distinguished. From **15.2(d)**, (V,W_c) is (n–1)-distinguished and (Y,W_c) is n-distinguished. We choose Z to be the union of all the subsets W_c $(c \in P)$. We note that the path components of Z are precisely the subsets W_c. Thus by **15.2(d)**, the pair (V,Z) is (n–1)-distinguished and (Y,Z) is n-distinguished. This completes the proof of the assertion $(_2)$.

The ***proof of*** $(_1)$ is similar to that of $(_2)$ and we shall not give the detail here. Finally the proof of Theorem 5.1 is complete. ▮

5.7 Proposition

Let $p : E \longrightarrow B$ be a map and $\mathcal{U}$ an open cover of B, of subsets which are distinguished with respect to p. Suppose further that for every $U,V \in \mathcal{U}$, and arbitrary x in $U \cap V$, there exists $W \in \mathcal{U}$ such that $x \in W$ and $W \subset U \cap V$.

Then $(p,\mathcal{U})$ is a local weak equivalence.

This observation is similar to $(_2)$ in the proof of Theorem **5.1** above, and is required in order to deduce the proof of Theorem **5.2**.

Proof of 5.7 Let $V, Y \in \mathcal{U}$ with $V \subset Y$. Let Z be any subset of V containing precisely one point of every path component of V. Then for any $c \in Z$, (V,c) and (Y,c) are distinguished pairs by assumption. Thus by **15.2(d)**, (V,Z) and (Y,Z) are distinguished. Moreover, the inclusion map $Z \subset V$ is a surjective function on path components. By **16.8**, (Y,V) is a distinguished pair. Condition **4.2**$(_2)$ on local weak equivalences is easily seen to be satisfied, and our result is proved. ▮

5.8 ***Proof of Theorem*** 5.2

Let x be any point of B. We must prove that (B,x) is a distinguished pair. Now pick any $C \in \mathcal{U}$ such that $x \in C$ ($\mathcal{U}$ covers B). By **5.7**, $(p,\mathcal{U})$ is a local weak equivalence and so by **5.1**, (B,C) is distinguished. Thus by **16.9(b)** it follows that (B,x) is a distinguished pair. This completes the proof of **5.2**. ▮

The following result generalizes a theorem in Gray's book [G_1; 16,23 on p140], and [**Mc**; Theorem 6 on p467] of McCord.

5.9 Theorem

Let n be a non-negative integer. Let $p : E \longrightarrow B$ be a map and $\mathcal{U}$ an open cover of B such that for every $U,V \in \mathcal{U}$ and any $x \in U \cap V$, there exists $W \in \mathcal{U}$ with $W \subset U \cap V$ and $x \in W$. Suppose further that for each $U \in \mathcal{U}$ the induced map $p^{-1}(U) \longrightarrow U$ is a n-equivalence.

Then $p : E \longrightarrow B$ is a n-equivalence.

Proof For any pair $U,V \in \mathcal{U}$ with $U \subset V$, by 16.2 it follows that the pair (U,V) is n-distinguished. Thus $(p,\mathcal{U})$ is a local n-equivalence. Thus by Theorem 5.1, for every U in $\mathcal{U}$, (B,U) is a n-distinguished pair. Our result follows by 16.1 . ∎

6. ADJUNCTION OF n–EQUIVALENCES AND QUASIFIBRATIONS

The first construction of push-outs of quasifibrations was made by Dold and Lashof [**DL**] while constructing the universal quasifibration of an associative H-space. After this, several similar constructions follow. The technique is discussed in detail by Hardie [**Ha**]. This method will now be generalized to the case of n-equivalences. Towards this end, Theorem **5.1** or the special case [$\mathbf{M_2}$; Theorem 1.2] of it is fundamental. The new push-out theorems have many applications, especially **6.1**, **6.3** and **6.8**(c), as can be seen in the remainder of this chapter and in Chapter III. Theorem **6.2** is an affirmative answer to a question posed by J.-P. Meyer in a letter to K. A. Hardie.

We work with free spaces, that is to say that spaces will not be assumed to have *fixed* base points. In order to formulate the theorems we recall the notation for double mapping cylinder and push-out from Section 1.

6.0 Notation The commutative diagram **A** can be considered to be a cotriad in the category $\underline{\text{Top}}^2$ of pairs and pair maps.

$$-\ \mathbf{A}\ - \qquad \begin{array}{ccccc} E_1 & \xleftarrow{\ g_1\ } & E_0 & \xrightarrow{\ g_2\ } & E_2 \\ \downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\ B_1 & \xleftarrow[\ f_1\ ]{} & B_0 & \xrightarrow[\ f_2\ ]{} & B_2 \end{array}$$

We recall from Section 1, that a cotriad $E_1 \longleftarrow E_0 \longrightarrow E_2$ has a push-out E and a double mapping cylinder E'. The $\underline{\text{Top}}^2$ -cotriad determines a unique map $p : E \longrightarrow B$ between push-outs and a unique map of double mapping cylinders, $p' : E' \longrightarrow B'$.

Let V_1 be the image in B' of $B_1 + B_0 \times [0,1)$ under the canonical map

$$\eta : B_1 + B_0 \times I + B_2 \longrightarrow B'.$$

Similarly, let V_2 be the image in B' of $B_0 \times (0,1] + B_2$ under η. Let $V_0 = V_1 \cap V_2$. Define similar subsets U_i of E'. For each $i \in \{0, 1, 2\}$ there are retractions $U_i \longrightarrow E_i$ and $V_i \longrightarrow B_i$ which are homotopy equivalences. ■

We now formulate the three main theorems that are proved in this section. For Theorem 6.1 we have the following setup.

We fix a subspace $T \subset B_0$. For each $x \in T$ there are subsets $F_j^x \subset p_j^{-1}(f_j\, x)$, $j \in \{1, 2\}$, and $F_0^x \subset g_1^{-1}(F_1^x) \cap p_0^{-1}(x) \cap g_2^{-1}(F_2^x)$. Let us assume that for $x, y \in T$, $F_j^x = F_j^y$ whenever $f_j(x) = f_j(y)$.

The double mapping cylinder of the cotriads $f_1(T) \longleftarrow T \longrightarrow f_2(T)$ is denoted by T', and for each $z \in T'$, F^z is the canonical image in $p'^{-1}(z)$ of F_0^z or F_j^z, according as $z \in V_0$ or $z \in V_j \setminus V_0$.

6.1 Theorem

Suppose that the inclusion map $T \subset B_0$ is a surjective function on the sets of path components. Suppose further that the following conditions hold, for each $x \in T$, $j \in \{1, 2\}$.

(1) $p_0 : (E_0, F_0^x) \longrightarrow (B_0, x)$ is a n-equivalence.

(2) $p_j : (E_j, F_j^x) \longrightarrow (B_j, f_j(x))$ p_j is a (n+1)-equivalence.

(3) The induced maps $h_j : F_0^x \longrightarrow F_j^x$ are n-equivalences.

Then: **(a)** For the map $p' : E' \longrightarrow B'$ of double mapping cylinders, $(E', F^z) \longrightarrow (B', z)$ is a (n+1)-equivalence for each $z \in T'$.

(b) If moreover, f_1 and g_1 are cofibrations and every $F_0^x \longrightarrow F_1^x$ is a homeomorphism, for the map $p : E \longrightarrow B$ of push-outs, we have the following.

For each $x \in T$, $(E,G) \longrightarrow (B,y)$ is a (n+1)-equivalence, G being the homeomorphic image of F_2^x in E and $y = p(G)$.

6.2 Theorem

Suppose that in diagram A every p_i is a quasifibration and each of the pair maps (g_j,f_j) is a weak equivalence of fibres.

(a) Then the map $p' : E' \longrightarrow B'$ of double mapping cylinders is a quasifibration.

(b) Suppose furthermore that g_1 and f_1 are cofibrations and that p_0 is the pull-back of p_1 over f_1. Then the map $p : E \longrightarrow B$ of push-outs is a quasifibration.

For the proofs of Theorems **6.1** and **6.2**, we make use of Theorem **6.3** and Proposition **6.4** below. The *homotopy fibres* approach as in **6.3**, is similar to and supplements work done by V. Puppe [**Pu**].

6.3 Theorem

Suppose that in diagram A, for each $j \in \{1, 2\}$, the $\underline{Top}^2$-morphism (g_j,f_j) is a n-equivalence of homotopy fibres. Then for every $i \in \{0, 1, 2\}$, we have the following:

(a) The $\underline{Top}^2$-morphism $p_i \longrightarrow p'$ to the double mapping cylinder is a n-equivalence of homotopy fibres. (See **1.15** for this concept).

(b) If moreover g_1 and f_1 are cofibrations, the $\underline{Top}^2$-morphism $p_i \longrightarrow p$ to the push-out is a n-equivalence of homotopy fibres.

Proof (of Theorem 6.3)

(a) For each $i \in \{0, 1, 2\}$, the map $U_i \longrightarrow V_i$ induced by p', is homotopy equivalent to p_i. Thus by **16.5** and with our assumption on homotopy fibres, each pair (V_j, V_0) is (n+1)-distinguished with respect to p'. Also the sets V_i are open. Thus for $\mathcal{U} = \{V_0, V_1, V_2\}$, the pair $(p', \mathcal{U})$ is a local n-equivalence.

By Theorem **5.1** or alternately [$\mathbf{M}_2$; Theorem 1.2], it follows that each pair (B', V_i), $i \in \{0, 1, 2\}$, is (n+1)-distinguished with respect to p'. By **16.5** we have that for each $i \in \{0, 1, 2\}$, the Top^2-morphism from the object $U_i \longrightarrow V_i$ to p' is a n-equivalence of homotopy fibres. Since p_i is homotopy equivalent to $U_i \longrightarrow V_i$, it follows that in fact $p_i \longrightarrow p'$ is a n-equivalence of homotopy fibres. This completes the proof of **(a)**.

The statement **(b)** follows from **(a)** and **1.14** . ∎

6.4 Proposition

Under the conditions of Theorem **6.1** we have that for each $j \in \{1, 2\}$, the Top^2-morphism $(g_j, f_j) : p_0 \longrightarrow p_j$ is a n-equivalence of homotopy fibres.

Proof Fix any $x \in T$. In diagram B, α (respectively, α_j) is the natural map into the homotopy fibre of p_0 over x (resp. p_j over $f_j(x)$), and β_j is the natural induced map (see **1.8**). The diagram is commutative.

– B –

$$\begin{array}{ccc} F_0^x & \xrightarrow{\ h_j\ } & F_j^x \\ \alpha \downarrow & & \downarrow \alpha_j \\ H & \xrightarrow[\ \beta_j\ ]{} & H_j \end{array}$$

By **16.5**, Condition **6.2**($_1$) ensures that α_0 is a (n–1)-equivalence.

Similarly from Condition 6.1($_2$), α_j is a n-equivalence. Due to 6.1($_3$), h_j is a n-equivalence and therefore $\alpha_j \circ h_j$ is a n-equivalence. Thus by 16.6(c), β_j is a n-equivalence. Our result follows since every path component of B_0 contains a point of T. ▮

6.5 *Proof of Theorem* 6.1

(a) By 6.4, each of the $\underline{\text{Top}}^2$-morphisms $p_0 \longrightarrow p_j$, $j \in \{1, 2\}$, is a n-equivalence of homotopy fibres. Consequently by Theorem 6.3, each of the $\underline{\text{Top}}^2$-morphisms $p_i \longrightarrow p'$ is a n-equivalence of homotopy fibres, $i = \{0, 1, 2\}$. For a given $x \in T'$ let r be the constant map $F^x \longrightarrow x$. Then the $\underline{\text{Top}}^2$-morphism $r \longrightarrow p'$ is a n-equivalence of homotopy fibres. By 16.5, the map $(E', F^x) \longrightarrow (B', x)$ induced by p is a $(n+1)$-equivalence. Thus (a) follows.

(b) The condition on fibres ensures that the quotient map $F_2^x \longrightarrow G$ is a weak equivalence. The result now follows from (a) and 1.14. ▮

6.6 *Proof of Theorem* 6.2

(a) From 6.1(a) it follows that for each $z \in \eta(B_0 \times I)$, η is as in 6.0, (B', z) is a distinguished pair. Let C be the union of all the path components of B' that meet the set $\text{Im}\,(B_0 \longrightarrow B')$. Then by comparison with the mapping path fibration of p', it follows that C is a distinguished subset of B'. The subspace $B' \backslash C$ is distinguished since it did not really participate in the adjunction process. This settles the proof of (a).

(b) The pull-back condition ensures that the $\underline{\text{Top}}^2$-morphism $p' \longrightarrow p$ is a weak equivalence of fibres. The result now follows from (a) and 1.14. ▮

6.7 Notation

Now suppose that diagram A is homotopy commutative. Let $k(-,t)$ and $l(-,t)$ be homotopies from $f_1 \circ p_0$ to $p_1 \circ g_1$, and from $f_2 \circ p_0$ to $p_2 \circ g_2$ respectively. Then we can define a map $q : E' \longrightarrow B'$ between the double mapping cylinders as follows :

$$q[e,t] = \begin{cases} [p_1(e),\ 0] & t = 0\ ,\ e \in E_1 \\ [k(e,\ 1-4t),\ 0] & 0 < t \leq \frac{1}{4}\ ,\ e \in E_0 \\ [p_0(e),\ 2t - \frac{1}{2}] & \frac{1}{4} \leq t \leq \frac{3}{4}\ ,\ e \in E_0 \\ [l(e,\ 4t-3),\ 1] & \frac{3}{4} \leq t < 1\ ,\ e \in E_0 \\ [p_2(e),\ 1] & t = 1\ ,\ e \in E_2 \end{cases}$$

Note that if $e \in E_0$ then $[p_1(e),0] = [k(e,1),0]$ and $[p_2(e),1] = [l(e,1),1]$. So q is well-defined. By using the pasting lemma, q is found to be continuous. Section 1 can be consulted for more detail on the double mapping cylinder.

6.8 Proposition

Let n be a positive integer. If in diagram A, p_1 and p_2 are n-equivalences and p_0 is a (n–1)-equivalence, then we have the following results :

(a) The map q defined in 6.7 is a n-equivalence.

(b) If diagram A is commutative with stationary homotopies, then p' is a n-equivalence.

(c) If diagram A is commutative with stationary homotopies, and f_1 and g_1 are cofibrations, then p is a n-equivalence.

Proof (a) We consider the following subsets of B':

$$W_0 = \eta\,(\,B_0 \times \{\tfrac{1}{2}\}\,);\ \ W_1 = \eta\,(\,B_1 + B_0 \times [0,\tfrac{1}{2}]\,);\ \ W_2 = \eta\,(\,B_0 \times [\tfrac{1}{2},1] + B_2\,)\,.$$

The map η is defined in 6.0. For each $i \in \{0, 1, 2\}$, let $X_i = q^{-1}(W_i)$. Then q is the push-out of the cotriad Top^2-cotriad of diagram **C**. The horizontal arrows are inclusions and are cofibrations. The vertical arrows are the pull-backs of q over the inclusions $W_i \subset B'$.

$$
\text{– C –} \qquad
\begin{array}{ccccc}
X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
W_1 & \longleftarrow & W_0 & \longrightarrow & W_2
\end{array}
$$

For each $i \in \{0, 1, 2\}$, q_i is homotopy equivalent to p_i. Thus q_0 is a $(n-1)$-equivalence, and q_j is a n-equivalence for each $j \in \{1, 2\}$. We obtain or result by application of **6.1(b)**, taking any set $T = W_0$ and taking the spaces F_i^x to be one point sets. Theorem **6.1** does not take care of the path components of B' which are disjoint from W_1, but such path components did not really take part in the process of formation of double mapping cylinders.

(b) In the case of strict commutativity of diagram **A** with stationary homotopies, p' is homotopic to q.

(c) This follows by (b) and 1.14. ▮

6.9 The reduced double mapping cylinder.

Suppose that we have a cotriad such as the one below, and let B' be its double mapping cylinder.

$$B_1 \xleftarrow{f_1} B_0 \xrightarrow{f_2} B_2$$

Let x be an abitrary element of B_0. The *reduced double mapping cylinder* of the cotriad with respect the point x is the space B_x obtained from B' by collapsing the subspace $\eta(x \times I)$.

6.10 Proposition

The quotient map $\beta_x : B' \longrightarrow B_x$ is a weak equivalence.

Proof There is a quotient map $\alpha = \eta \circ \beta_x : B_1 + B_0 \times I + B_2 \longrightarrow B_x$. Let V_1 and V_2 be, respectively, the images under α of the subsets $B_1 + B_0 \times [0,1)$ and $B_0 \times (0,1] + B_2$, and let $V_0 = V_1 \cap V_2$. Note that each V_i is deformable onto a subset homeomorphic to B_i, so that the induced map $\beta_x^{-1}(V_i) \longrightarrow V_i$ is a weak equivalence. Furthermore, each V_i is an open subset. Thus our claim follows by Theorem **5.9** . ▮

6.11 Remark

There is a forgetful functor from the category of pointed spaces and pointed maps to the category of free spaces and free maps. We define the concept n-equivalence in the pointed category via this forgetful functor. Then **6.10** guarantees that the results in this section that hold for double mapping cylinders, are also valid for reduced double mapping cylinders when working in the pointed category. For the remainder of Chapter II and in Chapter III we shall work in the pointed category, taking advantage of this fact.

7. RELATIVE HOMEOMORPHISMS

In this section we prove a result, Theorem 7.1, on relative homeomorphisms, similar to the triad connectivity theorem of Blakers and Massey [**BM**$_2$]. Subsequent to the work of Blakers and Massey, a number of essentially different treatments of triad connectivity has appeared, such as those of [**Mo**], [**N**], [**DKP**], [**Sw**], [**G**$_1$] and [**BH**]. Theorem 7.1 is obtained by adjoining n-equivalences, and its merit lies in its simplicity.

When forming quasifibrations via adjunction, then usually two of the vertical arrows are trivial fibrations. The following diagram is a typical one for such constructions.

$$\begin{array}{ccccc} F \times C & \longleftarrow & F \times E & \longrightarrow & E_1 \\ \downarrow & & \downarrow & & \downarrow \\ C & \longleftarrow & E & \longrightarrow & B \end{array}$$

Here C is the cone of E. The Dold-Lashof construction [**DL**] is, for example, a recursive formation of push-outs of diagrams of this form with $E = E_1$. The basic method of this section is to replace the trivial fibrations by their restrictions to the wedges $F \vee C$ or $F \vee E$ in the product spaces.

We shall be working throughout this section with pointed spaces and maps, see **6.11**.

7.0 Definition

For a map $p : (X,A) \longrightarrow (Z,B)$, we have a commutative square as in diagram **A**. The map of pairs p is said to be a ***relative homeomorphism*** if diagram **A** is a push-out square.

$$
\begin{array}{ccc}
A & \xrightarrow{\subset} & X \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Z
\end{array}
\qquad \text{– A –}
$$

Note how Z is uniquely determined from the other data if p is required to be a relative homeomorphism. The main theorem of this section follows.

7.1 Theorem

Suppose that $p_0 : A \longrightarrow B$ is a m-equivalence and (X,A) is a (n–1)-connected pair. Let us further assume that at least one of the following two conditions hold.

(1) (X,A) is an NDR-pair,

(2) A is an open subset of X and $p_0(A)$ is an open subset of B.

Then the relative homeomorphism $p : (X,A) \longrightarrow (Z,B)$ is a (m+n–1)-equivalence.

The proof of this theorem is finalized at the end of this section, and every result that follows is relevant to the ultimate proof. The more prominent of these are 7.2 and 7.6.

7.2 Proposition

Suppose G is a subspace of a space F such that the inclusion $G \subset F$ is a m-equivalence. Let C_n be the subset $F \times * \cup G \times S^n$ of $F \times S^n$. Then the inclusion $C_n \subset F \times S^n$ is a (m+n)-equivalence.

Proof We proceed by induction on n. For $n = 0$, as free spaces, $C_0 = F + G$, a disjoint union, and $F \times S^0 = F + F$. By assumption $i : G \longrightarrow F$ is a m-equivalence. Thus the inclusion $C_0 \subset F \times S^0$ is a $(m + 0)$-equivalence, and the case $n = 0$ is proved. Now let us assume that the statement is true for $n = t - 1$, where $t \geq 1$, and show that it also holds for $n = t$.

By assumption then, in the $\underline{\text{Top}}^2$-cotriad of diagram B, the inclusion map α is a (q+t–1)-equivalence. The map 1 is the identity map of F, β is the inclusion, h is the projection and $h' = h \circ \alpha$. Note that β is (a homotopy equivalence and hence) a weak equivalence.

– B –

$$\begin{array}{ccccc} (F \times *) \cup (G \times E^t) & \overset{\supset}{\longleftarrow} & (F \times *) \cup (G \times S^{t-1}) & \overset{h'}{\longrightarrow} & F \\ \beta \big\downarrow & & \alpha \big\downarrow & & 1 \big\downarrow \\ F \times E^t & \overset{\supset}{\longleftarrow} & F \times S^{t-1} & \overset{h}{\longrightarrow} & F \end{array}$$

The push-out of the cotriad is a map $F \times * \cup G \times S^t \longrightarrow F \times S^t$, and is a (q + t)-equivalence by **6.8(c)**. This proves the statement for $n = t$ and thus completes the proof. ▮

7.3 Corollary

For C_n as in **7.2** above, the map $\mu : (C_n, F \times *) \longrightarrow (S^n,*)$ which is the restriction of the projection $\tau : F \times S^n \longrightarrow S^n$ onto the first factor, is a (n + m)-equivalence.

Proof Condition **3.2** ($_2$) on path components is easily checked to be true. The triangle in diagram **C** is commutative. The map γ of pairs is a weak equivalence.

– C –

$$\begin{array}{ccc} (C_n, F \times *) & \overset{\alpha}{\longrightarrow} & (F \times S^n, F \times *) \\ & \mu \searrow \quad \swarrow \tau & \\ & (S^n, *) & \end{array}$$

By **7.2** and the five-lemma applied to the homotopy ladder of the map of pairs α, α is a (m+n)-equivalence (of pairs) and so our result follows. ▮

7.4 Remark. Suppose that G is a subspace of F such that the inclusion $G \subset F$ is a m-equivalence. Let W_n be a bouquet of n-spheres. A subspace D_n of $F \times W_n$ is defined as $D_n = F \times * \cup G \times W_n$. Then the following facts can be proved along the same route that we followed to prove **7.2** and **7.3** .

(a) The inclusion $D_n \subset F \times W_n$ is a (m + n)-equivalence.

(b) The restriction of the projection $F \times W_n \longrightarrow W_n$ to D_n is a (m + n)-equivalence $(D_n,F) \longrightarrow (W_n,*)$. ■

The notion of relative CW-complex that we use in **7.5** and beyond, is discussed in the book [**Wd**] of G. Whitehead. When working with a relative CW-complex, for brevity we shall use the term *cell* ambiguously to mean *relative cell.*

7.5 Proposition

Suppose that $p_0 : A \longrightarrow B$ is a m-equivalence and for some relative CW-complex (X,A), we have a relative homeomorphism $p : (X,A) \longrightarrow (Z,B)$.

Then $p : X \longrightarrow Z$ is a m-equivalence.

Proof By inductive limit considerations it suffices to prove that for every non-negative integer r, the statement $\mathcal{P}_r$ below is true.

$\mathcal{P}_r$: The map of r-skeleta, $p_r : X_r \longrightarrow Z_r$ is a m-equivalence.

We proceed by mathematical induction. The statement $\mathcal{P}_0$ is obviously true. Now suppose that $\mathcal{S}_r$ is true for all $r \leq t-1$ for some integer $t \geq n$.

In diagram D, the arrows pointing to the left are inclusions. The map g is the attaching maps of the t-cells of (X,A) and $f = p_{t-1} \circ g$. The left and centre vertical arrows are

$$
\text{– D –}\qquad
\begin{array}{ccccc}
\vee E^t & \xleftarrow{\;\supset\;} & \vee S^{t-1} & \xrightarrow{\;g\;} & X_{t-1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{t-1}} \\
\vee E^t & \xleftarrow{\;\supset\;} & \vee S^{t-1} & \xrightarrow{\;f\;} & Z_{t-1}
\end{array}
$$

identity maps. By the induction assumption $\mathcal{P}_{t-1}$, the vertical arrow p_{t-1} is a m-equivalence. Thus the pushout of this cotriad is precisely our map $p_r : X_t \longrightarrow Z_t$, and by 6.8(c), this map is a m-equivalence, and $\mathcal{P}_t$ follows. ∎

7.6 Lemma

Let $p_0 : A \longrightarrow B$ be a fibration with $(m-1)$-connected fibres, and let (X,A) be a relative CW-complex having only cells of dimension n.

Then the relative homeomorphism $p : (X,A) \longrightarrow (Z,B)$ is a $(m+n-1)$-equivalence.

Proof In diagram E, the arrows pointing to the left are inclusions. C is a bouquet of n-dimensional balls and S is the corresponding bouquet of boundaries of the balls in C. The left and centre vertical arrows are relative homeomorphisms. The space F is the fibre of p_0 over the base point $*$. The map g is such that $g|S$ attaches the cells of (X,A) while $g|F$ is the obvious embedding. With $f = p_0 \circ g|S$, diagram E is commutative.

$$
\text{– E –}\qquad
\begin{array}{ccccc}
C \vee F & \xleftarrow{\;\supset\;} & S \vee F & \xrightarrow{\;g\;} & A \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_0} \\
C & \xleftarrow{\;\supset\;} & S & \xrightarrow{\;f\;} & B
\end{array}
$$

The push–out of this cotriad is precisely our map $p : X \longrightarrow Z$. We show that:

$(_1)$...each of the $\underline{\text{Top}}^2$-morphisms constituted by the horizontal arrows in diagram E, is a $(m+n-2)$-equivalence of homotopy fibres.

To prove $({}_1)$, we make use of **6.4**, and we must choose a subset T as in **6.0**. For $n = 1$ we let $T = S$, and for $n > 1$, we let $T = \{*\}$. Then our assertion $({}_1)$ follows by **6.4**. Thus by **6.3**, the Top^2-morphism $p_0 \longrightarrow p$ is a (m+n–2)-equivalence of homotopy fibres. The lemma now follows by **16.5**. ∎

7.7 Lemma

Let $p_0 : A \longrightarrow B$ be a m-equivalence, and let (X,A) be a relative CW-complex having only cells of dimension n.

Then the relative homeomorphism $p : (X,A) \longrightarrow (Z,B)$ is a (m+n–1)-equivalence.

Proof The right hand side square of diagram **F** is a mapping path fibration factorization and the arrows pointing to the left are inclusions.

$$
-\mathbf{F}- \qquad
\begin{array}{ccccc}
X & \xleftarrow{\ \supset\ } & A & \xrightarrow{\ s\ } & A' \\
\downarrow & & \downarrow & & \downarrow r \\
Z & \longleftarrow & B & \xrightarrow{\ 1\ } & B
\end{array}
$$

The fibres of r are (m–1)-connected. Thus by **7.6** the push-out $p' : (X',A') \longrightarrow (Z,B)$ of the Top^2-cotriad of diagram **F** is a (m+n–1)-equivalence. The map s is a weak equivalence and by **7.5**, $q : X \longrightarrow X$ is a weak equivalence. Thus the relative homeomorphism $q : (X,A) \longrightarrow (X',A')$, is a weak equivalence Since $p = p' \circ q$, the result follows. ∎

7.8 Proposition

Let $p_0 : A \longrightarrow B$ be a m-equivalence, and let (X,A) be a relative CW-complex having only cells of dimension n and higher.

Then the relative homeomorphism $p : (X,A) \longrightarrow (Z,B)$ is a (m+n–1)-equivalence.

Proof Let $p_r : X_r \longrightarrow Z_r$ be the map of r-skeleta induced by p. By a direct limit argument it suffices to prove that for every $r \geq n$, the following statement is true.

$\mathcal{S}_r$: $$p_r : (X_r,A) \longrightarrow (Z_r,B) \text{ is a } (m+n-1)\text{-equivalence.}$$

We proceed by induction on r. The statement $\mathcal{S}_n$ is precisely Lemma **7.7**. Now we assume that $\mathcal{S}_r$ is true for all r such that $n \leq r < t$ for some t and we set out to prove $\mathcal{S}_t$. By **7.5**, $X_{t-1} \longrightarrow Z_{t-1}$ is a m-equivalence. Thus by **7.7**, $(X_t,X_{t-1}) \longrightarrow (Z_t,Z_{t-1})$ is a $(m+t-1)$-equivalence. Now $\mathcal{S}_t$ follows by application of **16.9(b)** to the map of triples,

$$(X_t,X_{t-1},A) \longrightarrow (Z_t,Z_{t-1},B).$$

This completes the induction process and hence the proof of the proposition. ∎

7.9 Proposition

Suppose $X \supset A \longrightarrow B$ is a cotriad satisfying condition $7.1(_2)$. Let D be the double mapping cylinder and P the push-out of the cotriad.

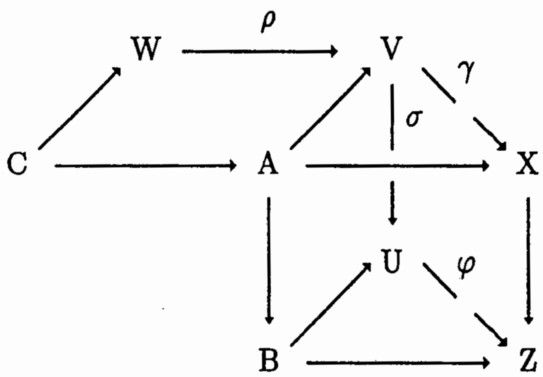
Then the natural map $\beta : D \longrightarrow P$ is a weak equivalence.

Proof The canonical map from the disjoint union $X + A + B$ onto P, maps the subspaces X, A and B onto open subsets of P. Let us denote these subspaces of P by U_1, U_2, and U_3 respectively, and let $\mathcal{U} = \{U_1, U_2, U_3\}$. Then for each $i = 1, 2, 3$, the map $\beta^{-1}(U_i) \longrightarrow U_i$ is a weak equivalence. Thus our result follows by Theorem **5.9**. ∎

7.10 ***Proof of Theorem*** 7.1

Let $p : (X,A) \longrightarrow (Z,B)$ be as in the formulation of 7.1. There is CW-complex W having a subcomplex C such that the pair (W,C) has no cells of dimension less than n, and a map $g : (W,C) \longrightarrow (X,A)$ such that each of the maps $W \longrightarrow X$ and $C \longrightarrow A$ is a weak equivalence (a CW-resolution in the terminology of Gray $[G_1]$). Then we form relative homeomorphisms $\rho : (W,C) \longrightarrow (V,A)$ and $\sigma : (V,A) \longrightarrow (U,B)$. By the definition of push-out, there are maps $\gamma : V \longrightarrow X$ and $\varphi : U \longrightarrow Z$ making diagram G commutative.

– G –



The map φ can be seen to be the push-out of the $\underline{\text{Top}}^2$-cotriad in diagram H. By **7.5**, $\rho : W \longrightarrow V$ is a weak equivalence. Since $\gamma \circ \rho$ is a weak equivalence, it follows that also γ is a weak equivalence.

– H –

$$\begin{array}{ccccc} V & \hookleftarrow & A & \longrightarrow & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow & & \downarrow \\ X & \hookleftarrow & A & \longrightarrow & B \end{array}$$

By **6.8(b)**, the map of double mapping cylinders of the $\underline{\text{Top}}^2$-cotriad is a weak equivalence $\varphi' : U' \longrightarrow Z'$. In diagram H each row is a $\underline{\text{Top}}$-cotriad which satisfy a condition of the type $(_1)$ or $(_2)$ in the formulation of Theorem **7.1** . Under such a condition the natural map from double mapping cylinder to push-out is a weak equivalence (**1.14** and **7.9** respectively). Thus $\varphi : U \longrightarrow Z$ is a weak equivalence. Recall that also $\gamma : V \longrightarrow X$ is a weak equivalence. So we obtain a commutative square, diagram I, in which the horizontal arrows are weak equivalences.

$$
\begin{array}{ccc}
(V,A) & \xrightarrow{\ \gamma\ } & (X,A) \\
\sigma \downarrow & & \downarrow p \\
(Y,B) & \xrightarrow[\ \varphi\ ]{} & (Z,B)
\end{array}
\qquad -\mathbf{I}-
$$

Note that via the relative homeomorphism ρ, (V,A) inherits a relative CW-structure with no cells of dimension less than n. Consequently by **7.8**, σ is a (m+n–1)-equivalence. Thus p is a (m+n–1)-equivalence. ∎

Chapter III : MAPS OF FINITE DIMENSIONAL CW-COMPLEXES

In Section **8** we generalize the concept of n–equivalence in the setting of isomorphisms modulo a Serre class of abelian groups. We also formulate a number of theorems, the proofs of which are given in Chapter V.

In Section **9** some results which are important for maps of finite dimensional spaces, are proved. These together with the clutching construction of Section **10** are applied in Section **11**. The basic result **11.1** gives sufficient conditions for a map of pairs to be a generalized n-equivalence. This result is used to study a specific class of maps in Section **12** and Section **13**.

8. GENERALIZED n-EQUIVALENCES

The idea of classes of abelian groups was invented and harnessed by Serre [**Se**]. A detailed discussion of Serre classes, and localization of abelian groups and 1-connected topological spaces appears in Chapter V. In the current section we shall only quote some key results required for the applications in the sections **9** to **13**.

Let us denote by P a fixed subset of primes, and let P′ be the complementary set of primes. Torsion abelian groups can be studied effectively by separate consideration of the different primary components. The P-component is obtained as the tensor product as $\mathbb{Z}$-modules, of the torsion group with a suitable subring of the rationals $\mathbb{Q}$. This subring is in fact the one generated by the subset $\{\frac{1}{p} : p \in P'\}$ of $\mathbb{Q}$ and we denote it by $\mathbb{Z}$. Localization is discussed in Section **18**.

The concept of a Serre class of abelian groups (see Section **19** for a detailed discussion) is indispensible when studying phenomena such as torsion in abelian groups. This notion gives rise to a generalization of the concepts of monomorphism and epimorphism, and consequently we are in a position to generalize the concept of n-equivalence as defined in Section **3**. In fact, once we have the correct definition, it is quite easy to verify a similarly generalized version of [$\mathbf{M_2}$; Theorem 1.2]. Among other results, we formulate such a theorem after the necessary definitions have been made. The proofs appear in Chapter V.

The class of all torsion abelian groups with vanishing p-component for every $p \in P$, which we shall denote by $\mathcal{C}$ throughout this section, is a Serre class of abelian groups. Spaces are assumed to be pointed.

8.1 Definition

Let $f: (X,A) \longrightarrow (Y,B)$ be a map of pairs of spaces and n an integer, $n \geq 1$. Then f is said to be a (P,n)-***equivalence*** if :

(1) X, Y, A, B are 1-connected,

(2) The homomorphism $f_*: \pi_k(X,A) \longrightarrow \pi_k(Y,B)$ is a $\mathcal{C}$-isomorphism for $2 \leq k \leq n-1$, and $\mathcal{C}$-epimorphism for $k = n$.

The map f is said to be a (P,n)-* ***equivalence*** if it is a (P,n)-equivalence, and the homomorphism $f_*: \pi_n(X,A) \longrightarrow \pi_n(Y,B)$ is a $\mathcal{C}$-isomorphism.

A map $g: X \longrightarrow Y$ of spaces is said to be a (P,n)-***equivalence*** [respectively, a (P,n)-* ***equivalence***] if for every $x \in X$ the map $(X,x) \longrightarrow (Y,y)$ is a (P,n)-equivalence [respectively, a (P,n)-*equivalence].

A map which is a (P,n)-equivalence for all integers n is called a P-***equivalence***.

8.2 Remark

(a) For a pair (X,A) of 1-connected spaces, it follows from the exact homotopy sequence of the pair that $\pi_1(X,A)$ is a one-point set and $\pi_2(X,A)$ is abelian.

(b) Let $f: X \longrightarrow Y$ be a fibration between 1-connected spaces and let B be a 1-connected subspace of Y, with $A = f^{-1}(B)$. Then $\pi_1(A)$ is abelian, although A may fail to be simply connected.

(c) The class $\mathcal{C}$ is perfect and complete in the terminology of [**Hu**], or an acyclic ideal of abelian groups in the sense of [**Sr**]. Thus $\mathcal{C}$ admits a generalized Hurewicz isomorphism theorems and a generalized Whitehead theorem, see **20.5**, **20.6**, **20.7**.

(d) From change-of-base-points considerations [Sr; Lemma 2 on p380], it follows that an unpointed map $g : X \longrightarrow Y$ is a (P,n)-equivalence if and only if for *some* $x \in X$, the map $(X,x) \longrightarrow (Y,y)$ with $y = g(x)$, is a (P,n)-equivalence of maps of pairs of spaces. ■

The results **8.3, 8.4** and **8.5** formulated below, are proved in Chapter V.

8.3 Lemma

Let G and H be abelian groups. Suppose that H is $\mathcal{C}$-isomorphic to G and that $G \otimes \mathcal{Z}$ is a finitely generated $\mathcal{Z}$-module. Let $\alpha : G \longrightarrow H$ be $\mathcal{C}$-epimorphism. Then α is a $\mathcal{C}$-isomorphism. (See **19.7** for the proof.) ■

We shall require the following generalization of the mapping path fibration factorization.

8.4 Lemma

Let $p : E \longrightarrow B$ be any map between 1-connected spaces such that $p_* : \pi_2(E) \longrightarrow \pi_2(B)$ is $\mathcal{C}$-surjective. Then there is a fibration $p_1 : E_1 \longrightarrow B$ and a P-equivalence $p_0 : E \longrightarrow E_0$ such that p_1 has P-local fibres and $p_1 \circ p_0 = p$. (See **20.4** for the proof.) ■

Theorem **8.5** which follows is precisely the generalization modulo the set of primes, of the theorem [**M**$_2$; Theorem 1.2] of May.

8.5 Theorem

Let B be a space with open subsets V_1 and V_2 such that $B = V_1 \cup V_2$. Let V_0 be the set $V_1 \cap V_2$. Let $p : E \longrightarrow B$ be a map, and for each $i = \{0, 1, 2\}$, let $U_i = p_1^{-1}(V_i)$. Suppose that for each $j \in \{1, 2\}$, $(U_j, U_0) \longrightarrow (V_j, V_0)$ is a (P,n+1)-equivalence.

Then for each $i \in \{0, 1, 2\}$ the map $(E, U_i) \longrightarrow (B, V_i)$ is a (P,n+1)-equivalence. (See **21.4** for proof.) ■

From **8.5** the proof of the following theorem, Theorem **8.6**, follows in exactly the same way that we obtained **6.1** from [**M_2**; Theorem 1.2] and we skip the proof.

8.6 Theorem

Suppose that in the commutative diagram B of maps of pairs of topological spaces, the following conditions hold.

(1) p_0 is a (P,n)-equivalence.

(2) For each $j \in \{1, 2\}$, p_j is a (P,n+1)-equivalence.

(3) Each of the maps $A_0 \longrightarrow A_j$ is a (P,n)-equivalence.

(4) g_1 and f_1 are cofibrations.

$$-\mathrm{B}- \qquad \begin{array}{ccccc} (E_1, A_1) & \xleftarrow{\;g_1\;} & (E_0, A_0) & \xrightarrow{\;g_2\;} & (E_2, A_2) \\ \downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\ (B_1, *) & \xleftarrow{\;f_1\;} & (B_0, *) & \xrightarrow{\;f_2\;} & (B_2, *) \end{array}$$

Then the map of push-outs $(E, A_2) \longrightarrow (B, *)$ is a (P, n + 1)-equivalence. ∎

9. RESULTS RELATED TO THE HUREWICZ ISOMORPHISM THEOREM.

Some results required for further applications of n-equivalences, especially to maps of finite dimensional CW-complexes, are presented here. Such results are deduced from properties of commutative diagrams in the category Ab of abelian groups and homomorphisms.

Spaces and maps are assumed to be pointed. We fix a set of primes P and use the symbol $\mathcal{C}$ to denote the Serre class of all torsion abelian groups for which the p-component vanishes for all $p \in P$. $\mathbb{Z}$ is the subring of $\mathbb{Q}$ generated by the subset $\{\frac{1}{n} : n$ is a prime not belonging to $P\}$.

9.1 Proposition

Suppose that in the commutative diagram **A**, which is a diagram in Ab, the rows are exact and κ is a $\mathcal{C}$-isomorphism.

If β is a $\mathcal{C}$-monomorphism, then β_1 is a $\mathcal{C}$-monomorphism.

– A –
$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow{\kappa} & & \downarrow{\lambda} & & \downarrow{\mu} \\ A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 \end{array}$$

Proof Let us first prove the statement under for the special case that P is the full set of all primes. Then $\mathcal{C}$ is the zero class consisting of only the trivial group. In this case, κ is an isomorphism and $\alpha_1 = \lambda\ \alpha\ \kappa^{-1}$. Since β is a monomorphism, $\alpha = 0$ by exactness. But then also $\alpha_1 = 0$, and consequently β_1 is a monomorphism.

The general case follows from the special case by application of the exact functor $- \otimes \mathbb{Z}$, in view of 19.3. ∎

9.2 Proposition

Suppose that in diagram B, which is a diagram in <u>Ab</u>, the rows are exact. Furthermore, suppose that θ is a $\mathcal{C}$-isomorphism, η is a $\mathcal{C}$-monomorphism and χ is a $\mathcal{C}$-epimorphism. Then ξ is a $\mathcal{C}$-epimorphism.

– B –

$$\begin{array}{ccccccc}
Z & \xrightarrow{\zeta} & M & & & & \\
\chi\downarrow & & \downarrow\xi & & & & \\
B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\
 & & \uparrow\xi & & \uparrow\theta & & \uparrow\eta \\
 & & M & \xrightarrow{\mu} & N & \xrightarrow{\nu} & P
\end{array}$$

Proof Let us first prove the statement under the assumption that (P is the full set of primes and consquently) $\mathcal{C}$ is the zero class. The proof is a diagram chase.

Let c be an arbitrary element of C. Then,

$$0 = \delta\,\gamma\,(c) = \delta\,\theta\,\theta^{-1}\,\gamma\,(c) = \eta\,\nu\,\theta^{-1}\,\gamma\,(c)\,.$$

Since η is a monomorphism, $\nu\,\theta^{-1}\,\gamma\,(c) = 0$. By exactness at N, $\theta^{-1}\,\gamma\,(c) = \mu\,(m)$ for some $m \in M$. Now,

$$\gamma\,\xi\,(m) = \theta\,\mu\,(m) = \theta\,[\theta^{-1}\,\gamma\,(c)] = \gamma\,(c),$$

and consequently $c - \xi\,(m) \in \ker \gamma$. Due to exactness at C, $c - \xi\,(m) = \beta\,(b)$ for some $b \in B$. Since χ is an epimorphism, $b = \chi\,(z)$ for zome $z \in Z$. Now

$$\xi\,[\zeta\,(z) + m] = \xi\,\zeta\,(z) + \xi\,(m) = \beta\,\chi\,(z) + \xi\,(m) = \beta\,(b) + \xi\,(m) = c,$$

and this proves surjectivity of ξ (in the special case).

The general case follows from the special case by application of the exact functor $- \otimes \mathcal{Z}$, in view of 19.3. ▮

9.3 Proposition

Let m be an integer, $m > 1$. Suppose that we have 1-connected spaces, $Y \supset D \supset G$, such that the injection $(D,G) \longrightarrow (Y,G)$ is a (P,m)-*equivalence.

Then $H_k(D,G) \longrightarrow H_k(Y,G)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$.

Proof In the commutative diagram C we have exact homotopy and homology sequences, and the vertical arrows are Hurewicz homomorphisms.

$$-\mathrm{C}- \qquad \begin{array}{ccccccc} \pi_{m+1}(Y,D) & \longrightarrow & \pi_m(D,G) & \longrightarrow & \pi_m(Y,G) & \longrightarrow & \pi_m(Y,D) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{m+1}(Y,D) & \longrightarrow & H_m(D,G) & \longrightarrow & H_m(Y,G) & \longrightarrow & H_m(Y,D) \end{array}$$

From the exact homotopy sequence of (Y,D,G) it follows that $\pi_k(Y,D) \in \mathcal{C}$ for every $k \leq m$. By the generalized relative Hurewicz theorem 20.6, $H_k(Y,D) \in \mathcal{C}$ for each $k \leq m$, and $\pi_{m+1}(Y,D) \longrightarrow H_{m+1}(Y,D)$ is a $\mathcal{C}$-isomorphism. By 9.1 applied to the right hand end of the displayed ladder, $H_m(D,G) \longrightarrow H_m(Y,G)$ is a $\mathcal{C}$-monomorphism.

Since $H_k(Y,D) \in \mathcal{C}$ for every $k \leq m$, it follows that $H_k(D,G) \longrightarrow H_k(Y,G)$ is a $\mathcal{C}$-isomorphism for every $k \leq m-1$, and $H_m(D,G) \longrightarrow H_m(Y,G)$ is a $\mathcal{C}$-epimorphism. ∎

9.4 Lemma

Let m be an integer, $m > 1$. Suppose that $\epsilon : (D,G) \longrightarrow (Y,F)$ is a map of pairs of 1-connected spaces such that :

(1) $\epsilon : (D,G) \longrightarrow (Y,F)$ is a (P,m)-*equivalence, and

(2) $G \longrightarrow F$ is a (P,m)-equivalence.

Then $\epsilon_* : H_k(D,G) \longrightarrow H_k(Y,F)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$.

If furthermore, $H_m(G) \in \mathcal{C}$, then $H_k(D) \longrightarrow H_k(Y)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$.

Proof We can assume that $D \subset Y$ (replacing Y by the mapping cylinder of ϵ if necessary) and $G \subset F$, and that ϵ is inclusion. Then ϵ can be factorized as:

$(_x)$ $$(D,G) \to (Y,G) \to (Y,F).$$

From the homotopy ladder of the second map in $(_x)$ it follows by $(_2)$ and the generalized five-lemma that $(Y,G) \to (Y,F)$ is a (P,m)-*equivalence. This together with $(_1)$ implies that $(D,G) \to (Y,G)$ is a (P,m)-*equivalence. Thus by **9.3** we have

$(_y)$ $H_k(D,G) \to H_k(Y,G)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$.

We apply the generalized Whitehead theorem **20.7** to the map $G \to F$. Then from the homology ladder of the second map in $(_x)$, by the generalized five-lemma we obtain,

$(_z)$ $H_k(Y,G) \to H_k(Y,F)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$.

Now the first part of the lemma follows by combining $(_y)$ and $(_z)$.

Suppose further that $H_m(G) \in \mathcal{C}$. Then in the homology ladder of the map of pairs $\epsilon : (D,G) \to (Y,F)$ as shown in diagram D below, also $H_m(F) \in \mathcal{C}$.

$$\text{– D –}\qquad \begin{array}{ccccccccc} H_m(G) & \longrightarrow & H_m(D) & \longrightarrow & H_m(D,G) & \longrightarrow & H_{m-1}(G) & \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ H_m(F) & \longrightarrow & H_m(Y) & \longrightarrow & H_m(Y,F) & \longrightarrow & H_{m-1}(F) & \to \end{array}$$

Hence, and due to the first part of the lemma that we have proved, the second part of the lemma follows by the generalized five-lemma. ∎

The main result of this section now follows. This technical lemma is a key tool towards the proof of the basic theorem, 11.1, on maps of pairs of finite dimensional CW-complexes.

9.5 Proposition Let q and m be integers with $1 < m < q$. Suppose that we have maps ϵ and ρ as shown below, for which the induced maps $G \longrightarrow F$ coincide.

$$(D,G) \xrightarrow{\ \epsilon\ } (Y,F) \xleftarrow{\ \rho\ } (E,G)$$

Suppose further that :

(1) $\epsilon : (D,G) \longrightarrow (Y,F)$ is a (P,q)-equivalence,

(2) $\rho : E \longrightarrow Y$ is a (P,q)-equivalence,

(3) $\rho : (E,G) \longrightarrow (Y,F)$ is a (P,m)-*equivalence,

(4) For each $i > m$, $H_i(D), H_i(Y) \in \mathcal{C}$,

(5) For each $i \geq m$, $H_i(G) \in \mathcal{C}$.

Then $\rho : (E,G) \longrightarrow (Y,F)$ is a (P,q)-equivalence.

Proof In diagram E, the upper square is part of the homotopy ladder of the map of pairs ϵ, and the ladder in the bottom is part of the homotopy ladder of the map of pairs ρ.

– E –

$$\begin{array}{ccccccc}
\pi_{i+1}(D,G) & \longrightarrow & \pi_i(G) & & & & \\
\chi \downarrow & & \downarrow \xi_i & & & & \\
\pi_{i+1}(Y,F) & \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(Y) & \longrightarrow & \pi_i(Y,F) \\
 & & \uparrow \xi_i & & \uparrow \varphi_i & & \uparrow \psi_i \\
 & & \pi_i(G) & \longrightarrow & \pi_i(E) & \longrightarrow & \pi_i(E,G)
\end{array}$$

Taking $i = m$, by 9.2 it follows that ξ_m is a $\mathcal{C}$-epimorphism. Moreover, conditions (2) and (3) together implies that the map $G \longrightarrow F$ induced by ρ is a $(P,m-1)$-*equivalence. But then (each of the maps) $G \longrightarrow F$ is a (P, m)-equivalence. So by 9.4, $H_k(D) \longrightarrow H_k(Y)$ is a $\mathcal{C}$-isomorphism for every $k \leq m$. This and (4) ensures that by the generalized Whitehead theorem 20.7, $D \longrightarrow Y$ is a P-equivalence.

This together with $(_1)$ implies by the generalized five-lemma that $G \longrightarrow F$ is a $(P, q-1)$-equivalence. This fed into the homotopy ladder of ρ together with $(_2)$ yields our result. ∎

10. THE CLUTCHING CONSTRUCTION TO APPROXIMATE A FIBRATION

We assume spaces to be pointed and maps to be base point preserving. We shall fix a set of primes P and use the symbol $\mathcal{C}$ to denote the Serre class of all torsion abelian groups for which the p-component vanishes for all $p \in P$.

The main result, 10.1, shows how a fibration over a finite dimensional CW-complex can be approximated by the push-out of some cotriad such as that in the Dold-Lashof construction. The other result puts a limit on the homology of the total space of a fibration.

10.1 Proposition

Suppose that B is a 0-connected CW-complex of dimension $n \geq 1$ and let C be its (n–1)-skeleton. Let $f: W \longrightarrow C$ be the attaching map of all the n-cells of B simultaneously. W is a bouquet of (n–1)-dimensional spheres. Let V be the bouquet of cones of the spheres in W. Suppose that $p: E \longrightarrow B$ is a fibration with fibre $F = p^{-1}(*)$. The pull-back of p over the inclusion $C \subset B$ is denoted by $q_1 : U_1 \longrightarrow C$. Let $q_0 : F \times W \longrightarrow W$ and $q_2 : F \times V \longrightarrow V$ be the trivial fibrations. Then

(a) there is a map $g : F \times W \longrightarrow U_1$, resulting in a Top^2-morphism $q_0 \longrightarrow q_1$ which is a weak equivalence of fibres.

(b) the push-out $q : U \longrightarrow B$ of the Top^2-cotriad in diagram A, is a quasifibration which factorizes through p via a weak equivalence $\mu : U \longrightarrow E$.

– A –

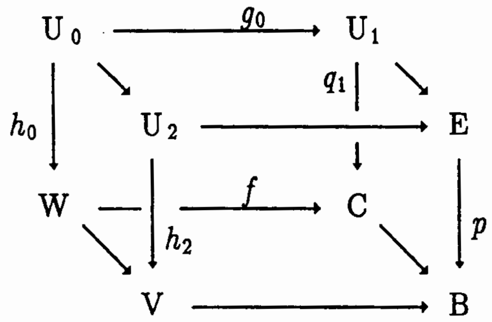
$$\begin{array}{ccccc} F \times V & \longleftarrow & F \times W & \xrightarrow{\;g\;} & U_1 \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_1} \\ V & \longleftarrow & W & \xrightarrow{\;f\;} & C \end{array}$$

Proof The space B is obtained as the push-out of the cotriad :

(1) $$V \supset W \longrightarrow C.$$

In diagram **B**, the square in the bottom is the push-out of (1). Each vertical face of the box is a pull-back square. The pull-backs via the different routes from W to B coincide.

– **B** –



The fibration h_2 is fibre homotopically trivial since V is contractible. So we have a weak equivalence $\epsilon_2 : F \times V \longrightarrow U_2$ such that $h_2 \circ \epsilon_2$ is coincides with the projection map $q_2 : F \times V \longrightarrow V$. Also there exists a map $\epsilon_0 : F \times W \longrightarrow U_0$ satisfying pull-back conditions. Taking $g = g_0 \circ \epsilon_0$, we settle **(a)**.

By 6.2 **(b)**, the push-out $q : U \longrightarrow B$ of the Top²-cotriad in diagram **A** is a quasifibration. From push-out properties, there is a map $\mu : U \longrightarrow E$ such that $p \circ \mu = q$. Each quasi-fibre of q is mapped into the corresponding fibre of p by a weak equivalence. The inclusion map $p^{-1}(*) \longrightarrow E$ induces a surjective function on path components since B is path-connected. By 19.1 applied to the maps of pairs $(U, q^{-1}(*)) \longrightarrow (E, p^{-1}(*))$, it follows that $\mu : U \longrightarrow E$ is a weak equivalence. So **(b)** also follows. ∎

Lemma 10.2 below, can be proved by using the Serre spectral sequence of a fibration. Nevertheless, Proposition 10.1 enables us to give a straightforward proof by induction.

10.2 Lemma

Let B be a 0-connected CW-complex of dimension d and E a space. Suppose that we have a fibration $p : E \longrightarrow B$ with fibre F such that $H_k(F) \in \mathcal{C}$ whenever $k > m$. Then $H_k(E) \in \mathcal{C}$ whenever $k > m + d$.

Proof We prove it by induction on d. It is obviously true for the case $d = 0$. Suppose that the statement is true for $d = 0; 1; 2; \ldots ; n-1$, and that B is a n-dimensional CW-complex.

We apply **10.1** and refer to the $\underline{\text{Top}}^2$-cotriad A. $F \times V$ is homotopy equivalent to F and by the induction assumption, $H_k(F \times W) \in \mathcal{C}$ for all $k \geq m+n-1$. From the exact homology sequence of the pair $(F \times V, F \times W)$, it follows that

$$H_k(F \times V, F \times W) \in \mathcal{C} \text{ for all } k > m+n-1.$$

By the excision theorem, the map $(F \times V, F \times W) \longrightarrow (U,U_1)$ delivers isomorphisms of relative homology groups in all dimensions. Thus $H_k(U,U_1) \in \mathcal{C}$ for all $k > m+n-1$. From the induction hypothesis applied to the exact homology sequence of the pair (U,U_1), it follows that $H_k(U) \in \mathcal{C}$ for all $k > m+n$. By the generalized Whitehead theorem, **20.7**, the homomorphisms $\mu_* : H_k(U) \longrightarrow H_k(E)$ induced by μ, are $\mathcal{C}$-isomorphisms for all $k \geq 1$. Thus, $H_k(E) \in \mathcal{C}$ for all $k > m+n$. ▮

11. FIBRATIONS OF FINITE CW-COMPLEXES.

We work with pointed spaces and pointed maps. The base point, generally denoted by $*$, is suppressed most of the time. The category is denoted by <u>Top</u>. The main result, Theorem 11.1 gives sufficient conditions for a map $p:(E,F) \longrightarrow (B,*)$, with all the spaces finite dimensional, to be a n-equivalence. This result seems to be a new contribution to the theory of fibrations.

The theorem is given in the generalized setting of isomorphisms modulo a Serre class of abelian groups. By P we denote a fixed set of primes, while $\mathcal{C}$ denotes the Serre class of all torsion abelian groups with vanishing p-component for every prime $p \in P$. The subring of $\mathbb{Q}$, generated by the inverses of the primes in the complement P' of P, is denoted by $\mathbb{Z}$.

11.1 Theorem

Let n, k and q be positive integers. Let G be a CW-complex of dimension less than n, such that the inclusion $G_k \subset G$, of the k-skeleton of G is a (P,q)-equivalence. Let B be a n-dimensional CW-complex. Let E be a space with $G \subset E$, and $p:(E,G) \longrightarrow (B,*)$ a map. Suppose further that the spaces B, E and G are 1-connected, and:

(1) $p:(E,G) \longrightarrow (B,*)$ is a $(P, n+k)$-*equivalence, and

(2) $H_i(E) \in \mathcal{C}$ for all $i > n+k$.

Under these conditions we have the following:

(a) If $G_k = G$, then $p:(E,G) \longrightarrow (B,*)$ is a P-equivalence.

(b) If $B = S^n$, then $p:(E,G) \longrightarrow (S^n,*)$ is a $(P, n+q)$-equivalence.

The proof of this theorem appears at the end of the section.

11.2 Proposition. Let n, k and q be integers, $k < n$. Suppose that G_k and B are CW-complexes of dimensions k and n respectively. We assume that B has no cells of dimension 1. Let G be a space containing G_k such that the inclusion $G_k \subset G$ is a (P,q)-equivalence. Let $r: Y \longrightarrow B$ be a fibration such that for the fibre $F = r^{-1}(*)$, there is a (P, n+k−1)- *equivalence, $i: G \longrightarrow F$.

Then there is a space D containing G and a map $\epsilon: (D,G) \longrightarrow (Y,F)$ for which the induced map $G \longrightarrow F$ coincides with i such that:

(a) If $G_k = G$, then $\epsilon: (D,G) \longrightarrow (Y,F)$ is a P-equivalence.

(b) If $B = S^n$, then $\epsilon: (D,G) \longrightarrow (Y,F)$ is a (P, n+q)-equivalence.

Proof Let us denote the **(a)**-part of the proposition by $\mathcal{S}_n$. We prove the statements $\mathcal{S}_n$ inductively. The inductive step is performed in such a way that the proof of the **(b)**-part is obtained as a by-product. For $n = 1$, we note that B is a one-point space. We choose ϵ to be the map $D = G \longrightarrow F = Y$. Thus $\mathcal{S}_1$ is true.

Now let us suppose that $\mathcal{S}_n$ is true for all $1 \leq n < t$. We now set out to prove the truth of $\mathcal{S}_t$. Let us assume that B is t-dimensional, and let B_{t-1} be its (t−1)-skeleton. Let $f: W \longrightarrow B_{t-1}$ be the attaching map of the t-cells of B, where W is a bouquet of (t−1)-dimensional spheres. [In the **(b)**–case, $B_{t-1} = *$ and W is one sphere.] We apply **10.1** to r and obtain the Top^2-cotriad with objects q_i as in the lower part of diagram A below.

– A –

$$\begin{array}{ccccc}
G \times V & \stackrel{\supset}{\longleftarrow} & G \times W & & \\
\downarrow \delta_2 & & \downarrow \delta_0 & & \\
F \times V & \stackrel{\supset}{\longleftarrow} & F \times W & \stackrel{g}{\longrightarrow} & U_1 \\
\downarrow q_2 & & \downarrow q_0 & & \downarrow q_1 \\
V & \stackrel{\supset}{\longleftarrow} & W & \stackrel{f}{\longrightarrow} & B_{t-1}
\end{array}$$

The cotriad can be augmented as shown, δ_0 and δ_2 being of the form $i \times 1$, such that **A** is commutative. By S_{t-1} applied to the fibration q_1, there exists a (P,t−1+q)-equivalence $\epsilon_1 : (D_1,G) \longrightarrow (U_1,F)$. [In the **(a)**-case, q is considered to be ∞, and in the **(b)**-case, we take $\epsilon_1 = i$.] Because the induced map $G \longrightarrow F$ coincides with i which is a (P, t−1+k)-equivalence, it follows by the five-lemma applied to the homotopy ladder of the map of pairs ϵ_1, that $D_1 \longrightarrow U_1$ is a (P, t−1+k)-equivalence. The map ϵ_1 can be factorized, by **8.4**, as a P-equivalence $\gamma : D_1 \longrightarrow T$ followed by a fibration $\beta : T \longrightarrow U_1$ with P-local fibres. Then β is a (t−1+k)-equivalence. Since $G_k \times W$ is a CW-complex of dimension t−1+k, $g \circ (\delta_0 | G_k \times W)$ can be lifted over α_1. Thus we obtain the commutative diagram B.

$$\text{– B –} \qquad \begin{array}{ccc} G \times W \supset G_k \times W & \xrightarrow{\;g_1\;} & T \\ \downarrow \delta_0 & & \downarrow \beta \\ F \times W & \xrightarrow{\;g\;} & U_1 \end{array}$$

Now let K be the subspace $G \times * \cup G_k \times W$ of $G \times W$. Then g_1 can be extended over K to give a commutative diagram C.

$$\text{– C –} \qquad \begin{array}{ccc} G \times W \supset K & \xrightarrow{\;g_2\;} & T \\ \downarrow \delta_0 & & \downarrow \beta \\ F \times W & \xrightarrow{\;g\;} & U_1 \end{array}$$

Let $L = G \times * \cup G_k \times V$. Let T' be the mapping cone of g_2 and $r : T' \longrightarrow T$ the retraction. Taking $\alpha_1 = \beta \circ r$, we obtain a <u>Top</u>^{2}–cotriad as in diagram D.

$$\text{– D –} \qquad \begin{array}{ccccc} L & \xleftarrow{\;\supset\;} & K & \xrightarrow{\;\subset\;} & T' \\ \downarrow \alpha_2 & & \downarrow \alpha_0 & & \downarrow \alpha_1 \\ F \times V & \xleftarrow{\;\supset\;} & F \times W & \xrightarrow{\;g\;} & U_1 \end{array}$$

For each $i \in \{0, 1, 2\}$, let $q_i \circ \alpha_i = s_i$. Then we obtain a Top^2-cotriad,

(1) $$s_2 \longleftarrow s_0 \longrightarrow s_1 .$$

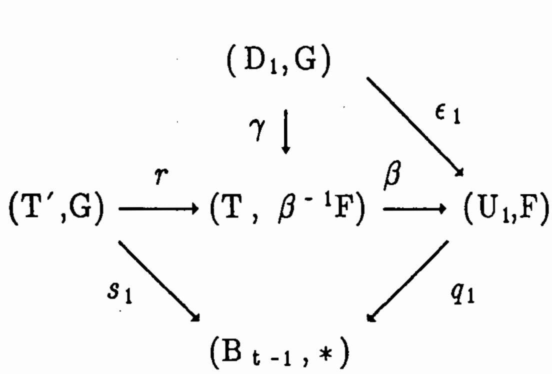
In the cotriad, $s_2 : (L,G) \longrightarrow (V,*)$ is a P-equivalence and by **7.4(b)**, $s_0 : (K,G) \longrightarrow (W,*)$ is a (P, t+q−1)-equivalence. For the subspace G of T′, a simple computation (quite clear in the **(b)**–case, the **(a)**–case is treated at the end of this proof) reveals that

(2) $$s_1 : (T',G) \longrightarrow (B_{t-1},*) \text{ is a P-equivalence.}$$

Thus by **8.6** the push-out $s : (D,G) \longrightarrow (B,*)$ of the Top^2-cotriad (1), is a (P, t+q)-equivalence. The map s factorizes through the push-out q of the q_i-cotriad of diagram **A**, and by **10.1**, q factorizes through r. This provides us with the required map ϵ.

Since $s : (D,G) \longrightarrow (B,*)$ is a (P, n+q)-equivalence and $r : (Y,F) \longrightarrow (B,*)$ is a (P, n+q)-equivalence, $\epsilon : (D,G) \longrightarrow (Y,F)$ turns out to be a (P, n+q)-equivalence. Certainly the map $G \longrightarrow F$ induced by ϵ coincides with i. This completes the induction [except for (2)] and thus the proof of the lemma.

Proof of (2). Note that this only need to be proved in the **(a)**–case, thus we can assume q to be ∞. In the commutative diagram **E**, r is the retraction of the mapping cylinder. We refer to maps as maps of pairs unless we explicitly indicate otherwise.



— **E** —

By assumption, ϵ_1 is a P-equivalence. By the fibration property **3.4(a)**, β is a P-equivalence. Thus γ is a P-equivalence. Since the map $D_1 \longrightarrow T$ is a P-equivalence, it follows that the map $G \longrightarrow \beta^{-1}(F)$ induced by γ is a P-equivalence. The latter map coincides with the map $G \longrightarrow \beta^{-1}(F)$ induced by r. Since the retraction $T' \longrightarrow T$ is a homotopy equivalence, it thus follows that r is a P-equivalence. The 'fibration' q_1 is a P-equivalence. Thus s_1 is a P-equivalence. ∎

11.3 Remark. For the case (a) in the proof above, it follows by **10.2** that

$$H_i(D) \in \mathcal{C} \text{ for every } i > n + k . \qquad (3)$$

By an argument similar to the proof of **10.2**, it follows that (3) also holds for the (b)–case, provided that $H_i(G) \in \mathcal{C}$ for every $i \geq n + k$.

11.4 *Proof of Theorem* 11.1

Note that in case (a) with $G_k = G$, we consider the number q to be infinity. Then we can prove **11.1(a)** and **11.1(b)** simultaneously. Let $E \xrightarrow{\rho} Y \xrightarrow{r} B$ be the mapping path fibration factorization of p with $F = r^{-1}(*)$. The map ρ is a weak equivalence. From **21.3(a)** it follows that the induced map $i : G \longrightarrow F$ is a (P,n+k–1)-*equivalence. By **11.2**, there exists a (P,n+q)-equivalence $\epsilon : (D,G) \longrightarrow (Y,F)$. By application of **9.5** to the triad

$$(D,G) \xrightarrow{\ \epsilon\ } (Y,F) \xleftarrow{\ \rho\ } (E,G) ,$$

the map $\rho : (E,G) \longrightarrow (Y,F)$ a (P,n + q)-equivalence. Since $r : (Y,F) \longrightarrow (B,*)$ is a weak equivalence, it follows that $p : (E,G) \longrightarrow (B,*)$ is a (P, n+q)-equivalence. ∎

12. RELATIVE WHITEHEAD PRODUCTS AND ADJUNCTION

We now pursue our study of relative homeomorphisms in the special case of a collapse map $c : (F \cup e^n, F) \longrightarrow (S^n, *)$. The map is extended to some space E in such a way that we get isomorphisms $\pi_r(E,F) \longrightarrow \pi_r(S^n,*)$ beyond the range of the relative homeomorphism theorem, 7.1. The author could not find results of this nature in the literature. The only ones entertained in the literature are those which lead to quasifibrations. Examples of such quasifibrations can be found in the work of Mimura [**Mi**], Hilton and Roitberg [**HR**], and Hardie and Porter [**HP**].

We also show that we obtain these isomorphisms irrespective of the manner in which we extend the map c. The adjunction technique that we use here is in-between that of Dold and Lashof [**DL**] or Hardie [**Ha**] on one hand and that of Section 7 on the other hand. The main result, Theorem 12.1, is particularly useful when studying reduced products of spheres as we show in Section 13. In [HP] a map $(S^n_3, S^n) \longrightarrow (S^{2n}, *)$ is studied by quasifibration methods. Such methods are sharpened in this section.

12.0 The basic construction. We work with pointed spaces and pointed maps. Let P be a fixed set of primes, and let $\mathcal{C}$ denote the class of all torsion abelian groups with torsion coprime to the elements of P. Let us fix integers n,k and q with $n > 1$ and $q \geq k > 1$. Also we assume that if n is even, then $k \neq n-1$. This will ensure that :

(1) $\qquad \pi_{n+k}(S^n)$ is a finite group.

We fix a space F, which admits a (P,q)-equivalence $i : S^k \longrightarrow F$, S^k being the k-sphere.

Let us assume i is an inclusion. We can replace F by the mapping cylinder of i if necessary. Let $g : S^{n-1} \longrightarrow F$ be a map and $\alpha \in \pi_{n-1}(F)$ its homotopy class. G is the mapping cone of g and $\beta \in \pi_n(G,F)$ is the homotopy class of the characteristic map of the pair (G,F). For a map $\zeta : S^k \times S^{n-1} \longrightarrow F$ we denote by $E(\zeta)$ the push-out space of the cotriad:

$$S^k \times E^n \xleftarrow{\ \supset\ } S^k \times S^{n-1} \xrightarrow{\ \zeta\ } F .$$

We can now formulate the main result of this section.

12.1 Theorem

Suppose we have a map $\zeta : S^k \times S^{n-1} \longrightarrow F$ of type (i,g).

(a) There exists a (P,q+n)-equivalence $p : (E,F) \longrightarrow (S^n,*)$.

(b) If moreover, $H_r(F) = \mathcal{C}$ for all $r \geq k+n$, then every extension $q : (E,F) \longrightarrow (S^n,*)$ of the relative homeomorphism $(G,F) \longrightarrow (S^n,*)$ is a (P,q+n)-equivalence.

The remainder of this section is devoted to proving this result. We recall some constructions made in Section 7. Let us reserve the symbol E_1 to denote the subspace of $F \times E^n$ which can be obtained as the push-out of the following cotriad.

$$S^k \times E^n \xleftarrow{\ \supset\ } S^k \times * \xrightarrow{\ \subset\ } F \times *$$

The restriction of the projection $F \times E^n \longrightarrow E^n$ to E_1 is denoted by p_1, and the pull-back of p_1 over the inclusion map $S^{n-1} \subset E^n$ is denoted by $p_0 : E_0 \longrightarrow S^{n-1}$.

12.2 Remark

(a) By a generalized version of **7.4**, $p_0 : (E_0,F) \longrightarrow (S^{n-1},*)$ is a (P, n+q−1)-equivalence.

(b) The map $p_1 : (E_1,F) \longrightarrow (E^n,*)$ is a weak equivalence.

12.3 *Proof of Theorem* **12.1(a).** We prepare a cotriad in order to apply **8.6**. There is an obvious extension of ζ to a map $\zeta' : E_0 \longrightarrow F$. Then we have a cotriad in $\underline{\text{Top}}^2$ as in **B**.

– B –

$$\begin{array}{ccccc} E_1 & \longleftarrow & E_0 & \xrightarrow{\ \zeta'\ } & F \\ \downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\ E^n & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

$E(\zeta)$ is precisely the space obtained as the push-out of the cotriad $E_1 \xleftarrow{\supset} E_0 \xrightarrow{\zeta} F \times *$ in the upper row of diagram B. By **12.2**, $(E_0,F) \longrightarrow (S^{n-1},*)$ is a (P, q+n–1)-equivalence and $(E_1,F) \longrightarrow (E^n,*)$ is a weak equivalence. The horizontal arrows induce P-equivalences between the fibres over the base points. By **8.6**, the push-out of the cotriad of diagram **B** is a (P,q+n)-equivalence $(E,F) \longrightarrow (S^n,*)$. ∎

12.4 Proposition

Suppose $p : (E,F) \longrightarrow (S^n,*)$ is as above. Suppose we have a map $q : (E,F) \longrightarrow (S^n,*)$ which coincides with p on G. Then $q : (E,F) \longrightarrow (S^n,*)$ is a (P,n+k)-*equivalence.

Proof By Theorem 7.1, $(G,F) \longrightarrow (S^n,*)$ is a (k+n–1)-equivalence. Thus the homomorphism $\pi_{n+k-1}(G,F) \longrightarrow \pi_{n+k-1}(S^n,*)$ is surjective. By commutativity of the triangle in diagram **C**, it follows that $\pi_{n+k-1}(E,F) \longrightarrow \pi_{n+k-1}(S^n,*)$ is an epimomorphism.

– C –

$$\begin{array}{ccc} \pi_r(G,F) & \longrightarrow & \pi_r(S^n,*) \\ \downarrow & \nearrow_{q_*} & \\ \pi_r(E,F) & & \end{array}$$

So q_* of diagram **C** is an epimorphism between groups which are $\mathcal{C}$-isomorphic due to the existence of the map p. Moreover these groups are finitely generated by condition 12.0(1).

Thus by **8.3**, q_* is an isomorphism in dimension $n+k-1$. For $r = n+k$ we consider part of the exact homotopy sequence of the triple (E,G,F) as in diagram **D** .

$$-\mathbf{D}- \qquad \pi_{n+k}(G,F) \longrightarrow \pi_{n+k}(E,F) \longrightarrow \pi_{n+k}(E,G)$$

Here we have $\pi_{n+k}(E,G) \cong \mathbb{Z}$. Moreover, $\pi_{n+k}(E,F) \longrightarrow \pi_{n+k}(S^n,*)$ is a $\mathcal{C}$-isomorphism and therefore $\pi_{n+k}(E,F)$ is finite by assumption **12.0**(1). Thus $\pi_{n+k}(G,F) \longrightarrow \pi_{n+k}(E,F)$ is an epimorphism. By commutativity of the triangle in **C**, $\pi_{n+k}(G,F) \longrightarrow \pi_{n+k}(S^n,*)$ is $\mathcal{C}$-surjective. It now follows, similarly as in the case of dimension $n+k-1$, that q_* is a $\mathcal{C}$-isomorphism in dimension $n + k$. ∎

Theorem 12.1(b) now follows by 12.4 and Theorem 11.1(b). ∎

13. JAMES'S MAPS OF REDUCED PRODUCTS OF SPHERES

We now prove the result of James [J_2] on maps of reduced product spaces of spheres, by means of adjunction of n-equivalences. A partial result in this direction has been obtained by Hardie and Porter [HP] by means of quasifibrations. For completeness and notation, we include a brief review of the work of Toda [T] on the CW-structure of S^k_∞ and we recall the relative Whitehead product in [$\mathbf{BM_1}$].

13.0 Notation. We fix an integer $k > 1$ and make the convention of denoting the k-dimensional sphere by S and the 2k-sphere by T. For any $r \in \mathbb{N}$, the r-sphere is denoted by S^r. The James reduced product of a space X is denoted by X_∞. The subspace of words of length at most r is denoted by X_r. If $r = 1$ then X_1 is written simply as X. Consider the map $k : S_2 \longrightarrow T_\infty$ which is the composition of a collapse map, a homeomorphism and an inclusion :

$$S_2 \longrightarrow S_2/S_1 \equiv T \longrightarrow T_\infty .$$

Let $h : S_\infty \longrightarrow T_\infty$ be any extension of k. We shall assume that h is cellular with respect to the standard CW-structure on the spaces. Then h induces maps :

$$h : (S_{2q+1}, S) \longrightarrow (T_q, *) .$$

Note the ambiguity of use of the symbol h. If k is odd, let P_q denote the set of all primes. If k is even, let P_q denote the set consisting of the prime 2 together with all primes which are bigger than 2q+1. We state the main result of this section.

13.1 Theorem

For every non-negative integer q, $h : (S_{2q+1}, S) \longrightarrow (T_q,*)$ is a P_q-equivalence.

13.2 CW-structure of S_k. In line with Section 12, let $F = S_{2q+1}$, let $i : S \longrightarrow S_{2q+1}$ be the inclusion, and $\alpha \in \pi_{2kq+2k-1}(S_{2q+1})$ the homotopy class of the attaching map of the cell (S_{2q+2}, S_{2q+1}). In diagram A, ∂ and ∂' are boundary homomorphisms in the exact homotopy sequences of the triple (S_{q+2}, S_{q+1}, S_q) and, respectively, the pair (S_{q+2}, S_{q+1}). The homomorphism ψ is an injection in the exact homotopy sequences of the pair (S_{q+1}, S_q).

— A —

$$\begin{array}{ccc} \pi_m(S_{q+2}, S_{q+1}) & \xrightarrow{\ \partial\ } & \pi_{m-1}(S_{q+1}, S_q) \\ \partial' \big\downarrow & \nearrow \psi & \\ \pi_{m-1}(S_{q+1}, *) & & \end{array}$$

Let α be a generator of $\pi_m(S_{q+2}, S_{q+1})$ with $m = k(q+2)$ and let $\iota \in \pi_k(S_{q+1})$ be the homotopy class of the inclusion map $S \longrightarrow S_{q+1}$. Then by Toda [T] we have the following description of the characteristic map of the pair (S_{q+2}, S_{q+1}).

13.3 Proposition

$\partial(\alpha) = r[\iota, \alpha_1]$ where α_1 is a generator of $\pi_{m-k}(S_{q+1}, S_q)$, $m = (q+2)k$. The bracket denotes the relative Whitehead product as in [**BM**$_1$]. Furthermore, $r = q+2$ for even k, and $r = 1$ for both k and q odd. ▮

13.4 Representation of Whitehead products

Let $C = E^k$ and $D = E^n$ where $n = 2k(q+1)$. Let $B = \partial(C \times D)$ be the boundary of the topological manifold $C \times D$. Then B is homeomorphic to the sphere of dimension $(2q+3)k-1$. We represent $[r\iota, \alpha_1]$ by some map $f : (V, \partial V) \longrightarrow (S_{q+1}, S_q)$ as described in [**BM**$_1$]. Here V is the subspace $\partial C \times D \cup C \times D_+$ of B with $D_+ = \{z \in D : z_1 \geq 0\}$, z_1 being the first co-ordinate of z.

Since $\partial(\alpha)$ is in the image of ψ, ψ as in diagram A, the map f can be extended to a map $f_0 : B \longrightarrow S_{q+1}$ and in such a way that the complement $B \setminus V$ of V is mapped into S_q. Let Q be the quotient space obtained from B by making the following identifications. A point x of $B \subset (C \times D)$ can be represented in a unique way as $x = (x_1,x_2)$ with $x_1 \in C$ and $x_2 \in D$. We identify two points x and y if $x_1; y_1 \in \partial C$ and $x_2 = y_2$. Then Q is homeomorphic to the space $Q' = (S \times S^{n-1}) \cup (* \times E^n)$. By the definition of f, we notice that the map f_0 can be factorized as

$$B \xrightarrow{f_1} Q' \xrightarrow{f_2} S_{q+1} .$$

Let $\zeta = f_2 | S \times S^{n-1}$, but consider ζ to be a map into S_q. Then ζ is of the type (r,g) where r is a self-map of S of degree r followed by the inclusion $S \subset S_q$, and g is a representative of α_1. The space obtained as the push-out of the cotriad :

$$S \times E^n \longleftarrow S \times S^{n-1} \longrightarrow S_q ,$$

is homotopy equivalent to S_{q+2}. In the notation of Section 12, S_{q+2} is the space $E(\zeta)$. Thus by **12.1** we have :

13.5 Theorem

Let $q : (S_{q+2},S_q) \longrightarrow (S^{qk+k},*)$ be any map which extends the relative homeomorphism $(S_{q+1},S_q) \longrightarrow (S^{qk+k},*)$. Then q is a (R,t)-equivalence, where $t = (q+3)k - 1$ and R is the set of all primes which are not factors of r, with r as in **13.3**. ∎

Note that in particular, when q is odd, then R contains P_x where $x = \frac{q+1}{2}$. The following result is the first step in the inductive proof of Theorem **13.1**. A proof of this case by means of adjunction of quasifibrations can be found in [HP].

13.6 Theorem. The map $h : (S_3,S) \longrightarrow (T,*)$ is a P_1-equivalence.

Proof By **13.4**, h_1 is a $(P_1, 4k-1)$-equivalence. The theorem now follows by **11.1(a)**. ∎

13.7 ***Proof of Theorem*** **13.1**

Let us recall the statement of the theorem and denote it by $\mathcal{T}_{q+1}$:

For every non-negative integer q, the map $h_{q+1} : (S_{2q+3},S) \longrightarrow (T_{q+1},*)$ is a P_{q+1}-equivalence.

We prove the statements $\mathcal{T}_{q+1}$ by induction. $\mathcal{T}_1$ is precisely the result **13.6**. Now assume that $\mathcal{T}_n$ is true for all positive integers n less than $q + 1$.

Let $l : (T_{q+1},T_q) \longrightarrow (S^t,*)$ be the relative homeomorphism, where $t = 2k(q + 1)$. By **7.1**, l is a m_1-equivalence, with $m_1 = 2k(q+2) - 1$. Let $\eta = l \circ h'$ as in diagram **B**, where h' is induced by h. Then η is a map $(S_{2q+3},S_{2q+1}) \longrightarrow (S^t,*)$ which by **13.5** is a (P_{q+1}, m_2)-equivalence with $m_2 = (2q + 4)k - 1 = m_1$. By commutativity of the triangle **B**, it follows that h' is a $(P_{q+1}, m_1 - 1)$-*equivalence.

– B –

$$\begin{array}{ccc} (S_{2q+3}, S_{2q+1}) & \xrightarrow{\ \eta\ } & (S^t,*) \\ {\scriptstyle h'}\downarrow & \nearrow{\scriptstyle l} & \\ (T_{q+1}, T_q) & & \end{array}$$

We now apply the generalized five-lemma to the homotopy ladder of the map of triples $(S_{2q+3}, S_{2q+1}, S) \longrightarrow (T_{q+1}, T_q, *)$. Note that $P_{q+1} \subset P_q$. Invoking the induction assumption, it follows that the map of pairs $(S_{2q+3},S) \longrightarrow (T_{q+1},*)$ is a $(P_{q+1}, m_2 - 1)$-equivalence. If $k \geq 3$, then $m_1 - 1 > (2q + 3)k$, and $\mathcal{T}_{q+1}$ follows by **11.1(a)**.

For the case $k = 2$ we still need to ascertain that the homomorphism,

(1) $$\varphi : \pi_{4q+6}(S_{2q+3}, S) \longrightarrow \pi_{4q+6}(T_{q+1}, *) ,$$

is $\mathcal{C}_{q+1}$-injective before **11.1(a)** is applicable. Consider the diagram **C** similar to **B**.

– C –

$$\begin{array}{ccc} (S_{2q+3}, S_{2q+1}) & \xrightarrow{\ \eta\ } & (S^t, *) \\ {\scriptstyle h^{\sim}} \big\downarrow & \nearrow {\scriptstyle l^{\sim}} & \\ (T_{q+2}, T_q) & & \end{array}$$

By **13.5**, the map $l^{\sim}$ is a $(P_{q+1}, 4q + 11)$-equivalence. Furthermore, η is a $(P_{q+1}, 4q + 7)$-equivalence. Thus $h^{\sim}$ is a $(P_{q+1}, 4q + 7)$-equivalence. In diagram **D**, the horizontal row represents the homotopy sequence of the triple (T_{q+2}, T_{q+1}, T_q). In view of **18.3** and exactness of the P_{q+1}-localization functor, we can regard the groups as being P_{q+1}-localized. Under this assumption we must prove that the homomorphism of (1) is injective.

– D –

$$\begin{array}{ccccc} & & \pi_{4q+7}(S_{2q+3}, S_{2q+1}) & & \\ & & {\scriptstyle \alpha} \big\downarrow & \searrow {\scriptstyle h^{\sim}_{*}} & \\ \pi_{4q+8}(T_{q+2}, T_{q+1}) & \xrightarrow{\ \partial\ } & \pi_{4q+7}(T_{q+1}, T_q) & \xrightarrow{\ i\ } & \pi_{4q+7}(T_{q+2}, T_q) \\ \big\downarrow & \nearrow {\scriptstyle \sigma} & & & \\ \pi_{4q+7}(T_{q+1}) & & & & \end{array}$$

Since $h^{\sim}_{*}$ is an epimorphism and the sequence $\cdot \xrightarrow{\partial} \cdot \xrightarrow{i} \cdot$ is exact, it follows that $\pi_{4q+7}(T_{q+1}, T_q) = \text{Im } \alpha + \text{Im } \partial$. By commutativity of the lower triangle in diagram **D**, $\text{Im } \partial \subset \text{Im } \sigma$. Thus $\pi_{4q+7}(T_{q+1}, T_q) = \text{Im } \alpha + \text{Im } \sigma$. Therefore in the homotopy sequence, in (2) below, of (T_{q+1}, T_q), we have $\tau[\pi_{4q+7}(T_{q+1}, T_q)] = \tau[\text{Im } \alpha]$.

(2) $$\pi_{4q+7}(T_{q+1}) \xrightarrow{\ \sigma\ } \pi_{4q+7}(T_{q+1}, T_q) \xrightarrow{\ \tau\ } \pi_{4q+6}(T_q)$$

Thus we can modify the homotopy ladder of the map $(S_{2q+3}, S_{2q+1}, S) \longrightarrow (T_{q+1}, T_q, *)$, and obtain an exact ladder, the first part of which is shown in diagram **E**.

$$
-\mathbf{E}- \qquad
\begin{array}{ccccc}
\pi_{4q+7}(S_{2q+3}, S_{2q+1}) & \longrightarrow & \pi_{4q+6}(S_{2q+1}, S) & \longrightarrow & \pi_{4q+6}(S_{2q+3}, S) \rightarrow \\
\bar{\alpha} \downarrow & \bar{\partial} & \downarrow & & \downarrow \varphi \\
\mathrm{Im}\ \alpha & \longrightarrow & \pi_{4q+6}(T_q, *) & \longrightarrow & \pi_{4q+6}(T_{q+1}, *) \rightarrow
\end{array}
$$

The ladder is assumed to start with the arrow $\bar{\alpha}$ going (dimensionally) downward. The arrow $\bar{\partial}$ is the restriction of the boundary homomorphism $\partial : (T_{q+1}, T_q) \longrightarrow (T_q, *)$. By the five-lemma applied to this diagram, it follows that φ is injective. Again $\mathcal{T}_{q+1}$ follows by **11.1(a)**. The induction is completed, and hence so is the proof of the Theorem **13.1**. ∎

The following conclusion follows from **13.1** by a simple direct limit argument.

13.8 Theorem [J_2; Theorem 1.2 and Theorem 1.3]

The map $h : (S_\infty, S) \longrightarrow (T_\infty, *)$ is a weak equivalence if k is odd, otherwise it is a $\{2\}$-equivalence. ∎

Chapter IV : ELEMENTARY PROPERTIES OF n-EQUIVALENCES

We prove the five-lemma and use it to derive some properties of maps of pairs and triples. In particular we make a study of the action of the group $\pi_1(X)$ on the set pointed set $\pi_2(X,A)$, for a pair (X,A) of spaces. A key result, cited several times, is Theorem **16.5** which compares a n–equivalence with its mapping path fibration.

Section **17** contains a detailed proof of a lemma on homotopy lifting over a n–equivalence which is required for the proof of Theorem **5.1**.

14. EXACTNESS

The contents of this section are well-known and we include it for completeness. The five-lemma is applied when studying exact *ladders* arising from a map of pairs or triples of spaces. We have to work in a category more general than groups since the lower homotopy sets are not groups functorially. One instance of such a study can be found in the work of Hardie and Kamps [HK] on Mayer-Vietoris-type phenomena. We prove the five-lemma in such a setting. A generalization, in the sense of Serre classes of abelian groups [Se], is proved in Chapter V. The category in which we shall work is pSet, pointed sets and base point-preserving functions. The base point will be denoted by u. The category of groups and group homomorphisms is denoted by Gp.

14.1 Definition

A sequence of morphisms $C \xrightarrow{j} D \xrightarrow{k} E$ in pSet is said to be *exact* if $j(C) = k^{-1}(u)$. If the given sequence was part of a longer sequence in pSet, then the longer sequence is said to be exact at the object D if $j(C) = k^{-1}(u)$. A sequence is said to be exact if it is exact at every object except perhaps at the first or last object of the sequence, if such exist.

14.2 Remark. Every group can be considered to be a pointed set, with the identity element as base point, and then group homomorphisms are pointed maps.

14.3 Definition

A sequence of morphisms $C \xrightarrow{j} D \xrightarrow{k} E$ in pSet is said to be (E2)-*exact*, see [HK], if it is exact and C is a group with a left action $\nu : C \times D \longrightarrow D$ on D satisfying the identity :

(1) $$j(x) = \nu(x,u) .$$

For the results in this section we assume that we have the following situation. There is a commutative square, diagram **A**, in <u>Gp</u>. Suppose the diagram is embedded, via the forgetful functor, in the commutative diagrams **B**, and **C** in <u>pSet</u>. Suppose also that in these diagrams the rows are exact.

– **A** –

$$\begin{array}{ccc} B & \xrightarrow{i} & C \\ \beta \downarrow & & \downarrow \gamma \\ W & \xrightarrow{i_1} & X \end{array}$$

14.4 Proposition

Suppose that in diagram **B**, we have (E2)-exactness at the object Y. Suppose β is surjective and $\epsilon^{-1}(u) = \{u\}$. Then,

(a) $j_1^{-1}(\delta D) \subset \gamma C$.

(b) γ is surjective if δ is surjective.

– **B** –

$$\begin{array}{ccccccc} B & \xrightarrow{i} & C & \xrightarrow{j} & D & \xrightarrow{k} & E \\ \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\ W & \xrightarrow{i_1} & X & \xrightarrow{j_1} & Y & \xrightarrow{k_1} & Z \end{array}$$

Proof. **(a)** (E2)-exactness provides us with a group action $\nu : X \times Y \longrightarrow Y$. Suppose that $x \in X$ is such that $j_1(x) \in \delta(D)$. We must show that $x \in \gamma(C)$.

There exists $d \in D$ such that $\delta(d) = j_1(x)$. But then,

$$\epsilon\, k(d) = k_1\, \delta(d) = k_1\, j_1(x) = u \in Z\,.$$

Since $\epsilon^{-1}(u) = \{u\}$ it follows that $k(d) = u \in E$. By exactness at D, it follows that for some $c \in C$, $d = j(c)$, and then,

$$j_1\, \gamma(c) = \delta\, j(c) = \delta(d) = j_1(x).$$

Furthermore, $j_1(x^{-1}\cdot\gamma(c)) = \nu(x^{-1}\cdot\gamma c,\ u)$

$$
\begin{aligned}
&= \nu(x^{-1}, \nu(\gamma c,\ u))\\
&= \nu(x^{-1}, j_1\gamma(c))\\
&= \nu(x^{-1}, j_1(x))\\
&= \nu(x^{-1}, \nu(x,u))\\
&= \nu(x^{-1}\cdot x, u) = \nu(u,u) = u\,.
\end{aligned}
$$

Thus by exactness at X, there exists $w \in W$ such that $x^{-1}\cdot\gamma(c) = i_1(w)$. Since β is surjective, $w = \beta(a)$ for some $a \in B$. Then,

$$\gamma(\, i(a)^{-1}) = [\,\gamma i(a)\,]^{-1} = [\,i_1\beta(a)\,]^{-1} = [\,x^{-1}\cdot\gamma(c)\,]^{-1} = \gamma(c)^{-1}\cdot x\,,$$

and so, $$\gamma(\,c\cdot i(a)^{-1}) = \gamma(c)\cdot\gamma(\,i(a)^{-1}) = \gamma(c)\cdot\gamma(c)^{-1}\cdot x = x\,.$$

This proves **(a)**. The statement **(b)** follows immediately from **(a)**. ∎

14.5 Proposition

Suppose that in diagram B, $\epsilon^{-1}(u) = \{u\}$ and β is surjective, and that we have (E2)-exactness at the object D. Then the induced function $\gamma^{-1}(u) \longrightarrow \delta^{-1}(u)$ is surjective.

Proof Suppose that $d \in \delta^{-1}(u)$. Then $\epsilon\, k(d) = k_1\,\delta(d) = k_1(u) = u$. But $\epsilon^{-1}(u) = \{u\}$, and so, $k(d) = u$. By exactness at D, there exists $c \in C$ such that $j(c) = d$.

Now $j_1\,\gamma(c) = \delta\, j(c) = \delta(d) = u$, and by exactness at X, $\gamma(c) = i_1(w)$ for some $w \in W$. Furthermore $w = \beta(b)$ for some $b \in B$, since β is surjective. But $\gamma i(b) = i_1\beta(b) = \gamma(c)$. So taking $x = [i(b)]^{-1} = i(b^{-1})$, then $j(x) = j\,i(b^{-1}) = u$, and $cx \in \ker\gamma$.

Let $\mu : C \times D \longrightarrow D$ be the action due to (E2)-exactness. Then,

$$j(cx) = \mu(cx,u) = \mu[c,\mu(x,u)] = \mu[c, j(x)] = \mu\,[c,u] = j(c) = d.$$

Thus we have found an element cx of $\ker\gamma$ which is mapped by j onto d. ∎

14.6 Proposition. Suppose that in diagram A, β is an epimorphism and the other three arrows are monomorphisms (regard i and i_1 as inclusion maps).

Then the function $\varphi : C/B \longrightarrow X/W$ between the sets of left cosets, is a monomorphism.

Proof Our information fits diagram B if we take ϵ to be a homomorphism between two trivial groups, and let $\delta = \varphi$. The functions j and j_1 are coset projections. So by **14.5**, $\varphi^{-1}(W) = B$ (here B and W are regarded as the base points of C/B and X/W respectively). Now suppose we have elements $gB, fB \in C/B$ for which $\varphi(gB) = \varphi(fB)$. Then $\gamma(g)W = \varphi(gB) = \varphi(fB) = \gamma(f)W$. This implies that $\gamma(f)^{-1}\gamma(g) \in W$.

Now $\gamma(f)^{-1}\gamma(g) = \gamma(f^{-1})\gamma(g) = \gamma(f^{-1}g)$. Thus $\gamma(f^{-1}g)B = u$. Since $\varphi^{-1}(W) = B$, it follows that $(f^{-1}g)B = B$, i.e. $f^{-1}g \in B$. This implies that $gB = fB$. Thus φ is injective. ▮

14.7 Proposition. Suppose that in diagram C, β and δ are bijective, that α is surjective, and that ϵ is injective. Suppose further that at the objects D and Y, the exactness is of the type (E2).

– C –

$$\begin{array}{ccccccccc} A & \xrightarrow{h} & B & \xrightarrow{i} & C & \xrightarrow{j} & D & \xrightarrow{k} & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ V & \xrightarrow{h_1} & W & \xrightarrow{i_1} & X & \xrightarrow{j_1} & Y & \xrightarrow{k_1} & Z \end{array}$$

Then γ is bijective.

Proof This follows by **4.4(b)** and **14.5**. ▮

Proposition **14.7** is known as the *five-lemma*. The results of this section are applied so often that we shall at times use the term five-lemma ambiguously when refering to any of the results **14.4(b)**, **14.5** or **14.7**, or a combination of these.

15. THE HOMOTOPY SEQUENCE OF A PAIR OF SPACES

Attention is given here to the exact homotopy sequence of a pair of spaces, especially in the lower dimensions. We obtain a number of results of the nature of the five-lemma as applied to maps of pairs of spaces. The lower homotopy sets are not groups functorially. However the homotopy sequence does live at least in the category pSet of pointed sets and base point-preserving functions.

The action of the fundamental group on the higher homotopy groups were first studied by Eilenberg [**Ei**]. Hereafter the role of group actions in algebraic topology was recognized. A variety of actions are discussed in [**Wd**]. The basics of abstract group actions can be found in the textbook [**Ro**] by Rose.

The action of the group $\pi_1(X,b)$ on the set $\pi_1(X,A,b)$ yields a description of $\pi_1(X,A,b)$ in terms of other homotopy sets, similar to the result [**JT**; Theorem 1.2] of James and Thomas and [**Ru**; Theorem 1.3.1] of Rutter. This action does not seem to have been given sufficient attention in the study of quasifibrations or weak equivalences. We shall take advantage of this description, and circumvent long arguments with pointwise diagram chasing. Some counter-examples are included to support the results obtained.

15.1 The homotopy sequence of a pair

For a space X and $b \in A \subset X$, there is for the pointed pair (X,A,b), an exact ***homotopy sequence*** in pSet (see [**Wd**] for example),

$$\pi_{r+1}(X,A,b) \xrightarrow{\partial_{r+1}} \pi_r(A,b) \xrightarrow{i_r} \pi_r(X,b) \xrightarrow{j_r} \pi_r(X,A,b) \xrightarrow{\partial_r} .$$

The sequence extends to the left indefinitely and on the right it terminates in the pointed set $\pi_0(X,b)$ of path components of X, with the path component containing b as the base point. The part of the sequence to the left of and including the object $\pi_1(X,b)$ belongs to the category Gp in fact, and except perhaps for the last three of these, the other groups are in fact abelian. Note that ∂_n; i_n; j_n are natural transformations, so that a map $(X,A,b) \longrightarrow (Y,B,b')$ of pointed pairs, determines a commutative ladder :

$$\begin{array}{ccccccccc}
\to \pi_{n+1}(X,A,b) & \xrightarrow{\partial} & \pi_n(A,b) & \xrightarrow{i} & \pi_n(X,b) & \xrightarrow{j} & \pi_n(X,A,b) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to \pi_{n+1}(Y,B,b') & \longrightarrow & \pi_n(B,b') & \longrightarrow & \pi_n(Y,b') & \longrightarrow & \pi_n(Y,B,b') & \to
\end{array}$$

We shall refer to this diagram as the ***homotopy ladder*** of p. There is also a similar ***homology ladder*** for a such a map p. The homology ladder does not have the same relevance to weak equivalences as does the homotopy ladder. Nevertheless it has a role to play in, for instance, Section 9. There are similar ladders for maps of triples of spaces.

15.2 The action of $\pi_1(X,b)$ on $\pi_1(X,A,b)$.

Let (X,A,b) be a fixed pair of pointed spaces. The pointed set $\pi_1(X,A,b)$ is defined as the set of all homotopy classes of maps $(I,\partial I,0) \longrightarrow (X,A,b)$, with $\partial I = \{0, 1\}$. We shall restrict our discussion to the case where X is path-connected. For a path $g : I \longrightarrow X$ in X, we define a path g^- in X by the formula, $g^-(t) = g(1-t)$. If f is another path in X and if $f(0) = g(1)$, then we define the path $g+f$ by setting

$$(g+f)(t) = \begin{cases} g(2t) & 0 \leq t \leq \frac{1}{2} \\ f(2t-1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

This composition gives rise to the fundamental group $\pi_1(X,b)$ of a pointed space (X,b).

The detail hereof can be found in [**Ma**]. In a similar manner, the composition determines a left action (which we shall denote by juxtaposition) of $\pi_1(X,b)$ on $\pi_1(X,A,b)$. We shall omit the verification of the axioms of group action since it is very similar to that for the fundamental group multiplication. From the definition, the action can be seen to be natural in the following sense. For a map $p:(X,A,b) \longrightarrow (Y,B,b')$, the action is compatible with the following functions induced by p.

$$p_* : \pi_1(X,b) \longrightarrow \pi_1(Y,b') \text{ and } p_* : \pi_1(X,A,b) \longrightarrow \pi_1(Y,B,b')$$

More precisely, for $\alpha \in \pi_1(X,b)$ and $\zeta \in \pi_1(X,A,b)$, $p_*(\alpha\zeta) = (p_*\alpha)(p_*\zeta)$.

In order to study the orbits of this action and the isotropy subgroups of elements, we fix the following terminology. Let $\{C_i : i \in L\}$ be the set of all the (distinct) path components of A. For each $i \in L$, we fix an element $c_i \in C_i$ and a path w_i in X from b to c_i. Let G_i denote the subgroup $\mathrm{Im}\,[\pi_1(C_i,c_i) \longrightarrow \pi_1(X,c_i)]$ of $\pi_1(X,c_i)$. For each $i \in L$, the subset of $\pi_1(X,A,b)$ determined by the paths terminating in C_i, will be denoted by F_i. Then the collection $\{F_i : i \in L\}$ is a partition of $\pi_1(X,A,b)$.

It is easy to see that every subset F_i is closed under the action. We now show that :

(1) Each F_i is an orbit of the action.

Let w be any path in X from b to c_i (every member of F_i contains such a path). Let α be the member of $\pi_1(X,b)$ represented by the loop $w + w_i^-$. Then $[w] = \alpha\cdot[w_i]$, and (1) follows [the bracket denotes the homotopy class in $\pi_1(X,A,b)$].

We now determine the isotropy subgroup $\mathrm{Fix}\,[w_i]$ of the element $[w_i]$. $\mathrm{Fix}\,[w_i]$ is defined as the subgroup of all elements of $\pi_1(X,b)$ fixing $[w_i]$.

Let v be any loop in X based at the point b, and let $\alpha \in \pi_1(X,b)$ be its homotopy class. Suppose that $\alpha \cdot [w_i] = [w_i]$. Then there is a homotopy,

$$H: (I \times I, \partial I \times I, 0 \times I) \longrightarrow (X,A,b),$$

such that:

$$H(s,0) = (v + w_i)(s) \text{ for all } s \in I,$$

$$H(s,1) = w_i(s) \text{ for all } s \in I,$$

$$H(1,t) \in A \text{ for all } t \in I, \text{ and}$$

$$H(0,t) = b \text{ for all } t \in I.$$

The formula $h(t) = H(1,t)$ defines a loop h in C_i based at c_i. From the properties of H, it follows that the loop $v + w_i + h + w_i^-$ represents the identity element of $\pi_1(X,b)$. Let β be the element of the group G_i represented by the loop h^-. Then α is the image of β under the change-of-base point function [Sr; Lemma 2 on p380],

$$W_i : \pi_1(X,A,c_i) \longrightarrow \pi_1(X,A,b),$$

determined by the path w_i. Thus, whenever $\alpha \in \text{Fix}[w_i]$, then $\alpha \in W_i(G_i)$. In fact, our argument is reversible, and $\text{Fix}[w_i]$ is precisely the subgroup $W_i(G_i) \subset \pi_1(X,b)$. So, as a left $\pi_1(X,b)$-set, $\pi_1(X,A,b)$ is equivalent to the disjoint union of the sets of left cosets $\pi_1(X,b) / W_i(G_i)$. That is, $\pi_1(X,A,b)$ can be written as:

15.2 (a) $$\cup_{i \in L} [\, \pi_1(X,b) / W_i(G_i) \,].$$

It is interesting to note that we can have an isomorphism $p_* : \pi_1(X,A,a) \longrightarrow \pi_1(Y,B,b)$ although the **15.2(a)**-descriptions for two sets may look completely different. The following example illustrate this phenomenon.

15.2(b) Example. Let $p:(\mathbb{R},\mathbb{Z}) \longrightarrow (S,1)$ be the exponential map where S is the set of complex numbers with unit modulus. The homomorphism $\pi_1(\mathbb{R},\mathbb{Z}) \longrightarrow \pi_1(S,1)$ is an isomorphism between infinite sets although $\pi_1(\mathbb{R})$ and $\pi_0(S)$ are 1-element sets. ∎

Also, from the decomposition **15.2(a)**, the following result can be deduced immediately.

15.2(c) Proposition

Let $p: X' \longrightarrow X$ be a map of path connected spaces, and let us denote for every $A \subset X$, the set $p^{-1}(A)$ by A'. Suppose that $Y \subset X$ and $\{Y_i\}_{i \in L}$ is the collection of all path components of Y. Then the conditions (1) and (2) below are equivalent.

(1) The homomorphism $p_*: \pi_1(X',Y',x) \longrightarrow \pi_1(X,Y,px)$ is surjective (or bijective) for every $x \in Y'$.

(2) For every $i \in L$ and for every $x \in Y'$, $p_*: \pi_1(X',Y_i',x) \longrightarrow \pi_1(X,Y_i,px)$ is surjective (or bijective). ∎

Since I is path connected we can drop the assumption of path connectedness in **15.2(c)** and deduce the following more general result.

15.2(d) Proposition

Let $p: X' \longrightarrow X$ be a map and let us denote for every $A \subset X$, the set $p^{-1}(A)$ by A'. Suppose that $Y \subset X$ and $\{Y_i\}_{i \in L}$ is the collection of all path components of Y. Then the conditions (1) and (2) below are equivalent.

(1) $p:(X',Y') \longrightarrow (X,Y)$ is a n-equivalence.

(2) $p_*: \pi_1(X',Y_i') \longrightarrow \pi_1(X,Y_i)$ is a n-equivalence for all $i \in L$. ∎

15.3 Lemma. Let $p : (X,A) \longrightarrow (Y,B)$ be a map of pairs, with X and Y path connected, and $b \in A$.

(a) If $\pi_1(X,b) \longrightarrow \pi_1(Y,pb)$ and $\pi_0(A) \longrightarrow \pi_0(B)$ are surjective, then $\pi_1(X,A,b) \rightarrow \pi_1(Y,B,pb)$ is surjective.

(b) If $\pi_1(A,c) \longrightarrow \pi_1(B,pc)$ is surjective for all $c \in A$, if $\pi_1(X,b) \longrightarrow \pi_1(Y,pb)$ and $\pi_0(A) \longrightarrow \pi_0(B)$ are injective, then $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,pb)$ is injective.

Proof **(a)** Since $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective, the induced function between the sets of orbits of the actions is surjective. The other assumption implies that a given orbit in $\pi_1(X,A,b)$ is mapped onto an orbit in $\pi_1(Y,B,pb)$, and thus $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,pb)$ is surjective.

(b) Injectivity of $\pi_0(A) \rightarrow \pi_0(B)$ implies that the set of $\pi_1(X,b)$-orbits of $\pi_1(X,A,b)$ are mapped injectively into the set of $\pi_1(Y,pb)$-orbits of $\pi_1(Y,B,pb)$. By **14.6**, each $\pi_1(X,b)$-orbit is also mapped injectively into the corresponding orbit of $\pi_1(Y,pb)$. Thus $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,pb)$ is injective. ▮

15.4 Lemma

Let $p : (X,A) \longrightarrow (Y,B)$ be a map of pairs, with X and Y path connected. Suppose $b \in A$ and $b' = p(b)$.

(a) If $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,b')$ is surjective, then $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective.

(b) If $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,b')$ is injective and $\pi_1(X,b) \longrightarrow \pi_1(Y,b')$ is surjective, then $\pi_0(A) \longrightarrow \pi_0(B)$ is injective.

Proof **(a)** follows since the function $\pi_0(A) \longrightarrow \pi_0(B)$ represents the map on orbit spaces of the action.

(b) If $\pi_1(X,b) \longrightarrow \pi_1(Y,b')$ is surjective, then any orbit of $\pi_1(X,A,b)$ maps onto an orbit of $\pi_1(Y,B,b')$. Now if $\pi_0(A) \longrightarrow \pi_0(B)$ fails to be injective, then there is an orbit of $\pi_1(Y,B,b')$ which has more than one orbit of $\pi_1(X,A,b)$ mapping onto it. Consequently then $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,b')$ fails to be injective. Thus $\pi_0(A) \longrightarrow \pi_0(B)$ is injective. ∎

Example **15.2(b)** shows that in **15.4(b)**, the second condition is necessary.

15.5 Lemma

Let $p : (X,A) \longrightarrow (Y,B)$ be a map of unpointed pairs and suppose that the following hold.

(1) For every $b \in A$ with $b' = p(b)$, we have $\pi_1(X,A,b) \longrightarrow \pi_1(Y,B,b')$ is surjective ;

(2) $\pi_0(A) \longrightarrow \pi_0(B)$ is bijective ;

(3) $\pi_0(A) \longrightarrow \pi_0(X)$ is surjective.

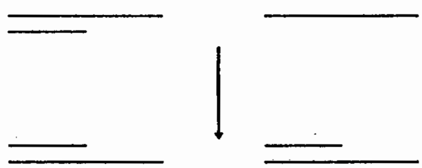
Then $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective.

Proof In view of **3.3(c)**, we can assume that Y is path connected. Suppose X_1 is a path component of X. Let $A_1 = A \cap X_1$. Then the injection $\pi_1(X_1,A_1,c) \longrightarrow \pi_1(X,A,c)$, for $c \in A_1$, is bijective (since I is path connected) and $\pi_0(A_1) \longrightarrow \pi_0(A)$ is injective. Thus due to (1), $\pi_1(X_1,A_1,c) \longrightarrow \pi_1(Y,B,p(d))$ is surjective. By **15.4(a)** then, we have a surjective function $\pi_0(A_1) \longrightarrow \pi_0(B)$. So $\pi_0(A_1) \longrightarrow \pi_0(B)$ is in fact bijective. Then by (2) and since X_1 is path connected, $A \setminus A_1 = \phi$. Finally, (3) implies that $X \setminus X_1 = \phi$, and it follows that X is path connected. ∎

In conclusion we have two examples. The first example shows that, with regard to **15.4(a)**, we will have to be cautious when the spaces X and Y fail to be path connected.

15.6 Example

Let $X = Y = [0,2] \cup [3,5] \subset \mathbb{R}$. We take $A = [0,1]$ and $B = [0,1] \cup [3,4]$. Let $p(x) = x$ for every $x \in X$.

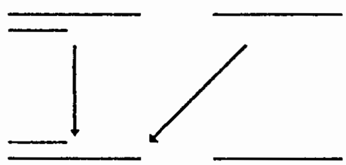


Then $\pi_n(X,b) \longrightarrow \pi_n(Y,b')$ and $\pi_{n+1}(X,A,b) \longrightarrow \pi_{n+1}(Y,B,b')$ are bijections for all $n \geq 0$ and for all $b \in A$ with $b' = p(b)$. However $\pi_0(A) \longrightarrow \pi_0(B)$ is not surjective. ■

The next example shows the necessity of condition ($_3$) in **15.5**.

15.7 Example

Choose $X = Y = [0,2] \cup [3,5] \subset \mathbb{R}$ and $A = B = [0,1]$. Let $p : (X,A) \longrightarrow (Y,B)$ be the map defined by the formula, $p(x) = \min\{x, 2\}$.



Then $\pi_n(A,b) \longrightarrow \pi_n(B,b')$ and $\pi_{n+1}(X,A,b) \longrightarrow \pi_{n+1}(Y,B,b')$ are bijections for all $n \geq 0$, all $b \in A$ with $p(b) = b'$. Yet $\pi_0(X) \longrightarrow \pi_0(Y)$ is neither one-to-one nor onto. ■

16. THE FIVE-LEMMA APPLIED TO MAPS OF PAIRS AND TRIPLES

We prove a number of results in the spirit of the title of this section. Proposition **16.5** can be considered the most prominent result. The proposition compares an n-equivalence with its mapping path fibration. Further, it relates the concepts n-equivalence for a map of pairs of spaces, and (n–1)-equivalence of homotopy fibres for a $\underline{\text{Top}}^2$-morphism.

16.1 Proposition

Suppose that $p : (X,A) \to (Y,B)$ is a map, and for the pair (Y,B), $\pi_0(B) \longrightarrow \pi_0(Y)$ is surjective. Let n be a non-negative integer. If $(X,A) \longrightarrow (Y,B)$ and $A \longrightarrow B$ are n-equivalences, then $X \longrightarrow Y$ is a n-equivalence.

Proof Surjectivity of $\pi_0(A) \longrightarrow \pi_0(B)$ and $\pi_0(B) \longrightarrow \pi_0(Y)$, together with **3.2($_2$)**, implies that $\pi_0(X) \longrightarrow \pi_0(Y)$ is surjective. If $n = 0$ the proof ends here.

So let us assume that $n > 0$. Then, by **15.5**, $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective. So in fact, $\pi_0(X) \longrightarrow \pi_0(Y)$ is bijective. For the remaining part of the proof we can assume by **3.3(c)** that X and Y are path connected.

Diagram **A** shows part of the ladder formed by the map p of pairs in homotopy.

$$\begin{array}{ccccccccccc} & \pi_r(A,b) & \xrightarrow{i} & \pi_r(X,b) & \xrightarrow{j} & \pi_r(X,A,b) & \xrightarrow{\partial} & \pi_{r-1}(A,b) & \longrightarrow & \pi_{r-1}(X,b) \\ -\mathbf{A}- & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \pi_r(B,b') & \rightarrow & \pi_r(Y,b') & \rightarrow & \pi_r(Y,B,b') & \rightarrow & \pi_{r-1}(B,b') & \rightarrow & \pi_{r-1}(Y,b') \end{array}$$

The result follows by successive application, for $r = 1, 2, \ldots, n-1$ of **14.7** to diagram **A**, and then **14.5**, taking $r = n$. The fact that we took a base point in A does not matter since our assumptions imply that $\pi_0(A) \longrightarrow \pi_0(X)$ is surjective. ∎

16.2 Lemma. Let $p:(X,A) \longrightarrow (Y,B)$ be a map and n a positive integer. If the map $X \longrightarrow Y$ is a (n+1)-equivalence and $A \longrightarrow B$ is a n-equivalence, then $(X,A) \longrightarrow (Y,B)$ is a (n+1)-equivalence.

Proof Since $X \longrightarrow Y$ is a 1-equivalence, $\pi_0(X) \longrightarrow \pi_0(Y)$ is a bijection. Condition **3.2($_2$)** can be seen to be satisfied, since $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective and $\pi_0(X) \longrightarrow \pi_0(Y)$ is injective. In view of **3.3(c)** we can assume that X (and hence Y) are path connected. By **15.3(a)**, $\pi_1(X,A) \longrightarrow \pi_1(Y,B)$ is surjective. For $n = 0$ the proof ends here.

For $n > 0$, the result follows by **15.3(b)** and the five-lemmas, **14.7** and **14.5** applied (as in the proof of **16.1**) to the homotopy ladder of p. ∎

16.3 Lemma

Let n be a non-negative integer. Suppose that $p:(X,A) \longrightarrow (Y,B)$ is a map for which $X \to Y$ is a n-equivalence and $(X,A) \longrightarrow (Y,B)$ is a (n+1)-equivalence.

Then $A \longrightarrow B$ is a n-equivalence.

Proof Let Y' be any path component of Y. Let $B' = B \cap Y'$, $X' = p^{-1}(Y')$ and $A' = A \cap X'$. Then from **3.3(c)**, the pull-back $r:(X',A') \longrightarrow (Y',B')$ is a $(n+1)$-equivalence. Now let C be any path component of X'. Due to path connectedness of the interval I, the function $r_*: \pi_1(C, C \cap A) \longrightarrow \pi_1(Y',B')$ is surjective. Thus by **15.4(a)**, $\pi_0(C \cap A) \longrightarrow \pi_0(B')$ is surjective. From this it follows that $\pi_0(A) \longrightarrow \pi_0(B)$ is surjective. For $n = 0$ the proof stops here.

If $n > 0$ then by **15.4(b)**, $\pi_0(C \cap A) \longrightarrow \pi_0(B')$ is injective. Moreover, X' is path connected, so that $C = X'$ and $C \cap A = A'$. Thus it follows that $\pi_0(A') \longrightarrow \pi_0(B')$ is injective. Together with the bijectivity of $\pi_0(X) \longrightarrow \pi_0(Y)$, this implies that $\pi_0(A) \longrightarrow \pi_0(B)$ is injective. From here on we can assume that X and Y are path connected. The result now follows by application of the five-lemmas to diagram A. ▮

16.4 Notation

For a map $p : X \longrightarrow Y$, the mapping path fibration factorizations of $X \longrightarrow Y$ will be denoted by $X \xrightarrow{g} Y\hat{} \xrightarrow{f} Y$. For $U \subset Y$, we denote the inverse image of U with respect to f by $U\hat{}$.

Suppose now that p is a map of pairs of spaces $p : (X,A) \longrightarrow (Y,B)$. We recall from **3.5(a)** that $f: (Y\hat{},B\hat{}) \longrightarrow (Y,B)$ is a weak equivalence. Since $f: (Y\hat{},B\hat{}) \longrightarrow (Y,B)$ satisfies condition **3.2(2)** and $p = f \circ g$, the map $g : (X,A) \longrightarrow (Y\hat{},B\hat{})$ satisfies **3.2(2)** if and only if $p : (X,A) \longrightarrow (Y,B)$ does. In fact, for a given integer k, $g : (X,A) \longrightarrow (Y\hat{},B\hat{})$ is a k-equivalence if and only if $p : (X,A) \longrightarrow (Y,B)$ is a k-equivalence.

Furthermore, we notice that the map $g : X \longrightarrow X\hat{}$ is homotopy equivalence and hence a weak equivalence. ■

16.5 Theorem

Given a non-negative integer n, the following three statements are equivalent for a map

$$p : (X,A) \longrightarrow (Y,B)\,.$$

(1) $p : (X,A) \longrightarrow (Y,B)$ is a (n+1)-equivalence.

(2) The map of A into the inverse image $p\hat{}^{\,-1}(B) = B\hat{}$ of B with respect to the

mapping path fibration $p\hat{}$ of p, is a n-equivalence.

($_3$) The Top^2-morphism of inclusions from $A \longrightarrow B$ to $X \longrightarrow Y$ is a n-equivalence of homotopy fibres (in the terminology of **1.15**).

Proof In view of **3.4**, condition ($_1$) is equivalent to condition ($_0$) stated below.

($_0$) The map $p : (X',A) \longrightarrow (Y',B)$, where Y' is the union of all the path components C for which $C \cap B \neq \phi$ and $X' = p^{-1}(Y')$, is a (n+1)-equivalence.

Furthermore it is clear that the conditions ($_2$) and ($_3$) above, does not involve the path components of Y for which $Y \cap B = \phi$. Thus it suffices to prove Theorem **16.5** under the assumption that the inclusion $B \subset Y$ is a surjective map of path components. We prove the equivalences ($_1$) $\Leftrightarrow$ ($_2$) and ($_2$) $\Leftrightarrow$ ($_3$).

($_1$) $\Rightarrow$ ($_2$): From the observations of **16.4**, condition ($_1$) implies that $g : (X,A) \longrightarrow (Y\hat{},B\hat{})$ is a (n+1)-equivalence. Since $X \longrightarrow Y\hat{}$ is a weak equivalence, the implication we want to prove now follows by **16.3**.

($_2$) $\Rightarrow$ ($_1$): From the observations of **16.4**, this implication follows by **16.2**.

The equivalence of ($_2$) and ($_3$) follows by comparison of the fibre homotopy sequences associated with the two maps. We give the detail.

Let $A \xrightarrow{\gamma} C \xrightarrow{\varphi} B$ be the mapping path fibration factorization of $A \longrightarrow B$. Then C can be considered a subspace of $B\hat{}$ with $\varphi = f_0 \mid C$ (with $f_0 : B\hat{} \longrightarrow B$ being the pull-back of f). Let b be any point in B with $F = \varphi^{-1}(b)$ and $G = f^{-1}(b)$. Then from the observations in **16.4**, applied to the commutative triangle below, it follows that $(C,F) \longrightarrow (B\hat{},G)$ is a weak equivalence.

$$\begin{array}{ccc} (C,F) & \xrightarrow{\;c\;} & (B\hat{},G) \\ & \varphi\searrow \quad \swarrow f_0 & \\ & (B,b) & \end{array}$$

Since $\gamma : A \longrightarrow C$ is a weak equivalence, it follows that the condition $(_2)$, (which is that $A \longrightarrow B\hat{}$ is a n-equivalence), is equivalent to the condition $(_4)$ below.

$(_4)$ $C \longrightarrow B\hat{}$ is a n-equivalence.

Now note that condition $(_3)$ means precisely that (for an arbitrary point b of B) the following condition holds.

$(_5)$ $F \longrightarrow G$ is a n-equivalence.

Now for the map $(C,F) \longrightarrow (B\hat{},G)$, the implication $(_4) \Rightarrow (_5)$ follows by **16.3** and the converse follows by **16.1**. ▮

The proof of the following result is a simple exercise on the level of the elementary theory of functions between sets, and we omit it.

16.6 Proposition

Suppose that diagram **B** below is a commutative triangle in the category <u>Top</u> .

– C –

$$\begin{array}{ccc} A & \xrightarrow{\;\alpha\;} & B \\ & \gamma\searrow \quad \swarrow\beta & \\ & C & \end{array}$$

(a) If α and β are n-equivalences, then γ is a n-equivalence.

(b) If γ is a n-equivalence and β is a (n+1)-equivalence, then α is a n-equivalence.

(c) If α is a n-equivalence and γ is a (n+1)-equivalence, then β is a (n+1)-equivalence. ▮

In what follows, we shall work with a map of triples $p : (X,X_1,A) \longrightarrow (Y,Y_1,B)$. Such a map induces three different maps of pairs. For these maps of pairs we obtain results analogous to the first three results of this section.

16.7 Notation regarding maps of triples.

Suppose that $p : (X,X_1,A) \longrightarrow (Y,Y_1,B)$ is a map of triples. Let us denote the maps $A \longrightarrow B$, $X_1 \longrightarrow Y_1$ and $X \longrightarrow Y$ induced by p, by the symbols p_0, p_1 and p_2 respectively. Then set inclusion provides us with Top^2-morphisms $\lambda : p_0 \longrightarrow p_1$ $\mu : p_1 \longrightarrow p_2$ and $\nu : p_0 \longrightarrow p_2$ such that $\mu \circ \lambda = \nu$. So we have a commutative diagram C in the Top^2.

– C –

$$\begin{array}{ccc} p_0 & \xrightarrow{\lambda} & p_1 \\ & {}_{\nu}\searrow \quad \swarrow_{\mu} & \\ & p_2 & \end{array}$$

This notation is used in the proofs of the following two lemmas, 16.8 and 16.9. ■

16.8 Lemma

Let $n > 0$. Suppose that p is a map of triples, $(X,X_1,A) \longrightarrow (Y,Y_1,B)$, such that :

(1) $(X,A) \longrightarrow (Y,B)$ is a (n+2)-equivalence ;

(2) $(X_1,A) \longrightarrow (Y_1,B)$ is a (n+1)-equivalence ;

(3) $\pi_0(B) \longrightarrow \pi_0(Y_1)$ is surjective.

Then $(X,X_1) \longrightarrow (Y,Y_1)$ is a (n+2)-equivalence.

Proof By 16.5 [(1) ⇒ (3)], the Top^2-morphism λ (in diagram C above) is a n-equivalence of homotopy fibres and ν is a $(n+1)$-equivalence of homotopy fibres. Thus by 16.6(c) and since $\pi_0(B) \longrightarrow \pi_0(Y_1)$ is surjective, it follows that μ is a $(n+1)$-equivalence of homotopy fibres. Our result follows by 16.5[(3) ⇒ (1)], applied to μ. ■

16.9 Lemma. Suppose that p is a map of triples, $(X,X_1,A) \longrightarrow (Y,Y_1,B)$. Then for a non-negative integer n we have :

(a) If $(X,A) \longrightarrow (Y,B)$ is a (n+1)-equivalence and $(X,X_1) \longrightarrow (Y,Y_1)$ is a (n+2)-equivalence, then $(X_1,A) \longrightarrow (Y_1,B)$ is a (n+1)-equivalence.

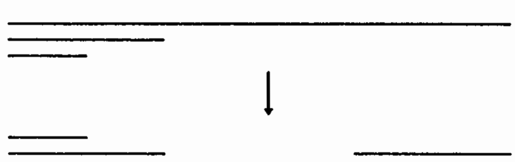
(b) If $(X_1,A) \longrightarrow (Y_1,B)$ and $(X,X_1) \longrightarrow (Y,Y_1)$ are (n+1)-equivalences, then $(X,A) \longrightarrow (Y,B)$ is a (n+1)-equivalence.

Proof **(a)** By **16.5**[$(_1) \Rightarrow (_3)$], the Top^2-morphism ν (diagram **C** in **16.7**) is a n-equivalence of homotopy fibres and μ is a (n + 1)-equivalence of homotopy fibres. Thus by **16.6(b)** it follows that λ is a n-equivalence of homotopy fibres. Our result follows by application of **16.5** [$(_3) \Rightarrow (_1)$] applied to λ.

(b) By **16.5**, the morphisms λ and μ are n-equivalences of homotopy fibres. Thus by **16.6(a)**, it follows that ν is a n-equivalence of homotopy fibres. Our result follows by Theorem **16.5** applied to ν. ∎

16.10 Example.

We show how **16.8** fails in the absence of condition **16.8**$(_3)$. Let $X = Y = [0,5] \subset \mathbb{R}$, $X_1 = [0,2]$, $Y_1 = [0,2] \cup [3,5]$, $A = B = [0,1]$, and $p(x) = x$ for every $x \in X$. Then $\pi_1(X,X_1)$ has cardinality 1, while that of $\pi_1(Y,Y_1)$ is 2. Note that condition **3.2**$(_2)$ holds for each pair map in this triple.



∎

17. PROOF OF LEMMA 5.5

An important result on homotopy lifting, Lemma **5.5**, used in the proof of Theorem 5.1, the globalization theorem for n-equivalences, is proved in this section. [$\mathbf{M_2}$; Corollary 2.4]. Lemma **5.5** forms part of the treatment in Dold and Thom's paper [**DT**] and in a modified form in the work [$\mathbf{M_2}$] of May. The proof of the lemma given here is in detail, showing maps and homotopies explicitly. We rely on the pasting lemma [**Ms**; p 108 Theorem 7.3], without repeated reference to it, to ensure continuity of functions defined piecewise.

Throughout this section, $p : E \longrightarrow B$ will denote a map of topological spaces. For a subspace U of B, $p^{-1}(U)$ will be denoted by U'. We work with base point-free spaces. When required, base points will be specifically mentioned. The following notation is used throughout this section.

17.1 Notation

The boundary of a topological manifold A is denoted by ∂A.

I is the unit interval $[0,1]$.

We fix an integer $n \geq 2$ and denote the unit n-cube I^n by K. The following subsets of K are important:

For $r = 0; 1$, $I_r = I^{n-1} \times \{r\}$, and J is the closure of $\partial K \backslash I_1$ i.e. $J = I_0 \cup [\ \partial(I_0) \times I\ ]$.

Denote by c the centre of the centre of I_1, and by d the centre of $I_1 \times I$.

$$c = (\ \tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}; \dots ; \tfrac{1}{2}; 1\ ) \quad — \quad \text{n co-ordinates},$$

$$d = (\ \tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}; \dots ; \tfrac{1}{2}; 1; \tfrac{1}{2}\ ) \quad — \quad \text{n+1 co-ordinates.} \qquad \blacksquare$$

17.2 Some homeomorphisms. For a point $x \in K\backslash\{c\}$, we let $x^{\wedge}$ be the point in J which is collinear with the pair of points $\{x, c\}$, and $x^{-} = \| x - c \| \cdot \| x^{\wedge} - c \|^{-1}$. If $x = c$, we define $x^{-} = 0$. Then $0 \leq x^{-} \leq 1$. For $i \in \{0; 1\}$, let Z_i be the subspace $K \times \{i\} \cup J \times I$ of $K \times I$. We define homeomorphisms $h_i : K \times \{i\} \to Z_i$ by the formulae :

$$h_0(x,0) = \begin{cases} (2x - c,\ 0) & 0 \leq x^{-} \leq \frac{1}{2} \\ (x^{\wedge},\ 2x^{-} - 1) & \frac{1}{2} \leq x^{-} \leq 1, \end{cases}$$

$$h_1(x,1) = \begin{cases} (2x - c,\ 1) & 0 \leq x^{-} \leq \frac{1}{2} \\ (x^{\wedge},\ 2 - 2x^{-}) & \frac{1}{2} \leq x^{-} \leq 1. \end{cases}$$

Now let J_1 be the closure in $K \times I$ of the subspace $\partial(K \times I) \setminus (I_1 \times I)$. Then the following formula defines a self-homeomorphism of J_1.

$$h_2(z) = \begin{cases} h_0(z) & z \in K \times \{0\} \\ h_1^{-1}(z) & z \in Z_1 \end{cases}$$

Note that h_2 maps ∂J_1 homeomorphically onto itself. The resulting self-homeomorphism h_3 of ∂J_1 can be extended to a self-homeomorphism h_4 of $I_1 \times I$, the closed n-cell complementary to J_1 in $\partial(K \times I)$. Pasted together, h_2 and h_4, regarded as maps into $K \times I$, provide us with a self-homeomorphism h of $\partial(K \times I)$. This in turn yields a self-homeomorphism H of $K \times I$ by regarding $K \times I$ as the cone on $\partial(K \times I)$. ∎

We repeat the formulation of **Lemma 5.5** :

Let $p : (X,U) \longrightarrow (Y,V)$ be a map of pairs of spaces. Then the following three conditions are equivalent.

(a) Given maps $f:(J,\partial J) \longrightarrow (X,U)$ and $g:(K,I_1) \longrightarrow (Y,V)$ together with a homotopy $d_t : J \longrightarrow Y$ from $p \circ f$ to the restriction $g|J$ of g to J, such that $d_t(\partial J) \subset V$ for all $t \in I$, there exists an extension $F:(K,I_1) \longrightarrow (X,U)$ of f, and a homotopy $D_t : K \longrightarrow Y$ from $p \circ F$ to g, extending d_t such that $D_t(I_1) \subset V$ for all $t \in I$.

(b) Given maps $\varphi:(J,\partial J) \longrightarrow (X,U)$ and $\gamma:(K,I_1) \longrightarrow (Y,V)$ with $p \circ \varphi = \gamma|J$, then there exists an extension $\Phi:(K,I_1) \longrightarrow (X,U)$ of φ, and a homotopy $\Delta_t : K \longrightarrow Y$ from $p \circ \Phi$ to γ, such that Δ_t is stationary on J and $\Delta_t(I_1) \subset V$ for all $t \in I$.

(c) For every $e \in U$ and $b = p(e)$, the function $p_*: \pi_r(X,U,e) \longrightarrow \pi_r(Y,V,b)$ is injective for $r = n-1$ and surjective for $r = n$.

Proof of Lemma 5.5

We prove the sequence of implications (b) ⇒ (a) ⇒ (c) ⇒ (b). Firstly, (b) ⇒ (a). Let us assume (b), and suppose that f, g, d_t are as in (a). We observe that d_t and g jointly define a map $G_1 : Z_1 \longrightarrow Y$ by the rule :

$$G_1(x,t) = \begin{cases} d_t(x) & (x,t) \in J \times I \\ g(x) & (x,t) \in K \times \{1\} . \end{cases}$$

Now we define a map $\gamma : K \longrightarrow Y$ as $\gamma(x) = G_1 \circ h_1(x,1)$, and let $\varphi = f$. Then $\gamma(I_1) \subset V$ and $p \circ \varphi = \gamma|J$. Applying (b) to the data φ, γ, we obtain Φ and Δ_t. The formula $\Delta(x,t) = \Delta_t(x)$ defines a map $\Delta : K \times I \longrightarrow Y$. Let $D = \Delta \circ H$. Let $G_0 : Z_0 \longrightarrow X$ be the map defined by the following formula :

$$G_0(x,t) = \begin{cases} f(x) & (x,t) \in J \times I \\ \Phi(x) & (x,t) \in K \times \{0\} . \end{cases}$$

We define a map $F : K \longrightarrow X$ by the formula $F(x) = G_0 \circ h_0(x,0)$.

Then F together with the homotopy D_t obtained from D, fulfill the requirements of (a). Thus we have proved (b) ⇒ (a). In retrospect we note that γ absorbs, via h_1, the information on d_t. This is released again after (b) has been exploited, by precomposition with H. This rectification of the homotopy calls for an adjustment of Φ to get the map F, which is in line with D_t.

(a) ⇒ (c) : Assume condition (a) holds. We shall first prove the injectivity part of (c). In the case $n = 2$ we note that for elements $\gamma_1; \gamma_2 \in \pi_1(X,U,e)$, path composition (one of them traversed in the opposite sense) associates with these path classes, a third path class $\gamma \in \pi_1(X,U,a)$ for some $a \in U$. The association is such that $\gamma_1 = \gamma_2$ if and only if $\gamma = 0$. So even in this case of not having groups, the following two conditions are equivalent :

($_x$) $\pi_{n-1}(X,U,e) \longrightarrow \pi_{n-1}(Y,V,b)$ is injective for every $e \in U$ and $b = p(e)$.

($_y$) Given any $e \in U$ and $\alpha \in \pi_{n-1}(X,U,e)$, then $\alpha = 0$ whenever $p_*(\alpha) = 0$.

We use ($_y$) to prove the injectivity. Let $\alpha \in \pi_{n-1}(X,U,e)$ be such that in $\pi_{n-1}(Y,V,b)$, $p_*(\alpha) = 0$. We can represent α by a map $f: (J,\partial J) \longrightarrow (X,U)$. Since $p_*(\alpha) = 0$, the map $p \circ f$ can be extended to a map $g : (K,I_1) \longrightarrow (Y,V)$. We apply (a), taking d_t to be the constant homotopy. Existence of the map F guaranteed by (a), means that $\alpha = 0$. So the injectivity follows.

To prove surjectivity, let $\beta \in \pi_n(Y,V,b)$ be an arbitrary element. We represent β by a map $g : (K,\partial K,J) \longrightarrow (Y,V,b)$. Let $f: J \longrightarrow X$ be the constant map onto the point $e \in U$. Let $d_t : J \longrightarrow V$ be the constant (b-valued) homotopy between the maps $g|J$ and $p \circ f$.

Now by (a), there exists F and D_t. The map F represents an element γ of $\pi_n(X,U,e)$ and D_t ensures that in $\pi_n(Y,V,b)$, $p_*(\gamma) = \beta$. So surjectivity of p_* follows, and we have proved the implication (a) ⇒ (c).

Finally we prove the implication **(c) ⇒ (b)**, and this will complete the proof of our result. Now suppose we are given maps $\varphi : (J,\partial J) \longrightarrow (X,U)$ and $\gamma : (K,I_1) \longrightarrow (Y,V)$ such that $p \circ \varphi = \gamma|J$.

Let s be the vertex (1; 1; ... ; 1) of K, let $\varphi(s) = e$ and $p(e) = b$. Then in the group $\pi_{n-1}(Y,V,b)$, $[\gamma|J] = 0$. Since p_* is injective, in $\pi_{n-1}(X,U,e)$ we have $[\varphi] = 0$. So there exists an extension $\Phi_1 : (K,I) \longrightarrow (X,U)$ of φ.

Let $C = \{x \in K : x_i \geq \frac{1}{2}$ for all $1 \leq i \leq n\}$. Let M be the closure $(K \setminus C)^-$ of $K \setminus C$ in K, and let $M_1 = M \cap J$. Let B be the union of all the (n–1)-faces of K which do not contain s. Given any point $x \in K \setminus \{s\}$, let x_s be the (unique) point in B which is collinear with the pair of points x; s. Define $g : K \longrightarrow Y$ and $F_1 : K \longrightarrow X$ as below.

$$g(x) = \begin{cases} \gamma(s) & x \in C \\ \gamma(2x - x_s) & x \in M \end{cases}$$

$$F_1(x) = \begin{cases} \Phi_1(s) & x \in C \\ \Phi_1(2x - x_s) & x \in M \end{cases}$$

These functions can routinely be shown to be continuous. Note that $g|J = p \circ F_1|J$ and that $F_1(C) = \{e\}$. Now we define $G : (J_1, \partial J_1) \longrightarrow (Y,V)$ by the formula below, with J_1 as in **17.2**.

$$J_1 = K \times \{0,1\} \cup J \times I \subset K \times I .$$

$$G(x,t) = \begin{cases} p \circ F_1(x) & (x,t) \in K \times 0 \\ g(x) & (x,t) \in J \times I \cup K \times 1 . \end{cases}$$

Since the pair $(J_1, \partial J_1)$ is homeomorphic to $(K,\partial K)$, G represents an element $\alpha \in \pi_n(Y,V,b)$. By surjectivity in condition (c), there is an element $\beta \in \pi_n(X,U,e)$ such that $p(\beta) = -\alpha$. Represent β by a map $F_2 : (K,\partial K,J) \longrightarrow (X,U,e)$. Now we define a map $F : (K,\partial K) \longrightarrow (X,U)$ by :

$$F(x) = \begin{cases} F_2(2x - s) & \text{for } x \in C \\ F_1(x) & \text{otherwise} . \end{cases}$$

Then F is continuous since F_1 is continuous and $F_1(C) = \{e\}$. We obtain a homotopy E_t as follows. Let $\Gamma : (J_1,\partial J_1) \longrightarrow (Y,V)$ be defined by the formula :

$$\Gamma(x,t) = \begin{cases} p \circ F(x) & (x,t) \in K \times \{0\} \\ g(x) & (x,t) \in J \times I \cup K \times \{1\} . \end{cases}$$

Then $[\Gamma]$, as an element of $\pi_n(Y,V)$, is zero by the choice of F_2 and its incorporation in the definition of Γ via F. Therefore Γ can be extended to a map

$$E : (K \times I , I_1 \times I) \longrightarrow (Y,V) .$$

We obtain the solution to the problem of (b) by the following quotient construction. Let A and A_s be the subsets of $K \times I$ given below.

$$A = (J \cap C) \times I \cup C \times \{1\} ; \quad A_s = \{s\} \times I$$

Let $r_0 : A \longrightarrow A_s$ be defined by the formula, for $(x,t) \in A \subset K \times I$, $r_0(x,t) = (s,t)$.

From **17.2** we recall that $Z_1 = K \times \{1\} \cup J \times I$. Extend r_0 to a map $r_1 : Z_1 \longrightarrow Z_1$ by mapping a point $(x,t) \in (Z_1 \backslash A)^-$ (closure in $K \times I$), onto the point $(2x - x_S, t)$. Extend r_1 to a relative homeomorphism

$$r_2 : (J_1, Z_1) \longrightarrow (J_1, Z_1),$$

and extend r_2 to a relative homeomorphism

$$r : (K \times I, J_1) \longrightarrow (K \times I, J_1).$$

Since $E\, r^{-1}(y)$ is a one-point set for every $y \in K \times I$, it follows by a theorem on quotient maps, [**Ms**; p 139 Theorem 11.1], that there exists $\Delta : K \times I \longrightarrow Y$ such that $\Delta \circ r = E$. From Δ we obtain a homotopy Δ_t in an obvious way. From r, we obtain an induced map $r_3 : K \times \{0\} \longrightarrow K \times \{0\}$.

Let $q : K \longrightarrow K$ be the obvious map determined by r_3. Then there exists $\Phi : K \longrightarrow X$ such that $\Phi \circ q = F$. The maps Φ and Δ comply to the specifications as in (a). ∎

Chapter IV : LOCALIZATION OF ABELIAN GROUPS AND 1-CONNECTED SPACES

The main purpose of chapter is to prove results formulated in Section **8**. The most important one among these, is Theorem **8.5**, which is a generalized version of the result [**M$_2$**; Theorem 1.2] of May. Although quasifibrations have been applied in a generalized form, there seems to be no explicit formulation of results of this nature. Chapter V aims to fill this gap in the theory.

18. LOCALIZATION OF ABELIAN GROUPS

When studying torsion abelian groups one can separate the different primary components. A particular component can be extracted by tensoring the group with the underlying abelian group of a suitable subring of $\mathbb{Q}$. Here we have a discussion of a localization theory on the category Ab of abelian groups. Some of the results here are mentioned in [HMR]. The cited book covers a more general theory namely the localization theory on the category of nilpotent groups (and of related topological spaces). The results we have here are elementary, and are included for completeness.

The group operation will be denoted by $+$ (addition). By P we denote a fixed set of primes. The complementary set of primes is denoted by P'. The subring of $\mathbb{Q}$ generated by the set $\{n^{-1} : n \in P'\}$ is denoted by $\mathbb{Z}$.

18.1 Definition.

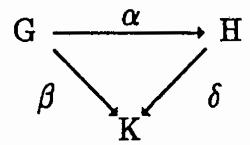
For a positive integer n, the n'***th multiple*** endomorphism of the abelian group G is the homomorphism $\mu : G \longrightarrow G$ defined by the rule : $\mu(g) = ng$. An abelian group is said to to ***have*** n'***th roots*** if the n'th multiple endomorphism is an epimorphism.

An abelian group is said to be P-***local*** if for every $n \in P'$, the n'th multiple endomorphism is an automorphism.

A morphism $\alpha : G \longrightarrow H$ of Ab is said to be a P-***localizing*** homomorphism if H is P-local and for every P-local group K, the function $\alpha^* : \underline{Ab}(H,K) \longrightarrow \underline{Ab}(G,K)$, obtained by pre-composing with α, is a bijection. This means that, given any P-local abelian group K and a homomorphism $\beta : G \longrightarrow K$, there exists a unique homomorphism

$\delta : H \longrightarrow K$, such that diagram A commutes.

– A –



18.2 Remarks. **(a)** The additive group $\mathcal{Z}^+$ of the ring $\mathcal{Z}$, is P-local. The abelian group $\mathcal{Z}^+$ will also be denoted by the same symbol $\mathcal{Z}$.

(b) Let G be any abelian group, let $G \otimes \mathcal{Z}$ be the tensor product of abelian groups, and $\iota(G) : G \longrightarrow G \otimes \mathcal{Z}$ the canonical homomorphism, $\iota(g) = g \otimes 1$. Then we have an endofunctor $(-) \otimes \mathcal{Z}$ on <u>Ab</u>, and ι is a natural transformation from the identity functor of <u>Ab</u> to the functor $(-) \otimes \mathcal{Z}$.

18.3 Proposition

(a) For any abelian group G, the natural homomorphism $\iota : G \longrightarrow G \otimes \mathcal{Z}$ is a P-localizing homomorphism.

(b) An abelian group G is P-local if and only if $\iota : G \longrightarrow G \otimes \mathcal{Z}$ is an isomorphism.

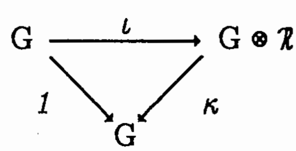
Proof **(a)** We first show that $G \otimes \mathcal{Z}$ is P-local. For any $n \in P'$, let μ be the n'th multiple endomorphism of $G \otimes \mathcal{Z}$, and ν the inverse of the n'th multiple endomorphism of $\mathcal{Z}$. Then the endomorphism $1 \otimes \nu$ of $G \otimes \mathcal{Z}$, is the inverse of μ. Thus μ is an isomorphism.

We now show that ι is a P-localizing homomorphism. Suppose that $\alpha : G \longrightarrow H$ is a morphism of <u>Ab</u> and H is a P-local group. Let m be any integer which is co-prime to the members of P. Then the m'th multiple endomorphism of H is an automorphism. Hence there is a well-defined function of sets $G \times \mathcal{Z} \longrightarrow H$, mapping $g \times \frac{r}{m}$ onto $\frac{1}{m} \cdot \alpha(rg)$.

This function is bilinear, and therefore determines a homomorphism $\alpha' : G \otimes \mathcal{Z} \longrightarrow H$, satisfying the condition $\alpha' \circ \iota = \alpha$. If $\beta : G \otimes \mathcal{Z} \longrightarrow H$ is a homomorphism for which $\beta \circ \iota = \alpha$, then β coincides with α' on $\iota(G)$. Since $\iota(G)$ is a generating set of $G \otimes \mathcal{Z}$ as an $\mathcal{Z}$-module, it follows that $\beta = \alpha'$. So ι P-localizes.

(b) If $\iota : G \longrightarrow G \otimes \mathcal{Z}$ is an isomorphism, then since by **(a)** $G \otimes \mathcal{Z}$ is P-local, it follows that G is a P-local group.

Conversely, suppose that G is P-local. Then for 1 the identity function of G, since ι P-localizes, there exists a left inverse $\kappa : G \otimes \mathcal{Z} \longrightarrow G$ to ι. So ι is injective.



On the other hand, $G_0 = \{x \otimes \frac{r}{n} : x \in G\ ,\ \frac{r}{n} \in \mathcal{Z}\}$ generates the group $G \otimes \mathcal{Z}$. Since G has n'th roots, it follows that $\frac{1}{n}\cdot x \in G$. Thus $\frac{r}{n}\cdot x \in G$, and $\iota(\frac{r}{n}\cdot x) = (\frac{r}{n}\cdot x) \otimes 1 = x \otimes \frac{r}{n}$. Therefore ι is surjective, and so ι is an isomorphism. ∎

18.4 Proposition

For a homomorphism $\alpha : G \longrightarrow H$ between P-local abelian groups,

(a) the kernel K is P-local

(b) the cokernel C is P-local.

Proof We have an exact sequence $0 \to K \to G \to H \to C \to 0$ in the category Ab. In diagram **B**, the vertical arrows are the canonical P-localizing homomorphisms as in **18.2(b)**. The diagram is commutative by naturality of ι. Since the abelian group $\mathcal{Z}$ is torsion free, the functor $(-) \otimes \mathcal{Z}$ preserves exactness. Thus the bottom row of diagram **B** is exact.

$$-\mathbf{B}- \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & \downarrow{\scriptstyle \iota_4} & & \downarrow \\ 0 & \longrightarrow & K \otimes \mathbb{Z} & \longrightarrow & G \otimes \mathbb{Z} & \longrightarrow & H \otimes \mathbb{Z} & \longrightarrow & C \otimes \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Note that the ladder can be augmented to both the left and the right by using the trivial groups. Since G and H are P-local, it follows by **18.3(b)** that ι_1 and ι_3 are isomorphisms. Thus by **14.7**, ι_1 and ι_4 are isomorphisms. Our results follow by **18.3(b)**. ▮

The proof of the following two results are similar to the one we have just given, and we omit the proofs.

18.5 Proposition

Given a short exact sequence of abelian groups, with any two of the groups P-local, then the third group is also P-local. ▮

18.6 Proposition

For an exact sequence in <u>Ab</u> as shown below, C is P-local if the other four groups are P-local.

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E\,. \quad ▮$$

19. SERRE CLASS OF ABELIAN GROUPS

The concept of a Serre class of abelian groups is indispensible when studying torsion phenomena in abelian groups. The pioneering work on this topic is found in the paper [Se] by the inventor. The textbooks [Hu], [Sp] and [HMR] also offer brief treatments of the topic. The notion of Serre class of abelian groups gives rise to a generalization of the concepts of monomorphism and epimorphism. This in turn calls for generalizations of concepts of exactness, and results such as the the five-lemma. We include a proof of the latter result.

As in Section 18, we use the following notation. $\underline{\mathrm{Ab}}$ denotes the category of abelian groups. The group operation will always be denoted by $+$ (addition). P is a fixed set of primes. The complementary set of primes will be denoted by P'. The subring of $\mathbb{Q}$ generated by the set $\{n^{-1} : n \in P'\}$ is denoted by $\mathbb{Z}$. Tensoring an abelian group G with $\mathbb{Z}$, regarded as an abelian group, annihilates every torsion element of G the order of which are relatively prime to the elements of P.

19.1 Definition.

A class $\mathcal{C}$ of abelian groups is said to be a ***Serre class of abelian groups*** if the class is non-empty, is closed with respect to taking subgroups of members, and has the property that whenever two of the objects in a short exact sequence in $\underline{\mathrm{Ab}}$ belong to $\mathcal{C}$, then also the third object belongs to $\mathcal{C}$.

Let $\mathcal{C}$ be any Serre class of abelian groups. A homomorphism of abelian groups is said to be a ***$\mathcal{C}$-monomorphism*** if its kernel belongs $\mathcal{C}$, a ***$\mathcal{C}$-epimorphism*** if its cokernel belongs to the class $\mathcal{C}$, and a ***$\mathcal{C}$-isomorphism*** if it is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism. ∎

19.2 Examples. **(a)** The class Φ of all finite abelian groups, and the class $\mathcal{F}$ of all finitely generated abelian groups are Serre classes.

(b) The class of all torsion abelian groups with vanishing p-component for every $p \in P$, which we shall denote by $\mathcal{C}(P)$, is a Serre class of abelian groups.

(c) For an integer n which is relatively prime to the members of P, the n'th multiple endomorphism on an abelian group is a $\mathcal{C}(P)$-isomorphism.

(d) The P-localizing homomorphism $\iota : G \longrightarrow G \otimes \mathcal{Z}$ is a $\mathcal{C}(P)$-isomorphism. Let us prove this statement :

The kernel of ι is the subgroup of G, consisting of all torsion elements with order co-prime to the elements of P, and is thus a $\mathcal{C}(P)$-group. Thus ι is a $\mathcal{C}(P)$-monomorphism. G is the surjective image, $q : F \longrightarrow G$, of a free abelian group F on a set S. Certainly then $F \otimes \mathcal{Z}$ is a free $\mathcal{Z}$-module on S, and the localizing homomorphism $\iota_1 = \iota(F)$ is the inclusion of S.

$$\begin{array}{ccccc} F & \xrightarrow{\iota_1} & F \otimes \mathcal{Z} & \dashrightarrow & \operatorname{coker} \iota_1 \\ {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow \\ G & \xrightarrow[\iota]{} & G \otimes \mathcal{Z} & \dashrightarrow & \operatorname{coker} \iota \end{array}$$

The cokernel of ι_1 is a direct sum of $|S|$ copies of $\mathcal{Z}/\mathbb{Z}$. Thus $\operatorname{coker} \iota_1 \in \mathcal{C}(P)$. Since $- \otimes \mathcal{Z}$ is an exact functor, $r = q \otimes \mathcal{Z}$ is an epimorphism. Therefore the homomorphism, $\operatorname{coker} \iota_1 \longrightarrow \operatorname{coker} \iota$, induced by the square of unbroken arrows, is surjective. Since $\mathcal{C}(P)$ is closed with respect to forming quotients, it follows that $\operatorname{coker} \iota_1 \in \mathcal{C}(P)$. ∎

19.3 Proposition

Let $h : H \longrightarrow G$ be a group homomorphism and $h' = h \otimes \mathcal{Z} : H \otimes \mathcal{Z} \longrightarrow G \otimes \mathcal{Z}$ the homomorphism assigned to it by the functor $- \otimes \mathcal{Z}$. Then we have the following.

(a) The only groups which are both P-local and in $\mathcal{C}(P)$ are the trivial groups.

(b) h is a $\mathcal{C}(P)$-monomorphism if and only if h' is a monomorphism.

(c) h is a $\mathcal{C}(P)$-epimorphism if and only if h' is an epimorphism.

Proof. **(a)** Let $G \in \mathcal{C}(P)$ and suppose that $0 \neq x \in G$. Then x has finite order n and n is divisible by a prime p in the complement of P. Let $z = \frac{n}{p}\cdot x$. Then z has order p and lies in the kernel of the p'th multiple endomorphism of G. Consequently, this endomorphism is not an isomorphism, and G fails to be P-local. So **(a)** is proved.

By **19.2(d)**, every $\iota(G)$ is a $\mathcal{C}(P)$-isomorphism. Therefore, h is a $\mathcal{C}(P)$-epimorphism [$\mathcal{C}(P)$-monomorphism] if and only if h' is a $\mathcal{C}(P)$-epimorphism [$\mathcal{C}(P)$-monomorphism]. In view of (a) above, (b) follows from **18.4(a)**, and (c) from **18.4(b)**. ∎

19.4 Remark.

We include the following concept for completeness. A sequence $C \xrightarrow{j} D \xrightarrow{k} E$ of morphisms in <u>Ab</u> is said to be *C-exact* if $kj(C) \in \mathcal{C}$ and $(\mathrm{Im}\,j + \mathrm{Ker}\,i) / \mathrm{Im}\,j \in \mathcal{C}$. A longer sequence is said to be *C-exact* if every pair of consecutive arrows forms a $\mathcal{C}$-exact sequence. ∎

The sequences that we deal with, are exact and we shall prove a generalized five-lemma only for this special case. We have the following analogues of the results in Section **14**.

19.5 Proposition

Let $\mathcal{C}$ be any Serre class of abelian groups. Suppose that in diagram **B** which is a commutative diagram in <u>Ab</u>, we have exact rows. Suppose further that α is $\mathcal{C}$-surjective and δ is $\mathcal{C}$-injective.

– B –

$$\begin{array}{ccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D \\ \downarrow{\scriptstyle\alpha} && \downarrow{\scriptstyle\beta} && \downarrow{\scriptstyle\gamma} && \downarrow{\scriptstyle\delta} \\ V & \xrightarrow[e]{} & W & \xrightarrow[f]{} & X & \xrightarrow[g]{} & Y \end{array}$$

(a) If γ is $\mathcal{C}$–surjective, then β is $\mathcal{C}$–surjective.

(b) If β is $\mathcal{C}$–injective, then γ is $\mathcal{C}$–injective.

Proof Given diagram B, we form the diagram C below, with homomorphisms as shown.

– C –

$$\begin{array}{ccccccc} A \times V & \xrightarrow{i\times V} & B \times V & \xrightarrow{j_1} & C & \xrightarrow{k} & D \\ \downarrow{\scriptstyle\alpha_1} && \downarrow{\scriptstyle\beta_1} && \downarrow{\scriptstyle\gamma_1} && \downarrow{\scriptstyle\delta_1} \\ V & \xrightarrow[e]{} & W & \xrightarrow[f_1]{} & X \times D & \xrightarrow[g\times D]{} & X \times D \end{array}$$

$$\alpha_1(a,v) = \alpha(a) + v$$

$$\beta_1(b,v) = \beta(b) + e(v)$$

$$\gamma_1(c) = (\ \gamma(c), k(c)\ )$$

$$\delta_1(d) = (\ \delta(d), d\ )$$

$$j_1(b,v) = j(b)$$

$$f_1(w) = (\ f(w), 0\ )\ .$$

The diagram is commutative, the rows are exact, α_1 is an epimorphism and δ_1 is a monomorphism.

(a) By **14.4(a)**, coker $\beta_1 \longrightarrow$ coker γ_1 is a monomorphism. The image of this homomorphism is the subgroup $(fW \times 0 + \gamma_1 C)/\gamma_1 C$ of coker γ_1.

$$(fW \times 0 + \gamma_1 C)/\gamma_1 C \cong (fW \times 0)/(fW \times 0 \cap \gamma_1 C)$$

$$\cong fW/\gamma \ker k\ .$$

There is an isomorphism and an epimorphism, respectively,

$$k^{-1}(\ker \delta)/\ker k \longrightarrow \ker \delta \quad \text{and} \quad k^{-1}(\ker \delta)/\ker k \longrightarrow fW/\gamma \ker k .$$

Thus $fW/\gamma \ker k$ and hence $\operatorname{coker} \beta_1$ belong to the class $\mathcal{C}$.

We note that the inclusion $\alpha A \subset V$ is a $\mathcal{C}$-isomorphism, thus so is $e\,\alpha A \subset e V$. Since $e\,\alpha A = \beta\, i A \subset \beta B$, the subgroup $\beta B + eV / \beta B$ belongs to $\mathcal{C}$. Now we have :

$$\begin{aligned} \operatorname{coker} \beta_1 &= W/\beta_1 B \\ &= W/(\beta B + eV) \\ &\cong (W/\beta B) \;/\; [(\beta B + eV)/\beta B]. \end{aligned}$$

From this it follows that $W/\beta B \in \mathcal{C}$.

(b) Note that the image of $\ker \beta_1$ under the projection $B \times V \longrightarrow B$, is precisely the subgroup $\beta^{-1}(eV)$. Thus the image of the homomorphism $\ker \beta_1 \longrightarrow \ker \gamma_1$ is the same as the image of the subgroup $\beta^{-1}(eV)$ of B under the homomorphism j. By **14.5**, the induced homomorphism $\ker \beta_1 \longrightarrow \ker \gamma_1$ is an epimorphism. Thus $\beta^{-1}(eV) \longrightarrow \ker \gamma_1$ is an epimorphism. The kernel of the latter homomorphism is the subgroup $\ker j = iA$ of B. Thus $\beta^{-1}(eV) / iA \cong \ker \gamma_1$. But since β is a $\mathcal{C}$-monomorphism, $\beta^{-1}(eV) / iA$ is $\mathcal{C}$-isomorphic to $eV / \beta\, iA$. Also $eV / \beta iA = eV / e\alpha A$. This group is a quotient of $V/\alpha A \in \mathcal{C}$. Thus $\ker \gamma_1 \in \mathcal{C}$. Moreover, $(\ker \gamma + \ker k)/\ker k$ can be embedded into the group $\ker \delta$. This follows by the first isomorphism theorem. Thus $(\ker \gamma + \ker k)/\ker k$ belongs to $\mathcal{C}$. But,

$$(\ker \gamma + \ker k)/\ker k \cong \ker \gamma /(\ker \gamma \cap \ker k) \cong \ker \gamma /\ker \gamma_1 .$$

So it follows that $\ker \gamma \in \mathcal{C}$. ∎

The following result simply merges the latter two, and is called the *generalized five–lemma.*

19.6 Lemma

Suppose that **D** is a commutative diagram in <u>Ab</u> the rows are exact.

$$-\mathbf{D}- \qquad \begin{array}{ccccccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

Suppose further that β and δ are $\mathcal{C}$-isomorphisms, while α is a $\mathcal{C}$-epimorphism and ϵ is a $\mathcal{C}$-monomorphism. Then γ is a $\mathcal{C}$-isomorphism. ▮

19.7 We now turn to **Lemma** 8.3, which is the following.

Let G and H be abelian groups such that $G \otimes \mathbb{Z}$ and $H \otimes \mathbb{Z}$ are isomorphic, and are finitely generated as $\mathbb{Z}$-modules. Then any $\mathcal{C}(P)$-epimorphism $G \longrightarrow H$ is a $\mathcal{C}(P)$-isomorphism.

For the proof of Lemma 8.3, we need the following result, **19.8(d)**, which can be proved to hold if the ring $\mathbb{Z}$ is replaced by any principal ideal domain, but for brevity we will be contented with the given special case. An exposition of modules over a principal ideal domain can be found in [HH].

19.8 Proposition

(a) Let $\mathcal{A}$ be any ring and P be a free $\mathcal{A}$-module of finite rank. Then every surjective $\mathcal{A}$-homomorphism of P onto itself is an $\mathcal{A}$-isomorphism.

(b) Let M be a cyclic $\mathbb{Z}$-module. If M contains a non-zero $\mathbb{Z}$-torsion element, then M is finite.

(c) Let M be a finitely generated $\mathcal{R}$-module. Then the torsion submodule T of M is finite and M/T is free of finite rank.

(d) Let M be a finitely generated $\mathcal{R}$-module and let $\alpha : M \longrightarrow M$ be a surjective $\mathcal{R}$-endomorphism. Then α is injective.

Proof **(a)** is obvious.

(b) Suppose that $M = \mathcal{R}x$, for some $x \in M$, and that rx is $\mathcal{R}$-torsion, for some $r \in \mathcal{R}$, but $rx \neq 0$. Then x itself is $\mathcal{R}$-torsion, and consequently x is $\mathbb{Z}$-torsion. Thus there is a least positive integer m such that $mx = 0$. We prove finiteness of M by showing that each element of M is of the form kx with k being a non-negative integer less than m. We first show that whenever e is a non-zero integer such that $\frac{1}{e} \in \mathcal{R}$, then e is relatively prime to m. Suppose now that the greatest common divisor of e and m is the positive integer c. Then $m = nc$ and $e = dc$ for some $n,d \in \mathbb{Z}$. But now $nx = \frac{d}{e}mx = \frac{d}{e} \cdot 0 = 0$, and due to minimality of m it follows that $c = 1$, and e is relatively prime to m. Now let s be an arbitrary element of $\mathcal{R}$. Then s can be written as a quotient of integers $s = \frac{a}{e}$ such that e is relatively prime to m. Thus there exists integers u, v such that $eu + mv = 1$. Thus $s = au + \frac{a}{e}vm$, and so $sx = aux + \frac{a}{e}vmx$. Since $mx = 0$, $sx = aux$. By the division algorithm $au = qm + r$, where q and r are integers with $0 \leq r \leq m - 1$. This yields $sx = aux = rx$ and completes the proof of (b).

(c) We note that an element of M is $\mathcal{R}$-torsion if and only if it is $\mathbb{Z}$-torsion. The finiteness of T can now be proved by induction on the number of generators of M. If M is cyclic then by (b) above, T is finite.

Now let us assume that for an integer n, whenever a module is generated by a subset of fewer than n generators, then its torsion sub-module is a finite set. Suppose that M is generated by a subset $S = \{ x_1; x_2; \dots ; x_n \}$. Let M_1 be the sub-module generated by $\{x_1\}$. Then M/M_1 is generated by $n-1$ elements. Thus M_1 and M/M_1 have finite torsion sub-modules T_1 and T_2 respectively. The rationalization functor is an exact functor. We obtain the commutative diagram below, in which the vertical arrows are the rationalizing homomorphisms. Thus the lower row is also a short exact sequence of abelian groups.

$$\begin{array}{ccccc} M_1 & \longrightarrow & M & \longrightarrow & M/M_1 \\ \iota_1 \downarrow & & \iota_2 \downarrow & & \downarrow \iota_3 \\ M_1 \otimes \mathbb{Q} & \longrightarrow & M \otimes \mathbb{Q} & \longrightarrow & M/M_1 \otimes \mathbb{Q} \end{array}$$

The kernel of a rationalizing homomorphism is precisely the torsion subgroup. Thus ι_1 and ι_3 are Φ-monomorphisms, where Φ is the Serre class of finite abelian groups, see **19.2(a)**. Thus, by **19.8**, ι_2 is a Φ-monomorphism and consequently T is finite. This completes the induction and hence the proof of the finiteness of T for a finitely generated module M.

The module M/T is torsion-free, and finitely generated since M is finitely generated. A proof that a finitely generated torsion-free module over a principal ideal domain is a free module (of finite rank) can be found in [**HH**].

(c) Let T be the torsion submodule of M. Then $\alpha T \subset T$, and α induces a homomorphism $\alpha_1 : T \longrightarrow T$ and an epimorphism $\alpha_2 : M/T \longrightarrow M/T$. These fit into the exact ladder in diagram **E** below.

– E –

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & M/T & \longrightarrow & 0 \\ \downarrow & & \downarrow \alpha_1 & & \downarrow \alpha & & \downarrow \alpha_2 & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & M/T & \longrightarrow & 0 \end{array}$$

Since M/T is free of finite rank it follows by **(a)** that the epimorphism α_2 is a monomorphism. By **14.4(b)** it follows thus that α_1 is an epimorphism. Since by **(b)**, T is finite, it follows that α_1 is a monomorphism as well. By **(a)**, α_2 is an isomorphism. By the five-lemma it follows that α is an isomorphism. ∎

Proof of **8.3**. The lemma follows from **19.8(c)** and by **19.3(b,c)**. ∎

20. LOCALIZATION OF 1-CONNECTED SPACES

Phenomena in topology, related to classes of groups were discovered by Serre [**Se**]. The need for localizing maps for spaces, with respect to a given set of prime numbers, arose from the work [**Se**] of Serre. The first example of such a localizing map was exhibited by Sullivan [**Su**].

20.1 Notation

Let us denote by P a fixed set of primes and by $\mathcal{C}$ the class of all torsion abelian groups with vanishing p-component for every $p \in P$. In this section we use the P-localization theory 1-connected CW-complexes as presented in [**HMR**] to define a localization functor on the homotopy category $\underline{h\mathrm{Top}}$ of pointed 1-connected spaces.

Since we work with pointed spaces, constructions such as double mapping cylinders will be considered to be reduced. The compatibility of this approach with the adjunction theorems in Section 6, is attended to in **6.9 – 6.11**.

The P-localization endofunctor of the category $\underline{H_1}$, of 1-connected pointed spaces of the homotopy type of CW-complexes and pointed homotopy classes of maps, as discussed in [**HMR**; Chapter II] is denoted by $\mathcal{L}$, and the natural transformation from the identity functor on $\underline{H_1}$ to $\mathcal{L}$ is denoted by λ. The functor $\mathcal{L}$ preserves P-torsion and anihillates P′-torsion.

We show how $\mathcal{L}$ and λ together with CW-approximation, gives rise to a functor $\mathcal{M}$ on $\underline{h\mathrm{Top}}$ and a natural transformation μ from the identity functor of $\underline{h\mathrm{Top}}$ to $\mathcal{M}$, such that μ P-localizes homotopy groups.

20.2 The localization functor. For every 1-connected space X we fix a CW-approximation $fX : X_1 \longrightarrow X$ and choose a representative $gX : X_1 \longrightarrow X_2$ of $\lambda(X)$. This provides us with a map $iX : X \longrightarrow X\hat{}$, where $X\hat{}$ is the double mapping cylinder of the cotriad (f,g). The inclusion map iX is a cofibration. Given a map $p : X \longrightarrow Y$, we can choose a map $p_1 : X_1 \longrightarrow Y_1$, unique up to homotopy, such that $fY \circ p_1$ is homotopic to $p \circ fX$. Let $p_2 : X_2 \longrightarrow Y_2$ be a representative of the homotopy class $\mathcal{L}(p_1)$. Then diagram A is commutative. This provides us, see 6.7, with a map $p\hat{} : X\hat{} \longrightarrow Y\hat{}$. The homotopy class of $p\hat{}$ is uniquely determined by that of p.

– A –

$$\begin{array}{ccccc} X_2 & \xleftarrow{\;gX\;} & X_1 & \xrightarrow{\;fX\;} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ Y_2 & \xleftarrow[\;gY\;]{} & Y_1 & \xrightarrow[\;fY\;]{} & Y \end{array}$$

For the homotopy class $[p]$ of p, we choose $\mathcal{M}[p]$ to be the homotopy class of $p\hat{}$, and $\mu(X)$ to be the homotopy class of iX. Then we have a functor $\mathcal{M}$ on <u>hTop</u>, and a natural transformation μ from the identity functor of <u>hTop</u> to $\mathcal{M}$. Note that for any maps p_1 and p_2 making diagram A homotopy commutative, diagram B is *strictly* commutative.

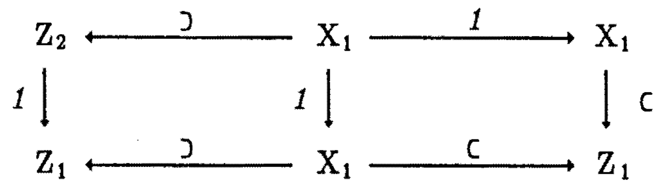
– B –

$$\begin{array}{ccc} X & \xrightarrow{\;iX\;} & X\hat{} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\hat{}} \\ Y & \xrightarrow{\;iY\;} & Y\hat{} \end{array}$$

20.3 Proposition. The natural maps i localize homotopy groups.

Proof Let Z_1 and Z_2 be the mapping cylinders of fX and gX respectively. The push-out of the <u>Top</u>^{2}-cotriad of diagram C is the inclusion map $h : Z_2 \longrightarrow X\hat{}$.

– C –



By **6.8(c)**, the map h is a weak equivalence. The inclusion $j: X_1 \longrightarrow Z_2$ is a P-localizing map and therefore so is $h \circ j$. Furthermore fX is a weak equivalence and $iX \circ fX$ is homotopic to $h \circ j$. Thus iX is a P-localizing map. ∎

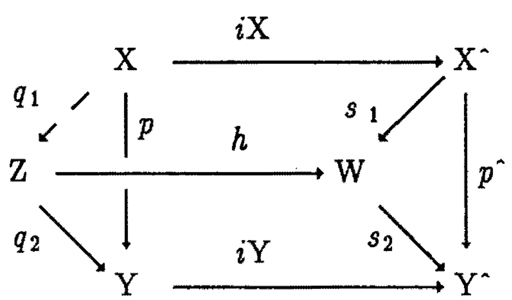
20.4 We now prove ***Lemma* 8.4**, the formulation of which is the following:

Given any map $p: X \longrightarrow Y$ between 1-connected spaces, we can factorize p as the composition of a P-equivalence $q_1: X \longrightarrow Z$ followed by a fibration $q_2: Z \longrightarrow Y$ with P-local fibres.

Proof Consider the commutative diagram B. We form the mapping path fibration factorization of $p\hat{}$, $X\hat{} \xrightarrow{s_1} W \xrightarrow{s_2} Y\hat{}$.

By **18.6** applied to the fibre homotopy sequence of s_2, it follows that the fibres of s_2 are P-local. We choose q_2 to be the pull-back of s_2 over $iY: Y \longrightarrow Y\hat{}$. The unique map φ such that $p = q_2 \circ \varphi$, as guaranteed by pull-back, is taken as q_1.

– D –



Comparison of the fibre homotopy sequences of q_2 and s_2 via the generalized five-lemma shows that $h: Z \longrightarrow W$ is a P-equivalence since the <u>Top</u>^{2}-morphism $(h, iY): q_2 \longrightarrow s_2$ is a P-equivalence of fibres. Thus q_1 is a P-equivalence. ∎

We require generalizations of the Hurewicz isomorphisms theorem and of the Whitehead theorem. We quote the following results without giving proofs.

20.5 The generalized absolute Hurewicz isomorphism theorem [**Hu**; Theorem 8.1 p305]. Let X be a 1-connected space and let $m \geq 2$ be an integer. Suppose that $\pi_i(X) \in \mathcal{C}$ for all $1 < i < m$. Then $H_i(X) \in \mathcal{C}$ for all $1 < i < m$ and the Hurewicz homomorphism $\pi_m(X) \longrightarrow H_m(X)$ is a $\mathcal{C}$-isomorphism while $\pi_{m+1}(X) \longrightarrow H_{m+1}(X)$ is a $\mathcal{C}$-epimorphism. ∎

The localization theory admits a version of the generalized relative Hurewicz isomorphism theorems, stronger than [**Hu**; Theorem 9.1 p306]. It is interesting to note that at the time of publication of [**Hu**], the localization theory for spaces was not known yet.

20.6 The generalized relative Hurewicz isomorphism theorem

Let (Y,X) be a pair of 1–connected spaces and let $m \geq 2$ be an integer. Suppose that $\pi_i(Y,X) \in \mathcal{C}$ for each i in the range $1 < i < m$. Then $H_i(Y,X) \in \mathcal{C}$ for all $1 < i < m$, and the Hurewicz homomorphism $\pi_m(Y,X) \longrightarrow H_m(Y,X)$ is a $\mathcal{C}$-isomorphism. ∎

From **20.6** we can deduce the generalized Whitehead theorem **20.7** below, just like [**Hu**; Theorem 10.1 p307] is deduced from [**Hu**; Theorem 9.1 p306]. We omit the proof.

20.7 The generalized Whitehead theorem

Let $f: X \longrightarrow Y$ be a map of 1-connected spaces and let $m \geq 2$ be an integer. Then the folowing two conditions are equivalent.

(1) The homomorphism $f_*: \pi_i(X) \longrightarrow \pi_i(Y)$ is a $\mathcal{C}$-isomorphism whenever $1 < i < m$ and a $\mathcal{C}$-epimorphism for $i = m$.

(2) The homomorphism $f_* : H_i(X) \longrightarrow H_i(Y)$ is a $\mathcal{C}$-isomorphism whenever $1 < i < m$ and a $\mathcal{C}$-epimorphism for $i = m$. ■

21. ADJUNCTION OF GENERALIZED n-EQUIVALENCES

Let P be a fixed set of primes. Adjunction of P-equivalences are commonplace in the literature. We can mention for example [Sf_3], [HR] and [G_3]. The purpose of this section is to prove Theorem **8.5**. The latter result is a variation of Theorem **6.3**, replacing n-equivalence by (P,n)-equivalences (for definitions see Section **8**).

Let $\mathcal{C}$ be the class of all torsion abelian groups with vanishing p-component for every prime $p \in P$. By $\mathcal{Z}$ we denote the subring of the rationals generated by the set $\{\frac{1}{p} : p$ is a prime and $p \notin P\}$.

21.1 Proposition

Let $\mathcal{K}$ be any Serre class of abelian groups. Suppose that in diagram A, f_1 and g_1 are cofibrations.

– A –

$$\begin{array}{ccccc} E_1 & \xleftarrow{\;g_1\;} & E_0 & \xrightarrow{\;g_2\;} & E_2 \\ \Big\downarrow p_1 & & \Big\downarrow p_0 & & \Big\downarrow p_2 \\ B_1 & \xleftarrow{\;f_1\;} & B_0 & \xrightarrow{\;f_2\;} & B_2 \end{array}$$

Suppose further that the maps p_i induces $\mathcal{K}$-isomorphisms in homology of all dimensions. Then the same is true for the push-out q of the $\underline{\text{Top}}^2$-cotriad of diagram A .

Proof The maps $(E_1,E_0) \longrightarrow (E,E_2)$ and $(B_1,B_0) \longrightarrow (B,B_2)$ of NDR-pairs are excisions. Thus, for every positive integer r, the homomorphisms

$$H_r(E_1,E_0) \longrightarrow H_r(E,E_2) \text{ and } H_r(B_1,B_0) \longrightarrow H_r(B,B_2),$$

are isomorphisms. From the ladder consisting of the exact homology sequences of the pairs (E_1,E_0) and (B_1,B_0), it follows by the generalized five-lemma and our assumptions, that $H_r(E_1,E_0) \longrightarrow H_r(B_1,B_0)$ is a $\mathcal{K}$-isomorphism for every positive integer r. Via the excisions then, $H_r(E,E_2) \longrightarrow H_r(B,B_2)$ is a $\mathcal{K}$-isomorphism for all r. In the homology ladder of the map of pairs $(E,E_2) \longrightarrow (B,B_2)$, it follows by the generalized five-lemma that $H_r(E) \longrightarrow H_r(B)$ is a $\mathcal{K}$-isomorphism for every positive integer r. ▮

21.2 Proposition

Suppose that in diagram A, the maps p_i are all P-equivalences. Then,

(a) the map of double mapping cylinders $p' : E' \longrightarrow B'$ is a P-equivalence.

(b) if moreover f_1 and g_1 are cofibrations, the push-out $p : E \longrightarrow B$ is a P-equivalence.

Proof We prove **(b)** first. By the generalized Whitehead theorem, 20.7, each p_i is a $\mathcal{C}$-isomorphism of homology groups in all dimensions. By 21.1, for every positive integer r, $p_* : H_r(E) \longrightarrow H_r(B)$ is a $\mathcal{C}$-isomorphism. The spaces E and B are 1-connected. Again from 20.7, it follows that $E \longrightarrow B$ is a P-equivalence. This proves **(b)**.

(a) Let $p_3 : E_3 \longrightarrow B_3$ be the map between mapping cylinders arising from with the $\underline{\text{Top}}^2$-morphism $(g_1,f_1) : p_0 \longrightarrow p_1$ shown in Diagram A. Let $g_3 : E_0 \longrightarrow E_3$ and $f_3 : B_0 \longrightarrow B_3$ be the inclusion maps. Then g_3 and f_3 are cofibrations and the map p' coincides with the push-out of the $\underline{\text{Top}}^2$-cotriad shown below.

$$p_3 \xleftarrow{(g_3,f_3)} p_0 \xrightarrow{(g_1,f_1)} p_1$$

Thus from **(b)** it follows that p' is a P-equivalence. ▮

21.3 Remark. **(a)** Having the localization theory of Section 20, we can easily prove analogous to 16.5, the equivalence of the following three conditions for a map $p : (X,A) \longrightarrow (Y,B)$ of pairs of 1-connected spaces, and a positive integer n :

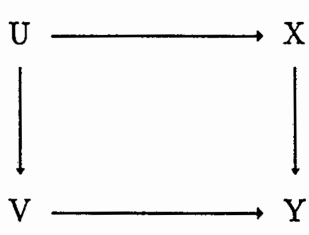
(1) The map $p : (X,A) \longrightarrow (Y,B)$ of pairs is a (P,n+1)-equivalence.

(2) For the map $A \longrightarrow B^{\wedge}$ of A into the inverse image of B with respect to the mapping path fibration, the homomorphism $\pi_r(A) \longrightarrow \pi_r(B)$ is a $\mathcal{C}$-isomorphism for $1 \leq r \leq n-1$, and a $\mathcal{C}$-isomorphism for $r = n$.

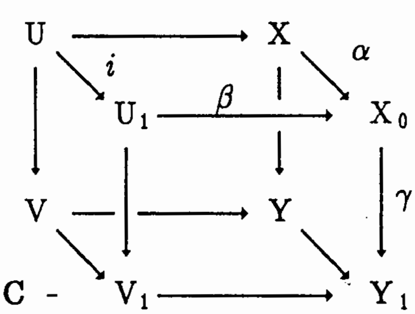
(3) The <u>Top</u>^{2}-morphism of inclusions from $A \longrightarrow B$ to $X \longrightarrow Y$ is a (P,n)-*equivalence of homotopy fibres.*

The latter concept means that for the induced map $H \longrightarrow K$ of homotopy fibres, the homomorphism $\pi_r(H) \longrightarrow \pi_r(K)$ is $\mathcal{C}$-isomorphic for $r \leq n-1$, and $\mathcal{C}$-epimorphic for $r = n$. Note that H and K may fail to be simply connected, but their fundamental groups are abelian, see **8.2(b)**.

(b) Suppose we have a commutative square such as diagram B below. We assume all spaces to be 1-connected. Then the localization theory of Section **20** provides us with a box, diagram C, of which all faces other than the front face, are commutative. The front face is homotopy commutative. $\alpha = iX$.



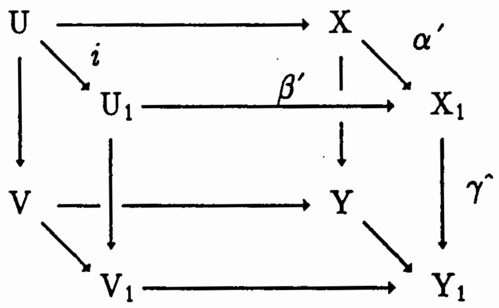
– B –



– C -

We can replace γ by its mapping fibration and adjust the other two maps, α and β, involved here. This yields a diagram D as shown below. Since i is a cofibration and $\gamma^{\wedge}$ is a fibration, by the homotopy lifting extension theorem, see [**Wd**; Theorem 7.16 on p35],

– D –



we can replace the map β', which make the front face homotopy commutative, by a map $\beta^{\sim}$ such that the front face is commutative and $\beta^{\sim}$ is homotopic to β' relative to U. But then the resulting box is strictly commutative and the map α' is a P-equivalence. ∎

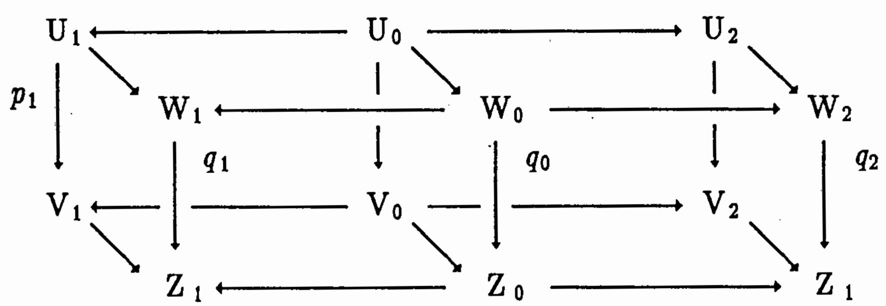
We now prove Theorem 8.5, the $\mathcal{C}$-generalization of [**M₂**; Theorem 1.2].

21.4 We repeat the formulation of ***Theorem*** **8.5**. Let B be a space with open subsets V_1 and V_2 such that $B = V_1 \cup V_2$. Let us denote the set $V_1 \cap V_2$ by V_0. Let $p : E \longrightarrow B$ be a map, and for each $i = \{0, 1, 2\}$, let $U_i = p_1^{-1}(V_i)$. Suppose that for each $j \in \{1, 2\}$, $(U_j, U_0) \longrightarrow (V_j, V_0)$ is a (P,n+1)-equivalence.

Then for each $i \in \{0, 1, 2\}$ the map $(E, U_i) \longrightarrow (B, V_i)$ is a (P,n+1)-equivalence.

Proof of Theorem **8.5**. From **21.3(b)**, for each $j \in \{1, 2\}$ the <u>Top</u>^{2}-morphism from the object $U_0 \longrightarrow V_0$ to $U_j \longrightarrow V_j$ is a (P,n)-equivalence of homotopy fibres. We make the P-localizing construction for a map of pairs as discussed in **21.3 (b)**. The processes for the two maps $(U_j, U_0) \longrightarrow (V_j, V_0)$ can be done simultaneously. This yields a commutative diagram **E** in <u>Top</u>.

– **E** –



Since each of the slanted arrows in E is a P-equivalence, by the generalized five-lemma it follows that the $\underline{\text{Top}}^2$-morphism from $W_0 \longrightarrow Z_0$ to $W_j \longrightarrow Z_j$ is a (P,n)-equivalence of homotopy fibres. The homotopy fibres of the maps q_i are P-local. Therefore the (P,n)-equivalence of homotopy fibres is in fact a n-equivalence of homotopy fibres. We apply Theorem **6.3** to the maps q_i. This asserts that for each $i \in \{0, 1, 2\}$, the $\underline{\text{Top}}^2$-morphism $q_i \longrightarrow q'$ ($q' : W' \longrightarrow Z'$ being the map of double mapping cylinders), is a n-equivalence of homotopy fibres.

Each of the maps of double mapping cylinders $U' \longrightarrow W'$ and $V' \longrightarrow Z'$ of the $\underline{\text{Top}}^2$-cotriads in the top and bottom are P-equivalences by **21.2(a)**. Thus the $\underline{\text{Top}}^2$-morphism from the map of double mapping cylinders p' to q' is a P-equivalence of homotopy fibres. Thus for each $i \in \{0, 1, 2\}$, the morphism $p_i \longrightarrow p'$ is a (P,n)-equivalence of homotopy fibres.

Finally, the natural maps $U' \longrightarrow E$ and $V' \longrightarrow B$ are weak equivalences by **5.9** since the subsets V_i and U_i are open. Thus for each $i \in \{0, 1, 2\}$, the $\underline{\text{Top}}^2$-morphism $p_i \longrightarrow p$ is a (P,n)-equivalence of homotopy fibres. Our result now follows by **21.3(a)**. ▮

Appendix : Toda's maps of reduced products of spheres

This section is devoted to proving the result [T; 2.11] of Toda. The proof of this result is slightly more complicated and we have to employ cohomology and the Wang cohomology sequence of a fibration. Otherwise, the proof follows along the same lines as that of James's results in Section **13**. We shall quote verbatim from [**Wd**] the theorem of Wang's cohomology sequence, A.1. The proof of this theorem is omitted, but we note that the proof in the given source is by fairly elementary methods.

Let us fix a prime p. For an even integer k we let S be the k-dimensional sphere, and T the pk-dimensional sphere. For an integer q we denote by P_q the set of all primes bigger than $p(q+1)-1$ together with p. Let $\mathbb{Z}_q$ be the subring of $\mathbb{Q}$ generated by the reciprocals of the integers which are relatively prime to the elements of P_q. Spaces and maps are considered to be pointed.

Consider the following map, which is the composition of a collapsing map followed by a homeomorphism :

$$(1) \qquad S_{qp} \longrightarrow S_{qp} / S_{qp-1} \equiv S^{qpk}.$$

We shall denote the spaces S_{qp+p-1}, S_{qp-1} and S^{qpk} by X, Y and Z respectively. Let $h : X \longrightarrow Z$ be any map extending the map in (1) above.

A.1 **Theorem** [Wd; Corollary 1.2 p317]

If $p : E \longrightarrow S^m$ is a fibration with fibre F, then for any abelian group $\mathcal{G}$ there is an exact sequence as below, in which i^* is an injection.

$$\to H^{q-1}(F;\mathcal{G}) \xrightarrow{\theta^*} H^{q-m}(F;\mathcal{G}) \longrightarrow H^q(X;\mathcal{G}) \xrightarrow{i^*} H^q(F;\mathcal{G}) \xrightarrow{\theta^*} \quad \blacksquare$$

The next result **A.2**, also quoted without proof from [**Wd**], is required to compute the integral cohomology ring of S_r. We describe the cohomology in **A.3** without showing the computations.

A.2 Theorem [**Wd**; Theorem 1.12 p319]

If in **A.1**, $\mathcal{G}$ is the additive group of a commutative ring with unit, then θ^* is a derivation of the graded ring $H^*(F;\mathcal{G})$. ■

A.3 The cohomology of S_r [**Wd**; 324–326]

The abelian group $H^*(S_\infty)$ is freely generated by a set $\{x_0, x_1, x_3, \dots\}$, $x_i \in H^{ki}(S_\infty)$ with $x_0 = 1$, and the cup product multiplication $(\cdot)$, satisfies the relations :

$$x_q \cdot x_m = \binom{q+m}{q} x_{q+m} .$$

In particular, we have : $x_1{}^q = q!\, x_q$. For a positive integer r, the cohomology algebra of S_r is the quotient algebra obtained from $H^*(S_\infty)$ by mapping every element of gradation bigger than rk onto zero. Thus the abelian group $H^*(S_\infty)$ is freely generated by a set :

$$\{ a_0, a_1, a_3, \dots , a_r \} ,$$

satisfying similar (truncated) cup product identities. ■

A.4 Proposition

The map $\eta : (X,Y) \longrightarrow (Z,*)$ induced by h is a $(P_{qp}, (q+1)pk - 1)$-equivalence.

Proof We can assume that h is a fibration with fibre F over $*$ and such that $Y \subset F$. From **13.4** we know that η is a $(P_q, pqk + k - 1)$-equivalence. From this it follows that the inclusion $Y \subset F$ is an $(P_q, pqk + k - 2)$-equivalence. By the generalized Whitehead theorem, it follows that $H_r(F;\mathbb{Z}_q) = 0$ whenever $p(q-1)k + 1 \leq r \leq (pq+1)k - 2$.

In particular, $H_{qpk-1}(F;\mathbb{Z}_q) = 0 = H_{qpk}(F;\mathbb{Z}_q)$. Thus by the universal coefficient theorem it turns out that $H^{qpk}(F;\mathbb{Z}_q) = 0$. So for the injection

$$i^* : H^*(X;\mathbb{Z}_q) \longrightarrow H^*(F;\mathbb{Z}_q),$$

we have that $i^*(x_{qp}) = 0$. But then for every $0 < r < p$, we have that $(qp + r)^{-1} \in \mathbb{Z}_q$. In particular,

$$x_{qp+1} = (qp + 1)^{-1} x_{qp} \cdot x_1 ,$$

Furthermore since i^* is a ring homomorphism, $i^*(x_{qp+1}) = 0$. Similarly, by induction we obtain $i^*(x_{qp+r}) = 0$ whenever $0 < r < p$. This means that i^* is the zero homomorphism in dimensions $qpk - 1$ up to $(q + 1)pk - 1$. So the cohomology Wang sequence yields short exact sequences,

$$H^{t-1}(F;\mathbb{Z}_q) \longrightarrow H^{t-qpk}(F;\mathbb{Z}_q) \longrightarrow H^t(X;\mathbb{Z}_q),$$

for $qpk \leq t \leq (q + 1)pk - 1$. Since the $\mathbb{Z}_q$-modules $H^t(X,\mathbb{Z}_q)$ are free, the sequences are split and result in isomorphisms,

$$H^{t-qpk}(F;\mathbb{Z}_q) \cong H^t(X;\mathbb{Z}_q) \oplus H^{t-1}(F;\mathbb{Z}_q).$$

The modules $H^{t-qpk}(F;\mathbb{Z}_q)$ and $H^t(X;\mathbb{Z}_q)$ are isomorphic and are finitely generated. Thus $H^{t-1}(F;\mathbb{Z}_q) = 0$. So in the Wang cohomology sequence we have that

$$\alpha^* : H^{t-qpk}(F;\mathbb{Z}_q) \rightarrow H^t(X;\mathbb{Z}_q)$$

is an isomorphism. By the universal coefficient theorem, it follows that

$$\alpha_* : H_t(X;\mathbb{Z}_q) \longrightarrow H_{t-qpk}(F;\mathbb{Z}_q)$$

is an isomorphism. Thus $H_{t-1}(F;\mathbb{Z}) = 0$. So,

$$H_t(F;\mathbb{Z}) = 0 \text{ for } (q - 1)pk + 1 \leq t < (q + 1)pk - 2 .$$

Again by the generalized Whitehead theorem, it follows that the inclusion map $Y \subset F$ is a $(P_q,\ qpk + pk - 2)$-equivalence, and our result follows. ▮

From here on, the proof of Toda's result follows almost verbatim as did James's. We made a special case of James's since the proof of the latter does not require the cohomology algebra or the Wang exact sequence.

A.5 Proposition

Let $f_q : (X, S_{p-1}) \longrightarrow (T_q, *)$ be any map extending the following map

$$S_p \longrightarrow S_p / S_{p-1} \equiv T \subset T_q .$$

Then f is a P_q-equivalence. ▪

REFERENCES

[Ag] Aguade, J. *Fiberings of spheres by spheres mod p.* Quart. J. Math. Oxford (2) **31** (1980) 129–137.

[An] Anderson, D.W. *Fibrations and geometric realization.* Bull. Amer. Math. Soc. **85** (5) (1978) 765–788.

[AP] Arkhangel'skii, A.V. and Ponomarev, V.I. *Fundamentals of General Topology, Problems and Exercises.* D. Reidel Publishing Company; Dordrecht Holland 1984.

[Ba] Baues, H.J. *Relative Homotopiegruppen bei Abbildungskegeln.* Compositio Mathematica **32** (2) (1976) 193–183.

[Br] Brown, R. *Topology: A Geometric Account of General Topology, Homotopy Types and The Fundamental Groupoid.* Ellis Horwood Limited; Chichester West Sussex England 1988.

[BH] Brown, R. and Higgins, P.J., *Colimit theorems for relative homotopy groups.* J. Pure Appl. Alg. 22 (1981) 11–41.

[BM_1] Blakers, A.L. and Massey, W.S. *The homotopy groups of a triad* II. Ann. of Math. **55** (2) (1952) 192–201.

[BM_2] Blakers, A.L. and Massey, W.S. *Products in homotopy theory.* Ann. of Math. 58 (2) (1952) 295–324.

[CS] Cartan, H. and Serre, J.-P. *Espaces fibres et groupes d'homotopie I. Constructions generales.* C. R. Acad. Sci. Paris **234** (1952) 288–290.

[CW] Caruso, J. and Waner, S. *An approximation to* $\Omega^n\Sigma^n X$. Trans. Amer. Math. Soc. **265** (1) (1981) 147–162.

[DKP] tom Dieck, T., Kamps, K. H. and Puppe, V., *Homotopietheorie*, Lecture Notes in Mathematics **157**, Springer–Verlag Berlin 1970.

[DL] Dold, A. and Lashof, R. *Principal quasifibrations and fibre homotopy equivalence of bundles.* Illinois J. Math. **3** (1959) 285–305.

[DE] Dyer, E. and Eilenberg, S. *Globalizing fibrations by schedules.* Fund. Math. **130** (1988) 125–136.

[Ed] Edwards, D.A. *The shape of a map.* Fund. Math. **96** (**3**) (1977) 195–210.

[Ei] Eilenberg, S. *On the relation between the fundamental group of a space and the higher homotopy groups.* Fund. Math. **32** (1939) 167–175.

[Fe] Ferry, S.C. *UV^k-equivalent compacta.* Lecture Notes in Mathematics **1283** (1987) 88–114.

[Fo] Fox, R.H. *On homotopy type and deformation retracts.* Ann. of Math. **44** (2) (1943) 40–50.

[G_1] Gray, B. *Homotopy theory: An Introduction to Algebraic Topology.* Academic Press; New York 1975.

[G_2] Gray, B. *On the homotopy groups of mapping cones.* Proc. London Math. Soc. (3) **26** (1973) 497–520.

[G_3] Gray, B. *On the iterated suspension.* Topology 27 (3) (1988) 301–310.

[Ha] Hardie, K.A. *Quasifibration and adjunction.* Pacific. J. Math. **35** (1970) 389–397.

[HK] Hardie, K.A. and Kamps, K.H. *Exact sequence interlocking and free homotopy.* Cahiers de Topologie et Geometrie Differentielle Categoriques Vol XXVI-1 (1985) 3–31.

[HP] Hardie, K.A. and Porter, G.H. *The slash product homotopy operation.* Proc. London Math. Soc., third series 11 (**3**) (1961) 588–600.

[HH] Hartley, B. and Hawkes, T.O. *Rings, Modules and Linear Algebra.* Chapman and Hall Mathematical Series, Chapman and Hall Ltd.; London 1970.

[HMR] Hilton, P., Mislin, G. and Roitberg, J. *Localization of nilpotent groups and spaces.* North-Holland; 1975.

[HR] Hilton, P., and Roitberg,. J. *Note on quasifibrations and fibre bundles.* Illinois J. Math. **15** (1) (1971) 1–8.

[Hu] Hu, S.-T. *Homotopy theory.* Academic Press; New York 1959.

[HM] Hubbuck, J.R. and Mimura, M. *The number of mod p A(p)–spaces.* Illinois J. Math. **33** (1) (1989) 162–169.

[Hh] Huebsch, W. *On the covering homotopy theorem.* Ann. of Math. **61** (1955) 555–563.

[Hz] Hurewicz, W. *On the concept of a fibre space.* Proc. Nat. Acad. Sci. U.S.A. **41** (1955) 956–961.

[Hr] Husemoller, D. *Fibre bundles.* Graduate Texts in Mathematics **20**, Springer-Verlag; New York, Heidelberg, Berlin 1975.

[J_1] James, I.M. *Reduced product spaces.* Ann. of Math. **62** (1955) 170–197.

[J_2] James, I.M. *The suspension triad* of *a sphere.* Ann. of Math. **63** (1956) 407–429.

[JT] James, I.M. and Thomas, E. *An approach to the enumeration problem for non-stable vector bundles.* Journal of Mathematics and Mechanics **14** (**3**) (1965) 485–506.

[Ma] Massey, W.S. *Algebraic Topology, an Introduction,* (4th corrected printing). Graduate Texts in Mathematics 70, Springer-Verlag; New York 1977.

[M_1] May, J.P. *Classifying spaces and fibrations.* Memoirs. Amer. Math. Soc. **155**; Providence 1975.

[M_2] May, J.P. *Weak equivalence and quasifibration,* Piccinini, R. (ed.), *Groups of Self–Homotopy Equivalences and Related Topics,* Lecture Notes in Mathematics **1425**, Springer-Verlag; Berlin 1990.

[Mc] McCord, M.C. *Singular homology groups and homotopy groups of finite topological spaces.* Duke Mathematical Journal **33** (1966) 465–474.

[Me] Meyer, J.-P. *Bar and cobar constructions* II. Journal of Pure and Applied Algebra **43** (**2**) (1986) 179–210.

[Mi] Mimura, M. *On the mod p H–structures of spherical fibrations.* Manifolds Tokyo 1973, edited by Akio Hattori, Tokyo University Press; 1973.

[Mo] Moore, J.C., *Some applications of homology theory to homotopy theory.* Ann. of Math. **58** (1953) 325–350.

[MP] Morgan, C. and Piccinninni, R. *Fibrations.* Expo. Math. **4** (1986) 217–242.

[Ms] Munkres, J.R. *Topology, a first course.* Prentice-Hall Inc.; Englewood Cliffs New Jersey 1975.

[N] Namioka, I. *Maps of pairs in homotopy theory.* Proc. London Math. Soc. **12** (**3**) (1962) 725–738.

[Po] Porter, R.D. *Introduction to Fibre Bundles.* Lecture Notes in Pure and Applied Mathematics Vol **31**, Marcel Dekker, Inc.; New York, Basel 1977.

[Pu] Puppe, V. *A remark on homotopy fibrations.* Manuskripta Math. **12** (1974) 113–130.

[Q_1] Quillen, D.G. *Higher algebraic K–theory* I. Lecture Notes in Mathematics Vol. **341**, Springer–Verlag; Berlin (1973) 85–147.

[Q_2] Quillen, D.G. *Homotopy Properties of the Poset of nontrivial p–Subgroups of a Group.* Advances in Math. **28** (1978) 101–128.

[Ro] Rose, J.S. *A Course on Group Theory.* Cambridge University Press; London 1978.

[Ru] Rutter, J.W. *A homotopy classification of maps into an induced fibre space.* Topology **6** (1967) 379–403.

[Se] Serre, J.-P. *Groupes d'homotopie et classes de groupes abeliens.* Ann. of Math. **58** (**2**) (1953) 258–294.

[Sr] Spanier, E.H. *Algebraic topology.* McGraw Hill Inc.; New York 1966.

[Ss] Srinivas, V. *Algebraic K–theory.* Progress in Mathematics Volume **90**, Birkhauser; Boston, Basel, Berlin 1991.

[Sf_1] Stasheff, J.D. *A classification theorem for fibre spaces.* Topology **2** (1963) 239–246.

[Sf_2] Stasheff, J.D. *Homotopy associativity of H–spaces* I. Trans. Amer. Math. Soc. **108** (1963) 275–292.

[Sd_1] Steenrod, N.E. *The classification of sphere bundles.* Ann. of Math. **45** (1944) 295–311.

[Sd_2] Steenrod, N.E. *The topology of fibre bundles.* Princeton Univ. Press; Princeton New Jersey 1951.

[Sd_3] Steenrod, N.E. *A convenient category of topological spaces.* Michigan Math. J. 14 (1967) 133–152.

[Sa] Sugawara, M. *On a condition that a space is an H–space.* Math. J. Okayama Univ. 6 (1957) 109–129.

[Sn] Sullivan, D. *Geometric topology part I, Localization, Periodicity, and Galois symmetry,* MIT (June 1970), (mimeographed notes)

[Sw] Switzer, R.M, *Algebraic Topology - Homotopy and Homology,* Springer-Verlag, Berlin and New York 1975.

[T] Toda, H. *On the double suspension* E^2. J. Inst. Poly. Osaka City Univ. 7 (1956) 103–145.

[Wa] Waner, S. *Equivariant classifying spaces and fibrations.* Trans. Amer. Math. Soc. **258** (2) (1980) 385–405.

[WW] Weidner, M. and Welker, V. *The combinatorics and the homology of the poset of subgroups of p-power index.* Journal of Pure and Applied Algebra 90 (1993) 253–274.

[Wd] Whitehead, G.W. *Elements of homotopy theory.* Graduate Texts in Mathematics **61**, Springer–Verlag; New York 1978.

[W_1] Whitehead, J.H.C. *Simplicial spaces, nuclei and m–groups.* Proc. London Math. Soc. **45** (2) (1939) 243–327.

[W_2] Whitehead, J.H.C. *The Mathematical Works of J.H.C. Whitehead, vol II: Complexes and manifolds.* Pergamon Press; London, New York 1962.

[Wy] Whitney, H. *Sphere spaces.* Proc. Nat. Acad. Sci. USA **21** (1935) 464–468.

[Wong] Wong, Shiu-chun. *The fibre of the iterated Freudenthal suspension.* Math. Z. **215** (1994) 377–414.