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# Solutions in multi-actor projects with collaboration and strategic incentives 

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# Solutions in multi-actor projects with collaboration and strategic incentives 

ANDRIES VAN BEEK

# Solutions in multi-actor projects with collaboration and strategic incentives 

Proefschrift<br>ter verkrijging van de graad van doctor aan Tilburg University op gezag van de rector magnificus, prof. dr. W.B.H.J. van de Donk, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de Aula van de Universiteit op

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Solutions in multi-actor projects with collaboration and strategic incentives

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## Acknowledgments

Thank you for opening this dissertation and browsing through the first few pages. Do not worry, you do not actually have to read the full thesis. I will not quiz you. I also summarized (in English and in Dutch) my four years of work in four and two pages, respectively, for academics and for non-experts. I will let you decide yourselves which category you belong to. Apologies in advance if I forgot your name in the upcoming two pages. Please let me know if you would like me to return your gift.

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## Introduction

This dissertation focuses on the mathematical analysis of projects involving decisions by multiple actors. Projects involve a set of different parties, firms, or stakeholders, often referred to as players. These players all have their own capabilities, requirements, and incentives, but their (monetary) outcome is dependent on the decisions of other players as well. Game theory is a mathematical tool to analyze the interactive decision-making process, generally paired with a method to 'resolve' the conflict situation. The way in which players interact in such a situation is commonly divided in two categories, distinguishing between cooperative and competitive (non-cooperative) behavior. The different models in this dissertation follow a similar division between collaborative projects and problems with strategic individual behavior, or a combination thereof. Though in most chapters of this dissertation games are not explicitly considered, all models are in some way related to game theory.

Models within a cooperative framework study situations in which groups of players can cooperate by reaching a mutual agreement on a joint plan of action to maximize their joint payoff (see, e.g., Peleg and Sudhölter (2007)). This is generally paired with a specification of how to allocate this payoff. Cooperative games with transferable utility assign a (joint) value to every possible subset of players (called 'coalitions'), not only to the group of players as a whole (the 'grand' coalition). In principle, this game serves as a conservative and consistent benchmark to properly address the allocation problem for the grand coalition, taking into account coalitional incentives. By 'solving' the game, one finds allocations of the total joint value of the grand coalition to the players.

This stands in contrast to non-cooperative models, in which strategic players are interested in maximizing their individual payoffs, taking into account the strategic behavior of other players based on individual incentives only (see, e.g., Fujiwara-Greve (2015)). Regarding solutions, (variants of) Nash equilibria (Nash, 1951) are the main topic of interest.

Chapters 2 and 3 are written within a cooperative framework. Specifically, Chapter 2 considers projects consisting of a number of tasks to be carried out by a set of players. Each task can only be carried out by a subset of all players, so players may have to cooperate to complete a project. In this context, a solution represents a measure of influence each player has on the completion of the project. In Chapter 3, players cooperate on the construction of a new infrastructure. Collaborating on a joint infrastructure that meets the requirements of all players, rather than constructing a separate infrastructure for each individual player, leads to cost savings. Then, a solution provides a way to allocate the joint construction costs to the players.

Chapter 4 considers two-stage models, in which a non-cooperative first stage is followed by a cooperative second stage. Conceptually, the specific format of the cooperative stage is determined by strategic decisions in the first stage. An example of such a 'biform' model is given in Example 1.2.3.

Chapters 5 and 6 analyze non-cooperative models. The former is concerned with auctions, in which players compete by strategically submitting their (sealed) bids to obtain items. The solution considered is that of a Nash equilibrium, with a specific focus on the efficiency of an equilibrium in utilitarian welfare terms, i.e., comparing the total valuation of the bidders in an equilibrium with their maximal total valuation in an optimal assignment of items to players. Finally, Chapter 6 analyzes a general class of non-cooperative games, called bimatrix games, focusing on a new solution concept that refines the notion of Nash equilibria.

Societal relevance is clear from numerous practical situations with multi-actor decisionmaking corresponding to the (theoretical) models developed in the different chapters of this dissertation, both in a cooperative and a non-cooperative setting.

For example, Chapter 3 presents a model for collaborative infrastructure construction projects, developed primarily with a practically relevant application in mind: CO2 transport infrastructure for industrial decarbonization. We extensively study a concrete case of a prospective CO2 transport infrastructure for carbon capture, uti-
lization and storage in the port of Rotterdam and the adjoining industry area. An important element of this model is the way in which it incorporates heterogeneity in the requirements of potential users of this infrastructure. Through this, and using a model inspired by cooperative game theory, we propose a well-substantiated method to allocate the total infrastructure construction costs to the users. Appropriate cost allocation, such that all players can only benefit from collaboration on the infrastructure construction project, can be a key enabler for the successful realization of such projects.

As another example, in a non-cooperative setting, Chapter 5 considers auctions with a corrupt auctioneer. Corruption in auctions, where auctioneers manipulate the submitted bids (referred to as 'bid rigging') to their own benefit, occurs in practice, especially in the public sector. However, even though this bid rigging has been studied and observed in practice, its impact on social welfare is still poorly understood. We contribute to this understanding by initiating the study of social welfare loss caused by corrupt auctioneers in fundamental auction settings.

### 1.1 Overview

This section briefly summarizes the five main chapters of the dissertation, each of which is based on a separate research article.

In Chapter 2, based on Van Beek et al. (2021), we define and axiomatically characterize a new proportional influence measure for sequential projects with imperfect reliability. We consider a model in which a finite set of players aims to complete a project, consisting of a finite number of tasks, which can only be carried out by certain specific players. Moreover, we assume the players to be imperfectly reliable, i.e., players are not guaranteed to carry out a task successfully. To determine which players are most important for the completion of a project, we use a proportional influence measure, where players' influence on the completion of each task within the project is measured in proportion to the likelihood that they complete it successfully. This chapter provides two characterizations of this influence measure. The most prominent property in the first characterization is task decomposability. This property describes the relationship between the influence measure of a project and the measures of influence one would obtain if one divides the tasks of the project over multiple independent smaller projects. Invariance under replacement is the most prominent property of the
second characterization. If in a certain task group a specific player is replaced by a new player who was not in the original player set, this property states that this should have no effect on the allocated measure of influence of any other original player.

Chapter 3, based on Van Beek et al. (2023a), provides a multi-actor perspective on the realization of new infrastructures, motivated by the necessity for infrastructures to support the ongoing climate and energy transition in general, and CO2 transport infrastructures for carbon capture, utilization and storage in particular. We develop a general model to represent infrastructures that allows for a unique decomposition into 'elementary infrastructure components' based on heterogeneous user requirements. Notably, it incorporates a cost function with a very generic and adaptable structure, for which we can still explicitly determine the costs of each individual component. As a direct consequence an intuitive cost allocation rule is obtained: equal component cost sharing. This allocation rule is in line with existing game-theoretic concepts and satisfies the desirable properties of advantageous scaling and coalitional rationality. Advantageous scaling guarantees that the costs allocated to each existing user do not increase if the number of users grows larger and coalitional rationality ensures that there is no subgroup of infrastructure users that would have a financial reason to object to the cost allocation. Additionally, we examine the application of our model to a prospective CO2 transport infrastructure for CCUS in the port of Rotterdam and the adjoining industry area.

Chapter 4, based on Van Beek et al. (2023b), analyzes applications of biform games to linear production (LP) and sequencing processes. Biform games apply to problems in which strategic decisions are followed by a cooperative stage, where the specific format of the cooperative stage is determined by these strategic decisions. The cooperative stage corresponding to a strategy combination is then 'solved', leading to a unique payoff allocation vector. By associating a payoff vector with each possible strategy combination, the induced strategic game is determined. In biform LP-processes, we allow firms to compete for resources, rather than assuming the resource bundles are simply given. With strategy dependent resource bundles that can be obtained from two locations, we show that the induced strategic game has a pure Nash equilibrium, using the Owen set or any game-theoretic solution concept that satisfies anonymity to solve the second-stage cooperative LP-game. In biform sequencing processes, we no longer assume an initial processing order is given. Instead, this initial order is strate-
gically determined by allowing players to request their preferred position in the initial order. Solving the second-stage cooperative sequencing game using a gain splitting rule, we fully determine the set of pure Nash equilibria of the induced strategic game.

In Chapter 5, based on Van Beek et al. (2022), we initiate the study of the social welfare loss (in utilitarian welfare terms) caused by corrupt auctioneers, both in single-item and multi-unit auctions. In our model, the auctioneer may collude with the winning bidders by letting them lower their bids in exchange for a (possibly bidder-dependent) fraction $\gamma \in[0,1]$ of the surplus: the difference between their bid and the highest losing bid. We consider different corruption schemes. In the most basic one, all winning bidders lower their bid to the highest losing bid. We show that this setting is equivalent to a $\gamma$-hybrid auction in which the payments are a convex combination of first-price and second-price auction payments. More generally, we consider corruption schemes that can be related to $\gamma$-approximate first-price auctions ( $\gamma-F P A$ ), where the payments recover at least a $\gamma$-fraction of the first-price payments. Our goal is to obtain a precise understanding of the robust price of anarchy of such auctions. If no restrictions are imposed on the bids, we establish a bound on the robust price of anarchy of $\gamma$-FPA which is tight for the single-item and the multi-unit auction setting. On the other hand, if bidders cannot overbid, a more fine-grained landscape of the price of anarchy emerges, depending on the auction setting and the equilibrium notion. Interestingly, we derive (almost) tight bounds for both auction settings and both pure Nash equilibria and coarse correlated equilibria.

Finally, Chapter 6 proposes a new refinement of Nash equilibria for bimatrix games. Most existing refinements are based on a thought experiment which imposes a certain 'imperfection' on the choices or payoffs of individual players. The equilibrium refinement proposed in this chapter deviates from the existing refinements by considering a thought experiment in which the imperfections occur on a 'system' level, instead of those corresponding (directly) to individual players. Imperfections are interpreted as the blocking of actions. If an imperfection occurs, the chosen actions are blocked for all players simultaneously, rather than for individual players. The idea behind this is that, after players submit their strategies, some entity converts these strategies into actions leading to payoffs. In this new thought experiment, with small probability, this entity makes an error that blocks the chosen actions instead of implementing them, and chooses a random combination of the remaining actions. Put differently, either
the chosen actions are executed for all players, or no player actually plays their chosen action. In this way, there is an entanglement in the errors. We therefore refer to an equilibrium based on this thought experiment as an entangled equilibrium. Focusing on bimatrix games, we show that the set of entangled equilibria is a non-empty subset of the set of (mixed) Nash equilibria. Further, we discuss a geometric-combinatorial approach to determine all entangled equilibria of $2 \times n$ bimatrix games. Importantly, solving a $2 \times n$ bimatrix game for entangled equilibria requires relatively little extra work compared to finding Nash equilibria for the bimatrix game.

### 1.2 Preliminaries

This section introduces some of the basic notation and fundamental concepts used throughout this dissertation, distinguishing between cooperative and non-cooperative models, and restricting to preliminaries required in more than one chapter. Further chapter-specific preliminaries are provided in the corresponding chapters.

### 1.2.1 Cooperative models

Let $N$ denote a finite and non-empty set of players. Then, $x \in \mathbb{R}^{N}$ denotes a (column) vector of $|N|$ real numbers, specifying for each player $i \in N$ a real number $x_{i}$. Further, $e_{i}$ with $i \in N$ denotes the unit vector in $\mathbb{R}^{N}$ of which the $i$-th element equals one and all other elements are equal to zero.

Within the framework of cooperative models, a common goal is to determine an allocation of the joint value of the player set $N$ as a whole to the individual players, for which some solution concept, or simply 'solution', is used. In Chapters 2, 3, and 4 , solution concepts will be defined on different domains.

A prominent example of such a domain is the class of (cooperative) transferable utility games. For this, let $2^{N}$ denote the collection of subsets of $N$. The non-empty subsets of $N$ are referred to as coalitions, and $N$ is called the grand coalition. A transferable utility game (TU-game) is a tuple ( $N, v$ ), where $v: 2^{N} \rightarrow \mathbb{R}$ is referred to as the characteristic function. The characteristic function is typically based on specific modeling assumptions. By convention, $v(\emptyset)=0$. The number $v(S)$ in principle provides the highest total value (often monetary, e.g., payoff or net profit) a coalition $S \in 2^{N} \backslash\{\emptyset\}$ can jointly generate without the help of the players in $N \backslash S$. The class of TU-games with player set $N$ is denoted by $T U^{N}$.

On this domain, the Shapley value (Shapley, 1953) is a one-point solution concept that assigns to each TU-game ( $N, v$ ) with some fixed player set $N$ a unique (payoff) vector $\Phi(v) \in \mathbb{R}^{N}$. The Shapley value is defined as the average of the so-called marginal vectors. To find these marginal vectors, we consider all possible orders in which cooperation between the players is established. Formally, such an order is described by a bijection $\sigma:\{1,2, \ldots,|N|\} \rightarrow N$, where $\sigma(k)$ is the player in position $k$ in the order. The collection of all orders is denoted by $\Pi(N)$.

Let $\sigma \in \Pi(N)$ and let $v \in T U^{N}$. Then, the corresponding marginal vector, $m^{\sigma}(v) \in \mathbb{R}^{N}$, is defined by

$$
m_{\sigma(k)}^{\sigma}(v)=v(\{\sigma(1), \ldots, \sigma(k-1), \sigma(k)\})-v(\{\sigma(1), \ldots, \sigma(k-1)\})
$$

for any $k \in\{1, \ldots,|N|\}$. The Shapley value $\Phi: T U^{N} \rightarrow \mathbb{R}^{N}$ is defined as the average of all marginal vectors. Formally,

$$
\Phi(v)=\frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v)
$$

for all $v \in T U^{N}$.
As a special case, let $v \in T U^{N}$ with $N=\{1,2\}$. Then, we have

$$
\begin{equation*}
\Phi_{i}(v)=v(\{i\})+\frac{v(N)-v(\{1\})-v(\{2\})}{2} \tag{1.1}
\end{equation*}
$$

for any $i \in N$.

To illustrate the Shapley value, consider an example of so-called glove games.

## Example 1.2.1

The concept of glove games is simple: players have a number of left-hand and righthand gloves, and each pair of gloves (i.e., one left-hand and one right-hand glove) has a certain value. An individual glove does not have value. By cooperating, players can form pairs of gloves together and share the joint payoff. Concretely, let $N=\{1,2,3\}$ and suppose players 1 and 2 both have three left-hand gloves and six right-hand gloves, where player 3 has eighteen left-hand gloves and six right-hand gloves. Each pair of gloves is worth 1 . Then, the corresponding glove game $v$ is determined by the number of pairs of gloves a coalition can generate, as given in Table 1.1.

| $S$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v(S)$ | 3 | 3 | 6 | 6 | 12 | 12 | 18 |

Table 1.1 The glove game $v$ of Example 1.2.1

Suppose the total joint payoff of 18 for the grand coalition is allocated to the players using the Shapley value of $v$. Note that there are $3!=6$ possible orders in $\Pi(N)$.

For example, consider $\sigma=\left(\begin{array}{ll}2 & 3\end{array}\right)$. Player 2 is first in the order, and receives $m_{2}^{\sigma}(v)=v(\{2\})-v(\emptyset)=3-0=3$. Then, player 3 joins and is assigned $m_{3}^{\sigma}(v)=v(\{2,3\})-v(\{2\})=12-3=9$. Finally, for player 1 we have $m_{1}^{\sigma}(v)=v(N)-v(\{2,3\})=18-12=6$.

In this way, one can determine all six marginal vectors, as given in Table 1.2.
$\left.\begin{array}{c|ccc}\sigma & m_{1}^{\sigma}(v) & m_{2}^{\sigma}(v) & m_{3}^{\sigma}(v) \\ \hline\left(\begin{array}{lll}1 & 2 & 3\end{array}\right) & 3 & 3 & 12 \\ (1 & 3 & 2\end{array}\right)$

Table 1.2 Marginal vectors for all orderings corresponding to the glove game $v$ of Example 1.2.1

Determining the average of these marginals vectors, we obtain that

$$
\Phi(v)=\frac{1}{6}(27,27,54)=(4.5,4.5,9) .
$$

### 1.2.2 Non-cooperative models

This section focuses on strategic games in normal form. Such games are denoted by $G=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}\right\}_{i \in N}\right)$, where $N$ (again) represents a finite set of players and the strategy set of player $i$ is denoted by $X^{i}$ for all $i \in N$. The set of all strategy combinations is $X=\Pi_{i \in N} X^{i}$. Let $x \in X$ and let $i \in N$. The strategy player $i$ chooses is denoted by $x^{i} \in X^{i}$ and the strategy combination chosen by all other players in $N \backslash\{i\}$ is denoted by $x^{-i} \in X^{-i}$, with $X^{-i}=\Pi_{j \in N \backslash\{i\}} X^{j}$. For any player $i \in N$, $\pi_{i}: X \rightarrow \mathbb{R}$ is the payoff function of this player.

The most prominent solution concept for such strategic games is the concept of Nash equilibria (Nash, 1951). A strategy combination $x \in X$ is a Nash equilibrium of a strategic game $G=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}\right\}_{i \in N}\right)$ if

$$
\pi_{i}(x) \geq \pi_{i}\left(y^{i}, x^{-i}\right)
$$

for all $i \in N$ and all $y^{i} \in X^{i}$. In words, a strategy combination is a Nash equilibrium if no player has an incentive to unilaterally deviate (i.e., change strategy, given the strategies of all other players). The set of all Nash equilibria of $G$ is denoted by $E(G)$.

## Example 1.2.2

Consider a situation in which two different stores give away gloves. Let $N=\{1,2,3\}$ and suppose these players compete to obtain gloves at these stores. At store 1, there are six left-hand gloves and twelve right-hand gloves. At store 2, there are eighteen left-hand gloves and six right-hand gloves. However, the stores are far apart, so players can only visit one of the two. Hence, $X^{i}=\{1,2\}$ for all $i \in N$. If multiple players choose to visit the same store, the owner divides the gloves available at this store over these players, giving an equal number of left-hand and right-hand gloves to each player.

For example, consider $x=(1,1,2)$, i.e., players 1 and 2 both go to the first store and obtain three left-hand gloves and six right-hand gloves each, while player 3 visits the second store and obtains eighteen left-hand gloves and six right-hand gloves. Players 1 and 2 can both form three pairs of gloves, while player 3 can form six pairs of gloves. Since each pair of gloves is worth 1, the resulting payoff vector is $\pi(x)=(3,3,6)$. In this way, the payoffs corresponding to each strategy combination are readily determined, as presented in Table 1.3. For such tables, the row always represents the strategy of player 1 , where player 2 chooses a column, and the matrix is determined by the choice of player 3 .


Table 1.3 The strategic game $G$ of Example 1.2.2

Clearly, $(1,1,1) \notin E(G)$, since, e.g., $\pi_{1}(1,1,1)<\pi_{1}(2,1,1)$ implies that player 1 has
an incentive to unilaterally deviate. Similarly, $(2,2,2) \notin E(G)$. All other strategy combinations are Nash equilibria of the strategic game $G$, i.e.,

$$
E(G)=\{(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1)\} .
$$

The following example illustrates how cooperative and non-cooperative game theory can be combined in a single two-stage model, by defining a biform glove game. Specifically, strategic decisions of choosing a store are made by the players in the non-cooperative first stage, after which the player can cooperate and form pairs of gloves together in the cooperative second stage. By allocating the total joint payoff of the grand coalition in the second stage, one assigns a payoff vector to each strategy combination, thereby defining the induced strategic game. This is a simple example of a biform model as considered in Chapter 4.

## Example 1.2.3

Consider the situation of Example 1.2.2, where, for any $i \in\{1,2,3\}, X^{i}=\{1,2\}$ represents the strategic choice of $i$ for a store that gives away gloves, where at store 1 there are six left-hand gloves and twelve right-hand gloves and at store 2 there are eighteen left-hand gloves and six right-hand gloves. Similar to before, available gloves are divided equally over players that choose to visit the same store.

However, now assume that after competing for gloves and focusing on individual incentives only, the three players cooperate and form pairs of gloves together. Moreover, assume that they allocate the joint payoff using the Shapley value, as illustrated in Example 1.2.1. For example, again consider $x=(1,1,2)$, so that players 1 and 2 both obtain three left-hand gloves and six right-hand gloves, while player 3 obtains eighteen left-hand gloves and six right-hand gloves. This corresponds to the starting situation of glove game $v$ of Example 1.2.1. Allocating the joint payoff using the Shapley value therefore yields $\pi(x)=\Phi(v)=(4.5,4.5,9)$. Of course, various symmetry arguments can be applied in the process to determine the payoff vectors based on the Shapley value corresponding to other strategy combinations.

In this model, each strategy combination $x \in X$ leads to a new cooperative game, for which a unique payoff vector is determined using the Shapley value. Hence, through this cooperative stage, each strategy combination ultimately induces a payoff allocation vector. Table 1.4 represents the induced strategic game $G$.


Table 1.4 The induced strategic game $G$ of Example 1.2.3

One readily verifies that the set of Nash equilibria of the strategic game $G$ is given by

$$
E(G)=\{(1,1,2),(1,2,1),(2,1,1)\} .
$$

## Axiomatic characterizations of a proportional influence measure for sequential projects with imperfect reliability

### 2.1 Introduction

Projects are omnipresent in society. Whether it concerns an improvised explosive device (IED) that needs to be developed, moved and placed (see Lindelauf (2011) for a more detailed breakdown of the tasks in a typical IED-project) or a project involving architecture, engineering and construction, there is an analogy. Generally speaking, projects consist of tasks, and each task can only be carried out by certain players, leading to a so-called task structure.

In the context of organized crime, an intelligence agency may be interested in determining who is the most important player for the completion of a project, in order to determine who should be eliminated or apprehended. Regarding projects involving architecture, engineering and construction, completing the project may lead to some (monetary) reward allocation that should fairly reflect the contribution of the various players. As a common feature, the question at hand is which players are most influential for the completion of a project. To this aim, we define and axiomatically characterize a new proportional influence measure based on the task structure of a project, where for each task the players' influence on the completion of the project is measured in proportion to the likelihood that they carry it out successfully.

A significant part of quantitative research on projects is concerned with project planning. For example, Brown et al. (2009) develop interdiction actions that maximally delay completion of a (nuclear) weapons project. In a similar context, Hermans et al. (2019) use game theory to analyze how to allocate intelligence resources and evaluate their performance regarding the timely detection of such covert projects. Estévez-Fernández et al. (2007) use so-called project games to analyze projects in which certain tasks can be delayed or expedited, leading to costs or rewards, respectively, that need to be allocated. The distribution of shared costs in delayed projects is also analyzed by Bergantiños and Sánchez (2002) and Brânzei et al. (2002).

In this chapter, based on Van Beek et al. (2021), we do not focus on project planning, but on measuring the relative control of each player in the completion of a project. A related, but different approach to measuring the relative influence of players in projects on the basis of a network structure, is given in Husslage et al. (2015). In the context of construction projects, Nasirzadeh et al. (2016) use cooperativebargaining theory for quantitative risk allocation between a client and a contractor. There is scarce literature explicitly using the task structure of a project to measure the relative influence of players. Further, the imperfect reliability of players, as illustrated below, is often not incorporated in the literature. This chapter serves as a starting point for quantitative research on the relative influence of imperfectly reliable players on the completion of a project, based on the task structure of this project.

Before elaborating on the new influence measure and its axiomatic characterizations, it is important to discuss what type of projects we are analyzing. The general definition of a task structure concerns projects for which a finite player set attempts to carry out a finite number of tasks. We assume that for the completion of a project, each task must be carried out successfully. Each task can only be carried out by a certain set of players, called the task group. The tasks of a project can be carried out sequentially. For example, this implies that if a player is a member of all task groups, this player could attempt to complete the project alone. We emphasize 'attempt to' here; when a player attempts to carry out a task, we generally do not assume this player is guaranteed to carry out the task successfully. Concretely, each player-task combination has a certain fixed success probability, also referred to as the reliability of the player for that specific task. Depending on the context, reliability can reflect, e.g., operational risk, trustworthiness, or quality of a player in a broader sense. The reliability of players is a key aspect of our model. Thus, we focus mainly on sequential projects with imperfect (player) reliability.

In sequential projects with imperfect reliability, the success probabilities of the individual players in each task group also lead to success probabilities of the tasks, which in turn yields a success probability of the project as a whole. For the latter, we assume that the successes of different tasks are independent. In our model, all players in a task group can attempt to carry out the corresponding task. The probability that a task is not carried out successfully equals the probability that all players independently fail to carry out this task. This can be interpreted in two ways. First, it is possible that all players in a task group attempt to carry out the task in parallel, where the task is considered successful if at least one player manages to carry out the task. Alternatively, a project might be such that some (random) player in the task group attempts to carry out the task, after which another (random) player in the task group attempts the task only if the first player has failed. This process continues until the task is either carried out successfully, or all players (in the task group) have failed. We do not consider, e.g., additional set-up or delay costs associated with attempting to carry out the task multiple times.

We define a proportional influence measure that allocates the final success probability of a project over the players, to determine to which extent each player contributes to the completion of a project. For each task players can carry out, the increase in value of their influence measure corresponds to their relative success probability within the task group. Hence, the allocated value per task increases with the success probability of the player (for this task), but decreases with the 'total' success probability of the other players in the corresponding task group. This essentially reflects a balance between a player's reliability and a player's replaceability.

We axiomatically characterize our proportional influence measure by proving it is the only allocation mechanism for sequential projects with imperfect reliability that satisfies two sets of logically independent properties. The first characterization is on the domain of projects with a fixed player set and its main property concerns the task decomposability of a project. It describes the relationship between the influence measure of a project and the measures of influence one would obtain if one divides the tasks of the project over multiple independent smaller projects. For the second characterization, we consider the domain of projects with a varying set of players. The characterization is mainly based on the property invariance under replacement of players. This property relates projects with different player sets. In particular, it
prescribes the relation when in a certain task group a specific player is replaced by a new player who was not in the original player set. We also observe that both characterizations still work on the smaller domain of sequential projects with perfect reliability.

A potential application lies within the framework of construction projects. Matthews and Howell (2005) propose an Integrated Project Delivery (IPD) method that emphasizes cooperation between various parties who share risk and reward. Despite the fact that IPD is regarded as an effective method, Teng et al. (2019) point out that the number of construction projects using IPD is limited, in part due to the lack of a fair mechanism to allocate profit. For IPD projects that fit the assumptions of our framework, the proportional influence measure could be used as such a mechanism.

An alternative way to analyze sequential projects with imperfect reliability is to define an appropriate cooperative game and use existing game-theoretic solution concepts. We sketch a path for future research in this direction in the final section of this chapter.

Section 2.2 formally introduces projects with imperfect reliability and the proportional influence measure, as well as their counterparts in case of perfect reliability. Section 2.3 discusses the first characterization, based on task decomposability of a project. The second characterization, based on invariance under replacement of players, is covered in Section 2.4. Section 2.5 concludes.

### 2.2 Projects and the proportional influence measure

A sequential project with imperfect reliability $P$ can be summarized by the tuple

$$
P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)
$$

where $N$ denotes the finite set of players and $T$ the finite set of tasks to be carried out by these players, and, for each $k \in T, p^{k} \in[0,1]^{N}$ denotes a probability vector such that $p_{i}^{k}, i \in N$, represents the success probability of player $i$ to carry out task $k$. If $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ is such that $p_{i}^{k} \in\{0,1\}$ for all $k \in T$ and $i \in N$ (as considered by Lindelauf (2011)), we refer to $P$ as a project with perfect reliability.

Additionally, for each $k \in T, N^{k}=\left\{i \in N \mid p_{i}^{k}>0\right\}$ is called the task group of task $k$. We assume that each task must be carried out successfully for the completion
of a project. Consequently, we also assume that $N^{k} \neq \emptyset$ for all $k \in T$ (in particular, this implies that for all project with perfect reliability there is at least one $i \in N$ with $p_{i}^{k}=1$ for all $\left.k \in T\right)$.

The class of all projects is denoted by $\mathcal{P}$. If we restrict to the class of projects with some fixed player set $N$, we emphasize this in the notation using $\mathcal{P}^{N}$. This distinction of domains becomes relevant in the characterizations of the proportional influence measure later on. The subclass of $\mathcal{P}$ of all projects with perfect reliability is denoted by $\mathcal{S}$. Similarly, $\mathcal{S}^{N}$ denotes the corresponding subclass of $\mathcal{P}^{N}$ with a fixed player set $N$.

Importantly, we assume that the tasks of a project can be carried out sequentially, meaning that if a player is in several task groups, this player can attempt all corresponding tasks. Further, all players in a task group can attempt to carry out the corresponding task. We implicitly assume that there is no interdependence in the success probabilities of players in a task group; the success probability is fixed for any player-task combination, independent of, e.g., another player in the task group failing to carry out the task. Hence, the probability that a task is not carried out successfully is equal to the probability that all players independently fail to carry out this task. Also assuming independence between the success of tasks, the success probability of a project can then be found by simply multiplying the success probabilities of each individual task. We denote the probability that a project $P \in \mathcal{P}$ is completed by $q(P)$. This can be seen as the 'quality' of the player set (with respect to carrying out the tasks in the project), or simply as the success probability of the project. Formally, we define the success probability of a project as a function $q: \mathcal{P} \rightarrow[0,1]$ such that

$$
q(P)=\prod_{k \in T}\left(1-\prod_{i \in N}\left(1-p_{i}^{k}\right)\right)
$$

for any $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$. Since $N^{k} \neq \emptyset$ for all $k \in T$, note that we have $q(P)=1$ for all $P \in \mathcal{S}$.

We now define a solution concept for sequential projects with imperfect reliability. We call $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ a solution concept on $\mathcal{P}^{N}$. For $i \in N, f_{i}(P)$ represents a measure of influence of player $i$ on the completion of the project. Similarly, $f$ is a solution concept on $\mathcal{P}$ if it assigns a vector in $\mathbb{R}^{N}$ to any $P \in \mathcal{P}^{N}$ for any finite player set $N$. We propose a proportional influence measure for sequential projects with imperfect
reliability, denoted by $\rho$, in which the total probability of success $q(P)$ is allocated among the players in the following way. First, by the nature of a project, $q(P)$ is shared equally among the tasks. Second, for each task $k \in T, q(P) /|T|$ is allocated to players proportional to their individual task-specific success probabilities provided by $p^{k}$. Intuitively, for each task, the influence measure allocates more to players with higher reliability (i.e., success probability). Further, the influence measure allocated to a player decreases with the 'total' success probability of the other players in the corresponding task group, as the player is then more replaceable.

## Definition 2.2.1

The proportional influence measure $\rho: \mathcal{P}^{N} \rightarrow[0,1]^{N}$ is defined by setting

$$
\rho_{i}(P)=\frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ and any $i \in N$.
Note that we define $\rho$ as a solution concept on $\mathcal{P}^{N}$ here. For the second characterization, we define properties of a solution concept on $\mathcal{P}$. In this case, we also interpret $\rho$ as a solution concept on $\mathcal{P}$, simply by using the definition above for any finite $N$.

## Example 2.2.1

Consider a project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ that consists of two tasks, to be carried out by a set of three players, with $N=\{1,2,3\}, T=\{a, b\}$, and $p^{a}=(0.8,0.9,0)$ and $p^{b}=(0.8,0,1)$. Clearly,

$$
q(P)=\left(1-\left(1-p_{1}^{a}\right)\left(1-p_{2}^{a}\right)\left(1-p_{3}^{a}\right)\right)\left(1-\left(1-p_{1}^{b}\right)\left(1-p_{2}^{b}\right)\left(1-p_{3}^{b}\right)\right)=0.98
$$

Consequently,

$$
\rho(P)=\frac{0.98}{2}\left(\frac{0.8}{1.7}+\frac{0.8}{1.8}, \frac{0.9}{1.7}, \frac{1}{1.8}\right) \approx(0.45,0.26,0.27)
$$

Clearly, for a project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{S}^{N}$ with perfect reliability, we have

$$
\rho_{i}(P)=\frac{1}{|T|} \sum_{k \in T: i \in N^{k}} \frac{1}{\left|N^{k}\right|}
$$

for all $i \in N$.

## Example 2.2.2

Consider the project of Example 2.2.1, but now with perfect reliability, i.e., $P=$ $\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ with $N=\{1,2,3\}, T=\{a, b\}$, and $p^{a}=(1,1,0)$ and $p^{b}=(1,0,1)$. Then,

$$
\rho(P)=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)=(0.5,0.25,0.25)
$$

### 2.3 Characterization using task decomposability

In this section, we present our first axiomatic characterization of the proportional influence measure. The most prominent property of a solution concept in this characterization considers the effect of dividing the tasks of a project over multiple smaller projects that we 'solve' independently. In particular, it relates the solution of the 'original' project to the solutions of the smaller projects.

We use the following four properties of a solution concept $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ to axiomatically characterize $\rho$ on $\mathcal{P}^{N}$.

We say that $f$ satisfies efficiency if $f$ allocates the total probability that a project is completed.

Efficiency (EFF) $f$ satisfies EFF on $\mathcal{P}^{N}$ if $\sum_{i \in N} f_{i}(P)=q(P)$ for all $P \in \mathcal{P}^{N}$.
For any $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ we define $Z(P)=N \backslash \bigcup_{k \in T} N^{k}$ as the set of null players with respect to $P$. These null players do not have a positive success probability for any task. We say that $f$ satisfies the null player property if all null players with respect to $P$ are allocated zero value.

Null player (NUL) $f$ satisfies $N U L$ on $\mathcal{P}^{N}$ if $f_{i}(P)=0$ for all $P \in \mathcal{P}^{N}$ with $i \in Z(P)$.

Proportionality only applies to projects with a single task. For such projects, this property states that the value allocated to players is proportional to their success probability.

## Proportionality (PROP) $f$ satisfies $P R O P$ on $\mathcal{P}^{N}$ if

$$
\frac{f_{i}(P)}{p_{i}^{l}}=\frac{f_{j}(P)}{p_{j}^{l}}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $T=\{l\}$ and all $i, j \in N^{l}$.
Finally, task decomposability only applies to projects $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ with $|T|>1$. For such projects, this property describes the relationship between the solution of this project and the solutions of two projects $P_{T^{1}}$ and $P_{T^{2}}$ over which the tasks in $T$ of the original project are divided into two disjoint sets $T^{1}$ and $T^{2}$. We do not impose a specific order in which the tasks need to be carried out, so $T^{1}$ and $T^{2}$ can be two arbitrary subsets that partition $T$. In general, we denote smaller projects with $P_{S}=\left(N, S,\left\{p^{k}\right\}_{k \in S}\right) \in \mathcal{P}^{N}$ for any $S \subseteq T$. Note that for all $P_{S}, P_{S^{1}}, P_{S^{2}} \in \mathcal{P}^{N}$ with $S \subseteq T, S^{1} \cup S^{2}=S$ and $S^{1} \cap S^{2}=\emptyset$, we have

$$
\begin{equation*}
q\left(P_{S}\right)=q\left(P_{S^{1}}\right) q\left(P_{S^{2}}\right) \tag{2.1}
\end{equation*}
$$

We say that $f$ satisfies task decomposability if the values allocated by $f$ for the original project can be written as a certain weighted average of the values allocated by $f$ for the two corresponding smaller projects. These weights contain the relative number of tasks to be carried out and the success probabilities of the smaller projects. In particular, the higher the relative number of tasks, the higher the weight of that project. Further, the weight increases as the success probability of the other smaller project increases. The intuition behind this is that the original project is only completed if both smaller projects are completed, meaning completion of one smaller project should only count if the other smaller project is also successful.

Task decomposability (DEC) $f$ satisfies $D E C$ on $\mathcal{P}^{N}$ if

$$
f(P)=\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) f\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) f\left(P_{T^{2}}\right)
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$ and all $P_{T^{1}}, P_{T^{2}} \in \mathcal{P}^{N}$ with $\left|T^{1}\right| \geq 1,\left|T^{2}\right| \geq 1, T^{1} \cup T^{2}=T$ and $T^{1} \cap T^{2}=\emptyset$.

## Example 2.3.1

Reconsider the project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ with $N=\{1,2,3\}, T=\{a, b\}, p^{a}=$ $(0.8,0.9,0)$ and $p^{b}=(0.8,0,1)$, as described in Example 2.2.1. We decompose this project into two smaller projects $P_{T^{1}}$ and $P_{T^{2}}$ with $T^{1}=\{a\}$ and $T^{2}=\{b\}$, and note that $q\left(P_{T^{1}}\right)=0.98$ and $q\left(P_{T^{2}}\right)=1$. Task decomposability is satisfied by $\rho$ in this example, since

$$
\begin{aligned}
\frac{1}{2} \cdot 1 \cdot \rho\left(P_{T^{1}}\right)+\frac{1}{2} \cdot 0.98 \cdot \rho\left(P_{T^{2}}\right) & =\frac{1}{2} \frac{0.98}{1}\left(\frac{0.8}{1.7}, \frac{0.9}{1.7}, 0\right)+\frac{0.98}{2} \frac{1}{1}\left(\frac{0.8}{1.8}, 0, \frac{1}{1.8}\right) \\
& =\frac{0.98}{2}\left(\frac{0.8}{1.7}+\frac{0.8}{1.8}, \frac{0.9}{1.7}, \frac{1}{1.8}\right) \\
& =\rho(P)
\end{aligned}
$$

Next, we show that $\rho$ is the only solution concept for sequential projects with imperfect reliability that satisfies the four properties defined above. To do so, we first derive a consequence of the DEC property towards decomposing a project into single-task projects.

## Lemma 2.3.1

Let $f$ be a solution concept on $\mathcal{P}^{N}$ that satisfies DEC. Then,

$$
\begin{equation*}
f(P)=\frac{1}{|T|} \sum_{k \in T} q\left(P_{T \backslash\{k\}}\right) f\left(P_{\{k\}}\right) \tag{2.2}
\end{equation*}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$.
Proof. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ with $|T|>1$. We give a proof by induction on $|T|$. First, consider the base case $T=\{k, l\}$ with $k \neq l$. By DEC, we have

$$
f(P)=\frac{1}{2} q\left(P_{\{k\}}\right) f\left(P_{\{l\}}\right)+\frac{1}{2} q\left(P_{\{l\}}\right) f\left(P_{\{k\}}\right)
$$

as required.
Next, assume the induction hypothesis that equation (2.2) holds for all $P=$ $\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|=t$ for a given integer $t \geq 2$. Then, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ with $|T|=t+1$ and let $l \in T$. We get

$$
f(P)=\frac{t}{t+1} q\left(P_{\{l\}}\right) f\left(P_{T \backslash\{l\}}\right)+\frac{1}{t+1} q\left(P_{T \backslash\{l\}}\right) f\left(P_{\{l\}}\right)
$$

$$
\begin{aligned}
& =\frac{t}{t+1} q\left(P_{\{l\}}\right) \frac{1}{t} \sum_{k \in T \backslash\{l\}} q\left(P_{T \backslash\{k, l\}}\right) f\left(P_{\{k\}}\right)+\frac{1}{t+1} q\left(P_{T \backslash\{l\}}\right) f\left(P_{\{l\}}\right) \\
& =\frac{1}{t+1} \sum_{k \in T \backslash\{l\}} q\left(P_{T \backslash\{k\}}\right) f\left(P_{\{k\}}\right)+\frac{1}{t+1} q\left(P_{T \backslash\{l\}}\right) f\left(P_{\{l\}}\right) \\
& =\frac{1}{t+1} \sum_{k \in T} q\left(P_{T \backslash\{k\}}\right) f\left(P_{\{k\}}\right)
\end{aligned}
$$

where we use the fact that $f$ satisfies DEC in the first equality, the induction hypothesis in the second equality, and (2.1) in the third equality.

## Theorem 2.3.2

Let $f$ be a solution concept on $\mathcal{P}^{N}$. Then, $f=\rho$ if and only if $f$ satisfies EFF, NUL, PROP, and DEC.

Proof. We first show that $\rho$ satisfies the four properties. EFF and NUL are obvious from the definition of $\rho$.

PROP: Consider $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $T=\{l\}$ and let $i, j \in N^{l}$. Then,

$$
\frac{\rho_{i}(P)}{p_{i}^{l}}=\frac{q(P) \frac{p_{i}^{l}}{\sum_{r \in N}^{p_{r}^{l}}}}{p_{i}^{l}}=\frac{q(P)}{\sum_{r \in N} p_{r}^{l}}=\frac{q(P) \frac{p_{j}^{l}}{\sum_{r \in N} p_{r}^{l}}}{p_{j}^{l}}=\frac{\rho_{j}(P)}{p_{j}^{l}} .
$$

DEC: Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ be such that $|T|>1$ and let $i \in N$. Let $T^{1}$ and $T^{2}$ be such that $\left|T^{1}\right| \geq 1,\left|T^{2}\right| \geq 1, T^{1} \cup T^{2}=T$ and $T^{1} \cap T^{2}=\emptyset$ and consider $P_{T^{1}}$ and $P_{T^{2}}$. Then,

$$
\begin{aligned}
\rho_{i}(P) & =\frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{q(P)}{|T|}\left(\sum_{k \in T^{1}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}+\sum_{k \in T^{2}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}\right) \\
& =\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) \frac{q\left(P_{T^{1}}\right)}{\left|T^{1}\right|} \sum_{k \in T^{1}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) \frac{q\left(P_{T^{2}}\right)}{\left|T^{2}\right|} \sum_{k \in T^{2}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) \rho_{i}\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) \rho_{i}\left(P_{T^{2}}\right) .
\end{aligned}
$$

Next, let $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ satisfy the four properties. We show that $f(P)=\rho(P)$ for all $P \in \mathcal{P}^{N}$. We first focus on projects with one task. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ with $T=\{k\}$. Let $i \in N \backslash N^{k}$. By NUL, we have $f_{i}(P)=0=\rho_{i}(P)$. Next, let $i \in N^{k}$. Since $f$ satisfies PROP, we know that for any $j \in N^{k}$, we have

$$
f_{j}(P)=\frac{p_{j}^{k}}{p_{i}^{k}} f_{i}(P)
$$

Using EFF, we get

$$
q(P)=\sum_{j \in N} f_{j}(P)=\sum_{j \in N^{k}} f_{j}(P)=\sum_{j \in N^{k}} \frac{p_{j}^{k}}{p_{i}^{k}} f_{i}(P)=f_{i}(P) \frac{\sum_{j \in N} p_{j}^{k}}{p_{i}^{k}}
$$

From this, we may conclude

$$
f_{i}(P)=q(P) \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}=\rho_{i}(P)
$$

Next, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$. By Lemma 2.3.1,

$$
\begin{aligned}
f_{i}(P) & =\frac{1}{|T|} \sum_{k \in T} q\left(P_{T \backslash\{k\}}\right) f_{i}\left(P_{\{k\}}\right) \\
& =\frac{1}{|T|} \sum_{k \in T} q\left(P_{T \backslash\{k\}}\right) q\left(P_{\{k\}}\right) \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\rho_{i}(P)
\end{aligned}
$$

for any $i \in N$, also using (2.1) in the third equality. This concludes the proof.

Finally, we show that the four characterizing properties mentioned in Theorem 2.3.2 are logically independent. Since $\rho$ is the only one-point solution concept satisfying all four properties, it suffices to find, for any subset of three properties, an alternative solution concept $f$ with $f \neq \rho$ that satisfies these properties. For the following solution concepts, it is clear that $f \neq \rho$. The single property that $f$ does not satisfy is indicated by, e.g., 'No EFF'.

No EFF: Consider $f(P)=2 \rho(P)$ for all $P \in \mathcal{P}^{N}$. Clearly, the scaling only affects the validity of EFF.

No NUL: Let $N=\{1, \ldots,|N|\}$. Consider the solution concept $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ that first shares the success probability equally among the tasks. For each task, there are two options for how its value is allocated. In case player 1 is not a member of the corresponding task group, the value is allocated only to player 1 . If player 1 is in the corresponding task group, the value is allocated to all players in this task group, proportional to their success probabilities. Formally, for all $P \in \mathcal{P}^{N}, f$ is defined by

$$
f_{1}(P)=\frac{q(P)}{|T|}\left(\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|+\sum_{k \in T: 1 \in N^{k}} \frac{p_{1}^{k}}{\sum_{j \in N} p_{j}^{k}}\right)
$$

and

$$
f_{i}(P)=\frac{q(P)}{|T|} \sum_{k \in T: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}
$$

for any $i \in N \backslash\{1\}$.
Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$. EFF is satisfied, since

$$
\begin{aligned}
\sum_{i \in N} f_{i}(P) & =\frac{q(P)}{|T|}\left(\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|+\sum_{i \in N} \sum_{k \in T: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}\right) \\
& =\frac{q(P)}{|T|}\left(\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|+\sum_{k \in T: 1 \in N^{k}} \sum_{i \in N} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}\right) \\
& =\frac{q(P)}{|T|}\left(\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|+\sum_{k \in T: 1 \in N^{k}} 1\right) \\
& =\frac{q(P)}{|T|}\left(\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|+\left|\left\{k \in T: 1 \in N^{k}\right\}\right|\right) \\
& =\frac{q(P)}{|T|}|T| \\
& =q(P) .
\end{aligned}
$$

For PROP, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $T=\{l\}$ and let $i \in N^{l}$. First, if $1 \notin N^{l}$, then $f_{i}(P)=0$. Alternatively, if $1 \in N^{l}$, then

$$
\frac{f_{i}(P)}{p_{i}^{l}}=\frac{\frac{q(P)}{|T|} \sum_{k \in T: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}}{p_{i}^{l}}=\frac{q(P) \frac{p_{i}^{l}}{\sum_{j \in N} p_{j}^{l}}}{p_{i}^{l}}=\frac{q(P)}{\sum_{j \in N} p_{j}^{l}}
$$

is constant, independent of $i$. Here, since $\left\{k \in T: 1 \notin N^{k}\right\}=\emptyset$, the first equality also holds if $i=1$. The second equality follows from the fact that $T=\{l\}$.

Finally, to see DEC is satisfied, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$, let $T^{1}$ and $T^{2}$ such that $\left|T^{1}\right| \geq 1,\left|T^{2}\right| \geq 1, T^{1} \cup T^{2}=T$ and $T^{1} \cap T^{2}=\emptyset$, and consider $P_{T^{1}}$ and $P_{T^{2}}$. We distinguish between player 1 and the other players in $N$. First, let $i \in N \backslash\{1\}$. Then,

$$
\begin{aligned}
f_{i}(P) & =\frac{q(P)}{|T|} \sum_{k \in T: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{q(P)}{|T|} \sum_{k \in T^{1}: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}}+\frac{q(P)}{|T|} \sum_{k \in T^{2}: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{\left|T^{1}\right|}{|T|} \frac{q(P)}{q\left(P_{T^{1}}\right)} \frac{q\left(P_{T^{1}}\right)}{\left|T^{1}\right|} \sum_{k \in T^{1}: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& +\frac{\left|T^{2}\right|}{|T|} \frac{q(P)}{q\left(P_{T^{2}}\right)} \frac{q\left(P_{T^{2}}\right)}{\left|T^{2}\right|} \sum_{k \in T^{2}: 1 \in N^{k}} \frac{p_{i}^{k}}{\sum_{j \in N} p_{j}^{k}} \\
& =\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) f_{i}\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) f_{i}\left(P_{T^{2}}\right)
\end{aligned}
$$

using (2.1) in the final equality. Analogously, one can show

$$
f_{1}(P)=\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) f_{1}\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) f_{1}\left(P_{T^{2}}\right),
$$

using the fact that $\left|\left\{k \in T: 1 \notin N^{k}\right\}\right|=\left|\left\{k \in T^{1}: 1 \notin N^{k}\right\}\right|+\left|\left\{k \in T^{2}: 1 \notin N^{k}\right\}\right|$ for the additional term in $f_{1}(P)$.

No PROP: Fix a representation function $g: 2^{N} \backslash\{\emptyset\} \rightarrow N$ such that $g(S) \in S$ for all $S \in 2^{N} \backslash\{\emptyset\}$. Consider the solution concept $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ that allocates value to only one player in each task group $N^{k}, k \in T$, defined by

$$
f(P)=\frac{q(P)}{|T|} \sum_{k \in T} e_{g\left(N^{k}\right)}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$. For each task $k \in T, f$ allocates $q(P) /|T|$ to exactly one player in $N^{k}$, so EFF and NUL are clearly satisfied. Next, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$, let $T^{1}$ and $T^{2}$ such that $\left|T^{1}\right| \geq 1,\left|T^{2}\right| \geq 1, T^{1} \cup T^{2}=T$ and $T^{1} \cap T^{2}=\emptyset$, and consider $P_{T^{1}}$ and $P_{T^{2}}$. DEC is satisfied, since

$$
\begin{aligned}
f(P) & =\frac{q(P)}{|T|}\left(\sum_{k \in T^{1}} e_{g\left(N^{k}\right)}+\sum_{k \in T^{2}} e_{g\left(N^{k}\right)}\right) \\
& =\frac{\left|T^{1}\right|}{|T|} \frac{q(P)}{q\left(P_{T^{1}}\right)} \frac{q\left(P_{T^{1}}\right)}{\left|T^{1}\right|} \sum_{k \in T^{1}} e_{g\left(N^{k}\right)}+\frac{\left|T^{2}\right|}{|T|} \frac{q(P)}{q\left(P_{T^{2}}\right)} \frac{q\left(P_{T^{2}}\right)}{\left|T^{2}\right|} \sum_{k \in T^{2}} e_{g\left(N^{k}\right)} \\
& =\frac{\left|T^{1}\right|}{|T|} q\left(P_{T^{2}}\right) f\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} q\left(P_{T^{1}}\right) f\left(P_{T^{2}}\right) .
\end{aligned}
$$

No DEC: Consider the solution concept $f: \mathcal{P}^{N} \rightarrow \mathbb{R}^{N}$ that equals $\rho$ if the number of tasks is equal to one and allocates the success probability of a project equally to all non-null players if the number of tasks if larger than one. Formally, $f$ is defined by

$$
f_{i}(P)= \begin{cases}\rho_{i}(P) & \text { if }|T|=1 \\ \frac{q(P)}{|N \backslash Z(P)|} & \text { if } i \in N \backslash Z(P) \text { and }|T|>1 \\ 0 & \text { if } i \in Z(P) \text { and }|T|>1\end{cases}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ and any $i \in N$. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|>1$. Then, $q(P)$ is allocated equally to the non-null players, so EFF and NUL are satisfied when restricting to project with multiple tasks. PROP cannot be violated here either, as this property only applies to projects with a single task. Next, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ such that $|T|=1$. $\rho$ satisfies all four properties for all $P \in \mathcal{P}^{N}$, so in particular for single-task projects. Hence, EFF, NUL, and PROP are satisfied.

To conclude this section, we analyze the results for the special case of sequential projects with perfect reliability. Despite the fact that the class $\mathcal{S}^{N}$ of projects is smaller than $\mathcal{P}^{N}$, the proportional influence measure $\rho$ is still the only solution concept on this subdomain that satisfies the four properties of Theorem 2.3.2.

For the sake of completeness, we provide the explicit reformulation of the four properties on $\mathcal{S}^{N}$. Let $f: \mathcal{S}^{N} \rightarrow \mathbb{R}^{N}$ be a solution concept on $\mathcal{S}^{N}$.

Efficiency (EFF) $f$ satisfies EFF on $\mathcal{S}^{N}$ if $\sum_{i \in N} f_{i}(P)=1$ for all $P \in \mathcal{S}^{N}$.
Null Player (NUL) $f$ satisfies $N U L$ on $\mathcal{S}^{N}$ if $f_{i}(P)=0$ for all $P \in \mathcal{S}^{N}$ with $i \in Z(P)$.

Proportionality (PROP) $f$ satisfies $P R O P$ on $\mathcal{S}^{N}$ if $f_{i}(P)=f_{j}(P)$ for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{S}^{N}$ such that $T=\{l\}$ and all $i, j \in N^{l}$.

Task decomposability (DEC) $f$ satisfies $D E C$ on $\mathcal{S}^{N}$ if

$$
f(P)=\frac{\left|T^{1}\right|}{|T|} f\left(P_{T^{1}}\right)+\frac{\left|T^{2}\right|}{|T|} f\left(P_{T^{2}}\right)
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{S}^{N}$ such that $|T|>1$ and all $P_{T^{1}}, P_{T^{2}} \in \mathcal{S}^{N}$ with $\left|T^{1}\right| \geq 1,\left|T^{2}\right| \geq 1, T^{1} \cup T^{2}=T$ and $T^{1} \cap T^{2}=\emptyset$.

## Theorem 2.3.3

Let $f$ be a solution concept on $\mathcal{S}^{N}$. Then, $f=\rho$ if and only if $f$ satisfies EFF, NUL, $P R O P$, and DEC.

The proof of this theorem is analogous to the proof of Theorem 2.3.2, simplified by the fact that, for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{S}^{N}, q(P)=1$ and $p_{i}^{k}=1$ for all $i \in N^{k}$ with $k \in T$. The solution concepts used to prove the logical independence of the four properties can be adapted accordingly as well.

### 2.4 Characterization using invariance under replacement

Our second characterization is most prominently concerned with the behavior of a solution concept when in a certain task group a specific player is replaced by a player who was not in the original player set. Clearly, this property requires the player set to change and the domain of solution concepts under consideration will become $\mathcal{P}$ instead of $\mathcal{P}^{N}$.

The axiomatic characterization of $\rho$ on $\mathcal{P}$ is based on the following four properties of a solution concept $f$ on $\mathcal{P}$.

The first two properties in this characterization are efficiency and the null player property.

Efficiency (EFF) $f$ satisfies EFF on $\mathcal{P}$ if $f$ satisfies EFF on $\mathcal{P}^{N}$ for all finite $N$.

Null player (NUL) $f$ satisfies NUL on $\mathcal{P}$ if $f$ satisfies $N U L$ on $\mathcal{P}^{N}$ for all finite $N$.

The next property only applies to projects $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ such that the collection $\left\{N^{k}\right\}_{k \in T}$, also called the task structure of a project, is a partition of $N \backslash Z(P)$. In this setting, partition proportionality states that, for each player-task combination (i.e., also across tasks), the value allocated by $f$ to each (non-null) player is proportional to the relative success probability of that player in the only task group to which the player belongs.

Partition proportionality (PAP) $f$ satisfies $P A P$ on $\mathcal{P}$ if

$$
\frac{\sum_{r \in N} p_{r}^{l}}{p_{i}^{l}} f_{i}(P)=\frac{\sum_{r \in N} p_{r}^{m}}{p_{j}^{m}} f_{j}(P)
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ such that $\left\{N^{k}\right\}_{k \in T}$ is a partition of $N \backslash Z(P)$, and all $i \in N^{l}$ and $j \in N^{m}$ with $l, m \in T$.

The final property of invariance under replacement states that when in a certain task group a specific player is replaced by exactly one player who was not in the original player set and who has the same success probability for that task, this does not affect any of the other non-null players. Before formally defining the property itself, we first introduce the general definition of a replicate project $\bar{P}_{i} \in \mathcal{P}$ corresponding to a project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$, in which a player $i \in N$ is replaced by a 'new' player in one task group. Here, we emphasize that this new player takes over the attempt to carry out exactly one task from player $i$. However, the specific task group in which $i$ is replaced is not important for the definition of the IUR property, so this task group is not explicitly reflected in the notation.

## Definition 2.4.1

Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ be a project and let $i \in N^{l}$ with $l \in T$. Then, a replicate project $\bar{P}_{i} \in \mathcal{P}$ with replica repl $(i)$ for player $i$ is defined by $\bar{P}_{i}=\left(\bar{N}, T,\left\{\bar{p}^{k}\right\}_{k \in T}\right)$,
with $\bar{N}=N \cup\{\operatorname{repl}(i)\}, \bar{p}_{\text {repl }(i)}^{l}=p_{i}^{l}, \bar{p}_{i}^{l}=0, \bar{p}_{j}^{l}=p_{j}^{l}$ for all $j \in N \backslash\{i\}$, and, for all $k \in T \backslash\{l\}, \bar{p}_{\text {repl }(i)}^{k}=0$ and $\bar{p}_{i}^{k}=p_{i}^{k}$ for all $i \in N$.

Invariance under replacement (IUR) $f$ satisfies IUR on $\mathcal{P}$ if

$$
f_{j}(P)=f_{j}\left(\bar{P}_{i}\right)
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$, all $i \in N \backslash Z(P)$ and $j \in N \backslash(Z(P) \cup\{i\})$, and all replicate projects $\bar{P}_{i}$.

## Example 2.4.1

Reconsider the project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ with $N=\{1,2,3\}, T=\{a, b\}, p^{a}=$ $(0.8,0.9,0)$ and $p^{b}=(0.8,0,1)$, as described in Example 2.2.1, where we derived

$$
\rho(P)=\frac{0.98}{2}\left(\frac{0.8}{1.7}+\frac{0.8}{1.8}, \frac{0.9}{1.7}, \frac{1}{1.8}\right) .
$$

Note that player 1 is in both task groups. We now consider the replicate project $\bar{P}_{1}$ in which player 1 is replaced by a new player 4 in the second task group, so $\bar{P}_{1}=\left(\bar{N}, T,\left\{\bar{p}^{k}\right\}_{k \in T}\right)$ with $\bar{N}=\{1,2,3,4\}, \bar{p}^{a}=(0.8,0.9,0,0)$ and $\bar{p}^{b}=(0,0,1,0.8)$. Since $q\left(\bar{P}_{1}\right)=0.98$, we have

$$
\rho\left(\bar{P}_{1}\right)=\frac{0.98}{2}\left(\frac{0.8}{1.7}, \frac{0.9}{1.7}, \frac{1}{1.8}, \frac{0.8}{1.8}\right) .
$$

Indeed, we find that IUR is satisfied in this example, as $\rho_{2}\left(\bar{P}_{1}\right)=\rho_{2}(P)$ and $\rho_{3}\left(\bar{P}_{1}\right)=$ $\rho_{3}(P)$. Note that we also have $\rho_{1}\left(\bar{P}_{1}\right)+\rho_{4}\left(\bar{P}_{1}\right)=\rho_{1}(P)$.

In Example 2.4.1, the value allocated to the player who is replaced (in one task group) in the project is equal to the sum of the values allocated to this player and the replica in the replicate project. In fact, this holds for any solution concept $f$ that satisfies EFF, NUL, and IUR on $\mathcal{P}$.

## Lemma 2.4.2

Let $f$ be a solution concept on $\mathcal{P}$ that satisfies EFF, NUL, and IUR. Then,

$$
f_{i}(P)=f_{i}\left(\bar{P}_{i}\right)+f_{\text {repl }(i)}\left(\bar{P}_{i}\right)
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$, all $i \in N \backslash Z(P)$, and all replicate projects $\bar{P}_{i}$ with replica repl $(i)$ for player $i$.

Proof. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ and let $i \in N^{l}$ with $l \in T$. Let $\bar{P}_{i}=$ $\left(\bar{N}, T,\left\{\bar{p}^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ be the replicate project in which replica repl $(i)$ replaces player $i$ for task $l$. First, note that the success probabilities of $P$ and $\bar{P}_{i}$ are equal, since

$$
\begin{aligned}
q(P) & =\prod_{k \in T}\left(1-\prod_{j \in N}\left(1-p_{j}^{k}\right)\right) \\
& =\left(1-\left(1-p_{i}^{l}\right) \prod_{j \in N \backslash\{i\}}\left(1-p_{j}^{l}\right)\right) \prod_{k \in T \backslash\{l\}}\left(1-\prod_{j \in N}\left(1-p_{j}^{k}\right)\right) \\
& =\left(1-\left(1-\bar{p}_{r e p l(i)}^{l}\right) \prod_{j \in \bar{N} \backslash\{i, r e p l(i)\}}\left(1-\bar{p}_{j}^{l}\right)\right) \prod_{k \in T \backslash\{l\}}\left(1-\prod_{j \in \bar{N}}\left(1-\bar{p}_{j}^{k}\right)\right) \\
& =\prod_{k \in T}\left(1-\prod_{j \in \bar{N}}\left(1-\bar{p}_{j}^{k}\right)\right) \\
& =q\left(\bar{P}_{i}\right)
\end{aligned}
$$

where we use $\bar{p}_{r e p l(i)}^{k}=0$ for all $k \in T \backslash\{l\}$ in the third equality and $\bar{p}_{i}^{l}=0$ in the fourth equality. It follows that

$$
\begin{aligned}
f_{i}(P) & =q(P)-\sum_{j \in N \backslash(Z(P) \cup\{i\})} f_{j}(P) \\
& =q\left(\bar{P}_{i}\right)-\sum_{j \in N \backslash(Z(P) \cup\{i\})} f_{j}\left(\bar{P}_{i}\right) \\
& =f_{i}\left(\bar{P}_{i}\right)+f_{\text {repl }(i)}\left(\bar{P}_{i}\right),
\end{aligned}
$$

where we use the fact that $f$ satisfies EFF and NUL in the first and third equality, and we use $q(P)=q\left(\bar{P}_{i}\right)$ and the fact that $f$ satisfies IUR in the second equality.

We now show that $\rho$ is the only solution concept for sequential projects with imperfect reliability satisfying the four properties defined above.

## Theorem 2.4.3

Let $f$ be a solution concept on $\mathcal{P}$. Then, $f=\rho$ if and only if $f$ satisfies EFF, NUL, PAP, and IUR.

Proof. We first show that $\rho$ satisfies the four properties. Clearly, EFF and NUL directly follow from the definition of $\rho$.

PAP: Consider $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ such that $\left\{N^{k}\right\}_{k \in T}$ is a partition of $N \backslash Z(P)$. Let $i \in N^{l}$ with $l \in T$. Then,

$$
\frac{\sum_{r \in N} p_{r}^{l}}{p_{i}^{l}} \rho_{i}(P)=\frac{\sum_{r \in N} p_{r}^{l}}{p_{i}^{l}} \frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{i}^{k}}{\sum_{r \in N} p_{r}^{k}}=\frac{\sum_{r \in N} p_{r}^{l}}{p_{i}^{l}} \frac{q(P)}{|T|} \frac{p_{i}^{l}}{\sum_{r \in N} p_{r}^{l}}=\frac{q(P)}{|T|}
$$

is constant, independent of $i$ and $l$, where the second equality follows from the fact that $i \notin N^{k}$ and hence $p_{i}^{k}=0$ for all $k \in T \backslash\{l\}$ since $\left\{N^{k}\right\}_{k \in T}$ is a partition of $N \backslash Z(P)$.

IUR: Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$, let $i \in N^{l}$ with $l \in T$, let $j \in N \backslash(Z(P) \cup\{i\})$, and let $\bar{P}_{i}=\left(\bar{N}, T,\left\{\bar{p}^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ be a replicate project. Since $q(P)=q\left(\bar{P}_{i}\right)$ and, for all $k \in T, \sum_{r \in N} p_{r}^{k}=\sum_{r \in \bar{N}} \bar{p}_{r}^{k}$ and $p_{j}^{k}=\bar{p}_{j}^{k}$, we have

$$
\rho_{j}(P)=\frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{j}^{k}}{\sum_{r \in N} p_{r}^{k}}=\frac{q\left(\bar{P}_{i}\right)}{|T|} \sum_{k \in T} \frac{\bar{p}_{j}^{k}}{\sum_{r \in \bar{N}} \bar{p}_{r}^{k}}=\rho_{j}\left(\bar{P}_{i}\right) .
$$

Next, let $f$ be a solution concept on $\mathcal{P}$ satisfying the four properties. We show that $f(P)=\rho(P)$ for all $P \in \mathcal{P}$.

Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$. To be able to use the PAP property, which holds specifically for projects with a task structure that is a partition of all non-null players, we first construct a project $\bar{P}=\left(\bar{N}, T,\left\{\bar{p}^{k}\right\}_{k \in T}\right)$ without null players and in which all players have a strictly positive probability to successfully carry out exactly one task only. To define $\bar{P}$, first choose mutually disjoint sets of replica players $\left\{R_{i}\right\}_{i \in N}$ such that

$$
\left|R_{i}\right|=\left|\left\{k \in T \mid p_{i}^{k}>0\right\}\right|
$$

for all $i \in N$. Note that $R_{i}=\emptyset$ if and only if $i \in Z(P)$. Set

$$
\bar{N}=\bigcup_{i \in N} R_{i}
$$

Let $k \in T$. To define $\bar{p}^{k} \in[0,1]^{\bar{N}}$, choose a bijection

$$
g_{i}: R_{i} \rightarrow\left\{k \in T \mid p_{i}^{k}>0\right\}
$$

for all $i \in N$ and set, for all $r \in \bar{N}$ and $k \in T$

$$
\bar{p}_{r}^{k}= \begin{cases}p_{i}^{k} & \text { if } r \in R_{i} \text { and } g_{i}(r)=k \\ 0 & \text { otherwise }\end{cases}
$$

Note that $Z(\bar{P})=\emptyset$ and that the task groups $\bar{N}^{k}, k \in T$, partition $\bar{N}$. Moreover, obviously, $q(P)=q(\bar{P})$.

For $r \in R_{i}$, let $k(r)$ denote the unique task group this player belongs to, i.e., $k(r)=g_{i}(r)$. Fix $t \in \bar{N}$. Then,

$$
\begin{aligned}
q(\bar{P}) & =\sum_{r \in \bar{N}} f_{r}(\bar{P}) \\
& =\sum_{r \in \bar{N}} \frac{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(r)}}{\bar{p}_{r}^{k(r)}} f_{r}(\bar{P}) \frac{\bar{p}_{r}^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(r)}} \\
& =\frac{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(t)}}{\bar{p}_{t}^{k(t)}} f_{t}(\bar{P}) \sum_{r \in \bar{N}} \frac{\bar{p}_{r}^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(r)}} \\
& =\frac{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(t)}}{\bar{p}_{t}^{k(t)}} f_{t}(\bar{P}) \sum_{k \in T} \frac{\sum_{s \in \bar{N}} \bar{p}_{s}^{k}}{\sum_{s \in \bar{N}} \bar{p}_{s}^{k}} \\
& =\frac{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(t)}}{\bar{p}_{t}^{k(t)}} f_{t}(\bar{P})|T|,
\end{aligned}
$$

where we use the fact that $f$ satisfies EFF in the first equality, that $f_{t}(\bar{P}) \sum_{s \in \bar{N}} \bar{p}_{s}^{k(t)} / \bar{p}_{t}^{k(t)}=f_{r}(\bar{P}) \sum_{s \in \bar{N}} \bar{p}_{s}^{k(r)} / \bar{p}_{r}^{k(r)}$ for all $r \in \bar{N}$ since $f$ satisfies PAP in the third equality, and that $\bar{p}_{r}^{k}=0$ for all $k \in T \backslash\{k(r)\}, r \in \bar{N}$ in the fourth equality. Hence, since $q(\bar{P})=q(P)$,

$$
\begin{equation*}
f_{t}(\bar{P})=\frac{q(P)}{|T|} \frac{\bar{p}_{t}^{k(t)}}{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(t)}} \tag{2.3}
\end{equation*}
$$

Now, let $i \in N \backslash Z(P)$. Then,

$$
f_{i}(P)=\sum_{r \in R_{i}} f_{r}(\bar{P})
$$

$$
\begin{aligned}
& =\sum_{r \in R_{i}} \frac{q(P)}{|T|} \frac{\bar{p}_{r}^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_{s}^{k(r)}} \\
& =\frac{q(P)}{|T|} \sum_{r \in R_{i}} \frac{p_{i}^{k(r)}}{\sum_{s \in N} p_{s}^{k(r)}} \\
& =\frac{q(P)}{|T|} \sum_{k \in T: p_{i}^{k}>0} \frac{p_{i}^{k}}{\sum_{s \in N} p_{s}^{k}} \\
& =\frac{q(P)}{|T|} \sum_{k \in T} \frac{p_{i}^{k}}{\sum_{s \in N} p_{s}^{k}} \\
& =\rho_{i}(P),
\end{aligned}
$$

where we use the fact that $f$ satisfies EFF, NUL, and IUR to apply IUR and in particular Lemma 2.4 .2 repeatedly in the first equality, equation (2.3) in the second equality, the definition of $\bar{p}_{r}^{k}$ for every replica $r \in R_{i}$ in the third equality, and the one-to-one correspondence between $R_{i}$ and $\left\{k \in T \mid p_{i}^{k}>0\right\}$ in the fourth equality.

Finally, let $i \in Z(P)$. Since $f$ satisfies NUL, we get $f_{i}(P)=0=\rho_{i}(P)$.

We conclude that $f_{i}(P)=\rho_{i}(P)$ for any $i \in N$.

Similar to the previous section, we show that the aforementioned four properties are logically independent. For each subset of three properties, we define an alternative solution concept $f$ on $\mathcal{P}$ with $f \neq \rho$ that satisfies these properties.

No EFF: Consider $f(P)=2 \rho(P)$ for all $P \in \mathcal{P}$.

No NUL: For any finite $N$, fix a representation function $g_{N}: 2^{N} \backslash\{\emptyset\} \rightarrow N$ such that $g_{N}(S) \in S$ for all $S \in 2^{N} \backslash\{\emptyset\}$. Consider the solution concept $f$ on $\mathcal{P}$ that equals $\rho$ if there are no null players and allocates all value to one specific null player otherwise, formally defined by

$$
f(P)= \begin{cases}\rho(P) & \text { if } Z(P)=\emptyset \\ q(P) e_{g_{N}(Z(P))} & \text { if } Z(P) \neq \emptyset\end{cases}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ with $Z(P) \neq \emptyset$. Then,
$f(P)$ allocates $q(P)$ (to a single null player), so EFF is satisfied for such projects. Next, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ with $Z(P) \neq \emptyset$ and such that $\left\{N^{k}\right\}_{k \in T}$ is a partition of $N \backslash Z(P)$, and let $i \in N \backslash Z(P)$. Note that $f_{i}(P)=0$, so PAP is not violated either. For IUR, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ with $Z(P) \neq \emptyset$, let $i \in N \backslash Z(P)$ and $j \in N \backslash(Z(P) \cup\{i\})$, and let $\bar{P}_{i} \in \mathcal{P}$ be a corresponding replicate project as defined in Definition 2.4.1. Importantly, $Z(P) \neq \emptyset$ implies that $Z\left(\bar{P}_{i}\right) \neq \emptyset$ as well. Hence, $f\left(\bar{P}_{i}\right)=q\left(\bar{P}_{i}\right) e_{g_{N}\left(Z\left(\bar{P}_{i}\right)\right)}$. Further, $j \notin Z\left(\bar{P}_{i}\right)$, so that $f_{j}\left(\bar{P}_{i}\right)=0=f_{j}(P)$, even if $g_{N}\left(Z\left(\bar{P}_{i}\right)\right) \neq g_{N}(Z(P))$. This shows that $f$ satisfies IUR as well, when restricting to projects with null players. Finally, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ with $Z(P)=\emptyset$, in which case $f(P)=\rho(P) . \rho$ satisfies all four properties for all $P \in \mathcal{P}$, so in particular for all projects without null players. Hence, EFF, PAP, and IUR are satisfied.

No PAP: Let $2^{T}$ denote the collection of subsets of $T$. Fix a representation function $g: 2^{T} \backslash\{\emptyset\} \rightarrow T$ such that $g(S) \in S$ for all $S \in 2^{T} \backslash\{\emptyset\}$. Consider the solution concept $f$ on $\mathcal{P}$ that only allocates value to players in one fixed task group, defined by

$$
f_{i}(P)=q(P) \frac{p_{i}^{g(T)}}{\sum_{r \in N} p_{r}^{g(T)}}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ and any $i \in N$. EFF is clearly satisfied. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ with $Z(P) \neq \emptyset$ and let $i \in Z(P)$. Then, $p_{i}^{g(T)}=0$ and hence $f_{i}(P)=0$, so NUL is satisfied. Next, let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$, let $i \in N \backslash Z(P)$ and $j \in N \backslash(Z(P) \cup\{i\})$, and let $\bar{P}_{i} \in \mathcal{P}$ be a corresponding replicate project as defined in Definition 2.4.1. Regardless of whether player $i$ is replaced in task $g(T)$ or in some other task, it holds that $\bar{p}_{j}^{g(T)}=p_{j}^{g(T)}$ and that $\sum_{r \in \bar{N}} \bar{p}_{r}^{g(T)}=\sum_{r \in N} p_{r}^{g(T)}$. Since $q\left(\bar{P}_{i}\right)=q(P)$ as well, we have $f_{j}\left(\bar{P}_{i}\right)=f_{j}(P)$, meaning IUR is satisfied.

No IUR: Fix a representation function $g: 2^{T} \backslash\{\emptyset\} \rightarrow T$ such that $g(S) \in S$ for all $S \in 2^{T} \backslash\{\emptyset\}$. Let $T_{i}=\left\{k \in T \mid p_{i}^{k}>0\right\}$. Consider the solution concept $f$ on $\mathcal{P}$ for which the value allocated to players is fully determined by (their relative success probability in) one specific task group the player appears in, formally defined by

$$
f_{i}(P)= \begin{cases}q(P) \frac{p_{i}^{g\left(T_{i}\right)} / \sum_{r \in N} p_{r}^{g\left(T_{i}\right)}}{\sum_{j \in N \backslash Z(P)}\left(p_{j}^{g\left(T_{j}\right)} / \sum_{r \in N} p_{r}^{g\left(T_{j}\right)}\right)} & \text { if } i \notin Z(P), \\ 0 & \text { if } i \in Z(P)\end{cases}
$$

for all $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ and any $i \in N$. By definition, we clearly see that EFF and NUL are satisfied. Let $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}$ such that $\left\{N^{k}\right\}_{k \in T}$ is a partition of $N \backslash Z(P)$ and let $i \in N \backslash Z(P)$. Then, $N^{g\left(T_{i}\right)}$ gives the only task group that contains $i$, and

$$
\frac{\sum_{r \in N} p_{r}^{g\left(T_{i}\right)}}{p_{i}^{g\left(T_{i}\right)}} f_{i}(P)=\frac{q(P)}{\sum_{j \in N \backslash Z(P)}\left(p_{j}^{g\left(T_{j}\right)} / \sum_{r \in N} p_{r}^{g\left(T_{j}\right)}\right)}
$$

is constant, independent of $i$. Hence, PAP is satisfied.

To conclude this section, we note that similar to the first characterization, the proportional influence measure $\rho$ is still the only solution concept that satisfies the four properties on the subdomain $\mathcal{S}$ of sequential projects with perfect reliability. Here, we omit the (direct) reformulations of the properties on $\mathcal{S}$. Similar to Section 2.3, the proof of Theorem 2.4.4 directly follows from the proof of Theorem 2.4.3.

## Theorem 2.4.4

Let $f$ be a solution concept on $\mathcal{S}$. Then, $f=\rho$ if and only if $f$ satisfies EFF, NUL, PAP, and IUR.

### 2.5 Extensions and a game-theoretic approach

The proportional influence measure implicitly assumes that all tasks have equal importance, since in its definition the success probability $q(P)$ is shared equally among the tasks. This can be justified by the fact that all tasks need to be carried out for the project to be completed. In practice, however, some tasks may be more costly or time-consuming, which could warrant an unequal division. The proportional influence measure can be modified quite straightforwardly to capture this, by assigning weights to each task. Extending the characterizations to account for these weights only requires adaptations to the properties task decomposability and partition proportionality.

Further, we assume that the tasks of a project can be carried out sequentially, and that all players in a task group can attempt to carry out this task. In certain contexts, this final assumption could imply that several players within a task group attempt to
carry out the task simultaneously, in which case it might happen that the task is carried out more than once. If all tasks are carried out more than once, the same project can be repeated, i.e., completed more than once. Assuming perfect reliability, Lindelauf (2011) proposes a 'project power measure' for such repeated projects by dividing players in three categories and assigning project power equally to each player within the same category. To allow for more differentiation between players, a modification of the proportional influence measure could be used as a solution concept for repeated projects as well, even with imperfect reliability. Essentially, instead of multiplying the proportional influence measure by $q(P)$, the probability that a project is completed, one could multiply by the expected number of completed projects. With minor adaptations to the properties efficiency and task decomposability, the characterizations can be adjusted to fit this context as well.

The proportional influence measure is a solution concept based directly on the task structure of a project. This solution concept accounts for the success probabilities of all players in $N$, but does not explicitly take into account the ability of subcoalitions of $N$ to complete the project. To analyze this interesting topic, we can model the situation corresponding to some $P \in \mathcal{P}^{N}$ as a cooperative (transferable utility) game, in which appropriate values for coalitions are quantified using the characteristic function $v_{P}: 2^{N} \rightarrow \mathbb{R}$. Based on such a game, the influence of all players in $N$ can then be measured using a game-theoretic solution concept like the Shapley value.

To give an impression of how to define an appropriate associated game, let $P=$ $\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{P}^{N}$ be a sequential project with imperfect reliability. One possible corresponding game $v_{P}$ could be defined by setting the value of a coalition $S \in 2^{N} \backslash\{\emptyset\}$ equal to the success probability of that coalition by means of players in $S$ only:

$$
v_{P}(S)=\prod_{k \in T}\left(1-\prod_{i \in S}\left(1-p_{i}^{k}\right)\right)
$$

for any $S \in 2^{N} \backslash\{\emptyset\}$. Note that $v_{P}(N)=q(P)$.

## Example 2.5.1

Consider the project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ of Example 2.2.1, with $N=\{1,2,3\}$, $T=\{a, b\}, p^{a}=(0.8,0.9,0)$ and $p^{b}=(0.8,0,1)$. The game $v_{P}$ is given in Table 2.1.

This game is a consistent representation from which we can derive an allocation of influence, thereby enabling a comparison of the relative importance of players in the

| $S$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{P}(S)$ | 0.64 | 0 | 0 | 0.784 | 0.8 | 0.9 | 0.98 |

Table 2.1 The game $v_{P}$ of Example 2.5.1
completion of the project. In this example, we do so using the Shapley value, denoted by $\Phi\left(v_{P}\right)$. The marginal vectors used to determine $\Phi\left(v_{P}\right)$ are given in Table 2.2.
$\left.\begin{array}{c|ccc}\sigma & m_{1}^{\sigma}\left(v_{P}\right) & m_{2}^{\sigma}\left(v_{P}\right) & m_{3}^{\sigma}\left(v_{P}\right) \\ \hline\left(\begin{array}{ll}1 & 2\end{array} 3\right) & 0.64 & 0.144 & 0.196 \\ (13 & 1 & 2\end{array}\right)$

Table 2.2 Marginal vectors for all orderings corresponding to the game $v_{P}$ of Example 2.5.1

The Shapley value is determined as the average of these marginals vectors, which yields $\Phi\left(v_{P}\right)=(0.504,0.234,0.242)$. Note that this allocation is quite similar to the proportional influence measure $\rho(P) \approx(0.45,0.26,0.27)$.

Restricting to projects with perfect reliability, the corresponding cooperative game $v_{P}$ becomes a simple game (i.e., $v_{P}(S) \in\{0,1\}$ for all $S \in 2^{N}, v_{P}(S) \leq v_{P}(T)$ for all $S, T \in 2^{N} \backslash\{\emptyset\}$ with $S \subseteq T$, and $v(N)=1$ ). Then, one could also consider solution concepts defined specifically on the class of simple games (often referred to as 'power indices'). Two prominent examples of such solution concepts are the Deegan-Packel (DP) index (Deegan and Packel, 1978), and the Public Good (PG) index (Holler, 1982). Both are based on so-called minimal winning coalitions.

Denoting the class of simple games with player set $N$ by $S I^{N}$, the set of minimal winning coalitions of some $v \in S I^{N}$ is defined by

$$
M W C(v)=\left\{S \in 2^{N} \backslash\{\emptyset\} \mid v(S)=1 \text { and } T \subsetneq S \Rightarrow v(T)=0\right\} .
$$

The set of all minimal winning coalitions that contain a player $i \in N$ is denoted by $M W C_{i}(v)$, i.e.,

$$
M W C_{i}(v)=\{S \in M W C(v) \mid i \in S\}
$$

The DP-index gives equal value to each minimal winning coalition. Within such a coalition, the value is distributed equally over the players. Hence, the Deegan-Packel index $D P: S I^{N} \rightarrow[0,1]^{N}$ is formally defined by

$$
D P_{i}(v)=\frac{1}{|M W C(v)|} \sum_{S \in M W C_{i}(v)} \frac{1}{|S|}
$$

for all $i \in N$ and all $v \in S I^{N}$.
The Public Good index only takes into account the number of times each player is in a minimal winning coalition; the size of a minimal winning coalition does not directly influence this index. Formally, the Public Good index $P G: S I^{N} \rightarrow[0,1]^{N}$ is defined by

$$
P G_{i}(v)=\frac{\left|M W C_{i}(v)\right|}{\sum_{j \in N}\left|M W C_{j}(v)\right|}
$$

for all $i \in N$ and all $v \in S I^{N}$.

## Example 2.5.2

Consider the project $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right)$ of Example 2.5.1, but now with perfect reliability, as also considered in Example 2.2.2. The corresponding game $v_{P}$ is readily found on the basis of Table 2.1, by replacing every positive number with a one.

Recall from Example 2.2 .2 that $\rho(P)=(0.5,0.25,0.25)$. One readily finds that $\Phi\left(v_{P}\right)=\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$. Further, note that $M W C\left(v_{P}\right)=\{\{1\},\{2,3\}\}$, so that $D P\left(v_{P}\right)=$ $\frac{1}{2}\left(1, \frac{1}{2}, \frac{1}{2}\right)=(0.5,0.25,0.25)$ and $P G\left(v_{P}\right)=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The proportional influence measure, the Deegan-Packel index, and the Public Good index do not generally yield the same allocation of influence. However, when restricting to projects $P=\left(N, T,\left\{p^{k}\right\}_{k \in T}\right) \in \mathcal{S}^{N}$ for which the task structure $\left\{N^{k}\right\}_{k \in T}$ partitions $N$, they do coincide. To see this, there are two key insights related to the fact that in that case each player is in exactly one task group. First, all minimal winning coalitions are of the same size, namely the number of tasks. Second, the fraction of minimal winning coalitions that contain a player is equal to the reciprocal of the size of the task group to which this player belongs. This readily leads to the following result.

## Proposition 2.5.1

Let $P \in \mathcal{S}^{N}$ and let $v_{P}$ be the corresponding cooperative game. Then,

$$
\rho(P)=D P\left(v_{P}\right)=P G\left(v_{P}\right)
$$

# Cost allocation in CO2 transport for CCUS hubs: a multi-actor perspective 

### 3.1 Introduction

The importance of reducing CO2 emissions to limit human-induced climate change is widely acknowledged (IPCC, 2022). Transforming energy-intensive industries plays a key role in this reduction of CO2 emissions (or 'decarbonization'), as they account for $20 \%$ of all CO2 emissions (IEA, 2020). New and adapted infrastructures are necessary for a successful and timely industrial transformation (de Bruyn et al., 2020; Janipour et al., 2020; Fu et al., 2018). Hydrogen transport and distribution, heat distribution, reinforced electricity grids and CO2 transport infrastructures are primary topics in national, European and worldwide climate change plans.

In this chapter, based on Van Beek et al. (2023a), we provide a multi-actor perspective on these necessary new infrastructures. We develop a generic model to represent infrastructures and analyze the nature of the corresponding construction costs. In particular, this model is 'component-based' to account for the heterogeneous requirements of the users of an infrastructure. By decomposing an infrastructure into a unique component structure, we explicitly differentiate between users based on whether they require, e.g., CO2 conditioning for re-use, or offshore transport. Importantly, we identify and adopt a cost function with a very general structure, for which we can explicitly determine the costs of each individual component. Although the generality of the cost function allows for a wide range of applications within infrastructure construction problems, it is developed primarily with a specific application
in mind, namely CO2 transport infrastructures for carbon capture, utilization and storage (CCUS). This is a potentially effective, but also heavily debated pathway towards decarbonization. A significant part of this chapter is dedicated to a discussion of such a CO2 transport infrastructure in the port of Rotterdam and the adjoining industry area. The case study most prominently illustrates the type of application our model allows.

Before moving to the more general contributions of our research, we start with a discussion of these carbon capture related technologies and explain the need for a multi-actor perspective in the CO2 transport infrastructure application. Carbon capture and storage (CCS) is the process of capturing waste carbon dioxide, transporting it to a storage site, and depositing it. The aim of CCS is to prevent the release of large quantities of CO2 into the atmosphere. Carbon capture and utilization (CCU) is the process of capturing carbon dioxide to be recycled for further usage. CCU aims to convert the captured carbon dioxide into more valuable substances or products. Detz and van der Zwaan (2019) and IEA (2020) expect a future increase of CO2 demand by industry for, e.g., production of blue hydrogen and methane. In new large scale carbon capture projects CCS and CCU are often combined into CCUS projects. CCUS recently gained more traction. Plans for significant investments in CCUS technologies can be found in European industrial transformation strategies (EC, 2020), in the World Energy Outlook (IRENA, 2021), but also in national plans such as in the UK and in the Netherlands. IEA (2020) dedicate a special technology outlook on the role of CCUS in decarbonization of energy-intensive industries. They conclude that CO2 reduction targets probably cannot be achieved without the carbon capture option, and that many technologies necessary for CCU and especially CCS seem to be fully developed. In many regions, CCUS could be a cost effective solution for decarbonization of such industries. However, IEA (2020) warns: "Infrastructure to transport and store CO2 safely and reliably is essential for rolling out CCUS technologies." They recommend the further development of shared CO2 transport and storage infrastructures in a regional industrial hub or cluster. This cluster approach is not a coincidence. Energy-intensive industries often use the same spatial characteristics (e.g., harbor), each other's products (e.g., intermediates), or shared infrastructures (e.g., steam network). The existence of these clusters provides both opportunities and barriers for the necessary industry transition (see, e.g., Janipour et al. (2020) and Fu et al. (2018)). Accordingly, Quarton and Samsatli (2020) explain that significant
stakeholder collaboration is required for CCUS investments and implementation. In case of CCUS technologies, both CO2 emitters and CO2 users could benefit from a CO2 transport infrastructure, but they do not have the same requirements regarding, e.g., the quality of the CO 2 or the transport capacity to a storage site. Given these differences in user requirements, it is important to consider how to allocate the total construction costs of such an infrastructure to the different users. This requires a multi-actor perspective, explicitly taking into account the heterogeneity of different (potential) users.

Of course, appropriate cost allocation is not the only enabler of such a CO 2 transport infrastructure, but we believe it is one of the key aspects of collaboration on infrastructure construction. It is desirable that the cost allocation method keeps existing users on board, since a significant majority of cluster participants need to go along with infrastructural investments for its successful realization. If all users are asked for cost contributions based on, e.g., only total transport capacity of the network, some of them might very well decide not to participate in the CO2 network. A 'no' from a subgroup of CO2 network users in a regional industrial cluster will increase investment costs for remaining users, which in return can result in a 'no' from another group of potential CO2 network users. On top of maintaining interested parties on board, appropriate cost allocation methods can even increase the number of participants of the CO2 network and thus speed-up the decarbonization of an industrial cluster. Indeed, SER (2019) mentions that decarbonization investments from heavy emitters can create traction or a 'piggyback effect' on small emitters. A piggyback effect for new entrants may only be admissible if their entrance is also beneficial to the existing group of users. This is the case if the cost allocation method satisfies the property advantageous scaling: the costs allocated to each existing user do not increase if the number of users grows larger. Further, to keep the existing users of the infrastructure on board, it is essential that partial cooperation is not profitable. This is ensured by coalitional rationality. This property implies that the cost allocation is stable against coalitional deviations: no subgroup of the existing users has a financial reason to object to the cost allocation, because none of these subgroups can benefit from splitting off from the group of existing users. The equal component cost sharing rule we propose for infrastructure construction problems in general and CO2 transport infrastructures in particular satisfies both advantageous scaling and coalitional rationality. It is based on the idea that participants (equally) pay only for those infrastructure components that they use.

Having established the need for a multi-actor perspective on CO2 transport infrastructures specifically, we now discuss the more general contributions of this chapter. User requirements towards new infrastructures usually differ over several characteristics, like transport radius, capacity, and conditioning. That is why we introduce so-called component-based infrastructure cost problems in which we construct an infrastructure that satisfies all user requirements over an arbitrary number of characteristics. Our modeling is based on the mathematical ability to uniquely decompose an infrastructure into a component structure such that each infrastructure component is either required by a user in its entirety, or not required at all. Each Elementary Infrastucture Component (EIC) will represent a unique part of the potential infrastructure that may or may not be required by the users, where we remark that EICs need not always directly correspond to physical components of the infrastructure, they are used to decompose the entire 'system'. Consequently, the infrastructure that is required by a group of users is given by a set of EICs.

We identify and adopt a generic function to determine the total infrastructure construction costs for a given set of requirements. This cost function is constructed in such a way that it can be easily adapted to many different types of infrastructures. In particular, it allows for both continuous and discrete (categorical) characteristics, and for costs to be assigned to any combination of characteristics. Discrete characteristics can represent characteristics that, e.g., only depend on whether or not a user requires a certain specification. Importantly, despite the generic structure of the cost function, we are able to derive the costs of each individual EIC. The (minimal) costs for any group of users can then straightforwardly be determined as the sum of the costs of the required EICs. Hence, using our component-based analysis of infrastructures, we are able to uniquely decompose an infrastructure into EICs, assign a cost to each EIC, and for each EIC we can pinpoint exactly which users need this component. From this, as a natural follow-up, a way of allocating costs is derived: the equal component cost sharing rule. For each required component, the costs are divided equally among all users that require this component.

This allocation mechanism is completely in line with existing fairness principles and procedures as provided within the game-theoretic allocation literature. In particular, conceptually, our allocation proposal follows Littlechild and Thompson (1977), who develop a model for allocating the costs of an aircraft landing strip to different types of users, in such a way that users only pay for the part of the strip that they
require. Their model and the corresponding cost allocation method are based on a single characteristic of the infrastructure, namely the length of the landing strip. Kuipers et al. (2013) introduce highway games for allocating construction costs of a highway to its users, taking into account that users may require different parts of the highway and ensuring that users only pay for those highway stretches that they require. There are several extensions or adaptations of highway games in which user requirements differ over two characteristics, see, e.g., Sudhölter and Zarzuelo (2017) for an overview of highway games and properties of different types of allocation methods. Our allocation mechanism in a collaborative infrastructure construction setting adds to this work by allowing user requirements to differ over any positive number of characteristics, instead of only two.

To further reflect on the role of game theory in the analysis of industrial clusters, Gedai et al. (2012) argue that game-theoretical models could help understand decision making in industrial clusters. Massol et al. (2018) use cooperative games to model CCS deployment specifically. They investigate the policy and economic conditions needed for a largest possible adoption of CCS technologies and networks, and to determine the break-even price for CCS adoption. Tan et al. (2016) and Andiappan et al. (2016) introduce cost allocation methods for newly developed multi-company industrial clusters, based on models from cooperative and non-cooperative game theory. In particular, both papers look at optimizing and allocating total cluster costs, including multiple infrastructures, for new sites.

We emphasize, however, that the main theoretical contribution and novelty of this chapter lies in the component-based modeling of infrastructures and in the definition of a cost function with a very generic structure, for which we can still determine the costs per component. The equal component cost sharing rule then simply is a natural follow-up that satisfies the desirable properties of advantageous scaling and coalitional rationality.

To complement the theoretical results in the technical section of the chapter, we subsequently study the specific case of a CO2 transport infrastructure for CCUShubs in the port of Rotterdam area in some detail. Here, we show how to transform a general description of such an infrastructure and its cost drivers into a componentbased infrastructure cost problem and we apply the equal component cost sharing rule. We also demonstrate the workings of the properties of advantageous scaling and coalitional rationality. Finally, since the cost parameters in our case study are based
on rough estimates, we analyze the behavior of the equal component cost sharing rule in various additional cost scenarios. In particular, we consider two scenarios in which either the fixed or the variable costs turn out to be higher than expected. Additionally, we consider a scenario in which the cost function changes more fundamentally, where certain costs depend on two characteristics instead of one. In each scenario, we find that the changes to the cost allocation accurately reflect which players 'should' be most affected by the changes. For example, distant emitters are most affected by increased variable transport costs, and additional costs to condition CO 2 for reuse are only allocated to those that actually require conditioning for re-use. The final scenario also demonstrates the adaptability of the cost function in a more general way.

The chapter is organized as follows. Section 3.2 discusses the CO2 transport infrastructure case as a specific application of our model, considering its users and their requirements towards this infrastructure. In Section 3.3 we formally define our component-based infrastructure model, discussing the EICs, the cost function, and the equal component cost sharing rule and its properties. In Section 3.4 we apply the general model to our case study.

### 3.2 CO2 transport infrastructure: users, requirements and costs

Currently there are (at least) 12 regional open CO2 transport hubs under development globally. In this section, we provide a qualitative description of the case of a CO2 transport infrastructure inspired by a large CCUS project in an industrial cluster in the Netherlands, specifically the Porthos initiative in the Rotterdam port area (Porthos, 2022). After a brief overview of the basic elements of a CO2 transport infrastructure, we consider the potential users of this specific infrastructure and their heterogeneous requirements, and how these requirements drive the infrastructure construction costs.

The port authority of Rotterdam and two partners have set out to develop an open, collaborative and long term CO2 network that facilitates transport, storage and reuse of CO2. A sketch of such a CCUS transport infrastructure network can be found in Figure 3.1. The basic idea is that captured CO2 is gathered onshore at the dif-


Figure 3.1 Schematic overview of CCUS hub adapted from IEA (2020)
ferent industry sites through a network of feeders and (a) main transport pipeline(s). After the gathering of the CO2 there are roughly two options. The gathered CO2 is either transported through a main transport pipeline towards the shore where it will be conditioned (pressure and temperature) for offshore transport towards identified storage fields close to the Rotterdam harbour industrial complex (see, e.g., EBN and Gasunie (2018) for potential CO2 storage fields in the Dutch North Sea), or it will be conditioned (pressure, temperature and purity) for re-use purposes and transported through a transport pipeline to sites that use CO2 as feedstock. The capacity of the main pipelines determines the capacity of the transport network. The blue lines in Figure 3.1 represent the CO2 transport network for a CCUS hub.

The following CO2 infrastructure users are in scope for this open network. First, the heavy emitters: the large scale (petro)chemical producers that see no short term cost effective alternative to CO2 emission reduction. They benefit from a large scale CO2 infrastructure with long term storage facilities. Second, the distant emitters: heavy emitters to be found a bit further away from the Rotterdam port area, such as in the Moerdijk area or the Zeeland chemical industrial cluster (DNVGL, 2020). They
have similar requirements towards a CO2 transport network as heavy emitters, but they would require a larger onshore transport radius of the pipelines as the identified offshore storage fields are closest to the Rotterdam industrial complex. Small emitters: small petrochemical producers that do not aim for a large transport capacity. For them investing in carbon capture technologies might become attractive because of the availability of shared CO2 transport infrastructure. Next to the emitters, there are also potential users of the captured CO2. Hydrogen producers and greenhouses: these users of CO2 also require a smaller capacity and they are not interested in offshore transport to storage facilities, but they do have some more conditioning requirements for the CO 2 , e.g., its purity needs to be higher than for common onshore transport purposes. Hence, there are two levels of conditioning that can be required for onshore transport: standard conditioning for onshore transportation and conditioning for re-use purposes. Table 3.1 gives a summary of the potential users and their heterogeneous requirements towards the CO 2 transport infrastructure.

| user | onshore <br> transport radius | offshore <br> transport | capacity | conditioning |
| :---: | :--- | :--- | :--- | :--- |
| heavy emitters | Large Rotterdam area | yes | large | standard |
| distant emitters | Zeeland area | yes | large | standard |
| small emitters | Large Rotterdam area | yes | small | standard |
| greenhouses | Large Rotterdam area | none | small | highly purified |
| hydrogen producers | Small Rotterdam area | none | small | highly purified |

Table 3.1 Stylized description of the user requirements for a regional CO2 transport infrastructure

Next, we discuss how these requirements drive the infrastructure construction costs. A more detailed argumentation is given at the start of Appendix 3.A. The costs consist of fixed (system) and variable (pipeline) costs. Following Serpa et al. (2011), we express a linear relationship between the costs and the pipeline length (i.e., required transport radius), where the specific cost parameters depend on the terrain (onshore or offshore) and the transport capacity. Further, because conditioning mostly occurs in separate stations, we assume that conditioning requirements only influence the fixed portion of the costs, together with the capacity. Finally, offshore transport requires more advanced conditioning than standard onshore transport (so there is actually a third option for the conditioning). These conditioning costs are included in the fixed costs for offshore transport.

With this application in mind, Section 3.3 presents a general framework for the component-based analysis of such infrastructures and their costs, with a natural cost allocation rule as a direct result. Finally, we remark that several users in Table 3.1 are not yet part of the core group of interested participants, while they could benefit from the use of such a transport infrastructure. Both DNVGL (2020) and the Rotterdam port authority have sketched potential future connections to the more distant emitters and future CO2 users in their CO2 transport infrastructure plans. The potential addition of users is a reason to look for cost allocation methods that satisfy the advantageous scaling property: adding new users to the project might raise the total construction costs, but it will not result in a cost allocation increase for any of the original users. Further, coalitional rationality ensures that there is no financial reason for any subgroup of users to split off from the group and carry on independently. Both properties will be formally defined in Section 3.3 .3 , where we also show that they are satisfied by the cost allocation rule we propose.

### 3.3 Component-based analysis of infrastructure construction problems

In this section, we introduce the theoretical framework to perform a componentbased analysis of infrastructure construction problems, also from a cost allocation perspective.

### 3.3.1 Elementary infrastructure components in infrastructure problems

The situation in which players have different requirements for certain characteristics of an infrastructure that must be constructed, is referred to as an infrastructure problem. Formally, we define such a problem with the tuple

$$
(N, M, X)
$$

where $N$ represents a finite player set, $M=\{1, \ldots, m\}$ represents a finite set of characteristics, and $X$ represents a requirement matrix of which the rows correspond to $N$ and the columns to $M$, such that the cell in the $i$-th row and $k$-th column, $X_{i}^{k} \in \mathbb{R}_{+}$, indicates the value that player $i \in N$ requires for characteristic $k \in M$.

Let ( $N, M, X$ ) be an infrastructure problem. We denote the column of $X$ with respect to $k \in M$ by $X^{k}$. For any $k \in M, Z^{k}=\left\{X_{i}^{k} \mid i \in N\right\}$ is defined as the set of unique values in $X^{k}$, and $n^{k}=\left|Z^{k}\right|$ denotes the number of distinct requirements for characteristic $k$. Then, $\tilde{X}^{k} \in \mathbb{R}_{+}^{n^{k}}$ is the vector containing the elements of $Z^{k}$ sorted in increasing order. To emphasize, for any $k \in M$ and $\alpha_{k} \in\left\{1, \ldots, n^{k}\right\}, \tilde{X}_{\alpha_{k}}^{k}$ represents the $\alpha_{k}$-th lowest value required by the players of the $k$-th characteristic. Moreover, we set $\tilde{X}_{0}^{k}=0$ for all $k \in M$.

Using this notation, an Elementary Infrastructure Component (EIC) is defined by

$$
C^{\alpha_{1}, \ldots, \alpha_{m}}=\prod_{k=1}^{m}\left[\tilde{X}_{\alpha_{k}-1}^{k}, \tilde{X}_{\alpha_{k}}^{k}\right]
$$

where $\alpha_{k} \in\left\{1, \ldots, n^{k}\right\}$ for any $k \in M$. Essentially, the origin (i.e., the point of the EIC with the lowest values of all characteristics) of $C^{\alpha_{1}, \ldots, \alpha_{m}}$ is ( $\tilde{X}_{\alpha_{1}-1}^{1}, \ldots, \tilde{X}_{\alpha_{m}-1}^{m}$ ) and the 'end point' (i.e., the point of the EIC with the highest values of all characteristics) is $\left(\tilde{X}_{\alpha_{1}}^{1}, \ldots, \tilde{X}_{\alpha_{m}}^{m}\right)$. In this way, we define a total of $\prod_{k=1}^{m} n^{k}$ EICs. We let $\mathcal{C}=\left\{C^{\alpha_{1}, \ldots, \alpha_{m}} \mid \alpha_{k} \in\left\{1, \ldots, n^{k}\right\}, k \in M\right\}$ denote the collection of all EICs. We remark that if $\tilde{X}_{1}^{k}=0$ for some $k \in M$, one dimension of certain components starts and ends at the same point: $\tilde{X}_{0}^{k}=\tilde{X}_{1}^{k}=0$. This does not lead to any issues.

Each EIC represents a unique part of the potential infrastructure, in such a way that each player either requires the entire EIC, or does not require it at all. Comparing two EICs with the same values for all characteristics except one, the EIC with a higher value for one characteristic comes on top of the EIC with lower value for this characteristic, it does not replace the EIC with lower value. To clarify this in the context of a CO2 transport network, consider Figure 3.2. The figure illustrates a 2-player infrastructure problem with two characteristics, radius and capacity, in which player $A$ requires a network with large capacity in a short radius, while player $B$ requires a smaller capacity, but a longer radius. In this example, the EIC corresponding to the top-left square represents the additional capacity required by player $A$, compared to player $B$, within the short radius. That is why player $A$ requires both the 'small capacity' and the 'large capacity' square. Importantly, not all of the components in $\mathcal{C}$ are necessarily required by the player set $N$. In the example, we define an EIC corresponding to a large capacity over the longer radius as well, but this is clearly not required by either of the players.

Concretely, we say that an EIC is required by the player set $N$ if and only if there is at least one player in $N$ who requires the corresponding characteristic values for all


Figure 3.2 Example of an infrastructure problem with two players $(A$ and $B)$ and two characteristics (radius and capacity). In each square, the set of players who require this square is given.
characteristics. So, the corresponding (minimal) set $A(N)$ of all EICs required by $N$ is formally defined by

$$
\begin{equation*}
A(N)=\left\{C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C} \mid \exists i \in N \text { such that } \forall k \in M \text { we have } X_{i}^{k} \geq \tilde{X}_{\alpha_{k}}^{k}\right\} \tag{3.1}
\end{equation*}
$$

The set $A(N)$ is also referred to as the minimal infrastructure that $N$ requires. In Figure 3.2, $A(N)$ is given by the three colored squares.

### 3.3.2 Component-based infrastructure cost problems

In this section, we discuss the costs corresponding to an infrastructure problem, thereby defining the component-based infrastructure cost problem. To be able to determine the exact costs of a required infrastructure, we first define the general cost function $\kappa: Z^{1} \times \cdots \times Z^{m} \rightarrow \mathbb{R}$ that gives the construction costs of a so-called boxlike infrastructure. Here, it is important to clarify the difference between a minimal and a boxlike infrastructure. In a minimal infrastructure, determined by $A(N)$ for player set $N$, the required value of one characteristic may decrease as another characteristic increases. In Figure 3.2 for example, the required capacity decreases from large to small as the radius increases from short to long. In a boxlike infrastructure, all characteristic values are fixed. In the example, $\kappa$ would be used to reflect the costs of a network with a fixed capacity level and a fixed radius, it is not suitable to directly determine the costs of a network with capacity levels that vary depending on the ra-
dius. Graphically, $\kappa$ determines the costs corresponding to any rectangle drawn from the bottom left corner, but not (directly) the total costs corresponding to the three colored squares in Figure 3.2.

However, the crucial feature of the general cost function $\kappa$ is that through its definition one can derive a closed-form formula for the construction costs incurred due to the 'presence' of each EIC individually. Using this, we find the minimal infrastructure construction costs by summing the costs of all EICs in $A(N)$.

Before formally defining $\kappa$, it is good to briefly discuss its general structure. The cost function allows for both continuous and discrete (categorical) characteristics. Where a continuous characteristic (like onshore transport radius) could take any nonnegative value, a discrete characteristic can take a restricted set of values (like 0 and 1 , depending on whether a player requires CO2 conditioning for re-use). On top of characteristics that are discrete by nature, such discrete characteristics may also be useful to represent characteristics that are in fact continuous, but for which the (unit) costs are grouped into certain categories, or an exact cost function is uncertain or difficult to determine in practice. By dividing the required values of such a characteristic into categories, it suffices to only find cost estimates for when the characteristic is, say, 'small' or 'large'. For example, cost estimates exist for a CO2 transport infrastructure with specific capacity levels (e.g., $2.5 \mathrm{Mt} / \mathrm{y}$ and $10 \mathrm{Mt} / \mathrm{y}$ ), but a continuous cost function for any capacity level may be significantly harder to determine (and perhaps unnecessary).

When quantifying costs related to qualitative characteristics, these characteristics should have a specific, ordinal structure, in such a way that if a player requires a certain level of a characteristic, this player also requires all lower levels of the characteristic. We assume requirements are not substitutable, both within a characteristic and across characteristics.

For every combination of continuous and discrete characteristics, we define a coefficient based on the discrete characteristics that determines the slope of a linear relation between the costs and the product of the continuous characteristics for this particular combination. We then sum over all combinations of characteristics to obtain the total cost function. Of course, not every specific combination of characteristics necessarily leads to additional costs, so many of the coefficients may be zero.

## Definition 3.3.1

Let $(N, M, X)$ be an infrastructure problem and partition $M$ into a set of continuous characteristics $M_{C}$ and a set of discrete characteristics $M_{D}$, so that $M_{C} \cup M_{D}=M$ and $M_{C} \cap M_{D}=\emptyset$. Then, we let

$$
I=(N, M, X, \kappa)
$$

denote a component-based infrastructure cost problem, with

$$
\kappa\left(z_{1}, \ldots, z_{m}\right)=\sum_{K \subseteq M_{C}} \sum_{L \subseteq M_{D}} \beta_{K}\left(\left\{z_{k}\right\}_{k \in L}\right) \prod_{k \in K} z_{k}
$$

where $z_{k} \in Z^{k}$ for all $k \in M$.
Note that each cost coefficient $\beta_{K}$ is essentially a function of the discrete characteristics in $L$, so that the cost coefficient takes different values depending on the specific values of the characteristics in $L$. For cost coefficients that do not depend on any discrete characteristic, we set $\beta_{K}(\emptyset)=\beta_{K}$.

## Example 3.3.1

To illustrate a component-based infrastructure cost problem $I=(N, M, X, \kappa)$, we now consider a general infrastructure problem based on Table 3.1, in which users have different requirements for four characteristics of a regional CO2 transport infrastructure. In particular, the characteristics are the onshore and offshore transport radius, the capacity of the network, and the conditioning of the CO2, respectively represented by $M=\{1,2,3,4\}$. As discussed previously, cost estimates may only be available for specific capacity levels rather than continuously for any capacity level. Therefore, the third characteristic will be treated as a discrete characteristic. The same holds for the fourth characteristic, so that $M_{C}=\{1,2\}$ and $M_{D}=\{3,4\}$. Written out in full, the cost function has 16 terms:

$$
\begin{array}{rllll}
\kappa\left(z_{1}, z_{2}, z_{3}, z_{4}\right) & =\beta_{\emptyset} & +\beta_{\emptyset}\left(z_{3}\right) & +\beta_{\emptyset}\left(z_{4}\right) & +\beta_{\emptyset}\left(z_{3}, z_{4}\right) \\
& +\beta_{\{1\}} z_{1} & +\beta_{\{1\}}\left(z_{3}\right) z_{1} & +\beta_{\{1\}}\left(z_{4}\right) z_{1} & +\beta_{\{1\}}\left(z_{3}, z_{4}\right) z_{1} \\
& +\beta_{\{2\}} z_{2} & +\beta_{\{2\}}\left(z_{3}\right) z_{2} & +\beta_{\{2\}}\left(z_{4}\right) z_{2} & +\beta_{\{2\}}\left(z_{3}, z_{4}\right) z_{2} \\
& +\beta_{\{1,2\}} z_{1} z_{2}+\beta_{\{1,2\}}\left(z_{3}\right) z_{1} z_{2}+\beta_{\{1,2\}}\left(z_{4}\right) z_{1} z_{2}+\beta_{\{1,2\}}\left(z_{3}, z_{4}\right) z_{1} z_{2}
\end{array}
$$

where $z_{k} \in Z^{k}$ for all $k \in M$. Many of the coefficients (and thereby the corresponding terms in $\kappa$ ) may be equal to zero, as will be discussed in Example 3.3.2.

Clearly, cost function $\kappa$ has a very generic structure, that can be adapted to reflect various infrastructure contexts. Despite this, it still allows the costs to be analyzed on a per component basis. Specifically, to analyze the costs of more sophisticated infrastructures, for which the requirement of a characteristic may vary for different values of other characteristics, we can derive the costs $\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)$ of each individual EIC $C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C}$. These costs can be iteratively determined on the basis of cost function $\kappa$, or, equivalently, using the closed-form expression (3.2), presented in Theorem 3.3.2, based on the inclusion-exclusion principle. In this theorem, we show that using (3.2) for the costs of each EIC, the costs of any boxlike infrastructure equal the sum of the costs of all EICs within this 'box'. We illustrate the workings of all elements of this theorem in Example 3.3.2.

## Theorem 3.3.2

Let $I=(N, M, X, \kappa)$ be a component-based infrastructure cost problem. Let the costs $\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)$ of an EIC $C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C}$ be given by

$$
\begin{equation*}
\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)=\sum_{M_{C}^{\alpha} \subseteq K \subseteq M_{C}} \sum_{M_{D}^{\alpha} \subseteq L \subseteq M_{D}} b^{\alpha_{1}, \ldots, \alpha_{m}}(K, L) \prod_{k \in K}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right), \tag{3.2}
\end{equation*}
$$

where $M_{C}^{\alpha}=\left\{k \in M_{C} \mid \alpha_{k}>1\right\}$ and $M_{D}^{\alpha}=\left\{k \in M_{D} \mid \alpha_{k}>1\right\}$, and

$$
\begin{equation*}
b^{\alpha_{1}, \ldots, \alpha_{m}}(K, L)=\sum_{T \subseteq M_{D}^{\alpha}}(-1)^{|T|} \beta_{K}\left(\left\{\tilde{X}_{\alpha_{k}}^{k}\right\}_{k \in L \backslash T},\left\{\tilde{X}_{\alpha_{k}-1}^{k}\right\}_{k \in T}\right) \tag{3.3}
\end{equation*}
$$

for all $M_{C}^{\alpha} \subseteq K \subseteq M_{C}$ and $M_{D}^{\alpha} \subseteq L \subseteq M_{D}$.
Then, for all $\zeta_{k} \in\left\{1, \ldots, n^{k}\right\}, k \in M$,

$$
\begin{equation*}
\kappa\left(\tilde{X}_{\zeta_{1}}^{1}, \ldots, \tilde{X}_{\zeta_{m}}^{m}\right)=\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\ k \in M}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)^{1} \tag{3.4}
\end{equation*}
$$

Proof. Let $\zeta_{k} \in\left\{1, \ldots, n^{k}\right\}$ for all $k \in M$. We show that the costs of the corresponding boxlike infrastructure, as determined by $\kappa\left(\tilde{X}_{\zeta_{1}}^{1}, \ldots, \tilde{X}_{\zeta_{m}}^{m}\right)$, equal the sum of the costs of all EICs within this box. In (3.2), for all $\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\}, k \in M$, we sum over all $K \subseteq M_{C}$ and $L \subseteq M_{D}$ such that $M_{C}^{\alpha}=\left\{k \in M_{C} \mid \alpha_{k}>1\right\} \subseteq K$

$$
1 \sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\ k \in M}} \text { is shorthand notation for } \sum_{\alpha_{1} \in\left\{1, \ldots, \zeta_{1}\right\}} \ldots \sum_{\alpha_{m} \in\left\{1, \ldots, \zeta_{m}\right\}} . \text { We use this notation }
$$

throughout the proof as well.
and $M_{D}^{\alpha}=\left\{k \in M_{D} \mid \alpha_{k}>1\right\} \subseteq L$, i.e., for all $k \in M$ such that $\alpha_{k}>1$ we have $k \in K \cup L$. Put differently, we only sum over $K \subseteq M_{C}$ and $L \subseteq M_{D}$ if $\alpha_{k}=1$ for all $k \in M \backslash(K \cup L)$. Consequently, we may rewrite (3.4) as

$$
\begin{aligned}
\kappa\left(\tilde{X}_{\zeta_{1}}^{1}, \ldots, \tilde{X}_{\zeta_{m}}^{m}\right) & =\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\
k \in M}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\sum_{K \subseteq M_{C}} \sum_{L \subseteq M_{D}} \sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\
k \in K \cup L}} b^{\alpha_{1}, \ldots, \alpha_{m}}(K, L) \prod_{k \in K}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right),
\end{aligned}
$$

where $\alpha_{k}=1$ for all $k \in M \backslash(K \cup L)$.
Recall that

$$
\kappa\left(\tilde{X}_{\zeta_{1}}^{1}, \ldots, \tilde{X}_{\zeta_{m}}^{m}\right)=\sum_{K \subseteq M_{C}} \sum_{L \subseteq M_{D}} \beta_{K}\left(\left\{\tilde{X}_{\zeta_{k}}^{k}\right\}_{k \in L}\right) \prod_{k \in K} \tilde{X}_{\zeta_{k}}^{k}
$$

Hence, we can now show that (3.4) holds by showing that

$$
\begin{aligned}
& \beta_{K}\left(\left\{\tilde{X}_{\zeta_{k}}^{k}\right\}_{k \in L}\right) \prod_{k \in K} \tilde{X}_{\zeta_{k}}^{k} \\
& =\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\
k \in K \cup L}} b^{\alpha_{1}, \ldots, \alpha_{m}}(K, L) \prod_{k \in K}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right) \\
& =\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\
k \in K \cup L}} \sum_{T \subseteq M_{D}^{\alpha}}(-1)^{|T|} \beta_{K}\left(\left\{\tilde{X}_{\alpha_{k}}^{k}\right\}_{k \in L \backslash T},\left\{\tilde{X}_{\alpha_{k}-1}^{k}\right\}_{k \in T}\right) \prod_{k \in K}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right)
\end{aligned}
$$

for all $K \subseteq M_{C}$ and all $L \subseteq M_{D}$, where we substitute (3.3) in the final equality.
Let $K \subseteq M_{C}$ and let $L \subseteq M_{D}$. Note that we can analyze the terms corresponding to continuous characteristics separately from those corresponding to discrete characteristics, by consecutively showing

$$
\begin{equation*}
\beta_{K}\left(\left\{\tilde{X}_{\zeta_{k}}^{k}\right\}_{k \in L}\right)=\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\ k \in L}} \sum_{T \subseteq M_{D}^{\alpha}}(-1)^{|T|} \beta_{K}\left(\left\{\tilde{X}_{\alpha_{k}}^{k}\right\}_{k \in L \backslash T},\left\{\tilde{X}_{\alpha_{k}-1}^{k}\right\}_{k \in T}\right) \tag{3.5}
\end{equation*}
$$

and

$$
\prod_{k \in K} \tilde{X}_{\zeta_{k}}^{k}=\sum_{\substack{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\} \\ k \in K}} \prod_{k \in K}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right)
$$

The latter can be straightforwardly shown by recursively using the telescoping sum $\sum_{\alpha_{k} \in\left\{1, \ldots, \zeta_{k}\right\}}\left(\tilde{X}_{\alpha_{k}}^{k}-\tilde{X}_{\alpha_{k}-1}^{k}\right)=\tilde{X}_{\zeta_{k}}^{k}-\tilde{X}_{0}^{k}=\tilde{X}_{\zeta_{k}}^{k}$ for all $k \in K$.

It remains to show that (3.5) holds. For this, we show that any $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ such that $\eta_{k} \in\left\{1, \ldots, \zeta_{k}\right\}$ for all $k \in L$ cancels out on the right-hand side of (3.5), except when $\eta_{k}=\zeta_{k}$ for all $k \in L$.

Let $\eta_{k} \in\left\{1, \ldots, \zeta_{k}\right\}$ for all $k \in L$. Let $U=\left\{k \in L \mid \eta_{k} \in\left\{1, \ldots, \zeta_{k}-1\right\}\right\}$ denote the set of discrete characteristics in $L$ that do not equal their $\zeta_{k}$-th value. Note that we always add the term $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ once, irrespective of $U$, by considering $\eta_{k}=\alpha_{k}$ for all $k \in L$ and $T=\emptyset$ on the right hand side of (3.5). Next, if $|U| \geq 1$, we subtract $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ exactly $|U|$ times, namely whenever $\eta_{l}=\alpha_{l}-1$ for some $l \in U, \eta_{k}=\alpha_{k}$ for all other $k \in U \backslash\{l\}$, and $T=\{l\}$ (so that $(-1)^{|T|}=-1$ ). Then, if $|U| \geq 2$, we add $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ exactly $\binom{|U|}{2}$ times, namely whenever $\eta_{l}=\alpha_{l}-1$ and $\eta_{m}=\alpha_{m}-1$ for $l, m \in U, \eta_{k}=\alpha_{k}$ for all other $k \in U \backslash\{l, m\}$, and $T=\{l, m\}$ (so that $(-1)^{|T|}=1$ ). This process continues until we arrive at $\eta_{k}=\alpha_{k}-1$ for all $k \in U$ and $T=U$, so that we add or subtract $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ once more, depending on $(-1)^{|U|}$.

More generally, $\beta_{K}\left(\left\{\tilde{X}_{\eta_{k}}^{k}\right\}_{k \in L}\right)$ is counted exactly $\sum_{r \in\{0, \ldots,|U|\}}(-1)^{r}\binom{|U|}{r}$ times. Hence, we can apply the binomial theorem

$$
(x+y)^{n}=\sum_{k \in\{0, \ldots, n\}}\binom{n}{k} x^{n-k} y^{k}
$$

for any non-negative integer $n$. In particular, we use $x=1, y=-1$, and $n=|U|$. We find zero whenever $|U| \geq 1$, and one for $|U|=0$. Hence, only one term does not cancel out on the right-hand side of (3.5), namely the term corresponding to $U=\emptyset$, and it is added exactly once. Since this term is $\beta_{K}\left(\left\{\tilde{X}_{\zeta_{k}}^{k}\right\}_{k \in L}\right)$, we see that (3.5) holds, which completes the proof.

## Example 3.3.2

We now illustrate Theorem 3.3.2 by means of an example of a component-based infrastructure cost problem with $M_{C}=\{1,2\}$ and $M_{D}=\{3,4\}$, and cost function

$$
\begin{equation*}
\kappa\left(z_{1}, z_{2}, z_{3}, z_{4}\right)=\beta_{\emptyset}\left(z_{3}, z_{4}\right)+\beta_{\{1\}}\left(z_{3}\right) z_{1}+\beta_{\{1,2\}} z_{1} z_{2} \tag{3.6}
\end{equation*}
$$

where $z_{k} \in Z^{k}$ for all $k \in M$. Going back to the CO 2 transport infrastructure context of Example 3.3.1, the first term in (3.6) corresponds to fixed costs depending on the
capacity level and the level of conditioning in the network. The second term represents costs that increase linearly in the onshore transport radius, where the slope of this linear relation depends on the capacity level. The final term represents variable costs based on the product of the onshore and offshore transport radius, for which the cost coefficient is constant, independent of the discrete characteristics. In Section 3.4, we provide a full case study using a more realistic cost function for such a CO 2 transport infrastructure.

We calculate the costs of all EICs $C^{\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}} \in \mathcal{C}$ with $\alpha_{k} \in\{1,2\}$ for all $k \in M$. Before elaborating on the calculations of a few instructive components, it is good to note that many of the EICs have zero costs. For example, consider the costs of $C^{1,2,1,2}$. In (3.2), we only sum over $K \subseteq\{1,2\}$ and $L \subseteq\{3,4\}$ such that $M_{C}^{\alpha}=\{2\} \subseteq K$ and $M_{D}^{\alpha}=\{4\} \subseteq L$. However, note that there is no non-zero coefficient in (3.6) for which both $\{2\} \in K$ and $\{4\} \in L$. Therefore, the costs of this EIC simply equal zero. A similar argument can be made for $C^{1,2,2,1}$ and $C^{2,1,1,2}$, and all EICs with $\alpha_{k}=2$ for three or four characteristics $k \in M$. All non-zero costs of the EICs are given in Table 3.2.

| EIC | $\lambda($ EIC $)$ |
| :--- | :---: |
| $C^{1,1,1,1}$ | $\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \tilde{X}_{1}^{1}+\beta_{\{1,2\}} \tilde{X}_{1}^{1} \tilde{X}_{1}^{2}$ |
| $C^{1,1,1,2}$ | $\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{2}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)$ |
| $C^{1,1,2,1}$ | $\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{1}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+\left(\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right)-\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right)\right) \tilde{X}_{1}^{1}$ |
| $C^{1,2,1,1}$ | $\beta_{\{1,2\}} \tilde{X}_{1}^{1}\left(\tilde{X}_{2}^{2}-\tilde{X}_{1}^{2}\right)$ |
| $C^{2,1,1,1}$ | $\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{2}^{1}-\tilde{X}_{1}^{1}\right)+\beta_{\{1,2\}} \cdot\left(\tilde{X}_{2}^{1}-\tilde{X}_{1}^{1}\right) \tilde{X}_{1}^{2}$ |
| $C^{1,1,2,2}$ | $\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{2}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{2}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)$ |
| $C^{2,1,2,1}$ | $\left(\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right)-\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right)\right)\left(\tilde{X}_{2}^{1}-\tilde{X}_{1}^{1}\right)$ |
| $C^{2,2,1,1}$ | $\beta_{\{1,2\}} \cdot\left(\tilde{X}_{2}^{1}-\tilde{X}_{1}^{1}\right)\left(\tilde{X}_{2}^{2}-\tilde{X}_{1}^{2}\right)$ |

Table 3.2 All non-zero costs of EICs $C^{\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}} \in \mathcal{C}$ with $\alpha_{k} \in$ $\{1,2\}$ for all $k \in M$ based on cost function (3.6)

For $C^{1,1,1,1}$, we have $M_{C}^{\alpha}=M_{D}^{\alpha}=\emptyset$, so we sum over all $K \subseteq\{1,2\}$ and $L \subseteq\{3,4\}$ in (3.2). However, $\lambda\left(C^{1,1,1,1}\right)$ will only consist of three terms, corresponding to the combinations of $K$ and $L$ with non-zero coefficients, namely $K=\emptyset$ and $L=\{3,4\}$, $K=\{1\}$ and $L=\{3\}$, and $K=\{1,2\}$ and $L=\emptyset$. Since $M_{D}^{\alpha}=\emptyset$, we restrict to $T=\emptyset$ in (3.3) for both $L=\{3\}$ and $L=\{3,4\}$. For $L=\emptyset$, we use the fact that

$$
b^{\alpha_{1}, \ldots, \alpha_{m}}(K, \emptyset)=\beta_{K}
$$

for any $K \subseteq M_{C}$. Further, we use the convention that the empty product equals one in the term corresponding to $K=\emptyset$, and that $\tilde{X}_{0}^{k}=0$ for all $k \in M$, which yields

$$
\begin{aligned}
& \lambda\left(C^{1,1,1,1}\right) \\
& =(-1)^{0} \beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+(-1)^{0} \beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{1}^{1}-\tilde{X}_{0}^{1}\right)+\beta_{\{1,2\}} \cdot\left(\tilde{X}_{1}^{1}-\tilde{X}_{0}^{1}\right)\left(\tilde{X}_{1}^{2}-\tilde{X}_{0}^{2}\right) \\
& =\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{1}^{1}-0\right)+\beta_{\{1,2\}} \cdot\left(\tilde{X}_{1}^{1}-0\right)\left(\tilde{X}_{1}^{2}-0\right) \\
& =\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \tilde{X}_{1}^{1}+\beta_{\{1,2\}} \tilde{X}_{1}^{1} \tilde{X}_{1}^{2} .
\end{aligned}
$$

To further illustrate the workings of (3.3), essentially representing the inclusion and exclusion of cost parameters, we consider the costs of EIC $C^{1,1,2,2}$ next. Since $M_{D}^{\alpha}=M_{D}=\{3,4\}$, we only consider $L=\{3,4\}$, for which $K=\emptyset$ gives the only non-zero coefficient. However, in this case $b^{1,1,2,2}(\emptyset, L)$ consists of four terms, corresponding to $T=\emptyset, T=\{3\}, T=\{4\}$, and $T=\{3,4\}$, respectively:

$$
\begin{aligned}
& \lambda\left(C^{1,1,2,2}\right) \\
& =(-1)^{0} \beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{2}^{4}\right)+(-1)^{1} \beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{2}^{4}\right)+(-1)^{1} \beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{1}^{4}\right)+(-1)^{2} \beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right) \\
& =\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{2}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{2}^{4}\right)-\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)
\end{aligned}
$$

Finally, we consider EIC $C^{2,1,2,1}$. Since $M_{C}^{\alpha}=\{1\}$ and $M_{D}^{\alpha}=\{3\}$, the only combination of $K$ and $L$ with a non-zero coefficient and $M_{C}^{\alpha} \subseteq K$ and $M_{D}^{\alpha} \subseteq L$ is $K=\{1\}$ and $L=\{3\}$, for which $b^{2,1,2,1}(\{1\},\{3\})=\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right)-\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right)$. Using this, we directly find the closed-form expression for the costs of this EIC, namely

$$
\lambda\left(C^{2,1,2,1}\right)=\left(\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right)-\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right)\right)\left(\tilde{X}_{2}^{1}-\tilde{X}_{1}^{1}\right)
$$

The remaining costs in Table 3.2 can be calculated in the same way. Indeed, we find that the costs of any boxlike infrastructure equal the sum of the costs of all EICs within this box. For example, one readily verifies that

$$
\begin{aligned}
& \lambda\left(C^{1,1,1,1}\right)+\lambda\left(C^{1,1,1,2}\right)+\lambda\left(C^{1,1,2,1}\right)+\lambda\left(C^{1,2,1,1}\right)+\lambda\left(C^{2,1,1,1}\right)+\lambda\left(C^{1,1,2,2}\right) \\
& +\lambda\left(C^{1,2,1,2}\right)+\lambda\left(C^{1,2,2,1}\right)+\lambda\left(C^{2,1,1,2}\right)+\lambda\left(C^{2,1,2,1}\right)+\lambda\left(C^{2,2,1,1}\right) \\
& +\lambda\left(C^{1,2,2,2}\right)+\lambda\left(C^{2,1,2,2}\right)+\lambda\left(C^{2,2,1,2}\right)+\lambda\left(C^{2,2,2,1}\right)+\lambda\left(C^{2,2,2,2}\right) \\
& =\beta_{\emptyset}\left(\tilde{X}_{2}^{3}, \tilde{X}_{2}^{4}\right)+\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right) \tilde{X}_{2}^{1}+\beta_{\{1,2\}} \tilde{X}_{2}^{1} \tilde{X}_{2}^{2} \\
& =\kappa\left(\tilde{X}_{2}^{1}, \tilde{X}_{2}^{2}, \tilde{X}_{2}^{3}, \tilde{X}_{2}^{4}\right) .
\end{aligned}
$$

To determine the total construction costs $c(N)$ of the minimally required infrastructure of player set $N$, we simply sum the costs of all EICs in $A(N)$, i.e.,

$$
c(N)=\sum_{C^{\alpha_{1}, \ldots, \alpha_{m}} \in A(N)} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)
$$

In a component-based infrastructure cost problem $I=(N, M, X, \kappa)$, non-negativity of the costs of each EIC is guaranteed if cost function $\kappa$ satisfies a natural condition. We describe such problems as component-based infrastructure cost problems with regular $\kappa$. The results in Section 3.3 .3 will use the non-negativity of each EIC's costs.

## Definition 3.3.3

Let $I=(N, M, X, \kappa)$ be a component-based infrastructure cost problem. Then, we say $\kappa$ is regular if

$$
b^{\alpha_{1}, \ldots, \alpha_{m}}(K, L) \geq 0
$$

for any $\alpha_{k} \in\left\{1, \ldots, n^{k}\right\}, k \in M$ and all $K, L$ such that $M_{C}^{\alpha} \subseteq K \subseteq M_{C}$ and $M_{D}^{\alpha} \subseteq L \subseteq M_{D}$, where $M_{C}^{\alpha}=\left\{k \in M_{C} \mid \alpha_{k}>1\right\}$ and $M_{D}^{\alpha}=\left\{k \in M_{D} \mid \alpha_{k}>1\right\}$.

In the context of Example 3.3.1 and using cost function (3.6), the regularity requirements on $\kappa$ boil down to the following conditions on the underlying cost coefficients. Next to non-negativity of all cost coefficients, regularity entails that the cost coefficient of onshore transport does not decrease as the capacity level increases. Further, the cost coefficient corresponding to the fixed costs cannot decrease when the capacity level increases and conditioning remains at the lowest level, or when the level of conditioning increases with the capacity fixed at its lowest level. Lastly, this fixed costs coefficient must be such that the additional costs of conditioning CO 2 for re-use do not decrease as the capacity level of the network grows, and vice versa. All requirements seem to be reasonable.

### 3.3.3 The equal component cost sharing rule

The equal component cost sharing rule $\gamma$ is an allocation mechanism that follows naturally from the component-based analysis of infrastructure construction costs. Specifically, $\gamma$ divides the total costs $c(N)$ corresponding to a component-based infrastructure cost problem $I=(N, M, X, \kappa)$ over the players in $N$, such that the costs of each required EIC are equally divided over the players who require that component.

To properly define the equal component cost sharing rule, we let $N^{\alpha_{1}, \ldots, \alpha_{m}}=$ $\left\{i \in N \mid X_{i}^{k} \geq \tilde{X}_{\alpha_{k}}^{k} \forall k \in M\right\}$ denote the set of players in $N$ who require component $C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C}$. Using this, $A(\{i\})=\left\{C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C} \mid i \in N^{\alpha_{1}, \ldots, \alpha_{m}}\right\}$ is defined as the set of EICs that a player $i \in N$ requires in $I$. Finally, we denote the class of all component-based infrastructure cost problems by $\mathcal{I}$.

## Definition 3.3.4

The equal component cost sharing rule $\gamma$ on $\mathcal{I}$ is defined by setting

$$
\gamma_{i}(I)=\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\})}} \frac{\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)}{\left|N^{\alpha_{1}, \ldots, \alpha_{m}}\right|}
$$

for all $I=(N, M, X, \kappa) \in \mathcal{I}$ and any $i \in N$.
For each player, the allocated costs are based only on the costs of the components that this player actually requires. Hence, the definition of $\gamma$ is such that players 'only pay for what they need'. Moreover, if players require the same set of components, they pay the same.

We will show that the equal component cost sharing rule satisfies the properties of coalitional rationality and advantageous scaling, if the component-based infrastructure cost problem is such that cost function $\kappa$ is regular.

Let $f$ be a cost sharing rule on $\mathcal{I}$. Intuitively, the coalitional rationality property states that for any component-based infrastructure cost problem $I=(N, M, X, \kappa) \in \mathcal{I}$, no coalition $S \subseteq N$ has a (financial) reason to object to the cost allocation $f(I)$, because no coalition can benefit from splitting off from the grand coalition $N$ : for any $S \subseteq N$, the total costs allocated to the players in $S$ are below or equal to the costs of the minimal infrastructure $A(S)$ that $S$ requires. Put differently, the cost allocation is stable against coalitional deviations. Note that this property is strongly related to the cooperative game-theoretic notion of the core (Gillies (1959), see Ray and Vohra (2015) for a more extensive discussion of the core).

A coalition $S$ requires an EIC if at least one player in $S$ requires the EIC. Formally, $A(S)=\left\{C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C} \mid S^{\alpha_{1}, \ldots, \alpha_{m}} \neq \emptyset\right\}$, where $S^{\alpha_{1}, \ldots, \alpha_{m}}=$ $\left\{i \in S \mid C^{\alpha_{1}, \ldots, \alpha_{m}} \in A(\{i\})\right\}$ denotes the set of players in $S$ who require EIC $C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C}$.

Coalitional rationality Let $f$ be a cost sharing rule on $\mathcal{I}$. Then, $f$ satisfies coalitional rationality on $\mathcal{I}$ if

$$
\sum_{i \in S} f_{i}(I) \leq \sum_{C^{\alpha_{1}, \ldots, \alpha_{m}} \in A(S)} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)
$$

for all $I=(N, M, X, \kappa) \in \mathcal{I}$ and all $S \subseteq N$.

## Theorem 3.3.5

The equal component cost sharing rule $\gamma$ satisfies coalitional rationality on $\mathcal{I}$, if $\kappa$ is regular.

Proof. Let $I=(N, M, X, \kappa) \in \mathcal{I}$ with regular $\kappa$ and let $S \subseteq N$. Then,

$$
\begin{aligned}
\sum_{i \in S} \gamma_{i}(I) & =\sum_{i \in S} \sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\})}} \frac{\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)}{\left|N^{\alpha_{1}, \ldots, \alpha_{m}}\right|} \\
& =\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in \mathcal{C}}}\left|S^{\alpha_{1}, \ldots, \alpha_{m}}\right| \frac{\lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)}{\left|N^{\alpha_{1}, \ldots, \alpha_{m}}\right|} \\
& \leq \sum_{\substack{C^{\alpha_{1}, \ldots, \alpha_{m} \in \mathcal{C}:} \\
S^{\alpha_{1}, \ldots, \alpha_{m} \neq \emptyset}}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right),
\end{aligned}
$$

where the inequality follows from the fact that the costs of each EIC are non-negative (since $\kappa$ is regular) and $S^{\alpha_{1}, \ldots, \alpha_{m}} \subseteq N^{\alpha_{1}, \ldots, \alpha_{m}}$ for all $\alpha_{k} \in\left\{1, \ldots, n^{k}\right\}, k \in M$.

Next, we define advantageous scaling for a cost sharing rule $f$ on $\mathcal{I}$. This property states that the costs allocated to each player can only decrease if the player set $N$ grows larger. Recall that $X_{j}, j \in N$, denotes the $j$-th row of matrix $X$.

Advantageous scaling Let $f$ be a cost sharing rule on $\mathcal{I}$. Then, $f$ satisfies advantageous scaling on $\mathcal{I}$ if

$$
f_{i}(\bar{I}) \leq f_{i}(I)
$$

for all $I=(N, M, X, \kappa) \in \mathcal{I}$ and $\bar{I}=(\bar{N}, M, \bar{X}, \kappa) \in \mathcal{I}$ such that $N \subseteq \bar{N}$ and $X_{j}=\bar{X}_{j}$ for all $j \in N$, and for all $i \in N$.

## Theorem 3.3.6

The equal component cost sharing rule $\gamma$ satisfies advantageous scaling on $\mathcal{I}$, if $\kappa$ is regular.

To see that $\gamma$ satisfies advantageous scaling if $\kappa$ is regular, consider $I=(N, M, X, \kappa) \in \mathcal{I}$ and $\bar{I}=(\bar{N}, M, \bar{X}, \kappa) \in \mathcal{I}$ with regular $\kappa$ such that $N \subseteq \bar{N}$ and $X_{j}=\bar{X}_{j}$ for all $j \in N$, and let $i \in N$. It is good to note that $\overline{\mathcal{C}}$ contains more components than $\mathcal{C}$ if a new player also brings new requirements. Since the costs of player $i$ can only be affected by EICs that $i$ requires, $i$ is not affected by new components that only the new players require. However, a new requirement may 'split' an existing component that $i$ requires into 'subcomponents'. We will not formally define such splits, but instead provide an intuitive argument why this never increases the costs of player $i$. Clearly, the costs of the original component equal the sum of the costs of the subcomponents. If a subcomponent is not required by a new player, the corresponding costs are divided over the same set of players in $N$ as before, so this does not affect the costs of $i$. If a subcomponent is required by a new player, the costs of this subcomponent are divided over more players than before, thereby lowering the costs of $i$ (here, we also use the non-negativity of the costs of each EIC). It follows that $\gamma_{i}(\bar{I}) \leq \gamma_{i}(I)$.

Before applying our cost sharing rule in a case study, we remark that one can define a cooperative cost game $\left(N, c_{I}\right)$ corresponding to a component-based infrastructure cost problem $I=(N, M, X, \kappa)$ by letting the value $c_{I}(S)$ reflect the total costs of the minimal infrastructure $A(S)$ that a coalition $S \subseteq N$ requires. Note that we consider cost games here, as opposed to the profit games considered in Section 1.2. However, all corresponding concepts can be straightforwardly translated to the cost setting.

Interestingly, Theorem 3.3.7 shows that the equal component cost sharing rule $\gamma(I)$ coincides with the Shapley value of the corresponding cost game. Further, if $\kappa$ is regular, this cost game is concave, where a cost game $c \in T U^{N}$ is concave if

$$
c(S \cup T)+c(S \cap T) \leq c(S)+c(T)
$$

for all $S, T \in 2^{N}$.

## Theorem 3.3.7

Let $I=(N, M, X, \kappa) \in \mathcal{I}$ be a component-based infrastructure cost problem and let
$c_{I} \in T U^{N}$ be the corresponding cost game, defined by

$$
c_{I}(S)=\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)
$$

for any $S \in 2^{N} \backslash\{\emptyset\}$. Then,

$$
\gamma(I)=\Phi\left(c_{I}\right)
$$

and $c_{I}$ is concave if $\kappa$ is regular.
Proof. Let $i \in N$ and recall that

$$
\Phi_{i}\left(c_{I}\right)=\frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m_{i}^{\sigma}\left(c_{I}\right)
$$

Let $\sigma \in \Pi(N)$. Then, the marginal costs of player $i$ in this order equal the total sum of the costs of every EIC that is required by $i$ and such that it is not required by any player that comes before $i$ in $\sigma$. To formalize this, let $P(\sigma, i)=\left\{j \in N \mid \sigma^{-1}(j)<\sigma^{-1}(i)\right\}$ denote the set of predecessors of player $i$ with respect to $\sigma \in \Pi(N)$. Then,

$$
\begin{aligned}
\Phi_{i}\left(c_{I}\right) & =\frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m_{i}^{\sigma}\left(c_{I}\right) \\
& =\frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \sum_{\substack{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\}):} \\
P(\sigma, i) \cap N^{\alpha_{1}, \ldots, \alpha_{m}}=\emptyset}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\frac{1}{|N|!} \sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\})}}\left|\left\{\sigma \in \Pi(N) \mid P(\sigma, i) \cap N^{\alpha_{1}, \ldots, \alpha_{m}}=\emptyset\right\}\right| \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\frac{1}{|N|!} \sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\})}} \frac{|N|!}{\mid N^{\alpha_{1}, \ldots, \alpha_{m} \mid}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(\{i\})}} \frac{1}{\mid N^{\alpha_{1}, \ldots, \alpha_{m} \mid}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =\gamma_{i}(I) .
\end{aligned}
$$

Next, we show that $c_{I}$ is concave, if $\kappa$ is regular. Recall that for any $S \in 2^{N}$ we have

$$
A(S)=\left\{C^{\alpha_{1}, \ldots, \alpha_{m}} \in \mathcal{C} \mid \exists i \in S \text { such that } \forall k \in M \text { we have } X_{i}^{k} \geq \tilde{X}_{\alpha_{k}}^{k}\right\}
$$

Let $S, T \in 2^{N}$. It is clear that $A(S \cup T)=A(S) \cup A(T)$, which yields

$$
\begin{aligned}
c(S \cup T)= & \sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S \cup T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
= & \sum_{C^{\alpha_{1}, \ldots, \alpha_{m}} \in A(S)} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)+\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& -\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S) \cap A(T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
= & c(S)+c(T)-\sum_{C^{\alpha_{1}, \ldots, \alpha_{m}} \in A(S) \cap A(T)} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) .
\end{aligned}
$$

For $c(S \cap T)$, however, any EIC in $A(S \cap T)$ has to be required by a player that is both in $S$ and in $T$, where in $A(S) \cap A(T)$ we also include EICs that are required by both $S$ and $T$, but not necessarily by the same player in both coalitions. Therefore, $A(S \cap T) \subseteq A(S) \cap A(T)$. Hence,

$$
\begin{aligned}
& c(S \cup T)+c(S \cap T) \\
& =c(S)+c(T)-\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S) \cap A(T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)+\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S \cap T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& \leq c(S)+c(T)-\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S) \cap A(T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right)+\sum_{C^{\alpha_{1}, \ldots, \alpha_{m} \in A(S) \cap A(T)}} \lambda\left(C^{\alpha_{1}, \ldots, \alpha_{m}}\right) \\
& =c(S)+c(T),
\end{aligned}
$$

where, in the inequality, we also use the fact that the costs of each EIC are nonnegative, since $\kappa$ is regular.

In fact, if $\kappa$ is regular, the concavity of the cost game directly implies that the Shapley value is in the core and that the (extended) Shapley value is a so-called population monotonic allocation scheme (Sprumont, 1990). The former directly implies coalitional rationality, where the latter implies advantageous scaling.

### 3.4 Component-based infrastructure cost problems applied to CO2 transport infrastructures

In this section, we apply the technical model of Section 3.3 to the specific case of CO2 transport infrastructure for CCUS-hubs in the port of Rotterdam area described in Section 3.2. This section follows the steps as sketched in Figure 3.3. We first convert the qualitative problem description of Section 3.2 into a component-based infrastructure cost problem, and determine the required set of EICs and their costs. Subsequently, we discuss the results with respect to cost allocation using the equal component cost sharing rule, also considering alternative scenarios for the cost parameters.


Figure 3.3 Steps for applying component-based analysis and equal component cost sharing rule to a multi-actor infrastructure construction problem

### 3.4.1 CO2 transport EICs and their costs

The general problem description of Section 3.2 can be transformed into a formal component-based infrastructure cost problem. The player set is represented more compactly using numbers: $N=\{1,2,3,4,5\}$, corresponding to the players 'Heavy

Emitters', 'Distant Emitters', 'Small Emitters', 'Greenhouses', and 'Hydrogen producers', respectively. Furthermore there are four infrastructure component characteristics for which the users have different requirements, hence $M=\{1,2,3,4\}$. Table 3.3, a quantified version of Table 3.1, gives the entries for our requirement matrix $X$.

|  | $M \rightarrow$ | onshore <br> transport <br> radius | offshore <br> transport <br> radius | capacity | conditioning |
| ---: | :--- | :---: | :---: | :---: | :---: |
| $N \downarrow$ |  | 1 | 2 | 3 | 4 |
| heavy emitters | 1 | 30 | 30 | 2 | 1 |
| distant emitters | 2 | 80 | 30 | 2 | 1 |
| small emitters | 3 | 30 | 30 | 1 | 1 |
| greenhouses | 4 | 30 | 0 | 1 | 2 |
| hydrogen producers | 5 | 10 | 0 | 1 | 2 |

Table 3.3 Transformation of Table 3.1 into entries for requirement matrix $X$

The first two characteristics, onshore and offshore transport radius, are expressed in kilometers. The third characteristic, transport capacity, is in this application represented by discrete options: 1 reflects a small capacity (around $2.5 \mathrm{Mt} / \mathrm{y}$ ), and a 2 reflects a larger network with higher capacity (around $10 \mathrm{Mt} / \mathrm{y}$ ). Finally, the fourth characteristic is related to the conditioning of CO2, is also discrete and can take two values: this characteristic is 2 if the corresponding player re-uses CO2 (and thereby requires additional conditioning w.r.t. pressure, temperature and purity), and 1 otherwise.

The elementary infrastructure components corresponding to this CO2 transport infrastructure are defined on the basis of the vectors of unique requirements (sorted in increasing order) for each characteristic, namely

$$
\tilde{X}^{1}=\left[\begin{array}{l}
10 \\
30 \\
80
\end{array}\right], \tilde{X}^{2}=\left[\begin{array}{c}
0 \\
30
\end{array}\right], \tilde{X}^{3}=\left[\begin{array}{l}
1 \\
2
\end{array}\right], \text { and } \tilde{X}^{4}=\left[\begin{array}{l}
1 \\
2
\end{array}\right] .
$$

Next, we present how these infrastructure characteristics drive the construction costs. This part continues with the findings on the CO2 transport cost drivers from Section 3.2: the CO 2 infrastructure construction cost function is a combination of onshore
costs and offshore costs; and those costs consist of a fixed and a variable portion. For a detailed derivation of the cost function parameter values, the $\beta$ 's in Section 3, we refer to Appendix 3.A.

The costs of onshore transportation are determined by a fixed and variable portion. The value of the fixed cost parameter is determined by the value of characteristics 3 and 4, i.e., capacity and level of conditioning. The variable cost parameter is determined by the requirement for characteristic 3, capacity, only. The higher the required transport capacity, the higher the costs per km. For readability purposes of this section, the cost function is first presented in a conditional breakdown, and the characteristic values $z_{3}$ and $z_{4}$ are included in the conditions only. Recall that $z_{3}=1$ and $z_{3}=2$ represent a small ( $2.5 \mathrm{Mt} / \mathrm{y}$ ) and large ( $10 \mathrm{Mt} / \mathrm{y}$ ) capacity, respectively, and $z_{4}=1$ and $z_{4}=2$ represent standard conditioning and conditioning for re-use, respectively. We have

$$
\kappa_{\text {onshore }}\left(z_{1}, z_{3}, z_{4}\right)= \begin{cases}6+0,6 z_{1} & \text { if } z_{3}=1 \text { and } z_{4}=1 \\ 8+0.75 z_{1} & \text { if } z_{3}=2 \text { and } z_{4}=1 \\ 42+0.6 z_{1} & \text { if } z_{3}=1 \text { and } z_{4}=2 \\ 94+0.75 z_{1} & \text { if } z_{3}=2 \text { and } z_{4}=2\end{cases}
$$

with $z_{1} \in\{10,30,80\}, z_{3} \in\{1,2\}$ and $z_{4} \in\{1,2\}$. Thus, an onshore transport radius of 10 kilometers ( $z_{1}=10$ ) with 2.5 Mton capacity $\left(z_{3}=1\right)$ and standard conditioning $\left(z_{4}=1\right)$, costs $6+0.6 \cdot 10=12$ million euros, while the same 10 kilometers with 2.5 Mton capacity and conditioning for re-use purpose ( $z_{4}=2$ ) cost $42+0.6 \cdot 10=48$ million euros.

Offshore transportation costs are also determined by a fixed and variable portion. The values of both the fixed and the variable cost parameter are determined only by the value of characteristic 3 (i.e., transport capacity), as the conditioning that partly determines the fixed costs is the same for all offshore transport. However, we only want to add (fixed) offshore costs if the players actually make use of offshore transportation, i.e., if their required offshore length is greater than 0 . For the variable cost portion this happens instantly as the cost parameter will be multiplied by the required offshore length. For the fixed costs portion we achieve this using the indicator variable $X^{5}$ with, for all $i \in N, X_{i}^{5}=1$ if $X_{i}^{2}>0$ and 0 otherwise. This is a simple
dummy variable that does not have a meaningful impact on the interpretation or structure of the model, while ensuring fixed costs for offshore transportation are only included when applicable. As a consequence, however, the resulting EICs will be based on 5 characteristics instead of 4 . Note that $\tilde{X}^{5}=\left[\begin{array}{ll}0 & 1\end{array}\right]^{\top}$. One can interpret these fixed costs as the required costs for transfer and compressor stations when going from onshore to offshore transportation. Formally, we have

$$
\kappa_{\text {offshore }}\left(z_{2}, z_{3}, z_{5}\right)= \begin{cases}1.2 z_{2}+38 z_{5} & \text { if } z_{3}=1 \\ 1.6 z_{2}+64 z_{5} & \text { if } z_{3}=2,\end{cases}
$$

with $z_{2} \in\{0,30\}, z_{3} \in\{1,2\}$ and $z_{5} \in\{0,1\}$. Note that here the first term represents the variable costs and the second term the fixed.

For the total cost function of our CO2 transport infrastructure application we have

$$
\begin{aligned}
\kappa\left(z_{1}, z_{2}, z_{3}, z_{4}, z_{5}\right) & =\kappa_{\text {onshore }}\left(z_{1}, z_{3}, z_{4}\right)+\kappa_{\text {offshore }}\left(z_{2}, z_{3}, z_{5}\right) \\
& = \begin{cases}6+0.6 z_{1}+1.2 z_{2}+38 z_{5} & \text { if } z_{3}=1 \text { and } z_{4}=1 \\
8+0.75 z_{1}+1.6 z_{2}+64 z_{5} & \text { if } z_{3}=2 \text { and } z_{4}=1 \\
42+0.6 z_{1}+1.2 z_{2}+38 z_{5} & \text { if } z_{3}=1 \text { and } z_{4}=2 \\
94+0.75 z_{1}+1.6 z_{2}+64 z_{5} & \text { if } z_{3}=2 \text { and } z_{4}=2\end{cases}
\end{aligned}
$$

with $z_{1} \in\{10,30,80\}, z_{2} \in\{0,30\}, z_{3} \in\{1,2\}, z_{4} \in\{1,2\}$ and $z_{5} \in\{0,1\}$. It can be shown that $\kappa$ is regular, so that the costs of each EIC are non-negative.

Up to this point, we only show a numerical breakdown of cost function $\kappa$ for the component-based infrastructure cost problem, by providing different cost functions depending on the capacity level and the level of conditioning. This is simply a more intuitive representation of a more general cost function as formulated in Definition 3.3.1. In particular, we have

$$
\begin{equation*}
\kappa\left(z_{1}, z_{2}, z_{3}, z_{4}, z_{5}\right)=\beta_{\emptyset}\left(z_{3}, z_{4}\right)+\beta_{\{1\}}\left(z_{3}\right) z_{1}+\beta_{\{2\}}\left(z_{3}\right) z_{2}+\beta_{\{5\}}\left(z_{3}\right) z_{5} \tag{3.7}
\end{equation*}
$$

with $z_{1} \in\{10,30,80\}, z_{2} \in\{0,30\}, z_{3} \in\{1,2\}, z_{4} \in\{1,2\}$ and $z_{5} \in\{0,1\}$. The coefficient values are derived in Appendix 3.A and summarized in Table 3.11. Using the more general notation, we have $\beta_{\emptyset}(1,1)=6, \beta_{\emptyset}(2,1)=8, \beta_{\emptyset}(1,2)=42, \beta_{\emptyset}(2,2)=94$,
$\beta_{\{1\}}(1)=0.6, \beta_{\{1\}}(2)=0.75, \beta_{\{2\}}(1)=1.2, \beta_{\{2\}}(2)=1.6, \beta_{\{5\}}(1)=38$ and $\beta_{\{5\}}(2)=64$.

Having completed the decomposition into EICs, Theorem 3.3.2 can now be used to derive the costs per EIC. From the vectors $\tilde{X}^{1}, \tilde{X}^{2}, \tilde{X}^{3}, \tilde{X}^{4}$ and $\tilde{X}^{5}$, we know there are $3 \cdot 2 \cdot 2 \cdot 2 \cdot 2=48$ EICs. As we will argue below, only 12 of these have non-zero costs. For these 12 EICs, Table 3.4 shows the final expression of the costs in terms of the coefficients. To clarify the final expressions, we further illustrate the calculations of two components. We omit some of the details that were already discussed in Example 3.3.2, since the calculations are largely analogous, even though we have a different cost function and an additional characteristic. Also, we are now able to fill in the actual values of the characteristics.

| EIC | $\lambda($ EIC $)$ | $=12$ |
| :--- | :---: | :--- |
| $C^{1,1,1,1,1}$ | $\beta_{\emptyset}(1,1)+\beta_{\{1\}}(1) \cdot 10$ | $=12$ |
| $C^{2,1,1,1,1}$ | $\beta_{\{1\}}(1) \cdot(30-10)$ | $=30$ |
| $C^{3,1,1,1,1}$ | $\beta_{\{1\}}(1) \cdot(80-30)$ | $=3.5$ |
| $C^{1,1,2,1,1}$ | $\beta_{\emptyset}(2,1)-\beta_{\emptyset}(1,1)+\left(\beta_{\{1\}}(2)-\beta_{\{1\}}(1)\right) \cdot 10$ | $=3$ |
| $C^{2,1,2,1,1}$ | $\left(\beta_{\{1\}}(2)-\beta_{\{1\}}(1)\right) \cdot(30-10)$ | $=7.5$ |
| $C^{3,1,2,1,1}$ | $\left(\beta_{\{1\}}(2)-\beta_{\{1\}}(1)\right) \cdot(80-30)$ | $=38$ |
| $C^{1,1,1,1,2}$ | $\beta_{\{5\}}^{\{3\}}(1) \cdot 1$ | $=26$ |
| $C^{1,1,2,1,2}$ | $\left(\beta_{\{5\}}(2)-\beta_{\{5\}}(1)\right) \cdot 1$ | $=36$ |
| $C^{1,2,1,1,1}$ | $\beta_{\{2\}}(1) \cdot 30$ | $=12$ |
| $C^{1,2,2,1,1}$ | $\left(\beta_{\{2\}}(2)-\beta_{\{2\}}(1)\right) \cdot 30$ | $=36$ |
| $C^{1,1,1,2,1}$ | $\beta_{\emptyset}(1,2)-\beta_{\emptyset}(1,1)$ | $=50$ |
| $C^{1,1,2,2,1}$ | $\beta_{\emptyset}(2,2)-\beta_{\emptyset}(1,2)-\beta_{\emptyset}(2,1)+\beta_{\emptyset}(1,1)$ |  |

Table 3.4 All non-zero costs of EICs for the case study of CO2 transport infrastructure for CCUS-hubs in the port of Rotterdam area

We first consider $\lambda\left(C^{1,1,1,1,1}\right)$. Since $M_{C}^{\alpha}=M_{D}^{\alpha}=\emptyset$, we sum over all $K \subseteq\{1,2,5\}$ and $L \subseteq\{3,4\}$ in (3.2). Since there are four combinations of $K$ and $L$ with non-zero coefficients in (3.7), $\lambda\left(C^{1,1,1,1,1}\right)$ would in principle consist of four terms. However, note that $\tilde{X}_{1}^{2}=\tilde{X}_{1}^{5}=0$, because the lowest required offshore transport radius is zero (and the binary characteristic indicating whether offshore transport is required is zero
as well then), and recall that $\tilde{X}_{0}^{k}=0$ for all $k \in M$. Consequently, we find

$$
\begin{aligned}
\lambda\left(C^{1,1,1,1,1}\right)= & \beta_{\emptyset}\left(\tilde{X}_{1}^{3}, \tilde{X}_{1}^{4}\right)+\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{1}^{1}-\tilde{X}_{0}^{1}\right)+\beta_{\{5\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{1}^{5}-\tilde{X}_{0}^{5}\right) \\
& +\beta_{\{2\}}\left(\tilde{X}_{1}^{3}\right) \cdot\left(\tilde{X}_{1}^{2}-\tilde{X}_{0}^{2}\right) \\
= & \beta_{\emptyset}(1,1)+\beta_{\{1\}}(1) \cdot(10-0)+\beta_{\{5\}}(1) \cdot(0-0)+\beta_{\{2\}}(1) \cdot(0-0) \\
= & \beta_{\emptyset}(1,1)+\beta_{\{1\}}(1) \cdot 10 \\
= & 12
\end{aligned}
$$

Next, consider $\lambda\left(C^{3,1,2,1,1}\right)$. Since $M_{C}^{\alpha}=\{1\}$ and $M_{D}^{\alpha}=\{3\}$, the only combination of $K$ and $L$ with a non-zero coefficient and $M_{C}^{\alpha} \subseteq K$ and $M_{D}^{\alpha} \subseteq L$ is $K=\{1\}$ and $L=\{3\}$. Therefore,

$$
\begin{aligned}
\lambda\left(C^{3,1,2,1,1}\right) & =b^{3,1,2,1,1}(\{1\},\{3\}) \cdot\left(\tilde{X}_{3}^{1}-\tilde{X}_{2}^{1}\right) \\
& =\left(\beta_{\{1\}}\left(\tilde{X}_{2}^{3}\right)-\beta_{\{1\}}\left(\tilde{X}_{1}^{3}\right)\right) \cdot\left(\tilde{X}_{3}^{1}-\tilde{X}_{2}^{1}\right) \\
& =\left(\beta_{\{1\}}(2)-\beta_{\{1\}}(1)\right) \cdot(80-30) \\
& =0.15 \cdot 50 \\
& =7.5
\end{aligned}
$$

This line of reasoning can also be used to show that only twelve out of 48 EICs have non-zero costs, since for all other EICs there does not exist a combination of $K$ and $L$ such that $M_{C}^{\alpha} \subseteq K \subseteq M_{C}$ and $M_{D}^{\alpha} \subseteq L \subseteq M_{D}$ with a non-zero coefficient.

Observe that if we sum all costs given in Table 3.4, we get

$$
266=\beta_{\emptyset}(2,2)+\beta_{\{1\}}(2) \cdot 80+\beta_{\{5\}}(2) \cdot 1+\beta_{\{2\}}(2) \cdot 30=\kappa(80,30,2,2,1) .
$$

More generally, the costs of any boxlike infrastructure equal the sum of the costs of all EICs within this box. Finally, we remark that $C^{1,1,2,2,1}$ is not required by any player, since no individual player requires both the highest capacity level and conditioning for re-use. Hence, $C^{1,1,2,2,1} \notin A(N)$. This component will therefore be omitted from now on. All other EICs are required by at least one player. This means $A(N)$ consists of exactly 11 elementary infrastructure components with non-zero costs.

Table 3.5 gives, for each of these 11 components, the EIC, the corresponding set of players $N(E I C)$ that requires it, the costs $\lambda$ (EIC), and an interpretation. It is clear from this EIC-decomposition that not all infrastructure components are required by
all players. The requirements from players 1 and 2 coincide in many EICs, while players 4 and 5 are only interested in the first two EICs and one EIC dedicated to re-use conditioning. This last one is not of interest to the other three players.

| EIC | $N($ EIC $)$ | $\lambda($ EIC $)$ | Interpretation |
| :--- | :---: | :---: | :--- |
| $C^{1,1,1,1,1}$ | $\{1,2,3,4,5\}$ | 12 | onsh. transp. [km 0-10] with small cap. <br> and no re-use cond. <br> onsh. transp. [km 10-30] with small cap. <br> $C^{2,1,1,1,1}$$\{1,2,3,4\}$ |
| $C^{3,1,1,1,1}$ | $\{2\}$ | 30 | and no re-use cond. <br> ansh. transp. [km 30-80] with small cap. <br> ond no re-use cond. |
| $C^{1,1,2,1,1}$ | $\{1,2\}$ | 3.5 | onsh. transp. [km 0-10] with increased cap. <br> and no re-use cond. |
| $C^{2,1,2,1,1}$ | $\{1,2\}$ | 3 | onsh. transp. [km 10-30] with increased cap. <br> and no re-use cond. |
| $C^{3,1,2,1,1}$ | $\{2\}$ | 7.5 | onsh. transp. [km 30-80] with increased cap. <br> and no re-use cond. |
| $C^{1,1,1,1,2}$ | $\{1,2,3\}$ | 38 | conversion to offsh. transp. with small cap. <br> $C^{1,1,2,1,2}$ <br> $C^{1,2,1,1,1}$ |
| $\{1,2\}$ | 26 | 36 | conversion to offsh. transp. with increased cap. <br> offsh. transp. [km 0-30] with small cap. <br> $C^{1,2,2,1,1}$$\{1,2\}$ |
| $C^{1,1,1,2,1}$ | $\{4,5\}$ | 36 | offsh. transp. [km 0-30] with increased cap. <br> cond. for re-use, small cap. |

Table 3.5 Each EIC with non-zero costs required in the CO2 transport network, with the set of players that requires it, its costs, and an interpretation. Onsh. = onshore, offsh. = offshore, transp. $=$ transport, cap. $=$ capacity, cond.$=$ conditioning.

If we sum the costs of these EICs, we find that $c(N)=216$ million euros for the construction costs of the minimal infrastructure that $N$ requires. One can also see that these 216 million euros can be split in 104 million euros onshore infrastructure construction costs and 112 million euros offshore infrastructure construction costs. The next section discusses how to allocate these costs to the CO2 transport infrastructure users.

### 3.4.2 Component-based cost allocations for CO2 transport infrastructure

In this section, we apply the equal component cost sharing rule $\gamma$ to allocate the total CO2 infrastructure construction costs $c(N)=216$ over its different users. The basic idea of equal component cost sharing is that the costs of each EIC in $A(N)$ are equally shared among the group of users of this component. For example, the costs of EIC $C^{1,1,1,1,1}$ are allocated equally to all players, whereas only player 2 pays for $C^{3,1,1,1,1}$, a component corresponding to the longer transport radius only the distant emitters require. In this way, combining the information from columns 2 and 3 in Table 3.5 directly provides the cost allocation that follows from this rule. All costs are rounded to (at most) one decimal place in this section, yielding

$$
\gamma(I)=(52.3,89.8,30.1,23.4,20.4)
$$

The remainder of this section demonstrates the use of the equal component cost sharing method in two ways. First, we vary the set of players that are using the CO2 transport network, in order to show the attractiveness of the rule's properties coalitional rationality and advantageous scaling. Second, we consider three additional cost scenarios to show the adaptability of the general cost function $\kappa$ and to show that one can easily attribute particular cost increases to specific players.

## Varying the player set

The player set $N$ consists of five typical potential users of CO2 infrastructure. Within this group of 5 , there are 2 logical groups that might find some common ground in their requirements and also in the things they do not need. The heavy, distant and small emitters each require offshore transport infrastructure, while they do not require any conditioning for re-use. The greenhouses and hydrogen producers need conditioning for re-use purposes, while they do not require any offshore transport. In Figure 3.4 we describe what would happen if either the 'offshorers' or the CO2 re-users would split off from $N$.

Let $S_{\text {offshore }}=\{1,2,3\}$ be the offshore subgroup that splits off from $N$ and constructs a CO2 transport infrastructure that only satisfies their subgroup's requirements. Compared to $N, S_{\text {offshore }}$ no longer requires $C^{1,1,1,2,1}$, so that $c\left(S_{\text {offshore }}\right)=$ $216-36=180$. Thus, if the offshore subgroup would build an infrastructure that only satisfies their requirements it would cost them 180 million euros. This is higher


Figure 3.4 Coalitional rationality: what would happen if the offshore or the re-use coalition would split off and decide to construct their own CO2 infrastructure?
than the approximately $52.3+89.8+30.1=172.2$ million euros that are allocated to them in the setting where an infrastructure is constructed for all players in $N$.

Alternatively if the re-use subgroup $S_{\text {re-use }}=\{4,5\}$ would split off, they require only $C^{1,1,1,1,1}, C^{2,1,1,1,1}$, and $C^{1,1,1,2,1}$, with total construction costs $c\left(S_{\text {re-use }}\right)=60$ million euros. This is significantly more than the approximately $23.4+20.4=43.8$ million euros in the case of constructing and sharing an CO 2 transport infrastructure for the total group of users $N$.

Due to the coalitional rationality property of the equal component cost sharing rule $\gamma$, these seemingly opposite subcoalitions $S_{\text {offshore }}=\{1,2,3\}$ and $S_{\text {re-use }}=\{4,5\}$ are not better of if they construct and share a CO2 transport infrastructure for only their subgroup. This does not only hold for these two coalitions, but for any potential coalitional deviation.

Next, we analyze the effect of adding players. For this, the new starting point of analysis is a situation in which only the 'local' players cooperate to construct a CO2 transport infrastructure and the distant emitter, player 2 , is not involved. The componentbased infrastructure problem $I=(N, M, X, \kappa)$ we have analyzed so far would instead
be a problem without player 2. Figure 3.5 illustrates what would happen if this local coalition would consider letting the distant emitter, player 2, join this local group.


Figure 3.5 Advantageous scaling: what would happen if the local coalition would extend its coalition and infrastructure design to the more distant CO2 transport user?

Formally, we then consider the component-based infrastructure problem $I_{\text {local }}=$ ( $N_{\text {local }}, M, X_{-2}, \kappa$ ), with $N_{\text {local }}=\{1,3,4,5\}$ and where $X_{-2}$ represents the matrix $X$ without its second row. Since the absence of player 2 has no effect on the requirements of any of the other players or on the cost function $\kappa$, we can use Table 3.5 to determine the total costs of the player set $N_{\text {local }}$ and the allocation of these costs according to the equal component cost sharing rule, and we find that $c\left(N_{\text {local }}\right)=178.5$. Applying the equal component cost sharing rule to this component-based CO2 infrastructure problem yields

$$
\gamma\left(I_{\text {local }}\right)=(88.5,44,25,21)
$$

Comparing $\gamma(I)$ to $\gamma\left(I_{\text {local }}\right)$, it becomes apparent that all players in the local group would benefit if the distant emitter, player 2 , joins the project. This addition to the player set from $N_{\text {local }}$ to $N$ would lead to a decrease in the costs allocated to all four 'local' players, as anticipated based on the advantageous scaling property. The difference is most notable for players 1 and 3 , since the first three players share
a requirement for costly offshore transportation. Therefore, players 1 and 3 benefit greatly from sharing the corresponding costs with player 2 . In case of player 1 , this effect is reinforced much further by the fact that player 2 is the only other player who requires high transport capacity.

## Varying the costs

CCUS is one of the key enabling technologies for accelerating and realizing decarbonization of industries. New insights in costs and cost drivers, especially of shared CO 2 transport infrastructures, will emerge every year. The parameters that we use are based on non-recent studies and the estimates are rough (cf. Appendix 3.A). That is why we extend our application with three extra cost parameter scenarios. This section not only shows the impact on cost allocations to the players, but it also shows that if better, more recent or more accurate data becomes available, one can easily update the cost model and its parameters. The following scenarios are considered:

- a scenario VAR $\uparrow 25 \%$ where all variable cost parameters values are 25 percent higher;
- a scenario FIX $\uparrow 25 \%$ where all fixed cost parameters are 25 percent higher;
- a scenario REUSE+VAR in which the conditioning for re-use also leads to increased variable onshore transportation costs.

In all three cases, the other parameter values remain the same. The adaptations of the cost function parameters used in $\kappa$ that follow from these three scenarios are given in Table 3.6. The third cost scenario REUSE+VAR shows the adaptability of the cost function of a component-based infrastructure cost problem. In this REUSE+VAR scenario, the conditioning for re-use $\left(z_{4}=2\right)$ does not just lead to fixed costs, but also increases the variable costs of onshore CO 2 transport. It means that parameter "Onshore VAR 2.5 Mton" is split into "Onshore VAR 2.5 Mton standard" with original value 0.6 , the variable costs for onshore transport with standard conditioning, and "Onshore VAR 2.5 Mton re-use", the new variable costs for onshore transport with re-use conditioning, with value 1. This new parameter value is for illustrative purposes only.

The extra variable cost driver in the third scenario leads to a change in the costs of the EIC $C^{1,1,1,2,1}$, since we need to account for the variable re-use conditioning costs. Since player 4 and 5 have different transport radius requirements, it also leads to one

| cost parameter description | baseline | VAR $\uparrow 25 \%$ | FIX $\uparrow 25 \%$ | REUSE + <br> VAR |
| :--- | :---: | :---: | :---: | :---: |
| Onshore VAR 2.5 Mton | 0.6 | 0.75 | 0.6 | - |
| Onshore VAR 2.5 Mton standard | - | - | - | 0.6 |
| Onshore VAR 2.5 Mton re-use | - | - | - | 1 |
| Onshore VAR 10 Mton | 0.75 | 0.94 | 0.75 | 0.75 |
| Onshore FIX 2.5 Mton standard | 6 | 6 | 7.5 | 6 |
| Onshore FIX 10 Mton standard | 8 | 8 | 10 | 8 |
| Onshore FIX 2.5 Mton re-use | 42 | 42 | 52.5 | 42 |
| Onshore FIX 10 Mton re-use | 94 | 94 | 117.5 | 94 |
| Offshore VAR 2.5 Mton | 1.2 | 1.5 | 1.2 | 1.2 |
| Offshore VAR 10 Mton | 1.6 | 2 | 1.6 | 1.6 |
| Offshore FIX 2.5 Mton | 38 | 38 | 47.5 | 38 |
| Offshore FIX 10 Mton | 64 | 64 | 80 | 64 |

Table 3.6 Cost parameters for different cost scenarios as used in the cost function $\kappa$ of the EIC application to CO2 transport infrastructure problems
new EIC (with non-zero costs) in $A(N): C^{2,1,1,2,1}$. This component is only required by player 4 and represents the extra 20 kilometer transport radius needed by player 4 , relative to player 5 . The costs per EIC in each of the three scenarios are outlined next to the baseline in Table 3.7.

Note that for cost scenarios FIX $\uparrow 25 \%$ and VAR $\uparrow 25 \%$, the costs of all EICs except $C^{1,1,1,1,1}$ and $C^{1,1,2,1,1}$ either increase $25 \%$, or do not change. This occurs because all these components correspond to a part of the infrastructure that leads to either only additional fixed costs (e.g., additional fixed costs for conditioning) or only additional variable costs (e.g., additional variable onshore transport costs). We also remark that the total costs are 243 million euros in both scenarios. This is merely a coincidence: in the baseline cost scenario the total costs $c(N)=216$ consist of exactly 108 fixed costs $(42+8-6+64)$ and 108 variable costs, so the total costs increase an equal amount in both scenarios.

Using Table 3.7, one can readily apply the equal component cost sharing rule to find the cost allocations given in Table 3.8.

Several interesting observations can be made here. The equality of the total costs for the scenarios FIX $\uparrow 25 \%$ and VAR $\uparrow 25 \%$ allows for a 'fair' comparison of the

| EIC | $N($ EIC $)$ | baseline | FIX $\uparrow 25 \%$ | VAR $\uparrow 25 \%$ | REUSE+VAR |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $C^{1,1,1,1,1}$ | $\{1,2,3,4,5\}$ | 12 | 13.5 | 13.5 | 12 |
| $C^{2,1,1,1,1}$ | $\{1,2,3,4\}$ | 12 | 12 | 15 | 12 |
| $C^{3,1,1,1,1}$ | $\{2\}$ | 30 | 30 | 37.5 | 30 |
| $C^{1,1,2,1,1}$ | $\{1,2\}$ | 3.5 | 4 | 3.875 | 3.5 |
| $C^{2,1,2,1,1}$ | $\{1,2\}$ | 3 | 3 | 3.75 | 3 |
| $C^{3,1,2,1,1}$ | $\{2\}$ | 7.5 | 7.5 | 9.375 | 7.5 |
| $C^{1,1,1,1,2}$ | $\{1,2,3\}$ | 38 | 47.5 | 38 | 38 |
| $C^{1,2,1,1,1}$ | $\{1,2,3\}$ | 36 | 36 | 45 | 36 |
| $C^{1,1,2,1,2}$ | $\{1,2\}$ | 26 | 32.5 | 26 | 26 |
| $C^{1,2,2,1,1}$ | $\{1,2\}$ | 12 | 12 | 15 | 12 |
| $C^{1,1,1,2,1}$ | $\{4,5\}$ | 36 | 45 | 36 | 40 |
| $C^{2,1,1,2,1}$ | $\{4\}$ | - | - | - | 8 |
| $c(N)$ |  | 216 | 243 | 243 | 228 |

Table 3.7 Costs per EIC in four different scenarios for the CO2 transport cost drivers

| player | baseline | FIX $\uparrow 25 \%$ | VAR $\uparrow 25 \%$ | REUSE + VAR |
| :--- | :--- | :--- | :--- | :--- |
| heavy emitters | 52.3 | $59.3(\uparrow 13.3 \%)$ | $58.4(\uparrow 11.7 \%)$ | $52.3(\uparrow 0 \%)$ |
| distant emitters | 89.8 | $96.8(\uparrow 7.8 \%)$ | $105.3(\uparrow 17.2 \%)$ | $89.8(\uparrow 0 \%)$ |
| small emitters | 30.1 | $33.5(\uparrow 11.5 \%)$ | $34.1(\uparrow 13.5 \%)$ | $30.1(\uparrow 0 \%)$ |
| greenhouses | 23.4 | $28.2(\uparrow 20.5 \%)$ | $24.5(\uparrow 4.5 \%)$ | $33.4(\uparrow 42.7 \%)$ |
| hydrogen producers | 20.4 | $25.2(\uparrow 23.5 \%)$ | $20.7(\uparrow 1.5 \%)$ | $22.4(\uparrow 9.8 \%)$ |

Table 3.8 Cost allocations based on the equal component cost sharing rule for the construction costs of a CO2 transport infrastructure in four different cost scenarios
corresponding cost allocations. The equal component cost sharing rule behaves as we would expect. It is clear that the players who require conditioning for re-use ( 4 and 5) are relatively most affected by an increase in fixed costs, since this is a relatively large part of their (allocated) costs. Similarly, the distant emitters are most affected by the increase in variable costs, due to their large onshore transport radius.

For the REUSE+VAR cost scenario, only the players who require conditioning for re-use purposes face an increase in allocated costs. The larger requirement for onshore transport radius of player 4 compared to player 5 means the former is affected more by the increase in variable costs.

## 3.A Cost drivers and parameter values for CO2 transport infrastructure application

In this appendix, we first review literature concerning the main cost drivers in the construction of a CO2 transport infrastructure. Then, we derive the cost parameter values used in Section 3.4 on the basis of two selected studies.
van der Linden (2019) estimates that for the regional CCUS cluster in Rotterdam investments in the range of 300 to 400 million euros are necessary. A year later, in a more extensive study, these costs have been re-estimated to 400 to 500 million euros (DNVGL, 2020). CO2 transport infrastructure is a significant (but not the only) portion of these roughly estimated investment costs. Capital cost estimates are highly variable (Akbilgic et al., 2015), depending, e.g., on which cost components are aggregated to calculate total capital costs. Though there are different factors driving the costs, often only estimates of total costs are presented. There is also a stream of literature developing detailed cost models for different CCUS network components, see Knoope (2015) for a review. These detailed models can be used to optimize pipeline configurations such as pipeline diameter and choice of materials based on, e.g., geological and economic conditions. These studies are, however, often focused on a single component (e.g., only considering compression cost drivers or offshore pipeline cost drivers), rather than an entire CO2 transport infrastructure like we aim to analyze. Two exceptions are summarized in Mallon et al. (2013): a 2011 report from the Zero Emmission Platform (ZEP, 2011) on CO2 transport costs and in the same year a technical study from JRC into CO2 pipeline costs (Serpa et al., 2011).

The cost drivers and the corresponding parameter values we consider in our application are mainly based on these two studies. We choose these sources for our cost estimates for various reasons: they are readily available, are most consistent with the cost driver relations found in the literature, and cover a range of CO2 transport infrastructure designs with varying characteristics. Although ZEP's and JRC's cost estimates as absolute numbers might no longer be accurate, they do provide insight in the relative changes in investment costs due to changes in type of terrain (onshore or offshore), transport capacity, or transport radius.

ZEP presents total transport cost estimates for different combinations of pipeline length and yearly transport capacity. They also perform a simple sensitivity analysis,
where they conclude that investment costs account for circa $90 \%$ of total CO2 transport costs, the relation between costs and transport length is almost linear, and the actual use of the transport pipeline does (almost) not influence the costs as long as it does not exceed its maximum capacity.

Serpa et al. (2011) state that the capital costs of a CO2 pipeline system consist of pipeline material and installation costs, and costs for system equipment such as pumping and filtering stations and (digital) control systems. They develop a heuristic and simplified approach to estimate CO2 pipeline costs. They express a linear relationship between costs and pipeline length, where the specific cost parameters depend on the terrain (onshore or offshore), the yearly transport capacity and the quality of CO 2 .

Next, we discuss the derivation of the cost parameter values used in the application in Section 3.4. Recall from Section 3.2 and Section 3.4.1 that, based on the aforementioned two studies, we consider costs consisting of fixed (system) and variable (pipeline) costs, for both onshore and offshore transportation. There is a linear relationship between the costs and the required transport radius, where the specific cost parameters depend on the terrain and the transport capacity.

Further, conditioning requirements only influence the fixed portion of the costs ${ }^{2}$, together with the capacity.

Since CCUS is one of the key enabling technologies for accelerating and realizing decarbonization of industries, new insights in costs and cost drivers, especially of shared CO2 transport infrastructures, will emerge regularly. The parameters that we use are based on non-recent studies and the estimates are rough. Therefore, we present our parameter derivation in such a way that if better, more recent or more accurate data becomes available, one can easily adapt the cost parameters.

## What parameters?

In Section 3.4.1, it becomes clear that we only need to find 10 different cost parameters for the corresponding total cost function. We have continuous characteristics 'onshore transport radius' and 'offshore transport radius', $M_{C}=\{1,2\}$, and discrete

[^0]characteristics 'transport capacity' and 'CO2 conditioning', $M_{D}=\{3,4\}$. Finally from the entries in the requirement matrix $X$ we see that both discrete variables only take two values.

## Onshore cost parameters

We start with the cost parameters for onshore transportation. In Table 3.9, we combine ZEP's and JRC's transport cost estimates into one table.

| Capacity (Mt/y) | per km (JRC) | 10 km feeders (ZEP) | 180 km spine (ZEP) |
| :---: | :---: | :---: | :---: |
| 2.5 | 0.59 | 11.5 | 147 |
| 5 | 0.64 |  |  |
| 10 |  | 15 | 226 |
| 15 | 0.83 |  |  |

Table 3.9 Onshore transport pipeline system cost estimates in million euros for different combinations of pipeline length and yearly transport capacity, based on Annex 3 from ZEP (2011) and Table 10 from Serpa et al. (2011).

JRC's data refer to pipeline costs only, while ZEP's data refer to total pipeline system costs. Since, we use this data only for inspirational and exemplary purposes, we combine this data without further corrections for potential price level differences. First, we interpolate the JRC data to find that for $10 \mathrm{Mt} / \mathrm{y}$ capacity the pipeline costs per km are approximately 0.735 million euros.

We combine JRC's variable cost estimates corresponding to $2.5 \mathrm{Mt} / \mathrm{y}$ and $10 \mathrm{Mt} / \mathrm{y}$ with ZEP's data to find approximations for the fixed costs, in the following way. The 10 km feeders are seen as a simple onshore transport system that is sufficiently conditioned for transportation to the shore - conditioning level 1. Hence, we estimate fixed costs for this type of pipeline at approximately $11.5-10 \cdot 0.59=5.6$ million euros for a small yearly capacity and $15-10 \cdot 0.735=7.65$ for a large capacity. The 180 km spine from ZEP is seen as a system that has a more advanced conditioning level due to the long distance, and is re-use ready - conditioning level 2. Again using the two variable cost parameters, the corresponding fixed costs are estimated in a similar way, which yields approximately 40.8 and 93.7 for small and large capacity, respectively.

## Offshore cost parameters

Next, we consider offshore system costs. Again, we use a combination of JRC and ZEP data.

| Capacity (Mt/y) | per km (JRC) | 180 km spine (ZEP) |
| :---: | :---: | :---: |
| 2.5 | 1.18 | 250 |
| 5 | 1.28 |  |
| 10 |  | 338 |
| 15 | 1.78 |  |

Table 3.10 Offshore transport pipeline system cost estimates in million euros for different combinations of pipeline length and yearly transport capacity, based on Annex 3 from ZEP (2011) and Table 10 from Serpa et al. (2011).

Using a similar linear interpolation approach as with the onshore cost parameter estimations, we find offshore variable costs of approximately 1.18 and 1.53 for small and large capacity, respectively, and use this to derive the corresponding offshore fixed costs estimates of approximately 37.6 and 62.6 .

## Parameters for application

The parameter estimates used in Section 3.4 are summarized under 'baseline' in Table 3.11. As explained previously, the derived cost function parameters are rough estimates based on rather non-recent sources. To avoid the impression of working with accurate cost estimates, we round up all estimates. In this table, we also present the first two scenarios in which the cost parameters differ, as discussed in Section 3.4.2.

| cost parameter description | name in $\kappa$ | baseline | VAR $\uparrow 25 \%$ | FIX $\uparrow 25 \%$ |
| :--- | :--- | :---: | :---: | :---: |
| Onshore VAR 2.5 Mton | $\beta_{\{1\}}(1)$ | 0.6 | 0.75 | 0.6 |
| Onshore VAR 10 Mton | $\beta_{\{1\}}(2)$ | 0.75 | 0.94 | 0.75 |
| Onshore FIX 2.5 Mton Cond1 | $\beta_{\emptyset}(1,1)$ | 6 | 6 | 7.5 |
| Onshore FIX 10 Mton Cond1 | $\beta_{\emptyset}(2,1)$ | 8 | 8 | 10 |
| Onshore FIX 2.5 Mton Cond2 | $\beta_{\emptyset}(1,2)$ | 42 | 42 | 52.5 |
| Onshore FIX 10 Mton Cond2 | $\beta_{\emptyset}(2,2)$ | 94 | 94 | 117.5 |
| Offshore VAR 2.5 Mton | $\beta_{\{2\}}(1)$ | 1.2 | 1.5 | 1.2 |
| Offshore VAR 10 Mton | $\beta_{\{2\}}(2)$ | 1.6 | 2 | 1.6 |
| Offshore FIX 2.5 Mton | $\beta_{\{5\}}(1)$ | 38 | 38 | 47.5 |
| Offshore FIX 10 Mton | $\beta_{\{5\}}(2)$ | 64 | 64 | 80 |

Table 3.11 Cost parameters, in million euros, for different cost scenarios as used in the cost function $\kappa$ of the component-based infrastructure cost problem applied to CO2 transport infrastructure problems.

## 4

## Competition and cooperation in linear production and sequencing processes

### 4.1 Introduction

Cooperative and non-cooperative game theory are often presented as two opposing branches of the same field, where players either cooperate, form coalitions, and reach joint decisions, or do not cooperate and decide only on their own individual strategies. In this chapter, based on Van Beek et al. (2023b), we analyze hybrid models that combine elements from these two branches of game theory in a single two-stage model. In particular, we incorporate strategy dependence into linear production (LP) processes and sequencing processes.

Brandenburger and Stuart (2007) create a two-stage model called a biform game to analyze strategic moves in business, where a non-cooperative first stage is followed by a cooperative second stage. The non-cooperative stage concerns a strategic decision like whether to invest in innovation, and these strategic decisions made by the players then determine the competitive environment in which some cooperative game is played. They use a cooperative solution procedure based on the core to find a unique allocation in the cooperative phase for all possible strategy combinations. These allocations per strategy combination are used as the payoff vectors for the so-called induced non-cooperative game. This induced game can then be analyzed as a standard strategic game, for which the existence of (pure) Nash equilibria is investigated.

Although certainly inspired by the work of Brandenburger and Stuart (2007), we deviate from the original biform games both in model and in application. Rather than studying business strategy, we define biform games to analyze two well-investigated operations research (OR) problems. We introduce a strategic component to cooperative 'OR-games' (see Borm et al. (2001) for a survey) corresponding to LP-processes and sequencing processes. In particular, we analyze the influence of a strategy dependent resource bundle and a strategy dependent initial processing order, respectively. Further, we will not use the core to determine the allocation in the cooperative game. The core is a powerful solution concept that is very suitable for the analysis of cooperative games in a competitive environment, but a disadvantage of the core is that it usually does not prescribe a unique allocation vector, where a unique payoff vector is required for the induced strategic game. Brandenburger and Stuart (2007) solve this problem by taking a weighted average of the extreme points of the core. The weights are determined by so-called confidence indices, reflecting the degree of confidence each player has in their performance in the cooperative game. Finding these confidence indices is somewhat arbitrary, perhaps even more so in an OR-game setting. Therefore, we choose payoff vectors based on the Owen set (Owen, 1975) and gain splitting rules as introduced by Hamers et al. (1996). Though examples can be constructed for which the Owen set contains infinitely many allocation vectors, it generally prescribes a unique allocation vector. A gain splitting rule always leads to a unique payoff vector. Note that the Owen set is a subset of the core in LP-games, and any non-negative gain splitting rule yields a core element in sequencing games, so we certainly do not disregard or disqualify the core as a solution concept. By choosing these context specific core selectors, however, we aim to let the allocation of value in the cooperative game more aptly reflect the specific problem at hand.

Before introducing both models in more detail, it is important to discuss the general purpose of this chapter, also considering the existing literature. Two-stage (or multi-stage) models are not uncommon in the game-theoretic literature, but the twostages are often both cooperative or both non-cooperative. Combining cooperative and non-cooperative game theory into the same model is still less common, though it has numerous advantages. Among other things, such a combination is able to incorporate externalities, by having the value created by a coalition in the cooperative stage also depend on the strategic choices of players who are not in that coalition, through their strategies in the non-cooperative stage. In this way, the combination
allows for a more explicit modeling of externalities than in cooperative models like partition function form games (Kóczy, 2018).

Before the aforementioned paper on biform games, Hart and Moore (1990) consider a two-stage model to compare transactions within firms to those between firms (with a focus on the optimal assignment of assets), in which a non-cooperative stage is followed by a cooperative game. Instead of the core, they use the Shapley value (Shapley, 1953) to solve the cooperative game. In the context of LP-processes, Anupindi et al. (2001) analyze a model (extended by, among others, Granot and Sošić (2003)) in which independent retailers face stochastic demand for an identical product and unilaterally order their inventories in a non-cooperative stage (before the demand is realized), followed by a cooperative stage in which incurred surpluses are transshipped to retailers that incurred shortages. This leads to additional profits to be allocated to the retailers. Next to the existence of Nash equilibria, an important focus of Anupindi et al. (2001) is an issue that ties to mechanism design: how to determine an allocation mechanism that, in this decentralized system, achieves the efficiency of a centralized system. This is outside the scope of this chapter.

Further, biform games have been used to analyze, among other things, supply network formation (Hennet and Mahjoub, 2010), smart grid communications (Kim, 2012), stochastic programming with recourse (Summerfield and Dror, 2013), and the impact of surplus division on investment incentives (Feess and Thun, 2014). Nonetheless, literature related to biform games is relatively limited in quantity, especially with regard to sequencing processes. The objective of this chapter is not to give an exhaustive theoretical analysis of a model with direct practical applications. However, we want to illustrate that the conceptual idea of biform games can be applied to a great variety of problems, and by analyzing and reflecting on both LP and sequencing processes, we aim to inspire others to continue in this field of research.

An LP-process in a general setting, as presented in Owen (1975), can be used to model situations in which a set of players is able to pool a set of resources used in the manufacturing of a set of products. How much of each resource players need to manufacture a product is described by a linear technology matrix, the availability of resources per player is determined by resource bundles, and the prices of products by some price vector. From this, a cooperative game can be derived, called a linear production game (LP-game). Without the need to formally define the game, the

Owen set derives a tailor-made set of payoff allocations directly from the LP-process on the basis of the values of players' resource bundles, as determined by the shadow prices obtained by solving the dual program of the linear programming problem faced by the grand coalition. Van Gellekom et al. (2000) provide a detailed axiomatic characterization of the Owen set.

In general, this line of research does not incorporate strategy dependence in the LP-process. Hennet and Mahjoub (2010) use a biform model in the context of LPprocesses, but do this through a strategy dependent price vector, and analyze this in the context of the role of a player in a supply chain. We adapt the original definition of LP-processes such that the resource bundles of individual players are determined strategically, which is more in line with the aforementioned work of Anupindi et al. (2001).

It is often assumed that resources are owned completely by the players (firms) at the start of an LP-process. This assumption is quite restrictive, since in practice the firms are often dependent on their supply chain to obtain these resources. This is a situation that lends itself well for analysis with a biform model. Starting with the non-cooperative first stage, players compete to obtain resources and we assume that they can only settle on one source. One might think of a situation in which firms can obtain a scarce or hard to produce resource, like fossil fuels or complicated electronics, from different sources. There may be significant costs and preparation time involved with settling on a source, for example due to a need for lobbying to access a resource in another country, or to train or financially support manufacturers of such a resource. Using a second supplier would therefore come at overly high additional costs, making it financially unattractive. In other types of practical situations, having an exclusive supplier may be contractually required or desirable for, e.g., branding purposes.

The resource bundle available at a source is often restricted, meaning this bundle has to be split between firms if multiple firms decide to settle on the same location. The competition for resource bundles gives rise to the first-stage strategic game that determines the exact LP-process in which the firms end up. Once each firm has obtained a resource bundle, it may be of interest to the firms to cooperate, as they might have a surplus of one resource and a deficit in another resource needed in the manufacturing process. This is modeled in the second-stage LP-process, which is solved using a payoff vector based on the Owen set. We refer to this model as a biform linear production (BLP) process.

The induced strategic games that arise from BLP-processes are shown to exhibit some interesting properties. The existence of a pure Nash equilibrium is guaranteed in the 'standard case': a finite number of players, two locations of sources, and players who choose the same source split the corresponding resource bundle equally. We restrict our analysis to pure Nash equilibria in this chapter, henceforth simply referred to as Nash equilibria. The payoff based on the Owen set is then contrasted with one-point game-theoretic solution concepts like the Shapley value and the nucleolus (Schmeidler, 1969) that explicitly make use of the corresponding LP-games with respect to each strategy combination. This approach is somewhat more similar to the aforementioned model of Hart and Moore (1990), albeit applied to an entirely different setting. We show that the existence of a Nash equilibrium is still guaranteed, provided that the solution concept based on the Owen set is replaced by an anonymous game-theoretic solution concept of the corresponding LP-games. Finally, we discuss the effect of changes in the structure of the BLP-process on the existence of Nash equilibria. We consider modifications with unequal resource bundle splitting, or with more than two locations. First, in case of unequal splitting, the existence of a Nash equilibrium is no longer guaranteed if there are three or more players. Second, with more than two locations, even for two players a Nash equilibrium need not exist.

Next to the analysis of LP-processes, we also apply the biform framework to sequencing processes. A sequencing process involves determining in what order a finite number of jobs should be lined up in front of one or more machines to minimize the joint costs incurred by the set of jobs as a whole. These costs are often a linear function of the completion times of the jobs, where different jobs generally have different cost parameters and processing times. This creates a measure of 'urgency' for the jobs. For linear cost functions, the sequencing problem is optimized by processing the jobs in (weakly) decreasing order of urgency, also defined as a Smith order (Smith, 1956). If we are given some initial order of jobs in the queue (often said to be representing initial processing rights), this order is generally not optimal. Solving the sequencing problem then yields a rearrangement of the jobs leading to maximal joint cost savings. A natural question is how to allocate these cost savings over the various jobs, which is particularly relevant when different jobs belong to different agents (e.g., different companies processing jobs on the same machines). By treating jobs as agents, or players, this question can be answered using game theory.

The first so-called sequencing game was developed by Curiel et al. (1989) for the deterministic one-machine sequencing problem. The work on sequencing games has been extended in several ways (see also Curiel et al. (2002) for a survey), where in the vast majority of the literature on sequencing games, an initial order is assumed to be given. Exceptions to this are Klijn and Sánchez (2006) and Hall and Liu (2016), who analyze sequencing games without an initial order. Still, the games defined here are cooperative games with a single stage. Multi-stage sequencing situations were proposed by Curiel (2010), in which each stage corresponds to a cooperative game, where the order arrived at after the first stage becomes the initial order of the second stage, and so on. In this multi-stage sequencing situation, however, an initial order is still assumed to be given for the first stage.

There are many scenarios in which the initial processing rights are not naturally fully determined, but can somehow be strategically influenced by the players. For example, when there is limited capacity for vaccination during a pandemic, this capacity may be centrally assigned to different groups in a certain order. However, these groups may be able to influence this (initial) order by requesting to be processed with a certain priority. Therefore, we do not assume an initial order is simply given. In our model, the initial order is determined strategically, where all players have the opportunity to request their preferred position in the order. If two or more players request the same position, a tie-breaking rule is used to determine who is processed first. This tie-breaking rule can be based on, e.g., the urgency of players.

In this way, we create a biform sequencing $(B S)$ process. The first stage corresponds to a strategic game that determines the initial order, where the second stage is a (cooperative) sequencing process in which we assume that the cost savings of rearranging the initial order to an optimal order are allocated using a so-called gain splitting rule. We consider biform sequencing processes with and without additional costs associated with the strategic decision. Most prominently, we fully specify the set of (pure) Nash equilibria in biform sequencing processes without strategy dependent additional costs. Since in certain practical situations players may be able to influence their position in the initial processing order by incurring some costs (e.g., payment for priority service), we also discuss biform sequencing processes with such costs. Players incur additional costs based only on their obtained or requested position in the initial order. We still find a Nash equilibrium if the additional costs are associated with the
obtained position in the initial order, and provide an example of the absence of Nash equilibria in case these costs are associated with the requested position instead.

Section 4.2 analyzes biform linear production processes and Section 4.3 treats biform sequencing processes. Both sections contain a discussion of possible extensions of the models.

### 4.2 Biform linear production processes

Biform linear production (BLP) processes are an extension of standard LP-processes, in which the resource bundles of players are strategically determined. As mentioned in the introduction, the motivation behind this model is to analyze an LP-process where firms compete for resources that are scarce or hard to manufacture. An LP-process is described by the tuple

$$
L=\left(N, R, P, A,\left\{b^{i}\right\}_{i \in N}, c\right)
$$

where $N$ represents the finite set of players, $R$ the finite set of resources, $P$ the finite set of products, $A$ the $|R| \times|P|$ linear technology matrix of which the cell in the $r$-th row and $p$-th column corresponds to the number of units of resource $r$ needed to manufacture one unit of product $p, b^{i} \in \mathbb{R}_{+}^{R}$ represents the resource bundle of player $i \in N$, and $c \in \mathbb{R}_{+}^{P}$ represents the market prices for a unit of each product. We assume for each resource that at least one player owns a positive quantity of this resource, i.e., $\sum_{i \in N} b_{r}^{i}>0$ for all $r \in R$, where $b_{r}^{i}$ denotes the amount of resource $r \in R$ owned by player $i \in N$. Further, we assume that each product with a positive market price also requires a positive quantity of at least one resource. Formally, for all $p \in P$ such that $c_{p}>0$, there exists a resource $r \in R$ such that $A_{r p}>0$.

Let $L=\left(N, R, P, A,\left\{b^{i}\right\}_{i \in N}, c\right)$ be an LP-process. Then, the corresponding transferable utility LP-game $v_{L} \in T U^{N}$ is defined such that the value of coalition $T \in 2^{N} \backslash\{\emptyset\}$ is given by

$$
v_{L}(T)=\max _{y \in \mathbb{R}^{P}}\left\{c^{\top} y \mid A y \leq \sum_{i \in T} b^{i}, y \geq 0\right\}
$$

In words, the value of a coalition is the maximum revenue generated by the sale of products, where production is restricted by the sum of resource bundles available to the coalition. The value $v_{L}(T)$ of a coalition $T$ can also be found by solving the dual program instead. In the dual programs, the feasible region does not depend on the coalition $T$ at hand. We denote the corresponding feasible region by $F$, formally defined by

$$
F=\left\{z \in \mathbb{R}^{R} \mid z^{\top} A \geq c^{\top}, z \geq 0\right\}
$$

For a coalition $T \in 2^{N} \backslash\{\emptyset\}$, we then have

$$
v_{L}(T)=\min _{z \in F} z^{\top} \sum_{i \in T} b^{i}
$$

For any $z \in \mathbb{R}^{R}$ that solves the dual program, $z_{r}$ is the shadow price of resource $r \in R$ corresponding to this solution. We define the optimal region $O$ as the set of shadow prices within the feasible region that solve the minimization problem for the grand coalition, i.e.,

$$
O=\left\{z \in F \mid v_{L}(N)=z^{\top} \sum_{i \in N} b^{i}\right\}
$$

The Owen set is a solution concept that exploits the unique structure of an LPprocess to find an allocation vector without the need to explicitly derive the LPgame. On the domain specified at the start of this section, Van Gellekom et al. (2000) axiomatically characterize the Owen set. The Owen set is a polytope that is based on the shadow prices that solve the dual linear programming problem for the grand coalition. Formally, the Owen set (Owen, 1975) is defined by

$$
\operatorname{Owen}(L)=\left\{\left(z^{\top} b^{i}\right)_{i \in N} \in \mathbb{R}^{N} \mid z \in O\right\}
$$

The Owen set of an LP-process is a subset of the core of the corresponding LPgame. The elements of the Owen set are called Owen vectors. Though the Owen set commonly consists of a single vector, examples can be contrived in which it does not prescribe a unique allocation vector. This is due to the fact that the optimal region $O$ is a polytope that can contain more than one element. For any given set of resource bundles, we therefore use the centroid of $O$, i.e., the mean of all its extreme points, denoted by $\bar{z}$, as 'the' vector of shadow prices of the resources to select one specific Owen vector $\left(\bar{z}^{\top} b^{i}\right)_{i \in N}$.

Our model introduces a strategic element to standard LP-processes by letting players compete for resources, rather than assuming each player owns some resource bundle beforehand. We assume that resources can be obtained from two locations, sources 1 and 2 , with resource bundles $l_{1} \in \mathbb{R}_{+}^{R}$ and $l_{2} \in \mathbb{R}_{+}^{R}$, respectively. The strategic choice of the players will be to settle on source 1 or on source 2. This leads to the following definition of a BLP-process.

## Definition 4.2.1

A BLP-process is a tuple

$$
\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)
$$

in which we have $X^{i}=\{1,2\}$ for all $i \in N, l_{1}$ and $l_{2}$ are the respective resource bundles at the two locations, and for any $x \in X=\Pi_{i \in N}\{1,2\}$,

$$
L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right) .
$$

Here, in the LP-process $L(x), b^{i}(x)=\frac{1}{\left|S_{k}(x)\right|} l_{k}$, with $i \in N$ and $k \in\{1,2\}$ such that $x^{i}=k$, where $S_{k}(x)=\left\{j \in N \mid x^{j}=k\right\}$.

Using the notation $L(x)$, we emphasize that an LP-process is strategy dependent. We explicitly show what part of the LP-process (indirectly) becomes strategy dependent in our notation as well, using, e.g., $b^{i}(x)$ for the strategically determined resource bundle of player $i \in N$. If a set of players chooses the same source, the resource bundle available at this location is divided using an 'equal bundle splitting rule', i.e., each player gets an equal fraction of the available resource bundle. For this, we let $S_{1}(x)$ and $S_{2}(x)$ denote the set of all players who choose location 1 and 2 , respectively.

The next step is to determine the payoff vectors associated with a BLP-process, thereby defining the induced (finite) strategic game. In the following two sections, the payoff vectors of an induced strategic game are determined using 'the' Owen vector or some one-point game-theoretic solution concept that satisfies anonymity, respectively. In both cases, we are able to guarantee the existence of a Nash equilibrium in the induced strategic game.

### 4.2.1 BLP-processes using the Owen set

In this section, the strategic game induced by a BLP-process has payoffs based on the Owen set of the corresponding LP-processes. Let $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ be a BLP-process. For any given $x \in X$, we define $\bar{z}(x)$ to be the average of all extreme points of the optimal region $O(x)=\left\{z(x) \in F \mid v_{L(x)}(N)=z(x)^{\top} \sum_{i \in N} b^{i}(x)\right\}$, where $v_{L(x)}$ is the LP-game corresponding to LP-process $L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right)$. Moreover, let $\omega(x) \in \mathbb{R}^{N}$ be the Owen vector corresponding to $\bar{z}(x)$, given by

$$
\omega_{i}(x)=\bar{z}(x)^{\top} b^{i}(x)
$$

for all $i \in N$. The strategic game $G^{\mathcal{L}, \text { Owen }}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{L}, \text { Owen }}\right\}_{i \in N}\right)$ that is induced by this BLP-process $\mathcal{L}$ is now defined by setting

$$
\pi_{i}^{\mathcal{L}, \text { Owen }}(x)=\omega_{i}(x)
$$

for any $x \in X$ and all $i \in N$.

Importantly, there are only three possibilities for the average shadow prices of a BLP-process. The key observation is that $\sum_{i \in N} b^{i}(x)$ is the only strategy dependent factor that influences $\bar{z}(x)$ for any $x \in X$. If all players choose the same location as their source, then only the resource bundle at that location will be available to the grand coalition. We define $\bar{z}^{1}$ as the (average) vector of shadow prices corresponding to the strategy combination $x \in X$ for which $S_{1}(x)=N$, and $\bar{z}^{2}$ is defined similarly for $x$ such that $S_{2}(x)=N$. For all remaining strategy combinations, note that each location is chosen by at least one player, so that the corresponding total resource bundle is the sum of $l_{1}$ and $l_{2}$. All such strategy combinations then lead to the same (average) price vector, denoted by $\bar{z}^{1,2}$. We formalize this in the following lemma.

## Lemma 4.2.2

Let $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ be a BLP-process. Let $x \in X$. Then,
(i)

$$
\bar{z}(x)= \begin{cases}\bar{z}^{1} & \text { if } S_{1}(x)=N \\ \bar{z}^{2} & \text { if } S_{2}(x)=N \\ \bar{z}^{1,2} & \text { otherwise }\end{cases}
$$

(ii) For all $i \in N$,

$$
\pi_{i}^{\mathcal{L}, \text { Owen }}(x)= \begin{cases}\left(\bar{z}^{1}\right)^{\top} \frac{1}{|N|} l_{1} & \text { if } S_{1}(x)=N \\ \left(\bar{z}^{2}\right)^{\top} \frac{1}{|N|} l_{2} & \text { if } S_{2}(x)=N \\ \left(\bar{z}^{1,2}\right)^{\top} \frac{1}{\left|S_{1}(x)\right|} l_{1} & \text { if } i \in S_{1}(x) \text { and } S_{2}(x) \neq \emptyset \\ \left(\bar{z}^{1,2}\right)^{\top} \frac{1}{\left|S_{2}(x)\right|} l_{2} & \text { if } i \in S_{2}(x) \text { and } S_{1}(x) \neq \emptyset\end{cases}
$$

The following example illustrates a BLP-process and its corresponding induced game.

## Example 4.2.1

Consider a BLP-process $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ with

$$
N=\{1,2,3\}, X^{i}=\{1,2\} \text { for all } i \in N, l_{1}=\left[\begin{array}{c}
18 \\
48
\end{array}\right] \text { and } l_{2}=\left[\begin{array}{l}
90 \\
12
\end{array}\right]
$$

where for any $x \in X$, we have $L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right)$ with

$$
R=\left\{r_{1}, r_{2}\right\}, P=\left\{p_{1}, p_{2}\right\}, A=\left[\begin{array}{ll}
1 & 4 \\
2 & 2
\end{array}\right] \text { and } c=\left[\begin{array}{l}
3 \\
6
\end{array}\right]
$$

The feasible region is given by

$$
F=\left\{z \in \mathbb{R}^{2} \mid z_{1}+2 z_{2} \geq 3,4 z_{1}+2 z_{2} \geq 6, z \geq 0\right\}
$$

with extreme points $\left[\begin{array}{ll}3 & 0\end{array}\right]^{\top},\left[\begin{array}{ll}1 & 1\end{array}\right]^{\top}$ and $\left[\begin{array}{ll}0 & 3\end{array}\right]^{\top}$.

Consider $x=(1,1,1)$, so that the players (equally) divide only resource bundle $l_{1}$. In this case, the objective function for the dual program of $N$ is given by $z^{\top} \sum_{i \in N} b^{i}(x)=18 z_{1}+48 z_{2}$ for all $z \in F$. The unique extreme point of the feasible region that minimizes this objective function, is $\left[\begin{array}{ll}3 & 0\end{array}\right]^{\top}$. Since the optimal region $O(x)$ consists of a single vector, we simply get $\bar{z}(x)=\bar{z}^{1}=\left[\begin{array}{ll}3 & 0\end{array}\right]^{\top}$. For these shadow prices, the corresponding payoff in the induced strategic game is $\pi_{i}^{\mathcal{L}, \text { Owen }}(x)=\left(\bar{z}^{1}\right)^{\top} b^{i}(x)=\left(\bar{z}^{1}\right)^{\top} \frac{1}{3} l_{1}=\left[\begin{array}{ll}3 & 0\end{array}\right]\left[\begin{array}{ll}6 & 16\end{array}\right]^{\top}=18$ for all $i \in N$.

Similarly, $x=(2,2,2)$ leads to $\bar{z}(x)=\bar{z}^{2}=\left[\begin{array}{ll}0 & 3\end{array}\right]^{\top}$ and $\pi_{i}^{\mathcal{L}, \text { Owen }}(x)=\left(\bar{z}^{2}\right)^{\top} \frac{1}{3} l_{2}=$ 12 for all $i \in N$.

For any other strategy combination $x$, the total resource bundle available to the
grand coalition $N$ becomes $l_{1}+l_{2}=[10860]^{\top}$. Solving the corresponding minimization problem $\min _{z \in F} 108 z_{1}+60 z_{2}$ gives $\bar{z}(x)=\bar{z}^{1,2}=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\top}$. With $x=(1,1,2)$, we find

$$
\pi^{\mathcal{L}, \text { Owen }}(x)=\left(\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
\frac{18}{2} \\
\frac{48}{2}
\end{array}\right],\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
\frac{18}{2} \\
\frac{48}{2}
\end{array}\right],\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
90 \\
12
\end{array}\right]\right)=(33,33,102) .
$$

For each strategy combination, the resulting payoffs determined by the Owen set of the corresponding LP-process are given in Table 4.1. Recall that in such tables, we always let the row represent the strategy of player 1, where player 2 chooses a column, and the matrix is determined by the choice of player 3 .


Table 4.1 The strategic game $G^{\mathcal{L}, \text { Owen }}$ induced by the BLPprocess of Example 4.2.1

The set of Nash equilibria of the induced game $G^{\mathcal{L}, \text { Owen }}$ is

$$
E\left(G^{\mathcal{L}, \text { Owen }}\right)=\{(1,2,2),(2,1,2),(2,2,1)\} .
$$

Next, we show that any strategic game $G^{\mathcal{L}, \text { Owen }}$ induced by a BLP-process $\mathcal{L}$, has a Nash equilibrium. We do so using a comprehensive result from Konishi et al. (1997). In particular, they show that any strategic game satisfying four properties has a Nash equilibrium. These properties are presented in the general context of 'congestion games', in which players compete by choosing to use a certain facility in a facility set. The first property, (P1), states that this facility set is finite. Let $G=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}\right\}_{i \in N}\right)$ be a strategic game. Then, (P1) is defined as
(P1) there exists a finite set $K$ such that $X^{i}=K$ for all $i \in N$.
Second, (P2) concerns 'independence of irrelevant choices' and requires that the payoff of a player is not affected by any change in the strategy combination, as long as the set of players who choose the same facility as this player is not altered. Formally,

$$
\begin{align*}
\pi_{i}\left(x^{j}, x^{-j}\right) & =\pi_{i}\left(\tilde{x}^{j}, x^{-j}\right) \text { for any } i, j \in N \text { and any } x \in X \text { and } \tilde{x}^{j} \in X^{j}  \tag{P2}\\
\text { such that } x^{i} & \neq x^{j} \text { and } x^{i} \neq \tilde{x}^{j} .
\end{align*}
$$

The third property (P3), 'anonymity', implies, in combination with (P1), that only the number of players choosing each of the facilities within $K$ impacts the payoffs, their identities do not. Formally,
(P3) $\quad \pi_{i}(x)=\pi_{i}(y)$ for any $i \in N$ and any $x, y \in X$ such that $x^{i}=y^{i}$ and $\left|\left\{j \in N \mid x^{j}=k\right\}\right|=\left|\left\{j \in N \mid y^{j}=k\right\}\right|$ for any $k \in K$.

Finally, 'partial rivalry' (P4) means that if some player chose the same facility as player $i \in N$, but then deviates to a different facility, this will never decrease the payoff of $i$ :
(P4) $\quad \pi_{i}\left(x^{j}, x^{-j}\right) \leq \pi_{i}\left(\tilde{x}^{j}, x^{-j}\right)$ for any $i, j \in N, i \neq j$, and any $x \in X$ and $\tilde{x}^{j} \in X^{j}$ such that $x^{i}=x^{j}$ and $x^{i} \neq \tilde{x}^{j}$.

After formalizing the result in Proposition 4.2.3, we prove Theorem 4.2.4 by showing that $G^{\mathcal{L}, \text { Owen }}$ satisfies all four properties.

## Proposition 4.2.3 [Konishi et al. (1997)]

Let $G=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}\right\}_{i \in N}\right)$ be a strategic game that satisfies properties (P1), (P2), (P3) and (P4). Then, $E(G) \neq \emptyset$.

## Theorem 4.2.4

Let $\mathcal{L}$ be a BLP-process and let $G^{\mathcal{L}, \text { Owen }}$ be the induced strategic game. Then, $E\left(G^{\mathcal{L}, \text { Owen }}\right) \neq \emptyset$.

Proof. We prove the theorem by showing that $G^{\mathcal{L}, \text { Owen }}$ satisfies (P1)-(P4) from Proposition 4.2.3. For $G^{\mathcal{L}, \text { Owen }}$, we have $X^{i}=\{1,2\}$ for all $i \in N$, so (P1) is clearly satisfied.

Next, let $i, j \in N, x \in X$ and $\tilde{x}^{j} \in X^{j}$ such that $x^{i} \neq x^{j}$ and $x^{i} \neq \tilde{x}^{j}$. Note that since $X^{i}=\{1,2\}$ for all $i \in N$, if $x^{i} \neq x^{j}$ and $x^{i} \neq \tilde{x}^{j}$, then $x^{j}=\tilde{x}^{j}$, meaning (P2) is satisfied as well.

For (P3), let $i \in N, x, y \in X$ such that $x^{i}=y^{i}$ and $\left|S_{k}(x)\right|=\left|S_{k}(y)\right|$ for all $k \in\{1,2\}$. Note that $x^{i}=y^{i}$ implies that $i \in S_{k}(x) \Leftrightarrow i \in S_{k}(y)$ for any $k \in\{1,2\}$. Using Lemma 4.2.2(ii), it follows that $\pi_{i}^{\mathcal{L}, \text { Owen }}(x)=\pi_{i}^{\mathcal{L}, \text { Owen }}(y)$.

To show (P4) is satisfied, let $i, j \in N, i \neq j$, let $k \in\{1,2\}$, and let $x \in X$ such that $x^{i}=x^{j}=k$ and $\tilde{x}^{j} \neq k$. Since, in $x$, at least two distinct players choose $k$, we must have either $S_{k}(x)=N$, or $1<\left|S_{k}(x)\right|<|N|$. For $S_{k}(x)=N$, we have

$$
\pi_{i}^{\mathcal{L}, \text { Owen }}\left(x^{j}, x^{-j}\right)=\left(\bar{z}^{k}\right)^{\top} \frac{1}{|N|} l_{k}
$$

$$
\begin{aligned}
& \leq\left(\bar{z}^{1,2}\right)^{\top} \frac{1}{|N|} l_{k} \\
& \leq\left(\bar{z}^{1,2}\right)^{\top} \frac{1}{|N|-1} l_{k} \\
& =\pi_{i}^{\mathcal{L}, \text { Owen }}\left(\tilde{x}^{j}, x^{-j}\right)
\end{aligned}
$$

where the first inequality follows from the fact that $\bar{z}^{k}$ is a solution to $\min _{z \in F}\left\{z^{\top} l_{k}\right\}$ since $S_{k}(x)=N$, and the final equality follows from $\left|S_{k}\left(\tilde{x}^{j}, x^{-j}\right)\right|=\left|S_{k}(x)\right|-1$. We also use the latter for $1<\left|S_{k}(x)\right|<|N|$, which yields

$$
\pi_{i}^{\mathcal{L}, \text { Owen }}\left(x^{j}, x^{-j}\right)=\left(\bar{z}^{1,2}\right)^{\top} \frac{1}{\left|S_{k}(x)\right|} l_{k} \leq\left(\bar{z}^{1,2}\right)^{\top} \frac{1}{\left|S_{k}(x)\right|-1} l_{k}=\pi_{i}^{\mathcal{L}, \text { Owen }}\left(\tilde{x}^{j}, x^{-j}\right)
$$

This shows that (P4) is satisfied as well, which completes the proof.

### 4.2.2 BLP-processes using an anonymous solution

In this section, we focus on a modification of Theorem 4.2.4 using an anonymous one-point game-theoretic solution concept (from now on referred to as an anonymous solution) instead of the Owen set to determine the payoffs in the induced strategic game. We consider game-theoretic solutions on the class of LP-games with fixed player set $N$, defined by a function $f$ on this class such that $\sum_{i \in N} f_{i}(v)=v(N)$. Solution $f$ satisfies anonymity if for every LP-game $v$, any bijection $\sigma: N \rightarrow N$, and all $i \in N$, we have $f_{\sigma(i)}(v)=f_{i}\left(v^{\sigma}\right)$, where $v^{\sigma}(T)=v(\sigma(T))$ for all $T \subseteq N \backslash\{\emptyset\}$. Here, $\sigma(i)=j$ implies player $j$ in $v$ is 'named' $i$ in $v^{\sigma}$, and $\sigma(T)$ is the set of players in $v$ corresponding to coalition $T$ in $v^{\sigma}$, i.e., $\sigma(T)=\{j \in N \mid \exists i \in T$ with $\sigma(i)=j\}$. In the context of LP-games, this means that any difference in payoffs between players is explained by a difference in their resource bundles, not by their identities. A direct implication of the anonymity of $f$ is that for any LP-game $v$ with fixed player set $N$ and all $i, j \in N$ with $i$ and $j$ symmetric in $v$, i.e., with $v(T \cup\{i\})=v(T \cup\{j\})$ for any $T \subseteq N \backslash\{i, j\}$, it holds that $f_{i}(v)=f_{j}(v)$. Examples of prominent anonymous solutions include the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969).

Given a BLP-process $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ and an anonymous solution $f$, we define the corresponding induced strategic game as
$G^{\mathcal{L}, f}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{L}, f}\right\}_{i \in N}\right)$, where the payoff of player $i \in N$ equals

$$
\pi_{i}^{\mathcal{L}, f}(x)=f_{i}\left(v_{L(x)}\right)
$$

Lemma 4.2.5 states that, in $G^{\mathcal{L}, f}$, for any strategy combination in which two players choose the same location, these players have the same payoff. Further, the effect of unilaterally deviating from this specific location to the other location is the same for each player.

## Lemma 4.2.5

Let $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ be a BLP-process, let $f$ be an anonymous solution and let $G^{\mathcal{L}, f}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{L}, f}\right\}_{i \in N}\right)$ be the induced strategic game. Then,
(i) $\pi_{i}^{\mathcal{L}, f}(x)=\pi_{j}^{\mathcal{L}, f}(x)$ for any $i, j \in N$ and any $x \in X$ such that $x^{i}=x^{j}$.
(ii) $\pi_{i}^{\mathcal{L}, f}\left(\tilde{x}^{i}, x^{-i}\right)=\pi_{j}^{\mathcal{L}, f}\left(\tilde{x}^{j}, x^{-j}\right)$ for any $i, j \in N$, any $x \in X$, and any $\tilde{x}^{i} \in X^{i}$ and $\tilde{x}^{j} \in X^{j}$ such that $x^{i}=x^{j}, \tilde{x}^{i}=\tilde{x}^{j}$ and $x^{i} \neq \tilde{x}^{i}$.

## Proof.

(i) Let $i, j \in N$ and $x \in X$ such that $x^{i}=x^{j}$. Since this implies that $b^{i}(x)=b^{j}(x)$, it readily follows that $i$ and $j$ are symmetric in $v_{L(x)}$, so that $\pi_{i}^{\mathcal{L}, f}(x)=\pi_{j}^{\mathcal{L}, f}(x)$ by anonymity of $f$.
(ii) Let $i, j \in N, x \in X, \tilde{x}^{i} \in X^{i}$ and $\tilde{x}^{j} \in X^{j}$ such that $x^{i}=x^{j}, \tilde{x}^{i}=\tilde{x}^{j}$ and $x^{i} \neq \tilde{x}^{i}$. Then, $b^{i}\left(\tilde{x}^{i}, x^{-i}\right)=b^{j}\left(\tilde{x}^{j}, x^{-j}\right)$ and $b^{i}\left(\tilde{x}^{j}, x^{-j}\right)=b^{j}\left(\tilde{x}^{i}, x^{-i}\right)$. Consider the LP-games $v_{L\left(\tilde{x}^{i}, x^{-i}\right)}$ and $v_{L\left(\tilde{x}^{j}, x^{-j}\right)}$, and consider the bijection $\sigma: N \rightarrow N$ such that $\sigma(i)=j, \sigma(j)=i$, and $\sigma(k)=k$ for all $k \in N \backslash\{i, j\}$. Note that $v_{L\left(\tilde{x}^{i}, x^{-i}\right)}^{\sigma}=v_{L\left(\tilde{x}^{j}, x^{-j}\right)}$. Hence, applying anonymity of $f$ (in the second equality), we get

$$
f_{i}\left(v_{L\left(\tilde{x}^{i}, x^{-i}\right)}\right)=f_{\sigma(j)}\left(v_{L\left(\tilde{x}^{i}, x^{-i}\right)}\right)=f_{j}\left(v_{L\left(\tilde{x}^{i}, x^{-i}\right)}^{\sigma}\right)=f_{j}\left(v_{L\left(\tilde{x}^{j}, x^{-j}\right)}\right)
$$

In particular, this lemma implies that, in $G^{\mathcal{L}, f}$, if a player $i \in N$ can deviate profitably from some $x \in X$, then all players $j \in N$ with $x^{i}=x^{j}$ can deviate profitably.

## Corollary 4.2.6

Let $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ be a BLP-process, let $f$ be an anonymous solution and let $G^{\mathcal{L}, f}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{L}, f}\right\}_{i \in N}\right)$ be the induced strategic game. Let
$i, j \in N, x \in X, \tilde{x}^{i} \in X^{i}$ and $\tilde{x}^{j} \in X^{j}$ such that $x^{i}=x^{j}$ and $\tilde{x}^{i}=\tilde{x}^{j}$. If $\pi_{i}^{\mathcal{L}, f}(x)<\pi_{i}^{\mathcal{L}, f}\left(\tilde{x}^{i}, x^{-i}\right)$, then $\pi_{j}^{\mathcal{L}, f}(x)<\pi_{j}^{\mathcal{L}, f}\left(\tilde{x}^{j}, x^{-j}\right)$.

The following example illustrates that the Nash equilibria of $G^{\mathcal{L}, O w e n}$ and $G^{\mathcal{L}, f}$ induced by the same BLP-process $\mathcal{L}$ need not coincide. Hence, the existence of a Nash equilibrium in $G^{\mathcal{L}, f}$ does not follow from Theorem 4.2.4. However, using Corollary 4.2.6, its existence is still guaranteed, as formalized in Theorem 4.2.7.

## Example 4.2.2

Consider a BLP-process $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ with

$$
N=\{1,2\}, X^{1}=X^{2}=\{1,2\}, l_{1}=\left[\begin{array}{c}
100 \\
200
\end{array}\right] \text { and } l_{2}=\left[\begin{array}{c}
300 \\
50
\end{array}\right]
$$

where for any $x \in X$, we have $L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right)$ with

$$
R=\left\{r_{1}, r_{2}\right\}, P=\left\{p_{1}, p_{2}\right\}, A=\left[\begin{array}{ll}
5 & 5 \\
6 & 6
\end{array}\right] \text { and } c=\left[\begin{array}{l}
9 \\
9
\end{array}\right] .
$$

The feasible region is given by

$$
F=\left\{z \in \mathbb{R}^{2} \mid 5 z_{1}+6 z_{2} \geq 9, z \geq 0\right\}
$$

and has two extreme points, leading to $\bar{z}^{1}=\left[\begin{array}{ll}9 / 5 & 0\end{array}\right]^{\top}$ and $\bar{z}^{2}=\bar{z}^{1,2}=\left[\begin{array}{ll}0 & 3 / 2\end{array}\right]^{\top}$. The LP-games corresponding to each strategy combination are given in Table 4.2.

| $T$ | $\{1\}$ | $\{2\}$ | $N$ |
| :---: | :---: | :---: | :---: |
| $v_{L(1,1)}(T)$ | 90 | 90 | 180 |
| $v_{L(1,2)}(T)$ | 180 | 75 | 375 |
| $v_{L(2,1)}(T)$ | 75 | 180 | 375 |
| $v_{L(2,2)}(T)$ | 37.5 | 37.5 | 75 |

Table 4.2 The LP-game $v_{L(x)}$ for each $x \in X$ in Example 4.2.2

The induced strategic game $G^{\mathcal{L}, \text { Owen }}$, for which the payoffs can be calculated using Lemma 4.2.2(ii), is depicted in Table 4.3. To derive $G^{\mathcal{L}, \Phi}$, given in Table 4.4, the Shapley value $\Phi\left(v_{L(x)}\right)$ can be straightforwardly calculated using the standard equation (1.1) for the Shapley value of two-player games.

|  | 1 | 2 |
| :--- | :---: | :---: |
| 1 | $(90,90)$ | $(300,75)$ |
| 2 | $(75,300)$ | $(37.5,37.5)$ |

Table 4.3 The strategic game $G^{\mathcal{L}, \text { Owen }}$ induced by the BLPprocess of Example 4.2.2

|  | 1 | 2 |
| :--- | :---: | :---: |
| 1 | $(90,90)$ | $(240,135)$ |
| 2 | $(135,240)$ | $(37.5,37.5)$ |
|  |  |  |

Table 4.4 The strategic game $G^{\mathcal{L}, \Phi}$ induced by the BLP-process of Example 4.2.2

Note that $E\left(G^{\mathcal{L}, \text { Owen }}\right)=\{(1,1)\}$, whereas $E\left(G^{\mathcal{L}, \Phi}\right)=\{(1,2),(2,1)\}$.

## Theorem 4.2.7

Let $\mathcal{L}$ be a BLP-process, let $f$ be an anonymous solution and let $G^{\mathcal{L}, f}$ be the induced strategic game. Then, $E\left(G^{\mathcal{L}, f}\right) \neq \emptyset$.

Proof. Denote $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ and suppose $E\left(G^{\mathcal{L}, f}\right)=\emptyset$. Set $x^{0}=(1,1, \ldots, 1)$ and $x^{|N|}=(2,2, \ldots, 2)$. The proof by induction (on a finite set) is based on the following idea: using the assumption that there are no equilibria, and starting from strategy combination $x^{0}$, we show that for all strategy combinations $x \in X$ with $S_{1}(x) \neq \emptyset$ it is beneficial to unilaterally deviate for all players in $S_{1}(x)$. This reasoning, however, would lead to the conclusion that $x^{|N|}$ is a Nash equilibrium, yielding a contraction.

As $\mathcal{L}$ and $f$ are fixed, we use $\pi$ instead of $\pi^{\mathcal{L}, f}$ in the proof. For the base case, since $x^{0} \notin E\left(G^{\mathcal{L}, f}\right)$, Corollary 4.2.6 implies that $\pi_{i}\left(x^{1}\right)>\pi_{i}\left(x^{0}\right)$ for all $x^{1} \in X$ and $i \in N$ with $S_{2}\left(x^{1}\right)=\{i\}$. Note that this inequality implies that it is not beneficial for player $i$ to unilaterally deviate from $x^{1}$ by choosing location 1 instead of 2 .

Next, we assume the induction hypothesis that for some $k \in\{1, \ldots,|N|-1\}$, it holds that

$$
\begin{equation*}
\pi_{i}\left(x^{k}\right)>\pi_{i}\left(x^{k-1}\right) \tag{4.1}
\end{equation*}
$$

for all $x^{k-1}, x^{k} \in X$ and $i \in S_{1}\left(x^{k-1}\right)$ with $\left|S_{2}\left(x^{k-1}\right)\right|=k-1$ and $S_{2}\left(x^{k}\right)=$ $S_{2}\left(x^{k-1}\right) \cup\{i\}$. To emphasize, player $i$ unilaterally deviates from $x^{k-1}$ by choosing location 2 instead of 1.

Note that for an arbitrary $x^{k} \in X$ with $\left|S_{2}\left(x^{k}\right)\right|=k$, (4.1) implies that there is no profitable unilateral deviation for the players in $S_{2}\left(x^{k}\right)$. Since $E\left(G^{\mathcal{L}, f}\right)=\emptyset$, there must therefore be a strictly profitable deviation for a player in $S_{1}\left(x^{k}\right)$. Corollary 4.2.6 then implies that

$$
\begin{equation*}
\pi_{i}\left(x^{k+1}\right)>\pi_{i}\left(x^{k}\right) \tag{4.2}
\end{equation*}
$$

for all $x^{k}, x^{k+1} \in X$ and $i \in S_{1}\left(x^{k}\right)$ with $\left|S_{2}\left(x^{k}\right)\right|=k$ and $S_{2}\left(x^{k+1}\right)=S_{2}\left(x^{k}\right) \cup\{i\}$. This shows the induction hypothesis holds for $k+1$ as well.

Note, however, that for $k=|N|-1$, (4.2) yields $\pi_{i}\left(x^{|N|}\right)>\pi_{i}\left(x^{|N|-1}\right)$ for all $x^{|N|-1} \in X$ and $i \in N$ with $S_{1}\left(x^{|N|-1}\right)=\{i\}$, which in turn implies that $x^{|N|}=(2,2, \ldots, 2) \in E\left(G^{\mathcal{L}, f}\right)$, a contradiction. Hence, $E\left(G^{\mathcal{L}, f}\right) \neq \emptyset$.

For the sake of completeness, we remark that a proof along similar lines as the proof of Theorem 4.2.7 can also be used to prove Theorem 4.2.4. However, the result of Konishi et al. (1997) as stated in Proposition 4.2.3, used to prove Theorem 4.2.4, cannot be used to prove Theorem 4.2.7. In particular, property (P4) need not be satisfied. This is illustrated in the following example.

## Example 4.2.3

Reconsider the BLP-process $\mathcal{L}$ from Example 4.2.2, but suppose the induced strategic game $G^{\mathcal{L}, f}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{L}, f}\right\}_{i \in N}\right)$ is now determined by the anonymous solution $f$ given by $f_{i}\left(v_{L(x)}\right)=\frac{3}{|N|} v_{L(x)}(N)-2 \Phi_{i}\left(v_{L(x)}\right)$ for any $x \in X$ and any $i \in N$. Then, we have $\pi_{1}^{\mathcal{L}, f}(1,1)=90>82.5=\pi_{1}^{\mathcal{L}, f}(1,2)$, meaning ( P 4 ) is violated.

### 4.2.3 BLP-processes with unequal bundle splitting or three locations

In the preceding sections, we consider BLP-processes in which resources can be obtained from two locations, where all players who choose the same location get an equal fraction of the resource bundle. In this section, we consider modifications with unequal bundle splitting or more than two locations. For the former, we show that the existence of a Nash equilibrium is no longer guaranteed if there are three or more players. For the latter, even for two players a Nash equilibrium need not exist.

The equal bundle splitting rule is contingent on the firms holding equal sway over the sources. In practice, power dynamics may be such that one firm will obtain a larger part of the resource bundle if firms are forced to compete at a source. Using an alternative splitting rule, where each player gets some non-negative fraction of the resource bundle that is not necessarily equal to the fraction of the other players, affects the results. Allowing for arbitrary ways of splitting, the set of Nash equilibria of the induced strategic game can become empty for $|N|>2$, as demonstrated in Example 4.2.4.

To properly define a BLP-process $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ with unequal bundle splitting rules, let $\lambda(x)=\left\{\lambda_{i}(x)\right\}_{i \in N}$ denote the bundle splitting rule of $x \in X$, where we require that $\lambda_{i}(x) \geq 0$ for all $i \in N$, and $\sum_{i \in S_{k}(x)} \lambda_{i}(x)=1$ for all $k \in\{1,2\}$ with $S_{k}(x) \neq \emptyset$. We refer to any such bundle splitting rule as a non-negative bundle splitting rule. Then, for any $x \in X, L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right)$ is determined by $b^{i}(x)=\lambda_{i}(x) l_{k}$ for any $i \in N$ and $k \in\{1,2\}$ such that $x^{i}=k$.

## Example 4.2.4

We reconsider the BLP-process $\mathcal{L}$ of Example 4.2.1. Recall that $\bar{z}^{1}=\left[\begin{array}{ll}3 & 0\end{array}\right]^{\top}, \bar{z}^{2}=$ $\left[\begin{array}{ll}0 & 3\end{array}\right]^{\top}, \bar{z}^{1,2}=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\top}, l_{1}=\left[\begin{array}{ll}18 & 48\end{array}\right]^{\top}$ and $l_{2}=[9012]^{\top}$. We use an equal bundle splitting rule for all strategies $x \in X$, except $\lambda(1,2,2)=\left(1, \frac{1}{6}, \frac{5}{6}\right), \lambda(2,1,2)=\left(\frac{5}{6}, 1, \frac{1}{6}\right)$ and $\lambda(2,2,1)=\left(\frac{1}{6}, \frac{5}{6}, 1\right)$.

For the corresponding resource bundles, we get, e.g., $b^{1}(1,2,2)=l_{1}, b^{2}(1,2,2)=$ $\left[\begin{array}{ll}15 & 2\end{array}\right]^{\top}$ and $b^{3}(1,2,2)=\left[\begin{array}{ll}75 & 10\end{array}\right]^{\top}$. The shadow prices are not affected by the bundle splitting rule, so $\bar{z}_{1,2}=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\top}$ is still used to determine the payoff based on the Owen set for (among others) the three strategy combinations with unequal bundle splitting. For each strategy combination, the resulting payoffs determined by the Owen set of the corresponding LP-process are given in Table 4.5.

|  | 1 | 2 |  |
| :--- | :---: | :---: | :---: |
| 1 | $(18,18,18)$ | $(33,102,33)$ |  |
| 2 | $(102,33,33)$ | $(17,85,66)$ |  |
| 1 |  | 1 |  |


|  | 1 | 2 |
| :--- | :---: | :---: |
| 1 | $(33,33,102)$ | $(66,17,85)$ |
| 2 | $(85,66,17)$ | $(12,12,12)$ |
| 2 |  |  |

Table 4.5 The strategic game $G^{\mathcal{L}, \text { Owen }}$ induced by the BLPprocess with unequal bundle splitting of Example 4.2.4

Note that $E\left(G^{\mathcal{L}, \text { Owen }}\right)=\emptyset$. A key difference from Example 4.2.1 is that we now have, e.g., $\pi_{1}^{\mathcal{L}, \text { Owen }}(2,1,2) \neq \pi_{1}^{\mathcal{L}, \text { Owen }}(2,2,1)$.

The final observation in Example 4.2.4 shows that (P3) from Proposition 4.2.3 no longer holds. This does not yet occur when there are only two players, meaning Theorem 4.2.4 can still be generalized for $|N|=2$.

## Theorem 4.2.8

Let $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2},\{L(x)\}_{x \in X}\right)$ be a BLP-process with $|N|=2$, using nonnegative bundle splitting rules, and let $G^{\mathcal{L}, O w e n}$ be the induced strategic game. Then, $E\left(G^{\mathcal{L}, \text { Owen }}\right) \neq \emptyset$.

Proof. Similar to the proof of Theorem 4.2.4, we show that $G^{\mathcal{L}, \text { Owen }}$ satisfies (P1)(P4) from Proposition 4.2.3. Properties (P1) and (P2) are again clearly satisfied.

To show that (P3) is satisfied, let $i \in N, x, y \in X$ such that $x^{i}=y^{i}$ and $\left|S_{k}(x)\right|=\left|S_{k}(y)\right|$ for all $k \in\{1,2\}$. Since $|N|=2$, this implies that $x=y$ and hence $\pi_{i}^{\mathcal{L}, \text { Owen }}(x)=\pi_{i}^{\mathcal{L}, \text { Owen }}(y)$.

Finally, for (P4), let $i, j \in N, i \neq j$, let $k \in\{1,2\}$, and let $x \in X$ such that $x^{i}=x^{j}=k$ and $\tilde{x}^{j} \neq k$. Note that $S_{k}(x)=N$ since $|N|=2$, so that

$$
\begin{aligned}
\pi_{i}^{\mathcal{L}, \text { Owen }}\left(x^{i}, x^{j}\right) & =\left(\bar{z}^{k}\right)^{\top} \lambda_{i}\left(x^{i}, x^{j}\right) l_{k} \\
& \leq\left(\bar{z}^{1,2}\right)^{\top} \lambda_{i}\left(x^{i}, x^{j}\right) l_{k} \\
& \leq\left(\bar{z}^{1,2}\right)^{\top} l_{k} \\
& =\pi_{i}^{\mathcal{L}, \text { Owen }}\left(x^{i}, \tilde{x}^{j}\right),
\end{aligned}
$$

using $\lambda_{i}\left(x^{i}, x^{j}\right) \geq 0$ and $\lambda_{i}\left(x^{i}, x^{j}\right) \leq 1$ in the first and second inequality, respectively.

Alternatively, maintaining the equal bundle splitting rule, one can generalize the BLPprocess using a model with three locations. Even for $|N|=2$, the set of Nash equilibria of the induced strategic game can become empty, as shown in Example 4.2.5.

## Example 4.2.5

Consider a BLP-process with three locations $\mathcal{L}=\left(N,\left\{X^{i}\right\}_{i \in N}, l_{1}, l_{2}, l_{3},\{L(x)\}_{x \in X}\right)$ with

$$
N=\{1,2\}, X^{1}=X^{2}=\{1,2,3\}, l_{1}=\left[\begin{array}{c}
100 \\
200
\end{array}\right], l_{2}=\left[\begin{array}{c}
300 \\
50
\end{array}\right], \text { and } l_{3}=\left[\begin{array}{c}
200 \\
150
\end{array}\right]
$$

where for any $x \in X$, we have $L(x)=\left(N, R, P, A,\left\{b^{i}(x)\right\}_{i \in N}, c\right)$ with

$$
R=\left\{r_{1}, r_{2}\right\}, P=\left\{p_{1}, p_{2}\right\}, A=\left[\begin{array}{ll}
0 & 2 \\
5 & 1
\end{array}\right] \text { and } c=\left[\begin{array}{l}
3 \\
6
\end{array}\right] .
$$

Similar to the two-location setting, the resource bundles in $L(x)$ are determined using an equal bundle splitting rule, i.e., $b^{i}(x)=\frac{1}{\left|S_{k}(x)\right|} l_{k}$, with $k \in\{1,2,3\}$ such that $x^{i}=k$, where $S_{k}(x)=\left\{j \in N \mid x^{j}=k\right\}$.

The feasible region is given by

$$
F=\left\{z \in \mathbb{R}^{2} \mid 5 z_{2} \geq 3,2 z_{1}+z_{2} \geq 6, z \geq 0\right\}
$$

The shadow prices correspond to the extreme points of the feasible region. Specifically, we find

$$
\bar{z}(x)= \begin{cases}{\left[\begin{array}{ll}
0 & 6
\end{array}\right]^{\top}} & \text { if } x \in\{(2,2),(2,3),(3,2)\} \\
{\left[\begin{array}{ll}
2.7 & 0.6
\end{array}\right]^{\top}} & \text { otherwise }\end{cases}
$$

The induced strategic game is determined similar to the two-location setting. For $x=(1,1)$, this leads to $\pi_{1}^{\mathcal{L}, \text { Owen }}(x)=\pi_{2}^{\mathcal{L}, \text { Owen }}(x)=[2.70 .6] \frac{1}{2} l_{1}=195$. Alternatively, for $x=(2,3)$, the payoffs are $\pi_{2}^{\mathcal{L} \text {,Owen }}(x)=\left[\begin{array}{ll}0 & 6\end{array}\right] l_{2}=300$ and $\pi_{3}^{\mathcal{L}, \text { Owen }}(x)=\left[\begin{array}{ll}0 & 6\end{array}\right] l_{3}=$ 900 , respectively. In this way, we find the strategic game given in Table 4.6.

|  | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: |
| 1 | $(195,195)$ | $(390,840)$ | $(390,630)$ |
| 2 | $(840,390)$ | $(150,150)$ | $(300,900)$ |
| 3 | $(630,390)$ | $(900,300)$ | $(315,315)$ |
|  |  |  |  |

Table 4.6 The strategic game $G^{\mathcal{L}, \text { Owen }}$ induced by the BLPprocess with three locations of Example 4.2.5

The set of Nash equilibria of this strategic game is empty.

In particular, note that (P2) from Proposition 4.2 .3 is violated, since, e.g., $\pi_{1}^{\mathcal{L}, \text { Owen }}(2,1) \neq \pi_{1}^{\mathcal{L}, \text { Owen }}(2,3)$. Whether player 2 chooses location 1 or 3 affects the shadow prices and thereby the payoff of player 1 . This does not yet come into play when there are only two locations.

### 4.3 Biform sequencing processes

In this section, we introduce a strategic component to sequencing processes. We first recall the definition of a standard (cooperative) sequencing process, where an initial order is given. Such a process will form the second stage of a biform sequencing process, at which we arrive after an initial order is strategically determined in the first stage.

### 4.3.1 Sequencing processes

A sequencing process is summarized by the tuple

$$
Q=\left(N, \sigma_{0}, p, \alpha\right)
$$

Here, $N$ is the finite player set, where each player represents a job (these two terms will be used interchangeably). From the start, all jobs are lined up to be processed sequentially on a single machine. The initial processing order is denoted by $\sigma_{0}$, where an order is described by a bijection $\sigma:\{1,2, \ldots,|N|\} \rightarrow N$ and the collection of all such orders is denoted by $\Pi(N)$. In particular, $\sigma(k)=i$ indicates that job $i \in N$ is on the $k$-th place in the processing order. Vectors $p \in \mathbb{R}^{N}$ and $\alpha \in \mathbb{R}^{N}$, satisfying $p_{i}>0$ and $\alpha_{i}>0$ for all $i \in N$, denote the processing times and cost parameters of the players, respectively. The cost parameter of a player determines the linear relationship between this player's costs and the completion time of the corresponding job.

Let $Q=\left(N, \sigma_{0}, p, \alpha\right)$ be a sequencing process. Then, for any processing order $\sigma \in \Pi(N)$, the completion time of a job $i \in N$ is denoted by $C_{i}(\sigma)$, with

$$
C_{i}(\sigma)=\sum_{j \in N:} p_{j} .
$$

The corresponding individual costs of player $i$ w.r.t. $\sigma$ are given by $\alpha_{i} C_{i}(\sigma)$.

Clearly, an individual player's costs are lower when this player is closer to the head of the queue in the initial order, since fewer players are processed before this player in that case. To determine an optimal order, however, we are interested in minimizing the total costs $\sum_{i \in N} \alpha_{i} C_{i}(\sigma)$ of the set of jobs as a whole over all orders $\sigma \in \Pi(N)$. Intuitively, it is clear that jobs with a high cost parameter should be processed before those with low cost parameter, unless the processing time of the former is substantially higher than that of the latter. This creates a measure of urgency for each player, denoted by $u=\left(u_{i}\right)_{i \in N}$, where the urgency of player $i \in N$ is defined by $u_{i}=\frac{\alpha_{i}}{p_{i}}$. Smith (1956) proved that any optimal processing order $\hat{\sigma} \in \Pi(N)$ is such that the players are ordered in weakly decreasing order of urgency, i.e., for all $k \in\{1,2, \ldots,|N|-1\}$, we have

$$
\frac{\alpha_{\hat{\sigma}(k)}}{p_{\hat{\sigma}(k)}} \geq \frac{\alpha_{\hat{\sigma}(k+1)}}{p_{\hat{\sigma}(k+1)}}
$$

The set of misplaced pairs $M P\left(\sigma_{0}\right)$ within the given initial order $\sigma_{0} \in \Pi(N)$ comprises all pairs of players who are not ordered according to a Smith order, formally defined by

$$
M P\left(\sigma_{0}\right)=\left\{(i, j) \in N \times N \mid \sigma_{0}^{-1}(i)<\sigma_{0}^{-1}(j) \text { and } u_{i}<u_{j}\right\}
$$

By rearranging all misplaced pairs, one arrives at a Smith order. If two neighboring players $i, j \in N$ with $\sigma_{0}^{-1}(i)=\sigma_{0}^{-1}(j)-1$ are misplaced, i.e., $(i, j) \in M P\left(\sigma_{0}\right)$, these players can save costs by switching places. Specifically, the cost savings of such a pair of neighboring players are

$$
g_{i j}=\alpha_{j} p_{i}-\alpha_{i} p_{j}>0
$$

Note that $\alpha_{j} p_{i}$ corresponds to the individual cost decrease of player $j$, where $\alpha_{i} p_{j}$ is the individual cost increase of player $i$, so that $g_{i j}$ gives the joint cost savings of the pair. Importantly, note that these cost savings are independent of the exact position of the neighbors in the queue.

The maximal cost savings the grand coalition can make, are achieved through rearranging misplaced pairs in the initial order by consecutively switching neighboring players until the players are arranged in a Smith order. The corresponding maximal cost savings obtained by a Smith order $\hat{\sigma} \in \Pi(N)$ therefore equal

$$
\sum_{i \in N} \alpha_{i}\left(C_{i}\left(\sigma_{0}\right)-C_{i}(\hat{\sigma})\right)=\sum_{(i, j) \in M P\left(\sigma_{0}\right)} g_{i j} .
$$

Next, we need to determine how to allocate these cost savings to the players. For this, we use so-called gain splitting rules. The concept of such a rule is that whenever a pair of neighboring players makes a gain by switching places, this gain is only divided among these players. Curiel et al. (1989) propose and axiomatically characterize the Equal Gain Splitting ( $E G S$ ) rule, that divides each such gain equally over the two players involved. Now, we consider gain splitting rules $G S^{\lambda}$ defined by

$$
G S^{\lambda}(Q)=\sum_{(i, j) \in M P\left(\sigma_{0}\right)}\left(\lambda_{i j} e^{\{i\}}+\left(1-\lambda_{i j}\right) e^{\{j\}}\right) g_{i j}
$$

for any $\lambda \in \Lambda$. Here, $\Lambda=\{\lambda: N \times N \rightarrow[0,1] \mid \lambda(r, s)+\lambda(s, r)=1$ for all $r, s \in N, r \neq$ $s$, and $\lambda(r, r)=1\}$. With minor abuse of notation, we write $\lambda_{i j}$ instead of $\lambda(i, j)$. For the equal gain splitting rule we have $E G S(Q)=G S^{\lambda}(Q)$ for $\lambda \in \Lambda$ such that $\lambda_{r s}=\frac{1}{2}$ for all $r, s \in N, r \neq s$. The split core of $Q$ in Hamers et al. (1996) refers to the set $\left\{G S^{\lambda}(Q) \mid \lambda \in \Lambda\right\}$ of all allocation vectors corresponding to a gain splitting rule.

For an arbitrary gain splitting rule $G S^{\lambda}$, the corresponding net profit of player $i \in N$ in the sequencing process $Q=\left(N, \sigma_{0}, p, \alpha\right)$ is defined by

$$
\pi_{i}^{\lambda}(Q)=G S_{i}^{\lambda}(Q)-I C_{i}(Q)
$$

where the initial individual costs $I C_{i}(Q)$ of player $i$ w.r.t. $\sigma_{0}$ are given by $I C_{i}(Q)=$ $\alpha_{i} C_{i}\left(\sigma_{0}\right)$. To emphasize, negative values of the net profit vector correspond to net payments made by the players. Note that the net profit vector does not (directly) depend on the individual costs of the players after rearranging; it is determined by a non-negative weighted sum of positive gains made when rearranging players from the initial order to a Smith order, from which we subtract the individual costs in the initial order. Therefore, no player can be worse off after rearranging all misplaced pairs in the initial order, even if the player is processed later due to the rearrangement.

## Example 4.3.1

Let $N=\{1,2,3\}, p=(4,3,4)$ and $\alpha=(2,3,6)$. In this example, we use the equal gain splitting rule and we determine the net profit vectors $\pi^{E G S}\left(N, \sigma_{0}, p, \alpha\right)$ corresponding to each of the six possible initial orders. First, note that there is a unique Smith order, given by $\hat{\sigma}=(3,2,1)$. Next, consider $\sigma_{0}=(1,2,3)$. Then, the initial cost vector is given by $(8,21,66)$. There are three misplaced pairs, $(1,2),(1,3)$ and $(2,3)$, with $g_{12}=g_{23}=6$ and $g_{13}=16$, so $\operatorname{EGS}(N,(1,2,3), p, \alpha)=(11,6,11)$. Hence,

$$
\pi^{E G S}(N,(1,2,3), p, \alpha)=(11,6,11)-(8,21,66)=(3,-15,-55)
$$

The initial individual costs of players 1,2 , and 3 , were 8,21 , and 66 , respectively. After switching each misplaced neighbor pair and equally allocating the joint cost savings to the corresponding pair of players, player 1 has a net profit of 3 , and players 2 and 3 pay 15 and 55, respectively. Indeed, no player is worse off with respect to the initial order. Player 1 even has a positive net profit, since the gains of switching the pairs $(1,2)$ and $(1,3)$ are relatively high compared to the initial individual costs of player 1 .

Similarly, we get

$$
\begin{aligned}
& \pi^{E G S}(N,(1,3,2), p, \alpha)=(11,3,8)-(8,33,48)=(3,-30,-40), \\
& \pi^{E G S}(N,(2,1,3), p, \alpha)=(8,3,11)-(14,9,66)=(-6,-6,-55), \\
& \pi^{E G S}(N,(2,3,1), p, \alpha)=(0,3,3)-(22,9,42)=(-22,-6,-39),
\end{aligned}
$$

$$
\begin{aligned}
& \pi^{E G S}(N,(3,1,2), p, \alpha)=(3,3,0)-(16,33,24)=(-13,-30,-24) \\
& \pi^{E G S}(N,(3,2,1), p, \alpha)=(0,0,0)-(22,21,24)=(-22,-21,-24)
\end{aligned}
$$

The net profit vectors clearly depend on the initial order. In particular, note that if a player $i \in N$ gets closer to the head of the queue in the initial processing order, this leads to a strict improvement of the net profit of $i$.

In fact, the final observation in Example 4.3.1 holds whenever the gains of position switches are allocated using a gain splitting rule. If any (set of) player(s) is 'removed' from the set of players before $i$ in the initial order, the net profit of $i$ increases. To formalize this in Lemma 4.3.1, we define the set of predecessors $P(\sigma, i)$ of player $i \in N$ with respect to $\sigma \in \Pi(N)$ by $P(\sigma, i)=\left\{j \in N \mid \sigma^{-1}(j)<\sigma^{-1}(i)\right\}$.

## Lemma 4.3.1

Let $Q=\left(N, \sigma_{0}, p, \alpha\right)$ and $\tilde{Q}=\left(N, \tilde{\sigma}_{0}, p, \alpha\right)$ be sequencing processes and let $i \in N$ be such that $P\left(\sigma_{0}, i\right) \subsetneq P\left(\tilde{\sigma}_{0}, i\right)$. Then, for all $\lambda \in \Lambda$, we have

$$
\pi_{i}^{\lambda}(Q)>\pi_{i}^{\lambda}(\tilde{Q})
$$

Proof. Starting from $\tilde{\sigma}_{0}$, note that $\sigma_{0}$ can be reached through consecutive neighbor switches by first adequately rearranging $i$ 's predecessors, then some switches between $i$ and its neighboring predecessor, and finally by rearranging $i$ 's successors (i.e., all players for which $i$ is a predecessor). We show that switches of the first and third type do no affect the net profit of $i$, where any switch of the second type strictly improves the net profit of $i$. Note that at least one switch of this second type is required, since $P\left(\sigma_{0}, i\right)$ is a strict subset of $P\left(\tilde{\sigma}_{0}, i\right)$.

Let $Q^{\prime}=\left(N, \sigma_{0}^{\prime}, p, \alpha\right)$ and $Q^{\prime \prime}=\left(N, \sigma_{0}^{\prime \prime}, p, \alpha\right)$ be sequencing processes such that $P\left(\sigma_{0}^{\prime}, i\right)=P\left(\sigma_{0}^{\prime \prime}, i\right)$. Then, $\pi_{i}^{\lambda}\left(Q^{\prime}\right)=\pi_{i}^{\lambda}\left(Q^{\prime \prime}\right)$, since $C_{i}\left(\sigma_{0}^{\prime}\right)=C_{i}\left(\sigma_{0}^{\prime \prime}\right)$, and $(i, j) \in M P\left(\sigma_{0}^{\prime}\right) \Leftrightarrow(i, j) \in M P\left(\sigma_{0}^{\prime \prime}\right)$ and $(j, i) \in M P\left(\sigma_{0}^{\prime}\right) \Leftrightarrow(j, i) \in M P\left(\sigma_{0}^{\prime \prime}\right)$ for any $j \in N \backslash\{i\}$. This shows that rearranging the predecessors and successors of $i$ does not affect $i$ 's net profit.

It therefore suffices to show that any switch of the second type, between $i$ and its neighboring predecessor, leads to a strictly higher net profit of $i$. So, we may restrict to the sequencing processes $Q^{\prime}$ and $Q^{\prime \prime}$ with $\sigma_{0}^{\prime}$ and $\sigma_{0}^{\prime \prime}$ such that for some $l \in\{1, \ldots,|N|-1\}$ and $j \in N \backslash\{i\}$, we have $\sigma_{0}^{\prime}(l)=\sigma_{0}^{\prime \prime}(l+1)=j, \sigma_{0}^{\prime}(l+1)=\sigma_{0}^{\prime \prime}(l)=i$, and $\sigma_{0}^{\prime}(k)=\sigma_{0}^{\prime \prime}(k)$ for all $k \in\{1, \ldots,|N|\} \backslash\{l, l+1\}$.

First, if $u_{i}<u_{j}$, then $M P\left(\sigma_{0}^{\prime \prime}\right)=M P\left(\sigma_{0}^{\prime}\right) \cup\{(i, j)\}$ and, consequently,

$$
\begin{aligned}
\pi_{i}^{\lambda}\left(Q^{\prime \prime}\right)-\pi_{i}^{\lambda}\left(Q^{\prime}\right) & =G S_{i}^{\lambda}\left(Q^{\prime \prime}\right)-I C_{i}\left(Q^{\prime \prime}\right)-G S_{i}^{\lambda}\left(Q^{\prime}\right)+I C_{i}\left(Q^{\prime}\right) \\
& =\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)+\lambda_{i j} g_{i j} \\
& \geq \alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right) \\
& >0
\end{aligned}
$$

Next, for $u_{i}=u_{j}$, note that $M P\left(\sigma_{0}^{\prime \prime}\right)=M P\left(\sigma_{0}^{\prime}\right)$ and $G S_{i}^{\lambda}\left(Q^{\prime \prime}\right)=G S_{i}^{\lambda}\left(Q^{\prime}\right)$, so

$$
\begin{aligned}
\pi_{i}^{\lambda}\left(Q^{\prime \prime}\right)-\pi_{i}^{\lambda}\left(Q^{\prime}\right) & =G S_{i}^{\lambda}\left(Q^{\prime \prime}\right)-I C_{i}\left(Q^{\prime \prime}\right)-G S_{i}^{\lambda}\left(Q^{\prime}\right)+I C_{i}\left(Q^{\prime}\right) \\
& =\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right) \\
& >0
\end{aligned}
$$

Finally, for $u_{j}<u_{i}, M P\left(\sigma_{0}^{\prime}\right)=M P\left(\sigma_{0}^{\prime \prime}\right) \cup\{(j, i)\}$ and, consequently,

$$
\begin{aligned}
\pi_{i}^{\lambda}\left(Q^{\prime \prime}\right)-\pi_{i}^{\lambda}\left(Q^{\prime}\right) & =G S_{i}^{\lambda}\left(Q^{\prime \prime}\right)-I C_{i}\left(Q^{\prime \prime}\right)-G S_{i}^{\lambda}\left(Q^{\prime}\right)+I C_{i}\left(Q^{\prime}\right) \\
& =\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)-\left(1-\lambda_{j i}\right) g_{j i} \\
& =\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)-\left(1-\lambda_{j i}\right)\left(\alpha_{i} p_{j}-\alpha_{j} p_{i}\right) \\
& =\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)-\left(1-\lambda_{j i}\right)\left(\alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)-\alpha_{j} p_{i}\right) \\
& =\lambda_{j i} \alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right)+\left(1-\lambda_{j i}\right) \alpha_{j} p_{i} \\
& \geq \lambda_{j i} \alpha_{i}\left(C_{i}\left(\sigma_{0}^{\prime}\right)-C_{i}\left(\sigma_{0}^{\prime \prime}\right)\right) \\
& \geq 0 .
\end{aligned}
$$

Note that the first inequality is strict unless $\lambda_{j i}=1$, where the second inequality is strict unless $\lambda_{j i}=0$.

Hence, we may conclude that $\pi_{i}^{\lambda}\left(Q^{\prime \prime}\right)-\pi_{i}^{\lambda}\left(Q^{\prime}\right)>0$, and, consequently, $\pi_{i}^{\lambda}(Q)>$ $\pi_{i}^{\lambda}(\tilde{Q})$.

We remark that Lemma 4.3.1 holds specifically for payoff vectors defined using a gain splitting rule. If instead we allocate the cost savings using, e.g., the Shapley value of the corresponding cooperative sequencing game (Curiel et al., 1989), it can happen that a player's net profit becomes lower if the player 'moves' towards the head of the queue in the initial processing order, as demonstrated in Example 4.3.2.

Let $Q=\left(N, \sigma_{0}, p, \alpha\right)$ be a sequencing process. A key difference between payoff
vectors defined using a gain splitting rule and those based on a game-theoretic solution, is the assumption underlying sequencing games that not all rearrangements of an initial order are admissible for all coalitions. Specifically, it is assumed that rearrangements of $\sigma_{0}$ in which a player in some coalition $T \in 2^{N} \backslash\{\emptyset\}$ 'passes' a player in $N \backslash T$ are not allowed for $T$. Formally, an order $\sigma \in \Pi(N)$ is admissible for $T$ if $P(\sigma, i)=P\left(\sigma_{0}, i\right)$ for all $i \in N \backslash T$. The set of all admissible orders for $T$ is denoted by $\mathcal{A}_{Q}(T)$.

Then, the sequencing game $v_{Q} \in T U^{N}$ is defined by cost savings game

$$
v_{Q}(T)=\max _{\sigma \in \mathcal{A}_{Q}(T)} \sum_{i \in T} \alpha_{i}\left(C_{i}\left(\sigma_{0}\right)-C_{i}(\sigma)\right)
$$

for all $T \in 2^{N} \backslash\{\emptyset\}$. Note that $\mathcal{A}_{Q}(N)=\Pi(N)$, so that

$$
v_{Q}(N)=\sum_{i \in N} \alpha_{i}\left(C_{i}\left(\sigma_{0}\right)-C_{i}(\hat{\sigma})\right)=\sum_{(i, j) \in M P\left(\sigma_{0}\right)} g_{i j} .
$$

Further, for all $i \in N, \mathcal{A}_{Q}(\{i\})=\left\{\sigma_{0}\right\}$, so $v_{Q}(\{i\})=0$.

## Example 4.3.2

Let $N=\{1,2,3\}, p=(6,3,4)$ and $\alpha=(2,3,12)$. Note that $\hat{\sigma}=(3,2,1)$. Using the Shapley value, the corresponding net profit of player $i \in N$ in the sequencing process $Q=\left(N, \sigma_{0}, p, \alpha\right)$ is defined by

$$
\pi_{i}^{\Phi}(Q)=\Phi_{i}\left(v_{Q}\right)-I C_{i}(Q)
$$

Let $Q=\left(N, \sigma_{0}, p, \alpha\right)$ be a sequencing process with $\sigma_{0}=(1,2,3)$ and denote the corresponding sequencing game by $v_{(1,2,3)}$. We have $I C_{1}(Q)=12, I C_{2}(Q)=27$, and $I C_{3}(Q)=156$. Further, $g_{12}=12, g_{13}=64$, and $g_{23}=24$. To determine $v_{(1,2,3)}$, note that, e.g., $\mathcal{A}_{Q}(\{1,2\})=\left\{\sigma_{0},(2,1,3)\right\}$, whereas $\mathcal{A}_{Q}(\{1,3\})=\left\{\sigma_{0}\right\}$. Hence, $v_{(1,2,3)}(\{1,2\})=g_{12}=12$, but $v_{(1,2,3)}(\{1,3\})=0$. In this way, we find $v_{(1,2,3)}$ as given in Table 4.7. One readily verifies that $\Phi\left(v_{(1,2,3)}\right)=\frac{1}{3}(82,118,100)$, so that

$$
\pi^{\Phi}(N,(1,2,3), p, \alpha)=\frac{1}{3}(46,37,-368)
$$

Next, let $Q=\left(N, \sigma_{0}, p, \alpha\right)$ be a sequencing process with $\sigma_{0}=(2,1,3)$ and denote the corresponding sequencing game by $v_{(2,1,3)}$. This game is determined in a similar

| $S$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{(1,2,3)}(S)$ | 0 | 0 | 0 | 12 | 0 | 24 | 100 |
| $v_{(2,1,3)}(S)$ | 0 | 0 | 0 | 0 | 64 | 0 | 88 |

Table 4.7 The sequencing games $v_{(1,2,3)}$ and $v_{(2,1,3)}$ corresponding to the sequencing process of Example 4.3.2
manner and also given in Table 4.7. For the net profit vector, we find

$$
\pi^{\Phi}(N,(2,1,3), p, \alpha)=(40,8,40)-(18,9,156)=(22,-1,-116)
$$

Importantly, note that

$$
P((2,1,3), 2) \subsetneq P((1,2,3), 2),
$$

but

$$
\pi_{2}^{\Phi}(N,(2,1,3), p, \alpha)<\pi_{2}^{\Phi}(N,(1,2,3), p, \alpha)
$$

contradicting the result for net profit vectors defined using gain splitting rules of Lemma 4.3.1. Intuitively, the reason why this happens, is that when $\sigma_{0}=(1,2,3)$, player 2 needs to cooperate (i.e., be a member of the coalition) to make switching positions admissible for players 1 and 3 in the sequencing game $v_{(1,2,3)}$. Because player 2 is needed to 'enable' the gain $g_{13}$, player 2 gets a third of this gain in the corresponding Shapley value. Since this gain is relatively very high, it outweighs the decrease in the initial costs of player 2 for $\sigma_{0}=(2,1,3)$. Such an effect does not occur if the gains are allocated using a gain splitting rule.

### 4.3.2 Biform sequencing processes

Our model introduces a strategic element to the sequencing processes described in the previous section. We no longer assume an initial order is given. Instead, strategic individual choices for a position in the first stage of the biform sequencing (BS) process determine an initial order that is used in the cooperative sequencing process of the second stage. The finite strategy set of any player $i \in N$ is given by $X^{i}=\{1,2, \ldots,|N|\}$. For example, $x^{i}=1$ indicates that player $i$ requests to be processed first.

As it can happen that several players request to be in the same position, we need a tie-breaking rule. We interpret a tie-breaking rule as a decision made by some
entity that determines the initial order for any given strategy combination. This entity first simply assigns every position that is requested by exactly one player to this player. An unassigned position is called empty if it is not requested by any player and called undecided if it is requested by at least two players. Then, recursively, starting from the earliest undecided position, all players who requested this position are assigned to either this position or an empty one. At every step in the recursion, the players are assigned such that a player with a higher 'priority' is assigned to an earlier position among the empty ones and the requested one. This recursive process is illustrated in Example 4.3.3.

Given a strategy combination $x \in X$ and tie-breaking rule $\tau: X \rightarrow \Pi(N)$, the strategically determined initial processing order is denoted by $\sigma_{0}^{\tau}(x)$. We assume a tie-breaking rule $\tau$ is priority order based (РОВ). This means that the way players in a tie are prioritized can be directly induced from the unique priority order $\bar{\sigma}_{0}^{\tau}$ on all players, needed when all players would request exactly the same position.

We remark that a tie-breaking rule $\tau$ can also be regarded as a matching mechanism. Specifically, our tie-breaking rule fits well with a two-sided mechanism (see, e.g., Gale and Shapley (1962)), in which players prefer earlier positions (not necessarily being the one they request) and positions prefer players with higher priority according to $\bar{\sigma}_{0}^{\tau}$. Alternatively, one could consider other mechanisms to match players to (empty and undecided) positions, e.g., one-sidedly using a serial dictatorship with priority order $\bar{\sigma}_{0}^{\tau}$ (see, e.g., Svensson (1999)). In this way, the tie-breaking rule can be further substantiated using matching theory. This is, however, not in the scope of this chapter.

## Example 4.3.3

Consider a sequencing process with $N=\{1,2,3,4\}, \alpha=(2,4,4,3)$ and $p=(2,1,2,1)$, so that $u=(1,4,2,3)$. An initial order will be strategically determined using the priority order based tie-breaking rule $\tau$, fully determined by the priority order $\bar{\sigma}_{0}^{\tau}=$ $(2,4,3,1)$. Note that this priority order fits with the principle of 'highest urgency comes first'. By definition, this means that if, e.g., $x=(2,2,2,2)$, then $\sigma_{0}^{\tau}(x)=$ $(2,4,3,1)$.

If $x=(2,1,3,1)$, the first step is to assign players to the positions that were requested exactly once. After assigning position 2 to player 1 and position 3 to player $3, \tau$ is used to break the tie between players 2 and 4 , who compete for the undecided position 1 and the empty position 4. Player 2 has priority over player 4, which leads to $\sigma_{0}^{\tau}(x)=(2,1,3,4)$.

If $x=(1,1,2,2), \tau$ is first used to break the tie between players 1 and 2 , who can be assigned to undecided position 1 and empty positions 3 and 4 . Since player 2 has the highest priority, we get $\sigma_{0}^{\tau}(x)(1)=2$, after which player 1 is placed in the best remaining available position, $\sigma_{0}^{\tau}(x)(3)=1$. The next undecided position is position 2 , with position 4 empty. Since player 4 has priority over player 3, player 4 is assigned to position 2 and player 3 to position 4 . Hence, $\sigma_{0}^{\tau}(x)=(2,4,1,3)$.

If $x=(3,3,4,4)$, players 1 and 2 first compete for empty positions 1 and 2 and undecided position 3. After assigning 2 to position 1 and 1 to position 2, players 3 and 4 compete for empty position 3 and undecided position 4 . To emphasize, at each step in the recursion, players with higher priority are assigned to earlier positions among the empty position(s) and the undecided position, not necessarily (as close as possible to) the position they requested. As a consequence, $\sigma_{0}^{\tau}(x)=(2,1,4,3)$.

Now we are able to formally define a biform sequencing process and a corresponding induced strategic game based on a gain splitting rule $G S^{\lambda}$.

## Definition 4.3.2

A biform sequencing ( $B S$ ) process is a tuple

$$
\mathcal{Q}=\left(N,\left\{X^{i}\right\}_{i \in N}, \tau,\{Q(x)\}_{x \in X}\right)
$$

in which for all $i \in N$ we have $X^{i}=\{1,2, \ldots,|N|\}, \tau: X \rightarrow \Pi(N)$ is a POB tie-breaking rule, and for any $x \in X$,

$$
Q(x)=\left(N, \sigma_{0}^{\tau}(x), p, \alpha\right)
$$

is a corresponding sequencing process with initial order $\sigma_{0}^{\tau}(x)$. Given such a $B S$ process $\mathcal{Q}$ and a gain splitting rule $G S^{\lambda}$ with $\lambda \in \Lambda$, the corresponding induced strategic game is given by $G^{\mathcal{Q}, \lambda}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{Q}, \lambda}\right\}_{i \in N}\right)$, where for any $x \in X$ and all $i \in N$ we set

$$
\pi_{i}^{\mathcal{Q}, \lambda}(x)=\pi_{i}^{\lambda}\left(N, \sigma_{0}^{\tau}(x), p, \alpha\right) \cdot{ }^{1}
$$

## Example 4.3.4

Reconsider the 3-player sequencing process of Example 4.3.1. We now consider the BS-process $\mathcal{Q}=\left(\{1,2,3\},\left\{X^{i}\right\}_{i \in N}, \tau,\{Q(x)\}_{x \in X}\right)$ using the POB tie-breaking rule $\tau$ with priority order $\bar{\sigma}_{0}^{\tau}=(2,3,1)$. For all $x \in X, \sigma_{0}^{\tau}(x)$ is presented in Table 4.8.

[^1]|  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| 1 | $(2,3,1)$ | $(3,2,1)$ | $(3,1,2)$ |
| 2 | $(2,1,3)$ | $(3,2,1)$ | $(3,1,2)$ |
| 3 | $(2,3,1)$ | $(3,2,1)$ | $(3,2,1)$ |
|  | 1 |  |  |


|  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| 1 | $(2,3,1)$ | $(1,2,3)$ | $(1,3,2)$ |
| 2 | $(2,3,1)$ | $(2,3,1)$ | $(3,1,2)$ |
| 3 | $(2,3,1)$ | $(2,3,1)$ | $(2,3,1)$ |
|  | 2 |  |  |


|  | 1 |  | 2 |
| :---: | :---: | :---: | :---: |
| 3 |  |  |  |
| 1 | $(2,1,3)$ | $(1,2,3)$ | $(1,2,3)$ |
| 2 | $(2,1,3)$ | $(2,1,3)$ | $(2,1,3)$ |
| 3 | $(2,3,1)$ | $(3,2,1)$ | $(2,3,1)$ |
|  | 3 |  |  |

Table 4.8 The strategy dependent initial processing orders $\sigma_{0}^{\tau}(x)$ for each $x \in X$ in Example 4.3.4

Next, we analyze the induced strategic game $G^{\mathcal{Q}, E G S}=$ $\left(\left\{X^{i}\right\}_{i \in\{1,2,3\}},\left\{\pi_{i}^{\mathcal{Q}, E G S}\right\}_{i \in\{1,2,3\}}\right)$ based on the equal gain splitting rule. For each of the six possible initial orders, the corresponding net profits are given in Example 4.3.1. The induced game $G^{\mathcal{Q}, E G S}$ is given in Table 4.9.


Table 4.9 The induced strategic game $G^{\mathcal{Q}, E G S}$ of Example 4.3.4

It follows that

$$
E\left(G^{\mathcal{Q}, E G S}\right)=\{(1,1,2),(2,1,2),(3,1,2)\}
$$

Note that for all $x \in E\left(G^{\mathcal{Q}, E G S}\right)$ we have $\sigma_{0}^{\tau}(x)=\bar{\sigma}_{0}^{\tau}=(2,3,1)$, but there are (many)
$x \in X$ such that $\sigma_{0}^{\tau}(x)=(2,3,1)$ with $x \notin E\left(G^{\mathcal{Q}, E G S}\right)$. The equilibrium $(3,1,2)$ is quite special, since it corresponds to the strategy combination in which all players request the position they are entitled to according to the priority order $\bar{\sigma}_{0}^{\tau}$.

The 'special' type of equilibrium found in Example 4.3.4 exists for any BS-process.

## Theorem 4.3.3

Let $\mathcal{Q}=\left(N,\left\{X^{i}\right\}_{i \in N}, \tau,\{Q(x)\}_{x \in X}\right)$ be a BS-process, let $G S^{\lambda}$ be a gain splitting rule with $\lambda \in \Lambda$, and let $G^{\mathcal{Q}, \lambda}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{Q}, \lambda}\right\}_{i \in N}\right)$ be the induced strategic game. Let $x \in X$ be such that $x^{i}=\left(\bar{\sigma}_{0}^{\tau}\right)^{-1}(i)$ for all $i \in N$. Then, $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$.

Proof. Let $i \in N$ and set $k=\left(\bar{\sigma}_{0}^{\tau}\right)^{-1}(i)$, so that $x^{i}=k$. Consider $\tilde{x}^{i}=l$ with $l \in X^{i} \backslash\{k\}$. With respect to the strategy combination $\left(\tilde{x}^{i}, x^{-i}\right)$, only position $l$ is undecided and only position $k$ is empty. In particular, position $l$ is requested by player $i$ and the player $j \in N$ such that $\left(\bar{\sigma}_{0}^{\tau}\right)^{-1}(j)=l$. If $l<k$, the underlying priority order ranks $j$ before $i$, so $j$ is assigned to the earlier (undecided) position $l$, while $i$ is assigned to the later (empty) position $k$. If $l>k$, the underlying priority order ranks $i$ before $j$, so $i$ is assigned to the earlier position $k$ and $j$ to the later position $l$. In both cases, $\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right)=\sigma_{0}^{\tau}(x)$ and therefore $\pi_{i}^{\mathcal{Q}, \lambda}\left(\tilde{x}^{i}, x^{-i}\right)=\pi_{i}^{\mathcal{Q}, \lambda}(x)$. Consequently, $x \in E(G)$.

Next, we show that any induced strategic game $G^{\mathcal{Q}, \lambda}$ has exactly $|N|$ Nash equilibria. In particular, the proof of Theorem 4.3.4 clarifies that these equilibria are such that the player with the lowest priority (as determined by the underlying priority order) can request any position. All other players should request the position they are entitled to according to the priority order.

## Theorem 4.3.4

Let $\mathcal{Q}=\left(N,\left\{X^{i}\right\}_{i \in N}, \tau,\{Q(x)\}_{x \in X}\right)$ be a BS-process, let $G S^{\lambda}$ be a gain splitting rule with $\lambda \in \Lambda$, and let $G^{\mathcal{Q}, \lambda}=\left(\left\{X^{i}\right\}_{i \in N},\left\{\pi_{i}^{\mathcal{Q}, \lambda}\right\}_{i \in N}\right)$ be the induced strategic game. Then, $\left|E\left(G^{\mathcal{Q}, \lambda}\right)\right|=|N|$.

Proof. For ease of notation and without loss of generality, let $N=\{1,2, \ldots, n\}$ and let the priority order underlying the tie-breaking rule $\tau$ be given by $\bar{\sigma}_{0}^{\tau}=(1,2, \ldots, n)$, meaning the players are 'numbered' in decreasing order of priority. It suffices to prove the following two claims. Claim 1 shows there are at least $n$ equilibria, and Claim 2 shows there are at most $n$ equilibria.
$\underline{\text { Claim } 1}$ Let $x \in X$ be such that $x^{j}=j$ for all $j \in\{1,2, \ldots, n-1\}$. Then, $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$.
Claim 2 Let $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$. Then, $x^{j}=j$ for all $j \in\{1,2, \ldots, n-1\}$.
Proof Claim 1 Let $i \in N$ and $\tilde{x}^{i} \in X^{i}$. We will show that either $\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right)=$ $\sigma_{0}^{\tau}(x)=\bar{\sigma}_{0}^{\tau}$, or $P\left(\sigma_{0}^{\tau}(x), i\right) \subsetneq P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right)$. In both cases, this implies that $\pi_{i}^{\mathcal{Q}, \lambda}(x) \geq \pi_{i}^{\mathcal{Q}, \lambda}\left(\tilde{x}^{i}, x^{-i}\right)$. Note that in the latter case, this is a consequence of Lemma 4.3.1.

For $i=n$, we have that $\sigma_{0}^{\tau}\left(\tilde{x}^{n}, x^{-n}\right)=\sigma_{0}^{\tau}(x)$ for all $\tilde{x}^{n} \in\{1,2, \ldots, n\}$. This is obvious if $\tilde{x}^{n}=n$. If $\tilde{x}^{n}=k, k \neq n$, note that there is a unique undecided position $k$ and a unique empty position $n$ w.r.t. $\left(\tilde{x}^{n}, x^{-n}\right)$. Due to the fact that player $n$ has a lower priority than player $k$, the tie-breaking rule assigns position $k$ to player $k$ and position $n$ to $n$.

Next, let $i \in N \backslash\{n\}$, and let $\tilde{x}^{i}=k$ with $k \neq i$ be a possible unilateral deviation from $x$ for player $i$. Denote $x^{n}=l$. We distinguish several cases. In these cases, we assume without loss of generality that $l<n$, as Theorem 4.3.3 states that $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$ for $x^{n}=n$.

Case 1: $k=n$. Then, player $i$ is the only player who requests position $n$, so player $i$ is assigned to position $n$. It is clear that $P\left(\sigma_{0}^{\tau}(x), i\right) \subsetneq P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right)$.

Case 2: $l=i, k<n$. In this case, position $k$ is undecided and position $n$ is empty. If $k<i$, player $i$ has lower priority than player $k$, so player $i$ loses the tie and is assigned to position $n$. If $k>i$, player $i$ has higher priority than player $k$, so player $i$ is assigned to position $k$ and player $k$ is assigned to position $n$. Either way, note that $P\left(\sigma_{0}^{\tau}(x), i\right) \subseteq P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right)$ and $n \in P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right) \backslash P\left(\sigma_{0}^{\tau}(x), i\right)$, so that $P\left(\sigma_{0}^{\tau}(x), i\right) \subsetneq P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right)$.

Case 3: $l=k, k<n$. Then, we have a three-way tie, where position $k$ is undecided and positions $i$ and $n$ are empty. Player $n$ always has the lowest priority and therefore always ends up in position $n$. Player $i$ loses the tie against player $k$ if $k<i$ and wins the tie if $k>i$. Either way, player $i$ is assigned to position $i$ and player $k$ to position $k$, so that $\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right)=\sigma_{0}^{\tau}(x)$.

Case 4: $l \neq i, l>k, k<n$. Then, positions $k$ and $l$ are undecided and positions $i$ and $n$ are empty. The tie-breaking rule first assigns the players $k$ and $i$, who request the earliest undecided position $k$. Similar to the previous case, players $k$ and $i$ are
always assigned to positions $k$ and $i$, respectively. Player $n$ always loses the tie for position $l$. Again, we get $\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right)=\sigma_{0}^{\tau}(x)$.

Case 5: $l \neq i, l<k, k<n$. Again, positions $k$ and $l$ are undecided and positions $i$ and $n$ are empty. However, the tie-breaking rule now first assigns the players $l$ and $n$, who request the earliest undecided position $l$. Player $l$ is assigned to position $l$ and player $n$ to position $i$ if $l<i$, and vice versa if $l>i$. Player $i$ is then assigned to empty position $n$ if $k<i$ and to undecided position $k$ if $k>i$. Similar to the second case, we get $P\left(\sigma_{0}^{\tau}(x), i\right) \subsetneq P\left(\sigma_{0}^{\tau}\left(\tilde{x}^{i}, x^{-i}\right), i\right)$.

Proof Claim 2 Let $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$ and assume towards contradiction that $x^{j} \neq j$ for some $j \in\{1,2, \ldots, n-1\}$. Let $i$ be the smallest index for which this is true, i.e., $x^{1}=1, \ldots, x^{i-1}=i-1$, and $x^{i}=k$ with $k \neq i$. Note that player $i$ can still be assigned to position $i$ through tie-breaking, but only if position $i$ is empty. So, if $\sigma_{0}^{\tau}(x)^{-1}(i)=i$, then $x^{j} \neq i$ for all $j \in N$. In that case, player $n$ can deviate and be assigned to $\tilde{x}^{n}=i$. Alternatively, if $\sigma_{0}^{\tau}(x)^{-1}(i)>i$, player $i$ can deviate and be assigned to $\tilde{x}^{i}=i$, as player $i$ will then be the player with the highest priority who requests position $i$. In both cases, the deviating player makes sure to be assigned to position $i$, instead of some position strictly later than $i$. Using the fact that the set of players assigned to the first $i-1$ positions is fixed, we can apply Lemma 4.3.1 to argue that the corresponding deviation is strictly profitable, contradicting $x \in E\left(G^{\mathcal{Q}, \lambda}\right)$.

### 4.3.3 Biform sequencing processes with additional costs

As an extension of the previous model, one could analyze the influence of associating $\operatorname{costs} \gamma \in \mathbb{R}^{N}$ with the strategic choice for a certain position in the initial order. First, we consider a cost function that assigns fixed costs to each position in the strategically determined initial order. Here, players do not pay for their requested position, but for the position in which they actually end up. More formally, for any $x \in X$ and $k \in\{1,2, \ldots,|N|\}$, player $i \in N$ incurs fixed costs $\gamma(k)$ if $\sigma_{0}^{\tau}(x)^{-1}(i)=k$.

## Example 4.3.5

Reconsider the BS-process $\mathcal{Q}=\left(\{1,2,3\},\left\{X^{i}\right\}_{i \in N}, \tau,\{Q(x)\}_{x \in X}\right)$ of Example 4.3.4 using the POB tie-breaking rule $\tau$ with priority order $\bar{\sigma}_{0}^{\tau}=(2,3,1)$, where for each $x \in X, Q(x)=\left(\{1,2,3\}, \sigma_{0}^{\tau}(x), p, \alpha\right)$ is a corresponding sequencing process for which $\sigma_{0}^{\tau}(x)$ is given in Table 4.8, $p=(4,3,4)$ and $\alpha=(2,3,6)$. Introducing costs, the net profits as specified by $\pi^{\mathcal{Q}, E G S}$ in the corresponding induced strategic game $G^{\mathcal{Q}, E G S}$ will change. We emphasize that the costs are associated with the obtained position in
the strategically determined initial processing order here, meaning that the net profit vectors of two different strategy combinations that lead to the same initial order are still equal.

Let $\gamma=(20,10,0)$, i.e., the fixed costs of being first, second or third in the resulting initial processing order are 20, 10 or 0 , respectively. Consider $\sigma_{0}^{\tau}(x)=(3,2,1)$, for which we saw in Example 4.3 .1 that the net profit vector is $(-22,-21,-24)$ without fixed costs. By assigning the additional costs to each position in the initial processing order, this net profit vector becomes $(-22,-31,-44)$. In a similar way, we find the new net profits for each strategy combination. The new induced strategic game is given in Table 4.10.

|  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| 1 | (-22,-26,-49) | (-22,-31,-44) | (-23,-30,-44) |
| 2 | (-16,-26,-55) | (-22,-31,-44) | (-23,-30,-44) |
| 3 | (-22,-26,-49) | (-22,-31,-44) | (-22,-31,-44) |
|  |  | 1 |  |
|  | 1 | 2 | 3 |
| 1 | (-22,-26,-49) | (-17,-25,-55) | (-17,-30,-50) |
| 2 | (-22,-26,-49) | (-22,-26,-49) | (-23,-30,-44) |
| 3 | (-22,-26,-49) | (-22,-26,-49) | $(-22,-26,-49)$ |
|  |  | 2 |  |
|  | 1 | 2 | 3 |
| 1 | (-16,-26,-55) | (-17,-25,-55) | (-17,-25,-55) |
| 2 | (-16,-26,-55) | (-16,-26,-55) | (-16,-26,-55) |
| 3 | (-22,-26,-49) | (-22,-31,-44) | (-22,-26,-49) |
|  |  | 3 |  |

Table 4.10 The induced strategic game of Example 4.3.5 with fixed costs $\gamma=(20,10,0)$ associated with obtained positions

For this game, the set of equilibria is given by $\{(2,1,2),(3,1,2)\}$.
Example 4.3.5 shows that Theorem 4.3.4 cannot be generalized to this new setting of BS-processes with additional costs. The reason for this is that Lemma 4.3.1 cannot be generalized: the fixed costs may outweigh the benefits of obtaining an earlier position in the initial order. However, it can be shown that the existence of the specific equilibrium in which all players request the position they are entitled to according to the underlying priority order, is still guaranteed. So, Theorem 4.3.3 can be generalized to this setting.

As an alternative option, one can associate costs with the strategic choice itself. In this case, players pay a fixed amount depending on their requested position. The position in which a player actually ends up in the initial processing order does not play a role here. Formally, for any $x \in X$ and $k \in\{1,2, \ldots,|N|\}$, player $i \in N$ incurs fixed costs $\gamma(k)$ if $x^{i}=k$. Note that different strategy combinations that lead to the same initial processing order can now have different net profit vectors. In fact, the existence of an equilibrium is no longer guaranteed, as illustrated by Example 4.3.6.

## Example 4.3.6

Reconsider the 3-player biform sequencing process with fixed costs presented in Example 4.3.5, with one key difference: the costs $\gamma=(20,10,0)$ are now associated with the requested position rather than the obtained position in the initial order. The resulting induced strategic game is given in Table 4.11.

|  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| 1 | (-42,-26,-59) | (-42,-31,-44) | (-33,-30,-44) |
| 2 | (-16,-26,-75) | (-32,-31,-44) | (-23,-30,-44) |
| 3 | (-22,-26,-59) | (-22,-31,-44) | (-22,-21,-44) |
| 1 |  |  |  |
| 1 ( $42,-26,49)$ |  | 2 | 3 |
| 1 | (-42,-26,-49) | (-17,-25,-65) | (-17,-30,-50) |
| 2 | (-32,-26,-49) | (-32,-16,-49) | (-23,-30,-34) |
| 3 | $(-22,-26,-49)$ | (-22,-16,-49) | (-22,-6,-49) |
| 2 |  |  |  |
| 1 |  | 2 | 3 |
| 1 | (-26,-26,-55) | (-17,-25,-55) | (-17,-15,-55) |
| 2 | (-16,-26,-55) | (-16,-16,-55) | (-16,-6,-55) |
| 3 | (-22,-26,-39) | (-22,-31,-24) | (-22,-6,-39) |
|  |  | 3 |  |

Table 4.11 The induced strategic game of Example 4.3.6 with fixed costs $\gamma=(20,10,0)$ associated with requested positions

Note that the set of Nash equilibria is empty.


## Corruption in auctions: social welfare loss in hybrid multi-unit auctions

### 5.1 Introduction

In this chapter, based on Van Beek et al. (2022), we initiate the study of the social welfare loss (in utilitarian welfare terms) caused by corrupt auctioneers, both in singleitem and multi-unit auctions. We consider auction settings where a seller wants to sell some items and for this purpose recruits an auctioneer to organize an auction on their behalf. Such settings are widely prevalent in practice as they emerge naturally whenever the seller lacks the expertise or facilities to host the auction themselves. For example, individual sellers usually involve dedicated auctioneers or auction houses when they want to sell particular objects (such as real estate, cars, artwork, etc.). In private companies, the responsible finance officers are typically in charge of handling the procurement auctions (i.e., auctions with a single buyer and multiple sellers). Similarly, government procurement is often executed by some entity that acts on behalf of the government. The dilemma in such settings is that the incentives of the seller and the auctioneer are rather diverse in general: while the seller is interested in extracting the highest payments for the objects (or getting service at the lowest cost), auctioneers primarily care about maximizing their own gains from hosting the auction. Although undesirably, this asymmetry can incentivize the auctioneer to manipulate the auction to their own benefit through some fraudulent scheme.

Corruption in auctions, where an auctioneer engages in bid rigging (i.e., manipulation of the submitted bids) with one or several of the bidders, occurs especially
in the public sector (e.g., in construction and procurement auctions). For example, in 1999 the procurement auction for the construction of the new Berlin Brandenburg airport had to be rerun after investigations revealed that the initial winner was able to change the bid after illegally acquiring information about the application of one of their main competitors (The Wall Street Journal, 1999). As another example, in 1993 the New York City School Construction Authority caused a scandal when investigation revealed that they used a simple (but effective) bid-rigging scheme in a procurement auction setting (Olmstead, 1993):
> "In what one investigator described as a nervy scheme, that worker would unseal envelopes at a public bid opening, saving for last the bid submitted by the contractor who had paid him off. At that point, knowing the previous bids, the authority worker would misstate the contractor's bid, insuring that it was low enough to secure the contract but as close as possible to the next highest bid so that the contractor would get the largest possible price."

This kind of bid rigging, where the winning bid 'magically' aligns with the highest losing bid, is also known as magic number cheating (Ingraham, 2005). We refer the reader to Lengwiler and Wolfstetter (2010) and Menezes and Monteiro (2006) (and the references therein) for several other bid rigging examples. Despite the fact that this form of corruption occurs in practice, its negative impact is still poorly understood theoretically.

Our goal is to initiate the study of the social welfare loss caused by corrupt auctioneers in fundamental auction settings. We focus on a basic model that captures the magic number cheating mentioned above and generalizations thereof. Clearly, more sophisticated bid rigging models are conceivable and we hope that our work will trigger future work along these lines.

We capture corruption in auctions by adapting the payment scheme. For illustration purposes, consider the single-item auction setting and suppose the auctioneer runs a sealed bid first-price auction. After receiving all bids, the auctioneer approaches the highest bidder with the offer that they can lower their bid to the second highest bid in exchange for a bribe. If the highest bidder agrees, they win the auction and pay the second-highest bid for the items plus the corresponding bribe to the auctioneer. If the highest bidder disagrees, they still win the auction, but pay their bid for the item according to the first-price auction format. We assume that the bribe to be
paid to the auctioneer is a pre-determined fraction $\gamma \in[0,1]$ of the savings of the highest bidder, i.e., the auctioneer's bribe amounts to $\gamma$ times the difference between the highest and second highest bid. In case of the multi-unit auction setting, the procedure described above is adapted accordingly by offering all winning bidders to lower their bids to the highest losing bid.

Observe that the payment scheme described above essentially reduces to the winning bidders paying a convex combination of $\gamma$ times their bids and $(1-\gamma)$ times the highest losing bid. We will show that this setting is therefore equivalent to studying a hybrid auction ( $\gamma-H Y A$ ), where the items are assigned to the highest bidders (according to the respective single-item or multi-unit auction scheme) and the payments are a convex combination of the first-price and the second-price payments. By varying the parameter $\gamma \in[0,1], \gamma$-HYA thus interpolates between the respective second-price auction ( $\gamma=0$ ) and first-price auction $(\gamma=1)$ schemes.

More elaborate corruption schemes are of course conceivable. For example, it may be overly suspicious to lower all winning bids in a multi-unit auction to the 'magic number' (the highest losing bid). To avoid this, the auctioneer may want to announce different (bribed) bids for every winning bidder, which can still be captured by $\gamma$-HYA. Alternatively, the auctioneer might ask for a fixed amount rather than a fraction of the gains, the bidders may be heterogenous (in which case the auctioneer does not use the same parameter $\gamma$ for each bidder), and bidders may have moral objections against partaking in such a scheme and do not accept the bribe. To capture such more general corruption schemes, we also study what we term $\gamma$-approximate first-price auctions ( $\gamma-F P A$ ) in this chapter. Basically, these auctions implement a payment scheme that recovers at least a fraction of $\gamma \in[0,1]$ of the first-price payment rule. The $\gamma$-HYA also belongs to this class.

We study the inefficiency of equilibria of $\gamma$-FPA and $\gamma$-HYA, both in the singleitem and the multi-unit auction setting. More specifically, our goal is to obtain a precise understanding of the (robust) price of anarchy (POA) (see Koutsoupias and Papadimitriou (1999), Roughgarden (2015a), and Syrgkanis and Tardos (2013)), i.e., the worst-case ratio between the optimal social welfare and the (expected) social welfare of an equilibrium. Here, social welfare is measured in utilitarian welfare terms, i.e., as the total valuation of the bidders. We opt for the price of anarchy notion here because it is one of the most appealing and widely accepted measures to assess the efficiency of equilibria, especially in the context of social welfare analysis. The POA
of the first-price and second-price auction has been investigated intensively for both the single-item and the multi-unit auction setting. We focus on the analysis of the (robust) price of anarchy under the complete information setting, focusing on two equilibrium notions: pure Nash equilibria and the more general notion of coarse correlated equilibria. An important reason we opt to study coarse correlated equilibria is that it yields comprehensive results. Since coarse correlated equilibria generalize correlated equilibria, which in turn generalize (mixed) Nash equilibria, upper bounds on the price of anarchy for coarse correlated equilibria hold for these other equilibrium notions as well.

An assumption that often needs to be made to derive meaningful bounds is that the bidders cannot overbid (see Feldman et al. (2013) for a more general discussion of the no-overbidding assumption). Accordingly, we derive bounds on the price of anarchy distinguishing between the case when bidders can overbid and when they cannot overbid their actual valuations for the items.

Altogether, our bounds provide a complete picture of the POA of $\gamma$-FPA and in particular $\gamma$-HYA, for different equilibrium notions both in the single-item and the multi-unit auction setting and with and without overbidding.

Finally, we discuss existing literature to which this chapter contributes. This literature can be roughly divided into two categories: corruption in auctions and the price of anarchy.

There is a large body of research in economics studying collusion among bidders in auctions (see, e.g., Graham and Marshall (1987) and McAfee and McMillan (1992)). Collusion between the auctioneer and the bidders in the form of bid rigging (as considered in this chapter) has also been studied in the literature, but less intensively. Most existing works study certain aspects of equilibrium outcomes (e.g., equilibrium structure, auctioneer surplus, seller revenue, and optimal bribe schemes); for an overview of the existing works along these lines, see Lengwiler and Wolfstetter (2010) and the references therein.

The specific bid rigging model that we consider here was first studied by Menezes and Monteiro (2006) and a slight generalization thereof by Lengwiler and Wolfstetter (2000), both for the single-item auction setting. These works consider a so-called Bayesian (i.e., incomplete information) setting, where the valuations are independent draws from a common distribution function. Menezes and Monteiro (2006) prove the existence of symmetric equilibrium bidding strategies and derive an optimal bribe
function for the auctioneer. The authors also study a fixed-price bribe scheme, where the auctioneer charges a fixed amount that is independent of the gained surplus. Subsequently, Lengwiler and Wolfstetter (2010) study a more complex bid rigging scheme for the single-item auction setting, where the auctioneer additionally offers the second highest bidder to increase their bid. However, none of these works study the price of anarchy of corrupt auctions. Hence, the impact of this corruption on social welfare is still poorly understood.

In our view, the existence of such bid rigging schemes serves as suitable motivation to analyze the price of anarchy of the resulting auctions $\gamma$-HYA and $\gamma$-FPA. But, at the same time, we feel that the study of such hybrid auction formats is interesting in its own right, purely from an auction design perspective. For example, tight bounds on the price of anarchy (as a function of $\gamma$ ) provide insights on which payment rule should ideally be used to reduce inefficiency of equilibria.

Studying the price of anarchy in auctions has recently received considerable attention (see Roughgarden et al. (2017) for a survey). Significant work has gone into deriving bounds on the price of anarchy for various auction formats, both in the complete and incomplete information setting. The smoothness notion, originally introduced by Roughgarden (2015a) to analyze the robust price of anarchy of strategic games, turned out to be very useful in an auction context as well. Syrgkanis and Tardos (2013) build upon this notion and provide a powerful (smoothness-based) toolbox for the analysis of a broad range of auctions that fall into their composition framework.

With respect to the multi-unit auction setting, De Keijzer et al. (2013) use an adapted smoothness approach to derive bounds on the POA for the first-price and the second-price multi-unit auction (mostly focusing on an incomplete information setting without overbidding). The price of anarchy bounds corresponding to coarse correlated equilibria we derive are also based on a smoothness approach, and coincide with theirs for the extreme points $\gamma=0$ and $\gamma=1$. We use an adapted smoothness notion (inspired by De Keijzer et al. (2013) and Syrgkanis and Tardos (2013)) to derive our bounds, both in the overbidding and the no-overbidding setting.

The structure of this chapter is as follows. Section 5.2 treats the preliminaries required to read the remaining sections. In Section 5.3, we formally introduce the auction formats used to capture corruption. In Section 5.4 and Section 5.5, we provide numerous bounds on the price of anarchy of such (multi-unit) auctions, respectively
with and without overbidding. A more fine-grained landscape of price of anarchy bounds emerges when restricting to single-item auctions, as analyzed in Section 5.6. Finally, we discuss and reflect on the main results in Section 5.7.

### 5.2 Preliminaries

In this section, we recall notation, definitions, and assumptions corresponding to standard auction formats, equilibrium notions, and the price of anarchy.

### 5.2.1 Standard auction formats

We focus on the description of the multi-unit auction setting; the single-item auction setting follows as a special case (choosing $k=1$ below). Let $N=\{1, \ldots, n\}$ denote the set of bidders, with $n \geq 2$. In the multi-unit auction setting, there are $k \geq 1$ identical items (or goods) that we want to sell to the bidders (or players). Each bidder $i \in N$ has a non-negative and non-decreasing valuation function $v_{i}:\{0, \ldots, k\} \rightarrow \mathbb{R}_{+}$with $v_{i}(0)=0$, where $v_{i}(j)$ specifies $i$ 's valuation for receiving $j \in\{0, \ldots, k\}$ items. We assume that for each bidder the valuation function $v_{i}$ is submodular or, equivalently, that the marginal valuations are non-increasing, i.e., for every $j \in\{1, \ldots, k-1\}$,

$$
v_{i}(j)-v_{i}(j-1) \geq v_{i}(j+1)-v_{i}(j) .
$$

The valuation function $v_{i}$ is assumed to be known to all players for all $i \in N$, i.e., we consider the complete information setting. We use $\boldsymbol{v}=\left(v_{1}, \ldots, v_{n}\right)$ to denote the profile (or vector) of the valuation functions of the bidders. We assume that the bidders submit their bids according to the following standard format: Each bidder $i \in N$ submits a bid vector $\boldsymbol{b}_{i}=\left(b_{i}(1), \ldots, b_{i}(k)\right)$ of $k$ non-negative and non-increasing marginal bids, i.e., $b_{i}(j)$ specifies the additional amount $i$ is willing to pay for receiving $j$ instead of $j-1$ items, with $j \in\{1, \ldots, k\}$. The overall amount that $i$ bids for receiving $q \in\{1, \ldots, k\}$ items is thus $\sum_{j=1}^{q} b_{i}(j)$.

Consider a multi-unit auction setting and suppose the auctioneer uses an auction mechanism $\mathcal{M}$ to determine an assignment of the items and the respective payments of the bidders. Each bidder submits their bid vector $\boldsymbol{b}_{i}$ to the mechanism. Based on the bidding profile $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right)$, the mechanism $\mathcal{M}$ orders the submitted marginal bids non-increasingly (breaking ties in an arbitrary, deterministic manner) and assigns the
$k$ items to the bidders who submitted the $k$ highest marginal bids in the order. We use $\beta_{j}(\boldsymbol{b})$ to refer to the $j$-th lowest winning (marginal) bid in $\boldsymbol{b}$, i.e., $\beta_{k}(\boldsymbol{b}) \geq \ldots \geq \beta_{1}(\boldsymbol{b})$. We use $\boldsymbol{x}(\boldsymbol{b})=\left(x_{1}(\boldsymbol{b}), \ldots, x_{n}(\boldsymbol{b})\right)$ to refer to the resulting allocation, where $x_{i}(\boldsymbol{b})$ specifies the number of items that bidder $i \in N$ receives; $x_{i}(\boldsymbol{b})=0$ if $i$ does not receive any item. Each bidder who receives at least one item is called a winner. The highest losing bid is denoted by $\bar{p}(\boldsymbol{b})$. Formally, $\bar{p}(\boldsymbol{b})=\max \left\{b_{i}(j) \mid j \in\left\{x_{i}(\boldsymbol{b})+1, \ldots, k\right\}, i \in N\right\}$.

For each bidder $i \in N$, the payment of $i$ is denoted by $p_{i}(\boldsymbol{b})$. We adopt the convention that $p_{i}(\boldsymbol{b})=0$ whenever $i$ is not a winner. If $i$ is a winner, there are two standard payment schemes that determine the respective payments, namely

- First-price payment scheme: Every bidder $i \in N$ pays their bid for the received items, i.e., $p_{i}(\boldsymbol{b})=\sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)$;
- Second-price payment scheme: Every bidder $i \in N$ pays the highest losing bid $\bar{p}(\boldsymbol{b})$ for each received item, i.e., $p_{i}(\boldsymbol{b})=x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b})$.

Suppose we fix the payment scheme of mechanism $\mathcal{M}$ according to one of these schemes. We refer to mechanism $\mathcal{M}$ with the first-price payment or the second-price payment scheme, respectively, as FP-AUCTION or SP-AUCTION. We remark that in the multi-unit auction setting these auctions are usually referred to as discriminatory price auction and uniform price auction, respectively. However, here we stick to the given naming convention to align it with the common terminology of the single-item auction setting.

The utility $u_{i}^{v_{i}}(\boldsymbol{b})$ of bidder $i \in N$ is defined as the total valuation minus the payment for receiving $x_{i}(\boldsymbol{b})$ items, i.e., $u_{i}^{v_{i}}(\boldsymbol{b})=v_{i}\left(x_{i}(\boldsymbol{b})\right)-p_{i}(\boldsymbol{b})$; note that $u_{i}^{v_{i}}(\boldsymbol{b})=0$ by definition if bidder $i$ is not a winner. Whenever $v_{i}$ is clear from the context, we simply denote the utility of bidder $i$ by $u_{i}(\boldsymbol{b})$. We assume that each bidder strives to maximize their utility.

## Example 5.2.1

Let $\mathcal{M}_{1}$ be a FP-Auction and let $\mathcal{M}_{2}$ be a SP-Auction, with $N=\{1,2\}$ and $k=2$. The valuation function of player $1, v_{1}$, is such that $v_{1}(1)=3$ and $v_{1}(2)=6$. For player 2 , we have $v_{2}(1)=1$ and $v_{2}(2)=1.5$. Recall that $v_{1}(0)=v_{2}(0)=0$ by convention.

Suppose the two players submit bid vectors $\boldsymbol{b}_{1}=(3,1)$ and $\boldsymbol{b}_{2}=(4,1)$. Recall that these vectors represent marginal bids; e.g., player 2 bids 4 to receive a single item and

5 to receive both items. For this bidding profile $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \boldsymbol{b}_{2}\right)$, the two highest marginal bids are $b_{1}(1)$ and $b_{2}(1)$, so both players will be assigned one item, i.e., $\boldsymbol{x}(\boldsymbol{b})=(1,1)$. To emphasize, the assignment of items for a given bidding profile does not depend on the payment scheme of the mechanism, so this holds for both $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$. Note that $\bar{p}(\boldsymbol{b})=1$.

First, we consider $\mathcal{M}_{1}$. Then, $p_{1}(\boldsymbol{b})=b_{1}(1)=3$ and $p_{2}(\boldsymbol{b})=b_{2}(1)=4$, and $u_{1}(\boldsymbol{b})=v_{1}\left(x_{1}(\boldsymbol{b})\right)-p_{1}(\boldsymbol{b})=3-3=0$ and $u_{2}(\boldsymbol{b})=1-4=-3$. Alternatively, for $\mathcal{M}_{2}, p_{1}(\boldsymbol{b})=p_{2}(\boldsymbol{b})=\bar{p}(\boldsymbol{b})=1$, with $u_{1}(\boldsymbol{b})=2$ and $u_{2}(\boldsymbol{b})=0$.

Note that $b_{2}(1)>v_{2}(1)$, i.e., the bid of player 2 to receive one item exceeds the player's valuation for this item. This is referred to as overbidding.

A common assumption on the bidding profiles is the no-overbidding ( $N O B$ ) assumption: for all bidders it holds that their bid vectors do not exceed their valuations, i.e., $\sum_{j=1}^{q} b_{i}(j) \leq v_{i}(q)$ for any $q \in\{1, \ldots, k\}$ and all $i \in N$. We do not always make this assumption; it will be explicitly indicated when we do.

Finally, we adopt the convention that the following two standard assumptions must always be satisfied by a mechanism. For the mechanisms we consider throughout this chapter, it is clear these two assumptions are satisfied.

1. No positive transfers (NPT): The payment of each bidder is non-negative, i.e., $p_{i}(\boldsymbol{b}) \geq 0$ for all $i \in N$.
2. Individual rationality (IR): The payment of each bidder does not exceed their bid, i.e., $p_{i}(\boldsymbol{b}) \leq \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)$ for all $i \in N$.

If for every bidding profile $\boldsymbol{b}$ of non-negative and non-increasing marginal bids it holds that $\sum_{i \in N} p_{i}(\boldsymbol{b}) \leq \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})$, then we call the mechanism first-price dominated. Note that every mechanism that satisfies individual rationality must be first-price dominated, since $\sum_{i \in N} \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)=\sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})$, recalling that $\beta_{j}(\boldsymbol{b}), j \in\{1, \ldots, k\}$, refers to the $j$-th lowest winning marginal bid in $\boldsymbol{b}$.

### 5.2.2 Equilibrium notions and the price of anarchy

Below, we briefly review the different equilibrium notions used in this chapter. Please note that the following notions are all defined in the context of some (fixed) auction mechanism $\mathcal{M}$, but that this mechanism is often not explicitly reflected in the notation. Recall that the strategy space of bidding profiles is restricted by the fact that
marginal bids must be non-negative and non-increasing.
Let $\mathcal{M}$ be an auction mechanism. A bidding profile $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right)$ is a pure Nash equilibrium (PNE) if no bidder has an incentive to deviate unilaterally; more formally, $\boldsymbol{b}$ is a PNE if for every bidder $i \in N$ and every bidding profile of $i, \boldsymbol{b}_{i}^{\prime}$, it holds that $u_{i}(\boldsymbol{b}) \geq u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)$. Here we use the standard notation $\boldsymbol{b}_{-i}$ to refer to the bid vector $\boldsymbol{b}$ with the $i$-th component being removed; $\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)$ then refers to the bid vector $\boldsymbol{b}$ with the $i$-th component being replaced by $\boldsymbol{b}_{i}^{\prime}$.

The concept of pure Nash equilibria can be generalized by considering randomized bid vectors (leading to mixed Nash equilibria) and then generalized further by allowing for correlation among bidders (leading to correlated equilbria). We consider an even more general equilibrium notion in this chapter. Let $\boldsymbol{\sigma}$ be a joint (correlated) distribution over bidding profiles of the bidders (the profile of valuation functions $\boldsymbol{v}$ remains fixed). Then, $\boldsymbol{\sigma}$ is a coarse correlated equilibrium (CCE) if for every bidder $i \in N$ and every bid vector $\boldsymbol{b}_{i}^{\prime}$ it holds that $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})\right] \geq \mathbb{E}_{\boldsymbol{b}_{-i} \sim \boldsymbol{\sigma}_{-i}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right]$, where, e.g., $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})\right]$ represents the expected utility of player $i$ when a random bidding profile $\boldsymbol{b}$ is drawn according to $\boldsymbol{\sigma}$. Hence, $\boldsymbol{\sigma}$ is a CCE if no bidder can improve their expected utility by deviating unilaterally to some fixed bid vector $\boldsymbol{b}_{i}^{\prime}$. To emphasize, the draw from a joint distribution $\boldsymbol{\sigma}$ does not influence this deviation $\boldsymbol{b}_{i}^{\prime}$ (contrary to correlated equilibria, as discussed in more detail below). This is reflected by the notation $\boldsymbol{b}_{-i} \sim \sigma_{-i}$ in the expectation, where $\sigma_{-i}$ refers to the same joint distribution $\boldsymbol{\sigma}$, but then with the $i$-th component removed. We elaborately discuss an example of a coarse correlated equilibrium in the proof of Theorem 5.4.4.

We now provide a brief intuitive explanation behind CCE, also clarifying the contrast with correlated equilibria (CE). Suppose some entity draws a random bidding profile $\boldsymbol{b}$ according to some correlated distribution $\boldsymbol{\sigma}$, and proposes to each bidder $i \in N$ a bid vector $\boldsymbol{b}_{i}$. Then, $\boldsymbol{\sigma}$ is a CE if no bidder can obtain an increase in expected utility by deviating from the proposed bid vector, where in the (possibly mixed) deviation the bidder takes into account the proposed bid vector and, through the correlation, information about the distribution of the bid vectors of other bidders. For CCE, however, a bidder only considers submitting a fixed deviation (bid vector) without taking into account information from the draw of the joint distribution. Conditioning on this information may determine that some deviation leads to an increase in expected utility, where this would not have been the case when 'unconditionally' submitting a fixed deviation. Therefore, the set of CE is a subset of the set of CCE.

We define the social welfare of a bidding profile $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right)$ as the overall valuation obtained by the bidders, i.e., $\mathrm{SW}(\boldsymbol{b})=\sum_{i \in N} v_{i}\left(x_{i}(\boldsymbol{b})\right)$. Note that although social welfare is defined independently of the payments, this expression can be rewritten on the basis of utilities and payments as well. The expected social welfare of a joint distribution $\sigma$ over bidding profiles is then defined by

$$
\mathbb{E}[\mathrm{SW}(\boldsymbol{\sigma})]=\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\mathrm{SW}(\boldsymbol{b})]=\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{i \in N} v_{i}\left(x_{i}(\boldsymbol{b})\right)\right]=\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{i \in N}\left(u_{i}(\boldsymbol{b})+p_{i}(\boldsymbol{b})\right)\right] .
$$

We use $\boldsymbol{x}^{*}(\boldsymbol{v})$ to refer to an assignment that maximizes the social welfare with respect to the valuation functions $\boldsymbol{v}=\left(v_{1}, \ldots v_{n}\right)$; i.e.,

$$
\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)=\sum_{i \in N} v_{i}\left(x_{i}^{*}(\boldsymbol{v})\right)
$$

is the maximum social welfare achievable for the bidders, independent of their bids. The assignment $\boldsymbol{x}^{*}(\boldsymbol{v})$ is also called a social optimum.

The price of anarchy is defined as the maximum ratio of the social welfare of the social optimum and the (expected) social welfare of an equilibrium in an auction mechanism $\mathcal{M}$. For its formal definition, let $\boldsymbol{v}=\left(v_{1}, \ldots, v_{n}\right)$ be a profile of valuation functions and let $\operatorname{PNE}(\boldsymbol{v})$ and $\operatorname{CCE}(\boldsymbol{v})$ denote the set of pure Nash equilibria and coarse correlated equilibria with respect to $\boldsymbol{v}$, respectively. Then, the price of anarchy (Koutsoupias and Papadimitriou, 1999) with respect to pure Nash equilibria (or PNE-POA for short), is defined by

$$
\operatorname{PNE-POA}(\boldsymbol{v})=\sup _{\boldsymbol{b} \in \operatorname{PNE}(\boldsymbol{v})} \frac{\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)}{\operatorname{SW}(\boldsymbol{b})}
$$

Similarly, we define

$$
\operatorname{CCE}-\operatorname{POA}(\boldsymbol{v})=\sup _{\boldsymbol{\sigma} \in \operatorname{CCE}(\boldsymbol{v})} \frac{\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)}{\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})]}
$$

The price of anarchy of an auction mechanism $\mathcal{M}$ then refers to the worst-case price of anarchy over all possible (in our case submodular) valuation profiles, i.e.,

$$
\operatorname{PNE}-\operatorname{POA}(\mathcal{M})=\sup _{\boldsymbol{v}} \operatorname{PNE}-\operatorname{POA}(\boldsymbol{v})
$$

and

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M})=\sup _{v} \operatorname{CCE}-\operatorname{POA}(\boldsymbol{v}) .
$$

Before illustrating the price of anarchy in an example, it is good to note that this notion is intended to study auctions in which equilibria actually exist. Informally, the price of anarchy should be interpreted as 'every equilibrium is such that the ratio between the optimal social welfare and the social welfare of the equilibrium is at most the price of anarchy'; profiles of valuation functions such that no equilibria exist, are not taken into account. Equilibria do not always exist, especially when restricting to pure Nash equilibria. The existence of (pure Nash) equilibria may even depend on the tie-breaking rule, as will be illustrated in Example 5.3.2. Providing a full characterization of the sets of equilibria for the different auction settings and equilibrium notions considered in this chapter would certainly be interesting, but is beyond the scope of this chapter.

## Example 5.2.2

Reconsider Example 5.2.1, with $N=\{1,2\}, k=2, v_{1}(1)=3, v_{1}(2)=6, v_{2}(1)=1$, and $v_{2}(2)=1.5$. Clearly, the social optimum is $\boldsymbol{x}^{*}(\boldsymbol{v})=(2,0)$, with $\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)=6$. With respect to $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \boldsymbol{b}_{2}\right)=((3,1),(4,1))$, we have $\boldsymbol{x}(\boldsymbol{b})=(1,1)$ and $\mathrm{SW}(\boldsymbol{b})=4$.

For the first-price auction $\mathcal{M}_{1}$, it is clear that $\boldsymbol{b}$ is not a pure Nash equilibrium, as $u_{2}(\boldsymbol{b})=-3<0=u_{2}\left(\boldsymbol{b}_{1},(0,0)\right)$. For the second-price auction $\mathcal{M}_{2}$, however, bidding profile $\boldsymbol{b}$ is a pure Nash equilibrium, since neither bidder can strictly increase their utility by deviating unilaterally. To see this, recall that $u_{1}(\boldsymbol{b})=2$ and $u_{2}(\boldsymbol{b})=0$. No player can deviate such that their payment decreases, but they are still assigned an item. The highest losing bid will only become strictly lower than one if a player's marginal bids both become strictly lower than one, in which case this player is no longer assigned any item, yielding zero utility. Hence, lowering a marginal bid will either not make a difference, or lead to lower utility. Instead, a player could submit a different bid vector in an attempt to win both items. However, the highest losing bid, and thereby the payment per item, would then increase to at least 4 if player 1 deviates accordingly, and at least 3 in case player 2 deviates. Neither player could benefit from this.

This shows that $\operatorname{PNE}-\operatorname{POA}\left(\mathcal{M}_{2}\right)$ is at least $\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right) / \operatorname{SW}(\boldsymbol{b})=1.5$. In fact, the price of anarchy with respect to pure Nash equilibria of second-price auctions is unbounded. For example, consider $v_{1}(1)=3$ and $v_{1}(2)=6$ as before, but now with $v_{2}(1)=v_{2}(2)=0$, combined with $\boldsymbol{b}_{1}=(0,0)$ and $\boldsymbol{b}_{2}=(3,3)$. Then, $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \boldsymbol{b}_{2}\right)$ is a
pure Nash equilibrium with a social welfare of zero.
Note, however, that player 2 overbids here, since $b_{2}(1)>v_{2}(1)$ and $b_{2}(1)+b_{2}(2)>$ $v_{2}(2)$. Under the NOB assumption, this would not be allowed. This assumption has major implications for the price of anarchy. In particular, Birmpas et al. (2019) find a tight bound on PNE-POA $\left(\mathcal{M}_{2}\right)$ of 2.1885 under NOB.

### 5.3 Capturing corruption with hybrid and approximate first-price auctions

In this section, we formally describe the auction settings we consider to model corruption in auctions. Let $\mathcal{M}$ be an auction mechanism in which bidders submit their bid vectors $\boldsymbol{b}=\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right)$ in a 'sealed manner', i.e., at first only the auctioneer sees the bidding profile $\boldsymbol{b}$. After receipt of the bidding profile $\boldsymbol{b}$, the auctioneer runs a firstprice multi-unit auction to obtain the respective assignment $\boldsymbol{x}(\boldsymbol{b})=\left(x_{1}(\boldsymbol{b}), \ldots, x_{n}(\boldsymbol{b})\right)$ and payments $\boldsymbol{p}(\boldsymbol{b})=\left(p_{1}(\boldsymbol{b}), \ldots, p_{n}(\boldsymbol{b})\right)$, but does not reveal this outcome yet. The bidders might want to verify the 'soundness' of the outcome of the auction, so the final bids may have to be revealed eventually. Importantly, however, since the bids are sealed, the revealed bids do not have to equal the submitted ones. A corrupt auctioneer abuses this by approaching each winning bidder $i \in N$ individually with the offer that they can lower all their $x_{i}(\boldsymbol{b})$ winning bids to the highest losing bid $\bar{p}(\boldsymbol{b})$ (while receiving the same number of items), in exchange for a fixed fraction $\gamma \in[0,1]$ of the surplus gained by $i$. The bidder can either reject or accept this offer. If bidder $i$ rejects the offer, the allocation $x_{i}(\boldsymbol{b})$ and respective payment $p_{i}(\boldsymbol{b})$ remain unmodified. If bidder $i$ accepts the offer, $i$ receives the $x_{i}(\boldsymbol{b})$ items at a reduced price of $\bar{p}(\boldsymbol{b})$ each, but additionally pays a fee $f_{i}^{\gamma}$ of $\gamma$ times the surplus to the auctioneer. More formally, the total payment of a winning bidder $i$ who accepts the offer is

$$
p_{i}^{\gamma}(\boldsymbol{b})=x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b})+f_{i}^{\gamma}(\boldsymbol{b}) \text { where } f_{i}^{\gamma}(\boldsymbol{b})=\gamma \sum_{j=1}^{x_{i}(\boldsymbol{b})}\left(b_{i}(j)-\bar{p}(\boldsymbol{b})\right) .
$$

We also refer to this setting as the $\gamma$-corrupt auction.
As the final payments are dependent on $\gamma$, we (implicitly) assume that the bidders are aware of this parameter, much alike it is assumed that the bidders know the used payment scheme in other auction formats. Note that the change in the bid vector of
player $i$ conforms to the imposed bidding format, i.e., the modified marginal bids of bidder $i$ are still non-negative and non-increasing.

Proposition 5.3 .1 shows that, under the assumption that the final allocation remains invariant ${ }^{1}$, each winning bidder can only benefit from accepting the corrupt auctioneer's offer, independently of the parameter $\gamma$ and of what the other bidders do.

## Proposition 5.3.1

Let $\gamma \in[0,1]$ and let $\mathcal{M}$ be a $\gamma$-corrupt auction. Let $\boldsymbol{b}$ be a bidding profile and let $i \in N$ be a winning bidder. Then, $p_{i}^{\gamma}(\boldsymbol{b}) \leq p_{i}(\boldsymbol{b})$.

Proof. Observe that the total payment to be made by $i$ becomes

$$
p_{i}^{\gamma}(\boldsymbol{b})=x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b})+f_{i}^{\gamma}(\boldsymbol{b})=\gamma \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)+(1-\gamma) x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b})
$$

when $i$ accepts the offer. Clearly, each winning bid of $i$ satisfies $b_{i}(j) \geq \bar{p}(\boldsymbol{b})$, $j \in\left\{1, \ldots, x_{i}(\boldsymbol{b})\right\}$. Thus, $p_{i}^{\gamma}(\boldsymbol{b}) \leq \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)=p_{i}(\boldsymbol{b})$, where $p_{i}(\boldsymbol{b})$ is the payment that $i$ would have to pay when rejecting the offer. In fact, this inequality is strict unless all winning bids of $i$ are equal to $\bar{p}(\boldsymbol{b})$ or $\gamma=1$. In both these cases, the offer made by the auctioneer does not have any effect for $i$ (as there is no surplus generated in the former case, and no difference in the final payment of $i$ in the latter case).

Subsequently, we introduce our novel hybrid auction scheme, which we term $\gamma$-hybrid auction (or $\gamma$-HYA for short), for which we assume that each winning bidder always accepts the offer. $\gamma$-HYA uses the same allocation rule as in the standard multi-unit auction setting, but uses a convex combination of the first-price and second-price payment scheme (parameterized by $\gamma$ ), i.e.,

$$
\begin{equation*}
p_{i}^{\gamma}(\boldsymbol{b})=\gamma \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)+(1-\gamma) x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b}) . \tag{5.1}
\end{equation*}
$$

[^2]for any $i \in N$. Said differently, $\gamma$-HYA interpolates between SP-AUCTION $(\gamma=0)$ and FP-Auction $(\gamma=1)$ as $\gamma$ varies from 0 to 1 . We also use $p^{\gamma}(\boldsymbol{b})$ to refer to the above payment in the single-item auction setting. To emphasize, (5.1) coincides with the equation to determine the payment in $\gamma$-corrupt auctions presented in the proof of Proposition 5.3.1.

## Example 5.3.1

Let $\gamma \in[0,1]$ and let $\mathcal{M}$ be a $\gamma$-corrupt auction with $N=\{1,2\}$ and $k=2$. Consider Example 5.2 .1 and recall that $\boldsymbol{b}=((3,1),(4,1))$, with $\boldsymbol{x}(\boldsymbol{b})=(1,1)$ and $\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right) / S W(\boldsymbol{b})=6 / 4=1.5$. Having determined that both players will be assigned one item, the corrupt auctioneer now approaches both bidders individually, with the offer to lower their winning bid to the highest losing bid $\bar{p}(\boldsymbol{b})=1$. (Note that we implicitly assume ties are broken such that it is ensured that both bidders still win exactly one item.) In exchange for lowering the bid, the auctioneer demands a fraction $\gamma$ of the surplus of both players, being $b_{1}(1)-\bar{p}(\boldsymbol{b})=2$ for player 1 and 3 for player 2. By Proposition 5.3.1, we assume both players accept.

First, let $\gamma=0.75$, so that the auctioneer benefits more than the players. Then,

$$
p_{1}^{\gamma}(\boldsymbol{b})=x_{1}(\boldsymbol{b}) \bar{p}(\boldsymbol{b})+\gamma\left(b_{1}(1)-\bar{p}(\boldsymbol{b})\right)=1+0.75(3-1)=2.5
$$

and similarly $p_{2}^{\gamma}(\boldsymbol{b})=3.25$. Clearly, the payments also correspond to a $\gamma$-HYA with $\gamma=0.75$. Bidding profile $\boldsymbol{b}$ is not a pure Nash equilibrium. For example, if player 1 bids 2 instead of 3 for the first item, this does not change the assignment of items, but would lower bidder 1's payment by 0.75 .

Next, let $\gamma=0.25$. Analogously, or using (5.1) instead, we find $p_{1}^{\gamma}(\boldsymbol{b})=1.5$ and $p_{2}^{\gamma}(\boldsymbol{b})=1.75$. Again, $\boldsymbol{b}$ is not a pure Nash equilibrium.

The following proposition follows immediately from the discussion above and allows us to focus on the POA of $\gamma$-HYA to study $\gamma$-corrupt auctions in which each bidder accepts the corrupt auctioneer's offer.

## Proposition 5.3.2

Let $\gamma \in[0,1]$. Then, the $\gamma$-corrupt auction and $\gamma$-HYA are equivalent. Hence, these settings admit the same set of equilibria and have identical social welfare objectives, and therefore have same the price of anarchy.

Of course, studying the price of anarchy of an auction mechanism is only interesting if equilibria exist. Example 5.3.2 shows, in a single-item setting, that there can be pure

Nash equilibria for $\gamma$-HYA, and also shows that the existence of pure Nash equilibria may depend on the tie-breaking rule.

## Example 5.3.2

Consider a single-item $\gamma$-HYA with $\gamma \in(0,1]$ and $N=\{1,2\}$. Let $v_{1}(1)>v_{2}(1)>0$. We consider two tie-breaking rules: one that always favors player 1 and another that always favors player 2 . For the former, note that, e.g., $b_{1}=b_{2}=v_{2}(1)$ is a pure Nash equilibrium, with $\operatorname{SW}\left(b_{1}, b_{2}\right)=v_{1}(1)=\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)$. In fact, pure Nash equilibria are always efficient in any $\gamma$-HYA with $\gamma \in(0,1]$ and without overbidding (i.e., the price of anarchy of such auction mechanisms is one, as formalized in Theorem 5.5.1).

However, in case the tie-breaking rule favors player 2, it can be shown that no pure Nash equilibrium exists.

In our basic bid rigging model, all winning bidders lower their bids to the highest losing bid. As discussed in the introduction, while this magic number bidding phenomenon has been observed in real-life for single-item auctions, it might seem somewhat unrealistic in the multi-unit auction setting. To allow for additional, more general corruption schemes, we introduce so-called $\gamma$-approximate first-price auctions $(\gamma-F P A)$. The allocation is still determined as in $\gamma$-HYA, but the payment scheme is relaxed. We say that a mechanism $\mathcal{M}$ with payment rule $\boldsymbol{p}=\left(p_{1}(\boldsymbol{b}), \ldots, p_{n}(\boldsymbol{b})\right)$ is a $\gamma$-FPA for some $\gamma \in[0,1]$ if it always recovers at least a fraction of $\gamma$ of the first-price payments, i.e., for every bidding profile $\boldsymbol{b}$,

$$
\sum_{i \in N} p_{i}(\boldsymbol{b}) \geq \gamma \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})
$$

These auctions capture several additional corruption settings. For example, suppose some bidders never accept the offer of the auctioneer (say due to moral objections) and their payments thus remains the first-price payment. While this setting is not covered by $\gamma$-HYA, it is covered by $\gamma$-FPA. As another example, if the auctioneer handles a different fraction $\gamma_{i}$ for each bidder $i \in N$, the resulting auction is $\gamma$-FPA with $\gamma=\min _{i \in N} \gamma_{i}$. It is immediate that every $\gamma$-HYA is a $\gamma$-FPA. In Sections 5.4 and 5.5 , we derive bounds on the coarse correlated price of anarchy for the more general class of $\gamma$-FPA.

### 5.4 Multi-unit auctions with overbidding

In this section, we consider $\gamma$-approximate first-price auctions, for which we derive a tight bound on the coarse correlated price of anarchy in case players can overbid (i.e., without the no-overbidding assumption), using a general proof template based on the notion of smoothness. We first define our smoothness notion, which is adapted from the ones given in Syrgkanis and Tardos (2013) and De Keijzer et al. (2013). For this, recall that given a bidding profile $\boldsymbol{b}$, we let $\beta_{j}(\boldsymbol{b})$ refer to the $j$-th lowest winning bid under $\boldsymbol{b}$.

## Definition 5.4.1

Let $\mathcal{M}$ be a multi-unit auction. Then, $\mathcal{M}$ is $(\lambda, \mu)$-smooth for some $\lambda>0$ and $\mu \geq 0$ if for every valuation profile $\boldsymbol{v}$ and for each bidder $i \in N$ there exists a (possibly randomized) bidding strategy $\boldsymbol{\sigma}_{i}^{\prime}$ such that for every bidding profile $\boldsymbol{b}$ we have

$$
\sum_{i \in N} \mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right] \geq \lambda S W\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)-\mu \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b}) .
$$

We remark that in essence, this definition comes close to the weak smoothness definition in Syrgkanis and Tardos (2013), but relates more directly to the winning bids in the multi-unit auction setting. A similar definition is also used in De Keijzer et al. (2013), but there it is imposed on a per-player basis and used for the Bayesian setting.

Next, we derive a parameterized bound on the coarse correlated price of anarchy of smooth mechanisms, which forms the basis of the bounds we obtain for $\gamma$-FPA.

## Theorem 5.4.2

Let $\gamma \in[0,1]$ and let $\mathcal{M}$ be a $\gamma$-FPA which is $(\lambda, \mu)$-smooth, with $\mu \leq \gamma$. Then,

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{1}{\lambda}
$$

Proof. Let $\boldsymbol{v}$ be a valuation profile and let $\boldsymbol{\sigma}$ be a coarse correlated equilibrium. Let $i \in N$ and let $\boldsymbol{\sigma}_{i}^{\prime}$ be the bidding strategy of bidder $i$ as given by the smoothness definition. Exploiting the coarse correlated equilibrium condition for $i$, we have for every (deterministic) bid vector $\boldsymbol{b}_{i}^{\prime}$ that $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})\right] \geq \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right]$ and thus also

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})\right] \geq \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right]\right] . \tag{5.2}
\end{equation*}
$$

Using this, we obtain

$$
\begin{align*}
\mathbb{E}[\mathrm{SW}(\boldsymbol{\sigma})] & =\sum_{i \in N} \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})+p_{i}(\boldsymbol{b})\right] \\
& \geq \sum_{i \in N} \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right]+p_{i}(\boldsymbol{b})\right] \\
& \geq \sum_{i \in N} \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right]\right]+\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\gamma \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})\right] \\
& \geq \lambda \operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)+(\gamma-\mu) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})\right]  \tag{5.3}\\
& \geq \lambda \operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)
\end{align*}
$$

where the first inequality follows from (5.2), the second inequality from the fact that $\mathcal{M}$ is a $\gamma$-FPA, the third inequality from Definition 5.4.1, and the fourth inequality from $\mu \leq \gamma$. Using this, we find CCE-POA $(\boldsymbol{v}) \leq 1 / \lambda$.

As a final step to obtain an optimized price of anarchy bound as a function of $\gamma$, we use the following lemma from De Keijzer et al. (2013) (adapted to our setting).

## Lemma 5.4.3 [De Keijzer et al. (2013)]

Let $\mathcal{M}$ be a mechanism that is first-price dominated and let $\alpha>0$ be fixed arbitrarily. Then, for every valuation profile $\boldsymbol{v}$ and for every bidder $i \in N$ there exists a randomized bidding strategy $\boldsymbol{\sigma}_{i}^{\prime}$ such that for every bidding profile $\boldsymbol{b}$ we have

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right] \geq \alpha\left(1-\frac{1}{e^{1 / \alpha}}\right) v_{i}\left(\boldsymbol{x}_{i}^{*}(\boldsymbol{v})\right)-\alpha \sum_{j=1}^{\boldsymbol{x}_{i}^{*}(\boldsymbol{v})} \beta_{j}(\boldsymbol{b}) \tag{5.4}
\end{equation*}
$$

On the basis of Theorem 5.4.2 and Lemma 5.4.3, we derive the bound of Theorem 5.4.4 below on the coarse correlated price of anarchy of $\gamma$-FPA for any $\gamma \in(0,1]$. It is known that the price of anarchy is unbounded for second-price auctions $(\gamma=0)$. Importantly, we also find a matching lower bound, already in a single-item $\gamma$-HYA setting. Thus, we completely settle the coarse correlated price of anarchy of $\gamma$-FPA when overbidding is allowed. The corresponding bound is displayed in Figure 5.1.


Figure 5.1 Tight upper bound of Theorem 5.4.4 on the CCE-POA ( $y$ axis) for multi-unit $\gamma$-FPA with overbidding as a function of $\gamma$ ( $x$-axis)

## Theorem 5.4.4

Let $\gamma \in(0,1]$ and let $\mathcal{M}$ be a $\gamma-F P A$. Then,

$$
\begin{equation*}
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{1}{\gamma\left(1-e^{-1 / \gamma}\right)} \tag{5.5}
\end{equation*}
$$

Further, this bound is tight.

## Proof.

Upper bound: Let $\alpha>0$ and note that $\mathcal{M}$ is first-price dominated. Let $\boldsymbol{v}$ be a valuation profile and let $\boldsymbol{b}$ be a bidding profile. Summing inequality (5.4) over all players, we find

$$
\begin{aligned}
\sum_{i \in N} \mathbb{E}_{\boldsymbol{b}_{i}^{\prime} \sim \boldsymbol{\sigma}_{i}^{\prime}}\left[u_{i}\left(\boldsymbol{b}_{i}^{\prime}, \boldsymbol{b}_{-i}\right)\right] & \geq \alpha\left(1-\frac{1}{e^{1 / \alpha}}\right) \sum_{i \in N} v_{i}\left(\boldsymbol{x}_{i}^{*}(\boldsymbol{v})\right)-\alpha \sum_{i \in N} \sum_{j=1}^{\boldsymbol{x}_{i}^{*}(\boldsymbol{v})} \beta_{j}(\boldsymbol{b}) \\
& \geq \alpha\left(1-\frac{1}{e^{1 / \alpha}}\right) \operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)-\alpha \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})
\end{aligned}
$$

for some (possibly randomized) bidding strategy $\sigma_{i}^{\prime}$, where the second inequality follows from the fact that $\sum_{i \in N} \sum_{j=1}^{\boldsymbol{x}_{i}^{*}(\boldsymbol{v})} \beta_{j}(\boldsymbol{b}) \leq \sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})$ for any bidding profile $\boldsymbol{b}$, as $\beta_{1}(\boldsymbol{b}) \leq \ldots \leq \beta_{k}(\boldsymbol{b})$ and $\sum_{i \in N} \boldsymbol{x}_{i}^{*}(\boldsymbol{v})=k$. Hence, $\mathcal{M}$ is $\left(\alpha\left(1-e^{-1 / \alpha}\right), \alpha\right)$-smooth. Restricting to $0<\alpha \leq \gamma$, Theorem 5.4.2 implies

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{1}{\alpha\left(1-e^{-1 / \alpha}\right)}
$$

To minimize this upper bound, consider its derivative with respect to $\alpha$,

$$
-\frac{1}{\alpha^{2}\left(1-e^{-1 / \alpha}\right)^{2}}\left(1-e^{-1 / \alpha}-\alpha \frac{1}{\alpha^{2}} e^{-1 / \alpha}\right)=-\frac{1-\left(1+\frac{1}{\alpha}\right) e^{-1 / \alpha}}{\alpha^{2}\left(1-e^{-1 / \alpha}\right)^{2}} .
$$

As $(1+1 / \alpha) e^{-1 / \alpha}<1$ for all $\alpha>0$, the derivative is negative for all $\alpha>0$. Therefore, the bound is minimized by maximizing $\alpha \in(0, \gamma]$. Substituting $\alpha=\gamma$ yields the upper bound.

Matching lower bound: The upper bound is proven to be tight by generalizing an example used by Syrgkanis (2014) to provide a lower bound on the CCE-POA for the first-price single-item auction. Consider a single-item $\gamma$-HYA with $N=\{1,2\}$. We have $v_{1}=v$ for some $v>0$ and $v_{2}=0$. If both bidders bid 0 , the tie is broken in favor of bidder 2 , whereas bidder 1 wins the auction if bidders tie with any positive bid. We construct a coarse correlated equilibrium with a welfare loss that matches the upper bound.

Let $t$ be a random variable with support $\left[0,\left(1-e^{-1 / \gamma}\right) v\right]$ whose cumulative distribution function (CDF) $F$ and density function $f$ (which is well-defined for any $\left.t \in\left(0,\left(1-e^{-1 / \gamma}\right) v\right]\right)$, respectively, are given by

$$
F(t)=(1-\gamma)+\frac{v}{v-t} \gamma e^{-1 / \gamma} \quad \text { and } \quad f(t)=\frac{v}{(v-t)^{2}} \gamma e^{-1 / \gamma}
$$

Note that $F(0)=(1-\gamma)+\gamma e^{-1 / \gamma}$.
Consider a bidding profile $\boldsymbol{\sigma}=(t, t)$. Since ties are broken in favor of bidder 2 only for $t=0$, bidder 2 wins with probability $(1-\gamma)+\gamma e^{-1 / \gamma}$, which yields

$$
\frac{\operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)}{\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})]}=\frac{v}{(1-F(0)) v}=\frac{1}{1-(1-\gamma)-\gamma e^{-1 / \gamma}}=\frac{1}{\gamma\left(1-e^{-1 / \gamma}\right)}
$$

It remains to show that $\boldsymbol{\sigma}$ is a CCE. This is quite obvious for bidder 2, who either wins by bidding 0 , or loses if $t>0$. Given any positive bid from bidder 1 , the payment would be strictly greater than $v_{2}=0$, meaning bidder 2 could never profitably deviate.

For bidder 1, we show that any deviation to a fixed bid $b_{1}=b$ with $b \in\left(0,\left(1-e^{-1 / \gamma}\right) v\right]$ leads to an expected utility of at most $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right]$. To start with $\boldsymbol{\sigma}$ itself, note that bidder 1 wins whenever $t>0$, and since both bidders bid $t$,
we have a payment of $\gamma t+(1-\gamma) t=t$. Recalling that $v_{1}=v$, we get

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right] & =\int_{0}^{\left(1-e^{-1 / \gamma}\right) v}(v-t) f(t) d t \\
& =\int_{0}^{\left(1-e^{-1 / \gamma}\right) v} \frac{v}{v-t} \gamma e^{-1 / \gamma} d t \\
& =v \gamma e^{-1 / \gamma}[-\ln (v-t)]_{0}^{\left(1-e^{-1 / \gamma}\right) v} \\
& =v \gamma e^{-1 / \gamma}\left(\ln (v)-\ln \left(e^{-1 / \gamma} v\right)\right) \\
& =v \gamma e^{-1 / \gamma} \frac{1}{\gamma}=v e^{-1 / \gamma}
\end{aligned}
$$

By deviating to $b$, bidder 1 wins the item if $b \geq t$, and for each $t \in(0, b]$ pays $\gamma b+(1-\gamma) t$. Hence, the expected utility of bidder 1 becomes

$$
\mathbb{E}_{t \sim F(t)}\left[u_{1}(b, t)\right]=\int_{0}^{b}(v-\gamma b-(1-\gamma) t) f(t) d t
$$

To facilitate the calculations, note that

$$
\begin{aligned}
\int_{0}^{b} t f(t) d t & =\gamma v e^{-1 / \gamma} \int_{0}^{b} \frac{t}{(v-t)^{2}} d t \\
& =\gamma v e^{-1 / \gamma}\left[\frac{v}{v-t}+\ln (v-t)\right]_{0}^{b} \\
& =\left(\frac{v}{v-b}-1\right) \gamma v e^{-1 / \gamma}+\ln \left(\frac{v-b}{v}\right) \gamma v e^{-1 / \gamma}
\end{aligned}
$$

and

$$
\int_{0}^{b} f(t) d t=F(b)-F(0)=\left(\frac{v}{v-b}-1\right) \gamma e^{-1 / \gamma}
$$

Using this, we get

$$
\begin{aligned}
\mathbb{E}_{t \sim F(t)}\left[u_{1}(b, t)\right]= & (v-\gamma b) \int_{0}^{b} f(t) d t-(1-\gamma) \int_{0}^{b} t f(t) d t \\
= & (v-\gamma b-(1-\gamma) v)\left(\frac{v}{v-b}-1\right) \gamma e^{-1 / \gamma} \\
& -(1-\gamma) \ln \left(\frac{v-b}{v}\right) \gamma v e^{-1 / \gamma}
\end{aligned}
$$

$$
\begin{aligned}
& =\gamma(v-b)\left(\frac{v}{v-b}-1\right) \gamma e^{-1 / \gamma}-(1-\gamma) \ln \left(\frac{v-b}{v}\right) \gamma v e^{-1 / \gamma} \\
& =b \gamma^{2} e^{-1 / \gamma}-(1-\gamma) \ln \left(\frac{v-b}{v}\right) \gamma v e^{-1 / \gamma}
\end{aligned}
$$

Since $0<b<v$, note that $-\ln \left(\frac{v-b}{v}\right)$ is increasing in $b$. As $\gamma \in(0,1]$, this implies the entire function above is increasing in $b$. Hence, it can be upper bounded by substituting the upper bound of the support: $b=\left(1-e^{-1 / \gamma}\right) v$. This yields

$$
\begin{aligned}
\mathbb{E}_{t \sim F(t)}\left[u_{1}(b, t)\right] & \leq\left(1-e^{-1 / \gamma}\right) v \gamma^{2} e^{-1 / \gamma}-(1-\gamma) \ln \left(e^{-1 / \gamma}\right) \gamma v e^{-1 / \gamma} \\
& =\left(\left(1-e^{-1 / \gamma}\right) \gamma^{2}+(1-\gamma)\right) v e^{-1 / \gamma} \\
& =\left(\left(1-e^{-1 / \gamma}\right) \gamma^{2}+(1-\gamma)\right) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right]
\end{aligned}
$$

Therefore, $\mathbb{E}_{t \sim F(t)}\left[u_{1}(b, t)\right] \leq \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right]$ for any $b \in\left(0,\left(1-e^{-1 / \gamma}\right) v\right]$ if

$$
\left(1-e^{-1 / \gamma}\right) \gamma^{2}+(1-\gamma) \leq 1 \quad \Longleftrightarrow \quad \gamma\left(1-e^{-1 / \gamma}\right) \leq 1
$$

which holds for any $\gamma \in(0,1]$ as required. This shows that bidder 1 does not have any profitable deviation in the interval $\left(0,\left(1-e^{-1 / \gamma}\right) v\right]$. Finally, since $b=\left(1-e^{-1 / \gamma}\right) v$ already gives $F(b)=1$, any higher bid will only lead to a (strictly) higher payment (since $\gamma>0$ ), thereby being (strictly) worse than bidding $b=\left(1-e^{-1 / \gamma}\right) v$. Hence, deviations to a bid higher than this upper bound of the support of $F(t)$ need not be considered.

This shows that $\boldsymbol{\sigma}$ is a CCE for which the ratio of the social welfare of the social optimum and the expected social welfare of $\boldsymbol{\sigma}$ exactly coincides with the upper bound on the coarse correlated price of anarchy.

### 5.5 Multi-unit auctions without overbidding

In the previous section, we established a tight bound on the coarse correlated price of anarchy of $\gamma$-FPA when players are allowed to overbid. Especially when $\gamma$ gets small this has an extremely negative effect on the price of anarchy. In this section, we still focus on the multi-unit auction setting and show that the bound on the coarse correlated price of anarchy of $\gamma$-FPA improves significantly under the standard assumption of no-overbidding (NOB as defined in Section 5.2), most notably for lower
values of $\gamma$. Before deriving this improved bound, we first show that under NOB, pure Nash equilibria of (multi-unit) $\gamma$-HYA with $\gamma \in(0,1]$ are fully efficient.

The pure price of anarchy of $\gamma$-HYA without overbidding has been analyzed before for $\gamma=0$ and $\gamma=1$ : Birmpas et al. (2019) show that the PNE-POA is 2.1885 for the second-price multi-unit auction $(\gamma=0)$, while De Keijzer et al. (2013) show that the PNE-POA is 1 for the first-price multi-unit auction $(\gamma=1)$. Interestingly, we do not find a smooth interpolation between these two boundary points when analyzing the PNE-POA on the interval $\gamma \in[0,1]$. As it turns out, for $\gamma$-HYA the PNE-POA stays at 1 almost over the entire range, the only exception being at $\gamma=0$ where it is 2.1885 by the result of Birmpas et al. (2019).

## Theorem 5.5.1

Let $\gamma \in(0,1)$ and let $\mathcal{M}$ be a $\gamma-H Y A$ in which players cannot overbid. Then, pure Nash equilibria are always efficient, i.e., $\operatorname{PNE}-\operatorname{POA}(\mathcal{M})=1$.

This theorem follows from a minor adaption of the proof template used by De Keijzer et al. (2013) to show that pure Nash equilibria are always efficient for $\gamma=1$. It can be shown that the same result goes through whenever $\gamma>0$. Intuitively, when considering $\gamma>0$, there is a non-zero first price component and thus a player always has an incentive to lower their winning bid (to the highest losing bid). The reasoning that a player would increase their utility by lowering their winning bid holds in the same way.

Next, we reconsider the bound on the coarse correlated price of anarchy of Theorem 5.4.4, but now under NOB. Under this assumption, we no longer have to restrict to $\mu \leq \gamma$ when considering the coarse correlated price of anarchy of $(\lambda, \mu)$-smooth mechanisms. This leads to the following improvement on the bound of Theorem 5.4.2.

## Theorem 5.5.2

Let $\gamma \in[0,1]$ and let $\mathcal{M}$ be a $\gamma-F P A$ in which players cannot overbid and which is $(\lambda, \mu)$-smooth. Then,

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{\max \{1,1+\mu-\gamma\}}{\lambda}
$$

Proof. Let $\boldsymbol{v}$ be a valuation profile and let $\boldsymbol{\sigma}$ be a coarse correlated equilibrium. Similar to the proof of Theorem 5.4.2, we arrive at (5.3), namely

$$
\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})] \geq \lambda \mathrm{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)+(\gamma-\mu) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})\right]
$$

Then, we distinguish two cases.
Case 1: $\mu \leq \gamma$. In the proof of Theorem 5.4.2, we found CCE-POA $(\boldsymbol{v}) \leq 1 / \lambda$.
Case 2: $\mu>\gamma$. Exploiting that the no-overbidding assumption holds in this case, we get that $\sum_{j=1}^{k} \beta_{j}(\boldsymbol{b})=\sum_{i \in N} \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j) \leq \sum_{i \in N} v_{i}\left(x_{i}(\boldsymbol{b})\right)=\operatorname{SW}(\boldsymbol{b})$. Recalling that $\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})]=\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\mathrm{SW}(\boldsymbol{b})]$, we obtain

$$
\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})] \geq \lambda \operatorname{SW}\left(\boldsymbol{x}^{*}(\boldsymbol{v})\right)+(\gamma-\mu) \mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})]
$$

Rearranging terms yields CCE-POA $(\boldsymbol{v}) \leq(1+\mu-\gamma) / \lambda$. Combining both cases proves the claim.

Let $\gamma \in[0,1]$ and let $\mathcal{M}$ be a $\gamma$-FPA. On the basis of Theorem 5.5.2, we derive a stronger bound on the coarse correlated price of anarchy of $\mathcal{M}$, as stated in Theorem 5.5.3 below. This upper bound is significantly lower than its counterpart from the previous section, especially for lower values of $\gamma$.

The upper bound displayed in Figure 5.2 is based on the improved bound of Theorem 5.5.3 for lower values of $\gamma$, until it intersects with the bound of Theorem 5.4.4. Then, the latter is used for higher values of $\gamma$. In particular, we obtain $\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq-\mathcal{W}_{-1}\left(-e^{-2}\right) \approx 3.146$ for $\gamma=0$ and $\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq e /(e-1) \approx$ 1.582 for $\gamma=1$. Here, $\mathcal{W}_{-1}$ refers to the Lambert $\mathcal{W}$ function. This function solves $y e^{y}=x$ for $y$, where $x, y \in \mathbb{R}$ with $x \geq-1 / e$. Specifically, this function takes two values if $-1 / e \leq x<0$, denoted by $y=\mathcal{W}_{0}(x)$ and $y=\mathcal{W}_{-1}(x)$, where $\mathcal{W}_{-1}(x)$ refers to the lower branch of this function. It cannot be expressed in terms of elementary functions.

## Theorem 5.5.3

Let $\gamma \in[0,0.607 \ldots)^{2}$ and let $\mathcal{M}$ be a $\gamma-F P A$ in which players cannot overbid. Then,

$$
\begin{equation*}
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq-(1-\gamma) \mathcal{W}_{-1}\left(-\frac{1}{e^{(2-\gamma) /(1-\gamma)}}\right) \tag{5.6}
\end{equation*}
$$

[^3]

Figure 5.2 Upper bound of Theorems 5.4.4 85 5.5.3 on the CCEPOA ( $y$-axis) for multi-unit $\gamma$-FPA without overbidding as a function of $\gamma$ ( $x$-axis)

Proof. As established in the proof of Theorem 5.4.4, $\mathcal{M}$ is $\left(\alpha\left(1-e^{-1 / \alpha}\right), \alpha\right)$-smooth. By Theorem 5.5.2, we therefore have

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{\max \{1,1+\alpha-\gamma\}}{\alpha\left(1-e^{-1 / \alpha}\right)}
$$

Similar to the proof of Theorem 5.4.4, we minimize this upper bound with respect to $\alpha$. The proof of Theorem 5.4.4 shows that it is optimal to use $\alpha=\gamma$ when restricting to $\alpha \leq \gamma$. Using the no-overbidding assumption, we can also set $\alpha \geq \gamma$ and obtain

$$
\begin{equation*}
\mathrm{CCE}-\mathrm{POA} \leq \frac{1+\alpha-\gamma}{\alpha\left(1-e^{-1 / \alpha}\right)} \tag{5.7}
\end{equation*}
$$

This upper bound is minimized for

$$
\begin{equation*}
\alpha=-\frac{1}{\mathcal{W}_{-1}\left(-e^{-(2-\gamma) /(1-\gamma)}\right)+\frac{2-\gamma}{1-\gamma}} \tag{5.8}
\end{equation*}
$$

Substituting this into (5.7), we obtain the upper bound in (5.6). Importantly, the optimized bound in (5.6) is only valid if we have $\alpha \geq \gamma$, which does not hold for the entire range $\gamma \in[0,1]$ if we use (5.8). More concretely, we have $\alpha \geq \gamma$ for all $\gamma \leq 0.607 \ldots$ only. Thus, for $\gamma \leq 0.607 \ldots$ we can use (5.6) to bound the price of anarchy. For $\gamma \geq 0.607 \ldots$ the best we can do is to choose $\alpha=\gamma$ and obtain the same CCE-POA bound as in Theorem 5.4.4.

### 5.6 Single-item auctions without overbidding

In this section, we further improve the price of anarchy bounds for single-item $\gamma$-HYA. This setting allows to make more direct use of the (second-price) payments. For coarse correlated equilibria, we derive a strong price of anarchy bound for low values of $\gamma$, namely $1 /(1-\gamma)$. This bound can be complemented by the bounds for multi-unit auctions displayed in Figure 5.2. Finally, to improve upon this multi-unit bound for the higher range of $\gamma$, we derive two technically more involved bounds that work specifically in a two-player setting.

For the proofs of the coming theorems, we introduce some more notation. Given a bid vector $\boldsymbol{b}$, let $H B(\boldsymbol{b})=\max _{i \in N} \boldsymbol{b}_{i}$ and $S B(\boldsymbol{b})$ denote the highest and second highest bid in $\boldsymbol{b}$, respectively, and let $H B_{-i}(\boldsymbol{b})=\max _{j \in N \backslash\{i\}} \boldsymbol{b}_{j}$ be the highest bid excluding bid $\boldsymbol{b}_{i}, i \in N$. For a randomized bid vector $\boldsymbol{\sigma}$, let $H B(\boldsymbol{\sigma})$ be the random variable equal to the highest bid when the bids are distributed according to $\boldsymbol{\sigma}$. We sometimes write $\mathbb{E}[H B(\boldsymbol{\sigma})]$ for $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[H B(\boldsymbol{b})]$ (similarly for $S B(\boldsymbol{\sigma})$ and $H B_{-i}(\boldsymbol{\sigma})$ ). Finally, let $p^{\gamma}(\boldsymbol{b})$ denote the payment of the (only) winner w.r.t. $\boldsymbol{b}$ in a single-item $\gamma$-HYA, irrespective of that player's identity. If that identity becomes relevant, we indicate this using a subscript as before, e.g., by $p_{i}^{\gamma}(\boldsymbol{b})$.

### 5.6.1 Single-item auctions with $n$ players

To upper bound the price of anarchy of single-item $\gamma$-HYA, note that any upper bound for the multi-unit auction setting also holds for the single-item setting, so Theorems 5.4.4 and 5.5.3 still apply. We combine this with Theorem 5.6.1 stated below, which provides a good bound for small values of $\gamma$. Together, these three bounds (separated using different colors) yield the upper bound displayed in Figure 5.3 for the coarse correlated price of anarchy in the single-item auction setting.

## Theorem 5.6.1

Let $\gamma \in[0,1)$ and let $\mathcal{M}$ be a single-item $\gamma$-HYA in which players cannot overbid. Then,

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{1}{1-\gamma}
$$

Proof. Let player 1 be the player with highest valuation $v_{1}$, and if there are multiple players with the highest valuation the player in whose favor ties are broken when


Figure 5.3 Upper bound of Theorems 5.4.4, 5.5.3 85 5.6.1 on the CCE-POA (y-axis) for single-item $\gamma$-HYA without overbidding as a function of $\gamma$ (x-axis)
bidding $v_{1}$. Let $\boldsymbol{\sigma}$ be a coarse correlated equilibrium. We have

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\mathrm{SW}(\boldsymbol{b})]=\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right]+\sum_{i \in N \backslash\{1\}} \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{i}(\boldsymbol{b})\right]+\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[p^{\gamma}(\boldsymbol{b})\right] . \tag{5.9}
\end{equation*}
$$

Define $\mathcal{E}$ as the event that player 1 wins the auction with respect to $\sigma$, and let $\overline{\mathcal{E}}$ be the complement event that player 1 does not win the auction with respect to $\boldsymbol{\sigma}$.

Suppose player 1 deviates to $v_{1}$. Then, since no player overbids, player 1 wins under $\left(v_{1}, \boldsymbol{b}_{-1}\right)$ either with the single highest bid or because ties are broken in favor of player 1 by assumption. Note that this holds independently for $\mathcal{E}$ and $\overline{\mathcal{E}}$. By the CCE conditions, we thus have

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}(\boldsymbol{b})\right] & \geq \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[u_{1}\left(v_{1}, \boldsymbol{b}_{-1}\right)\right] \\
& =v_{1}-\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[p^{\gamma}\left(v_{1}, \boldsymbol{b}_{-1}\right)\right] \\
& =v_{1}-\left(\gamma v_{1}+(1-\gamma) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[H B_{-1}(\boldsymbol{b})\right]\right) \\
& =(1-\gamma) v_{1}-(1-\gamma) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[H B_{-1}(\boldsymbol{b})\right] .
\end{aligned}
$$

Substituting this inequality in (5.9), we obtain

$$
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\operatorname{SW}(\boldsymbol{b})] \geq(1-\gamma) v_{1}-(1-\gamma) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[H B_{-1}(\boldsymbol{b})\right]+\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{i \in N \backslash\{1\}} u_{i}(\boldsymbol{b})+p^{\gamma}(\boldsymbol{b})\right]
$$

Since we have to show that $v_{1} / \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\mathrm{SW}(\boldsymbol{b})] \leq 1 /(1-\gamma)$, which can be rewritten as $\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}[\operatorname{SW}(\boldsymbol{b})] \geq(1-\gamma) v_{1}$, the proof follows if we can show that

$$
\mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[\sum_{i \in N \backslash\{1\}} u_{i}(\boldsymbol{b})+p^{\gamma}(\boldsymbol{b})\right] \geq(1-\gamma) \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{\sigma}}\left[H B_{-1}(\boldsymbol{b})\right]
$$

Case 1: Suppose $\boldsymbol{b} \in \mathcal{E}$. Then player 1 wins the auction with respect to $\boldsymbol{b}$ and we have

$$
\sum_{i \in N \backslash\{1\}} u_{i}(\boldsymbol{b})+p^{\gamma}(\boldsymbol{b})=p_{1}^{\gamma}(\boldsymbol{b})=\gamma \boldsymbol{b}_{1}+(1-\gamma) H B_{-1}(\boldsymbol{b}) \geq H B_{-1}(\boldsymbol{b})
$$

Case 2: Suppose $\boldsymbol{b} \in \overline{\mathcal{E}}$. Then some other player $i^{\prime} \in N \backslash\{1\}$ wins the auction with respect to $\boldsymbol{b}$ and we have

$$
\sum_{i \in N \backslash\{1\}} u_{i}(\boldsymbol{b})+p^{\gamma}(\boldsymbol{b})=u_{i^{\prime}}(\boldsymbol{b})+p_{i^{\prime}}^{\gamma}(\boldsymbol{b})=v_{i^{\prime}}-p_{i^{\prime}}^{\gamma}(\boldsymbol{b})+p_{i^{\prime}}^{\gamma}(\boldsymbol{b})=v_{i^{\prime}} \geq \boldsymbol{b}_{i^{\prime}}=H B_{-1}(\boldsymbol{b}),
$$

where last inequality holds because $i^{\prime}$ does not overbid and the last equality holds because $i^{\prime}$ being the highest bidder implies that $\boldsymbol{b}_{i^{\prime}}=H B_{-1}(\boldsymbol{b})$. This concludes the proof.

### 5.6.2 Single-item auctions with 2 players

We now present a more fine-grained picture for the coarse correlated price of anarchy of $\gamma$-HYA in a 2 -player setting, for which the upper bound ultimately becomes a combination of three upper bounds, as represented by the three colors in Figure 5.4.

We already derived the bound we use for small values of $\gamma$ in Theorem 5.6.1, corresponding to the green graph in the figure. To derive the two remaining bounds, we take inspiration from an approach for first-price auctions by Feldman et al. (2016). The extra difficulty we have is bounding the second-price component. The first-price has a direct relation with winning the auction and so we can use the CCE conditions to bound it, but the second-price component is more difficult to get a grip on. However, we still derive two bounds that significantly improve on the bounds of Theorem 5.4.4 and Theorem 5.5.3.

We first consider the interval $\gamma \in\left[0, \frac{1}{2}\right]$. On this interval, the upper bound on the coarse correlated price of anarchy is the minimum of the upper bound from Theorem 5.6.1 and the upper bound of Theorem 5.6.2 below (represented by the orange graph in Figure 5.4), which holds specifically for $\gamma \in\left(0.217 \ldots, \frac{1}{2}\right]$. However, since the


Figure 5.4 Upper bound of Theorems 5.6.1 85 5.6.2 on the CCEPOA (y-axis) for single-item $\gamma$-HYA without overbidding and two bidders as a function of $\gamma$ (x-axis)
upper bound of Theorem 5.6 .2 is strictly higher than the bound of $1 /(1-\gamma)$ from Theorem 5.6 .1 for all $\gamma$ below their intersection point $\gamma=0.339 \ldots$, this interval is sufficiently wide.

## Theorem 5.6.2

Let $\gamma \in\left(0.217 \ldots, \frac{1}{2}\right]$ and let $\mathcal{M}$ be a 2-player single-item $\gamma$-HYA in which players cannot overbid. Then,

$$
\operatorname{CCE}-\operatorname{POA}(\mathcal{M}) \leq \frac{1}{1-(1-\gamma)\left(e^{-1 /(1-\gamma)}+e^{\left.-1-e^{-1 /(1-\gamma)}\right)}\right.}
$$

Proof. Without loss of generality we assume that player 1 has a valuation of 1 and player 2 has a valuation of $v \leq 1$. Consider some coarse correlated equilibrium $\boldsymbol{\sigma}$. Let $\alpha=\mathbb{E}\left[u_{1}(\boldsymbol{\sigma})\right]$ be the utility of player 1 and $\beta=\mathbb{E}\left[u_{2}(\boldsymbol{\sigma})\right]$ be the utility of player 2 in $\boldsymbol{\sigma}$. The maximum social welfare is clearly 1 , achieved when player 1 always wins. Lower bounding the expected welfare of an arbitrary $\boldsymbol{\sigma}$ and taking the reciprocal of this bound thus translates into an upper bound on the price of anarchy. We have

$$
\begin{equation*}
\mathbb{E}[\mathrm{SW}(\boldsymbol{\sigma})] \geq \alpha+\beta+\mathbb{E}\left[p^{\gamma}(\boldsymbol{\sigma})\right]=\alpha+\beta+\gamma \mathbb{E}[H B(\boldsymbol{\sigma})]+(1-\gamma) \mathbb{E}[S B(\boldsymbol{\sigma})] . \tag{5.10}
\end{equation*}
$$

We find the $v, \alpha$ and $\beta$ that minimize this expression to determine a lower bound on the expected social welfare. Let $F_{X}$ be the CDF of the random variable $X$ where $X \in\left\{H B, H B_{-1}, H B_{-2}, S B\right\}$. Then, by the CCE conditions and the fact that a CDF
is always bounded by 1 , we know that on their respective domains

$$
\begin{align*}
F_{H B_{-1}(\sigma)}(x) & \leq \min \left\{\frac{\alpha}{1-x}, 1\right\}, F_{H B_{-2}(\sigma)}(x) \leq \min \left\{\frac{\beta}{v-x}, 1\right\}  \tag{5.11}\\
F_{H B(\sigma)}(x) & \leq \min \left\{\frac{\alpha}{1-x}, \frac{\beta}{v-x}, 1\right\} \tag{5.12}
\end{align*}
$$

For example, if $F_{H B_{-1}(\boldsymbol{\sigma})}(x)>\frac{\alpha}{1-x}$, player 1 can bid $x$ instead and thereby obtain a utility strictly greater than $\frac{\alpha}{1-x} \cdot(1-x)=\alpha$. Since the current utility of player 1 is $\alpha$, this contradicts the CCE conditions.

Observe that for 2 players the following chain of equalities holds

$$
\begin{align*}
F_{S B(\boldsymbol{\sigma})}(x) & =\mathbb{P}[S B(\boldsymbol{\sigma}) \leq x] \\
& =\mathbb{P}\left[\min \left(H B_{-1}(\boldsymbol{\sigma}), H B_{-2}(\boldsymbol{\sigma})\right) \leq x\right] \\
& =\mathbb{P}\left[H B_{-1}(\boldsymbol{\sigma}) \leq x\right]+\mathbb{P}\left[H B_{-2](\boldsymbol{\sigma})} \leq x\right]-\mathbb{P}[H B(\boldsymbol{\sigma}) \leq x] \\
& =F_{H B_{-1}(\boldsymbol{\sigma})}(x)+F_{H B_{-2}(\boldsymbol{\sigma})}(x)-F_{H B(\boldsymbol{\sigma})}(x) . \tag{5.13}
\end{align*}
$$

We derive a more explicit expression for the expected payment using (5.13)

$$
\begin{align*}
& \mathbb{E}\left[p^{\gamma}(\boldsymbol{\sigma})\right] \\
& =\gamma \mathbb{E}[H B(\boldsymbol{\sigma})]+(1-\gamma) \mathbb{E}[S B(\boldsymbol{\sigma})] \\
& =\gamma \int_{0}^{1} 1-F_{H B(\boldsymbol{\sigma})}(x) d x+(1-\gamma) \int_{0}^{1} 1-F_{S B(\boldsymbol{\sigma})}(x) d x \\
& =\gamma \int_{0}^{1} 1-F_{H B(\boldsymbol{\sigma})}(x) d x+(1-\gamma) \int_{0}^{1} 1-F_{H B_{-1}(\boldsymbol{\sigma})}(x)-F_{H B_{-2}(\boldsymbol{\sigma})}(x)+F_{H B(\boldsymbol{\sigma})}(x) d x \\
& =(2 \gamma-1) \int_{0}^{1} 1-F_{H B(\boldsymbol{\sigma})}(x) d x+(1-\gamma) \sum_{i=1}^{2} \int_{0}^{1} 1-F_{H B_{-i}(\boldsymbol{\sigma})}(x) d x \tag{5.14}
\end{align*}
$$

Using the two bounds in (5.11) we can lower bound the two integrals in the summation by

$$
\begin{gathered}
\int_{0}^{1} 1-F_{H B_{-1}(\boldsymbol{\sigma})}(x) d x \geq \int_{0}^{1-\alpha} 1-\frac{\alpha}{1-x} d x=1-\alpha+\alpha \ln (\alpha) \\
\int_{0}^{1} 1-F_{H B_{-2}(\boldsymbol{\sigma})}(x) d x \geq \int_{0}^{v-\beta} 1-\frac{\beta}{v-x} d x=v-\beta+\beta \ln (\beta / v)
\end{gathered}
$$

Since $\gamma \leq 1 / 2$, we have $(2 \gamma-1) \leq 0$ in (5.14). To lower bound the social welfare,
we should therefore upper bound the expected highest bid. For this, note that due to the fact that players cannot overbid, player 2 never bids higher than $v$. Therefore, since $\gamma>0$, any bid of player 1 that is (strictly) above $v$ is (strictly) dominated by bidding $v$ instead. (Formally, player 1 should bid $v+\epsilon$ for any $\epsilon>0$. Since $\epsilon$ can be an arbitrarily small number, we ignore it in the remainder of the proof for notational convenience.) Hence, it is clear that $\mathbb{E}[H B(\boldsymbol{\sigma})] \leq v$. From this, it also follows that $\alpha \geq 1-v$, because bidding $v$ will yield a utility of at least $1-v$ for player 1 . Using this, and again lower bounding the two rightmost integrals of (5.14) using (5.11), we get

$$
\begin{aligned}
& \gamma \mathbb{E}[H B(\boldsymbol{\sigma})]+(1-\gamma) \mathbb{E}[S B(\boldsymbol{\sigma})] \\
& \geq(2 \gamma-1) v+(1-\gamma)(1-\alpha+\alpha \ln (\alpha)+v-\beta+\beta \ln (\beta / v))
\end{aligned}
$$

Substituting into (5.10) yields

$$
\begin{align*}
\mathbb{E}[\mathrm{SW}(\boldsymbol{\sigma})] & \geq \alpha+\beta+(2 \gamma-1) v+(1-\gamma)(1-\alpha+\alpha \ln (\alpha)+v-\beta+\beta \ln (\beta / v)) \\
& =\gamma(\alpha+\beta+v)+(1-\gamma)(1+\alpha \ln (\alpha)+\beta \ln (\beta)-\beta \ln (v)) \tag{5.15}
\end{align*}
$$

The derivative of this bound with respect to $\beta$ equals

$$
\gamma+(1-\gamma)(1+\ln (\beta)-\ln (v))=1+(1-\gamma) \ln (\beta / v)
$$

Note that this derivative is equal to zero for $\beta=v e^{-1 /(1-\gamma)}$, and that it is positive for greater $\beta$ and negative for smaller $\beta$. Therefore, the bound attains its minimum at $\beta=v e^{-1 /(1-\gamma)}$. Substituting this $\beta$ in (5.15) yields

$$
\begin{align*}
\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})] & \geq \gamma\left(\alpha+\left(1+e^{-\frac{1}{1-\gamma}}\right) v\right)+(1-\gamma)\left(1+\alpha \ln (\alpha)+v e^{-\frac{1}{1-\gamma}} \ln \left(e^{-\frac{1}{1-\gamma}}\right)\right. \\
& =\gamma\left(\alpha+\left(1+e^{-\frac{1}{1-\gamma}}\right) v\right)+(1-\gamma)(1+\alpha \ln (\alpha))-v e^{-\frac{1}{1-\gamma}} \tag{5.16}
\end{align*}
$$

Next, we take the derivative of (5.16) with respect to $v$, which gives

$$
\gamma\left(1+e^{-\frac{1}{1-\gamma}}\right)-e^{-\frac{1}{1-\gamma}}=\gamma-(1-\gamma) e^{-\frac{1}{1-\gamma}}
$$

This derivative is positive for all $\gamma>0.21781 \ldots$, so for all $\gamma \in(0.21781 \ldots, 1 / 2]$, we minimize the upper bound by setting $v$ to its lowest admissible value, being $v=1-\alpha$.

Substituting the optimal parameter settings $\beta=v e^{-1 /(1-\gamma)}=(1-\alpha) e^{-1 /(1-\gamma)}$ gives the following social welfare bound

$$
\begin{align*}
\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})] & \geq \gamma\left(\alpha+(1-\alpha)\left(1+e^{-\frac{1}{1-\gamma}}\right)\right)+(1-\gamma)(1+\alpha \ln (\alpha))-(1-\alpha) e^{-\frac{1}{1-\gamma}} \\
& =1-(1-\gamma)(1-\alpha) e^{-\frac{1}{1-\gamma}}+(1-\gamma) \alpha \ln (\alpha), \tag{5.17}
\end{align*}
$$

as a function of $\alpha$ only, which we optimize by setting its derivative with respect to $\alpha$ equal to zero. This yields

$$
\begin{aligned}
(1-\gamma) e^{-\frac{1}{1-\gamma}}+(1-\gamma)(1+\ln (\alpha))=0 & \Longleftrightarrow \ln (\alpha)=-1-e^{-\frac{1}{1-\gamma}} \\
& \Longleftrightarrow \alpha=e^{-1-e^{-1 /(1-\gamma)}}
\end{aligned}
$$

To facilitate the simplification of the formula of the final bound, we first substitute only $\ln (\alpha)$ in (5.17), after which $\alpha$ itself is substituted in the final step. We get

$$
\begin{align*}
\mathbb{E}[\operatorname{SW}(\boldsymbol{\sigma})] & \geq 1-(1-\gamma) e^{-\frac{1}{1-\gamma}}+(1-\gamma) \alpha e^{-\frac{1}{1-\gamma}}+(1-\gamma) \alpha\left(-1-e^{-\frac{1}{1-\gamma}}\right) \\
& =1-(1-\gamma) e^{-\frac{1}{1-\gamma}}-(1-\gamma) \alpha \\
& =1-(1-\gamma)\left(e^{-1 /(1-\gamma)}+e^{-1-e^{-1 /(1-\gamma)}}\right) \tag{5.18}
\end{align*}
$$

We divide 1 by (5.18) to get the upper bound on the price of anarchy presented as the orange graph in Figure 5.4.

A similar proof template, combined with some additional numerical analysis, can be used to also derive an upper bound on the coarse correlated price of anarchy for $\gamma \in\left[\frac{1}{2}, 1\right]$. The key change in the derivation is that in this case $(2 \gamma-1) \geq 1$, so we derive a lower bound instead for the leftmost integral of (5.14). A more detailed derivation is given in Van Beek et al. (2022).

In this way, we find the third and final upper bound, represented by the blue graph in Figure 5.4. Note that for $\gamma=1$ this bound coincides with the (tight) bound of approximately 1.229 in Feldman et al. (2016).


Figure 5.5 Overview of our upper bounds on the POA ( $y$ axis) for $\gamma$-FPA and $\gamma$-HYA, respectively, as a function of $\gamma(x-$ axis). (a) CCE-POA for multi-unit $\gamma$-FPA with overbidding (Theorem 5.4.4). (b) CCE-POA for multi-unit $\gamma$-FPA without overbidding (Theorems 5.4.4 \& 5.5.3). (c) CCE-POA for single-item $\gamma$-HYA without overbidding (Theorems 5.4.4, 5.5.3 \& 5.6.1). (d) CCE-POA for single-item $\gamma$-HYA without overbidding and two bidders (Theorems 5.6.1 \& 5.6.2).

### 5.7 Discussion

In this section, we discuss and reflect on the main results of this chapter. The main price of anarchy bounds we obtain are summarized in Figure 5.5. First, if the bidders can overbid we obtain a tight bound on the CCE-POA over the entire range of $\gamma \in(0,1]$ (Figure 5.5(a)). This bound shows that the POA increases from a small constant $e /(e-1)$ to infinity as $\gamma$ decreases from 1 to 0 . Put differently, for the type of
corruption considered in this paper, the negative impact on worst-case social welfare is larger for lower values of $\gamma$. Thinking about $\gamma$-HYA, we feel that this makes sense intuitively: as $\gamma$ approaches 0 , the auctioneer only withholds a small fraction of the surplus and the bidders are thus incentivized to exploit the corruption (as it comes at a low cost). In contrast, as $\gamma$ approaches 1 , the auctioneer charges a significant fraction of the surplus and while the bidders still have good reasons to join the corruption (as formalized in Proposition 5.3.1), they exploit it less drastically as it comes at a large cost.

Where we completely settle the CCE-POA of $\gamma$-FPA for both the single-item and multi-unit auction setting, a more fine-grained landscape of the price of anarchy emerges under the no-overbidding assumption. This is a standard assumption that often needs to be made to derive meaningful bounds on the POA. For example, it is well-known that the PNE-POA of the second-price single-item auction is unbounded if the bidders can overbid. On the other hand, it is one if bidders cannot overbid. In the second-price single-item auction, the no-overbidding assumption can also be motivated 'endogenously', since overbidding is a dominated strategy for each bidder. However, this does not hold in second-price multi-unit auctions.

In general, the impact that the no-overbidding assumption has on the price of anarchy is not well-understood. This aspect also relates to the price of undominated anarchy studied by Feldman et al. (2016). They prove a clear separation for the POA in single-item first-price auctions, depending on whether this assumption is made. Specifically, the CCE-POA increases from 1.229 (without overbidding) to $e /(e-1)$ (with overbidding). A similar separation holds for the multi-unit second-price auction setting, where the PNE-POA is $e /(e-1)$ (without overbidding, Markakis and Telelis (2015)) and 2.1885 (with overbidding, Birmpas et al. (2019)). Our bounds also contribute to this line of research by revealing that there is a substantial difference in the POA depending on whether or not bidders can overbid; e.g., compare the bounds depicted in (a) and (b) (multi-unit setting), or (a) and (c) (single-item setting) in Figure 5.5.

Looking at our proof techniques on a higher level, our upper bounds for $\gamma$-FPA are based on an adapted smoothness notion which relates directly to the highest marginal winning bids (i.e., first-price payments). In particular, our smoothness argument does not exploit the second-price payments of $\gamma$-HYA at all. As it turns out, this allows
us to derive tight bounds for $\gamma$-HYA and, more generally, for $\gamma$-FPA when bidders can overbid. Thus, our results reveal that the (approximate) first-price payments are the determining component of such composed payment schemes. In contrast, in single-item auctions when overbidding is not allowed, it becomes crucial to exploit the second-price payments of $\gamma$-HYA to obtain improved bounds. In a two-player setting, further improvements (as reflected by Figure 5.5(d)) can be achieved by directly using the cumulative distribution functions of equilibrium bids to derive these bounds. This triggers some interesting questions for future research regarding proof techniques to derive price of anarchy bounds.

Further, we remark that although we focus on the complete information setting in this chapter, the price of anarchy can also be studied in the incomplete information setting as introduced by Harsanyi (1967), where players have private valuation functions drawn from a common prior. Several of our upper bounds are based on an adapted smoothness approach for multi-unit auctions which can be extended to this incomplete information setting and Bayes-Nash equilibria (Roughgarden, 2015b). More specifically, these extensions could be proven along similar lines as in De Keijzer et al. (2013), where smoothness is used to bound the Bayes-Nash POA of (standard) multi-unit auctions.

On a more conceptual level, in this chapter we considered a basic bid rigging model where the auctioneer colludes with the winning bidders only. In this basic model, all winning bidders lower their bids to the highest losing bid, which might seem somewhat unrealistic in the multi-unit auction setting. However, note that $\gamma$-HYA can also incorporate non-uniform bid rigging, as illustrated in Example 5.7.1. Further, $\gamma$-FPA captures several additional corruption settings. Still, it would be interesting to study the price of anarchy of more complex bid rigging models. For example, the model introduced in Lengwiler and Wolfstetter (2010) (ideally generalized to the multi-unit auction setting) might be a natural next step.

## Example 5.7.1 [Non-uniform bid rigging in $\gamma$-HYA]

As before, we consider auctions in which the bidders submit their bid vectors $\boldsymbol{b}=$ $\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right)$ to the auctioneer who runs a first-price multi-unit auction. The auctioneer then approaches each winning bidder $i \in N$ individually with the offer that they can lower their $x_{i}(\boldsymbol{b})$ winning bids. However, in contrast to the basic model, the auctioneer and bidder $i$ agree to 'camouflage' their bid rigging by bidding the
highest losing bid $\bar{p}(\boldsymbol{b})$ plus a fraction $\alpha \in[0,1]$ of the surplus $b_{i}(j)-\bar{p}(\boldsymbol{b})$ for each $j \in\left\{1, \ldots, x_{i}(\boldsymbol{b})\right\}$. Note that this maintains the relative order among the winning bids and the magic number cheating becomes less obvious as the winning bids fluctuate more. The remaining surplus of $(1-\alpha)\left(b_{i}(j)-\bar{p}(\boldsymbol{b})\right)$ is then split, where the auctioneer withholds a fraction of $\beta \in[0,1]$. As before, bidder $i$ can either reject or accept the offer. But, also here, it is not hard to see that accepting the offer is a dominant strategy. The total payment of a winning bidder $i$ is then

$$
\begin{gathered}
p_{i}^{(\alpha, \beta)}(\boldsymbol{b})=\sum_{j=1}^{x_{i}(\boldsymbol{b})}\left(\bar{p}(\boldsymbol{b})+\alpha\left(b_{i}(j)-\bar{p}(\boldsymbol{b})\right)\right)+f_{i}^{(\alpha, \beta)}(\boldsymbol{b}), \quad \text { where } \\
f_{i}^{(\alpha, \beta)}(\boldsymbol{b})=\beta \sum_{j=1}^{x_{i}(\boldsymbol{b})}(1-\alpha)\left(b_{i}(j)-\bar{p}(\boldsymbol{b})\right)
\end{gathered}
$$

After simplifying, we obtain

$$
p_{i}^{(\alpha, \beta)}(\boldsymbol{b})=(\alpha+\beta(1-\alpha)) \sum_{j=1}^{x_{i}(\boldsymbol{b})} b_{i}(j)+(1-\alpha-\beta(1-\alpha)) x_{i}(\boldsymbol{b}) \bar{p}(\boldsymbol{b}) .
$$

If we define $\gamma=\alpha+\beta-\alpha \beta$, the above payments $p_{i}^{(\alpha, \beta)}$ are equivalent to $p_{i}^{\gamma}$ as defined in (5.1). Note also that this mapping satisfies $\gamma \in[0,1]$ for every $\alpha, \beta \in$ $[0,1]$. Put differently, given $\alpha, \beta \in[0,1]$, the price of anarchy of the above nonuniform bid rigging scheme is determined by the price of anarchy of $\gamma$-HYA with $\gamma=\alpha+\beta-\alpha \beta$.

# 6 <br> <br> Entangled equilibria for <br> <br> Entangled equilibria for bimatrix games 

 bimatrix games}

### 6.1 Introduction

This chapter introduces and analyzes a refinement of Nash equilbria (Nash, 1951). While the notion of Nash equilibria for strategic games is the most prevalent solution concept in non-cooperative game theory, it is not without drawbacks. The set of Nash equilibria for a strategic game can contain many outcomes, some of which are counterintuitive. Further, a Nash equilibrium need not be 'robust', in the sense that it may no longer be a Nash equilibrium after minor perturbations in the data of the game. To address this issue, Selten (1975) proposes the notion of perfectness of equilibria. This equilibrium notion, also referred to as trembling hand perfect equilibrium, is based on a thought experiment that takes into account the possibility that players unintentionally play the 'wrong' strategy (not necessarily corresponding to a strategy played in a Nash equilibrium) with small, but positive probability due to a 'slip of the hand'. As a consequence of these mistakes, each strategy is played with positive probability. This refinement of Nash equilibria was followed by the notion of properness (Myerson, 1978), strict perfectness (Okada, 1984), that of stable sets (Kohlberg and Mertens, 1986), and many others (see Van Damme (1991) and Govindan and Wilson (2008) for an overview).

The fact that a different strategy is played with small probability can alternatively be interpreted as a consequence of blocked actions, rather than mistakes in the execution of actions. With this interpretation, Kleppe et al. (2012) introduce fall back equilibria, based on a thought experiment in which players strategically choose
a back-up action in case their 'primary' action is blocked.
The majority of existing Nash equilibrium refinements are based on a thought experiment which imposes a certain 'imperfection', due to mistakes or blocked actions, on the choices or payoffs of individual players.

The refinement proposed in this chapter deviates from the existing refinements by considering a thought experiment in which the imperfections occur on a 'system' level, instead of those corresponding (directly) to individual players. Imperfections are interpreted as the blocking of actions. However, if an imperfection occurs, the chosen actions are blocked for all players simultaneously, rather than for individual players. The idea behind this is that after players submit their strategies, some entity converts these strategies into actions leading to payoffs. In this new thought experiment, with small probability, this entity makes an error that blocks the chosen actions instead of implementing them, and chooses a random combination of the remaining actions. Put differently, either the chosen actions are executed for all players, or no player actually plays their chosen action. In this way, there is an entanglement in the errors. We therefore refer to an equilibrium based on this thought experiment as an entangled equilibrium. We emphasize that the aim of this explorative chapter is not prescriptive but descriptive, in the sense that it investigates the workings and computational aspects of entangled equilibria rather than their quality. In this context, we note that there is no direct relationship between entangled equilibria on one hand and perfect or fall back equilibria on the other hand.

This thought experiment considers a type of entanglement that is not related to quantum entanglement. Nash equilibrium refinements have also been considered in the context of quantum games (for example, Pakuła (2008) analyzes perfect equilibria in quantum games). The entangled equilibrium notion we consider in this chapter allows for the analysis of entanglement between (actions of) players without relying on quantum game theory.

In this chapter, we focus on mixed extensions of two-person finite strategic games, a class of strategic games known as bimatrix games. Given such a bimatrix game, we define a perturbed game corresponding to the thought experiment described above and then define an entangled equilibrium as the limit of a sequence of Nash equilibria for the perturbed games, when these perturbations tend to zero. We show that the set of entangled equilibria is a non-empty subset of the set of Nash equilibria.

Further, an important part of this chapter is dedicated to the discussion of a geometric-combinatorial approach to determine all entangled equilibria of $2 \times n$ bimatrix games. This approach is based on the approach of Borm (1992) to find perfect and proper equilibria in $2 \times n$ bimatrix games. Importantly, solving a $2 \times n$ bimatrix game for entangled equilibria requires relatively little extra work compared to finding Nash equilibria for the bimatrix game.

The structure of this chapter is as follows. Section 6.2 discusses the preliminaries related to bimatrix games required to read the remaining sections. Section 6.3 provides the formal definition of entangled equilibria and discusses their existence and relation to Nash equilibria. Finally, Section 6.4 treats the geometric-combinatorial approach to derive all entangled equilibria for $2 \times n$ bimatrix games. Section 6.5 concludes.

### 6.2 Preliminaries

A bimatrix game is the mixed extension of a two-person finite strategic game, characterized by a pair $(A, B)$ of real-valued matrices of size $m \times n$. The two players are referred to as 1 and 2. Player 1 chooses a row, with index set $M=\{1, \ldots, m\}$, and player 2 chooses a column, indexed by $N=\{1, \ldots, n\}$. In the mixed extension, the players can choose mixed (or randomized) strategies, from sets denoted by $\triangle_{m}$ for player 1 and $\triangle_{n}$ for player 2 , and formally defined by

$$
\triangle_{m}=\left\{p \in \mathbb{R}^{m} \mid p \geq 0, \sum_{i \in M} p_{i}=1\right\} \quad \text { and } \quad \triangle_{n}=\left\{q \in \mathbb{R}^{n} \mid q \geq 0, \sum_{j \in N} q_{j}=1\right\}
$$

In the corresponding finite strategic game, the strategies are denoted by $e_{1}, \ldots, e_{m}$ and $f_{1}, \ldots, f_{n}$ for players 1 and 2 , respectively. For example, this means that in the mixed extension, $e_{i}$ with $i \in M$ corresponds to the unit vector in $\triangle_{m}$ for which $p_{i}=1$ and $p_{k}=0$ for all $k \in M \backslash\{i\}$. More generally, for $p \in \triangle_{m}$ and $i \in M, p_{i}$ is interpreted as the probability that player 1 selects $e_{i}$. The strategies $e_{1}, \ldots, e_{m}$ and $f_{1}, \ldots, f_{n}$ are referred to as pure strategies in the mixed extension.

The matrices $A$ and $B$ represent the payoffs of players 1 and 2 , respectively. Hence, if player 1 selects $e_{i}$ and player 2 selects $f_{j}$, with $i \in M$ and $j \in N$, then player 1 receives $A_{i j}$ and player 2 receives $B_{i j}$. In the mixed extension, we consider expected
payoffs, given by

$$
p^{\top} A q=\sum_{i \in M} \sum_{j \in N} p_{i} A_{i j} q_{j} \quad \text { and } \quad p^{\top} B q=\sum_{i \in M} \sum_{j \in N} p_{i} B_{i j} q_{j}
$$

for players 1 and 2, respectively, for any $(p, q) \in \triangle_{m} \times \triangle_{n}$. Hereafter, we omit the transpose sign, and simply denote the expected payoffs by $p A q$ and $p B q$.

Let $(A, B)$ be a $m \times n$ bimatrix game. Then, a strategy combination $(p, q) \in \triangle_{m} \times \triangle_{n}$ is a Nash equilibrium of $(A, B)$ if

$$
p A q \geq p^{\prime} A q \quad \text { and } \quad p B q \geq p B q^{\prime}
$$

for all $p^{\prime} \in \triangle_{m}$ and all $q^{\prime} \in \triangle_{n}$. In fact, this is equivalent to

$$
p A q \geq e_{i} A q \quad \text { and } \quad p B q \geq p B f_{j}
$$

for all $i \in M$ and all $j \in N$. The set of Nash equilibria of $(A, B)$ is denoted by $E(A, B)$. The existence of a (mixed) Nash equilibrium is guaranteed in bimatrix games (Nash, 1951).

Let $p \in \triangle_{m}$ and let $q \in \triangle_{n}$. The so-called pure best reply correspondence of player 1 to $q$ is defined by

$$
P B_{1}(q, A)=\left\{e_{i} \mid e_{i} A q \geq e_{k} A q \text { for all } k \in M\right\}
$$

and the pure best reply correspondence of player 2 to $p$ is

$$
P B_{2}(p, B)=\left\{f_{j} \mid p B f_{j} \geq p B f_{l} \text { for all } l \in N\right\}
$$

Similarly, the sets of best replies of player 1 (to $q$ ) and player 2 (to $p$ ) are given by

$$
B_{1}(q, A)=\left\{p \in \triangle_{m} \mid p A q \geq p^{\prime} A q \text { for all } p^{\prime} \in \triangle_{m}\right\}
$$

and

$$
B_{2}(p, B)=\left\{q \in \triangle_{n} \mid p B q \geq p B q^{\prime} \text { for all } q^{\prime} \in \triangle_{n}\right\},
$$

respectively. Note that

$$
B_{1}(q, A)=\operatorname{Conv}\left\{P B_{1}(q, A)\right\}
$$

and

$$
B_{2}(p, B)=\operatorname{Conv}\left\{P B_{2}(p, B)\right\}
$$

Clearly, a strategy combination $(p, q) \in \triangle_{m} \times \triangle_{n}$ is a Nash equilibrium if and only if $p \in B_{1}(q, A)$ and $q \in B_{2}(p, B)$.

### 6.3 Entangled equilibria for bimatrix games

In this section, we formally define the notion of an entangled equilibrium for a bimatrix game $(A, B)$, on the basis of perturbed games corresponding to $(A, B)$. Then, we show that the set of entangled equilibria is a subset of the set of Nash equilibria, and that the existence of an entangled equilibrium is guaranteed in any bimatrix game.

Let $(A, B)$ be an $m \times n$ bimatrix game. We first formally define the perturbed game $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ corresponding to $(A, B)$, for some small $\varepsilon>0$. In the perturbed game, the entries of $A^{\varepsilon}$ and $B^{\varepsilon}$ are based on the thought experiment that, with probability $\varepsilon$, the resulting pure strategy combination chosen by the players is blocked instead of played. If cell $(i, j)$ is blocked, then all cells in row $i$ and $j$ are blocked too. Thus, effectively, actions $i$ and $j$ are blocked. ${ }^{1}$ In Definition 6.3.1, we assume that, if a pure strategy combination is blocked, a replacing pure strategy combination is chosen randomly from those not corresponding to one of the blocked actions, with equal probability for all remaining pure strategy combinations.

Definition 6.3.1 [Perturbed game]
Let $(A, B)$ be an $m \times n$ bimatrix game and let $0<\varepsilon<1$. Then, the perturbed bimatrix game $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ is defined such that

$$
\begin{align*}
& A_{i j}^{\varepsilon}=(1-\varepsilon) A_{i j}+\frac{\varepsilon}{(m-1)(n-1)} \sum_{k \in M \backslash\{i\}} \sum_{l \in N \backslash\{j\}} A_{k l},  \tag{6.1}\\
& B_{i j}^{\varepsilon}=(1-\varepsilon) B_{i j}+\frac{\varepsilon}{(m-1)(n-1)} \sum_{k \in M \backslash\{i\}} \sum_{l \in N \backslash\{j\}} B_{k l}, \tag{6.2}
\end{align*}
$$

for any $i \in M$ and any $j \in N$.

[^4]Using this definition of perturbed games, an entangled equilibrium is defined as follows.

## Definition 6.3.2 [Entangled equilibrium]

Let $(A, B)$ be an $m \times n$ bimatrix game. A strategy combination $(p, q) \in \Delta_{m} \times \Delta_{n}$ is an entangled equilibrium of $(A, B)$ if there exists a sequence $\left\{\varepsilon_{k}\right\}_{k \in \mathbb{N}}$ converging to zero and a sequence $\left\{\left(p_{k}, q_{k}\right)\right\}_{k \in \mathbb{N}}$ converging to $(p, q)$, such that $\left(p_{k}, q_{k}\right) \in E\left(A^{\varepsilon_{k}}, B^{\varepsilon_{k}}\right)$ for all $k \in \mathbb{N}$. The set of entangled equilibria is denoted by $E E(A, B)$.

Example 6.3.1 illustrates both definitions.

## Example 6.3.1

Consider the $2 \times 2$ bimatrix game $(A, B)$ given by

$$
(A, B)={ }^{e_{1}} e_{2}\left[\begin{array}{cc}
f_{1} & f_{2} \\
1,0 & 0,0 \\
0,1 & 0,2
\end{array}\right] .
$$

The Nash equilibria of a bimatrix game can be determined graphically as the intersection points of the graphs of the best replies of players 1 and $2, B_{1}$ and $B_{2}$. For $(A, B)$, these graphs coincide: they are both given by the black lines in Figure 6.1 at $f_{2}$ from $e_{2}$ to $e_{1}$ and at $e_{1}$ from $f_{2}$ to $f_{1}$. Hence,

$$
E(A, B)=\left\{e_{1}\right\} \times \operatorname{Conv}\left\{f_{1}, f_{2}\right\} \cup \operatorname{Conv}\left\{e_{1}, e_{2}\right\} \times\left\{f_{2}\right\}
$$

The perturbed game $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ for some (small) $\varepsilon>0$ is determined using Definition 6.3.1. For example, consider $\left(e_{1}, f_{1}\right)$. Then, with probability $\varepsilon$ this strategy combination is blocked, and a strategy combination is chosen from the remaining strategies, being only $e_{2}$ and $f_{2}$ here. Hence, $A_{11}^{\varepsilon}=(1-\varepsilon) \cdot 1+\varepsilon \cdot 0=1-\varepsilon$ and $B_{11}^{\varepsilon}=(1-\varepsilon) \cdot 0+\varepsilon \cdot 2=2 \varepsilon$. The remaining payoffs can be computed in a similar manner. This yields

$$
\left(A^{\varepsilon}, B^{\varepsilon}\right)=\begin{gathered}
e_{1} \\
e_{2}
\end{gathered}\left[\begin{array}{cc}
f_{2} \\
1-\varepsilon, 2 \varepsilon & 0, \varepsilon \\
0,1-\varepsilon & \varepsilon, 2-2 \varepsilon
\end{array}\right]
$$



Figure 6.1 Graphical representation of best reply correspondences in $(A, B)$ (thick black lines) and $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ (green dotted line for player 1, green dashed line for player 2) of Example 6.3.1

For any $q \in \Delta_{2}$, substituting $q_{2}=1-q_{1}$ yields

$$
\begin{aligned}
& e_{1} A^{\varepsilon} q=(1-\varepsilon) q_{1}=q_{1}-\varepsilon q_{1}, \\
& e_{2} A^{\varepsilon} q=\varepsilon\left(1-q_{1}\right)=\varepsilon-\varepsilon q_{1} .
\end{aligned}
$$

Hence,

$$
B_{1}\left(q, A^{\varepsilon}\right)= \begin{cases}\left\{e_{2}\right\} & \text { if } 0 \leq q_{1}<\varepsilon \\ \Delta_{2} & \text { if } q_{1}=\varepsilon \\ \left\{e_{1}\right\} & \text { if } \varepsilon<q_{1} \leq 1\end{cases}
$$

Similarly, for any $p \in \Delta_{2}$ we substitute $p_{2}=1-p_{1}$ and find

$$
\begin{aligned}
& p B^{\varepsilon} f_{1}=2 \varepsilon p_{1}+(1-\varepsilon)\left(1-p_{1}\right)=1-p_{1}-\varepsilon+3 \varepsilon p_{1}, \\
& p B^{\varepsilon} f_{2}=\varepsilon p_{1}+(2-2 \varepsilon)\left(1-p_{1}\right)=2-2 p_{1}-2 \varepsilon+3 \varepsilon p_{1} .
\end{aligned}
$$

Hence,

$$
B_{2}\left(p, B^{\varepsilon}\right)= \begin{cases}\left\{f_{2}\right\} & \text { if } 0 \leq p_{1}<1-\varepsilon \\ \Delta_{2} & \text { if } p_{1}=1-\varepsilon \\ \left\{f_{1}\right\} & \text { if } 1-\varepsilon<p_{1} \leq 1\end{cases}
$$

We also represent these best replies graphically in Figure 6.1, using a dotted line for
player 1 and a dashed line for player 2. Clearly,

$$
E\left(A^{\varepsilon}, B^{\varepsilon}\right)=\left\{\left(e_{1}, f_{1}\right),\left((1-\varepsilon) e_{1}+\varepsilon e_{2}, \varepsilon f_{1}+(1-\varepsilon) f_{2}\right),\left(e_{2}, f_{2}\right)\right\}
$$

The interior intersection point converges to $\left(e_{1}, f_{2}\right)$. By Definition 6.3.2, we therefore have

$$
E E(A, B)=\left\{\left(e_{1}, f_{1}\right),\left(e_{1}, f_{2}\right),\left(e_{2}, f_{2}\right)\right\}
$$

In Example 6.3.1, we found that the set of entangled equilbria is a non-empty subset of the set of Nash equilibria. This holds for any $m \times n$ bimatrix game.

## Theorem 6.3.3

Let $(A, B)$ be an $m \times n$ bimatrix game. Then,

$$
E E(A, B) \subseteq E(A, B)
$$

## Proof.

Let $(p, q) \in E E(A, B)$. Then, there exists a sequence $\left\{\varepsilon_{k}\right\}_{k \in \mathbb{N}}$ converging to zero and a sequence $\left\{\left(p_{k}, q_{k}\right)\right\}_{k \in \mathbb{N}}$ converging to $(p, q)$ such that $\left(p_{k}, q_{k}\right) \in E\left(A^{\varepsilon_{k}}, B^{\varepsilon_{k}}\right)$ for all $k \in \mathbb{N}$. Put differently, we have $p_{k} A^{\varepsilon_{k}} q_{k} \geq p^{\prime} A^{\varepsilon_{k}} q_{k}$ and $p_{k} B^{\varepsilon_{k}} q_{k} \geq p_{k} B^{\varepsilon_{k}} q^{\prime}$ for all $p^{\prime} \in \Delta_{m}, q^{\prime} \in \Delta_{n}$ and all $k \in \mathbb{N}$. Using the convergence of $\left\{\varepsilon_{k}\right\}_{k \in \mathbb{N}},\left\{p_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{q_{k}\right\}_{k \in \mathbb{N}}$, this implies

$$
p A q=\lim _{k \rightarrow \infty} p_{k} A^{\varepsilon_{k}} q_{k} \geq \lim _{k \rightarrow \infty} p^{\prime} A^{\varepsilon_{k}} q_{k}=p^{\prime} A q
$$

for all $p^{\prime} \in \Delta_{m}$. Similarly, we find

$$
p B q=\lim _{k \rightarrow \infty} p_{k} B^{\varepsilon_{k}} q_{k} \geq \lim _{k \rightarrow \infty} p_{k} B^{\varepsilon_{k}} q^{\prime}=p B q^{\prime}
$$

for all $q^{\prime} \in \Delta_{n}$. This shows that $(p, q) \in E(A, B)$.

## Theorem 6.3.4

Let $(A, B)$ be an $m \times n$ bimatrix game. Then, $E E(A, B) \neq \emptyset$.

## Proof.

Let $\left\{\varepsilon_{k}\right\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to zero. Since any bimatrix game has a Nash equilibrium, there exists a sequence $\left\{\left(p_{k}, q_{k}\right)\right\}_{k \in \mathbb{N}}$ such that that $\left(p_{k}, q_{k}\right) \in E\left(A^{\varepsilon_{k}}, B^{\varepsilon_{k}}\right)$ for all $k \in \mathbb{N}$. Further, note that $\Delta_{m} \times \Delta_{n}$ is compact,
meaning this sequence has a subsequence that converges to, say, $(p, q) \in \Delta_{m} \times \Delta_{n}$ by the Bolzano-Weierstrass theorem. By definition, this implies $(p, q) \in E E(A, B)$.

### 6.4 Entangled equilibria for $2 \times n$ bimatrix games

In this section, we discuss a method to solve $2 \times n$ bimatrix games using a geometriccombinatorial approach. We follow the notation and methodology developed in Borm (1992), adapted to our setting. Importantly, solving a $2 \times n$ bimatrix game for entangled equilibria requires relatively little extra work compared to finding Nash equilibria of the bimatrix game. In particular, explicit computation of (equilibria in) the perturbed games is not required.

Let $(A, B)$ be a $2 \times n$ bimatrix game and let $j \in N$. We assign a label to each pure strategy $f_{j}$ of player 2 , indicating whether the set of pure best replies of player 1 to $f_{j}$ is $\left\{e_{1}\right\}$, denoted by label [1], $\left\{e_{2}\right\}$, with label [2], or $\left\{e_{1}, e_{2}\right\}$, with label [12]. The set of all pure strategies of player 2 with label [1] is denoted by $J([1])$. Formally, we have $J([1])=\left\{f_{j} \mid P B_{1}\left(f_{j}, A\right)=\left\{e_{1}\right\}\right\} . J([2])$ and $J([12])$ are defined similarly. Next, let $p \in \triangle_{2}$. Then, $P B_{2}(p,[1]), P B_{2}(p,[2])$, and $P B_{2}(p,[12])$ denote the sets of pure best replies of player 2 to $p$ with the corresponding label.

Throughout this section, we consistently use a superscript $\varepsilon$ to indicate that we consider definitions related to the perturbed game $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ corresponding to $(A, B)$. For example, we analogously define $J^{\varepsilon}([1])=\left\{f_{j} \mid P B_{1}\left(f_{j}, A^{\varepsilon}\right)=\left\{e_{1}\right\}\right\}$, and similarly $J^{\varepsilon}([2])$ and $J^{\varepsilon}([12])$ for the sets of pure strategies of player 2 with labels [2] and [12], respectively, in the perturbed game. Hereafter, we use more concise notation to indicate whether we consider (pure) best replies corresponding to the perturbed game or the original game, using $P B_{1}^{\varepsilon}(q)=P B_{1}\left(q, A^{\varepsilon}\right)$ and $P B_{1}(q)=P B_{1}(q, A)$ for any $q \in \triangle_{n}$. For example, $P B_{1}^{\varepsilon}\left(f_{j}\right)=\left\{e_{i} \mid e_{i} A^{\varepsilon} f_{j} \geq e_{k} A^{\varepsilon} f_{j}\right.$ for all $\left.k \in\{1,2\}\right\}$. Similarly, we let, e.g., $P B_{2}^{\varepsilon}(p,[1])=\left\{f_{j} \in J^{\varepsilon}([1]) \mid p B^{\varepsilon} f_{j} \geq p B^{\varepsilon} f_{l}\right.$ for all $\left.l \in N\right\}$ denote the set of pure best replies of player 2 to $p$ with label [1] in the perturbed game.

The geometric part of the method to find all Nash equilibria starts by drawing the $n$ lines $p \mapsto p B f_{j}, j \in N$, representing all possible payoffs to player 2 corresponding to the pure strategy $f_{j}$ as a function of $p$. Note that the piecewise linear maximum function fully describes the best reply correspondence of player 2 . The label assigned to each pure strategy, providing partial information about the best reply correspondence
of player 1, is also represented graphically above the corresponding line.
The piecewise linear maximum function consists of $t$ line segments for some $t \in N$, with $t+1$ extreme points. The strategies of player 1 corresponding to these extreme points are denoted by $p_{0}, p_{1}, \ldots, p_{t}$ and ordered 'from left to right' such that $p_{0}=e_{2}$ and $p_{t}=e_{1}$. Further, for each $k \in\{1, \ldots, t\}, I_{k}$ and $\bar{I}_{k}$ denote the open and closed interval of strategies between $p_{k-1}$ and $p_{k}$, respectively. For all $p^{\prime}, p^{\prime \prime} \in I_{k}$, we have $P B_{2}\left(p^{\prime}\right)=P B_{2}\left(p^{\prime \prime}\right)$, meaning $P B_{2}\left(I_{k}\right), P B_{2}\left(I_{k},[1]\right), P B_{2}\left(I_{k},[2]\right)$, and $P B_{2}\left(I_{k},[12]\right)$ can unambiguously be defined as well.

Towards the set of Nash equilibria, let $S(p)=\left\{q \in \triangle_{n} \mid(p, q) \in E(A, B)\right\}$ denote the set of solutions to some $p \in \triangle_{2}$. Note that $S(p)$ may be empty for specific $p$, and that $S(p)$ is a polytope, since it is a bounded set determined by a finite system of linear inequalities. The extreme points of $S(p)$ are the sets of pure solutions and coordination solutions, denoted by $P S(p)$ and $C S(p)$, respectively. The set of pure solutions is formally defined by $P S(p)=\left\{f_{j} \mid\left(p, f_{j}\right) \in E(A, B)\right\}$, and it is straightforwardly determined as

$$
P S(p)= \begin{cases}P B_{2}(p,[12]) & \text { if } p \in \triangle_{2} \backslash\left\{e_{1}, e_{2}\right\} \\ P B_{2}(p,[12]) \cup P B_{2}(p,[2]) & \text { if } p=e_{2} \\ P B_{2}(p,[12]) \cup P B_{2}(p,[1]) & \text { if } p=e_{1}\end{cases}
$$

The set of coordination solutions is formally defined by

$$
C S(p)=\left\{q(j, l) \mid f_{j} \in P B_{2}(p,[1]), f_{l} \in P B_{2}(p,[2])\right\},
$$

where for each $f_{j} \in J([1])$ and $f_{l} \in J([2])$ the strategy $q(j, l) \in \triangle_{n}$ is uniquely determined by requiring that $e_{1} A q(j, l)=e_{2} A q(j, l)$ and that $q_{k}(j, l)=0$ for all $k \in N \backslash\{j, l\}$. Then,

$$
S(p)=\operatorname{Conv}\{P S(p) \cup C S(p)\}
$$

Similar to before, $P S\left(I_{k}\right)$ and $C S\left(I_{k}\right)$, and hence $S\left(I_{k}\right)=\operatorname{Conv}\left\{P S\left(I_{k}\right) \cup C S\left(I_{k}\right)\right\}$, are well-defined for all $k \in\{1, \ldots, t\}$.

Since for all $k \in\{1, \ldots, t\}$ it holds that $S\left(I_{k}\right) \subseteq S\left(p_{k-1}\right)$ and $S\left(I_{k}\right) \subseteq S\left(p_{k}\right)$, it is generally most efficient to first determine the set of all Nash equilibria with respect to the intervals $I_{k}$ and to include the corresponding end points of the interval in the
equilibrium set. Next, the extreme points $p_{0}, p_{1}, \ldots, p_{t}$ need to be considered only if an additional pure or coordination solution arises compared to the intervals. In this way, the set of Nash equilibria can be determined in at most $2 t+1$ steps.

## Theorem 6.4.1 [Borm (1992)]

Let $(A, B)$ be a $2 \times n$ bimatrix game. The set of Nash equilibria is given by

$$
E(A, B)=\bigcup_{k=1}^{t} \bar{I}_{k} \times S\left(I_{k}\right) \cup \bigcup_{k=0}^{t}\left\{p_{k}\right\} \times S\left(p_{k}\right)
$$

In Example 6.4.1, we illustrate the methodology to find both the set of Nash equilibria and the set of entangled equilibria. For the latter, we follow the 'naive' approach of first explicitly calculating all Nash equilibria of the perturbed game, after which we find the set of entangled equilibria by letting $\varepsilon$ converge to zero. This clarifies the workings of entangled equilibria, but also shows the computations are quite intricate. After this example, we discuss how to systematically find the set of entangled equilibria more efficiently using the set of Nash equilibria of the 'original' bimatrix game.

## Example 6.4.1

Consider the $2 \times 4$ bimatrix game $(A, B)$ given by

$$
(A, B)={ }^{e_{1}} e_{2}\left[\begin{array}{cccc}
f_{1} & f_{2} & f_{3} & f_{4} \\
0,-15 & 0,0 & 0,6 & 6,6 \\
6,12 & 0,9 & 6,-6 & 0,-9
\end{array}\right]
$$

Before focusing on the set of entangled equilibria, we first illustrate the geometriccombinatorial approach to find the set of Nash equilibria, starting with a drawing of the lines $p \mapsto p B f_{j}, j \in\{1, \ldots, 4\}$, to represent the possible payoffs of player 2 corresponding to $f_{j}$ (on the vertical axis) as a function of $p$ (on the horizontal axis) in Figure 6.2. For example, the downward sloping line starting from the top left corresponds to $f_{1}$, with label [2], as indicated by [2] ${ }_{1}$. To find the set of Nash equilibria, we focus on the piecewise linear maximum function (i.e., the best reply correspondence of player 2). Note that the lines corresponding to $f_{1}$ and $f_{2}$ intersect in $p_{1}=\frac{1}{6} e_{1}+\frac{5}{6} e_{2}$, the lines corresponding to $f_{2}$ and $f_{3}$ intersect in $p_{2}=\frac{5}{7} e_{1}+\frac{2}{7} e_{2}$, and finally those corresponding to $f_{3}$ and $f_{4}$ intersect in $p_{3}=e_{1}$.


Figure 6.2 Graphical representation towards solving the $2 \times 4$ bimatrix game of Example 6.4.1

Using Theorem 6.4.1, and the fact that $q(4,3)=\frac{1}{2} f_{3}+\frac{1}{2} f_{4}$, we find

$$
\begin{aligned}
E(A, B)= & \left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup \operatorname{Conv}\left\{\frac{1}{6} e_{1}+\frac{5}{6} e_{2}, \frac{5}{7} e_{1}+\frac{2}{7} e_{2}\right\} \times\left\{f_{2}\right\} \\
& \cup\left\{e_{1}\right\} \times \operatorname{Conv}\left\{f_{4}, \frac{1}{2} f_{3}+\frac{1}{2} f_{4}\right\} .
\end{aligned}
$$

Next, we consider the set of entangled equilibria, which is determined by finding the set of Nash equilibria of the perturbed game $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ corresponding to $(A, B)$ and letting $\varepsilon$ converge to zero. Using Definition 6.3.1, we find

$$
\left(A^{\varepsilon}, B^{\varepsilon}\right)=\begin{gathered}
f_{1} \\
e_{1} \\
e_{2}
\end{gathered}\left[\begin{array}{cccc}
f_{2} & f_{3} & f_{4} \\
2 \varepsilon,-15+13 \varepsilon & 4 \varepsilon,-\varepsilon & 2 \varepsilon, 6-2 \varepsilon & 6-2 \varepsilon, 6-\varepsilon \\
6-4 \varepsilon, 12-8 \varepsilon & 2 \varepsilon, 9-10 \varepsilon & 6-4 \varepsilon,-6+3 \varepsilon & 0,-9+6 \varepsilon
\end{array}\right] .
$$

We graphically represent this perturbed game in Figure 6.3, using $\varepsilon=0.1$ for illustration purposes.

Globally speaking, the pictures in Figures 6.2 and 6.3 look alike. There are two important changes: the line corresponding to $f_{2}$ now has label [1] instead of [12], and the lines corresponding to $f_{3}$ and $f_{4}$ now intersect slightly left of $e_{1}$.


Figure 6.3 Graphical representation towards solving the perturbed $2 \times 4$ bimatrix game of Example 6.4.1

For the former, we now no longer have a pure solution $f_{2}$ on the middle interval between $p_{1}^{\varepsilon}=\frac{3+2 \varepsilon}{18-12 \varepsilon} e_{1}+\frac{15-14 \varepsilon}{18-12 \varepsilon} e_{2}$ and $p_{2}^{\varepsilon}=\frac{15-13 \varepsilon}{21-14 \varepsilon} e_{1}+\frac{6-\varepsilon}{21-14 \varepsilon} e_{2}$. Instead, since $f_{1}$ and $f_{3}$ both have label [2], combined with $f_{2}$ now having label [1], there are two coordination solutions on the boundaries of these intervals. For the coordination solution at $p_{1}^{\varepsilon}$, we have $q_{1}^{\varepsilon}(2,1)=\frac{2 \varepsilon}{6-4 \varepsilon}$ and $q_{2}^{\varepsilon}(2,1)=\frac{6-6 \varepsilon}{6-4 \varepsilon}$. At $p_{2}^{\varepsilon}$, we find $q^{\varepsilon}(2,3)=$ $\frac{6-6 \varepsilon}{6-4 \varepsilon} f_{2}+\frac{2 \varepsilon}{6-4 \varepsilon} f_{3}$.

The second prominent change is that the lines corresponding to $f_{3}$ and $f_{4}$ now intersect in $p_{3}^{\varepsilon}=\frac{3-3 \varepsilon}{3-2 \varepsilon} e_{1}+\frac{\varepsilon}{3-2 \varepsilon} e_{2}$ instead of $e_{1}$. The corresponding coordination solution is given by $q^{\varepsilon}(4,3)=\frac{3-\varepsilon}{6-4 \varepsilon} f_{3}+\frac{3-3 \varepsilon}{6-4 \varepsilon} f_{4}$.

Also, since $P B_{2}^{\varepsilon}\left(e_{1},[1]\right)=\left\{f_{4}\right\}$, we still have this pure solution at $e_{1}$. However, as $p_{3}^{\varepsilon} \neq e_{1}$, we no longer have a convex hull between the coordination solution and the pure solution. Instead, they form two separate components.

Finally, note that on the 'new' interval, between $p_{3}^{\varepsilon}$ and $e_{1}$, there are no solutions.
To ensure the ordering from left to right of the extreme points does not change, we require $0<\varepsilon<\frac{3}{4}$. For all such $\varepsilon$, we find

$$
\begin{aligned}
E\left(A^{\varepsilon}, B^{\varepsilon}\right) & =\left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup\left\{\frac{3+2 \varepsilon}{18-12 \varepsilon} e_{1}+\frac{15-14 \varepsilon}{18-12 \varepsilon} e_{2}\right\} \times\left\{\frac{2 \varepsilon}{6-4 \varepsilon} f_{1}+\frac{6-6 \varepsilon}{6-4 \varepsilon} f_{2}\right\} \\
& \cup\left\{\frac{15-13 \varepsilon}{21-14 \varepsilon} e_{1}+\frac{6-\varepsilon}{21-14 \varepsilon} e_{2}\right\} \times\left\{\frac{6-6 \varepsilon}{6-4 \varepsilon} f_{2}+\frac{2 \varepsilon}{6-4 \varepsilon} f_{3}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \cup\left\{\frac{3-3 \varepsilon}{3-2 \varepsilon} e_{1}+\frac{\varepsilon}{3-2 \varepsilon} e_{2}\right\} \times\left\{\frac{3-\varepsilon}{6-4 \varepsilon} f_{3}+\frac{3-3 \varepsilon}{6-4 \varepsilon} f_{4}\right\} \\
& \cup\left\{e_{1}\right\} \times\left\{f_{4}\right\}
\end{aligned}
$$

Letting $\varepsilon$ converge to zero, we find that the set of entangled equilibria is given by

$$
\begin{aligned}
E E(A, B) & =\left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup\left\{\frac{1}{6} e_{1}+\frac{5}{6} e_{2}\right\} \times\left\{f_{2}\right\} \\
& \cup\left\{\frac{5}{7} e_{1}+\frac{2}{7} e_{2}\right\} \times\left\{f_{2}\right\} \\
& \cup\left\{e_{1}\right\} \times\left\{\frac{1}{2} f_{3}+\frac{1}{2} f_{4}\right\} \\
& \cup\left\{e_{1}\right\} \times\left\{f_{4}\right\} .
\end{aligned}
$$

Clearly, the coordination solution $q^{\varepsilon}(4,3)$ converges to $q(4,3)$, and the extreme points $p_{1}^{\varepsilon}, p_{2}^{\varepsilon}$, and $p_{3}^{\varepsilon}$ converge to $p_{1}, p_{2}$, and $p_{3}$, respectively. Further, note that the 'new' coordination solutions $q^{\varepsilon}(2,1)$ and $q^{\varepsilon}(2,3)$ that arose due to the change in label of $f_{2}$ from [12] to [1] both converge to the original pure solution $f_{2}$.

Having built some intuition in Example 6.4.1, we now discuss how the geometriccombinatorial approach to determine the set of Nash equilibria can be adapted to systematically find the set of entangled equilibria without the cumbersome explicit computation of (equilibria in) the perturbed game. Each 'old' interior extreme point (i.e., $p_{1}, \ldots ., p_{t-1}$ ) will correspond to exactly one 'new' interior extreme point ( $p_{1}^{\varepsilon}, \ldots, p_{t-1}^{\varepsilon}$ ). Moreover, there may be at most two additional interior extreme points $e_{2}^{\varepsilon}$ and $e_{1}^{\varepsilon}$ at either side, i.e., close to $e_{2}$ and $e_{1}$, respectively. Although $e_{2}^{\varepsilon}$ and $e_{1}^{\varepsilon}$ need not be new interior extreme points, it is helpful to explicitly consider them in general. The exact correspondence between $p_{k}$ and $p_{k}^{\varepsilon}, k \in\{1, \ldots, t-1\}, e_{2}$ and $e_{2}^{\varepsilon}$, and $e_{1}$ and $e_{1}^{\varepsilon}$ is described below.

## Definition 6.4.2

Let $p=[r 1-r]^{\top} \in \triangle_{2}$ and let $0<\varepsilon<1$. Then, $p^{\varepsilon}=\left[r^{\varepsilon} 1-r^{\varepsilon}\right]^{\top}$ with

$$
\begin{equation*}
r^{\varepsilon}=\frac{(1-\varepsilon) r+\frac{\varepsilon}{n-1}(1-r)}{1-\varepsilon+\frac{\varepsilon}{n-1}} \tag{6.3}
\end{equation*}
$$

Note that $p^{\varepsilon} \in \triangle_{2} \backslash\left\{e_{1}, e_{2}\right\}$ for any $p \in \triangle_{2}$, since $0<r^{\varepsilon}<1$ for any $0 \leq r \leq 1$. Further, note that the ordering from left to right of the extreme points is maintained if $\varepsilon$ is such that $r^{\varepsilon}$ is strictly increasing in $r$, which holds for all $0<\varepsilon<\frac{n-1}{n}$.

Next, we define two subsets of the pure best reply correspondence of player 2, with respect to $e_{1}$ and $e_{2}$, in the 'original' bimatrix game. In Lemma 6.4.4, we show that these sets correspond to the pure best replies of player 2 to $e_{1}$ and $e_{2}$ in the perturbed game.

## Definition 6.4.3

Let $(A, B)$ be a $2 \times n$ bimatrix game. Then,

$$
\overline{P B_{2}}\left(e_{1}\right)=\left\{f_{j} \in P B_{2}\left(e_{1}\right) \mid B_{2 j} \leq B_{2 l} \text { for all } f_{l} \in P B_{2}\left(e_{1}\right)\right\}
$$

and

$$
\overline{P B_{2}}\left(e_{2}\right)=\left\{f_{j} \in P B_{2}\left(e_{2}\right) \mid B_{1 j} \leq B_{1 l} \text { for all } f_{l} \in P B_{2}\left(e_{2}\right)\right\}
$$

Similar to before, we denote, e.g., the corresponding set of pure best replies of player 2 to $e_{1}$ with label [1] by $\overline{P B_{2}}\left(e_{1},[1]\right)$.

To determine the set of Nash equilibria of the perturbed game, we are ultimately interested in the piecewise linear maximum function representing the best reply correspondence of player 2, i.e., the upper envelope of the line-label picture. The next lemma fully describes the structure of (the upper envelope of) the new line-label picture for the perturbed game.

## Lemma 6.4.4

Let $(A, B)$ be a $2 \times n$ bimatrix game and let $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ be the corresponding perturbed game. Then, for sufficiently small $\varepsilon$,
(i) $B f_{j}=B f_{l} \Longleftrightarrow B^{\varepsilon} f_{j}=B^{\varepsilon} f_{l}$ for all $j, l \in N$;
(ii) $P B_{2}^{\varepsilon}\left(p^{\varepsilon}\right)=P B_{2}(p)$ for all $p \in \triangle_{2}$;
(iii) $P B_{2}^{\varepsilon}\left(e_{1}\right)=\overline{P B_{2}}\left(e_{1}\right)$ and $P B_{2}^{\varepsilon}\left(e_{2}\right)=\overline{P B_{2}}\left(e_{2}\right)$.

## Proof.

(i) This statement follows directly from (6.2).
(ii) Let $p \in \triangle_{2}$. First, note that $P B_{2}^{\varepsilon}\left(p^{\varepsilon}\right) \subseteq P B_{2}(p)$. This follows directly from the fact that, for sufficiently small $\varepsilon, p B f_{j}>p B f_{k}$ implies $p^{\varepsilon} B^{\varepsilon} f_{j}>p^{\varepsilon} B^{\varepsilon} f_{k}$ for all $k \in N$.

Hence, it suffices to prove that $P B_{2}(p) \subseteq P B_{2}^{\varepsilon}\left(p^{\varepsilon}\right)$. For this, let $p=[r 1-r]^{\top} \in \triangle_{2}, p^{\varepsilon}=\left[r^{\varepsilon} 1-r^{\varepsilon}\right]^{\top} \in \triangle_{2} \backslash\left\{e_{1}, e_{2}\right\}$ as given in Definition 6.4.2, and let $j, l \in N$ such that $f_{j}, f_{l} \in P B_{2}(p)$. To prove that $f_{j}, f_{l} \in P B_{2}^{\varepsilon}\left(p^{\varepsilon}\right)$, it suffices to show that

$$
\begin{equation*}
r^{\varepsilon} B_{1 j}^{\varepsilon}+\left(1-r^{\varepsilon}\right) B_{2 j}^{\varepsilon}=r^{\varepsilon} B_{1 l}^{\varepsilon}+\left(1-r^{\varepsilon}\right) B_{2 l}^{\varepsilon} \tag{6.4}
\end{equation*}
$$

By definition of $P B_{2}(p)$, we have

$$
\begin{equation*}
r B_{1 j}+(1-r) B_{2 j}=r B_{1 l}+(1-r) B_{2 l} \tag{6.5}
\end{equation*}
$$

To prove (6.4), note that

$$
\begin{aligned}
& r^{\varepsilon}\left(B_{1 j}^{\varepsilon}-B_{1 l}^{\varepsilon}\right)+\left(1-r^{\varepsilon}\right)\left(B_{2 j}^{\varepsilon}-B_{2 l}^{\varepsilon}\right) \\
&= r^{\varepsilon}\left((1-\varepsilon)\left(B_{1 j}-B_{1 l}\right)+\frac{\varepsilon}{n-1}\left(B_{2 l}-B_{2 j}\right)\right) \\
&+\left(1-r^{\varepsilon}\right)\left((1-\varepsilon)\left(B_{2 j}-B_{2 l}\right)+\frac{\varepsilon}{n-1}\left(B_{1 l}-B_{1 j}\right)\right) \\
&= \frac{1}{1-\varepsilon+\frac{\varepsilon}{n-1}}\left((1-\varepsilon)^{2} r\left(B_{1 j}-B_{1 l}\right)+\frac{\varepsilon}{n-1}(1-\varepsilon)(1-r)\left(B_{1 j}-B_{1 l}\right)\right. \\
&+\frac{\varepsilon}{n-1}(1-\varepsilon) r\left(B_{2 l}-B_{2 j}\right)+\left(\frac{\varepsilon}{n-1}\right)^{2}(1-r)\left(B_{2 l}-B_{2 j}\right) \\
&+(1-\varepsilon)^{2}(1-r)\left(B_{2 j}-B_{2 l}\right)+\frac{\varepsilon}{n-1}(1-\varepsilon) r\left(B_{2 j}-B_{2 l}\right) \\
&\left.+\frac{\varepsilon}{n-1}(1-\varepsilon)(1-r)\left(B_{1 l}-B_{1 j}\right)+\left(\frac{\varepsilon}{n-1}\right)^{2} r\left(B_{1 l}-B_{1 j}\right)\right) \\
&= \frac{1}{1-\varepsilon+\frac{\varepsilon}{n-1}}\left((1-\varepsilon)^{2}\left(r\left(B_{1 j}-B_{1 l}\right)+(1-r)\left(B_{2 j}-B_{2 l}\right)\right)\right. \\
&\left.+\left(\frac{\varepsilon}{n-1}\right)^{2}\left(r\left(B_{1 l}-B_{1 j}\right)+(1-r)\left(B_{2 l}-B_{2 j}\right)\right)\right) \\
&= 0
\end{aligned}
$$

where in the first equality we substitute (6.2) with $m=2$, in the second equality we substitute (6.3), also using the fact that $\left(1-r^{\varepsilon}\right)\left(1-\varepsilon+\frac{\varepsilon}{n-1}\right)=$ $(1-\varepsilon)(1-r)+\frac{\varepsilon}{n-1} r$, and the final equality follows from (6.5).
(iii) We only prove $P B_{2}^{\varepsilon}\left(e_{1}\right)=\overline{P B_{2}}\left(e_{1}\right) ; P B_{2}^{\varepsilon}\left(e_{2}\right)=\overline{P B_{2}}\left(e_{2}\right)$ follows similarly. Since
$P B_{2}^{\varepsilon}\left(e_{1}^{\varepsilon}\right) \subseteq P B_{2}\left(e_{1}\right)$ for sufficiently small $\varepsilon$,

$$
P B_{2}^{\varepsilon}\left(e_{1}\right)=\left\{f_{j} \in P B_{2}\left(e_{1}\right) \mid B_{1 j}^{\varepsilon} \geq B_{1 l}^{\varepsilon} \text { for all } f_{l} \in P B_{2}\left(e_{1}\right)\right\} .
$$

Let $f_{j}, f_{l} \in P B_{2}\left(e_{1}\right)$. Then, since

$$
B_{1 j}^{\varepsilon}-B_{1 l}^{\varepsilon}=(1-\varepsilon)\left(B_{1 j}-B_{1 l}\right)+\frac{\varepsilon}{n-1}\left(B_{2 l}-B_{2 j}\right)=\frac{\varepsilon}{n-1}\left(B_{2 l}-B_{2 j}\right),
$$

it follows that

$$
P B_{2}^{\varepsilon}\left(e_{1}\right)=\left\{f_{j} \in P B_{2}\left(e_{1}\right) \mid B_{2 j} \leq B_{2 l} \text { for all } f_{l} \in P B_{2}\left(e_{1}\right)\right\}=\overline{P B_{2}}\left(e_{1}\right)
$$

Lemma 6.4.4 has several implications. First, Lemma 6.4.4(i) shows that coincident lines in the 'old' line-label picture for $(A, B)$ remain coincident lines in the 'new' picture for $\left(A^{\varepsilon}, B^{\varepsilon}\right)$.

Next, if two or more non-coincident lines intersect (on the upper envelope of the line-label picture) at $p \in \triangle_{2}$ in the old picture, then these lines all intersect again in the new picture at $p^{\varepsilon}$. In particular, this applies to each old interior extreme point, in relation to the new interior extreme points $p_{1}^{\varepsilon}, \ldots, p_{t-1}^{\varepsilon}$. As $\varepsilon$ converges to zero, the corresponding set $S^{\varepsilon}\left(p_{k}^{\varepsilon}\right)$ of solutions to $p_{k}^{\varepsilon}, k \in\{1, \ldots, t-1\}$, converges to some set of solutions, denoted by $S^{e}\left(p_{k}\right)$.

Now, consider the left boundary $p=e_{2}$ and suppose $\left|\left\{B_{1 j} \mid f_{j} \in P B_{2}\left(e_{2}\right)\right\}\right| \geq 2$, i.e., two or more non-coincident lines intersect at $e_{2}$. Then, by Lemma 6.4.4(ii),

$$
e_{2}^{\varepsilon}=\left[\begin{array}{cc}
\frac{\frac{\varepsilon}{n-1}}{1-\varepsilon+\frac{\varepsilon}{n-1}} & \frac{1-\varepsilon}{1-\varepsilon+\frac{\varepsilon}{n-1}}
\end{array}\right]^{\top}
$$

is the new interior extreme point in which all former pure best replies at $e_{2}$ are the new pure best replies. The set of solutions to which $S^{\varepsilon}\left(e_{2}^{\varepsilon}\right)$ converges is denoted by $S^{e}\left(e_{2}\right)$. Of course, there is still an extreme point at $e_{2}$ in the perturbed game. Hence, the old extreme point $e_{2}$ is essentially 'split in two'. Lemma 6.4.4(iii) shows that the set of solutions $S^{\varepsilon}\left(e_{2}\right)$ can be determined on the basis of $\overline{P B_{2}}\left(e_{2}\right)$. The set of solutions to which $S^{\varepsilon}\left(e_{2}\right)$ converges is denoted by $\bar{S}^{e}\left(e_{2}\right)$.

Next, reconsider $p=e_{2}$, but now suppose $\left|\left\{B_{1 j} \mid f_{j} \in P B_{2}(p)\right\}\right|=1$. Then, $e_{2}^{\varepsilon}$ is not actually a new (interior) extreme point. In that case, we refer to $e_{2}^{\varepsilon}$ as an 'artificial' extreme point. We will see later that in this case $S^{e}\left(e_{2}\right) \subseteq \bar{S}^{e}\left(e_{2}\right)$.

Similarly, with respect to $p=e_{1}$, we define a (possibly artificial) extreme point $e_{1}^{\varepsilon}=p_{t}^{\varepsilon}$, for which the corresponding set of solutions $S^{\varepsilon}\left(e_{1}^{\varepsilon}\right)$ converges to $S^{e}\left(e_{1}\right)$. Further, $\bar{S}^{e}\left(e_{1}\right)$ is defined analogously to $\bar{S}^{e}\left(e_{2}\right)$

Finally, it remains to consider the intervals. Note that Lemma 6.4.4(i) implies that the old set of pure best replies with respect to the interval $I_{k}$ (between $p_{k-1}$ and $p_{k}$ for $k \in\{1, \ldots, t\})$ exactly corresponds to the new set of pure best replies with respect to $I_{k}^{\varepsilon}$ (between $e_{2}^{\varepsilon}$ and $p_{1}^{\varepsilon}$ in case $k=1$, between $p_{k-1}^{\varepsilon}$ and $p_{k}^{\varepsilon}$ for $k \in\{2, \ldots, t-1\}$, and between $p_{t}^{\varepsilon}$ and $e_{1}^{\varepsilon}$ in case $k=t$ ). The set of (old) solutions to which the corresponding set of new solutions $S^{\varepsilon}\left(I_{k}^{\varepsilon}\right)$ converges, is denoted by $S^{e}\left(I_{k}\right)$.

We remark that in case an extreme point at a boundary is split into two extreme points, there is also a new interval, e.g., between $e_{1}^{\varepsilon}$ and $e_{1}$. However, we know that the set of solutions on an interval is always a subset of the sets of solutions to its extreme points. In the limit, such an interval therefore does not lead to additional entangled equilibria compared to the ones obtained from its two converging extreme points, and hence such intervals need not be considered.

To exactly determine $S^{e}\left(p_{k}\right), S^{e}\left(e_{2}\right), \bar{S}^{e}\left(e_{2}\right), S^{e}\left(e_{1}\right), \bar{S}^{e}\left(e_{1}\right)$, and $S^{e}\left(I_{k}\right)$, one additional consideration is necessary: how to determine the new labels. Again, the approach remains tractable, since, for sufficiently small $\varepsilon$, only the old labels [12] can change, and Lemma 6.4 .5 shows that all such labels change in the same way. For any $2 \times n$ bimatrix game $(A, B)$, the way all labels [12] change follows directly from the row sums of matrix $A$, denoted by

$$
a_{1}=\sum_{l \in N} A_{1 l} \quad \text { and } \quad a_{2}=\sum_{l \in N} A_{2 l} .
$$

## Lemma 6.4.5

Let $(A, B)$ be a $2 \times n$ bimatrix game and let $\left(A^{\varepsilon}, B^{\varepsilon}\right)$ be the corresponding perturbed game. Then, for sufficiently small $\varepsilon$,
(i) $J^{\varepsilon}([12])=J([12]), J^{\varepsilon}([1])=J([1])$, and $J^{\varepsilon}([2])=J([2])$, if $a_{1}=a_{2}$;
(ii) $J^{\varepsilon}([12])=\emptyset, J^{\varepsilon}([1])=J([1]) \cup J([12])$, and $J^{\varepsilon}([2])=J([2])$, if $a_{1}<a_{2}$;
(iii) $J^{\varepsilon}([12])=\emptyset, J^{\varepsilon}([1])=J([1])$, and $J^{\varepsilon}([2])=J([2]) \cup J([12])$, if $a_{1}>a_{2}$.

Proof. Let $j \in N$. Using equation (6.1), we have

$$
\begin{equation*}
A_{1 j}^{\varepsilon}-A_{2 j}^{\varepsilon}=(1-\varepsilon)\left(A_{1 j}-A_{2 j}\right)+\frac{\varepsilon}{n-1} \sum_{l \in N \backslash\{j\}}\left(A_{2 l}-A_{1 l}\right) \tag{6.6}
\end{equation*}
$$

First, suppose $f_{j} \notin J([12])$. Then, it follows from (6.6) that, for sufficiently small $\varepsilon$, $A_{1 j}>A_{2 j}$ implies $A_{1 j}^{\varepsilon}>A_{2 j}^{\varepsilon}$ and $A_{1 j}<A_{2 j}$ implies $A_{1 j}^{\varepsilon}<A_{2 j}^{\varepsilon}$. This shows that $P B_{1}\left(f_{j}\right)=P B_{1}^{\varepsilon}\left(f_{j}\right)$, i.e., all pure strategies with label [1] or [2] in $(A, B)$ keep that same label in $\left(A^{\varepsilon}, B^{\varepsilon}\right)$.

Next, suppose $f_{j} \in J([12])$. Then, $A_{1 j}=A_{2 j}$ and consequently

$$
A_{1 j}^{\varepsilon}-A_{2 j}^{\varepsilon}=\frac{\varepsilon}{n-1} \sum_{l \in N \backslash\{j\}}\left(A_{2 l}-A_{1 l}\right)=\frac{\varepsilon}{n-1} \sum_{l \in N}\left(A_{2 l}-A_{1 l}\right)=\frac{\varepsilon}{n-1}\left(a_{2}-a_{1}\right) .
$$

So, $A_{1 j}^{\varepsilon}=A_{2 j}^{\varepsilon}$ and $f_{j} \in J^{\varepsilon}([12])$ if $a_{1}=a_{2}, A_{1 j}^{\varepsilon}>A_{2 j}^{\varepsilon}$ and $f_{j} \in J^{\varepsilon}([1])$ if $a_{1}<a_{2}$, and $A_{1 j}^{\varepsilon}<A_{2 j}^{\varepsilon}$ and $f_{j} \in J^{\varepsilon}([2])$ if $a_{1}<a_{2}$.

In particular, Lemma 6.4.5 implies that there can be a new coordination solution in the perturbed game that does not directly correspond to an old coordination solution in the original game. However, such a new coordination solution will converge to the old pure solution corresponding to the old label [12].

Summarizing, Lemmas 6.4.4 and 6.4.5 fully describe the line-label picture for a perturbed game. A direct analysis of the solutions of the basis of this new picture using coordination solutions and pure solutions, in combination with a limit argument, leads to the following exact characterizations of the solutions corresponding to all interior extreme points and all intervals.

## Proposition 6.4.6

Let $(A, B)$ be a $2 \times n$ bimatrix game and let $k \in\{1, \ldots, t-1\}$. Then,

$$
S^{e}\left(p_{k}\right)= \begin{cases}\emptyset & \text { if } a_{1}<a_{2} \text { and } P B_{2}\left(p_{k},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } P B_{2}\left(p_{k},[1]\right)=\emptyset, \\ S\left(p_{k}\right) & \text { otherwise. }\end{cases}
$$

## Proposition 6.4.7

Let $(A, B)$ be a $2 \times n$ bimatrix game and let $k \in\{1, \ldots, t\}$. Then,

$$
S^{e}\left(I_{k}\right)= \begin{cases}\emptyset & \text { if } a_{1}<a_{2} \text { and } P B_{2}\left(I_{k},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } P B_{2}\left(I_{k},[1]\right)=\emptyset, \\ S\left(I_{k}\right) & \text { otherwise. }\end{cases}
$$

For the solutions corresponding to the possibly artificial interior extreme points $e_{2}^{\varepsilon}$ and $e_{1}^{\varepsilon}$, note that, e.g., $P S\left(e_{1}\right)$ also includes pure best replies with label [1], while we should restrict to those with label [12] for interior points. Therefore, we use, e.g., $P B_{2}\left(e_{1},[12]\right)$ instead of $P S\left(e_{1}\right)$ in Proposition 6.4.8.

## Proposition 6.4.8

Let $(A, B)$ be a $2 \times n$ bimatrix game. Then,

$$
S^{e}\left(e_{1}\right)= \begin{cases}\emptyset & \text { if } a_{1}<a_{2} \text { and } P B_{2}\left(e_{1},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } P B_{2}\left(e_{1},[1]\right)=\emptyset, \\ \operatorname{Conv}\left\{C S\left(e_{1}\right) \cup P B_{2}\left(e_{1},[12]\right)\right\} & \text { otherwise, }\end{cases}
$$

and

$$
S^{e}\left(e_{2}\right)= \begin{cases}\emptyset & \text { if } a_{1}<a_{2} \text { and } P B_{2}\left(e_{2},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } P B_{2}\left(e_{2},[1]\right)=\emptyset, \\ \operatorname{Conv}\left\{C S\left(e_{2}\right) \cup P B_{2}\left(e_{2},[12]\right)\right\} & \text { otherwise } .\end{cases}
$$

Finally, to describe $\bar{S}^{e}\left(e_{1}\right)$ and $\bar{S}^{e}\left(e_{2}\right)$, we define

$$
\overline{P S}\left(e_{1}\right)=\overline{P B_{2}}\left(e_{1},[1]\right) \cup \overline{P B_{2}}\left(e_{1},[12]\right)
$$

and

$$
\overline{P S}\left(e_{2}\right)=\overline{P B_{2}}\left(e_{2},[2]\right) \cup \overline{P B_{2}}\left(e_{2},[12]\right),
$$

and, for $i \in\{1,2\}$,

$$
\overline{C S}\left(e_{i}\right)=\left\{q(j, l) \in C S\left(e_{i}\right) \mid f_{j}, f_{l} \in \overline{P B_{2}}\left(e_{i}\right)\right\}
$$

and

$$
\bar{S}\left(e_{i}\right)=\operatorname{Conv}\left\{\overline{P S}\left(e_{i}\right) \cup \overline{C S}\left(e_{i}\right)\right\}
$$

## Proposition 6.4.9

Let $(A, B)$ be a $2 \times n$ bimatrix game. Then,

$$
\bar{S}^{e}\left(e_{1}\right)= \begin{cases}\operatorname{Conv}\left\{\overline{P B_{2}}\left(e_{1},[1]\right)\right\} & \text { if } a_{1}<a_{2} \text { and } \overline{P B_{2}}\left(e_{1},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } \overline{P B_{2}}\left(e_{1},[1]\right)=\emptyset, \\ \bar{S}\left(e_{1}\right) & \text { otherwise },\end{cases}
$$

and

$$
\bar{S}^{e}\left(e_{2}\right)= \begin{cases}\operatorname{Conv}\left\{\overline{P B_{2}}\left(e_{2},[2]\right)\right\} & \text { if } a_{1}<a_{2} \text { and } \overline{P B_{2}}\left(e_{2},[2]\right)=\emptyset, \text { or } \\ & \text { if } a_{1}>a_{2} \text { and } \overline{P B_{2}}\left(e_{2},[1]\right)=\emptyset, \\ \bar{S}\left(e_{2}\right) & \text { otherwise. }\end{cases}
$$

Combining the four propositions above, we can fully characterize the set of entangled equilibria for a $2 \times n$ bimatrix game in the following way.

## Theorem 6.4.10

Let $(A, B)$ be a $2 \times n$ bimatrix game. Then, the set of entangled equilibria of $(A, B)$ is given $b y^{2}$

$$
\begin{aligned}
E E(A, B)= & \bigcup_{k=1}^{t} \bar{I}_{k} \times S^{e}\left(I_{k}\right) \\
& \cup \bigcup_{k=1}^{t-1}\left\{p_{k}\right\} \times S^{e}\left(p_{k}\right) \\
& \cup\left\{e_{1}\right\} \times\left(S^{e}\left(e_{1}\right) \cup \bar{S}^{e}\left(e_{1}\right)\right) \\
& \cup\left\{e_{2}\right\} \times\left(S^{e}\left(e_{2}\right) \cup \bar{S}^{e}\left(e_{2}\right)\right) .
\end{aligned}
$$

Using the structure of $E E(A, B)$ as described in Theorem 6.4.10, we now illustrate how to efficiently modify the line-label picture to readily determine the set of entangled equilibria for each specific $2 \times n$ bimatrix game by using the format of the set of Nash equilibria with respect to the modified picture, thus avoiding explicit computations for perturbed games.

[^5]
## Example 6.4.2

Consider the $2 \times 7$ bimatrix game $(A, B)$ given by

$$
(A, B)=\begin{gathered}
e_{1} \\
e_{2}
\end{gathered}\left[\begin{array}{ccccccc}
f_{1} & f_{2} & f_{3} & f_{4} & f_{5} & f_{6} & f_{7} \\
0,-15 & 0,0 & 0,6 & 6,6 & 0,6 & 0,6 & 0,6 \\
6,12 & 0,9 & 6,-6 & 0,-9 & 0,6 & 3,6 & 3,-9
\end{array}\right]
$$

This game is graphically represented in Figure 6.4. Note that the $2 \times 4$ bimatrix game of Example 6.4.1 is a subgame of $(A, B)$. Similar to Example 6.4 .1 , one readily finds

$$
\begin{aligned}
E(A, B) & =\left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup \operatorname{Conv}\left\{\frac{1}{6} e_{1}+\frac{5}{6} e_{2}, \frac{1}{3} e_{1}+\frac{2}{3} e_{2}\right\} \times\left\{f_{2}\right\} \\
& \cup\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}\right\} \times \operatorname{Conv}\left\{f_{2}, f_{5}\right\} \\
& \cup \operatorname{Conv}\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}, e_{1}\right\} \times\left\{f_{5}\right\} \\
& \cup\left\{e_{1}\right\} \times \operatorname{Conv}\left\{f_{4}, f_{5}, q(4,3), q(4,6), q(4,7)\right\},
\end{aligned}
$$

where $q(4,3)=\frac{1}{2} f_{3}+\frac{1}{2} f_{4}, q(4,6)=\frac{1}{3} f_{4}+\frac{2}{3} f_{6}$, and $q(4,7)=\frac{1}{3} f_{4}+\frac{2}{3} f_{7}$.


Figure 6.4 Graphical representation towards solving the $2 \times 7$ bimatrix game of Example 6.4.2 for Nash equilibria

From this, we can straightforwardly derive the set of entangled equilibria. First, note that with respect to the row sums

$$
a_{1}=6<18=a_{2},
$$

i.e., $f_{2}$ and $f_{5}$ get label [1] instead of [12] in the perturbed games. We keep track of this in Figure 6.5 using $[1]_{\overline{2}}$ and $[1]_{\overline{5}}$.

The extreme point at $e_{1}$ is split into two extreme points, which is graphically reflected in Figure 6.5 by shifting the right boundary to the right, to $\bar{e}_{1}$, so that $e_{1}$ becomes an interior extreme point.


Figure 6.5 Graphical representation towards solving the $2 \times 7$ bimatrix game of Example 6.4.2 for entangled equilibria

The set of Nash equilibria corresponding to Figure 6.5 is

$$
\begin{aligned}
& \left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup\left\{\frac{1}{6} e_{1}+\frac{5}{6} e_{2}\right\} \times\{q(\overline{2}, 1)\} \\
& \cup\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}\right\} \times \operatorname{Conv}\{q(\overline{2}, 6), q(\overline{5}, 6)\} \\
& \cup \operatorname{Conv}\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}, e_{1}\right\} \times\{q(\overline{5}, 6)\} \\
& \cup\left\{e_{1}\right\} \times \operatorname{Conv}\{q(4,3), q(4,6), q(4,7), q(\overline{5}, 3), q(\overline{5}, 6), q(\overline{5}, 7)\} \\
& \cup\left\{\bar{e}_{1}\right\} \times \operatorname{Conv}\left\{f_{4}, q(4,7)\right\},
\end{aligned}
$$

where, e.g., $q(\overline{2}, 1)$ corresponds to the coordination solution between $f_{2}$ (with label [12] in the original game) and $f_{1}$. Using the fact that each new coordination solution converges to the old pure solution with label [12] in the original game, we readily find

$$
\begin{aligned}
E E(A, B) & =\left\{e_{2}\right\} \times\left\{f_{1}\right\} \\
& \cup\left\{\frac{1}{6} e_{1}+\frac{5}{6} e_{2}\right\} \times\left\{f_{2}\right\} \\
& \cup\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}\right\} \times \operatorname{Conv}\left\{f_{2}, f_{5}\right\} \\
& \cup \operatorname{Conv}\left\{\frac{1}{3} e_{1}+\frac{2}{3} e_{2}, e_{1}\right\} \times\left\{f_{5}\right\} \\
& \cup\left\{e_{1}\right\} \times \operatorname{Conv}\left\{f_{5}, q(4,3), q(4,6), q(4,7)\right\} \\
& \cup\left\{e_{1}\right\} \times \operatorname{Conv}\left\{f_{4}, q(4,7)\right\} .
\end{aligned}
$$

### 6.5 Concluding remarks

The perturbations considered in the thought experiment corresponding to entangled equilibria are not of the Kohlberg-Mertens type, and there is also no direct relationship between entangled equilibria and perfect equilibria. Example 6.5.1 illustrates that these two equilibrium sets can be disjoint. Consequently, the same holds when comparing entangled equilibria with the sets of proper or fall back equilibria (since Kleppe et al. (2012) show that, for bimatrix games, each proper equilibrium is a fall back equilibrium).

## Example 6.5.1

Consider the $2 \times 2$ bimatrix game $(A, B)$ given by

$$
\left.(A, B)={ }_{e_{2}} \begin{array}{c}
e_{1}
\end{array} \begin{array}{cc}
f_{1} & f_{2} \\
1,0 & 0,0 \\
0,2 & 0,1
\end{array}\right]
$$

Note that this is the $2 \times 2$ bimatrix game of Example 6.3 .1 in which $B_{21}$ and $B_{22}$ are interchanged. For the bimatrix game of Example 6.3.1, we found that $E E(A, B)=$ $\left\{\left(e_{1}, f_{1}\right),\left(e_{1}, f_{2}\right),\left(e_{2}, f_{2}\right)\right\}$. There, the only perfect equilibrium is $\left(e_{1}, f_{2}\right)$.

For the bimatrix game of this example, one readily finds that $E E(A, B)=\left(e_{1}, f_{2}\right)$, while $\left(e_{1}, f_{1}\right)$ is the only perfect equilibrium (since $e_{1}$ weakly dominates $e_{2}$ and $f_{1}$ weakly dominates $f_{2}$ ). This shows both equilibrium sets can also be disjoint.

The definitions and results of Section 6.3 can be extended in various ways. First, note that while Definition 6.3.1 assumes that if a strategy combination is blocked, a replacing pure strategy combination is chosen with equal probability for all remaining pure strategy combinations, Theorems 6.3.3 and 6.3.4 hold for non-uniform distributions as well. It may also be interesting to consider truly correlated strategies as replacement for the blocked actions. Further, note that the entangled equilibrium notion is well-defined for $n$-person finite strategic games in general.

One could also explore whether there exist generic relationships between entangled equilibria and existing solution concepts, to offer an alternative justification for these concepts.

Finally, another direction for further research is to examine characterizations of the structure of the set of entangled equilibria, in general or for specific classes of games (e.g., strictly competitive bimatrix games or generic games).

## Bibliography

Akbilgic, O., Doluweera, G., Mahmoudkhani, M., and Bergerson, J. (2015). A metaanalysis of carbon capture and storage technology assessments: Understanding the driving factors of variability in cost estimates. Applied Energy, 159:11-18.

Andiappan, V., Tan, R., and Ng, D. (2016). An optimization-based negotiation framework for energy systems in an eco-industrial park. Journal of Cleaner Production, 129:496-507.

Anupindi, R., Bassok, Y., and Zemel, E. (2001). A general framework for the study of decentralized distribution systems. Manufacturing $\mathcal{B}$ Service Operations Management, 3(4):349-368.
van Beek, A., Borm, P., and Quant, M. (2021). Axiomatic characterizations of a proportional influence measure for sequential projects with imperfect reliability. Axioms, 10(4), 247.
van Beek, A., Brokkelkamp, R., and Schäfer, G. (2022). Corruption in auctions: Social welfare loss in hybrid multi-unit auctions. In Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems, pages 1283-1291.
van Beek, A., Groote Schaarsberg, M., Borm, P., Hamers, H., and Veneman, M. (2023a). Cost allocation in CO2 transport for CCUS hubs: a multi-actor perspective. CentER Discussion Paper Nr. 2023-008.
van Beek, A., Malmberg, B., Borm, P., Quant, M., and Schouten, J. (2023b). Competition and cooperation in linear production and sequencing processes. Games and Economic Behavior, 139:117-132.

Bergantiños, G. and Sánchez, E. (2002). How to distribute costs associated with a delayed project. Annals of Operations Research, 109(1-4):159-174.

Birmpas, G., Markakis, E., Telelis, O., and Tsikiridis, A. (2019). Tight welfare guarantees for pure Nash equilibria of the uniform price auction. Theory of Computing Systems, 63(7):1451-1469.

Borm, P. (1992). On perfectness concepts for bimatrix games. Operations-ResearchSpektrum, 14(1):33-42.

Borm, P., Hamers, H., and Hendrickx, R. (2001). Operations research games: A survey. TOP, 9(2):139.

Brandenburger, A. and Stuart, H. (2007). Biform games. Management Science, 53(4):537-549.

Brânzei, R., Ferrari, G., Fragnelli, V., and Tijs, S. (2002). Two approaches to the problem of sharing delay costs in joint projects. Annals of Operations Research, 109(1-4):359-374.

Brown, G., Carlyle, M., Harney, R., Skroch, E., and Wood, K. (2009). Interdicting a nuclear-weapons project. Operations Research, 57(4):866-877.
de Bruyn, S., Jongsma, C., Kampman, B., Görlach, B., and Thie, J.-E. (2020). Energy intensive industries - challenges and opportunities in energy transition. Study requested by the European Parliament's committee on Industry, Research and Energy (ITRE).

Curiel, I. (2010). Multi-stage sequencing situations. International Journal of Game Theory, 39(1-2):151-162.

Curiel, I., Hamers, H., and Klijn, F. (2002). Sequencing games: a survey. In Borm, P. and Peters, H., editors, Chapters in Game Theory, pages 27-50. Springer.

Curiel, I., Pederzoli, G., and Tijs, S. (1989). Sequencing games. European Journal of Operational Research, 40(3):344-351.
van Damme, E. (1991). Stability and perfection of Nash equilibria, volume 339. Springer.

De Keijzer, B., Markakis, E., Schäfer, G., and Telelis, O. (2013). Inefficiency of standard multi-unit auctions. In Proceedings of the European Symposium on Algorithms (ESA), pages 385-396. Springer.

Deegan, J. and Packel, E. W. (1978). A new index of power for simplen-person games. International Journal of Game Theory, 7(2):113-123.

Detz, R. and van der Zwaan, B. (2019). Transitioning towards negative CO2 emissions. Energy Policy, 60. Article 110938.

DNVGL (2020). Meerjarenprogramma infrastructuur en klimaat [Multi-year program for energy and climate infrastructure]. Taskforce Infrastructuur Klimaatakkoord Industrie, Advisory report for Dutch ministry of Economic Affairs.

EBN and Gasunie (2018). Transport en opslag van CO2 in Nederland [Transport and storage of CO2 in the Netherlands]. Exploratory study at the request of Dutch ministry of Economic Affairs.

EC (2020). European commission communication: A new industrial strategy for europe. Communication COM/2020/102.

Estévez-Fernández, A., Borm, P., and Hamers, H. (2007). Project games. International Journal of Game Theory, 36(2):149-176.

Feess, E. and Thun, J.-H. (2014). Surplus division and investment incentives in supply chains: A biform-game analysis. European Journal of Operational Research, 234(3):763-773.

Feldman, M., Fu, H., Gravin, N., and Lucier, B. (2013). Simultaneous auctions are (almost) efficient. In Proceedings of the forty-fifth annual ACM symposium on Theory of computing, pages 201-210.

Feldman, M., Lucier, B., and Nisan, N. (2016). Correlated and coarse equilibria of single-item auctions. In Proceedings of the International Conference on Web and Internet Economics (WINE), pages 131-144. Springer.

Fu, Y., Kok, R. A., Dankbaar, B., Ligthart, P. E., and van Riel, A. C. (2018). Factors affecting sustainable process technology adoption: A systematic literature review. Journal of Cleaner Production, 205:226-251.

Fujiwara-Greve, T. (2015). Non-cooperative game theory. Springer.
Gale, D. and Shapley, L. (1962). College admissions and the stability of marriage. The American Mathematical Monthly, 69(1):9-15.

Gedai, E., Kóczy, L. A., and Zombori, Z. (2012). Cluster games: A novel, game theory-based approach to better understand incentives and stability in clusters. MPRA Paper nr 65095.
van Gellekom, J., Potters, J., Reijnierse, H., Engel, M., and Tijs, S. (2000). Characterization of the owen set of linear production processes. Games and Economic Behavior, 32(1):139-156.

Gillies, D. B. (1959). Solutions to general non-zero-sum games. Contributions to the Theory of Games, 4:47-85.

Govindan, S. and Wilson, R. (2008). Refinements of Nash equilibrium. In The New Palgrave Dictionary of Economics, pages 1-14. Palgrave Macmillan UK.

Graham, D. A. and Marshall, R. C. (1987). Collusive bidder behavior at single-object second-price and english auctions. Journal of Political Economy, 95(6):1217-1239.

Granot, D. and Sošić, G. (2003). A three-stage model for a decentralized distribution system of retailers. Operations Research, 51(5):771-784.

Hall, N. and Liu, Z. (2016). Capacity allocation games without an initial sequence. Operations Research Letters, 44(6):747-749.

Hamers, H., Suijs, J., Tijs, S., and Borm, P. (1996). The split core for sequencing games. Games and Economic Behavior, 15(2):165-176.

Harsanyi, J. C. (1967). Games with incomplete information played by "Bayesian" players, Parts I, II and III. Management Science, 14(3):159-182, 320-334, and 486-502.

Hart, O. and Moore, J. (1990). Property rights and the nature of the firm. Journal of Political Economy, 98(6):1119-1158.

Hennet, J.-C. and Mahjoub, S. (2010). Supply network formation as a biform game. IFAC Proceedings Volumes, 43(17):108-113.

Hermans, B., Hamers, H., Leus, R., and Lindelauf, R. (2019). Timely exposure of a secret project: Which activities to monitor? Naval Research Logistics, 66(6):451468.

Holler, M. J. (1982). Forming coalitions and measuring voting power. Political studies, 30(2):262-271.

Husslage, B., Borm, P., Burg, T., Hamers, H., and Lindelauf, R. (2015). Ranking terrorists in networks: A sensitivity analysis of Al Qaeda's 9/11 attack. Social Networks, 42:1-7.

IEA (2020). Special report on carbon capture utilization and storage. Energy Technology Perspectives 2020.

Ingraham, A. (2005). A test for collusion between a bidder and an auctioneer in sealed-bid auctions. Contributions in Economic Analysis \&3 Policy, 4(1):1-32.

IPCC (2022). Climate change 2022: Impacts, adaptation and vulnerability working group ii contribution to the sixth assessment report of the intergovernmental panel on climate change. [H.-O. Pörtner, D.C. Roberts, M. Tignor, E.S. Poloczanska, K. Mintenbeck, A. Alegría, M. Craig, S. Langsdorf, S. Löschke, V. Möller, A. Okem, B. Rama (eds.)].

IRENA (2021). World energy transitions outlook: $1.5^{\circ} \mathrm{C}$ pathway. International Renewable Energy Agency, Abu Dhabi.

Janipour, Z., de Nooij, R., Scholten, P., Huijbregts, M., and de Coninck, H. (2020). What are sources of carbon lock-in in energy-intensive industry? a case study into dutch chemicals production. Energy Research E3 Social Science, 60:1-9.

Kim, S. (2012). Biform game based cognitive radio scheme for smart grid communications. Journal of Communications and Networks, 14(6):614-618.

Kleppe, J., Borm, P., and Hendrickx, R. (2012). Fall back equilibrium. European Journal of Operational Research, 223(2):372-379.

Klijn, F. and Sánchez, E. (2006). Sequencing games without initial order. Mathematical Methods of Operations Research, 63(1):53-62.

Knoope, M. (2015). Costs, safety and uncertainties of CO2 infrastructure development. PhD thesis, Utrecht University.
Kóczy, L. Á. (2018). Partition function form games. Theory and Decision Library C, 48:312.

Kohlberg, E. and Mertens, J.-F. (1986). On the strategic stability of equilibria. Econometrica: Journal of the Econometric Society, pages 1003-1037.

Konishi, H., Le Breton, M., and Weber, S. (1997). Equilibria in a model with partial rivalry. Journal of Economic Theory, 72(1):225-237.

Koutsoupias, E. and Papadimitriou, C. (1999). Worst-case equilibria. In Proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS), pages 404-413. Springer.

Kuipers, J., Mosquera, M. A., and Zarzuelo, J. M. (2013). Sharing costs in highways: A game theoretic approach. European Journal of Operational Research, 228(1):158168.

Lengwiler, Y. and Wolfstetter, E. (2010). Auctions and corruption: An analysis of bid rigging by a corrupt auctioneer. Journal of Economic Dynamics and Control, 34(10):1872-1892.

Lengwiler, Y. and Wolfstetter, E. G. (2000). Auctions and Corruption. CESifo Working Paper Series 401, CESifo.

Lindelauf, R. (2011). Design and Analysis of Covert Networks, Affiliations and Projects. PhD thesis, CentER, Tilburg School of Economics and Management, Tilburg University.
van der Linden, N. (2019). Inventarisatie van de behoefte van de industrieclusters aan grootschalige infrastructuur voor transport van elektriciteit, waterstof, warmte en CO2 nodig voor het realiseren van klimaatdoelstellingen [Inventory of industry clusters' needs for large-scale infrastructure for transportation of electricity, hydrogen, heat and CO2 required to achieve climate goals]. TNO report P10550.

Littlechild, S. and Thompson, G. (1977). Aircraft landing fees: a game theory approach. The Bell Journal of Economics, 8:186-204.

Mallon, W., Buit, L., J. van Wingerden, H. L., and Eldrup, N. (2013). Costs of CO2 transportation infrastructures. Energy Procedia, 37:2969-2980.

Markakis, E. and Telelis, O. (2015). Uniform price auctions: Equilibria and efficiency. Theory of Computing Systems, 57(3):549-575.

Massol, O., Tchung-Ming, S., and Banal-Estañol, A. (2018). Capturing industrial CO2 emissions in Spain: Infrastructures, costs and break-even prices. Energy Policy, 115:545-560.

Matthews, O. and Howell, G. (2005). Integrated project delivery an example of relational contracting. Lean Construction Journal, 2(1):46-61.

McAfee, R. and McMillan, J. (1992). Bidding rings. American Economic Review, 82(3):579-99.

Menezes, F. M. and Monteiro, P. K. (2006). Corruption and auctions. Journal of Mathematical Economics, 42(1):97-108.

Myerson, R. (1978). Refinements of the Nash equilibrium concept. International Journal of Game Theory, 7:73-80.

Nash, J. (1951). Non-cooperative games. Annals of Mathematics, 54(2):286-295.
Nasirzadeh, F., Mazandaranizadeh, H., and Rouhparvar, M. (2016). Quantitative risk allocation in construction projects using cooperative-bargaining game theory. International Journal of Civil Engineering, 14(3):161-170.
Okada, A. (1984). Strictly perfect equilibrium points of bimatrix games. International Journal of Game Theory, 13:145-153.

Olmstead, L. (April 21, 1993). 2 managers held in bidding scheme at school agency. The New York Times.

Owen, G. (1975). On the core of linear production games. Mathematical Programming, 9(1):358-370.

Pakuła, I. (2008). Analysis of trembling hand perfect equilibria in quantum games. Fluctuation and Noise Letters, 8(01):L23-L30.

Peleg, B. and Sudhölter, P. (2007). Introduction to the theory of cooperative games. Vol. 34. Springer Science \& Business Media.

Porthos (2022). CO2 reduction through storage under the north sea. www.porthosco2.nl/en.

Quarton, C. J. and Samsatli, S. (2020). The value of hydrogen and carbon capture, storage and utilisation in decarbonising energy: Insights from integrated value chain optimisation. Applied Energy, 257:113936.

Ray, D. and Vohra, R. (2015). Chapter 5 - coalition formation. volume 4 of Handbook of Game Theory with Economic Applications, pages 239-326. Elsevier.

Roughgarden, T. (2015a). Intrinsic robustness of the price of anarchy. Journal of the ACM, 62(5).

Roughgarden, T. (2015b). The price of anarchy in games of incomplete information. ACM Transactions on Economics and Computation (TEAC), 3(1):1-20.

Roughgarden, T., Syrgkanis, V., and Tardos, E. (2017). The price of anarchy in auctions. Journal on Artificial Intelligence Research, 59(1):59-101.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics, 17(6):1163-1170.

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. International Journal of Game Theory, 4:25-55.

SER (2019). Nationale klimaataanpak voor regionale industriële koplopers [National climate approach for regional industry leaders]. SER advisory report: nr. 6, June 2019.

Serpa, J., Morbee, J., and Tzimas, E. (2011). Technical and economic characteristics of a CO2 transmission pipeline infrastructure. JRC Scientific and Technical reports.

Shapley, L. (1953). A value for n-person games. Contributions to the Theory of Games, 2(28):307-317.

Smith, W. (1956). Various optimizers for single-stage production. Naval Research Logistics Quarterly, 3(1-2):59-66.

Sprumont, Y. (1990). Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic Behavior, 2(4):378-394.

Sudhölter, P. and Zarzuelo, J. M. (2017). Characterizations of highway toll pricing methods. European Journal of Operational Research, 260(1):161-170.

Summerfield, N. and Dror, M. (2013). Biform game: Reflection as a stochastic programming problem. International Journal of Production Economics, 142(1):124129.

Svensson, L.-G. (1999). Strategy-proof allocation of indivisible goods. Social Choice and Welfare, 16(4):557-567.

Syrgkanis, V. (2014). Efficiency of mechanisms in complex markets. PhD thesis, Cornell University.

Syrgkanis, V. and Tardos, E. (2013). Composable and efficient mechanisms. In Proceedings of the ACM Symposium on Theory of Computing (STOC), pages 211220.

Tan, R., Andiappan, V., Wan, Y., Ng, R., and Ng, D. (2016). An optimizationbased cooperative game approach for systematic allocation of costs and benefits in interplant process integration. Chemical Engineering Research and Design, 106:4358.

Teng, Y., Li, X., Wu, P., and Wang, X. (2019). Using cooperative game theory to determine profit distribution in IPD projects. International Journal of Construction Management, 19(1):32-45.

The Wall Street Journal (August 19, 1999). State governments consider reopening bidding for Berlin airport. The Wall Street Journal.

ZEP (2011). The costs of CO2 transport. www.zeroemmissionsplatform.eu.

## Academic summary

This dissertation focuses on the mathematical analysis of projects involving decisions by multiple actors. Projects involve a set of different parties, firms, or stakeholders, often referred to as players. These players all have their own capabilities, requirements, and incentives, but their (monetary) outcome is dependent on the decisions of other players as well. Game theory is a mathematical tool to analyze the interactive decision-making process, generally paired with a method to 'resolve' the conflict situation. The way in which players interact in such a situation is commonly divided in two categories, distinguishing between cooperative and competitive (non-cooperative) behavior.

Models within a cooperative framework study situations in which groups of players can cooperate by reaching a mutual agreement on a joint plan of action to maximize their joint payoff. This is generally paired with a specification of how to allocate this payoff. Cooperative games with transferable utility assign a (joint) value to every possible subset of players (called 'coalitions'), not only to the group of players as a whole (the 'grand' coalition). In principle, this game serves as a conservative and consistent benchmark to properly address the allocation problem for the grand coalition, taking into account coalitional incentives. By 'solving' the game, one finds allocations of the total joint value of the grand coalition to the players.

This stands in contrast to non-cooperative models, in which strategic players are interested in maximizing their individual payoffs, taking into account the strategic behavior of other players based on individual incentives only. Regarding solutions, (variants of) Nash equilibria are the main topic of interest.

Chapters 2 and 3 are written within a cooperative framework. Chapter 4 considers two-stage models, in which a non-cooperative first stage is followed by a cooperative second stage. Chapters 5 and 6 analyze non-cooperative models.

In Chapter 2, we define and axiomatically characterize a new proportional influence measure for sequential projects with imperfect reliability. We consider a model in which a finite set of players aims to complete a project, consisting of a finite number of tasks, which can only be carried out by certain specific players. Moreover, we assume the players to be imperfectly reliable, i.e., players are not guaranteed to carry out a task successfully. To determine which players are most important for the completion of a project, we use a proportional influence measure, where players' influence on the completion of each task within the project is measured in proportion to the likelihood that they complete it successfully. This chapter provides two characterizations of this influence measure. The most prominent property in the first characterization is task decomposability. This property describes the relationship between the influence measure of a project and the measures of influence one would obtain if one divides the tasks of the project over multiple independent smaller projects. Invariance under replacement is the most prominent property of the second characterization. If in a certain task group a specific player is replaced by a new player who was not in the original player set, this property states that this should have no effect on the allocated measure of influence of any other original player.

Chapter 3, provides a multi-actor perspective on the realization of new infrastructures, motivated by the necessity for infrastructures to support the ongoing climate and energy transition in general, and CO2 transport infrastructures for carbon capture, utilization and storage in particular. We develop a general model to represent infrastructures that allows for a unique decomposition into 'elementary infrastructure components' based on heterogeneous user requirements. Notably, it incorporates a cost function with a very generic and adaptable structure, for which we can still explicitly determine the costs of each individual component. As a direct consequence an intuitive cost allocation rule is obtained: equal component cost sharing. This allocation rule is in line with existing game-theoretic concepts and satisfies the desirable properties of advantageous scaling and coalitional rationality. Advantageous scaling guarantees that the costs allocated to each existing user do not increase if the number of users grows larger and coalitional rationality ensures that there is no subgroup of infrastructure users that would have a financial reason to object to the cost allocation. Additionally, we examine the application of our model to a prospective CO2 transport infrastructure for CCUS in the port of Rotterdam and the adjoining industry area.

Chapter 4, analyzes applications of biform games to linear production (LP) and sequencing processes. Biform games apply to problems in which strategic decisions are followed by a cooperative stage, where the specific format of the cooperative stage is determined by these strategic decisions. The cooperative stage corresponding to a strategy combination is then 'solved', leading to a unique payoff allocation vector. By associating a payoff vector with each possible strategy combination, the induced strategic game is determined. In biform LP-processes, we allow firms to compete for resources, rather than assuming the resource bundles are simply given. With strategy dependent resource bundles that can be obtained from two locations, we show that the induced strategic game has a pure Nash equilibrium, using the Owen set or any game-theoretic solution concept that satisfies anonymity to solve the second-stage cooperative LP-game. In biform sequencing processes, we no longer assume an initial processing order is given. Instead, this initial order is strategically determined by allowing players to request their preferred position in the initial order. Solving the second-stage cooperative sequencing game using a gain splitting rule, we fully determine the set of pure Nash equilibria of the induced strategic game.

In Chapter 5, we initiate the study of the social welfare loss (in utilitarian welfare terms) caused by corrupt auctioneers, both in single-item and multi-unit auctions. In our model, the auctioneer may collude with the winning bidders by letting them lower their bids in exchange for a (possibly bidder-dependent) fraction $\gamma \in[0,1]$ of the surplus: the difference between their bid and the highest losing bid. We consider different corruption schemes. In the most basic one, all winning bidders lower their bid to the highest losing bid. We show that this setting is equivalent to a $\gamma$-hybrid auction in which the payments are a convex combination of first-price and secondprice auction payments. More generally, we consider corruption schemes that can be related to $\gamma$-approximate first-price auctions ( $\gamma$-FPA) , where the payments recover at least a $\gamma$-fraction of the first-price payments. Our goal is to obtain a precise understanding of the robust price of anarchy of such auctions. If no restrictions are imposed on the bids, we establish a bound on the robust price of anarchy of $\gamma$-FPA which is tight for the single-item and the multi-unit auction setting. On the other hand, if bidders cannot overbid, a more fine-grained landscape of the price of anarchy emerges, depending on the auction setting and the equilibrium notion. Interestingly, we derive (almost) tight bounds for both auction settings and both pure Nash equilibria and coarse correlated equilibria.

Finally, Chapter 6 proposes a new refinement of Nash equilibria for bimatrix games. Most existing refinements are based on a thought experiment which imposes a certain 'imperfection' on the choices or payoffs of individual players. The equilibrium refinement proposed in this chapter deviates from the existing refinements by considering a thought experiment in which the imperfections occur on a 'system' level, instead of those corresponding (directly) to individual players. Imperfections are interpreted as the blocking of actions. If an imperfection occurs, the chosen actions are blocked for all players simultaneously, rather than for individual players. The idea behind this is that, after players submit their strategies, some entity converts these strategies into actions leading to payoffs. In this new thought experiment, with small probability, this entity makes an error that blocks the chosen actions instead of implementing them, and chooses a random combination of the remaining actions. Put differently, either the chosen actions are executed for all players, or no player actually plays their chosen action. In this way, there is an entanglement in the errors. We therefore refer to an equilibrium based on this thought experiment as an entangled equilibrium. Focusing on bimatrix games, we show that the set of entangled equilibria is a non-empty subset of the set of (mixed) Nash equilibria. Further, we discuss a geometric-combinatorial approach to determine all entangled equilibria of $2 \times n$ bimatrix games. Importantly, solving a $2 \times n$ bimatrix game for entangled equilibria requires relatively little extra work compared to finding Nash equilibria for the bimatrix game.

## Academische samenvatting

Dit proefschrift richt zich op de wiskundige analyse van projecten waarbij beslissingen worden genomen door meerdere actoren. Bij projecten zijn verschillende partijen, bedrijven of belanghebbenden betrokken, vaak aangeduid als spelers. Deze spelers hebben elk hun eigen capaciteiten, benodigdheden en (financiële) prikkels, maar hun uitkomst is ook afhankelijk van de beslissingen van andere spelers. Speltheorie is een wiskundig hulpmiddel om het interactieve besluitvormingsproces te analyseren, meestal gepaard met een methode om de conflictsituatie 'op te lossen'. Het type interactie tussen de spelers in een dergelijke situatie wordt doorgaans onderverdeeld in twee categorieën, namelijk coöperatief en competitief (niet-coöperatief) gedrag.

Modellen binnen een coöperatief kader bestuderen situaties waarin groepen spelers kunnen samenwerken door een wederzijdse overeenkomst te sluiten over een gemeenschappelijk actieplan om hun gezamenlijke uitbetaling te maximaliseren. Dit gaat meestal gepaard met een specificatie van hoe deze uitbetaling moet worden verdeeld over de spelers. Coöperatieve spellen met overdraagbaar nut kennen een (gezamenlijke) waarde toe aan elke mogelijke subset van spelers ('coalities'), niet alleen aan de groep spelers als geheel (de 'grote coalitie'). Dit spel dient als referentiekader om het allocatieprobleem voor de grote coalitie consistent op te lossen, rekening houdend met de (financiële) uitkomsten van alle coalities. Door het spel op te lossen vindt men allocaties van de totale gezamenlijke waarde van de grote coalitie aan de spelers.

Dit staat in contrast met niet-coöperatieve modellen, waarin strategische spelers geïnteresseerd zijn in het maximaliseren van hun individuele uitbetalingen, rekening houdend met het strategische gedrag van andere spelers op basis van alleen de individuele prikkels. Wat oplossingen betreft, zijn (varianten van) Nash-evenwichten het belangrijkste onderwerp van interesse.

Hoofdstukken 2 en 3 zijn geschreven in een coöperatief kader. Hoofdstuk 4 behandelt modellen waarin een niet-coöperatieve eerste fase wordt gevolgd door een coöperatieve tweede fase. Hoofdstukken 5 en 6 analyseren niet-coöperatieve modellen.

In Hoofdstuk 2 definiëren we een nieuwe proportionele invloedsmaat voor sequentiële projecten met imperfecte betrouwbaarheid. We beschouwen een model waarin een eindige verzameling spelers een project wil voltooien dat bestaat uit een eindig aantal taken die alleen kunnen worden uitgevoerd door bepaalde specifieke spelers. Bovendien nemen we aan dat de spelers imperfect betrouwbaar zijn; spelers kunnen in het algemeen niet garanderen dat ze een taak succesvol uitvoeren. Om te bepalen welke spelers het belangrijkst zijn voor de voltooiing van een project, gebruiken we een proportionele invloedsmaat, waarbij de invloed van spelers op de voltooiing van elke taak binnen het project wordt gemeten in verhouding tot de kans dat ze de taak succesvol voltooien. Dit hoofdstuk geeft twee axiomatische karakteriseringen van deze invloedsmaat. De belangrijkste eigenschap van de eerste karakterisering is 'taakontleedbaarheid'. Deze eigenschap beschrijft de relatie tussen de invloedsmaat van een project en de invloedsmaat die je zou krijgen als je de taken van het project zou verdelen over meerdere onafhankelijke kleinere projecten. 'Invariantie bij vervanging' is de belangrijkste eigenschap van de tweede karakterisering. Als in een bepaalde taakgroep een specifieke speler wordt vervangen door een nieuwe speler die niet in de oorspronkelijke spelersgroep zat, stelt deze eigenschap dat dit geen effect mag hebben op de toegekende invloedsmaat van een andere oorspronkelijke speler.

Hoofdstuk 3 biedt een multi-actorperspectief op de totstandkoming van nieuwe infrastructuren, gemotiveerd door de behoefte aan infrastructuren ter ondersteuning van de huidige klimaat- en energietransitie in het algemeen, en CO2-transportinfrastructuren voor koolstofafvang, -gebruik en -opslag in het bijzonder. We ontwikkelen een algemeen model om infrastructuren te beschrijven dat een unieke decompositie in 'elementaire infrastructuurcomponenten' mogelijk maakt op basis van heterogene gebruikerseisen. In het bijzonder bevat dit model een kostenfunctie met een zeer generieke en aanpasbare structuur, waarvoor we nog steeds expliciet de kosten van elk individueel component kunnen bepalen. Hieruit volgt direct een intuïtieve kostenallocatieregel: de kosten per component worden gelijk verdeeld over de spelers die dit component nodig hebben. Deze toewijzingsregel is in lijn met bestaande speltheoretische concepten en voldoet aan twee wenselijke eigenschappen: 'voordelige schaling' en 'coalitionele rationaliteit'. Voordelige schaling garandeert dat de kosten die aan elke bestaande gebruiker worden toegewezen niet stijgen als het aantal gebruikers toeneemt. Coalitionele rationaliteit zorgt ervoor dat er geen subgroep van infrastructuurgebruikers is die een financiële reden zou hebben om bezwaar te maken tegen de
kostentoewijzing. Verder analyseren we de toepassing van ons model op een mogelijke CO2-transportinfrastructuur voor koolstofafvang, -gebruik en -opslag in de haven van Rotterdam en het aangrenzende industriegebied.

Hoofdstuk 4 analyseert toepassingen van 'biforme' spellen op lineaire productie (LP) processen en sequencingprocessen (gerelateerd aan wachtrijtheorie). Biforme spellen zijn van toepassing op problemen waarin strategische beslissingen worden gevolgd door een coöperatieve fase, waarbij de specifieke uitgangssituatie in de coöperatieve fase bepaald wordt door deze strategische beslissingen. De coöperatieve fase die overeenkomt met een strategiecombinatie wordt dan 'opgelost', wat leidt tot een unieke uitbetalingsallocatievector. Door aan elke mogelijke strategiecombinatie een uitbetalingsvector te koppelen, wordt het 'geïnduceerde strategische spel' bepaald. In biforme LP-processen stellen we bedrijven in staat om te concurreren voor grondstoffen, in plaats van aan te nemen dat de grondstoffenbundels simpelweg gegeven zijn. Met strategie-afhankelijke grondstoffenbundels die kunnen worden verkregen op twee locaties, tonen we aan dat het geïnduceerde strategische spel een zuiver Nash-evenwicht heeft, waarbij gebruik wordt gemaakt van de Owen-set of een willekeurig speltheoretisch oplossingsconcept dat voldoet aan anonimiteit om het coöperatieve LP-spel van de tweede fase op te lossen. Bij biforme sequencingprocessen gaan we er niet langer van uit dat een initiële verwerkingsvolgorde gegeven is. In plaats daarvan wordt deze initiële volgorde strategisch bepaald door spelers de mogelijkheid te geven om hun voorkeurspositie in de initiële volgorde aan te vragen. Door het coöperatieve spel van de tweede fase op te lossen met een 'winstsplitsingsregel', bepalen we de volledige verzameling pure Nash-evenwichten van het geïnduceerde strategische spel.

In Hoofdstuk 5 initiëren we de studie van het sociale welvaartsverlies (in utilitaire welvaartstermen) veroorzaakt door corrupte veilingmeesters, zowel bij veilingen met één item als bij veilingen met meerdere eenheden. In ons model kan de veilingmeester samenspannen met de winnende bieders door hen hun bod te laten verlagen in ruil voor een (mogelijk van de bieder afhankelijke) fractie $\gamma \in[0,1]$ van het surplus: het verschil tussen hun bod en het hoogste verliezende bod. We beschouwen verschillende corruptieschema's. In de meest basale variant verlagen alle winnende bieders hun bod tot het hoogste verliezende bod. We laten zien dat dit equivalent is aan een $\gamma$-hybride veiling waarin de betalingen een convexe combinatie zijn van de betalingen bij 'eerste prijs' (winnaar betaalt het winnende bod) en 'tweede prijs' (winnaar
betaalt het hoogste verliezende bod) veilingen. Ter veralgemenisering beschouwen we corruptieschema's die gerelateerd kunnen worden aan ' $\gamma$-approximate first-price auctions' ( $\gamma$-FPA), waarbij de betalingen ten minste een fractie $\gamma$ van de betalingen uit de eerste prijs veiling opleveren. Ons doel is om een nauwkeurig begrip te krijgen van de 'price of anarchy' (POA) van dergelijke veilingen. Als er geen beperkingen worden opgelegd aan de biedingen, vinden we de precieze waarde van de POA voor $\gamma$-FPA voor veilingen met één of meerdere items. Als bieders niet kunnen overbieden, ontstaat er een fijnmaziger landschap van de POA, afhankelijk van de veilingsetting en het evenwichtsconcept. We leiden scherpe limieten af voor beide veilingsettings en voor zowel pure Nash-evenwichten als 'coarse correlated' evenwichten.

Tenslotte stelt Hoofdstuk 6 een nieuwe verfijning van Nash-evenwichten voor bimatrixspellen voor. De meeste bestaande verfijningen zijn gebaseerd op een gedachteexperiment dat een zekere 'imperfectie' oplegt aan de keuzes of uitbetalingen van individuele spelers. De evenwichtsverfijning die in dit hoofdstuk wordt voorgesteld wijkt af van de bestaande verfijningen door een gedachte-experiment te beschouwen waarin de imperfecties op 'systeemniveau' optreden, in plaats van op het niveau van individuele spelers. Imperfecties worden geïnterpreteerd als het blokkeren van acties. Als een imperfectie optreedt, worden de gekozen acties geblokkeerd voor alle spelers tegelijk, in plaats van voor individuele spelers. Het idee hierachter is dat, nadat spelers hun strategieën hebben ingediend, een entiteit deze strategieën omzet in acties die leiden tot uitbetalingen. In dit nieuwe gedachte-experiment maakt deze entiteit met kleine waarschijnlijkheid een fout die de gekozen acties blokkeert in plaats van uitvoert, en kiest deze een willekeurige combinatie van de overgebleven acties. Anders gezegd, ofwel worden de gekozen acties uitgevoerd voor alle spelers, ofwel speelt geen enkele speler zijn gekozen actie. Op deze manier is er een verstrengeling in de fouten. Daarom noemen we een evenwicht gebaseerd op dit gedachte-experiment een 'entangled' (verstrengeld) evenwicht. Voor bimatrixspellen tonen we aan dat de verzameling van verstrengelde evenwichten een niet-lege deelverzameling is van de verzameling van (gemengde) Nash-evenwichten. Verder bespreken we een geometrisch-combinatorische aanpak om alle verstrengelde evenwichten van $2 \times n$ bimatrixspellen te bepalen. Belangrijk is dat het oplossen van een $2 \times n$ bimatrixspel voor verstrengelde evenwichten relatief weinig extra werk kost vergeleken met het vinden van Nash-evenwichten voor het bimatrixspel.

## Summary for non-experts

This dissertation focuses on the mathematical modeling of group decisions. Though there is great variety in the types of situations modeled in this dissertation, the commonality is that in all cases the situation somehow involves decisions made by multiple actors. All models are in some way related to game theory. In game theory, the actors are referred to as players. In general, these players all have their own capabilities, requirements, and incentives, but their (monetary) outcome is dependent on the decisions of other players as well. Game theory is a mathematical tool to analyze the interactive decision-making process, generally paired with a method to 'resolve' the conflict situation. The way in which players interact in such a situation is commonly divided in two categories, distinguishing between cooperative and competitive (non-cooperative) behavior. The different models in this dissertation follow a similar division between collaborative projects and problems with strategic individual behavior, or a combination thereof.

Cooperative models study situations in which groups of players can cooperate by reaching a mutual agreement on a joint plan of action, leading to joint payoffs, costs, or cost savings, generally paired with a specification of how these should be allocated to the different players. Such allocation mechanisms are often called 'solutions'. For example, Chapter 3 analyzes situations in which players cooperate on the construction of a new infrastructure. Collaborating on a joint infrastructure that meets the requirements of all players, rather than constructing a separate infrastructure for each individual player, leads to cost savings. Appropriate allocation of the joint costs, such that all players can only benefit from collaboration on the infrastructure construction project, can be a key enabler for the successful realization of such projects. The model of this chapter is developed primarily with a practically relevant application in mind: CO2 transport infrastructure for industrial decarbonization. We extensively study a
concrete case of a prospective CO2 transport infrastructure for carbon capture, utilization and storage in the port of Rotterdam and the adjoining industry area. An important element of this model is the way in which it incorporates heterogeneity in the requirements of potential users of this infrastructure. Through this, and using a model inspired by cooperative game theory, we propose a well-substantiated method to allocate the total infrastructure construction costs to the users.

On the other hand, there are non-cooperative models. In these models, strategic players focus on maximizing their individual payoffs, taking into account the strategic behavior of other players based on individual incentives only. The main solution concept studied in this context is that of a (Nash) equilibrium. Intuitively, some combination of strategies is an equilibrium if no players has a (financial) incentive to unilaterally deviate and choose a different strategy instead. An auction, in which players compete by strategically submitting their (sealed) bids to obtain items, is a prominent example of interactive decision-making by multiple players in a non-cooperative setting. Chapter 5 considers auctions with a corrupt auctioneer. Corruption in auctions, where auctioneers manipulate the submitted bids (referred to as 'bid rigging') to their own benefit, occurs in practice, especially in the public sector. However, even though this bid rigging has been studied and observed in practice, its impact on social welfare (measured as the total valuation of players for the items they win in the auction) is still poorly understood. We contribute to this understanding by initiating the study of social welfare loss caused by corrupt auctioneers in fundamental auction settings. Specifically, we try to bound the worst-case ratio between the optimal social welfare and the social welfare of an equilibrium, the so-called price of anarchy.

Chapter 2 considers collaborative projects consisting of a number of tasks to be carried out by a set of players. Each task can only be carried out by a subset of all players, so players may have to cooperate to complete a project. In this context, we define a new solution that measures the influence each player has on the completion of the project, and show this solution is the only one satisfying two sets of properties.

Chapter 4 considers two-stage models, in which a non-cooperative first stage is followed by a cooperative second stage. Conceptually, the specific format of the cooperative stage is determined by strategic decisions in the first stage.

Finally, Chapter 6 presents a new solution concept that refines the notion of Nash equilibria for a general class of non-cooperative games, called bimatrix games.

## Samenvatting voor niet-deskundigen

Dit proefschrift richt zich op het wiskundig modelleren van groepsbeslissingen. Hoewel de situaties die in dit proefschrift worden gemodelleerd sterk uiteenlopen, is de overeenkomst dat de situatie in alle gevallen beslissingen omvat die door meerdere actoren worden genomen. Alle modellen zijn op enigerlei wijze gerelateerd aan speltheorie. In speltheorie worden de actoren spelers genoemd. In algemene zin hebben deze spelers elk hun eigen capaciteiten, benodigdheden en (financiële) prikkels, maar hun uitkomst is ook afhankelijk van de beslissingen van andere spelers. Speltheorie is een wiskundig hulpmiddel om het interactieve besluitvormingsproces te analyseren, meestal gepaard met een methode om de conflictsituatie 'op te lossen'. Het type interactie tussen de spelers in een dergelijke situatie wordt doorgaans onderverdeeld in twee categorieën, namelijk coöperatief en competitief (niet-coöperatief) gedrag. De verschillende modellen in dit proefschrift volgen een vergelijkbare verdeling.

Coöperatieve modellen bestuderen situaties waarin groepen spelers kunnen samenwerken door een wederzijdse overeenkomst te sluiten over een gemeenschappelijk actieplan, wat leidt tot gezamenlijke uitbetalingen, kosten of kostenbesparingen, meestal gepaard met een specificatie van hoe deze moeten worden toegewezen aan de verschillende spelers. Zulke allocatiemechanismen worden vaak 'oplossingen' genoemd. In Hoofdstuk 3 worden situaties geanalyseerd waarin spelers samenwerken bij de aanleg van een nieuwe infrastructuur. Samenwerken aan een gezamenlijke infrastructuur die voldoet aan de eisen van alle spelers, in plaats van het bouwen van een aparte infrastructuur voor elke individuele speler, leidt tot kostenbesparingen. Een juiste allocatie van de gezamenlijke kosten, zodat alle spelers profiteren van de samenwerking aan het infrastructuurbouwproject, kan een belangrijke factor zijn voor de succesvolle realisatie van dergelijke projecten. Het model in dit hoofdstuk is voornamelijk ontwikkeld met een praktisch relevante toepassing in gedachten: CO2-transportinfrastructuur voor industriële decarbonisatie. We bestuderen een concrete casus van een mogelijke CO2-transportinfrastructuur voor koolstofafvang, -gebruik en -opslag in de haven van

Rotterdam en het aangrenzende industriegebied. Een belangrijk element van dit model is de manier waarop het rekening houdt met de heterogene eisen van potentiële gebruikers van deze infrastructuur incalculeert. Hierdoor, en door gebruik te maken van een model geïnspireerd op de coöperatieve speltheorie, stellen we een goed onderbouwde methode voor om de totale bouwkosten toe te wijzen aan de gebruikers.

Aan de andere kant zijn er niet-coöperatieve modellen. In deze modellen richten strategische spelers zich op het maximaliseren van hun individuele uitbetalingen, rekening houdend met het strategische gedrag van andere spelers op basis van hun individuele prikkels. Het belangrijkste oplossingsconcept voor dergelijke modellen is het (Nash) evenwicht. Intuïtief is een combinatie van strategieën een evenwicht als geen enkele speler een (financiële) prikkel heeft om eenzijdig van strategie af te wijken en in plaats daarvan een andere strategie te kiezen. Een veiling, waarin spelers concurreren door strategisch hun biedingen uit te brengen om items te verkrijgen, is een prominent voorbeeld van interactieve besluitvorming door meerdere spelers in een niet-coöperatieve context. Hoofdstuk 5 beschouwt veilingen met een corrupte veilingmeester. Corruptie bij veilingen, waarbij veilingmeesters de uitgebrachte ('gesloten') biedingen manipuleren, komt in de praktijk voor, vooral in de publieke sector. Hoewel dit manipuleren van biedingen in de praktijk is bestudeerd en waargenomen, is de invloed ervan op de sociale welvaart (gemeten als de totale waardering van spelers voor de items die ze winnen in de veiling) nog steeds onduidelijk. Wij dragen bij aan dit inzicht met de analyse van sociaal welvaartsverlies veroorzaakt door corrupte veilingmeesters in fundamentele veilingcontexten. Specifiek proberen we de slechtste verhouding tussen de optimale sociale welvaart en de sociale welvaart van een evenwicht, de zogenaamde 'prijs van anarchie', te begrenzen.

Hoofdstuk 2 behandelt collaboratieve projecten die bestaan uit een aantal taken die uitgevoerd moeten worden door een aantal spelers. Elke taak kan slechts worden uitgevoerd door een deelverzameling van alle spelers, dus het kan zijn dat spelers moeten samenwerken om een project te voltooien. In deze context definiëren we een nieuwe oplossing die de invloed meet van elke speler op de voltooiing van het project, en we laten zien dat deze oplossing de enige is die voldoet aan twee groepen eigenschappen.

Hoofdstuk 4 beschouwt tweefasenmodellen, waarin een niet-coöperatieve eerste fase wordt gevolgd door een coöperatieve tweede fase. De specifieke uitgangssituatie in de coöperatieve fase wordt bepaald door strategische beslissingen in de eerste fase.

Tot slot presenteert Hoofdstuk 6 een nieuw oplossingsconcept dat Nash-evenwichten verfijnt voor 'bimatrixspelen', een algemene klasse van niet-coöperatieve spelen.

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Andries van Beek (Houten, The Netherlands, 1995) received his Bachelor's degree in Econometrics and Operations Research from Tilburg University in 2016, followed by a Master's degree in Business Analytics and Operations Research in 2018 and a Research Master degree in Operations Research in 2019 from the same university. In September 2019, he became a PhD candidate in Operations Research at the Department of Econometrics and Operations Research at Tilburg University.

This dissertation focuses on the mathematical analysis of projects involving decisions by multiple players. These players all have their own capabilities, requirements, and incentives, but their (monetary) outcome is dependent on the decisions of other players as well. Game theory is a mathematical tool to analyze the interactive decision-making process, generally paired with a method to 'resolve' the conflict situation. The way in which players interact in such a situation is commonly divided in two categories, distinguishing between cooperative and competitive (non-cooperative) behavior. This dissertation first studies two models within a cooperative framework, starting with the definition and analysis of a new influence measure for general, collaborative projects. The second model applies to situations where players cooperate on the construction of a new joint infrastructure, with a specific focus on cost allocation for CO2 transport infrastructure. Next, two-stage models are considered, in which a noncooperative first stage is followed by a cooperative second stage. Subsequently, social welfare loss in auctions with a corrupt auctioneer is studied. Finally, a new solution concept is presented that refines the notion of Nash equilibria for a general class of non-cooperative games.

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[^0]:    ${ }^{2}$ In fact, it is not clear how exactly the conditioning requirements influence the infrastructure construction costs. Unfortunately, Knoope (2015) concludes in her CO2 infrastructure cost model review that cost models for pumping stations (related to conditioning of the CO 2 ) are not validated, and also EBN (2018) notes that there is no reference cost known for pumping stations. Mallon et al. (2013) apply a cost model to estimate the investments costs of pumping stations, but the actual investment costs seem to be 3 to 5 times higher than its estimates.

[^1]:    ${ }^{1}$ Note that if $x, y \in X$ are such that $\sigma_{0}^{\tau}(x)=\sigma_{0}^{\tau}(y)$, then $\pi_{i}^{\mathcal{Q}, \lambda}(x)=\pi_{i}^{\mathcal{Q}, \lambda}(y)$ for all $i \in N$.

[^2]:    ${ }^{1}$ Formally, there does not exist a tie-breaking rule that ensures that, for any bidding profile, the allocation of items to bidders does not change when all winning bidders lower their bids to the highest losing bid. To circumvent this tie-breaking issue, one can consider non-uniform bid rigging schemes instead, where bidders lower their bids in different ways, and such that they turn out higher than the highest losing bid. An example of such a bid rigging scheme that fits within our framework is given in Example 5.7.1.

[^3]:    ${ }^{2}$ Here, $0.607 \ldots$ corresponds to the exact (unrounded) intersection point between the bounds of (5.5) and (5.6). In the remainder of this chapter, several of such (intersection) points will be presented in a similar manner; it will be clear from context to which exact solution the point corresponds.

[^4]:    ${ }^{1}$ This in particular means that the perturbations here are not of the Kohlberg-Mertens type (c.f. Kohlberg and Mertens (1986)).

[^5]:    ${ }^{2}$ It is readily verified that if, e.g., $\left|\left\{B_{1 j} \mid f_{j} \in P B_{2}\left(e_{1}\right)\right\}\right|=1$, then $S^{e}\left(e_{1}\right) \subseteq \bar{S}^{e}\left(e_{1}\right)$.

