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Approximate Symmetries and Conservation Laws for Mechanical Systems Described by Mixed Derivative Perturbed PDEs

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Abstract. This article focuses on developing and applying approximation techniques to derive conservation laws for the Timoshenko–Prescott mixed derivatives perturbed partial differential equations (PDEs). Central to our approach is employing approximate Noether-type symmetry operators linked to a conventional Lagrangian one. Within this framework, this paper highlights the creation of approximately conserved vectors for PDEs with mixed derivatives. A crucial observation is that the integration of these vectors resulted in the emergence of additional terms. These terms hinder the establishment of the conservation law, indicating a potential flaw in the initial approach. In response to this challenge, we embarked on the rectification process. By integrating these additional terms into our model, we could modify the conserved vectors, deriving new modified conserved vectors. Remarkably, these modified vectors successfully satisfy the conservation law. Our findings not only shed light on the intricate dynamics of fourth-order mechanical systems but also pave the way for refined analytical approaches to address similar challenges in PDE-driven systems.

Keywords: beams oscillations, traveling-wave reduction, conserved vectors, Noether approach.

1 Introduction

Perturbated partial differential equations (PDEs) are foundational laws for delineating core principles across the mechanical and chemical engineering disciplines. Specifically, within mechanical engineering, fourth-order mixed-derivative perturbed PDEs are instrumental in elucidating the dynamics of beam oscillations, stress distribution in complex materials, and the vibrational behavior of mechanical structures.

On the other hand, in chemical engineering, PDEs are extensively applied in areas such as diffusion-reaction processes, mass and heat transfer in heterogeneous systems, and kinetic modeling of complex chemical reactions. By offering a mathematical framework for these diverse phenomena, perturbed PDEs fortify our understanding and predictive capabilities in both engineering domains.

Conversely, PDEs are used to anticipate the behavior of complex systems in biology and economics. In computer science, perturbed PDEs can be used to simulate complex interactions of nonlinear events. PDEs may be used to describe almost any natural phenomenon.

Perturbed PDEs play a critical role in engineering [1], offering valuable insights into the behavior of systems subjected to minute disturbances or uncertainties. Spanning domains from fluid mechanics to geotechnical engineering, they enable professionals to gauge the ripple effects of small changes in complex systems (e.g., the transition from laminar to turbulent flows), the dynamics of structural vibrations, and the dispersion of pollutants in environmental contexts [2]. Engineers employing

perturbed PDEs are better equipped to design robust, efficient, and safer systems, understand the nuanced impacts of variations, and ensure optimal performance under diverse conditions.

2 Literature Review

For perturbed PDEs, approximate symmetries and approximate conservation laws are the most important subjects. The significant contribution to the derivation of the Euler–Lagrange equation was given by Emmy Noether in 1918. The approximate conservation laws in [3, 4] were introduced via the approximate Noether-type symmetries associated with the Lagrangian of the perturbed equation. If one has obtained the approximate Noethe symmetries for the Euler–Lagrange equation by using Noether's theorem, it is very convenient to construct the approximate conservation laws. Several differential equations do not admit nontrivial exact approximate Lie symmetries. We recall that Lie's integration theorem [5] does not apply to generating group-invariant solutions. Noether's theorem [6, 7] is also useless for those perturbed differential equations where no nontrivial exact approximate Lie symmetries exist. These PDEs can be analyzed by splitting them into unperturbed and perturbed parts, provided the former admits exact approximate Lie symmetries.

Additionally, developing perturbed PDEs that do not satisfy nontrivial exact approximate Lie symmetries is significant using the perturbation method [8, 9]. They offered the approximate version of Lie's theorem.

The approximate version of Noether's theorem was developed by Govinder et al. [10]. For perturbed differential equations, portions of the unperturbed equation's local symmetries may (or may not) reemerge as approximate symmetries and new approximate symmetries may develop. Approximate symmetries are helpful tools for developing approximate solutions. We show an approximate point symmetry of the perturbed equation for every point symmetry of the unperturbed equation for algebraic and first-order ordinary differential equations (ODEs). This is not the case for second-order and higher-order ODEs. Some point symmetries of the original ODE may be unstable; they do not appear in the perturbed ODE's approximate point symmetry classification. We illustrate that such unstable point symmetries relate to higher-order approximation symmetries of the perturbed ODE and can be determined systematically.

In [11], it was shown that the association between approximate Lie Backlund symmetries and approximate conserved vectors may be utilized to construct approximation conservation

rules for perturbed equations that do not have a Lagrangian. Conservation laws may be constructed for unperturbed PDEs that accept partial Lagrangians using Noether-type symmetry operators. In [7], the authors looked at how to approximate Lie Backlund symmetries and how conserved vectors might be utilized to construct approximate Lagrangians for perturbed equations. This study extended the previous discovery to perturbed PDEs with mixed derivative components. We provide novel techniques for constructing approximation conservation laws of perturbed equations using approximate operators that are not always approximate symmetry operators of the perturbed differential equations. We note out that when we use Lagrangian or partial Lagrangian to approximate Noether's theorem conservation laws, the mixed derivative terms include differentiation by more than one of the independent variables. When we insert them into the approximation equation (divergence relation), several new terms emerge.

As a result, we have trivial conserved values that must be immediately entered into the conserved vectors derived using Noether's theorem in the first place. These concepts are essential to verify that conserved flows and symmetries have a relationship. [12, 13].

These techniques were used to find the approximate Noether-type symmetries for the Timoshenko–Prescott mixed derivatives perturbed PDEs [14, 15]. We mention that the Timoshenko–Prescott equation and the Rayleigh equation are algebraic equivalents but different from the

Euler-Bernoulli equation admits a higher dimensional Lie algebra. The beam equation, in addition to deflection, describes forces and moments and may thus be used to explain stresses. For the plane stress problems, a Timoshenko beam theory [16, 17] is proposed. The theory is built by a novel combination of critical components (kinematic displacements, stress and strain moments, and axially invariant plane stress) to calibrate the relationships between all these quantities [18].

Prescott equations [19] use average through-thickness displacement and average rotation variables to calibrate thin rods. We shall see mixed fourth-order terms in Timoshenko–Prescott mixed derivatives perturbed PDEs.

Approximately conserved vectors do not satisfy the divergence relation, and some extra terms arise. We observe that mixed derivatives terms arise in every conservation law. By taking differentials and adding extra terms to conserved vectors, we get modified conserved vectors that satisfy the divergence relation. This equation has many applications in engineering fields. The presence of fourth-order spatial derivatives is reminiscent of beam deflection equations [20], where fourth-order derivatives describe the curvature of a beam under various loads. The time derivatives could suggest that this equation models the beam's dynamic response, perhaps due to time-varying forces or boundary conditions. In many real-world engineering problems, the systems can be too complex to solve analytically.

Approximate symmetries can be used to reduce the complexity of the model, providing a more straightforward yet reasonably accurate representation. Conservation laws provide deep insights into the behavior of systems [21]. For instance, a conservation law associated with energy or momentum can clarify the system's inherent properties and responses to external influences. In engineering designs, understanding the symmetries and conservation laws can lead to better

control strategies or optimizations in the system, ensuring that it performs optimally under given conditions.

The paper is organized as follows. A concise background of Timoshenko–Prescott equations collected in the Research Methodology. It also contains the basic definitions, operators, and equations required to accomplish the work. The approximate Noether-type symmetries are computed in the Results, where the approximate conservation laws are also found by using approximate Noether-type symmetries. Finally, some concluding remarks are given in the Discussion.

3 Research Methodology

3.1 Scientific background

Nonlinear transport and shock waves have always played a significant role in studying fluid dynamics, particularly in understanding abrupt changes in properties such as pressure and density. Such phenomena can often be attributed to events such as supersonic jet flows or explosions. On the structural side, the oscillation of beams, plates, and shells forms a vital area of study in mechanical engineering, particularly in predicting the behavior of structures under various load conditions. These oscillations, arising from external forces or inherent structural properties, can determine the stability and longevity of structures, ranging from bridges to spacecraft components. Nonhomogeneous diffusion, often encountered in material science and chemical engineering, refers to the process in which the diffusion coefficient varies spatially. This can result from temperature gradients, composition differences, or other external conditions that affect the system. Regardless of the sudden onset of shock waves, rhythmic movement of architectural elements, or irregular spread of particles in a medium, these nonlinear and nonhomogeneous phenomena play a pivotal role in shaping the outcomes in their respective fields.

We consider the following r-th order($r \geq 1$) system of perturbed differential equations with n independent variables $x = (x^1, x^2, \dots\dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots\dots, u^m)$ with $\in$

$$E^\sigma\big(x, u, u_{(1)}, \dots\dots, u_{(r)}; \epsilon\big) = 0;\ \sigma = 1, 2, \dots, l. \quad (1)$$

Each E^σ represents a perturbed PDE of order to r, where $u_{(1)} = u_i^\alpha$, $u_{(2)} = u_{i_1 i_2}^\alpha$, $u_{(r)} = u_{i_1, i_2, \dots, i_r}^\alpha$, $\alpha = 1, 2, \dots, m$, and $i, i_1, \dots, i_r = 1, 2, \dots, n$ are the collections of all the first, second up to r-th order partial derivatives. We say that $u_{ij}^\alpha = u_{ji}^\alpha, u_2$ consisting only terms u_{ij}^α, for which $i \leq j$. Simultaneously, $u_{(3)}$ has only terms for $i \leq j \leq r$.

We observe that $u_i^\alpha = D_i(u^\alpha)$, $u_{ji}^\alpha = D_i D_j(u^\alpha)$, and so on, where D_i is a total differentiation operator with respect to x^i defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \dots, i = 1, 2, \dots \quad (2)$$

The Euler–Lagrange operator for each α is given as follows [22]:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \dots\dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1, i_2, \dots\dots, i_s}}; \quad (3)$$

for $\alpha = 1, 2, \dots, m$.

The k-th order approximate Lie–Bäcklund symmetry operator is defined as follows [23]:

$$\mathcal{H} = X_0 + \varepsilon X_1 + \cdots \dots\dots + \epsilon^k X_k; \quad (4)$$

$$\mathcal{H} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha};\ \xi^i, \eta^\alpha \in F; \quad (5)$$

where F is the space of the differential functions and $\xi^i = \xi_0^i + \epsilon\xi_1^i + \cdots \dots + \epsilon^k \xi_k^i$ $(i = 1, 2, \dots, n)$; $\eta^\alpha = \eta_0^\alpha + +\epsilon\eta_1^\alpha + \cdots \dots + \epsilon^k \eta_k^\alpha$ $(\alpha = 1, 2, \dots, m)$.

The operator in equation (5) originates from the following indefinite formal sum:

$$\mathcal{H} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1, i_2, \dots, i_s} \frac{\partial}{\partial u^\alpha_{i_1, i_2, \dots, i_s}}. \quad (6)$$

Additionally, the following can be considered:

$$X_b = \xi_b^i \frac{\partial}{\partial x^i} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_{b,i}^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{b,i_1,i_2}^\alpha \frac{\partial}{\partial u_{i_1,i_2}^\alpha} + \cdots,$$

where $b = 0, 1, \dots, k$; $\xi^i, \eta^\alpha \in F$, and $\zeta_{b,i}^\alpha = D_i\big(\widehat{W}_b^a\big) + +\xi_b^i u_{ij}^\alpha$; $\zeta_{b,i_1,i_2}^\alpha = D_{i_1} D_{i_2}\big(\widehat{W}_b^a\big) + \xi_b^i u_{ji_1}^\alpha$.

Now, $\widehat{W}_b^a$ is the Lie characteristic function:

$$\widehat{W}_b^a = \eta_b^\alpha - \xi_b^i u_j^\alpha,$$

where $\mathbb{W} = (\mathbb{W}^1, \mathbb{W}^2, \dots\dots, \mathbb{W}^m), \mathbb{W}^\beta \in F$ is an approximate characteristic of $\mathcal{H}$, for which the following can be considered for $i = 1, 2, \dots, n$:

$$\mathbb{W}^i = \widehat{W}_0^i + \varepsilon\widehat{W}_1^i + \cdots \dots + \epsilon^k \widehat{W}_k^i. \quad (7)$$

Equation (1) can be rewritten as follows:

$$E^\sigma = E^\sigma + \epsilon E^\sigma;\ \sigma = 1, 2, \dots, m. \quad (8)$$

If there exists and nonzero functions $\psi_\gamma^\sigma \in F$ and function $\mathfrak{L} = \mathfrak{L}\big(x, u, u_{(1)}, \dots\dots, u_{(l)}; \epsilon\big), r \geq l$, the last equation can be written as the following Euler–Lagrange type equation:

$$\frac{\delta\mathfrak{L}}{\delta u^\alpha} = \varepsilon\psi_\gamma^\sigma E_1^\gamma, \quad (9)$$

where $\psi_\gamma^\sigma = \psi_\gamma^\sigma\big(x, u, u_{(1)}, \dots\dots\dots, u_{(r-1)}\big)$ is an invertible matrix [22], then supplied $(\sigma, \gamma = 1, 2, \dots, m)$.

For some γ and $\mathfrak{L}$, $E_1^\gamma \neq 0$ is a partial Lagrangian of equation (8).

The function $\mathfrak{L}\big(x, u, u_{(1)}, \dots\dots, u_{(l)}; \epsilon\big)$ can be written in perturbated form as follows.

$$\mathfrak{L}\big(x, u, u_{(1)}, \dots\dots, u_{(l)}; \epsilon\big) = \mathfrak{L}_0\big(x, u, u_{(1)}, \dots\dots, u_{(l)}\big) + +\varepsilon\mathfrak{L}_1\big(x, u, u_{(1)}, \dots\dots, u_{(l)}\big).$$

If $E_1^\gamma = 0$, then $\mathfrak{L}$ in equation (9) is a standard Lagrangian that satisfies the following condition:

$$\frac{\delta\mathfrak{L}}{\delta u} = 0. \quad (10)$$

The approximate Lie–Bäcklund symmetry operator $\mathcal{H}$ given in (6) is an approximate Noether-type symmetry. It corresponds to the standard Lagrangian $\mathfrak{L} \in F$ in equation (9) if and only if there exists a vector $\mathfrak{B}^i = (B^1, B^2, \dots\dots, B^n)$, $\mathfrak{B}^i \in F$ defined as follows:

$$\mathfrak{B}^i = \hat{B}_0^i + \epsilon \hat{B}_1^i + \cdots . + \epsilon^k \hat{B}_k^i,$$

such that

$$\mathcal{H}(\mathfrak{L}) + \mathfrak{L} D_i(\xi^i) = D_i(\mathfrak{B}^i) + O(\epsilon^{k+1}). \quad (11)$$

The higher-order terms of ϵ can be ignored. Also, the terms with respect to zero and first-order of ε can be separated. Then, the following equations can be obtained:

$$\begin{aligned} X_0(\mathfrak{L}) + \mathfrak{L} D_i\left(\xi_0{}^i\right) \mathfrak{L} &= D_i\left(\hat{B}_0^i\right); \\ X_1(\mathfrak{L}) + \mathfrak{L} D_i\left(\xi_1{}^i\right) \mathfrak{L} &= D_i\left(\hat{B}_1^i\right). \end{aligned} \quad (12)$$

The zeroth-order system and the first-order system for the approximate Noether-type symmetry can be obtained for determining the equation (11).

The vector $\mathcal{J} = (\mathcal{J}^1, \mathcal{J}^2, \dots\dots \mathcal{J}^n)$ [3, 11]

$$\mathcal{J}^i = T_0^i + \epsilon T_1^i + \cdots . + \epsilon T_k^i, \quad (13)$$

is an approximate conserved vector of equation (1), if $\mathcal{J}^i$ satisfies the approximate the following equation:

$$D_i \mathcal{J}^i|_{(2.1)} = O(\epsilon^{k+1}). \quad (14)$$

This equation defines the approximate conservation law for equation (1).

According to [23], an approximate Lie–Bäcklund symmetry operator $\mathcal{H}$ given in equation (6) is an approximate Noether-type symmetry operator of an approximate partial Lagrangian L corresponding to approximate Euler–Lagrange type system of the form equation (9) if and only if the characteristic $\mathbb{W} = (\mathbb{W}^1, \mathbb{W}^2, \dots\dots, \mathbb{W}^m)$, $\mathbb{W}^\beta \in F$, defined in equation (7), is also the characteristic of conservation law $D_i \mathcal{J}^i = O(\epsilon^{k+1})$, where

$$\mathcal{J}^i = \mathfrak{B}^i - \mathfrak{L}\xi^i - \mathbb{W}^\alpha \frac{\delta \mathfrak{L}}{\delta u_i{}^\alpha} + \cdots \dots + O(\epsilon^{k+1}) \quad (15)$$

of the approximate Euler–Lagrange type equation (9) for $i = 1, 2, \dots, n$.

An approximate Lie–Bäcklund symmetry operator X given in equation (6) is associated with the approximate conserved vector $\mathcal{J}^i$ in representation (13) of equation (1) if the following relation holds for $i = 1, 2, \dots, n$ [12, 24]:

$$\mathcal{H}\mathcal{J}^i + D_j(\xi^j)\mathcal{J}^i - \mathcal{J}^i D_j(\xi^i) = 0. \quad (16)$$

The approximate symmetries and associated approximate conserved vectors satisfy this relation.

Remarkably, Sjöberg [25, 26] developed the double reduction theory, which states that a PDE with two independent variables can be reduced to an ordinary differential equation of order one less than the order of the PDE.

3.2 Timoshenko–Prescott equation

Layered orthotropic beams represent a specialized category within a broad spectrum of engineering materials. Such beams consist of multiple layers, each having distinct mechanical properties that vary in different directions, hence the term "orthotropic". This orthotropic nature often arises from the material's inherent structure, such as the grain direction in wood or the fiber orientation in fiber-reinforced composites. Owing to this unique structure, the oscillatory behaviors of these beams differ considerably from those of isotropic materials, which have uniform properties in all directions.

Studying oscillations in layered orthotropic beams is crucial for understanding their dynamic responses to external loads and disturbances. These oscillations can be influenced by factors such as the layer thickness, material properties of each layer, boundary conditions, and external forces. Correctly predicting these oscillatory behaviors is vital for designing and analyzing structures fabricated from these materials, ensuring their safety and functionality in real-world applications.

The Timoshenko–Prescott form of the beam equation is as follows [14, 15]:

$$\alpha\beta u_{xxxx} + u_{tt} - \beta(1+\varepsilon)u_{xttt} + \frac{\epsilon\beta u_{tttt}}{\alpha}. \quad (17)$$

It is a perturbed PDE with mixed derivatives of independent variables and provides a mathematical framework for modeling the behavior of such beams. This equation captures the intricate relationships among displacements, velocities, and accelerations within the beam, accounting for the complex interplay of forces resulting from the material's orthotropic nature.

When layered orthotropic beams are subjected to external excitations, the resultant wave patterns exhibit standing and traveling waves. The challenge often lies in reducing these traveling-wave components to better understand the beam's oscillatory behavior in isolation. Such traveling-wave reduction problems aim to simplify the analysis, enabling engineers and scientists to focus on the specific features of the beam's response.

The recent development in generating approximate conservative laws for systems governed by perturbed PDEs provides a significant advancement in the engineering domain. Using an approximate Noether-type symmetry operator, particularly for PDEs affected by mixed Timoshenko–Prescott derivatives, novel conservation vectors can be discovered.

Such an approach paves the way for creating sophisticated computational models, especially in fields where exact solutions are elusive.

It admits the standard Lagrangian of the following form:

$$\mathfrak{L} = \frac{1}{2}\alpha\beta u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}\beta(1+\epsilon)u_{xt}^2 + \frac{1}{2}\frac{\epsilon\beta u_{tt}^2}{\alpha}.$$

Thus, the approximate Euler–Lagrange type equation will be (10).

4 Results

4.1 Approximate Noether-type symmetries

The Timoshenko–Prescott form of the beam equation also finds potential applications in discrete mechanical-electrical dynamic systems, particularly those with inherent nonregularities [27].

One intriguing application lies in microelectromechanical systems (MEMS), miniaturized mechanical and electromechanical elements ranging from simple structures to complex interconnected systems. MEMS often combines mechanical components, such as beams and membranes, with electrical elements, such as sensors and actuators, fabricated using microfabrication technology.

Considering a MEMS device in which a microbeam [28] subjected to an electrically induced force is used as a resonating element, nonregularities can arise owing to manufacturing imperfections, material inconsistencies, or intentional design features. The Timoshenko–Prescott equation can be employed to describe the complex interplay of mechanical displacements and electrical disturbances in such a system. The nonlinear and higher-order derivative terms can capture the intricate dynamics of the system, particularly when subjected to rapid or high-frequency excitations.

Furthermore, piezoelectric materials, which possess the unique property of generating a voltage difference when subjected to mechanical stress, can be integrated into MEMS devices. The Timoshenko–Prescott form can aid in modeling the behavior of such systems, especially when the piezoelectric material is layered or possesses orthotropic properties.

The equation may provide insights into nonregular systems, such as circuits with nonlinear components or signal paths with discontinuities in signal processing. This equation captures spatial and temporal dynamics and can help predict unexpected resonances or transient behaviors.

The approximate Noether-type symmetries corresponding to standard Lagrangian $\mathfrak{L}$ satisfy equation (11). For $i = \{1, 2\}$, the following equation can be obtained:

$$(X_0 + \varepsilon X_1)\mathfrak{L} + D_i\left(\xi_0{}^i + \varepsilon\xi_1{}^i\right)\mathfrak{L} = D_i\left(\hat{B}_0^i + \varepsilon\hat{B}_1^i\right). \quad (18)$$

The zeroth-order and the first-order of ε separate it. These two systems of determining equations, zeroth-order and first-order, are as follows.

For the zeroth-order approximation:

$$\begin{cases} \xi_{0u}^2 = 0;\ \xi_{0u}^1 = 0;\ \eta_{0uu} - 2\xi_{0xu}^2 = 0; \\ \xi_{0t}^2 = 0;\ 2\eta_{0xu} - \xi_{0xx}^2 = 0;\ \hat{B}_{0t}^1 + \hat{B}_{0x}^2 = 0; \\ \xi_{0x}^1 = 0;\ \xi_{0tu}^1 + \eta_{0uu} - \xi_{0xu}^2 = 0;\ \eta_{0xx} = 0; \\ \eta_{0xt} = 0;\ \eta_{0tu} - \xi_{0xt}^2 = 0;\ -\eta_{0xu} + \xi_{0tu}^1 = 0; \\ \hat{B}_{0u}^2 = 0;\ \eta_{0t} - \hat{B}_{0u}^1 = 0;\ -\frac{1}{2}\xi_{0x}^2 + \frac{1}{2}\xi_{0t}^1 - \eta_{0u} = 0. \end{cases} \quad (19)$$

The following solution can be obtained:

$$\xi_0^1 = c_1;\ \xi_0^2 = c_2;$$
$$\eta_0 = c_3 t + c_4 x + c_5;$$
$$\hat{B}_0^1 = -c_3 - \int F_x(x,t)dt + G(x);$$
$$\hat{B}_0^2 = F(x,t).$$

Without loss of generality, it can be set $F(x,t) = 0$ and $G(x) = 0$. Therefore, the obtained Noether-type symmetries for unperturbed equations are as follows:

$$X_0^1 = \partial_t;\ X_0^2 = \partial_x;\ X_0^3 = t\partial_u;\ X_0^4 = x\partial_u;\ X_0^5 = \partial_u.$$

For the first-order approximation:

$$\begin{cases} \xi_{1u}^2 = 0;\ \xi_{1u}^1 = 0;\ \eta_{1uu} - 2\xi_{1xu}^2 = 0; \\ \xi_{1t}^2 = 0;\ 2\eta_{1xu} - \xi_{1xx}^2 = 0;\ \hat{B}_{1t}^1 + \hat{B}_{1x}^2 = 0; \\ \hat{B}_{1u}^2 = 0;\ \xi_{1tu}^1 + \eta_{1uu} - \xi_{1xu}^2 = 0;\ \eta_{1xx} = 0; \\ \xi_{1x}^1 = 0;\ \eta_{1tu} - \xi_{1xt}^2 = 0;\ \eta_{1t} - \hat{B}_{1u}^1 = 0; \\ \eta_{1xt} = 0; \xi_{1tx}^1 - \eta_{1xu} = 0; \frac{1}{2}\xi_{1t}^1 - \frac{1}{2}\xi_{1x}^2 - \eta_{1u} = 0. \end{cases} \quad (20)$$

Solving system of equations (20) allows for obtaining the following solutions:

$$\xi_1^1 = c_6;\ \xi_1^2 = c_7;$$
$$\eta_1 = c_8 t + c_9 x + c_{10};$$
$$\hat{B}_1^1 = -c_8 - \int C_x(x,t)dt + K(x);$$
$$\hat{B}_1^2 = C(x,t).$$

After setting $C(x,t) = 0$ and $K(x) = 0$, approximate Noether-type symmetries can be obtained using representation (4) for equation (17) such that

$$\mathcal{H} = (c_1\partial_t + c_2\partial_x + c_3 t\partial_u + c_4 x\partial_u + c_5\partial_u) + \\ +\epsilon(c_6\partial_t + c_7\partial_x + c_8 t\partial_u + c_9 x\partial_u + c_{10}\partial_u). \quad (21)$$

From it, the approximate Noether-type symmetries (by equating one by one constant) equal to one and other equal to zero as follows:

$$\mathcal{H}^1 = \partial_t;\ \mathcal{H}^2 = \partial_x;\ \mathcal{H}^3 = t\partial_u;\ \mathcal{H}^4 = x\partial_u; \\ \mathcal{H}^5 = \partial_u;\ \mathcal{H}^6 = \epsilon\partial_t;\ \mathcal{H}^7 = \epsilon\partial_x; \quad (22) \\ \mathcal{H}^8 = \epsilon t\partial_u;\ \mathcal{H}^9 = \epsilon x\partial_u;\ \mathcal{H}^{10} = \epsilon\partial_u.$$

4.2 Approximate conservation laws

The approximate conserved vectors corresponding to approximate Noether-type symmetry $\mathcal{H}^1$ in equation (22) can be obtained using equation (15). As a result,

$$\mathcal{J}_1^1 = -\frac{1}{2}\alpha\beta u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}\beta(1+\epsilon)u_{xt}^2 + \\ +\frac{1}{2}\frac{\epsilon\beta u_{tt}^2}{\alpha} + u_t\left[u_t - \frac{\epsilon\beta u_{ttt}}{\alpha} + \beta(1+\epsilon)u_{xxt}\right]; \quad (23) \\ \mathcal{J}_1^2 = u_t[-\alpha\beta u_{xxx} + \beta(1+\epsilon)u_{xtt}] + \\ +\alpha\beta u_{xt}u_{xx} - \beta(1+\epsilon)u_{xt}u_{tt}.$$

Therefore, the total divergence is as follows:

$$D_t\mathcal{J}_1^1 + D_x\mathcal{J}_1^2 = -\beta(1+\epsilon)u_{xt}u_{xtt} + \\ +\beta(1+\epsilon)u_{xxtt}u_t.$$

As we know, some extra terms arise that require more simplification. After some adjustments to the terms emerging in conservation laws, it can be obtained:

$$D_t\mathcal{J}_1^1 + D_x\mathcal{J}_1^2 = -D_t[\beta(1+\epsilon)u_{xt}^2] + \\ +D_x[\beta(1+\epsilon)u_t u_{xtt}]. \tag{24}$$

The modified approximate conserved vectors are stated as $\tilde{\mathcal{J}}_1^i$ with

$$\tilde{\mathcal{J}}_1^1 = \mathcal{J}_1^1 + \beta(1+\epsilon)u_{xt}^2;$$

$$\tilde{\mathcal{J}}_1^2 = \mathcal{J}_1^2 - \beta(1+\epsilon)u_t u_{xtt},$$

where $D_t\tilde{\mathcal{J}}_1^1 + D_x\tilde{\mathcal{J}}_1^2 = 0$.

The approximate conserved vectors corresponding to approximate Noether-type symmetry $\mathcal{H}^2$ in equation (22) are obtained using equation (15):

$$\mathcal{J}_1^2 = u_x\left[-u_t - \frac{\epsilon\beta u_{ttt}}{\alpha} + \beta(1+\epsilon)u_{xxt}\right] + \\ + \frac{\epsilon\beta u_{xt}u_{tt}}{\alpha} - \beta(1+\epsilon)u_{xt}u_{xx};$$

$$\mathcal{J}_2^2 = \frac{1}{2}\alpha\beta u_{xx}^2 + \frac{1}{2}u_t^2 - \frac{1}{2}\beta(1+\epsilon)u_{xt}^2 - \frac{1}{2}\frac{\epsilon\beta u_{tt}^2}{\alpha} + \\ +u_x[-\alpha\beta u_{xxx} + \beta(1+\epsilon)u_{xtt}].$$

Consequently, the following can be written:

$$D_t\mathcal{J}_2^1 + D_x\mathcal{J}_2^2 = D_t[\beta(1+\epsilon)u_{xxt}u_x] - \\ -D_x[\beta(1+\epsilon)u_{xt}^2].$$

The modified approximate conserved vectors are as follows:

$$\tilde{\mathcal{J}}_2^1 = \mathcal{J}_2^1 - \beta(1+\epsilon)u_{xxt}u_x; \\ \tilde{\mathcal{J}}_2^2 = \mathcal{J}_2^2 + \beta(1+\epsilon)u_{xt}^2. \tag{25}$$

In this equation, the approximate conserved vectors satisfy the approximate conservation law:

$$D_t\tilde{\mathcal{J}}_2^1 + D_x\tilde{\mathcal{J}}_2^2 = 0.$$

The approximate conserved vectors correspond to the approximate Noether-type symmetry $\mathcal{H}^3$.

After considering equation (15), the following expressions can be obtained:

$$\mathcal{J}_3^2 = -u + t\left[u_t + \frac{\epsilon\beta u_{ttt}}{\alpha} - \beta(1+\epsilon)u_{xxt}\right] - \frac{\epsilon\beta u_{tt}}{\alpha};$$

$$\mathcal{J}_3^2 = t[\alpha\beta u_{xxx} - \beta(1+\epsilon)u_{xtt}] + \alpha\beta u_{xt}.$$

They correspond to the following condition:

$$D_t\mathcal{J}_3^1 + D_x\mathcal{J}_3^2 = D_x[-t\beta(1+\epsilon)u_{xtt}].$$

The modified approximate conserved vectors $\tilde{\mathcal{J}}_3^i$ satisfy the approximate conservation law:

$$D_t\tilde{\mathcal{J}}_3^1 + D_x\tilde{\mathcal{J}}_3^2 = 0,$$

where

$$\tilde{\mathcal{J}}_3^1 = \mathcal{J}_3^1; \\ \tilde{\mathcal{J}}_3^2 = \mathcal{J}_3^2 + t\beta(1+\epsilon)u_{xtt}. \tag{26}$$

For approximate Noether-type symmetry $\mathcal{H}^4$ in equation (21), the approximate conserved vectors are

$$\mathcal{J}_4^1 = x\left[u_t + \frac{\epsilon\beta u_{ttt}}{\alpha} - \beta(1+\epsilon)u_{xxt}\right] + \frac{\epsilon\beta u_{tt}}{\alpha};$$
$$\mathcal{J}_4^2 = x(\alpha\beta u_{xxx} - \beta(1+\epsilon)u_{xtt}) - \alpha\beta u_{xx}.$$

There is an extra term arising in approximate conservation law such that

$$D_t\mathcal{J}_4^1 + D_x\mathcal{J}_4^2 = D_t[-x\beta(1+\epsilon)u_{xxt}].$$

Now, the modified approximate conserved vectors $\tilde{\mathcal{J}}_4^i$ satisfy the following approximate conservation law:

$$D_t\tilde{\mathcal{J}}_4^1 + D_x\tilde{\mathcal{J}}_4^2 = 0.$$

where

$$\tilde{\mathcal{J}}_4^1 = \mathcal{J}_4^1 + x\beta(1+\epsilon)u_{xxt}; \\ \tilde{\mathcal{J}}_4^2 = \mathcal{J}_4^2. \tag{27}$$

For approximate Noether-type symmetry $\mathcal{H}^5$ in the equation (12), the approximate conserved vectors are as follows:

$$\mathcal{J}_5^1 = u_t + \frac{\epsilon\beta u_{ttt}}{\alpha} - \beta(1+\epsilon)u_{xxt};$$
$$\mathcal{J}_5^2 = \alpha\beta u_{xxx} - \beta(1+\epsilon)u_{xtt}.$$

This time, there is an extra term arising in approximate conservation law such that

$$D_t\mathcal{J}_5^1 + D_x\mathcal{J}_5^2 = D_t[-\beta(1+\epsilon)u_{xxt}].$$

The modified approximate conserved vectors $\tilde{\mathcal{J}}_5^i$ satisfy the following approximate conservation law:

$$D_t\tilde{\mathcal{J}}_5^1 + D_x\tilde{\mathcal{J}}_5^2 = 0.$$

where

$$\tilde{\mathcal{J}}_5^1 = \mathcal{J}_5^1 + x\beta(1+\epsilon)u_{xxt}; \\ \tilde{\mathcal{J}}_5^2 = \mathcal{J}_5^2. \tag{28}$$

Similarly, the modified approximate conserved vectors corresponding to approximate Noether-type symmetries $\mathcal{H}^6$, $\mathcal{H}^7$, $\mathcal{H}^8$, $\mathcal{H}^9$, and $\mathcal{H}^{10}$ in equation (22) are listed as follows:

$$\tilde{\mathcal{J}}_6^1 = \epsilon\mathcal{J}_1^1 + \epsilon\beta(1+\epsilon)u_{xt}^2; \\ \tilde{\mathcal{J}}_6^2 = \epsilon\mathcal{J}_1^1 - \epsilon\beta(1+\epsilon)u_{xtt}u_t; \\ \tilde{\mathcal{J}}_7^1 = \epsilon\mathcal{J}_2^1 - \epsilon\beta(1+\epsilon)u_{xxt}u_x; \\ \tilde{\mathcal{J}}_7^2 = \epsilon\mathcal{J}_2^2 + \epsilon\beta(1+\epsilon)u_{xt}^2; \\ \tilde{\mathcal{J}}_8^1 = \epsilon\mathcal{J}_3^1, \tilde{\mathcal{J}}_8^2 = \epsilon\mathcal{J}_3^2 + \epsilon t\beta(1+\epsilon)u_{xtt}; \\ \tilde{\mathcal{J}}_9^1 = \epsilon\mathcal{J}_4^1 + \epsilon x\beta(1+\epsilon)u_{xxt};\ \tilde{\mathcal{J}}_9^2 = \epsilon\mathcal{J}_4^2; \\ \tilde{\mathcal{J}}_{10}^1 = \epsilon\mathcal{J}_4^1 + \epsilon\beta(1+\epsilon)u_{xxt};\ \tilde{\mathcal{J}}_{10}^2 = \epsilon\mathcal{J}_5^2. \tag{29}$$

All these modified approximate conserved vectors corresponding to approximate Noethor-type symmetries satisfy the approximate conservation laws $D_t\tilde{\mathcal{J}}^i = 0$ $(i = \{1, 2\})$.

5 Discussion

Developing a Timoshenko–Prescott form of the beam equation provides an array of potential engineering applications. At the heart of this formulation lies the perturbed partial differential equation with mixed derivatives of independent variables, which is a sophisticated mathematical structure with significant implications for real-world modeling and simulation.

The ability to derive approximate conservation laws for systems governed by such perturbed PDEs opens the door to tackling complex engineering problems where exact solutions might be difficult to obtain. Such approximate laws can be instrumental in systems modeling, where conservation principles (such as energy and momentum) play an essential role. The introduction of approximate Noether-type symmetry operators further enriches this framework, offering a systematic way to generate conservation vectors.

One direct application of this methodology is in structural engineering, specifically in modeling beams under various loadings and boundary conditions. The Timoshenko–Prescott beam equation can potentially model the behavior of advanced materials or complex loading scenarios more accurately than traditional beam theories. This could be crucial for predicting failures, understanding vibration modes, and optimizing structural designs.

Furthermore, in the broader spectrum of mechanical engineering, this methodology may find application in systems with small perturbations. For instance, perturbed PDEs are often used in fluid mechanics to study stability and transition to turbulence. Similarly, in aeroelasticity, understanding the behavior of structures (such as aircraft wings) under aerodynamic loading often involves perturbed differential equations.

Another potential application is in computational engineering. The presented results can be applied to develop new numerical algorithms for solving problems described by the Timoshenko–Prescott equations. Such algorithms could be more efficient or accurate than generic solvers, particularly for problems where perturbations play a significant role.

The scientific research presented delves deep into the realm of beam mechanics, which is a core subject in mechanical engineering. The Timoshenko–Prescott form of the beam equation provides a more generalized and robust description of beam behavior, accounting for shearing effects and rotational inertia, phenomena often omitted in simpler Euler–Bernoulli beam theories. The significance of this research for mechanical engineering can be summarized in advanced computational models, predictive capabilities, innovative material and structural design, and bridges to advanced mechanical phenomena.

While the current research provides a significant leap in understanding beam mechanics, the horizon of possibilities is vast. Making on this foundation, the mechanical engineering community can drive academic discourse and practical innovations that shape our world.

6 Conclusions

For systems characterized by perturbed or approximated partial differential equations that admit a standard Lagrangian, a unique method has been proposed to derive approximate conservation laws. This approach relies heavily on utilizing approximate Noether-type symmetry operators specific to perturbed PDEs.

The focus of this study was primarily on the PDE-affected mixed Timoshenko–Prescott derivatives, leading to the unearthing of new approximation conservation vectors for the said equation.

Notably, this methodology is a pioneering approach to generating conservation laws. It is observed that all conserved vectors that possess mixed derivatives in their highest-order equations incorporate additional terms. These terms serve as foundational elements for constructing modified approximation-conserved vectors.

With the necessary modifications to the full derivatives of the associated mixed components and incorporation of the approximate conserved vector, these modified approximate conserved vectors adhere to the approximation conservation law. The discoveries and methodologies outlined in this study have profound implications for mechanical engineering.

Understanding and applying these approximate conservation laws can significantly influence complex mechanical systems' design, analysis, and optimization. This research paves the way for more precise predictive systems modeling, particularly when traditional conservation laws may not offer adequate accuracy. In real-world scenarios, such as the design of structures, machinery, or transport systems, these insights can lead to safer, more efficient, and more resilient systems.

Engineers can apply this knowledge to make informed decisions in the presence of uncertainties or perturbations, thereby ensuring the reliability and longevity of mechanical systems. Therefore, this breakthrough advances the theoretical aspects of mechanical engineering and provides practitioners with robust tools for tangible real-world applications.

References

1. Biazar, J., Ghazvini, H. (2008). Homotopy perturbation method for solving hyperbolic partial differential equations. *Computers and Mathematics with Applications*, Vol. 56, pp. 453–458. http://dx.doi.org/10.1016/j.camwa.2007.10.032
2. Gualtieri, C., Chanson, H. (2021). Physical and numerical modelling of air-water flows: An introductory overview. *Environmental Modelling and Software*, Vol. 143, 105109. https://doi.org/10.1016/j.envsoft.2021.105109
3. Nass, A. M. (2019). Lie symmetry analysis and exact solutions of fractional ordinary differential equations with neutral delay. *Applied Mathematics and Computation*, Vol. 345, pp. 370–380. https://doi.org/10.1016/j.amc.2018.11.002

4. Yaşar, E., Yıldırım, Y., Khalique, C. M. (2016). Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation. *Results in Physics*, Vol. 6, pp. 322–328. https://doi.org/10.1016/j.rinp.2016.06.003
5. Olver, P. J. (1986). Applications of Lie Groups to Differential Equations. *In: Graduate Texts in Mathematics*, Vol. 107. Springer, New York, NY, USA. https://doi.org/10.1007/978-1-4684-0274-2
6. Johnpillai, A. G., Kara, A., Mahomed, F. M. (2006). A basis of approximate conservation laws for PDEs with a small parameter. *International Journal of Nonlinear Mechanics*, Vol. 41, pp. 830–837. https://doi.org/10.1016/j.ijnonlinmec.2006.04.009
7. Johnpillai, A. G., Kara, A. H. (2001). Variational formulation of approximate symmetries and conservation laws. *International Journal of Theoretical Physics*, Vol. 40, pp. 1501–1509. https://doi.org/10.1016/j.ijnonlinmec.2006.04.009
8. Gorgone, M., Oliveri, F. (2021). Approximate Noether symmetries of perturbed Lagrangians and approximate conservation laws. *Mathematics*, Vol. 9(22), 2900. https://doi.org/10.3390/math9222900
9. Jamal, S., Mnguni, N. (2018). Approximate conditions admitted by classes of the Lagrangian. *Applied Mathematics and Computation*, Vol. 335, pp. 65–74. https://doi.org/10.1016/j.amc.2018.04.020
10. Govinder, K. S., Heil, T. G., Uzer, T. (1998). Approximate Noether symmetries. *Physics Letters A*, Vol. 240, pp. 127–131. https://doi.org/10.1016/S0375-9601(98)00067-X
11. Kara, A. H., Mahomed, F. M., Unal, G. (1999). Approximate symmetries and conservation laws with applications. *International Journal of Theoretical Physics*, Vol. 38, pp. 2389–2399. https://doi.org/10.1023/A:1026684004127
12. Kara, A. H., Mahomed, F. M. (2000). Relationship between symmetries and conservation laws. *International Journal of Theoretical Physics*, Vol. 39, pp. 23–40. https://doi.org/10.1023/A:1003686831523
13. Kara, A. H., Mahomed, F. M. (2000). A basis of conservation laws for partial differential equations. *Journal of Nonlinear Mathematical Physics*, Vol. 9, pp. 60–72. https://doi.org/10.2991/jnmp.2002.9.s2.6
14. Thomson, W. T., Dahleh, M. D. (2014). *Theory of Vibrations with Applications*. Pearson New International Edition, Prentice Hall, NJ, USA.
15. Halder, A. K., Paliathanasis, A., Leach, P. G. L. (2020). Similarity solutions and conservation laws for the beam equations: A complete study. *Acta Polytechnica*, Vol. 60(2), https://doi.org/10.14311/AP.2020.60.0098
16. L'vov, G. I., Vereshchaka, S. M. (2005). Stress state of multilayer thin-walled elements with interfacial defects of structure. *Mechanics of Composite Materials*, Vol. 41, pp. 511–518. https://doi.org/10.1007/s11029-006-0004-9
17. Adámek, V. (2018). The limits of Timoshenko beam theory applied to impact problems of layered beams. International *Journal of Mechanical Sciences*, Vol. 145, pp. 128–137. https://doi.org/10.1016/j.ijmecsci.2018.07.001
18. Kennedy, G. J., Hansen, J. S., Martins, J. R. R. A. (2011). A Timoshenko beam theory with pressure corrections for layered orthotropic beams. *International Journal of Solids and Structures*, Vol. 48(16–17), pp. 2373–2382. https://doi.org/10.1016/j.ijsolstr.2011.04.009
19. Goldsmith, W., Cunningham, D. M. (1956). Kinematic phenomena observed during the oblique impact of a sphere on a beam. *Journal of Applied Mechanics*, Vol. 23(4), pp. 612–616. https://doi.org/10.1115/1.4011408
20. Elishakoff, I. (2009). An equation both more consistent and simpler than the Bresse-Timoshenko equation. *In: Gilat, R., Banks-Sills, L. (eds) Advances in Mathematical Modeling and Experimental Methods for Materials and Structures. Solid Mechanics and Its Applications*, Vol 168, pp. 249–254. Springer, Dordrecht. https://doi.org/10.1007/978-90-481-3467-0_19
21. . Khalique, C. M., Plaatjie, K. (2022). Symmetry methods and conservation laws for the nonlinear generalized 2D equal-width partial differential equation of engineering. *Mathematics*, Vol. 10(1), 24. https://doi.org/10.3390/math10010024
22. Sulaiman, T. A., Yusuf, A., Tchier, F., Inc, M., Tawfiq, F. M. O., Bousbahi, F. (2021). Lie-Bäcklund symmetries, analytical solutions and conservation laws to the more general (2 + 1)-dimensional Boussinesq equation. *Results in Physics*, Vol. 22, 103850. https://doi.org/10.1016/j.rinp.2021.103850
23. Johnpillai, A. G., Kara, A. H., Mahomed, F. M. (2009). Approximate Noether-type symmetries and conservation laws via partial Lagrangians for PDEs with a small parameter. *Journal of Computational and Applied Mathematics*, Vol. 223, pp. 508–518. https://doi.org/10.1016/j.cam.2008.01.020
24. Iqbal, A., Naeem, I. (2021). Generalised conservation laws, reductions and exact solutions of the K(m,n) equations via double reduction theory. *Pramana – J Phys*, Vol. 95, 30. https://doi.org/10.1007/s12043-020-02071-z
25. Sjöberg, A. (2009). On double reductions from symmetries and conservation laws. *Nonlinear Analysis: Real World Applications*, Vol. 10, pp. 3472–3477. https://doi.org/10.1016/j.nonrwa.2008.09.029
26. Sjöberg, A. (2007). Double reduction of PDEs from the association of symmetries with conservation laws with applications. *Applied Mathematics and Computation*, Vol. 184, pp. 608–616. https://doi.org/10.1016/j.amc.2006.06.059
27. Shunin, Y., Bellucci, S., Gruodis, A., Lobanova-Shunina, T. (2018). Nonregular Nanosystems: Theory and Applications. *In: Lecture Notes in Nanoscale Science and Technology*, Vol 26. Springer, Cham. https://doi.org/10.1007/978-3-319-69167-1
28. Younis, M. I., Abdel-Rahman, E. M., Nayfeh, A. (2003). A reduced-order model for electrically actuated microbeam-based mems. *Journal of Microelectromechanical Systems*, Vol. 12, pp. 672–680. https://doi.org/10.1109/JMEMS.2003.818069