Notes on Concavity, Convexity, Quasiconcavity and Quasiconvexity

Xavier Vilà*

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Abstract

This is just a quick and condensed note on the basic definitions and characterizations of concave, convex, quasiconcave and (to some extent) quasiconvex functions, with some examples.

Contents

1 Concave and convex functions

1.1 Definitions

Definition 1. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S is **concave** if for any two points $x^1, x^2 \in S$ and for any $\lambda \in [0, 1]$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \lambda f(x^1) + (1-\lambda) f(x^2)$$

f is called **strictly concave** if for any two points $x^1, x^2 \in S$ and for any $\lambda \in (0, 1)$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) > \lambda f(x^1) + (1-\lambda) f(x^2)$$

*Departament d'Economia i d'Història Econòmica (Universitat Autònoma de Barcelona) and Barcelona Graduate School of Economics.

Definition 2. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S is **convex** if for any two points $x^1, x^2 \in S$ and for any $\lambda \in [0, 1]$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \le \lambda f(x^1) + (1-\lambda) f(x^2)$$

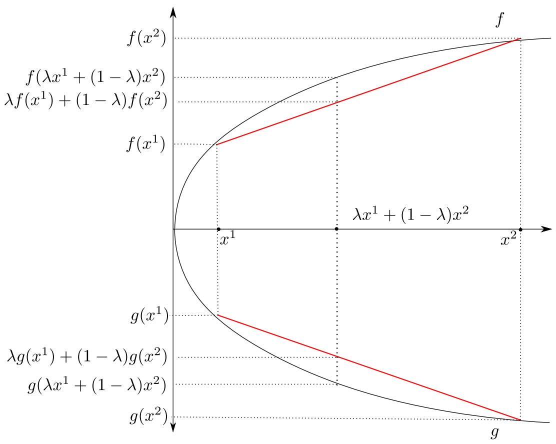
f is called **strictly convex** if for any two points $x^1, x^2 \in S$ and for any $\lambda \in (0, 1)$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) < \lambda f(x^1) + (1-\lambda) f(x^2)$$

Remark 3. A function is concave (convex) if the graph of the function is always above (below) any chord (line segment between two points in the graph).

Remark 4. f concave $\Leftrightarrow -f$ convex.

Example 5. Let $S = [0, \infty)$ and consider $f(x) = \sqrt{x}$ and $g(x) = -f(x) = -\sqrt{x}$



f is a concave function and g is a convex function.

1.2 Selected properties of concave functions

Theorem 6. *Let $f_1, f_2, \ldots, f_n$ be concave (convex) functions, and let $\alpha_1 \ge 0, \alpha_2 \ge 0, \ldots, \alpha_n \ge 0$. Then, the linear combination*

$$f = \alpha_1 f_1 + \cdots + \alpha_n f_n$$

is also concave (convex).

Proof. Consider any two points $x^1, x^2 \in S$ and any $\lambda \in [0, 1]$. Then,

$$\begin{aligned}
f\left(\lambda x^1 + (1-\lambda)\, x^2\right) &= \alpha_1 f_1\left(\lambda x^1 + (1-\lambda)\, x^2\right) + \cdots + \alpha_n f_n\left(\lambda x^1 + (1-\lambda)\, x^2\right) \ge \\
&\ge \alpha_1\left(\lambda f_1\left(x^1\right) + (1-\lambda)\, f_1\left(x^2\right)\right) + \cdots + \alpha_n\left(\lambda f_n\left(x^1\right) + (1-\lambda)\, f_n\left(x^2\right)\right) = \\
&= \lambda\left(a_1 f_1(x^1) + \cdots + a_n f(x^1)\right) + (1-\lambda)\left(a_1 f_1(x^2) + \cdots + a_n f(x^2)\right) = \\
&= \lambda f(x^1) + (1-\lambda)\, f(x^2)
\end{aligned}$$

The inequality at the end of the first line and beginning of the second line holds because all f_j ($j = 1 \ldots, n$) are concave functions and thereofere $f_j\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \lambda f_j(x^1) + (1-\lambda)\, f_j(x^2)$ ($j = 1 \ldots, n$), and also because $\alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_n \geq 0$. Thus, we have proved that

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \lambda f(x^1) + (1-\lambda)\, f(x^2)$$

and hence the linear combinations of concave functions is concave.
(with a similar proof for convexity.) □

Definition 7. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where $\alpha_0, \ldots \alpha_n \in \mathbb{R}$, is called an **affine function**. It is a **linear function** whenever $\alpha_0 = 0$.

Theorem 8. *Any affine function is both concave and convex.*

Proof. The proof follows from Theorem 6 above and from the fact that $f(x) = x_i$, $f(x) = -x_i$, and $f(x) = a_0$ are both concave and convex functions. □

Theorem 9. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a concave (convex) function, and let $g : \mathbb{R} \to \mathbb{R}$ be concave (convex) and increasing. Then $(f \circ g) : S \subset \mathbb{R}^n \to \mathbb{R}$ is a concave (convex) function.*

Proof. Consider any two points $x^1,\ x^2 \in S$ and any $\lambda \in [0, 1]$. Then,

$$\begin{aligned} g\left(f\left(\lambda x^1 + (1-\lambda)\, x^2\right)\right) &\geq g\left(\lambda f(x^1) + (1-\lambda)\, f(x^2)\right) \geq \\ &\geq \lambda g\left(f(x^1)\right) + (1-\lambda)\, g\left(f(x^2)\right), \end{aligned}$$

where the first inequality holds since f is concave and g increasing, and the second inequility follows from g being concave.
We have thus proved that

$$g\left(f\left(\lambda x^1 + (1-\lambda)\, x^2\right)\right) \geq \lambda g\left(f(x^1)\right) + (1-\lambda)\, g\left(f(x^2)\right),$$

that is,

$$(f \circ g)\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \lambda\, (f \circ g)\, (x^1) + (1-\lambda)\, (f \circ g)\, (x^2)$$

Thus, $(f \circ g)$ is a concave function.
(with a similar proof for convexity.) □

Remark 10. An increasing transformation of a concave (convex) function is not necessarily concave (convex). Consider $f(x) = x$ and $g(z) = z^3$.

1.3 Characterization of concave and convex functions by means of contour sets

Definition 11. Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ where S is a convex set. For any $\bar{x} \in \mathbb{R}$ the **upper contour set** ($U_f(\bar{x})$) and **lower contour set** ($L_f(\bar{x})$) of $\bar{x}$ according to f are defined as:

$$U_f(\bar{x}) = \{x \in S \mid f(x) \geq \bar{x}\}$$
$$L_f(\bar{x}) = \{x \in S \mid f(x) \leq \bar{x}\}$$

Theorem 12. *Let the function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S be concave. Then for any $\bar{x} \in \mathbb{R}$ the upper contour set $U_f(\bar{x})$ is either empty or a convex set.*
Analogously, if f is convex then the lower contour set $L_f(\bar{x})$ is either empty or a convex for any $\bar{x} \in \mathbb{R}$.

Proof. (*For concavity*)
Consider any two points $x^1, x^2 \in U_f(\bar{x})$ and any $\lambda \in [0,1]$. We need to prove that if f is concave then $U_f(\bar{x})$ is convex, that is, $\lambda x^1 + (1-\lambda)\,x^2 \in U_f(\bar{x})$ for any $\bar{x} \in \mathbb{R}$.
Since by assumption f is concave we have that

$$f\left(\lambda x^1 + (1-\lambda)\,x^2\right) \geq \lambda f(x^1) + (1-\lambda) f(x^2) \tag{1.1}$$

Now, since $x^1, x^2 \in U_f(\bar{x})$ we have that

$$\left.\begin{array}{lll} f(x^1) \geq \bar{x} & \Rightarrow & \lambda f(x^1) \geq \lambda\bar{x} \text{ for any } \bar{x} \\ f(x^2) \geq \bar{x} & \Rightarrow & (1-\lambda)\, f(x^2) \geq (1-\lambda)\,\bar{x} \text{ for any } \bar{x} \end{array}\right\} \overset{\text{adding up}}{\overbrace{\Longrightarrow}} \lambda f(x^1) + (1-\lambda)\, f(x^2) \geq \bar{x}$$

for any $\bar{x} \in \mathbb{R}$. Going back to (1.1) we conclude that

$$f\left(\lambda x^1 + (1-\lambda)\,x^2\right) \geq \lambda f(x^1) + (1-\lambda) f(x^2) \geq \bar{x}$$

for any $\bar{x} \in \mathbb{R}$, and thus $\lambda x^1 + (1-\lambda)\,x^2 \in U_f(\bar{x})$ for any $\bar{x} \in \mathbb{R}$ as we wanted to prove. □

Remark 13. Notice that this is only a necessary condition, not sufficient. Consider $f(x) = x^3$.

1.4 Characterization of concave and convex differentiable functions

1.4.1 Continuous differentiability

Definition 14. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is **continuously differentiable**, or $f \in C^1$, if all its partial derivatives exist and are continuous functions.

Definition 15. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is **twice continuously differentiable**, or $f \in C^2$, if all its partial first and second derivatives exist and are all continuous functions.

1.4.2 Principal minors and leading principal minors

Definition 16. A **principal submatrix of order *k*** $(1 \leq k \leq n)$ of an $n \times n$ matrix A is the matrix obtained by deleting any $n-k$ rows and the corresponding $n-k$ columns.

Definition 17. The determinant of a principal submatrix of order k is called a **principal minor of order *k*** of A, denoted Δ_k.

Claim 18. An $n \times n$ matrix A contains $\begin{pmatrix} n \\ n-k \end{pmatrix}$ principal minors of order $k\,(1 \leq k \leq n)$, which yields a total of[1] $\sum_{k=1}^n \begin{pmatrix} n \\ n-k \end{pmatrix} = 2^n - 1$ principal minors.

Definition 19. The **leading principal submatrix of order *k*** $(1 \leq k \leq n)$ of an $n \times n$ matrix is obtained by deleting the last $n-k$ rows and columns of the matrix.

[1] By Newton's Binomial Theorem.

Definition 20. The determinant of the leading principal submatrix of order k is called the **leading principal minor of order k** of A, denoted D_k.

Claim 21. An $n \times n$ matrix A contains exactly one leading principal minor for each order $k\,(1 \le k \le n)$, which yields a total of $\sum_{k=1}^{n} 1 = n$ leading principal minors.

Example 22. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then,

there are $\begin{pmatrix} n \\ n-k \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ principal minors of order $k = 1$: Δ_1^1, Δ_1^2, and Δ_1^3

$$\Delta_1^1 = |a_{11}|\,, \; \Delta_1^2 = |a_{22}|\,, \; \Delta_1^3 = |a_{33}|$$

there are $\begin{pmatrix} n \\ n-k \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$ principal minors of order $k = 2$: Δ_2^2, Δ_2^2, and Δ_2^3

$$\Delta_2^1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \; \Delta_2^2 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \; \Delta_2^3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

there are $\begin{pmatrix} n \\ n-k \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 1$ principal minor of order $k = 3$: Δ_3

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Notice that there is a total of $2^n - 1 = 2^3 - 1 = 7$ principal minors in total.
There is 1 leading principal minor of order $k = 1$: D_1

$$D_1 = |a_{11}|$$

there is 1 leading principal minor of order $k = 2$: D_2

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

there is 1 leading principal minor of order $k = 3$: D_3

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Proposition 23. *Let A be an $n \times n$ matrix. Then,*

- A is positive definite $\Leftrightarrow D_k > 0, \forall k\,(1 \le k \le n)$;
- A is negative definite $\Leftrightarrow \text{sign} D_k = \text{sign}(-1)^k, \forall k\,(1 \le k \le n)$;
- A is positive semidefinite $\Leftrightarrow \; \Delta_k \; \ge \; 0, \forall k\,(1 \le k \le n)$;
- A is negative semidefinite $\Leftrightarrow \; \text{sign}\Delta_k = \text{sign}(-1)^k$ or $\Delta_k = 0, \forall k\,(1 \le k \le n)$.

1.4.3 Characterization of concave and convex functions by means of their derivatives

Definition 24. Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. The vector of first partial derivatives of f,

$$Df(x) = \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n}\right),$$

is called de **Jacobian** of f.

Definition 25. Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. The matrix of second partial derivatives of f,

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix},$$

is called de **Hessian** of f.

Theorem 26. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Then, f is concave if and only if*

$$f(x^2) - f(x^1) \le Df(x^1)(x^2 - x^1), \forall x^1, x^2 \in S,$$

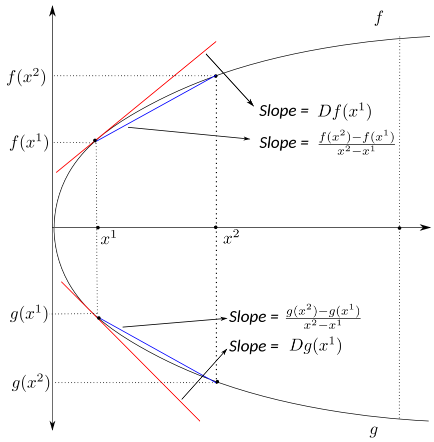
that is,

$$f(x^2) - f(x^1) \le \frac{\partial f(x^1)}{\partial x_1}\left(x_1^2 - x_1^1\right) + \cdots + \frac{\partial f(x^1)}{\partial x_n}\left(x_n^2 - x_n^1\right).$$

Similarly f is convex if and only if

$$f(x^2) - f(x^1) \ge Df(x^1)(x^2 - x^1), \forall x^1, x^2 \in S.$$

Example 27. Let $S = [0, \infty)$ and consider $f(x) = \sqrt{x}$ and $g(x) = -f(x) = -\sqrt{x}$



For f we have

$$\frac{f(x^2) - f(x^1)}{x^2 - x^1} \le Df(x^1) \Leftrightarrow f(x^2) - f(x^1) \le Df(x^1)(x^2 - x^1)$$

Thus, f is a concave function. Analogously, For g we have

$$\frac{g(x^2) - g(x^1)}{x^2 - x^1} \geq Dg(x^1) \Leftrightarrow g(x^2) - g(x^1) \geq Dg(x^1)(x^2 - x^1)$$

Thus, g is a convex function.

Remark 28. A function is concave (convex) if the graph of the function is always below (above) the graph of the tangent to the function.

Theorem 29. *Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Then,*

(i) *f is concave if and only if the Hessian matrix $D^2 f(x)$ is negative semidefinite for all $x \in S$;*

(ii) *f is strictly concave if the Hessian matrix $D^2 f(x)$ is negative definite for all $x \in S$;*

(iii) *f is convex if and only if the Hessian matrix $D^2 f(x)$ is positive semidefinite for all $x \in S$;*

(iv) *f is strictly convex if the Hessian matrix $D^2 f(x)$ is positive definite for all $x \in S$.*

We thus have:

$D^2 f(x)$ negative definite for all x	$\Rightarrow$	$f(x)$ is strictly concave
$\Downarrow$		$\Downarrow$
$D^2 f(x)$ negative semidefinite for all x	$\Leftrightarrow$	$f(x)$ is concave

with similar implications for convexity and positive definiteness.

Example 30. Consider the Cobb-Douglas function $f(x_1, x_2) = x_1^\alpha x_2^\beta$, with $\alpha, \beta > 0$ and $(x_1, x_2) \in \mathbb{R}^2_+$. For what values of α and β is this function concave ?

In this case we have:

$$Df(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}\right) = \left(\alpha x_1^{\alpha-1} x_2^\beta, \beta x_1^\alpha x_2^{\beta-1}\right)$$

and

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^\beta & \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1) x_1^\alpha x_2^{\beta-2} \end{pmatrix}$$

Notice that in this case,

there are $\begin{pmatrix} n \\ n-k \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ =2 principal minors of order $k = 1$: Δ_1^1, and Δ_1^2,

$$\Delta_1^1 = \alpha(\alpha-1) x_1^{\alpha-2} x_2^\beta, \ \Delta_1^2 = \beta(\beta-1) x_1^\alpha x_2^{\beta-2}$$

there are $\begin{pmatrix} n \\ n-k \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1$ principal minors of order $k = 2$: Δ_2,

$$\Delta_2 = \begin{vmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta} & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)x_1^{\alpha}x_2^{\beta-2} \end{vmatrix} =$$
$$= \left(\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}\right)\left(\beta(\beta-1)x_1^{\alpha}x_2^{\beta-2}\right) - \left(\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}\right)\left(\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}\right) =$$
$$= \left(\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2\right)\left(x_1^{2\alpha-2}x_2^{2\beta-2}\right) =$$
$$= \left((\alpha^2-\alpha)(\beta^2-\beta) - \alpha^2\beta^2\right)\left(x_1^{2\alpha-2}x_2^{2\beta-2}\right) =$$
$$= \left((\alpha^2\beta^2 - \alpha^2\beta - \alpha\beta^2 + \alpha\beta) - \alpha^2\beta^2\right)\left(x_1^{2\alpha-2}x_2^{2\beta-2}\right) =$$
$$= (\alpha\beta(-\alpha-\beta)+1)\left(x_1^{2\alpha-2}x_2^{2\beta-2}\right) =$$
$$= (1-(\alpha+\beta))\,\alpha\beta\left(x_1^{2\alpha-2}x_2^{2\beta-2}\right)$$

For the function to be concave, by Theorem 29, the Hessian must be negative semidefinite. This, by Proposition 23, occurs, when the principal minors of order 1 are all non-positive and the principal minor of order 2 is non-negative. That is,

$$\Delta_1^1 =\leq 0,\ \Delta_1^2 \leq 0,\ \text{and } \Delta_2 \geq 0$$

Notice that, given that $\alpha, \beta > 0$ and $x_1\, x_2 \geq 0$, we have:

$$\Delta_1^1 \leq 0 \Leftrightarrow \alpha \leq 1$$
$$\Delta_1^2 \leq 0 \Leftrightarrow \beta \leq 1$$
$$\Delta_2 \geq 0 \Leftrightarrow (\alpha+\beta) \leq 1$$

Thus, the function is concave if and only if it exhibits constant or decreasing returns to scale
Moreover, note that if $(x_1, x_2) \in \mathbb{R}^2_{++}$ then the Hessian is negative definite and thus the function is strictly concave.
Question: Can the function be convex ?

2 Quasiconcave and quasiconvex functions

2.1 Definitions

Definition 31. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S is **quasiconcave** if the upper contour set $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$.
Similarly, the function f is **quasiconvex** if the lower contour set $L_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$.

Remark 32. Notice that what was a necessary condition for concavity (convexity) according to Theorem 12 is now a necessary and sufficient condition for quasiconcavity (quasiconvexity), for it is the definition.

Definition 33. A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S is **quasiconcave** if for any two points $x^1,\ x^2 \in S$ and for any $\lambda \in [0,1]$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\,x^2\right) \geq \min\left\{f(x^1), f(x^2)\right\}$$

Similarly, the function f is **quasiconvex** f for any two points $x^1,\ x^2 \in S$ and for any $\lambda \in [0,1]$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\,x^2\right) \leq \max\left\{f(x^1), f(x^2)\right\}$$

Theorem 34. *Definition 31 and Definition 33 are equivalent.*

Proof. Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set S.

[Definition 31 ⇒ Definition 33]

Asuming that the upper contour set $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$, we need to prove that for any two points $x^1, x^2 \in S$ and for any $\lambda \in [0,1]$ we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \min\left\{f(x^1), f(x^2)\right\}$$

Consider any two points $x^1, x^2 \in S$ and take[2] $\bar{x} = \min\left\{f(x^1), f(x^2)\right\} = f(x^1)$. This means that

$$f(x^1) = \bar{x} \Rightarrow x^1 \in U_f(\bar{x})$$

Also, since $\min\left\{f(x^1), f(x^2)\right\} = f(x^1)$ we have that $f(x^2) \geq f(x^1) = \bar{x}$, which means that

$$f(x^2) \geq \bar{x} \Rightarrow x^2 \in U_f(\bar{x})$$

Since we are asuming that $U_f(\bar{x})$ is a convex set we have that

$$x^1 \in U_f(\bar{x}) \text{ and } x^2 \in U_f(\bar{x}) \Rightarrow \lambda x^1 + (1-\lambda)\, x^2 \in U_f(\bar{x}) \text{ for any } \lambda \in [0,1]$$

Then, by definition of upper contour set, $\lambda x^1 + (1-\lambda)\, x^2 \in U_f(\bar{x})$, which means that

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \bar{x},$$

and this is true for any $\lambda \in [0,1]$

Finally, since we have chosen $\bar{x} = \min\left\{f(x^1), f(x^2)\right\}$, we conclude that for any $\lambda \in [0,1]$, we have:

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \min\left\{f(x^1), f(x^2)\right\}$$

as we wanted to prove.

[Definition 31 ⇐ Definition 33]

Suppose that for any two points $x^1, x^2 \in S$ and for any $\lambda \in [0,1]$ we have $f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \min\left\{f(x^1), f(x^2)\right\}$. We need to prove that the upper contour set $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$.

Take any $x^1, x^2 \in U_f(\bar{x}) \subset S$. To prove convexity we need to verify that $\lambda x^1 + (1-\lambda)\, x^2 \in U_f(\bar{x})$ for any $\lambda \in [0,1]$.

Since $x^1, x^2 \in U_f(\bar{x})$ we have that $f(x^1) \geq \bar{x}$ and $f(x^2) \geq \bar{x}$.

Assume WLOG that $\min\left\{f(x^1), f(x^2)\right\} = f(x^1)$. Then, by assumption we have that for any $\lambda \in [0,1]$

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \min\left\{f(x^1), f(x^2)\right\} = f(x^1) \geq \bar{x}.$$

The above means that for any $\lambda \in [0,1]$

$$f\left(\lambda x^1 + (1-\lambda)\, x^2\right) \geq \bar{x} \Rightarrow \lambda x^1 + (1-\lambda)\, x^2 \in U_f(\bar{x})$$

thus proving that $U_f(\bar{x})$ is indeed a convex set. □

[2] We can choose this particular $\bar{x}$ because $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$. Also, we can assume WLOG that $\min\left\{f(x^1), f(x^2)\right\} = f(x^1)$

Remark 35. By Remark 32 above, concavity (convexity) implies, but is not implied by, quasiconcavity (quasiconvexity). Consider the function $f(x) = x^3$, it is quasiconcave (and quasiconvex) but not concave (nor convex).

Remark 36. f quasiconcave $\Leftrightarrow -f$ quasiconvex.

Theorem 37. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a quasiconcave function, and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then $(f \circ g) : S \subset \mathbb{R}^n \to \mathbb{R}$ is a quasiconcave function.*

Proof. Suppose that f is quasiconcave, that is, $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$. We need to prove that also $U_{(f\circ g)}(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$.
Take any $\bar{x} \in \mathbb{R}$. Since $g : \mathbb{R} \to \mathbb{R}$ is increasing there must exist a unique $\hat{x} \in \mathbb{R}$ such that

$$\bar{x} = g(\hat{x})$$

or, in other words, $\hat{x} = g^{-1}(\bar{x}) \in \mathbb{R}$. Then,

$$\begin{aligned} U_{(f\circ g)}(\bar{x}) &= \{x \in S \mid g(f(x)) \geq \bar{x}\} = \\ &= \left\{x \in S \mid f(x) \geq g^{-1}(\bar{x})\right\} = \\ &= \{x \in S \mid f(x) \geq \hat{x}\} = U_f(\hat{x}) \end{aligned}$$

Since f is quasiconcave we know that $U_f(\bar{x})$ is convex for any $\bar{x} \in \mathbb{R}$, and thus $U_f(\hat{x})$ in convex. Hence, $U_{(f\circ g)}(\bar{x}) = U_f(\hat{x})$ is convex, as we wanted to prove. □

Remark 38. Notice that the transformation function g does not need to be quasiconcave for this property to hold, unlike in the case of concave functions where the transformation needed to be a concave function.

Example 39. Any Cobb-Douglas function $f(x_1, x_2) = x_1^\alpha x_2^\beta$, with $\alpha, \beta > 0$ and $(x_1, x_2) \in \mathbb{R}^2_+$. is quasiconcave.
Indeed, as seen in Example 30, $\alpha + \beta \leq 1 \Rightarrow f$ concave, which by Remark 32 implies that f is quasiconcave.
Now, if $\alpha + \beta > 1$ consider the increasing function $g(z) = z^{\alpha+\beta}$ and the Cobb-Douglas function $h(x_1, x_2) = x_1^{\frac{\alpha}{\alpha+\beta}} x_2^{\frac{\beta}{\alpha+\beta}}$. Then, we have that $f = g \circ h$, that is,

$$g(h(x_1, x_2)) = \left(x_1^{\frac{\alpha}{\alpha+\beta}} x_2^{\frac{\beta}{\alpha+\beta}}\right)^{\alpha+\beta} = x_1^\alpha x_2^\beta = f(x_1, x_2)$$

The funcion h is concave as $\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1$ (Example 30), and therefore quasiconcave; and the function g is clearly increasing. Therefore, an increasing returns to scale Cobb-Douglas function ($\alpha + \beta > 1$) can be obtained as an increasing transformation of a quasiconcave function, thus being quasiconcave itself by Theorem 37.

Example 40. Any CES function $y = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$, $0 < \rho < 1$ is quasiconcave.

Indeed, if $0 < \rho < 1$ then both x_1^ρ and x_2^ρ are concave functions. Then, $\left(x_1^\rho + x_2^\rho\right)$ is also concave for beeing a linear combination of concave functions (Theorem 6) and thus quasiconcave (Remark 32). Finaly, since $g(z) = z^{\frac{1}{\rho}}$ is an increasing function we have that any CES function is an increasing transformation of a quasiconcave function and thus quasiconcave by Theorem 37.

2.2 Characterization of quasiconcave and quasiconvex differentiable functions

2.2.1 Bordered Hessian

Definition 41. Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. The matrix of first and second partial derivatives of f,

$$\begin{pmatrix} 0 & \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} & \cdots & \frac{\partial f(x)}{\partial x_k} \\ \frac{\partial f(x)}{\partial x_1} & \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_k} \\ \frac{\partial f(x)}{\partial x_2} & \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(x)}{\partial x_k} & \frac{\partial^2 f(x)}{\partial x_k \partial x_1} & \frac{\partial^2 f(x)}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_k \partial x_k} \end{pmatrix},$$

is called the **bordered Hessian of order *k*** of f. Let $D_k^2 f(x)$ denote the determinant of the bordered Hessian of order k.

Remark 42. Note that $D_k^2 f(x)$ is equal to the leading principal minor of order $k+1$ of the bordered Hessian of order n.

2.2.2 Characterization of quasiconcave and quasiconvex functions by means of their derivatives

Theorem 43. *Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Then, f is quasiconcave if and only if:*

$$f(x^2) \geq f(x^1) \Rightarrow Df(x^1)(x^2 - x^1) \geq 0, \forall x^1, x^2 \in S$$

Similarly f is quasiconvex if and only if :

$$f(x^2) \leq f(x^1) \Rightarrow Df(x^1)(x^2 - x^1) \leq 0, \forall x^1, x^2 \in S$$

Theorem 44. *Let Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function.*
(i) *If f is quasiconcave then*

$$D_1^2 f(x) \leq 0,\ D_2^2 f(x) \geq 0, \cdots, D_k^2 f(x) \geq 0 \text{ if } k \text{ is even and } D_k^2 f(x) \leq 0 \text{ if } k \text{ is odd}$$

for $k = 1, \dots n$ and for all $x \in S$;
(ii) *if f is quasiconvex then $D_k^2 f(x) \leq 0$ for $k = 1, \dots n$ and for all $x \in S$;*
(iii) *if*

$$D_1^2 f(x) < 0,\ D_2^2 f(x) > 0, \cdots, D_k^2 f(x) > 0 \text{ if } k \text{ is even and } D_k^2 f(x) < 0 \text{ if } k \text{ is odd}$$

for $k = 1, \dots n$ and for all $x \in S$, then f is quasiconcave;
(iv) *if $D_k^2 f(x) < 0$ for $k = 1, \dots n$, for all $x \in S$, then f is quasiconvex.*

Remark 45. Notice that the characterization of quasiconcave and quasiconvex functions are not comparable with that for concave and convex functions in Theorem 29. To clarify this let us define

$$\text{Condition } A \to D_1^2 f(x) \leq 0,\ D_2^2 f(x) \geq 0, \cdots, D_k^2 f(x) \geq 0 \text{ if } k \text{ is even and } D_k^2 f(x) \leq 0$$
$$\text{if } k \text{is odd for } k = 1, \dots, n \text{ and for all } x \in S$$
$$\text{Condition } B \to D_1^2 f(x) < 0,\ D_2^2 f(x) > 0, \cdots, D_k^2 f(x) > 0 \text{ if } k \text{ is even and } D_k^2 f(x) < 0$$
$$\text{if } k \text{is odd for } k = 1, \dots, n \text{ and for all } x \in S$$

Then, according to Theorem 44 we have

$$B \quad \Rightarrow \quad f \text{ quasiconcave} \quad \Rightarrow \quad A$$

Example 46. In Example 39 we have seen that any Cobb-Douglas function $f(x_1, x_2) = x_1^\alpha x_2^\beta$, with $\alpha, \beta > 0$ and $(x_1, x_2) \in \mathbb{R}_+^2$, is quasiconcave by showing that it can be expressed as an increasing transofrmation of a quasiconcave function. We are now going to see that it verifies item **(i)** in Theorem 44. To this purpose we compute the bordered Hessians $D_1^2 f(x)$ and $D_2^2 f(x)$ (since $n = 2$ there are no more bordered Hessians in this case)

$$D_1^2 f(x) = \begin{vmatrix} 0 & \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_1} & \frac{\partial^2 f(x)}{\partial x_1^2} \end{vmatrix} = \begin{vmatrix} 0 & \alpha x_1^{\alpha-1} x_2^\beta \\ \alpha x_1^{\alpha-1} x_2^\beta & \alpha(\alpha-1) x_1^{\alpha-2} x_2^\beta \end{vmatrix} = $$
$$= -\left(\alpha x_1^{\alpha-1} x_2^\beta\right)^2$$

$$D_2^2 f(x) = \begin{vmatrix} 0 & \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} \\ \frac{\partial f(x)}{\partial x_1} & \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial f(x)}{\partial x_2} & \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & \alpha x_1^{\alpha-1} x_2^\beta & \beta x_1^\alpha x_2^{\beta-1} \\ \alpha x_1^{\alpha-1} x_2^\beta & \alpha(\alpha-1) x_1^{\alpha-2} x_2^\beta & \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ \beta x_1^\alpha x_2^{\beta-1} & \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1) x_1^\alpha x_2^{\beta-2} \end{vmatrix} =$$
$$= \left(\alpha\beta + \alpha^2\beta + \alpha\beta^2\right) x_1^{3\alpha-2} x_2^{3\beta-2}$$

We note that, for $\alpha, \beta > 0$ and $(x_1, x_2) \in \mathbb{R}_+^2$,

$$\begin{aligned} D_1^2 f(x) &= -\left(\alpha x_1^{\alpha-1} x_2^\beta\right)^2 \leq 0 \\ D_2^2 f(x) &= \left(\alpha\beta + \alpha^2\beta + \alpha\beta^2\right) x_1^{3\alpha-2} x_2^{3\beta-2} \geq 0 \end{aligned} \tag{2.1}$$

Therefore, conlusion **(i)** in Theorem 44 (Condition A) is indeed satisfied.