ORIGINAL RESEARCH

Is the HYPE about strength warranted?

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Abstract

In comparing classical and non-classical solutions to the semantic paradoxes arguments relying on strength have been influential. In this paper I argue that non-classical solutions should preserve the proof-theoretic strength of classical solutions. Leitgeb's logic of HYPE is then presented as an interesting possibility to strengthen FDE with a suitable conditional. It is shown that HYPE allows for a non-classical Kripkean theory of truth, called KFL, that is strong enough for the relevant purposes and has additional attractive properties.

Keywords FDE · Conditional · Proof-theoretic strength · HYPE · Reflective closure

1 Introduction

The divide between classical and non-classical solutions to the liar paradox appears to be substantial. On the one hand there are arguments in favour of classical solutions purporting to show that giving up classical logic is a price too high to pay. On the other hand there are arguments in favour of non-classical solutions pointing to some truth theoretic principles that make the non-classical solutions more coherent. In this paper I take both kinds of arguments seriously and argue that a suggestion building on the non-classical logic called HYPE is able to combine positive aspects of both sides.

The area of interest is a Kripkean conception of type-free truth, broadly based on fixed-point constructions, not restricted to a specific interpretation. The discussion focusses on the question whether the best way of capturing such a conception within a formal theory of truth employs classical logic or a non-classical logic. The Kripkean conception of type-free truth is not only a philosophically motivated suggestion, but also a rigorously explored terrain so that several results are available to support the debate. The point of departure for the discussion concerns the classical theory KF and the non-classical theory PKF based on the non-classical logic of first-degree-

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entailment, FDE. One of the papers that has influenced the discussion strongly is Halbach and Horsten (2006).

The main arguments in favour of the classical side concern questions of loss in giving up classical logic. A specific version of this argument focusses on an explicit measure of this loss by considerations of proof-theoretic strength. The difference in proof-theoretic strength in the case of KF and PKF is a forceful example for such a loss. Whereas in the classical setting of KF Gentzen's proof establishes transfinite induction for all ordinals less than ε_0 for the language containing truth, in the case of PKF transfinite induction for truth is only provable for ordinals less than ω^ω.

In the paper I will rehearse the argument by recapitulating the necessary background for the discussion. This will be the focus of Sect. 2, where I will briefly repeat the dialectic situation, the relevant results and additionally some of the reasons why the classical theory KF itself is not optimal.

The third section is then concerned with the main argument against non-classical solutions based on the 'costs' of dropping classicality. Feferman's 'sustained ordinary reasoning' criticism is a starting point for the more concrete desideratum of proof-theoretic strength. First I argue that proof-theoretic strength is a desirable feature for a type-free theory of truth because of its close relationship with expressive power. This makes it not merely a purely technical aspect only of interest for specialized proof-theorists. More concretely for the case at hand I argue that the interaction of type-free truth with reflection is a powerful tool that should be preserved. Then I point out why simple recapture strategies are not adequate to answer such a challenge.

In the fourth section I locate the specific discussion within the broader debate of philosophy of logic. Starting from a tolerant position and circumventing philosophical arguments based on questionable epistemic or metaphysical assumptions I adopt an inferentialist understanding that fits within a naturalistic account. I argue that for the case at hand it is unlikely that a decision is forced by correctness considerations. Rather, the question what the best logic is for a type-free theory of truth is decided by pragmatic criteria like fruitfulness.

In the fifth section I will then show that HYPE is an attractive logic for a type-free theory of truth. Not only is it a logic that promises to be a suitable suggestion for a logic that can play the role for a unified picture, but also that it pays the bills in the specific case of proof-theoretic strength. Moreover, I provide an interpretation of the HYPE conditional that explains the failure of disquotation.

In the last section I argue that there are additional aspects that make HYPE not only an attractive alternative to classical logic, but that it even purports fruitful aspects that are difficult to recover in classical logic. The final paragraphs compare the HYPE strategy to alternative ways of regaining the strength of classical reasoning, especially Field's proposal in (2020).

2 Background: classical versus non-classical

The semantic paradoxes have been a driving force in the development of theories of type-free truth as well as influential for proponents of non-classical logics. By adopting naive truth principles and classical logic an inconsistency arises. To avoid a

trivialization something has to go, either truth principles or classical principles. It is not obvious what has to go and there is a plentitude of options.[1] In order to have a more manageable area, the attention will be restricted on Kripkean solutions. Kripke's 'Outline of a theory of truth' (Kripke, 1975) promises an original conception of type-free truth based on an intriguing philosophical story and is not without reason one of the most popular accounts.

Kripke's semantic theory provides models for type-free truth that are based on fixed-points for a monotone operator. These fixed points have the nice property that they satisfy a sentence A iff they satisfy $\mathsf{T}\ulcorner A \urcorner$. The construction itself treats truth as an indeterminate notion, i.e. either as a partial or overdetermined predicate or both, which makes a non-classical logic prima facie an attractive candidate. However, the fixed-points also provide classical models. In this case there is a tension between the inner and the outer logic, that also shows up in the classical version of KF and is one of the main problems for a classical solution.

2.1 KF versus PKF

The theories KF and PKF are adequate axiomatizations of Kripke's construction.[2] The former is formulated in classical logic and based on the closed-off fixed points, whereas the latter is formulated in FDE.[3] Whereas PKF contains compositional principles for all connectives, KF has to give up on compositional principles for negation as well as the conditional.[4]

These theories are well understood and suitable proof-theoretic analysis are available.[5] The relevant observation that started the discussion is that the theory PKF is proof-theoretically significantly weaker than its classical counterpart. The proof-theoretic analysis of KF carried out by Feferman in (1991) and Cantini in (1989) shows that KF proves transfinite induction for arithmetical formulas for all ordinals less than $\varphi_{\varepsilon_0}(0)$. In contrast as shown by Halbach and Horsten PKF is only able to prove transfinite induction for arithmetical formulas for all ordinals less than $\varphi_{\omega^\omega}(0)$. This can be traced back to the underlying logic since it is already possible to show that over the logic of FDE the Peano axioms PA formulated in the language of $\mathcal{L}_\mathsf{T}$ will only prove transfinite induction for ordinals less than ω^ω for formulas containing the truth predicate, whereas Gentzen's proof shows that over classical logic transfinite induction for all ordinals less than ε_0 is provable. Some philosophers have taken this to be a strong argument in favour of KF.[6] In the next section I argue that it is reasonable to understand this proof-theoretic deficiency as a serious drawback.

On the other hand there are good reasons to be not completely satisfied with KF itself. In the following I will say more about this, but I already mentioned the problem

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[1] Friedman and Sheard (1987) for example cover only a small fraction of the options in classical logic.

[2] For a discussion of axiomatizations compare Fischer et al. (2015).

[3] There are variations in K3, LP and symmetric strong Kleene.

[4] For details of the axioms see for example Halbach (2014).

[5] It is much less clear how the criterion of proof-theoretic strength is applicable for example in substructural logics.

[6] See for example Halbach (2014) and Halbach and Nicolai (2018).

of coherently combining a classical outer logic with a non-classical inner logic. Since PKF takes the outer logic to be the same as the inner logic it does not have similar problems and appears to be completely motivated. Now the challenge is to find a theory of truth that preserves the motivation of PKF, but has the strength of KF. I will argue in Sect. 5 that KFL, a Kripkean theory of truth based on the logic of HYPE, is a theory with attractive properties that is able to recover the strength by preserving the motivation.[7] An alternative is Field's proposal in 'The power of naive truth' (Field 2020) that I will discuss briefly.

2.2 The problems of KF

There are several drawbacks of the classical solution of KF that are well-known and I will repeat some of those based on a tension between the inner and outer logic. The fixed-point construction relies on a non-classical logic, whereas KF is formulated in classical logic. The semantics is only adequate by closing off the fixed-points and thereby including unintended sentences within the anti-extension. So whereas all sentences in the extension will be correctly identified as true, there will be sentences in the closed off anti-extension that are not intended. In the case of consistent fixed-points the situation is worse as paradoxical sentences will end up in the anti-extension of every closed off fixed-point and therefore will be derivable in the version of KF that includes the consistency axiom, (CONS) $\neg(\mathsf{T}\ulcorner\varphi\urcorner \wedge \mathsf{T}\ulcorner\neg\varphi\urcorner)$. The consistency axiom is intuitively well motivated and allows the use of modus ponens truth internally, i.e. $\mathsf{T}\ulcorner\varphi\urcorner \wedge \mathsf{T}\ulcorner\varphi \to \psi\urcorner \to \mathsf{T}\ulcorner\psi\urcorner$. In the case of the liar sentence λ, both λ and $\neg\mathsf{T}\ulcorner\lambda\urcorner$ are derivable in KF + (CONS). This is difficult to bring in line with our usual understanding of the workings of negation, truth and assertion.

But even without the presence of (CONS) the axiomatic theory of truth lacks some desirable features. For example it is incompatible with the two principles of truth introduction, T-Intro, and truth elimination , T-Elim. Even if it is compatible with one of them it becomes problematic. For example by the addition of T-Intro an internal inconsistency arises $\mathsf{T}(\lambda \wedge \neg\lambda)$.[8] A similar problem occurs by closing KF under global reflection. It is inconsistent in the presence of (CONS) and internally inconsistent without it. One way to drive the point home is to say that KF cannot claim its own truth.

Another aspect that plays a prominent role is the lack of unrestricted disquotation or naivety. Clearly KF does not support full disquotation, however disquotation for the positive fragment of the language is provable.[9] This excludes negated truth claims such as $\neg\mathsf{T}(t)$, and it is also not to be expected for truth to interact with the classical conditional in the correct way. Therefore disquotation as a schematic principle for all formulas is not available. However, considering all the instances of the principle that are derivable, then all the PKF instances will also be KF instances. It is however not obvious that one should expect disquotation to be applicable without restrictions as

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[7] For details of the first claim see Fischer et al. (2021).

[8] Compare Halbach and Horsten (2006).

[9] Compare Cantini (1989).

an open-ended schema and therefore I take the failure of unrestricted disquotation as a less serious problem.

An interesting attempt to solve these problems is Reinhardt's instrumentalist interpretation of KF in his (1986). According to this interpretation the core of KF is the inner part, i.e. the theorems that KF proves to be true, also called IKF. In contrast, negated truth theorems of KF are reinterpreted as an instrumental, auxiliary surrounding. PKF can be understood as a way to make this inner part explicit. However, there are two problems. One of them is the lack of proof theoretic strength of PKF showing that there are significant theorems in the inner part of KF which are not derivable in PKF. The other is that the non-significant part plays an irreducible role in derivations of significant theorems. So a straightforward interpretation of KF via Reinhardt's program appears to be inadequate, as has been shown by Halbach and Horsten (2006).

Recent attempts to the rescue of Reinhardt's program have been proposed by Castaldo and Stern (2022) building on the work by Nicolai (2018). Nicolai showed that the proof-theoretic problem could be solved by adding $\mathcal{L}_T$-TI$(< \varepsilon_0)$ to PKF, i.e. TI(A, α) for all formulas $A \in \mathcal{L}_T$ and for all ordinal notations less than ε_0. PKF$+\mathcal{L}_T$-TI$(< \varepsilon_0)$ is proof-theoretically equivalent to IKF. However, taking PKF$+\mathcal{L}_T$-TI$(< \varepsilon_0)$ as an explication of the inner part of KF appears to be ad hoc and a philosophically motivated system that has $\mathcal{L}_T$-TI$(< \varepsilon_0)$ as a theorem would be desirable, especially in light of justifying the auxiliary part of KF. Regarding the second problem it is known that in a Hilbert style system KF-derivations of significant statements cannot be restricted to significant derivation steps. Castaldo and Stern (2022) argue that the additional resources of a two-sided sequent calculus make it possible to proceed via significant sequents only, i.e. sequents only containing significant formulas.[10] It is the presence of the sequent arrow that allows such a move and it appears not possible (without additional complications) to replace it by an object linguistic conditional, which fully interacts with T.

As a preliminary conclusion I take it that classical solutions to the liar paradox do have a conceptual drawback. How serious this drawback is and whether it warrants a retreat to a non-classical solution is up for discussion. Within a tolerant approach the options have to be analysed.

3 Costs of non-classical solutions

Feferman famously rejected solutions to the semantic paradoxes built on non-classical logics such as K3 and Ł3 by claiming that "nothing like sustained ordinary reasoning can be carried on in either logic".[11] Feferman considers a few principles, such as modus ponens, that he takes to be desirable and that are not available in these specific non-classical logics. Feferman's remark is not directed against all non-classical suggestions

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[10] The strategy works at least for versions of KF with restricted forms of induction.

[11] (Feferman, 1984, P. 94)

and especially intuitionistic logic supports sustained ordinary reasoning according to Feferman.[12]

Besides these specific principles sustained ordinary reasoning is relevant "not only in everyday discourse, but also in mathematics and the sciences" (Feferman, 2008, p. 206). Since mathematics and sciences are classical, non-classical logics require a suitable form of recapture. Recapture is based on the idea that as a sublogic of classical logic it is possible to regain the classical rules at least for a restricted range of applications by adding the relevant principles. For example in the case of FDE it is sufficient to assume LEM and EFQ for the arithmetical part of the language to regain classical PA. This weak form of recapture is, however, at best a minimal requirement for discourses that are clearly separable from mathematical or scientific ones. It is not satisfactory for reasoning adequately with the additional vocabulary and especially not for mixed discourses. I will argue that the truth predicate should not be strictly separated from mathematical discourse, as it has fruitful applications. Therefore there are good reasons why stronger forms of recapture are desirable.

3.1 Truth, usefulness and strength

Another aspect of this sustained ordinary reasoning concerns patterns of reasoning. There are some well-known forms of arguments in mathematics based on classical logic as well as intuitionistic logic, that have a wide range of applications. It would be a serious drawback if these familiar patterns of reasoning could not be accounted for. An example that will be important for our case is Gentzen's argument form establishing transfinite induction for ordinals less than ε_0. This is also the point where sustained ordinary reasoning is connected to the proof-theoretic strength.

One example of the usefulness of truth and patterns of reasoning that one should be reluctant to give up upon is the way in which classical truth can be employed to establish the consistency of PA. The example of compositional typed truth is well known, but a recapitulation might be illustrative. One answer to the question why one should belief in the consistency of PA might employ truth within an argument sketched as follows: the axioms of PA are true and the rules are truth preserving, so all the theorems are true. Since $0 = 1$ is not true, it is also not derivable in PA and therefore PA is consistent.[13]

A theory in which the argument can be spelled out in detail is the typed compositional theory of arithmetical truth called CT.[14] It is well-known that CT proves the T-biconditionals for the arithmetical language and the global reflection principle for PA, $\forall x(\text{sent}_A(x) \wedge \text{pr}_{pa}(x) \rightarrow \mathsf{T}(x))$, which is sufficient to establish the consistency of PA. Important for our discussion is that the interaction of truth and classical principles allow us to form such a simple and accessible form of reasoning that is useful. The derivability of global reflection for the base theory is taken as a desideratum for a

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[12] Compare also his remark "if they [paraconsistent systems] do [account for sustained ordinary reasoning], they deserve serious consideration" in Feferman, 2008, p. 206.

[13] The debate about the role of such an argument is far reaching. Shapiro (1998) sketches it as an explanation for a student.

[14] For details compare Halbach (2014).

theory of truth by several authors.[15] For the type-free case Feferman (2008) adds the desideratum that global reflection should be provably true.

Before proceeding fully to the type-free setting let me use the opportunity to connect the previous with proof-theoretic strength. Although conceptually the truth predicate is useful within such a consistency argument, it is more difficult to understand its epistemic role. Since any first-order theory containing enough syntax theory and establishing the consistency of PA will itself be stronger than PA, the epistemic commitments are not directly reducible.[16] Now some natural questions are how much stronger is CT than PA and what is the role that truth plays here.

Fortunately the first question has been successfully investigated. Since CT is fully classical the same argument as Gentzen's original works for the language expansion $\mathcal{L}_T$, i.e. in classical logic the arithmetical axioms of PA formulated in a language expansion by a truth predicate prove $\mathcal{L}_T$-TI($< \varepsilon_0$). Interestingly for the proof of the lower bound intuitionistic logic is sufficient.[17]

So far the expressive power of truth was neglected as T was treated as an arbitrary predicate without any specific axioms attached. The power of truth is only realized once suitable axioms for truth are added. The T-biconditionals alone are weak in the sense that they do not increase the amount of transfinite induction for the arithmetical fragment. On the other hand the compositional axioms for truth do significantly increase the proof-theoretic strength. More specifically CT proves transfinite induction for all arithmetical formulas and for all ordinal notations for ordinals less than $\varphi_1(\varepsilon_0)$, i.e. the ε_0-th element in the enumeration of the ε-numbers, the fixed points of the function that takes α to ω^α. So one can not only prove the consistency statement for PA but also the stronger uniform reflection principle and more.[18]

This shows how much stronger CT is compared to PA, but the question concerning the role of truth has not been addressed so far. For the argument to work two kinds of axioms are required: induction instances for the language with the truth predicate,[19] and compositional truth principles.[20] The interplay is known to be sufficient to establish the consistency, but an illustration might be helpful to explain the rationale behind the argument. PA as a reflexive theory is able to prove the consistency statements for all its finite subtheories, but not for PA itself. If this external quantification over all $n \in \mathbb{N}$ could be turned into an internal quantification, then possibly an inductive argument proving the consistency could be carried out as well. Now the compositional principle of truth for the universal quantifier is a device to achieve exactly this.[21] The extra expressive resources of the truth predicate within the induction principle allow us to exclude the relevant non-standard models of PA. This illustrates the role of the truth predicate. In a broad understanding of the expressive function of truth as a device of

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[15] For example in Leitgeb (2007).

[16] Feferman's arithmetized completeness theorem establishes that for any recursively enumerable theory T and any theory S containing Q and proving the consistency statement, for a standard provability predicate, T will be relatively interpretable in S. Compare Lindström, 1997, Theorem 4, p. 99.

[17] Compare Troelstra and Schwichtenberg (2000) or Arai (2020).

[18] It is interdefinable with ACA, compare Halbach (2014).

[19] CT^- is conservative over PA, see for Halbach (2014).

[20] UTB is conservative over PA, see for Halbach (2014).

[21] The universal quantifier axiom for truth might be interpretable as a predicative form of the ω-rule.

generalizations this function could be arguably subsumed. The suggestion is now that the additional amount of transfinite induction for arithmetical formulas that is gained by the truth theoretic principles is a good indicator for the expressive power of truth.

For the consistency argument to work induction has to be expanded to the truth predicate. There is a very natural understanding of the structure of the natural numbers that allows an open-ended conception of induction.[22] In the open-ended conception induction should not be restricted to a fixed vocabulary, but should be applicable to arbitrary language expansions by a definite predicate. Although it is difficult to give a sharp general criterion of definiteness, there are clear cases for definite predicates, such as typed truth itself, as well as indefinite ones, such as vague predicates. In the case of type-free truth there is more room for scepticism about definiteness, especially in the presence of ω-inconsistent theories of truth, such as FS.

Nevertheless, under such a conception, there is the possibility for another desideratum. In an open-ended conception it is not only natural to expand the induction principle itself, but also all arithmetical theorems that hold for an arbitrary property. In Feferman (1991) this is captured with a substitution principle in his schematic version PA(P) of Peano arithmetic. The schematic PA(P) is based on a language expansion of the arithmetical language by a predicate P, where P is used to formulate an induction axiom and the instances are retrievable by a substitution principle that allows one to replace P in an arithmetical theorem by a formula of a language containing the arithmetical language. For our discussion the example of transfinite induction will be important. With Gentzen's method it is possible to derive in classical as well as intuitionistic logic within PA(P) the principles of transfinite induction $\mathsf{TI}(P, \alpha)$ for all ordinals $\alpha < \varepsilon_0$. In the open-ended conception one would expect $\mathsf{TI}(B, \alpha)$ to hold for all formulas B in a language expansion with definite predicates.[23] This form of transfinite induction for type-free truth is available in the case KF, but not in PKF. This contrasts with the intuition that the definiteness of PKF truth is directly acknowledgable, whereas KF truth seems to require a justification for the acknowledgment of its definiteness.

3.2 Truth and reflection

In the previous subsection one role of truth has been identified as enabling an internalization of induction arguments. A similar role is played by principles of uniform reflection.[24] This connection is significant for an understanding of Feferman's aims in his Feferman (1991). Feferman combined type-free truth and reflection in an attractive way to gain a suitable characterization of his form of 'predicativism' via the notion of reflective closure, without relying on the notion of an ordinal. It is the strength of combining classical reasoning with a type-free truth predicate that allows him to do this. Feferman's use of type-free truth will be used here as an example for a useful

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[22] See for example Parsons (2008), Feferman (1991, 2014) and Warren (2020).

[23] It is not an easy task to spell out criteria of definiteness. Compare 'The question—What is a definite property?—requires in each instance the mathematician's judgment' (Feferman, 2014, p. 7).

[24] Leigh (2016) contains a very informative analysis of the connections between truth and reflection.

application of type-free truth in the classical setting and function as a desideratum for a non-classical setting with a suitable type-free truth predicate.[25]

The method of reflective closure of a theory in Feferman (1991) unifies two promising strands of research in a fruitful way.[26] On the one hand there is the idea that reflection is an especially motivated way to expand theories. These forms of expansion have from an epistemic point of view a significant advantage in contrast to other expansions. On the other hand there is the insight that type-free truth provides a useful tool to make 'big' reflection steps. In combination this allows for a construction that does not rely on the structure of the ordinals.

The main example for a reflective closure for Feferman is arithmetic and the notion of a schematic theory PA(P) allows him to provide an interesting characterization of predicative theories 'given the natural numbers'. Conceptually the reflective closure of a theory is understood as a way of making explicit the implicit commitments of accepting a theory. Whereas the notion of implicit commitment, as a general requirement, has become the target of some forms of criticism[27] there is a more neutral reading that preserves its importance. By taking reflection principles as a warranted form of expanding arithmetical theories in the background of a predicativist conception, truth may play an important role. Since it is not possible to provide a satisfying account of reflection in this paper[28], let us accept it as a working hypothesis that a reflection strategy is sound and focus on the contribution of type-free truth.

Let us take a look at the case of the arithmetical theory PA. Gentzen showed that PA proves $\mathcal{L}_A$-TI($< \varepsilon_0$), but not $\mathcal{L}_A$-TI(ε_0) itself. By adding uniform reflection to PA, $\mathcal{L}_A$-TI(ε_0) becomes derivable and repeating Gentzen's argument shows that PA plus uniform reflection for PA proves $\mathcal{L}_A$-TI($< \varepsilon_1$). In general the addition of uniform reflection principles enables a step to the next ε ordinal. As a bottom-up approach for justifying transfinite recursion this reflection process proceeds along the ε hierarchy, but it does not guarantee that a fixed point of the enumeration of the ε numbers is reached and therefore does not justify transfinite induction for the first fixed-point $\varphi_2(0)$. In other words a justified reflective closure of PA under uniform reflection will prove $\mathcal{L}_A$-TI($< \varphi_2(0)$), but not $\mathcal{L}_A$-TI($\varphi_2(0)$) itself.

Now if reflective closure is a suitable strategy for establishing predicatively acceptable theories and if the Feferman-Schütte ordinal Γ_0 is accepted as the limit of predicativity, then a stronger form of reflection is needed, as $\varphi_2(0)$ is way below Γ_0. This is where truth comes into focus. Feferman (1991) provides a way to understand the truth predicate as a tool for these stronger forms of reflection, specifically the classical type-free truth predicate of KF is able to provide sufficiently strong forms of reflection. But let us walk through the conceptual steps one by one.

The aim is to go from PA all the way up to the limit of predicativity, in a warranted way. One way to rephrase the aim is to have transfinite induction for all arithmetical statements for all ordinals less than Γ_0. However, assuming that the reflection strategy

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[25] Field (2020) for example writes that 'there is an attractive project of "reflectively closing" theories by adding predicates for truth' p. 7.

[26] In his later work Feferman employs the method of unfolding as a way to make the reflective closure explicit.

[27] For example Dean (2014).

[28] For an attempt to justify expansions by uniform reflection see for Fischer (2021a).

is a suitable way, by uniform reflection on PA only a small portion, namely ordinals less than $\varphi_2(0)$ are accessible. It takes a different version of reflection to go beyond this level. Now here is where type-free truth comes into play. Feferman realized that in classical logic $\mathcal{L}_T\text{-TI}(< \varepsilon_0)$ in combination with type-free truth allows a significant increase of transfinite induction for arithmetical statements, namely $\mathcal{L}_A\text{-TI}(< \varphi_{\varepsilon_0}(0))$. Abstracting on the power of truth it is possible to show that $KF + \mathcal{L}_T\text{-TI}(< \alpha) \vdash \mathcal{L}_A\text{-TI}(< \varphi_\alpha(0))$, at least for the relevant limit ordinals α of the form γ_n, where the γ hierarchy is given by $\gamma_0 := \varphi_0(0) = 1$, $\gamma_{n+1} := \varphi_{\gamma_n}(0)$. The interesting fact about the γ_i's is that for every ordinal β less than Γ_0, there is an $n \in \mathbb{N}$ such that $\beta \prec \gamma_n$.[29] So by finite iterations all the ordinals less than Γ_0 are reachable.

What is missing now is a way to iterate the process in a suitable way. Feferman's solution makes use of the schematic version of PA(P) in which induction is understood as open-ended with a crucial twist. Ordinary schematic theories are formulated with a substitution principle that allows the substitution of P within theorems by formulas and provides ordinary theories relative to a language. The twist is that a suitable reformulation of the substitution principle allows us to substitute formulas containing P, so that the result is again a schematic theory enabling an iteration of the process. In order to bring truth into play the truth predicate is to be allowed in substitution instances. So for the reflective closure to work formulas containing T as well as P have to be allowed as instances in the substitution.

However, in order to avoid trivialization a restriction of the substitution rule is necessary. In this case substitution is only applicable for arithmetical theorems. Feferman's theory allows for the substitution of P in a theorem $A(P)$ by an arbitrary formula of the expanded language $\mathcal{L}_T(P)$, only if A is arithmetical. For the reflective closure the important point is that if $\text{TI}(P, < \alpha)$ is established, then it is also warranted to assume $\mathcal{L}_T\text{-TI}(< \alpha)$. This specific understanding of open-endedness allows Feferman to establish that the reflective closure of PA(P) corresponds to predicative systems, exemplified as ramified analysis up to Γ_0.

Feferman's interpretation of open-endedness is a nontrivial step. Conceptually the substitution is a generalization of the schematic open-endedness. Not only should it be allowed to use induction for arbitrary 'definite' predicates but also all arithmetical properties that hold for an arbitrary property should hold for any property expressed by a definite predicate. Problematic for the reflection story is the assumption that the classical KF truth predicate is 'definite'. Feferman admits at several places that there are drawbacks of KF that appear to undermine parts of his strategy.[30] His comments suggest that this is part of the reason for him to prefer his later suggestion of unfolding.

This is where the advantages of a non-classical truth predicate might be relevant. Feferman's story is connected to the open-endedness of induction and the warrant to make use of 'definite' notions within induction. In the classical version the 'definiteness' of type-free truth is a difficult topic. Especially KF-truth with its well-known problems leave it up for discussion, whether it is fully coherent. Although KF has standard models and induction for KF truth does not pose a problem, the justification for this fact is derived externally from the model construction. An internal motivation

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[29] Cantini (1989) provides an alternative proof-theoretic analysis of KF.

[30] Compare his remark in Feferman (2008) that it is artificial.

would be preferable. Additionally, there is the problem that KF truth is not truth-determinate, in the sense that for all sentences A in $\mathcal{L}_T$, $\mathsf{T}\ulcorner A\urcorner$ or $\mathsf{T}\ulcorner\neg A\urcorner$ and not both. And although truth is determinate, where determinateness of truth is understood as $(\mathsf{T}(t) \vee \neg\mathsf{T}(t)) \wedge \neg(\mathsf{T}(t) \wedge \neg\mathsf{T}(t))$, this is implied by the outer classical logic. In contrast, the arbitrary predicate 'P', for which substitution with formulas containing 'T' is allowed, is assumed to be determinate as well as truth-determinate.

The non-classical version PKF on the other hand does not suffer from similar problems and therefore the reflection strategy appears to be better motivated in this case.[31] But now the lack of proof-theoretic strength of the non-classical solution becomes relevant. The steps that are possible in the case of PKF formulated in the logic FDE are significantly smaller and therefore not sufficient. The question now becomes: Is it possible to provide a motivated form of reflection by truth that allows us to preserve the positive aspects of the classical version without importing the negative aspects?

3.3 Why simple recapture strategies are not sufficient

The previous remarks provide two reasons why 'simple' recapture strategies do not suffice.[32] Simple recapture strategies are understood here as strategies based on the idea that for the arithmetical language classical principles are available.[33] This includes simple versions of substructural solutions as well as restrictions of classical principles. In the case of FDE and PKF the addition of classical principles of identity imply the classical principles $\Rightarrow A, \neg A$ and $A, \neg A \Rightarrow$ for all arithmetical statements. Clearly this suffices to recapture classical arithmetic as a base theory. The same would also hold for stronger base theories such as ZF. In this way it is possible to recapture all of 'pure' mathematics. Those recapture strategies seem to work for a variety of logics that are compatible with classical logic and so especially for sublogics. So except for generality and simplicity concerns there is no loss in content.[34] 'Pure' mathematics can be carried out as before.

Still simple recapture is unsatisfactory. The first reason is connected with the open-ended conception of arithmetic. By identifying definiteness with determinateness, it would be possible to align this with simple recapture. If a predicate is known to be determinate, then the necessary determinateness principles can be added easily. This works easy enough for example for typed truth.

However, it is not obvious that any definite property has to be fully determinate. Partiality itself is not in conflict with induction as is witnessed by constructive mathematics as well as the treatment of partial functions in arithmetic. The main examples for indefinite predicates are vague predicates. But is type-free truth definite in the relevant sense? Since in the Kripkean picture the fixed-point models are based on the

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[31] Compare Fischer et al. (2019) for a philosophical story. Although the reflection is better motivated as in the classical case going up to Γ_0 seems to presuppose an external notion of ordinal. A strengthening of the substitution principle in the case of schematic PKF might allow for a ground up reflection approach.

[32] Compare also the argument in Nicolai (2022) with a similar conclusion.

[33] Compare Rosenblatt (2020) for the distinction of different forms of recapture.

[34] Compare Williamson (2017).

standard model, induction is usually taken for granted and the expansion of induction to a truth predicate is not argued for, especially in the case of KF. In PKF, however, only the weaker form of an induction rule is available. Without determinateness principles for type-free truth the full principle is not contained, but only a significantly weaker alternative in rule form is available, raising doubts about the possibility of sustained ordinary reasoning. Clearly, adding determinateness principles for type-free truth in PKF is not an option. So the simple recapture strategy appears to be compatible with an open-ended conception of arithmetic only if definiteness and determinateness are identified. This undermines the non-classical motivation to treat an indeterminate notion of type-free truth in an adequate way.

The second and even more important reason is the fruitfulness of the type-free truth predicate in the classical setting that is not present in the FDE setting. Especially the useful interaction of type-free truth and reflection strategies is not directly obtainable by a simple recapture strategy. Simple recapture strategies just cannot account for these forms of interaction of truth and mathematics. There might be conceptions of truth that provide a philosophical story of why there is no form of interaction, such as disquotational accounts. However, under the assumption that the reflection strategy involving a truth predicate is a reasonable strategy in the classical case, this form of interaction should be preserved in the non-classical case. In an open-ended conception the move from $\mathsf{TI}(P, < \varepsilon_0)$ to $\mathcal{L}_\mathsf{T}\text{-}\mathsf{TI}(< \varepsilon_0)$ is warranted. Simple recapture strategies just do not account for such an interaction.

A stronger form of recapture which appears natural is grounded recapture.[35] Semantically all the grounded sentences behave classically. Complexity considerations do not allow us to introduce classical principles for the class of grounded sentences in an acceptable way, since groundedness is Π_1^1-complete. But at least for provably grounded sentences one should expect recapture. Since KF can handle grounded sentences that get into the minimal fixed point below ε_0 we should expect recapture for these provably grounded sentences. This would be achieved by showing that the Tarskian hierarchy for all levels less than ε_0 can be truth defined.[36] Again the proof-theoretic analysis of PKF establishes that it can only provide recapture for levels up to ω^ω.

This also shows that it is short sighted to simply stick to the proof-theoretic measure in form of the arithmetical sentences derivable. It is not sufficient to close the proof-theoretic gap by adding the missing arithmetical principles, in this case $\mathcal{L}_A$-$\mathsf{TI}(< \varphi_{\varepsilon_0}(0))$. Although the truth predicate is obviously not necessary to introduce mathematical principles, it is exactly its usefulness to provide an intrinsic warrant for the introduction of arithmetical principles that proved desirable. Otherwise a different form of motivation for these principles is asked for, a form that does not involve truth, and probably does not share the positive epistemic aspects of reflection extensions.

Additionally, Feferman's conception was intended as a general way to form the reflective closure and PA is only one specific example. Independent of the specifics of the base theory, one can inquire about the reflective closure of such a base theory. So even if one starts with a strong theory such as ZFC, truth can be useful in determining the reflective closure. Formulated positively the reflective closure of a theory B shares

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[35] Again see Rosenblatt (2020) for such a distinction and an interesting discussion.

[36] See Fujimoto (2010) for more details.

the same assumptions of B, but provides an estimate of what is provable without making further assumptions. Formulated negatively it makes the implicit commitment in accepting B explicit. So even if in principle it is always possible to form the relevant extension of the mathematical theories there is a loss of uniformity since the addition of the relevant mathematical principles is a piecemeal process.

4 Tolerance, pluralism or one true logic

Let us take a step back and reconsider the more specific example of KF versus PKF within the broader picture. The basic methodology adopted in the paper follows Carnap's suggestion of tolerance, allowing for different approaches and evaluating them according to their virtues. The only requirement is that the systems are presented in sufficient detail to allow for a rigorous comparison. The KF/PKF setting satisfies these requirements.

Two obvious questions are in need of an answer. What are the criteria for deciding which logic to choose, and should a unique solution be expected. Let us start with the second question, although it is dependent and interconnected with the first. There are prima facie reasons why one is inclined to think that there is only one true logic. If a logic is correct for one discourse and there is a differing account for the same discourse, then the alternative cannot also be correct. For example, either LEM is correct in arithmetic or it is not.

This obvious fact has to be qualified. First of all there is the possibility of different areas of application. Although our main focus is not a relativistic form of pluralism for completely different areas of interest, there are still several forms of discourse to be distinguished in the Kripkean framework: the syntactical base discourse, the discourse of the object theory and the semantical part containing the truth predicate. The syntactical part is usually understood as completely determinate and definite and therefore classical principles are accepted for this form of discourse and in my opinion justifiably so. Also the use of an arithmetical base theory for these purposes is rather unproblematic.[37] The second form of discourse is in most of the more formal treatments of truth simply ignored and arithmetic itself treated as an example with the background assumption that it easily expands to other object theories. This is probably the case for object theories that only concern definite and determinate discourse. In order to keep the discussion simple also here only the arithmetic case is considered, but it should be remembered that other discourses, such as those containing vague predicates or quantum physics, might well provide independent reasons for giving up classicality.

For us the most interesting case is the semantic discourse containing the truth predicate. In the Kripkean conception the truth predicate itself is not fully truth-determinate. On the other hand Kripkean truth is definite in the sense that it allows for a standard interpretation of the arithmetical vocabulary and should therefore be allowed in induction arguments, at least within a suitable open-ended conception of induction. From

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[37] Although officially we should employ a direct theory of syntax, the structural equivalences of syntax and arithmetic make an identification acceptable. For more see Leigh and Nicolai (2013) and Mount and Waxman (2021).

this perspective the choice between a classical treatment and a non-classical treatment is a choice between a determinate understanding of truth versus an indeterminate understanding. At this point it is difficult to imagine evidence that would support the correctness of one option rather than the other and even the assumption of correctness itself seems questionable.

Also in the case of a fixed area of interest there are possibilities for a pluralism of different kind. An intriguing form of pluralism was put forward by Carnap in his principle of tolerance in Carnap (1937) and has been revived in Warren (2020).[38] This form of pluralism does allow a reconsideration of the question of correctness. Theoretical realms contain parts that are less directly connected to evidence, such that there is room for at least an epistemic form of logical pluralism. Although there might be only one correct logic, for all that is known two different logics could be equally good in all respects. This underdetermination might even be not only epistemic, but it might be the case that for these speculative parts of theorizing there really is no fact of the matter and no single 'correct' form of reasoning. This is nicely accommodated for in a conventionalist picture as given in Carnap (1937) and more recently defended in Warren (2020).

Once it is realized that the criterion of correctness is not providing sufficient criteria to determine a single correct logic, pragmatic or abductive criteria become more relevant. The usual candidates are fitness with evidence, simplicity, strength, fruitfulness, uniformity and more. Also with these criteria at hand it is rather implausible to expect a single best logic to emerge.[39] Hjortland (2017) draws such a conclusion and I agree with him in this respect. Even if it is possible to find an ordering along one dimension of the criteria, such as strength, it is far from clear how one should compare logics that are mutually better on different criteria.[40] Attempts to provide a general framework for deciding between logics appear rather difficult and therefore a case study seems more promising.

Whereas classical approaches fare better with respect to strength and uniformity with other sciences non-classical approaches on the other hand claim to fit better with linguistic evidence. Rather than trying to force a decision a tolerant stance allowing for different logics seems preferable. Especially in the presence of phenomena, that have no adequate representation within classical logic. Shapiro (2014) forcefully argues that there are areas of mathematics that are only available within an intuitionistic logic. The basic idea is that the less general principles of intuitionistic logic allow for consistent theories that would be inconsistent in classical logic. An example is given by a formalized Church-Turing principle stating that all functions are recursive.[41] These theories still deserve to be counted as part of an overall scientific project and a dogmatic restriction to classical logic would be an unnecessary exclusion of these parts

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[38] Unfortunately I cannot adequately treat the most well-known pluralist account by Beall and Restall (2006) based on different interpretations of 'cases' within a generalized Tarskian thesis concerning logical consequence in this paper.

[39] Williamson's argues for a unique solution given abductive criteria in his (2017). Hjortland provides a critique in his (2017).

[40] Suggestions for weighting and using ratios do not solve the problem. Obviously if one could assign concrete numbers, then an ordering could be established.

[41] Compare Shapiro (2014).

of science. Following a more naturalistic[42] approach in combination with tolerance also allows for innovations that are not possible within a restrictive classical approach.

In our case the weakening of the background logic opens the possibility of principles of type-free truth that are inconsistent with classical logic. Transparency is an obvious example, but there might be additional possibilities that are even relevant for the realm of mathematics itself. A concrete case at hand is the fact that the usual version of KF is incoherent with its own global reflection principle.[43] PKF on the other hand is compatible with global reflection in a straightforward way. The interaction of a type-free truth predicate with global reflection might allow for a better motivated and/or stronger form of reflection than uniform reflection. This might be an exemplary case of fruitfulness that is not available in a classical setting and which might have impact on the underlying arithmetic and not only on the truth theoretic part.

5 HYPE as a solution

The discussion so far shows that the logic of FDE lacks some of the resources of classical logic, making it unsuitable for several tasks. The major drawback in the case of PKF is its proof-theoretic weakness. A first diagnosis of the reasons for this weakness points towards the formulation of induction. Whereas in classical logic induction can be formulated either in conditional form or as a rule, in FDE only the rule form is available. The reason is simply the lack of a suitable conditional. Leitgeb (2019) has proposed an interesting conditional in order to extend FDE, calling it the logic HYPE. Although Leitgeb introduced the logic of HYPE as a logic for hyperintensional contexts, one of the motivating cases is the semantic case of the paradoxes, which is enough motivation to evaluate it here. Moreover, Leitgeb provides an interesting model based on Kripke's construction demonstrating that it is a suitable formalization of a Kripkean conception of truth. The theory of type-free truth over HYPE that is of interest here is called KFL. It is basically a theory of type-free truth for the conditional free fragment of the language. The theory KFL consists of the compositional principles of PKF for the language not containing $\rightarrow$. It is formulated over the logic HYPE instead of FDE. In the presence of the HYPE conditional induction is formulated again in conditional form, i.e. $\Rightarrow A(0) \land \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x)$.[44]

5.1 The logic of HYPE

HYPE is a nice logic when it comes to having an overall logic. It has expressive resources that go beyond FDE as well as intuitionistic logic. It is based on the logical symbols $\lor$, $\neg$, $\rightarrow$, $\forall$, $\bot$. Without the conditional, HYPE is basically FDE and without

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[42] Maddy (2008) provides an interesting version of how such a naturalistic account might look like.

[43] The claim is that KF plus global reflection results in an internal inconsistency. This is a well-known problem of KF and we have remarked this already in Fischer et al. (2017). Unfortunately we have referred to a non-standard version of KF due to Leigh (2016) that restricts its range of application to the positive part of the language and is therefore coherently combinable with global reflection. Still this does not undermine our main philosophical point since an argument is missing that the nonstandard version of KF is adequate.

[44] For details of the theory see the appendix or compare Fischer et al. (2021).

the negation, it is intuitionistic logic. Leitgeb (2019) describes HYPE as a logic suitable for the comparison of different logics.

The conditional is primarily governed by the following rules[45]:

$$(\text{L} \to) \; \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \quad (\text{R} \to) \; \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

These rules correspond to the rules of a multiconclusion version of intuitionistic logic.[46] Although the conditional in HYPE bears a close resemblance to a conditional in intuitionistic logic I hesitate to call it an 'intuitionistic' conditional, since even in an inferentialist picture, where it is assumed that the rules are meaning constitutive one should not underestimate the rules which indirectly determine the meaning of the conditional. These are basically the rules in which it is allowed to use instances containing the conditional.

In the following I try to provide some reasons for the claim that HYPE behaves nice enough to account for sustained ordinary reasoning. To answer Feferman's challenge one first checks that HYPE has all the desirable properties that K3 and Ł3 were missing. Modus ponens holds in the usual form, and moreover a deduction theorem for the HYPE conditional is available.[47] Obviously $A \to A$ is valid in HYPE, but also $A \wedge (A \to B) \to B$. Although $(A \to \neg A) \to \neg A$ does not hold for the HYPE negation, it does for the defined ('intuitionistic') negation $\sim A :\Leftrightarrow A \to \bot$, in the form $(A \to \sim A) \to \sim A$. Similarly, although neither $A \vee \neg A$ nor $A \vee \sim A$ holds, $\sim (A \wedge \sim A)$ holds but not $\neg(A \wedge \neg A)$. Interestingly, $\neg(A \wedge \sim A)$ also holds, which establishes a weak law of excluded middle that shows that HYPE does not have the disjunction property.[48]

The logic of HYPE has a nice Routley style semantics for which the proof-system is sound and complete.[49] Moreover, there are sequent style proof-systems with nice proof-theoretic properties and even a system with more liberal conditional introduction rules based on the notion of connections that allows for a cut-elimination at least in the propositional case.[50]

So far it has been argued that HYPE is up for the task of providing sustained ordinary reasoning for most purposes, but how does it fit into a scientific picture in which classical theories play a major role? In cases where a clean separation of the languages is present and where the law of excluded middle holds for one fragment, classical reasoning for these restricted parts is recoverable. For example in the case of arithmetic it is sufficient to have $s = t \vee \neg(s = t)$ as additional assumptions to have classical principles for the language of arithmetic. This form of weak recapture was

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[45] Naturally there are different ways to present a calculus for HYPE. For axiomatic presentations compare Leitgeb (2019), Speranski (2021). For more on sequent calculi see Fischer (2021b).

[46] See for example Negri and Plato (2001).

[47] Interestingly in a sequent system without cut, inversion for the right introduction rule for the conditional is problematic. Compare Fischer (2021b) for details.

[48] The observation that the disjunction property does not hold in HYPE is in Odintsov and Wansing (2021). See also Speranski, 2021, p. 24.

[49] For details see Speranski (2021).

[50] For details see Fischer (2021b).

already present in the case of FDE. This is sufficient for scientific areas not containing semantic or other partial or overdetermined predicates. However, I have argued that a stronger form of recapture is desirable.

Before moving to the central case of truth let me mention an additional positive aspect of the logic of HYPE. In his (2008) Feferman adds "a further criterion to a-h [Leitgeb's criteria in (2007)], namely that the logic of the ambient metatheory used to establish consistency of one's theory should be the same as the logic basic to that theory (i.e., its outer logic); this holds for both the systems KF and DT" (Feferman 2008, p. 214). Recapture in combination with the deduction theorem and modus ponens are sufficient to establish that the consistency proof can also be carried out in HYPE. If $ZFC \vdash_{cl} con_{kfl}$, then there is a finite $\Gamma \subseteq ZFC$, such that $\Gamma \vdash_{cl} con_{kfl}$ and $\vdash_{cl} \bigwedge \Gamma \rightarrow con_{kfl}$. With the assumption of LEM for $\in$, one can argue that $LEM_\in \vdash_h \bigwedge \Gamma \rightarrow con_{kfl}$ and then $ZFC \vdash_h con_{kfl}$.

5.2 Truth over HYPE

In contrast to classical logic, HYPE is a logic, that allows a treatment of indeterminate predicates. Kripkean type-free truth is a case at hand. Whereas Kripke's original treated truth as a partial predicate there are variations that also allow an interpretation as overdetermined, so besides truth value gaps one can allow for truth value gluts. The negation of HYPE requires both options, either in form of a four-valued treatment, or as a symmetric three valued variation.

For the arithmetical language $\mathcal{L}_A$ it is again assumed that it does not contain indeterminate predicates. In the case of $\mathcal{L}_A$ the classical rules of negation apply and the HYPE conditional does not contribute anything and one can just argue classically with the conditional defined via $\neg, \vee$.

By adding a truth predicate T to receive the language $\mathcal{L}_T$ the situation changes. Due to Curry's paradox the self-referential truth predicate for the language $\mathcal{L}_T$ cannot be fully disquotational without leading to inconsistency. A promising solution is to treat the truth predicate to be a truth predicate for the restricted language $\mathcal{L}_T^- := \mathcal{L}_T \setminus \{\rightarrow\}$. Without requiring truth to interact with the conditional there is no obstacle for a Kripkean construction. In this case all the PKF sequents restricted to $\mathcal{L}_T^-$ do form a consistent theory, that is called KFL. The T-biconditionals for $\mathcal{L}_T^-$ are sound with respect to the intended interpretation as established in Leitgeb, 2019, Theorem 42, p. 385ff.

Let us sketch a simplified version of Leitgeb's model construction.[51] The construction has two stages. The first stage starts with the complete lattice of all possible valuations of the truth predicate as the state space. On this fixed state space a jump operation J is defined. By monotonicity considerations and the Knaster-Tarski fixed-point theorem it is possible to see that some states are mapped to themselves by J, and so there are fixed point states s, for which $s = J(s)$. These fixed-point states will then form the state space of the intended model in the second stage.

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[51] The additional complexity in Leitgeb's case is due to the stronger result, concerning the set of sentences, for which transparency holds.

So the starting point is the state space S where every state is identified with a subset of SENT $:= \{\#A \mid A$ a sentence of $\mathcal{L}_T\}$, the set of all codes of sentences of the language $\mathcal{L}_T$, containing T as well as $\neg$ and $\rightarrow$. The arithmetical vocabulary is interpreted standardly in the model $\mathfrak{M}$. The jump is defined $J : S \rightarrow S$, such that for all $s \in S$ and for all $\#A \in$ SENT, $\#A \in J(s)$ iff $\mathfrak{M}, s \models A$ and $\#\neg A \in J(s)$ iff $\mathfrak{M}, s^* \not\models A$. Then the jump is monotone, i.e. for all s, s' if $s \leq s'$, then $J(s) \leq J(s')$. General fixed-point considerations show that there must be fixed points, i.e. states s, such that $J(s) = s$. Moreover, the set of fixed points FP is uncountable and the ordering $\leq$ restricted to FP provides again a complete lattice (FP, $\leq_{\mathrm{FP}}$). The resulting model is an interpretation of the theory KFL establishing not only the consistency of KFL, but also supporting the claim that it is a Kripkean type-free truth predicate.

Interestingly, the construction would not work as easily if one would also adapt the state space. If $\mathfrak{M}'$ is the model based on the state space $S' := \{J(s) \mid s \in S\}$, then for the Curry sentence χ it is the case that for all $s \in S$, $\mathfrak{M}, s \not\models \chi$, and so χ will be in no extension of the truth predicate in S'. Then, however, in all $J(s)$ of $\mathfrak{M}'$ it holds that $\mathfrak{M}', J(s) \models \chi$ trivially, because there is no state in which the antecedent holds. Understood as an operation on state spaces the jump is not monotonic and therefore also not guaranteed to reach a fixed point.

The jump operator is in a sense locally correct, but not globally. Also in the case of restricting the model to (FP, $\leq_{\mathrm{FP}}$) a jump will deliver unintended consequences. On the negative side this implies that full disquotation is not possible. Although all the disquotation instances of PKF are available, disquotation for the full language containing the conditional is out of reach. On the positive side it creates an opportunity to utilize the structure of the fixed-points. The classical KF and the FDE version PKF are local axiomatizations of the fixed-point construction in the sense that a single fixed-point is a model. In the case of KFL at least two fixed points are required, which are related by the involutive star mapping of the Routley semantics, for example the state space consisting of the minimal and the maximal fixed-point of the four valued construction will be a model of KFL. The non-local conditional might be a first step in fully utilising the structure of all fixed-points.

Now that the intended semantics is sketched, it is time to check that there is no loss in strength in comparison to the classical KF. Let us start with Gentzen's argument. In the classical case it is possible to argue for arbitrary language expansions by a definite and determinate predicate P, that over PA formulated in the language expansion $\mathsf{TI}(P, < \varepsilon_0)$ is derivable. This is due to the fact that induction is understood as open-ended and therefore applies to P. In the case of HYPE it is possible to retain a similar claim, but now for language expansions by a predicate P that is assumed to be definite. It is not necessary to require that P is also determinate. In the case of truth the observation is then basically that KFL proves $\mathcal{L}_T\text{-}\mathsf{TI}(< \varepsilon_0)$.[52]

Regarding the question of patterns of sustained ordinary reasoning this result indicates that HYPE is able to preserve relevant forms of reasoning. Additionally it does so by preserving the natural correspondence between determinacy and truth determinacy, which is lost in the classical case. Moreover, it nicely fits into the picture of open-

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[52] For details see Fischer et al. (2021).

endedness. In contrast to PKF the principle of transfinite induction on P, $\mathsf{TI}(P, < \varepsilon_0)$ allows for a substitution by a definite predicate.

This is not the end of the story as it is possible to show that the truth theoretic principles of KFL are sufficient to interpret the Tarskian hierarchy up to ε_0, i.e. $\mathsf{RT}^{\varepsilon_0}$ is truth definable in KFL.[53] This establishes a suitable form of provably grounded recapture. Employing the results of the proof-theoretic analysis of the Tarskian hierarchy, the arithmetical statements derivable in KFL and KF are the same.[54]

In HYPE it is possible to fruitfully combine the expressive power of type-free truth and reflection in the way suggested by Feferman. The KFL truth predicate can be used to form a reflective closure of the schematic theory $\mathrm{PA}(P)$, called KFL*. There is a result that shows that the reflective closure of $\mathrm{PA}(P)$ in the form of KFL* has the same proof-theoretic strength as the classical reflective closure KF* for arithmetical formulas.[55] In contrast to the case of KF*, however, the substitution rule in KFL* only applies to truth theoretic formulas that are provably determinate. So the substitution principle is better motivated in this case. It allows to substitute in an arithmetical theorem $A(P)$ the predicate P, which is assumed to be arbitrary, but determinate, by a formula of the expanded language $\mathcal{L}_\mathsf{T}(P)$ that is provably determinate. There is not the same kind of artificiality involved as in the case of KF*, where substitution is allowed for formulas containing a truth predicate even if it is not truth determinate. In the KFL* case the substitution is a natural expansion of the open-ended conception captured by a schematic theory. The iteration process therefore seems better motivated. This form of reflective closure preserves the proof-theoretic strength of the classical case as well as the coherent picture of the non-classical case, capturing the best of the two opposing conceptions.

It is also possible to show that KFL proves the truth of global reflection for PA i.e. $\mathrm{KFL} \vdash \mathsf{T}\ulcorner \forall x(\mathsf{sent}_A(x) \wedge \mathsf{pr}_{pa}(x) \rightarrow \mathsf{T}(x))\urcorner$. Since CT is a subtheory of KFL, the usual argument shows $\mathsf{sent}_A(x) \wedge \mathsf{pr}_{pa}(x) \Rightarrow \mathsf{T}(x)$.[56] Since the left hand side is purely arithmetical one can derive $\Rightarrow \forall x(\neg(\mathsf{sent}_A(x) \wedge \mathsf{pr}_{pa}(x)) \vee \mathsf{T}(x))$ in KFL and then use the T-biconditionals and the compositional clauses to derive the desired. Observe that there is some similarity to Feferman's argument in Feferman (2008), where he makes use of the fact that for his conditional to be determinate it is sufficient for the antecedent to be determinate and the consequent to be determinate under the assumption that the antecedent is true.

5.3 The conditional

So far the focus was on the positive aspects of the picture. Now it is time to address the most obvious problem, namely an adequate interpretation of the conditional. Prima facie one could be tempted to claim that KFL also combines the negative aspects of both approaches. On the one hand the simple and strong classical principles have to be sacrificed and on the other hand it is not even possible to retain full disquotation.

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[53] Nicolai (2018) provides the details for such an interpretation.

[54] See for Feferman (1991) and Halbach (2014).

[55] For details see Corollary 8 in Fischer et al. (2021).

[56] See for example Halbach (2014).

The first of these problems has already been dealt with by a suitable form of recapture. The second problem will be addressed in the following.

From the perspective of disquotationalism it appears prima facie to be a serious restriction to give up full naivety. But on a second look it appears less counterintuitive. What HYPE provides is a fully compositional type-free truth predicate that works fine for the language of the object theory containing the truth predicate itself. The conditional is an additional tool to argue with indeterminate predicates such as type-free truth. The conditional allows us to simulate hypothetical reasoning. What is not possible is that the truth predicate interacts fully with the object language containing the additional conditional. To integrate the conditional into the picture it is treated instrumentally and not part of the object theory. On such an instrumentalist reading the role of the conditional has to be clarified.

In an inferentialist picture the meaning is fully determined by the rules of inference. This is also possible with the conditional. What is not possible is to bring this in line with a truth-conditional picture. Of course it is a simplification if the inferentialist reading of the conjunction and disjunction is compatible with a truth functional interpretation, however this is not an essential component of inferentialism. There may be more logical tools available that are perfectly understandable via their rules of inference, but do not fit into a restricted truth-conditional framework.

Let us compare this instrumentalist interpretation to the instrumentalist reading of KF. The most promising strategy to reinterpret the role of KF in the classical setting as has been argued before is Reinhardt's attempt. Moreover, also in the case of KF the use of an auxiliary tool, the sequent arrow, is necessary to provide a coherent interpretation of KF. Since the sequent arrow is not part of the object language there are no expectations for it to interact with truth. This can hardly be taken as a significant advantage, since for a universal approach an object linguistic representation appears desirable. In contrast to the KF case the HYPE conditional is included in the object language. However, this should not blur the main purpose of its introduction as a tool. In a sense it is an object-linguistic representation of the sequent arrow, that in addition can be nested. From this perspective the HYPE conditional is not more problematic than the instrumentalist interpretation in the classical case, contrariwise it allows for additional expressive resources.

This interpretation is supported by an external perspective on the conditional of HYPE. In the semantic picture the conditional is a device that combines different fixed-points. It is not interpreted locally at a single state where truth is interpreted, but intensionally by considering all the states that are above it. It is not very surprising that a local notion of truth, one that works for single fixed points, is not able to fully interact with the conditional. In this case it is not unnatural that the T-biconditionals fail.[57]

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[57] Leitgeb (2019) argues that with the locality of truth and the non-locality of the conditional one should not expect the T-biconditionals to hold.

6 Concluding remarks

6.1 A positive outlook

Now that I addressed one of the main possible objections to truth over HYPE, I want to mention some possible additional reasons why truth over HYPE might be a worthwhile option. First, I want to give a positive outlook on the options of getting a modality involved that might in contrast to truth interact better with the intensional conditional. Second I want to explore the options of a reflection strategy based on global reflection rather than the reflective closure.

6.1.1 Modal variants

A two stages fixed-point construction promises to be an interesting expansion in the case of HYPE.[58] The idea would be to carry out Leitgeb's construction on an expanded language containing modal predicates. In the first stage the extension of the modal predicates are treated as empty along the construction. Again in a second step one would build a fixed point model by considering the state space of fixed points. On top of that one carries out a second global fixed point construction for the modal predicates, where in this case the truth extensions are kept fixed.[59] In order to do this it would be easier to work in the symmetric strong Kleene logic rather than FDE to keep fixed points with gluts and those with gaps separate.

In contrast to the local notion of truth a non-local notion of necessity promises to have some interesting interaction principles with the HYPE conditional. The modal notions would also allow for the extraction of more information from the lattice of fixed points. Whereas KFL has models with only two fixed points one can imagine that the modal notions would require richer structures.

6.1.2 Global reflection

A substantial portion of the paper concerned the costs of going non-classical. But HYPE also opens up new possibilities. One intriguing example is that it allows for a fruitful combination of self-referential concepts with global reflection principles. In the case of classical KF this is only possible with severe restrictions, for example to the positive part of the language.[60] In the case of PKF the lack of a suitable conditional only allows for global reflection in rule form. In the case of KFL it seems straightforward to formulate strong global reflection principles, such as $\forall x(\mathsf{sent}_{\mathcal{L}_{\mathsf{T}}^{\neg}}(x) \wedge \mathsf{pr}_{kfl}(x) \to \mathsf{T}(x))$. It would be interesting to investigate how strong this form of reflection is and what a reflective closure built on such a rule would result in.

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[58] Two stage fixed-point constructions are used for example in Gupta and Belnap (1993) for weak Kleene and in Field (2020).

[59] Global fixed-point constructions have been investigated in Halbach and Welch (2009) and Stern (2016).

[60] Compare Leigh's version of KF in Leigh (2016). The observation is due to Zicchetti (2022).

6.2 Alternatives

There are several alternatives for answering the challenge of proof-theoretic strength. Although adding $\mathcal{L}_T$-TI$(< \varepsilon_0)$ to PKF is sufficient to interpret RT$^{\varepsilon_0}$ and also to prove the same arithmetical statements as KF, it should not be considered as an attractive solution.[61] The addition appears ad hoc and it does not help in formulating the reflective closure, but rather relies on it. A little bit better motivated is the attempt to exploit reflection principles in combination with disquotational principles.[62] Although it has an attractive motivation in the non-classical case, it requires an external motivation for the construction to be carried out far enough to recapture KF, let alone KF*, which is a drawback for the proposed strategy.

The most attractive alternative is provided by Field in his Field (2020). Field's proposal is based on the notion of classicality for which he introduces a new predicate Scl, read as strongly classical. Sentences that are true and strongly classical are strongly classically true or $Strue$. Field's principles for Scl allow him to treat Scl as well as $Strue$ classically. The theory INT is then able to interpret the classical theory KF and its schematic version S-INT interprets KF*. Although Field's proposal is very attractive for naive truth I think that the HYPE proposal has several advantages.

First of all Field's proposal does not answer Feferman's challenge for naive truth itself. Sustained ordinary reasoning for naive truth itself is still not provided and Field admits this, but argues that the amount of provably grounded recapture for $Strue$ should be sufficient. Since $Strue$ is provably classical expanded induction allows transfinite induction for all ordinals less than ε_0 for formulas containing $Strue$.

Another reason is that the motivation for introducing a notion of Scl that is governed by classical principles is external. Also the concept of truth that is captured by $Strue$ is not motivated by the naive version itself, but by the KF version. It is an elegant way of formulating the principles of Scl, such that KF like principles for $Strue$ are recoverable. However, it appears not fully warranted to call it the power of naive truth. It is rather the power of classical truth within naive truth.

A final reason is that it does not have similar prospects for exploiting the fixed point structure as HYPE. INT is still based on a local Kripkean truth conception. The additional complications in the semantic construction are not owed to the extraction of additional information but rather to incorporating the strongly classical truth notion within a naive truth predicate.

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[61] For details see Nicolai (2018).

[62] Such an attempt was suggested in Fischer et al. (2019).

Appendix

In the appendix we provide one version of the system KFL based on a multi-conclusion sequent system for the underlying logic of HYPE.[63] The system **G1h$_{cd}$** consists of the following initial sequents and rules:

$$(\text{ID}_p) \quad A \Rightarrow A \qquad (\text{L}\bot) \quad \bot \Rightarrow$$

$$(\text{Cut}) \ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\text{LW}) \ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (\text{RW}) \ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$(\text{LC}) \ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (\text{RC}) \ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$(\text{L}\vee) \ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\text{R}\vee) \ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\text{L} \rightarrow) \ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad (\text{R} \rightarrow) \ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$(\text{ConCp}) \ \frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma} \qquad (\text{ClCp}) \ \frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma}$$

$$(\text{L}\forall) \ \frac{A(t), \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \qquad (\text{R}\forall) \ \frac{\Gamma \Rightarrow \Delta, A(y)}{\Gamma \Rightarrow \Delta, \forall x A}$$
$$y \notin \text{FV}(\Gamma, \Delta, \forall x A)$$

G1h$_{cd}^{=}$ is obtained by adding to **G1h$_{cd}$** the following initial sequents for equality.

$$\Rightarrow t = t \qquad\qquad\qquad\qquad (\text{Ref})$$
$$s = t, A(s) \Rightarrow A(t) \qquad\qquad\qquad (\text{Rep})$$

The base theory HYA is obtained by extending the logic **G1h$_{cd}^{=}$** with the basic axioms for $0, \mathsf{S}, +, \times$ (axioms Q1–2, Q4–7 of Hájek and Pudlák, 1993), and the recursive clauses for suitable additional function symbols.

Induction is formulated as

$$\Rightarrow A(0) \wedge \forall x (A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x), \qquad (\text{IND}^{\rightarrow}(\mathcal{L}))$$

for A a formula of $\mathcal{L}$.

In the presence of a truth predicate we distinguish the language $\mathcal{L}_{\mathsf{T}}^{\rightarrow}$ containing T and $\rightarrow$ from $\mathcal{L}_{\mathsf{T}}^{\neg}$ only containing T and the HYPE negation $\neg$, but not the HYPE con-

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[63] For more details compare Fischer et al. (2021).

ditional. In the following the syntactical predicates, such as sent are to be understood as syntactical predicates fo the language $\mathcal{L}_T^{\rightarrow}$ only.

Definition 1 (*The theory KFL*) KFL extends the base theory HYA formulated in $\mathcal{L}_T^{\rightarrow}$ – i.e. with the induction schema extended to $\mathcal{L}_T^{\rightarrow}$ – with the following truth initial sequents for the $\rightarrow$-free fragment $\mathcal{L}_T^-$:

$$\mathsf{ct}(x) \wedge \mathsf{ct}(y) \Rightarrow \mathsf{T}(x \,\dot{=}\, y) \leftrightarrow \mathsf{val}(x) = \mathsf{val}(y) \tag{KFL1}$$

$$\Rightarrow \mathsf{T}(\ulcorner \mathsf{T}\dot{x} \urcorner) \leftrightarrow \mathsf{T}(x) \tag{KFL2}$$

$$\mathsf{sent}(x) \Rightarrow \mathsf{T}(\dot{\neg}\, x) \leftrightarrow \neg\mathsf{T}(x) \tag{KFL3}$$

$$\mathsf{sent}(x) \wedge \mathsf{sent}(y) \Rightarrow \mathsf{T}(x \,\dot{\vee}\, y) \leftrightarrow \mathsf{T}(x) \vee \mathsf{T}(y) \tag{KFL4}$$

$$\mathsf{sent}(\dot{\forall}\, vx) \wedge \mathtt{var}(v) \Rightarrow \mathsf{T}(\dot{\forall}\, vx) \leftrightarrow \forall y(\mathsf{ct}(y) \to \mathsf{T}(x(y/v))) \tag{KFL5}$$

$$\mathsf{T}(x) \Rightarrow \mathsf{sent}(x) \tag{KFL6}$$

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