# NUMERICAL SOLUTIONS FOR FRACTIONAL ADVECTION-DIFFUSION PROBLEMS

BY

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*I dedicate my Dissertation to my loving father, my wife, my son, my two daughters, my sisters, my brothers, my brothers-in-law and my relatives. To every SAVIOR works hard for the freedom of Palestine!*

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# THESIS ABSTRACT

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**TITLE OF STUDY:** Numerical Solutions for Fractional Advection-Diffusion Problems

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We establish the existence and the uniqueness of the weak solution of a linear time-fractional advection diffusion equation, more precisely, for a time-fractional Fokker-Planck model problem. We study the behavior of the time derivatives of the continuous solution which is important for the numerical error analysis. We propose and analyze a numerical solution based on a time stepping Crank-Nicolson combined with finite elements in space. We also investigate a time-stepping $L1$ approximation scheme. The well-posedness and error analyses of both computational schemes are studied. Some numerical results are delivered at the end to confirm the theoretical convergence results. We use MATLAB in order to implement our schemes to check the results numerically.

CHAPTER 1

# INTRODUCTION

In this chapter we give some background of the model problem under consideration follows by the literature review on the numerical contribution. An outline of the thesis is explained in the last section.

## 1.1 Background of the model problem

Over the past few decades there has been an enormous growth in the number of papers devoted to experimental and theoretical aspects of anomalous diffusion. The landmark review by Metzler and Klafter in 2000 [39] has been particularly influential, promoting the description of anomalous diffusion within the framework of continuous time random walks and fractional calculus. There are now numerous applications utilizing this approach in physics, chemistry, biology and finance [4, 18, 48, 49]. A central theoretical result in [18] was the derivation of a time-fractional Fokker-Planck equation:

$$\partial_t u - \nabla \cdot \left(\kappa \nabla \partial_t^{1-\alpha} u - \vec{F} \partial_t^{1-\alpha} u\right) = g \quad \text{for } x \in \Omega \text{ and } 0 < t \leq T, \tag{1.1.1}$$

to describe the evolution of the probability density function $u(x,t)$ for subdiffusion in an external space-time dependent force field $F(x,t)$. The Riemann-Liouville fractional derivative of order $1-\alpha$ is defined by

$$\partial_t^{1-\alpha} v(x,t) = \partial_t \mathcal{I}^\alpha v(x,t),$$

where the Riemann-Liouville integral of order $\alpha$ is defined by

$$\mathcal{I}^\alpha v(t) := \int_0^t \omega_\alpha(t-s) v(s)\, ds, \qquad \alpha > 0,$$

with $\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(t) := \int_0^\infty x^{t-1} e^{-x}\, dx$ is the standard gamma function. Since $\omega_\alpha \in L_1(0,T)$,

$$\mathcal{I}^\mu : L_p\big((0,T), L_2(\Omega)\big) \to L_p\big((0,T), L_2(\Omega)\big) \quad \text{for } 1 \leq p \leq \infty, \tag{1.1.2}$$

is a bounded linear operator.

The authors in [18] derived the time-fractional Fokker-Planck equation in (1.1.1) from power law waiting time continuous time random walks biased by Boltzmann weights. The governing equation was derived from a generalized master equation and was shown to be equivalent to a subordinated stochastic Langevin equation.

In this thesis, the existence and uniqueness of the model problem (1.1.1) is investigated. Furthermore the behavior of the time derivatives of the weak solution is studied, proving estimates that play an important role in the error analysis of the numerical schemes. For the numerical solution of the model problem (1.1.1), the generalized Crank-Nicolson scheme for the time discretization is investigated. Formally, such a scheme is second-order accurate. However, it seems that, in the presence of a weakly singular kernel and the fractional derivative operator $\partial_t^{1-\alpha}$, we could prove only an $O(k^{1+\alpha})$ convergence for $0 < \alpha < 1$ over non-uniform time meshes, where $k$ denotes the maximum time step size. A fully discrete scheme based on combining finite elements in space and Crank-Nicolson in time is developed. The existence and uniqueness of the numerical solution of the fully discrete scheme is proved. Another numerical scheme that based on $L1$ approximation in time and finite elements in space is investigated. The error results of the second fully discrete scheme shows some advantages over the first one. Convergence rate of order $O(k^2)$ is shown.

## 1.2 Literature review

- *When* $F = 0$, numerical methods for (1.1.1) were proposed and analyzed by several authors. For time-stepping methods, Langlands [28] proposed backward Euler scheme for discretization the fractional derivative. Mclean and Mustapha [35] applied finite-difference time discretization combined with finite elements in space. For the discontinuous Galerkin in time and finite elements in space, we refer to Mclean and Mustapha [36]. Mustapha [41] investigated an implicit finite-difference Crank-Nicolson scheme combined with finite elements in space. For piecewise constant discontinuous Galerkin method to discretize the time, see McLean and Mustapha [38]. Later on Mustapha et al. [42] proposed and analyzed a time-stepping Petrov–Galerkin method combined with the continuous conforming finite elements method in space. Sweilan et al. [50] proposed a Crank-Nicolson finite difference method to solve the linear time-fractional diffusion equation, formulated with Caputo's fractional derivative. Zeng et al. [55] developed a new Crank–Nicolson finite elements method in which a novel time discretization called the modified $L1$ method was used to discretize the Riemann–Liouville fractional derivative.

For *space discretization*, Zhang et al. [56] considered a standard central difference approximation for the spatial discretization, for the time stepping, two new alternating direction implicit schemes based on the $L1$ approximation and backward Euler method were proposed to solve a two-dimensional anomalous subdiffusion equation with time fractional derivative. For semidiscrete spatial finite volume method to approximate solutions of anomalous subdiffusion equations in a two-dimensional convex domain, we refer to Karaa et al. [22]. Jin et al. [20] applied Galerkin finite elements method and lumped mass Galerkin, using piecewise linear functions to solve initial boundary value problem for a homogeneous time-fractional diffusion equation in a bounded convex polygonal domain. Karaa et al. [23] applied a piecewise-linear finite elements method to approximate the solution of time-fractional diffusion equations on bounded convex domains.

Indeed, $L1$ approximation scheme is one of the most popular techniques to approximate the time fractional derivative and it was proposed by many authors to solve various types of fractional diffusion problems [54, 31, 57, 10, 46, 53, 33, 44].

Various numerical methods have been presented for solving (1.1.1), usually for $F$ assumed to be a function of $x$ only. The starting point by rewriting it in the form

$$\mathcal{I}^{1-\alpha}(u') - \kappa_\alpha u_{xx} + \mu_\alpha^{-1}(Fu)_x = 0, \tag{1.2.1}$$

where the first term is a Caputo fractional derivative. Deng [12] transformed the equation into a system of fractional ODEs by dicretizing the spatial derivatives and using the properties of Riemann-Liouville and Caputo fractional derivatives and then applying a predictor–corrector approach combined with the method of lines. The authors in [7] adopted a similar approach for (1.1.1) and solved the resulting system of fractional ODEs using a second-order scheme. Chen et al. [8] studied the stability and convergence properties of three implicit finite difference techniques, in each of which $u_{xx}$ was approximated by the standard second-order difference approximation at the advanced time level. Regarding the investigating of a collocation method based on shifted Legendre polynomials in time and sinc functions in space, we refer to Saadmandin et al. [47]. Jiang [19] established monotonicity properties of the numerical solutions obtained by using these schemes and showed that the time-stepping preserves nonnegativity of the solution. Fairweather et al. [14] investigated the stability and convergence of an orthogonal spline collocation method in space with the backward Euler method in time, based on the $L1$ approximation of the Caputo derivative. Vong and Wang [52] analyzed a high order compact scheme for (1.2.1).

For general fractional convection-diffusion equation,

$$\mathcal{I}^{1-\alpha}u' - (au_x)_x + bu_x + cu = f, \tag{1.2.2}$$

with coefficients $a$, $b$, $c$ that may depend on $x$ and $t$, Cui [11] investigated a high-order approximation for the time-fractional derivative combined with a compact exponential finite difference scheme for approximating the convection and diffusion terms.

Recently, Gracia et al. [15] applied a standard finite difference method on a uniform mesh to solve (1.2.2). They proved that the rate of convergence of the maximum nodal error on any subdomain that is bounded away from t = 0 is higher than the rate obtained when the maximum nodal error is measured over the entire space-time domain.

- Case of *space-time dependent* forcing $F$ in *one space dimension*. Le et al. [29] proposed and analyzed piecewise-linear Galerkin finite elements method in space and implicit Euler method for time to solve (1.1.1).

- For the case of the space-time dependent forcing $F$ in *multi-dimension space.* Le et al. [30] presented a new stability and convergence analysis for the spatial discretization of (1.1.1) in a convex polyhedral domain, using continuous, piecewise-linear, finite elements. Their analysis used a novel sequence of energy arguments in combination with a generalized Gronwall inequality.

## 1.3 Thesis outline

In chapter 2 we show the existence and uniqueness of the weak solution for the model prolem (1.1.1). The regularity properties for the higher order time derivative of the weak solution is proved in chapter 3. In chapter 4, we propose and analyze a numerical solution for the model problem (1.1.1). We use Crank-Nicolson scheme in time and finite elements in space. The stability of the numerical scheme is shown and also the error bound is derived. Then, we develop a fully discrete scheme using finite elements in space and Crank-Nicolson in time. Existence and uniqueness of the solution of the fully discrete scheme is studied. In chapter 5, we develop another numerical scheme that based on $L1$ approximation in time and finite elements in space. We perform the error analysis for the new fully discrete scheme.

Some numerical results will be delivered in chapter 6 to illustrate the theoretical finding. We demonstrate the convergences of the numerical schemes under consideration for different values of the fractional exponent $\alpha$ as well as for different values of the grading mesh parameter $\gamma$.

CHAPTER 2

# PRELIMINARIES AND NOTATIONS

In the section 1 we state some definitions of spaces. Section 2 contained some classical inequalities that will be used in our analysis. The last section is devoted for fractional inequalities which will be used to prove our results.

## 2.1 Spaces

**Definition 2.1 ($L_2$ Space)** We denote by $L_2(\Omega)$ the space of all Lebesgue real-valued measurable functions $v$ defined on a bounded, convex domain $\Omega \subseteq \mathbb{R}^n$ for which $\|v\| < \infty$, where

$$\|v\| := \|v\|_{L_2} = \left( \int_\Omega v^2(x)\, dx \right)^{1/2}.$$

The space $L_2(\Omega)$ can be equipped with the inner product

$$\langle v, w \rangle := \int_\Omega v(x)\, w(x)\, dx.$$

**Definition 2.2 (Weak Solution)** We define the weak solution of a partial differential equation to be the solution $u$ that satisfies the weak formulation of the partial differential equation for any test function $v \in H_0^1(\Omega)$.

**Definition 2.3 (Weak Derivative)** Assume that $v \in L_{1,loc}(\Omega)$ and let $\alpha \in \mathbb{N}^n$ be a multi-index. Then $v \in L_{1,loc}(\Omega)$ is the $\alpha-$th weak partial derivative of $u$, written $D^\alpha u = v$ if

$$\int_\Omega u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v\phi\, dx, \quad \text{for every test function } \phi \in C_0^\infty(\Omega)$$

where $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

**Definition 2.4** We define $\mathbb{H}^r(\Omega)\,, r \geq 0$, to be the space of all functions whose weak derivatives of order $\leq r$ belong to $L_2(\Omega)$, i.e.,

$$\mathbb{H}^r(\Omega) = \{v \in L_2(\Omega) : D^m v \in L_2(\Omega) \;\; \text{for} \quad |m| \leq r\}.$$

The space $\mathbb{H}^r(\Omega)$ can be equipped with the norm

$$\|v\|_r := \|v\|_{\mathbb{H}^r} = \left( \sum_{|m| \leq r} \|D^m v\|^2 \right)^{1/2} .$$

**Definition 2.5 (Sobolev Space)** We define $W_p^k(\Omega)$ to be the usual Sobolev space of functions that belong to $L_p(\Omega)$, and also the weak partial derivatives of order $k$ or less belong to $L_p(\Omega)$.

**Definition 2.6 ($H_0^1$ Space)** We define the space $H_0^1$ by

$$H_0^1(\Omega) = \{v \in H^1 : trace(v) = 0\}$$

where $trace(v(x)) = v(x)$ for $x \in \partial\Omega$

**Definition 2.7** The associated function space $\dot{H}^r(\Omega) = \{\, v \in L_2(\Omega) : \|v\|_r < \infty \,\}$ is a subspace of the usual Sobolev space $H^r(\Omega)$ for $0 \leq r \leq 2$; in particular, $\dot{H}^0(\Omega) = L_2(\Omega)$ and $\dot{H}^1(\Omega) = H_0^1(\Omega)$. Also, $\dot{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ provided $\Omega$ is convex.

**Definition 2.8** *(* **Semi Group Property)** If $\alpha > 0$ and $\beta > 0$, then

$$\mathcal{I}^{\alpha+\beta} v = \mathcal{I}^{\alpha} \mathcal{I}^{\beta} v, \tag{2.1.1}$$

is satisfied at almost every point in $[0, T]$ for $v \in L_p(0, T)$, $1 \leq p < \infty$.

**Definition 2.9 (Equicontinuity)** Let $X, Y$ be two metric spaces, and $\mathcal{F}$ is a family of functions from $X$ to $Y$. Then $\mathcal{F}$ is equicontinuous at $x_0 \in X$, if for every $\epsilon > 0$ there exist a $\delta > 0$ such that if $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$ for all $x, x_0 \in X$ and for all $f \in \mathcal{F}$.

## 2.2 Classical inequalities

In this section, we display some inequalities that we will use in the next chapters.

- (**Cauchy-Schwarz Inequality)** If $v, w \in L_2(0, T)$, then $vw \in L_1(0, T)$ and

$$|\langle v, w \rangle| \leq \|v\| \|w\|.$$

- **(Geometric Arithmetic Mean Inequality)** If $a, b \in \mathbb{R}$, then for any $\epsilon > 0$,

$$ab \leq \frac{\epsilon\, a^2}{2} + \frac{b^2}{2\,\epsilon}.$$

**Definition 2.10 (Poincare's Inequality)** If $\Omega$ is a bounded domain in $\mathbb{R}^d$, then there exist a constant $C = C(\Omega)$ such that

$$\|v\| \leq C\|\nabla v\|, \quad \text{for all } v \in H_0^1(\Omega) \tag{2.2.1}$$

**Theorem 2.1** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ is differentiable almost everywhere with integrable derivative such that

$$f(t) = \int_a^t f'(x)\,dx + f(a) \qquad \text{for } \ t \in (a, b)$$

holds if and only if $f$ is absolutely continuous.

**Remark 2.2.1 ( Green's Formula)** Let $u \in C^2$ and $v \in C^1$, then

$$\int_\Omega \nabla u\, \nabla v dx = \int_\Gamma \frac{\partial u}{\partial n} v ds - \int_\Omega \Delta u\, v dx,$$

where $\frac{\partial u}{\partial n} = n.\nabla u$ is the exterior normal derivative of $u$ on $\Gamma$.

## 2.3 Fractional inequalities

For convenience we introduce the following notations, for $\mu \geq 0$, for $\phi \in L_2\big((0, T), L_2(\Omega)\big)$ and $0 \leq t \leq T$, by

$$\mathcal{Q}_1^\mu(\phi, t) = \int_0^t \langle \phi, \mathcal{I}^\mu \phi \rangle\, ds \quad \text{and} \quad \mathcal{Q}_2^\mu(\phi, t) = \int_0^t \|\mathcal{I}^\mu \phi\|^2\, ds.$$

These operators coincide when $\mu = 0$ because $\mathcal{I}^0\phi = \phi$, so we write $\mathcal{Q}^0 = \mathcal{Q}_1^0 = \mathcal{Q}_2^0$. The operator $\mathcal{Q}_1^\mu$ is non-negative, that is,

$$\mathcal{Q}_1^\mu(\phi, T) \geq 0, \tag{2.3.1}$$

assuming that $\phi$ is real-valued [43, Lemma 3.2]. The next four lemmas establish key inequalities satisfied by $\mathcal{Q}_1^\mu$ and $\mathcal{Q}_2^\mu$.

**Lemma 2.1 ([30], Lemma 3.2)** *If* $0 < \alpha < 1$ *and* $\epsilon > 0$, *and for* $\phi, \psi \in L_2\big((0,T), L_2(\Omega)\big)$ *for* $0 \leq t \leq T$ *then*

$$\left| \int_0^t \langle \phi, \mathcal{I}^\alpha \psi \rangle \, ds \right| \leq \frac{\mathcal{Q}_1^\alpha(\phi, t)}{4\epsilon(1-\alpha)^2} + \epsilon \, \mathcal{Q}_1^\alpha(\psi, t), \tag{2.3.2}$$

$$\mathcal{Q}_2^\alpha(\phi, t) \leq \frac{2t^\alpha}{1-\alpha} \, \mathcal{Q}_1^\alpha(\phi, t), \tag{2.3.3}$$

$$\mathcal{Q}_1^\alpha(\phi, t) \leq 2t^\alpha \, \mathcal{Q}^0(\phi, t), \tag{2.3.4}$$

$$\left| \int_0^t \langle \phi, \mathcal{I}^\alpha \psi \rangle \, ds \right| \leq \frac{t^\alpha \mathcal{Q}^0(\phi, t)}{2\epsilon(1-\alpha)^2} + \epsilon \, \mathcal{Q}_1^\alpha(\psi, t). \tag{2.3.5}$$

From [24] we have the following identity, for $m \geq 1$,

$$\partial_t^m \mathcal{I}^\mu \phi(t) = \mathcal{I}^\mu \partial_t^m \phi(t) + \sum_{j=0}^{m-1} (\partial_t^j \phi)(0) \omega_{\mu-j}(t) \quad \text{for} \phi \in W_1^m\big((0,t); L_2(\Omega)\big), \tag{2.3.6}$$

and for $0 < t \leq T$. Noting that if $\phi \in W_1^1\big((0,T); X\big)$ for a normed space $X$, then $\phi : [0,T] \to X$ is absolutely continuous.

**Lemma 2.2** *If $0 < \alpha \le 1$, then for $\phi \in W_1^1\big((0,t), L_2(\Omega)\big)$,*

$$\mathcal{Q}_2^\alpha(\phi,t) \le 2\int_0^t \omega_\alpha(t-s)\mathcal{Q}_1^\alpha(\phi,s)\,ds.$$

**Proof.** Let $\psi = \mathcal{I}^\alpha\phi$ ($\psi$ is absolutely continuous). Since $\psi(0) = 0$, the Caputo fractional derivative of $\psi$ is

$${}^{\mathrm{C}}\partial_t^\alpha\psi = \mathcal{I}^{1-\alpha}(\psi') = (\mathcal{I}^{1-\alpha}\psi)' - \psi(0)\omega_{1-\alpha} = (\mathcal{I}^1\phi)' = \phi.$$

Recalling the identity in [2, Corollary1],

$$2\big\langle\psi(t), {}^{\mathrm{C}}\partial_t^\alpha\psi(t)\big\rangle = {}^{\mathrm{C}}\partial_t^\alpha\big(\|\psi\|^2\big)(t) + \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\left(\int_0^s \frac{\psi'(q)\,dq}{(t-q)^\alpha}\right)^2 ds,$$

we conclude that

$$2\langle\phi, \mathcal{I}^\alpha\phi\rangle = 2\langle{}^{\mathrm{C}}\partial_t^\alpha\psi, \psi\rangle \ge {}^{\mathrm{C}}\partial_t^\alpha\big(\|\psi\|^2\big) = \mathcal{I}^{1-\alpha}(\|\mathcal{I}^\alpha\phi\|^2)', \tag{2.3.7}$$

and thus $\mathcal{I}^1\big(\|\mathcal{I}^\alpha\phi\|^2\big) = \mathcal{I}^2\big(\|\mathcal{I}^\alpha\phi\|^2\big)' = \mathcal{I}^{1+\alpha}\mathcal{I}^{1-\alpha}\big(\|\mathcal{I}^\alpha\phi\|^2\big)' \le 2\mathcal{I}^{1+\alpha}\big(\langle\phi, \mathcal{I}^\alpha\phi\rangle\big) = 2\mathcal{I}^\alpha\mathcal{I}^1\big(\langle\phi, \mathcal{I}^\alpha\phi\rangle\big)$, which is equivalent to the desired inequality.

Now let $\phi \in L_2\big((0,T), L_2(\Omega)\big)$, and choose a sequence $\phi_n \in W_1^1\big((0,T), L_2(\Omega)\big)$ such that

$$\int_0^T \|\phi_n(t) - \phi(t)\|^2\,dt \to 0 \qquad \text{as } \ n \to \infty.$$

Using (1.1.2) with $\mu = \alpha$ and $p = 2$, it follows that using the estimates (2.3.3),

(2.3.4)

$$\mathcal{Q}_1^\alpha(\phi_n,t)-\mathcal{Q}_1^\alpha(\phi,t)\le\int_0^t\langle\phi_n-\phi,\mathcal{I}^\alpha(\phi_n-\phi)\rangle\le 2t^\alpha\int_0^t\|\phi_n-\phi\|^2\,ds\to 0$$

and

$$\begin{aligned}\mathcal{Q}_2^\alpha(\phi_n,t)-\mathcal{Q}_2^\alpha(\phi,t)&=\int_0^t\|\mathcal{I}^\alpha\phi_n\|^2\,ds-\int_0^t\|\mathcal{I}^\alpha\phi\|^2\,ds\\&\le\int_0^t\|\mathcal{I}^\alpha\phi_n-\mathcal{I}^\alpha\phi\|^2\,ds=\int_0^t\|\mathcal{I}^\alpha(\phi_n-\phi)\|^2\,ds\\&\le Ct^{2\alpha}\int_0^t\|\phi_n-\phi\|^2\,ds\to 0\end{aligned}$$

Therefore

$$\mathcal{Q}_1^\alpha(\phi_n,t)\to\mathcal{Q}_1^\alpha(\phi,t)\qquad\text{and}\quad\mathcal{Q}_2^\alpha(\phi_n,t)\to\mathcal{Q}_2^\alpha(\phi,t),$$

uniformly for $t\in[0,T]$. ∎

A useful upper bound of a function $\phi\in W_1^1\big((0,t);L_2(\Omega)\big)$ will be proved next.

**Lemma 2.3** *Let $0<\alpha<1$ and $\phi\in W_1^1\big((0,t);L_2(\Omega)\big)$. If $\phi$ is absolutely continuous, then*

$$\|\phi(t)\|^2\le 2\omega_{2-\alpha}(t)\,\mathcal{Q}_1^\alpha(\phi',t).$$

**Proof.** Applying the operator $\mathcal{I}^1$ to both sides of (2.3.7) with $\phi'$ in place of $\phi$, and using $\mathcal{I}^\alpha\phi'(0)=0$, we observe that,

$$\mathcal{I}^{1-\alpha}\big(\|\mathcal{I}^\alpha\phi'\|^2\big)(t)\le 2\mathcal{Q}_1^\alpha(\phi',t).\tag{2.3.8}$$

Put $\psi(t) = \mathcal{I}^{\alpha}\phi'$. Since $\phi = \mathcal{I}^1\phi' = \mathcal{I}^{1-\alpha}\psi$,

$$\begin{aligned}
\|\phi(t)\|^2 &\leq \left(\int_0^t \omega_{1-\alpha}(t-s)\|\psi(s)\|\,ds\right)^2 \\
&\leq \int_0^t \omega_{1-\alpha}(t-s)\,ds \int_0^t \omega_{1-\alpha}(t-s)\|\psi(s)\|^2\,ds \\
&= \omega_{2-\alpha}(t)\,\mathcal{I}^{1-\alpha}\Big(\|\mathcal{I}^{\alpha}\phi'\|^2\Big)(t),
\end{aligned}$$

and hence, the desired result follows immediately after using (2.3.8). ∎

**Lemma 2.4 ([30], Lemma 3.1)** *If $0 \leq \mu \leq \nu \leq 1$, then*

$$\mathcal{Q}_2^{\nu}(\phi, t) \leq 2t^{2(\nu-\mu)}\mathcal{Q}_2^{\mu}(\phi, t).$$

**Lemma 2.5 ([13], Theorem 3.1)** *Let $\beta > 0$ and $T > 0$. Assume that $\mathsf{a}$ and $\mathsf{b}$ are non-negative and non-decreasing functions on the interval $[0, T]$. If $\mathsf{q} : [0, T] \to \mathbb{R}$ is an integrable function satisfying*

$$0 \leq \mathsf{q}(t) \leq \mathsf{a}(t) + \mathsf{b}(t)\int_0^t \omega_{\beta}(t-s)\mathsf{q}(s)\,ds \quad \textit{for } 0 \leq t \leq T,$$

*then*

$$\mathsf{q}(t) \leq \mathsf{a}(t)E_{\beta}\big(\mathsf{b}(t)t^{\beta}\big) \quad \textit{for } 0 \leq t \leq T.$$

**Lemma 2.6 ([26], Lemma 6.4)** *Assume that $0 < \alpha < 1$, $K \geq 0$, $\phi_N, a_N \geq$*

$0, a_N \leq a_{N+1}$ *for* $N \geq 1$, *and* $\delta = Kk^\alpha/\alpha$. *If,*

$$\phi_N \leq a_N + K \sum_{n=1}^{N} \omega_{N,n}^{(\alpha)} \phi_n, \quad \textit{where} \quad \omega_{N,n}^{(\alpha)} = \int_{I_n} (t_N - t)^{\alpha-1}\, dt \qquad \textit{for } t_N \in (0, T],$$

*then*

$$\phi_N \leq C(\delta, K, \alpha, T) a_N, \qquad \textit{for} \quad t_N \in (0, T]$$

CHAPTER 3

# EXISTENCE AND UNIQUENESS ANALYSIS

In the first section of this chapter we recall the time-fractional model problem, define the weak formulation follows by giving some definitions and notations that will be used later. In section two we consider the projected equation to our model problem (3.1.1) to prove some results that are needed in section three to show existence and uniqueness for the weak solution. In the last section we prove the well-posedness of the weak formulation of our model problem.

## 3.1 Introduction

Recall that our fractional PDE is of the form:

$$\partial_t u - \nabla \cdot \left(\kappa \nabla \partial_t^{1-\alpha} u - F \partial_t^{1-\alpha} u\right) = g \quad \text{for } x \in \Omega \text{ and } 0 < t \leq T. \tag{3.1.1}$$

for $x \in \Omega$ and $0 < t \leq T$. The spatial domain $\Omega \subseteq \mathbb{R}^d$ $(d \geq 1)$ is bounded and convex. The driving force $F$, as well as the the source term $g$, are assumed to be known functions of $x$ and $t$, while the generalized diffusivity $\kappa = \kappa(x) \geq c_0 > 0$ may depend only on $x$. We consider homogeneous Dirichlet boundary condition,

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega \text{ and } 0 \leq t \leq T, \tag{3.1.2}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega. \tag{3.1.3}$$

For an integer $m \geq 1$, the following regularity assumptions on the coefficients will be used:

$$\kappa \in L_\infty(\Omega), \qquad F \in C^{m+1}\big([0,T]; W^1_\infty(\Omega)^d\big). \tag{3.1.4}$$

When $\kappa = 1$, $F = \mathbf{0}$, and $g = 0$, problem (3.1.1) reduces to the fractional subdiffusion equation:

$$\partial_t u - \nabla^2 \partial_t^{1-\alpha} u = 0.$$

In this case, the solution of the problem (3.1.1)–(3.1.3) has the form

$$u(x,t) = \sum_{m=0}^{\infty} E_\alpha(-\lambda_m t^\alpha)\varphi_m(x),$$

where the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1+n\alpha),$$

and $\lambda_m > 0$ and $\varphi_m$ are the eigenvalues and the eigenfunctions associated with the operator $\Delta$ subject to the homogeneous Drichlet boundary conditions. This allows us to extend the classical method of separation of variables for the heat equation to construct an explicit solution for any initial data $u_0 \in L_2(\Omega)$. In our case such an explicit construction is no longer possible for the solution of (3.1.1). Therefore we proceed formally by integrating (3.1.1) in time, multiplying both sides by a test function $v$, and applying the first Green identity over $\Omega$ to arrive at the weak formulation

$$\langle u(t), v\rangle + \int_0^t \left\langle \kappa\nabla\partial_s^{1-\alpha}u(s) - F(s)\partial_s^{1-\alpha}u(s), \nabla v\right\rangle ds$$
$$= \langle u_0, v\rangle + \int_0^t \langle g(s), v\rangle\, ds \quad \text{for all } v \in H_0^1(\Omega), \quad (3.1.5)$$

Due to complexity, the well-posedness as well as the regularity analysis of the continuous solution was not investigated despite its importance, apart from the case [37] when $F = \mathbf{0}$.

In order to write the weak formulation (3.1.5) as a Volterra integral equation, we introduce the bounded linear operator:

$$K_1(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$$

defined by

$$\langle K_1(t)v, w\rangle = \langle \kappa\nabla v, \nabla w\rangle - \langle F(t)v, \nabla w\rangle \quad \text{for } v,\, w \in H_0^1(\Omega),$$

The variational problem (3.1.5), with the initial condition (3.1.3), can then be written as

$$u(t) + \int_0^t \left[K_1(s)\partial_s^{1-\alpha}u(s)\right] ds = f(t) \qquad (3.1.6)$$

with

$$f(t) = u_0 + \int_0^t g(s)\, ds.$$

Integrating in time by parts, we find that

$$\int_0^t K_1(s)\partial_s^{1-\alpha}u(s)\,ds = K_1(t)\mathcal{I}^\alpha u(t) - \int_0^t K_1'(s)\mathcal{I}^\alpha u(s)\,ds$$
$$= \int_0^t \left(\omega_\alpha(t-s)K_1(t) - \int_s^t \omega_\alpha(z-s)K_1'(z)\,dz\right)u(s)\,ds,$$

with $K_1'(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by

$$\langle K_1'(t)v, w\rangle = -\langle F'(t)v, \nabla w\rangle,$$

noting that $\mathcal{I}^\alpha u(0) = 0$ by Theorem 3.5.

Therefore, $u$ satisfies

$$u(t) + \int_0^t K(t,s)u(s)\,ds = f(t) \quad \text{for } 0 \le t \le T, \tag{3.1.7}$$

where the kernel $K(t,s) : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by

$$K(t,s) = \omega_\alpha(t-s)K_1(t) - \int_s^t \omega_\alpha(z-s)K_1'(z)\,dz \quad \text{for } 0 \le s < t \le T. \tag{3.1.8}$$

We apply the Galerkin method in Section 3.2 to project the problem (3.1.7) to a finite dimensional subspace $X \subseteq H_0^1(\Omega)$, obtaining an approximate solution $u_X : [0,T] \to X$. By using delicate energy arguments and a fractional Gronwall inequality, we establish *a priori* estimates for $u_X$ that are uniform with respect to the dimension of $X$, allowing us in Section 3.3 ((3.4) and (3.6)) to prove the existence and uniqueness of a weak solution $u$ to the model problem

(3.1.1)–(3.1.3) assuming that (3.1.4) holds true for $m = 1$.

In the remaining part of this section, we introduce some notations and state some technical results that will be used later. Let

$$(\mathcal{M}^j \phi)(t) = t^j \phi(t),$$

and note the commutator properties (for any integer $j \geq 1$ and any real $\mu \geq 0$)

$$\partial_t^j \mathcal{M} - \mathcal{M} \partial_t^j = j \partial_t^{j-1}, \quad \partial_t \mathcal{M}^j - \mathcal{M}^j \partial_t = j \mathcal{M}^{j-1}, \quad \mathcal{M}\mathcal{I}^\mu - \mathcal{I}^\mu \mathcal{M} = \mu \mathcal{I}^{\mu+1}. \tag{3.1.9}$$

The following identities then follow by induction on $m$.

**Lemma 3.1** *For $0 \leq q \leq m$ and $\mu > 0$, there exist constant coefficients $a_j^{m,q}$, $b_j^{m,q}$, $c^{m,\mu}$ and $d_j^{m,\mu}$ such that*

$$\partial_t^q \mathcal{M}^m = \mathcal{M}^m \partial_t^q + \sum_{j=1}^{q} a_j^{m,q} \mathcal{M}^{m-j} \partial_t^{q-j}, \quad \mathcal{M}^m \partial_t^q = \partial_t^q \mathcal{M}^m + \sum_{j=1}^{q} b_j^{m,q} \partial_t^{q-j} \mathcal{M}^{m-j},$$

$$\mathcal{I}^\mu \mathcal{M}^m = \mathcal{M}^m \mathcal{I}^\mu + \sum_{j=1}^{m} c_j^{m,\mu} \mathcal{M}^{m-j} \mathcal{I}^{\mu+j}, \quad \mathcal{M}^m \mathcal{I}^\mu = \mathcal{I}^\mu \mathcal{M}^m + \sum_{j=1}^{m} d_j^{m,\mu} \mathcal{I}^{\mu+j} \mathcal{M}^{m-j},$$

*Later on, we set $a_0^{m,q} = b_0^{m,q} = c_0^{m,\mu} = d_0^{m,\mu} = 1$ and*

$$\tilde{a}_j^{m,q} = a_{q-j}^{m,q}, \quad \tilde{b}_j^{m,q} = b_{q-j}^{m,q}, \quad \tilde{c}_j^{m,\mu} = c_{m-j}^{m,\mu}, \quad \tilde{d}_j^{m,\mu} = d_{m-j}^{m,\mu}.$$

Given a real number $\mu \geq 0$, let

$$(B^\mu_\psi \phi)(t) = \psi(t)\,\mathcal{I}^\mu\phi(t) - \int_0^t \psi'(s)\,\mathcal{I}^\mu\phi(s)\,ds. \tag{3.1.10}$$

**Lemma 3.2** *If $\psi \in W^1_\infty\big((0,T); L_\infty(\Omega;\mathbb{R}^d)\big)$ and $\phi \in W^1_1\big((0,T); L_2(\Omega)\big)$, then there exists a constant $C$ (depending only on $\psi$, $\mu$ and $T$) such that for $0 \leq t \leq T$,*

$$\mathcal{Q}^0(B^\mu_\psi\phi, t) \leq C\mathcal{Q}^\mu_2(\phi, t), \tag{3.1.11}$$

$$\mathcal{Q}^0(\mathcal{M}B^\mu_\psi\phi, t) + \mathcal{Q}^0(\mathcal{I}^1 B^\mu_\psi\phi, t) \leq Ct^2\mathcal{Q}^\mu_2(\phi, t), \tag{3.1.12}$$

$$\mathcal{Q}^0\big((\mathcal{M}B^\mu_\psi\phi)', t\big) \leq C\mathcal{Q}^\mu_2\big((\mathcal{M}\phi)', t\big) + C\mathcal{Q}^\mu_2(\mathcal{M}\phi, t) + C\mathcal{Q}^\mu_2(\phi, t). \tag{3.1.13}$$

**Proof.** The assumption on $\psi$ implies that

$$\|(B^\mu_\psi\phi)(t)\|^2 \leq C\|(\mathcal{I}^\mu\phi)(t)\|^2 + C\int_0^t \|(\mathcal{I}^\mu\phi)(s)\|^2\,ds,$$

and (3.1.11) follows after integrating in time. By the Cauchy–Schwarz inequality,

$$\|(\mathcal{M}B^\mu_\psi\phi)(t)\|^2 + \|(\mathcal{I}^1 B^\mu_\psi\phi)(t)\|^2 \leq t^2\|(B^\mu_\psi\phi)(t)\|^2 + t\int_0^t \|(B^\mu_\psi\phi)(s)\|^2\,ds,$$

and (3.1.12) follows after integrating in time. The third identity in (3.1.9) implies that

$$\mathcal{M}B^\mu_\psi\phi = \psi\big(\mathcal{I}^\mu\mathcal{M}\phi + \mu\mathcal{I}^{\mu+1}\phi\big) - \mathcal{M}\mathcal{I}^1(\psi'\mathcal{I}^\mu\phi)$$

and therefore, differentiating with respect to $t$,

$$(\mathcal{M}B^{\mu}_{\psi}\phi)' = \psi'\big(\mathcal{I}^{\mu}\mathcal{M}\phi + \mu\mathcal{I}^{\mu+1}\phi\big) + \psi\big((\mathcal{I}^{\mu}\mathcal{M}\phi)' + \mu\mathcal{I}^{\mu}\phi\big) - (\mathcal{I}^{1} + \mathcal{M})(\psi'\mathcal{I}^{\mu}\phi).$$

Thus, noting that $(\mathcal{I}^{\mu}\mathcal{M}\phi)' = \mathcal{I}^{\mu}(\mathcal{M}\phi)'$ by (2.3.6), with

$$\|\mathcal{I}^{\mu+1}\phi(t)\|^2 = \|\mathcal{I}^{1}(\mathcal{I}^{\mu}\phi)(t)\|^2 \le t\mathcal{Q}^{\mu}_{2}(\phi, t)$$

and

$$\|\mathcal{I}^{1}(\psi'\mathcal{I}^{\mu}\phi)(t)\|^2 \le Ct\mathcal{Q}^{\mu}_{2}(\phi, t),$$

we have

$$\begin{aligned}\|(\mathcal{M}B^{\mu}_{\psi}\phi)'(t)\|^2 \le{}& C\|\mathcal{I}^{\mu}(\mathcal{M}\phi)(t)\|^2 + C\|\mathcal{I}^{\mu}(\mathcal{M}\phi)'(t)\|^2 \\ &+ C\|(\mathcal{I}^{\mu}\phi)(t)\|^2 + Ct\mathcal{Q}^{\mu}_{2}(\phi, t),\end{aligned}$$

so (3.1.13) follows after integrating in time. ∎

## 3.2 The projected equation

Suppose that $X$ is a finite-dimensional subspace of $H^1_0(\Omega)$, equipped with the induced norm: $\|v\|_X = \|v\|_{H^1_0(\Omega)}$. We define a bounded linear operator

$$K_X(t, s) : X \to X$$

by

$$\langle K_X(t,s)v, w\rangle = \langle K(t,s)v, w\rangle \quad \text{for } v,\, w \in X \text{ and } 0 \le s \le t \le T,$$

and let $f_X(t)$ denote the $L_2$-projection of $f(t)$ onto $X$, that is,

$$\langle f_X(t), w\rangle = \langle f(t), w\rangle \quad \text{for } w \in X \text{ and } 0 \le t \le T.$$

In this way, we arrive at a finite dimensional reduction of the Volterra equation (3.1.7),

$$u_X(t) + \int_0^t K_X(t,s)u_X(s)\, ds = f_X(t) \quad \text{for } 0 \le t \le T. \tag{3.2.1}$$

In the next theorem, we outline a self-contained proof of existence and uniqueness under relaxed assumptions on the coefficients in the fractional PDE (3.1.1). Such as results for scalar-valued kernels are proved by Linz [32, §3.4], Becker [3], and Brunner [6].

We assume $Y = C([0,T]; X)$, equipped with the norm $\|v\|_Y = \max_{0\le t\le T} \|v(t)\|_X$. For the remaining part of this chapter, $C$ denotes a generic constant that may depend on the coefficients in (3.1.1): $\Omega$, $\alpha$, $\eta$ and the integer $m$ in (3.1.4). However, any dependence on the subspace $X$ is indicated explicitly by writing $C_X$.

**Theorem 3.1** *Assume that the coefficients in* (3.1.1) *satisfy*

$$\kappa \in L_\infty(\Omega), \quad F \in W^1_\infty\big((0,T); L_\infty(\Omega)^d\big).$$

*Assume, in addition, that the source term $g : (0, T] \to L_2(\Omega)$ satisfies*

$$\|g(t)\| \leq M t^{\eta-1} \quad \textit{for } 0 < t \leq T, \tag{3.2.2}$$

*where $M$ and $\eta$ are positive constants, and that the initial data $u_0 \in L_2(\Omega)$. Then, the weakly-singular Volterra integral equation (3.2.1) has a unique solution $u_X \in Y$. Moreover,*

$$\|u_X\|_Y \leq C_X \|f_X\|_Y \leq C_X(\|u_0\| + M).$$

**Proof.** Our assumptions on $u_0$ and $g$ ensure that $f_X \in Y$. The kernel (3.1.8) has the form

$$K(t,s) = \omega_\alpha(t-s)G(t,s),$$

where for $0 \leq s \leq t \leq T$

$$G(t,s) = K_1(t) - \Gamma(\alpha)(t-s)\int_0^1 \omega_\alpha(y)K_1'\big(s + (t-s)y\big)\,dy.$$

Our assumptions on the coefficients ensure that $G$ is continuous mappings from the closed triangle

$$\triangle = \{\,(t,s) : 0 \leq s \leq t \leq T\,\}$$

into the space of bounded linear operators $H_0^1(\Omega) \to H^{-1}(\Omega)$. Likewise,

$$K_X(t,s) = \omega_\alpha(t-s)G_X(t,s),$$

where $G_X(t,s) : X \to X$ is defined for $(t,s) \in \triangle$ by

$$\langle G_X(t,s)v, w\rangle = \langle G(t,s)v, w\rangle \quad \text{and} \qquad \text{for } v,\, w \in X.$$

Since $X$ is finite dimensional, $G_X$ is continuous function from $\triangle$ into the space of bounded linear operators $X \to X$. Hence, there is a positive constant $\gamma_X$ such that

$$\|K_X(t,s)v\|_X \le \gamma_X \omega_\alpha(t-s)\|v\|_X \qquad \text{for } (t,s) \in \triangle \text{ and } v \in X,$$

and we can define the Volterra operator $\mathcal{K}_X : Y \to Y$ by

$$\mathcal{K}_X v(t) = \int_0^t K_X(t,s)v(s)\,ds \quad \text{for } 0 \le t \le T \text{ and } v \in Y.$$

We see that $\|\mathcal{K}_X v\|_Y \le \gamma_X \omega_{1+\alpha}(T)\|v\|_Y$. In fact, using the semigroup property (2.1.1), we obtain the following estimate for the operator norm of the $n$th power of $\mathcal{K}_X$,

$$\|\mathcal{K}_X^n\|_{Y\to Y} \le \gamma_X^n \max_{0\le t\le T} \int_0^t \omega_{n\alpha}(t-s)\,ds = \gamma_X^n \omega_{1+n\alpha}(T) \quad \text{for } n = 1,\, 2,\, 3,\, \ldots.$$

It follows that the sum

$$\mathcal{R}_X = \sum_{n=1}^{\infty} (-1)^n \mathcal{K}_X^n$$

defines a bounded linear operator with

$$\|\mathcal{R}_X\|_{Y\to Y} \le \sum_{n=1}^{\infty} \omega_{1+n\alpha}(T)\gamma_X^n = E_\alpha(\gamma_X T^\alpha) - 1.$$

This operator is the resolvent for $\mathcal{K}_X$, that is,

$$u_X + \mathcal{K}_X u_X = f_X \quad \text{if and only if} \quad u_X = f_X - \mathcal{R}_X f_X,$$

implying the existence and uniqueness of $u_X \in Y$, [32], as well as the *a priori* estimate. ∎

For second-kind Volterra equation, it is known that if $f_X$ admits an expansion in powers of $t$ and $t^\alpha$, then so does $u_X$; see Lubich [34, Corollary 3], and also Brunner, Pedas and Vainikko [5, Theorem 2.1] (with $\nu = 1-\alpha$). In order to prove that a similar result holds for systems of Volterra equations. We define $C_\alpha^m = C_\alpha^m\big([0,T];X\big)$ to be the space of continuous functions $v:[0,T]\to X$ that are $C^m$ on the half-open interval $(0,T]$ and for which the seminorm

$$|v|_{j,\alpha} = \sup_{0<t\le T} t^{j-\alpha}\|v^{(j)}(t)\|_X \quad \text{is finite for } 1 \le j \le m.$$

We define $C_\alpha^m$ into a Banach space by defining the norm

$$\|v\|_{m,\alpha} = \|v\|_Y + \sum_{j=1}^{m} |v|_{j,\alpha}.$$

**Theorem 3.2** *Assume that* (3.1.4) *holds for some integer* $m \ge 1$*. If the initial* *data* $u_0 \in L_2(\Omega)$ *and the source term* $g : (0,T] \to X$ *is* $C^m$ *with*

$$\|g^{(i-1)}(t)\| \leq Mt^{\alpha-i} \quad \textit{for } 1 \leq i \leq m,$$

*then* $u_X \in C^m_\alpha$ *and*

$$\|u_X\|_{m,\alpha} \leq C_X \|f_X\|_{m,\alpha} \leq C_X(\|u_0\| + M).$$

**Proof.** Our assumptions on $u_0$ and $g$ imply that $f_X \in C^m_\alpha$. The substitution

$$z = s + (t-s)y$$

in (3.1.8) shows that if $j + k \leq m$ and $0 \leq s < t \leq T$, then

$$\left\|\partial_t^k(\partial_t + \partial_s)^j K(t,s)v\right\|_{H^{-1}(\Omega)} \leq C_X(t-s)^{\alpha-1-k}\|v\|_{H^1_0(\Omega)} \quad \text{for } v \in H^1_0(\Omega),$$

and, since $X$ is finite dimensional,

$$\left\|\partial_t^k(\partial_t + \partial_s)^j K_X(t,s)v\right\|_X \leq C_X(t-s)^{\alpha-1-k}\|v\|_X \quad \text{for } v \in X.$$

It follows that the Volterra operator $\mathcal{K}_X : C^m_\alpha \to C^m_\alpha$ is compact [51, Theorem 6.1], and by 3.1 the homogeneous equation

$$u_X + \mathcal{K}_X u_X = 0$$

has only the trivial solution $u_X = 0$. Hence, the inhomogeneous equation

$$u_X + \mathcal{K}_X u_X = f_X$$

is well-posed not only in $Y$ but also in $C_\alpha^m$. ∎

In the preceding theorem, $u_X^{(i)}(t)$ is bounded in $H_0^1(\Omega)$, however these bounds depend on $X$. As a result, the aim is to obtain bound of $\|u_X(t)\|$ and of $\|\nabla u_X(t)\|$ independently of $X$. Our proof relies on a sequence of technical lemmas. For convenience we rescale the time variable, if necessary, so that

$$\kappa(x) \geq 1 \quad \text{for } x \in \Omega. \tag{3.2.3}$$

In this way, $\langle \kappa \nabla v, \nabla v \rangle \geq \|\nabla v\|^2$ for $v \in H_0^1(\Omega)$, and we see that for (real-valued) $\phi \in C\big([0,T]; H_0^1(\Omega)\big)$

$$\int_0^t \langle \kappa \mathcal{I}^\mu \nabla \phi, \nabla \phi \rangle \, ds \geq \int_0^t \langle \mathcal{I}^\mu \nabla \phi, \nabla \phi \rangle \, ds = \mathcal{Q}_1^\mu(\nabla \phi, t), \tag{3.2.4}$$

see [35].

Since (3.1.6) is equivalent to (3.1.7), if $v \in X$ then

$$\begin{aligned} \left\langle \int_0^t K_X(t,s) u_X(s) \, ds, v \right\rangle &= \int_0^t \big\langle K_1(s) \partial_s^{1-\alpha} u_X, v \big\rangle \, ds \\ &= \big\langle \kappa (\mathcal{I}^\alpha \nabla u_X)(t), \nabla v \big\rangle - \big\langle (B_1 u_X)(t), \nabla v \big\rangle, \end{aligned}$$

where

$$B_1\phi(t) = \int_0^t \Big( F(s)\partial_s^{1-\alpha}\phi(s) \Big)\, ds. \tag{3.2.5}$$

Assuming $\phi \in C_\alpha^1\big([0,T];X\big)$, and integrating by parts and use the notation (3.1.10) to write

$$\vec{B}_1 = B_F^\alpha. \tag{3.2.6}$$

Thus, the solution of (3.2.1) satisfies

$$\langle u_X(t), v\rangle + \langle \kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v\rangle - \big\langle (B_1 u_X)(t), \nabla v\big\rangle = \langle f_X(t), v\rangle \tag{3.2.7}$$

for $v \in X$, which yields the following estimates (with $C$ independent of $X$).

**Lemma 3.3** *For $0 \le t \le T$, the solution $u_X$ of the Volterra equation* (3.2.1) *satisfies the* a priori *estimates*

$$\mathcal{Q}_1^\alpha(u_X,t) + \mathcal{Q}_2^\alpha(\nabla u_X,t) \le Ct^\alpha \mathcal{Q}^0(f_X,t)$$

*and*

$$\mathcal{Q}^0(u_X,t) + \mathcal{Q}_1^\alpha(\nabla u_X,t) \le C\mathcal{Q}^0(f_X,t).$$

**Proof.** From (3.2.7),

$$\big\langle u_X(t), v\big\rangle + \langle \kappa\nabla\mathcal{I}^\alpha u_X(t), \nabla v\rangle \le \tfrac{1}{2}\|\nabla v\|^2 + \tfrac{1}{2}\|B_1 u_X(t)\|^2 + \tfrac{1}{2}\|v\|^2 + \big\langle f_X(t), v\big\rangle.$$

Choosing $v = \mathcal{I}^\alpha u_X(t)$, and using

$$\langle \kappa \nabla \mathcal{I}^\alpha u_X(t), \nabla v \rangle = \langle \kappa \nabla v, \nabla v \rangle \geq \|\nabla v\|^2,$$

after canceling the term $\frac{1}{2}\|\nabla v\|^2$ and integrating in time, we see that

$$\mathcal{Q}_1^\alpha(u_X, t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(\nabla u_X, t) \leq \tfrac{1}{2}\mathcal{Q}^0(B_1 u_X, t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(u_X, t) + \int_0^t \big\langle f_X(s), \mathcal{I}^\alpha u_X(s) \big\rangle \, ds. \quad (3.2.8)$$

Using the representation (3.2.6) and the inequality in (3.1.11),

$$\mathcal{Q}^0(\vec{B}_1 u_X, t) \leq 2\mathcal{Q}^0(B_F^\alpha u_X, t) \leq C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^1(u_X, t) \leq C\mathcal{Q}_2^\alpha(u_X, t),$$

where, in the final step, we used Lemma 2.4.

Using (2.3.5) with $\phi = f_X$, $\psi = u_X$ and $\epsilon = 1/2$, we deduce that

$$\mathcal{Q}_1^\alpha(u_X, t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(\nabla u_X, t) \leq C\mathcal{Q}_2^\alpha(u_X, t) + Ct^\alpha \mathcal{Q}^0(f_X, t) + \tfrac{1}{2}\mathcal{Q}_1^\alpha(u_X, t).$$

Hence, applying Lemma 2.2 with $\phi = u_X$, the function

$$\mathsf{q}(t) = \mathcal{Q}_1^\alpha(u_X, t) + \mathcal{Q}_2^\alpha(\nabla u_X, t)$$

satisfies

$$\mathsf{q}(t) \leq Ct^\alpha \mathcal{Q}^0(f_X, t) + C \int_0^t \omega_\alpha(t-s)\mathcal{Q}_1^\alpha(u_X, s) \, ds.$$

Since $\mathcal{Q}_1^\alpha(u_X, s) \le \mathsf{q}(s)$, Lemma 2.5 implies the first estimate.

To show the second estimate, use

$$-\big\langle (B_1 u_X)(t), \nabla v \big\rangle = \big\langle \nabla \cdot B_1 u_X(t), v \big\rangle$$

in (3.2.7) to obtain

$$\langle u_X(t), v\rangle + \big\langle \kappa \nabla \mathcal{I}^\alpha u_X(t), \nabla v \big\rangle \le \tfrac{1}{2}\|v\|^2 + \tfrac{3}{2}\|\nabla \cdot (B_1 u_X)(t)\|^2 + \tfrac{3}{2}\|f_X(t)\|^2.$$

Choosing $v = u_X(t)$, integrating in time, and using (3.2.4), we have

$$\tfrac{1}{2}\mathcal{Q}^0(u_X, t) + \mathcal{Q}_1^\alpha(\nabla u_X, t) \le C\mathcal{Q}^0(\nabla \cdot B_1 u_X, t) + C\mathcal{Q}^0(f_X, t).$$

Since

$$\begin{aligned}\nabla \cdot (B_F^\alpha u_X)(t) &= \big(\nabla \cdot F(t)\big)\mathcal{I}^\alpha u_X(t) + F(t) \cdot \mathcal{I}^\alpha \nabla u_X(t) \\ &\quad - \int_0^t \Big( \big(\nabla \cdot F'(s)\big)\mathcal{I}^\alpha u_X(s) + F'(s) \cdot \mathcal{I}^\alpha \nabla u_X(s) \Big)\, ds, \quad (3.2.9)\end{aligned}$$

we see that

$$\begin{aligned}\|\nabla \cdot (B_F^\alpha u_X)(t)\|^2 \le{} & C\|\mathcal{I}^\alpha u_X(t)\|^2 \\ & + C\|\mathcal{I}^\alpha \nabla u_X(t)\|^2 + C\int_0^t \Big( \|\mathcal{I}^\alpha u_X(s)\|^2 + \|\mathcal{I}^\alpha \nabla u_X(s)\|^2 \Big)\, ds.\end{aligned}$$

Implying that

$$\mathcal{Q}^0(\nabla \cdot B_F^\alpha u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t)$$

and therefore, by Lemma 2.4,

$$\mathcal{Q}^0(\nabla \cdot B_1 u_X, t) \le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t).$$

Now letting

$$\mathsf{q}(t) = \mathcal{Q}^0(u_X, t) + \mathcal{Q}_1^\alpha(\nabla u_X, t),$$

it follows using Lemma 2.2 and (2.3.4) that

$$\begin{aligned}
\mathsf{q}(t) &\le C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t) + C\mathcal{Q}^0(f_X, t) \\
&\le C\mathcal{Q}^0(f_X, t) + C\int_0^t \omega_\alpha(t-s)\Big(\mathcal{Q}_1^\alpha(u_X, s) + \mathcal{Q}_1^\alpha(\nabla u_X, s)\Big)\, ds \\
&\le C\mathcal{Q}^0(f_X, t) + Ct^\alpha \mathcal{I}^\alpha q(t).
\end{aligned}$$

We may now apply Lemma 2.5 to complete the proof. ∎

The function $\mathcal{M}u_X(t) = tu_X(t)$ satisfies a similar estimate to the first one in Lemma 3.3, but with an additional factor $t^2$ on the right-hand side.

**Lemma 3.4** *The solution $u_X$ of* (3.2.1) *satisfies*

$$\mathcal{Q}_1^\alpha(\mathcal{M}u_X, t) + \mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X, t) \le Ct^{2+\alpha}\mathcal{Q}^0(f_X, t) \quad \textit{for } 0 \le t \le T.$$

**Proof.** Multiplying both sides of (3.2.7) by $t$, and applying the third identity in (3.1.9), we find that (since $\kappa$ is independent of $t$)

$$\langle \mathcal{M}u_X, v\rangle + \big\langle \kappa(\mathcal{I}^\alpha \mathcal{M} + \alpha\mathcal{I}^{\alpha+1})\nabla u_X, \nabla v\big\rangle = \langle \mathcal{M}B_1 u_X, \nabla v\rangle + \langle \mathcal{M}(f_X), v\rangle, \quad (3.2.10)$$

whereas integrating (3.2.7) in time gives

$$\langle \kappa\mathcal{I}^{\alpha+1}\nabla u_X, \nabla v\rangle = \langle \mathcal{I}^1 B_1 u_X, \nabla v\rangle + \big\langle \mathcal{I}^1(f_X - u_X), v\big\rangle,$$

so, after eliminating $\langle \kappa\mathcal{I}^{\alpha+1}\nabla u_X, \nabla v\rangle$,

$$\begin{aligned}
&\langle \mathcal{M}u_X, v\rangle + \langle \kappa\mathcal{I}^\alpha\mathcal{M}\nabla u_X, \nabla v\rangle = \\
&\qquad \langle (\mathcal{M} - \alpha\mathcal{I}^1)B_1 u_X, \nabla v\rangle + \langle (\mathcal{M} - \alpha\mathcal{I}^1)(f_X) + \alpha\mathcal{I}^1 u_X, v\rangle \\
&\qquad \le \tfrac{1}{2}\|\nabla v\|^2 + \tfrac{1}{2}\|B_2 u_X\|^2 + \tfrac{1}{2}\|v\|^2 + \big\langle (\mathcal{M} - \alpha\mathcal{I}^1)f_X + \alpha\mathcal{I}^1 u_X, v\big\rangle,
\end{aligned}$$

where $B_2\phi = (\mathcal{M} - \alpha\mathcal{I}^1)B_1\phi$.

Choosing $v = \mathcal{I}^\alpha\mathcal{M}u_X$, we have

$$\langle \kappa\mathcal{I}^\alpha\mathcal{M}\nabla u_X, \nabla v\rangle = \langle \kappa\nabla v, \nabla v\rangle \ge \|\nabla v\|^2$$

so, after canceling the term $\frac{1}{2}\|\nabla v\|^2$ and integrating in time,

$$\begin{aligned}\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)+\tfrac{1}{2}\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t)\le \tfrac{1}{2}\mathcal{Q}^0(B_2u_X,t)&+\tfrac{1}{2}\mathcal{Q}_2^\alpha(\mathcal{M}u_X,t)\\&+\int_0^t\big\langle(\mathcal{M}-\alpha\mathcal{I}^1)f_X,\mathcal{I}^\alpha\mathcal{M}u_X\big\rangle\,ds\\&+\alpha\int_0^t\big\langle\mathcal{I}^1u_X,\mathcal{I}^\alpha\mathcal{M}u_X\big\rangle\,ds.\end{aligned}$$

Using (2.3.5), we find that

$$\int_0^t\big\langle(\mathcal{M}-\alpha\mathcal{I}^1)f_X,\mathcal{I}^\alpha\mathcal{M}u_X\big\rangle\,ds\le Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big)+\tfrac{1}{4}\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)$$

and

$$\int_0^t\big\langle\mathcal{I}^1u_X,\mathcal{I}^\alpha\mathcal{M}u_X\big\rangle\,ds\le Ct^\alpha\mathcal{Q}^0(\mathcal{I}^1u_X,t)+\tfrac{1}{4}\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t),$$

so

$$\begin{aligned}\mathcal{Q}_1^\alpha(\mathcal{M}u_X,t)+\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X,t)&\le\mathcal{Q}^0(B_2u_X,t)\\&\quad+Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big)+Ct^\alpha\mathcal{Q}^0(\mathcal{I}^1u_X,t).\end{aligned}$$

Since $B_2=(\mathcal{M}-\alpha\mathcal{I}^1)B_F^\alpha$ the estimate (3.1.12) gives

$$\mathcal{Q}^0(B_2u_X,t)\le Ct^2\mathcal{Q}_2^\alpha(u_X,t)$$

where, in the last step, we used Lemma 2.4 with $\mu=\alpha$ and $\nu=1$. We easily verify that

$$\mathcal{Q}^0\big((\mathcal{M}-\alpha\mathcal{I}^1)f_X,t\big) \le Ct^2\mathcal{Q}^0(f_X,t),$$

and by Lemma 2.4 with $\mu = 0$ and $\nu = 1$,

$$\mathcal{Q}^0(\mathcal{I}^1 u_X,t) = \mathcal{Q}^1_2(u_X,t) \le t^2\mathcal{Q}^0(u_X,t).$$

Thus, the function

$$\mathsf{q}(t) = \mathcal{Q}^\alpha_1(\mathcal{M}u_X,t) + \mathcal{Q}^\alpha_2(\mathcal{M}\nabla u_X,t)$$

satisfies

$$\mathsf{q}(t) \le Ct^2\mathcal{Q}^\alpha_2(u_X,t) + 2\mathcal{Q}^\alpha_2(\mathcal{M}u_X,t) + Ct^{2+\alpha}\mathcal{Q}^0(f_X,t) + Ct^{2+\alpha}\mathcal{Q}^0(u_X,t).$$

By (2.3.3) and Lemma 3.3,

$$t^2\mathcal{Q}^\alpha_2(u_X,t) + t^{2+\alpha}\mathcal{Q}^0(u_X,t) \le Ct^{2+\alpha}\mathcal{Q}^0(u_X,t) \le Ct^{2+\alpha}\mathcal{Q}(f_X,t),$$

and therefore, using Lemma 2.2 with $\phi = \mathcal{M}u_X$,

$$\mathsf{q}(t) \le Ct^{2+\alpha}\mathcal{Q}^0(f_X,t) + C\mathcal{I}^\alpha q(t).$$

The result now follows by applying Lemma 2.5. ∎

**Lemma 3.5** *The solution $u_X$ of* (3.2.1) *satisfies, for $0 \le t \le T$,*

$$\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)', t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)', t\big) \le Ct^\alpha \mathcal{Q}^0(f_X, t) + Ct^\alpha \mathcal{Q}^0\big((\mathcal{M}f_X)', t\big).$$

**Proof.** By differentiating (3.2.10) with respect to $t$, we have

$$\big\langle(\mathcal{M}u_X)', v\big\rangle + \big\langle\kappa\nabla(\mathcal{I}^\alpha\mathcal{M}u_X)', \nabla v\big\rangle = \big\langle B_5 u_X - \alpha\kappa\mathcal{I}^\alpha\nabla u_X, \nabla v\big\rangle + \big\langle(\mathcal{M}f_X)', v\big\rangle, \tag{3.2.11}$$

where $B_5\phi = (\mathcal{M}B_1\phi)'$.

Hence,

$$\begin{aligned}\big\langle(\mathcal{M}u_X)', v\big\rangle + \big\langle\kappa\nabla(\mathcal{I}^\alpha\mathcal{M}u_X)', \nabla v\big\rangle &\le \tfrac{1}{2}\|\nabla v\|^2 + \|B_5 u_X\|^2 + \tfrac{1}{2}\|v\|^2 \\ &\quad + C\|\mathcal{I}^\alpha\nabla u_X\|^2 + \big\langle(\mathcal{M}f_X)', v\big\rangle.\end{aligned}$$

Putting $v = \mathcal{I}^\alpha(\mathcal{M}u_X)'$, we can cancel $\frac{1}{2}\|\nabla v\|^2$ because $v = (\mathcal{I}^\alpha\mathcal{M}u_X)'$ by (2.3.6). Thus, by integrating in time and using (2.3.5) to show

$$\int_0^t \big\langle(\mathcal{M}f_X)', \mathcal{I}^\alpha(\mathcal{M}u_X)'\big\rangle\, ds \le Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big) + \tfrac{1}{2}\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)', t\big),$$

and using (3.2.4), we arrive at the estimate

$$\begin{aligned}\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)', t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)', t\big) &\le 2\mathcal{Q}^0(B_5 u_X, t) + \mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)', t\big) \\ &\quad + C\mathcal{Q}_2^\alpha(\nabla u_X, t) + Ct^\alpha\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big).\end{aligned}$$

Since $B_5 u_X = (\mathcal{M} B_F^\alpha u_X)'$, it follows from (3.1.13) that

$$\mathcal{Q}^0(B_5 u_X, t) \le C \mathcal{Q}_2^\alpha\big((\mathcal{M} u_X)', t\big).$$

By Lemmas 2.4, 3.3 and 3.4,

$$\begin{aligned} \mathcal{Q}_2^\alpha(\mathcal{M} u_X, t) + \mathcal{Q}_2^\alpha(u_X, t) &\le C t^\alpha \mathcal{Q}_1^\alpha(\mathcal{M} u_X, t) + C t^\alpha \mathcal{Q}_1^\alpha(u_X, t) \\ &\le C(t^{2+2\alpha} + t^{2\alpha}) \mathcal{Q}^0(f_X, t) \end{aligned}$$

and $\mathcal{Q}_2^\alpha(\nabla u_X, t) \le C t^\alpha \mathcal{Q}^0(f_X, t)$. Hence, the function

$$\mathsf{q}(t) = \mathcal{Q}_1^\alpha\big((\mathcal{M} u_X)', t\big) + \mathcal{Q}_2^\alpha\big((\mathcal{M} \nabla u_X)', t\big)$$

satisfies

$$\mathsf{q}(t) \le C t^\alpha \mathcal{Q}^0(f_X, t) + C t^\alpha \mathcal{Q}^0\big((\mathcal{M} f_X)', t\big) + C \mathcal{Q}_2^\alpha\big((\mathcal{M} u_X)', t\big).$$

Finally, by Lemma 2.2,

$$\mathcal{Q}_2^\alpha\big((\mathcal{M} u_X)', t\big) \le C \int_0^t \omega_\alpha(t-s) \mathcal{Q}_1^\alpha\big((\mathcal{M} u_X)', s\big)\, ds \le C \mathcal{I}^\alpha q(t),$$

and the desired estimate follows by Lemma 2.5. ∎

**Lemma 3.6** *The solution* $u_X$ *of* (3.2.1) *satisfies*

$$\mathcal{Q}^0\big((\mathcal{M}u_X)',t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)',t\big) \le C\mathcal{Q}^0(f_X,t) + C\mathcal{Q}^0\big((\mathcal{M}f_X)',t\big)$$

*for* $0 \le t \le T$.

**Proof.** Using

$$-\big\langle B_5 u_X, \nabla v\big\rangle = \big\langle \nabla\cdot B_5 u_X(t), v\big\rangle$$

in (3.2.11), we obtain

$$\begin{aligned}\big\langle(\mathcal{M}u_X)',v\big\rangle + \big\langle\kappa\mathcal{I}^\alpha(\mathcal{M}\nabla u_X)',\nabla v\big\rangle &\le \tfrac{1}{2}\|v\|^2 + 2\|\nabla\cdot B_5 u_X\|^2\\ &\quad + \|(\mathcal{M}f_X)'\|^2 - \alpha\langle\kappa\mathcal{I}^\alpha\nabla u_X,\nabla v\rangle.\end{aligned}$$

Choosing $v = (\mathcal{M}u_X)'$, integrating in time, and using (3.2.4) yields

$$\begin{aligned}\tfrac{1}{2}\mathcal{Q}^0\big((\mathcal{M}u_X)',t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)',t\big) &\le 2\mathcal{Q}^0\big(\nabla\cdot B_5 u_X,t\big) + \mathcal{Q}^0\big((\mathcal{M}f_X)',t\big)\\ &\quad - \alpha\int_0^t \big\langle(\mathcal{M}\nabla u_X)'(s),\kappa\mathcal{I}^\alpha\nabla u_X(s)\big\rangle\,ds.\end{aligned}$$

Recall from (3.2.9) that $\nabla\cdot B_F^\alpha\phi = B_{\nabla\cdot F}^\alpha\phi + B_F^\alpha\nabla\phi$, where we have used the notation

$$B_F^\alpha\nabla\phi = F(t)\cdot\mathcal{I}^\alpha\nabla\phi - \int_0^t F'(s)\cdot\mathcal{I}^\alpha\nabla\phi(s)\,ds.$$

Thus,

$$\nabla \cdot B_5 u_X = \nabla \cdot \big(\mathcal{M} B_1 u_X\big)' = \big(\mathcal{M} \nabla \cdot B_1 u_X\big)' = \big(\mathcal{M} \nabla \cdot B_F^\alpha u_X\big)'$$
$$= \big(\mathcal{M} B_{\nabla \cdot F}^\alpha u_X\big)' + \big(\mathcal{M} B_F^\alpha \nabla u_X\big)'$$

By (2.3.2),

$$\int_0^t \big\langle (\mathcal{M}\nabla u_X)'(s), \kappa \mathcal{I}^\alpha \nabla u_X(s)\big\rangle \, ds \le \tfrac{1}{2} \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)', t\big) + C\mathcal{Q}_1^\alpha(\nabla u_X, t),$$

and thus the function $\mathsf{q}(t) = \mathcal{Q}^0\big((\mathcal{M}u_X)', t\big) + \mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)', t\big)$ satisfies

$$\begin{aligned}
\mathsf{q}(t) &\le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)', t\big) + C\mathcal{Q}_2^\alpha(\mathcal{M}u_X, t) + C\mathcal{Q}_2^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)', t\big) \\
&\quad + C\mathcal{Q}_2^\alpha(\mathcal{M}\nabla u_X, t) + C\mathcal{Q}_2^\alpha(\nabla u_X, t) + C\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big) + C\mathcal{Q}_1^\alpha(\nabla u_X, t) \\
&\le C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)', t\big) + Ct^\alpha \mathcal{Q}_1^\alpha(\mathcal{M}u_X, t) + Ct^\alpha \mathcal{Q}_1^\alpha(u_X, t) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)', t\big) \\
&\quad + Ct^{2+\alpha}\mathcal{Q}^0(f_X, t) + Ct^\alpha \mathcal{Q}^0(f_X, t) + C\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big) + C\mathcal{Q}^0(f_X, t),
\end{aligned}$$

where, in the second step, Lemmas 2.2, 3.3, and 3.4 are used. A further application of Lemmas 3.3, and 3.4,

$$\mathsf{q}(t) \le C\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big) + C\mathcal{Q}^0(f_X, t) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)', t\big) + C\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)', t\big)$$

and we can use Lemma 2.2 to bound $\mathcal{Q}_2^\alpha\big((\mathcal{M}u_X)',t\big)+\mathcal{Q}_2^\alpha\big((\mathcal{M}\nabla u_X)',t\big)$ by

$$C\int_0^t \omega_\alpha(t-s)\Big(\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)',s\big)+\mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)',s\big)\Big)\,ds \le C\mathcal{I}^\alpha q(t),$$

where we used $\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)',s\big) \le Ct^\alpha \mathcal{Q}^0\big((\mathcal{M}u_X)',s\big)$, which holds by Lemma 2.4. Finally, we apply Lemma 2.5 to obtain the desired estimate. ∎

Using the previous lemmas we are able to prove the main result of this section.

**Theorem 3.3** *Assume that the coefficients satisfy* (3.1.4) *for $m=1$, that $u_0 \in L_2(\Omega)$ and that the source term $g$ satisfies* (3.2.2). *Then, the solution $u_X$ of the projected Volterra equation* (3.2.1) *satisfies (with $C$ independent of $X$)*

$$\|u_X(t)\|^2 + t^\alpha\|\nabla u_X(t)\|^2 \le C\big(\|u_0\|^2 + M^2t^{2\eta}\big) \quad \textit{for } 0\le t\le T.$$

**Proof.** Applying Lemma 2.3 with $\phi=\mathcal{M}u_X$, we see that Lemma 3.5 gives

$$\begin{aligned} t^2\|u_X(t)\|^2 = \|\mathcal{M}u_X(t)\|^2 &\le Ct^{1-\alpha}\mathcal{Q}_1^\alpha\big((\mathcal{M}u_X)',t\big) \\ &\le Ct\mathcal{Q}^0(f_X,t) + Ct\mathcal{Q}^0\big((\mathcal{M}f_X)',t\big). \end{aligned}$$

Define $g_X:[0,T]\to X$ by $\langle g_X(t),v\rangle = \langle g(t),v\rangle$ for $v\in X$, so that $f_X = u_0 + \mathcal{I}^1 g_X$ and $(\mathcal{M}f_X)' = f_X + \mathcal{M}f_X' = f_X + \mathcal{M}g_X$. We find using (3.2.2) that

$$\begin{aligned} \mathcal{Q}^0(f_X,t)+\mathcal{Q}^0\big((\mathcal{M}f_X)',t\big) &\le C\int_0^t\Big(\|u_0\|^2+\|\mathcal{I}^1 g\|^2+\|\mathcal{M}g\|^2\Big)\,ds \\ &\le Ct\big(\|u_0\|^2+M^2t^{2\eta}\big), \end{aligned} \tag{3.2.12}$$

so the estimate for the first term $\|u_X(t)\|^2$ follows at once. Similarly, applying Lemma 2.3 with $\phi = (\mathcal{M}\nabla u_X)'$ followed by Lemma 3.6, we have

$$\begin{aligned} t^{2+\alpha}\|\nabla u_X(t)\| = t^\alpha\|\mathcal{M}\nabla u_X(t)\|^2 &\le Ct\mathcal{Q}_1^\alpha\big((\mathcal{M}\nabla u_X)', t\big) \\ &\le Ct\mathcal{Q}^0(f_X, t) + Ct\mathcal{Q}^0\big((\mathcal{M}f_X)', t\big), \end{aligned}$$

implying the estimate for the second term $t^\alpha\|\nabla u_X(t)\|^2$. ∎

## 3.3 The weak solution

We will now prove that the model problem (3.1.1)–(3.1.3) is well-posed. In addition to the achieved estimates in section 3.2, the following local Hölder continuity properties of $u_X$ is needed.

**Lemma 3.7** *If $0 < \delta \le t_1 < t_2 \le T$, then*

$$\|u_X(t_2) - u_X(t_1)\|^2 \le C\delta^{-2}t_2\big(\|u_0\|^2 + M^2 t_2^{2\eta}\big)(t_2 - t_1)$$

*and*

$$\|\mathcal{I}^\alpha\nabla u_X(t_2) - \mathcal{I}^\alpha\nabla u_X(t_1)\| \le C\big(\|u_0\| + Mt_2^\eta\big)\big[\delta^{\alpha-2}(t_2 - t_1) + \delta^{-\alpha/2}(t_2 - t_1)^\alpha\big].$$

**Proof.** The Cauchy–Schwarz inequality implies that

$$\|u_X(t_2) - u_X(t_1)\|^2 = \left\|\int_{t_1}^{t_2} u_X'(s)\,ds\right\|^2 \le (t_2 - t_1)\int_{t_1}^{t_2} \|u_X'(s)\|^2\,ds,$$

and by the second inequality of Lemma 3.3, together with Lemma 3.6,

$$\int_{t_1}^{t_2} \|u_X'(s)\|^2\,ds = \int_{t_1}^{t_2} s^{-2}\|(\mathcal{M}u_X)'(s) - u_X(s)\|^2\,ds$$
$$\le 2\delta^{-2}\int_0^{t_2}\big(\|(\mathcal{M}u_X)'\|^2 + \|u_X\|^2\big)\,ds = 2\delta^{-2}\big[\mathcal{Q}^0\big((\mathcal{M}u_X)', t_2\big) + \mathcal{Q}^0(u_X, t_2)\big]$$
$$\le C\delta^{-2}\big[\mathcal{Q}^0\big(\mathcal{M}f_X)', t_2\big) + \mathcal{Q}^0(f_X, t_2)\big].$$

The first result now follows from (3.2.12). To prove the second, we write

$$\mathcal{I}^\alpha\nabla u_X(t_2) - \mathcal{I}^\alpha\nabla u_X(t_1) = \int_0^{t_1-\delta/2}\big[\omega_\alpha(t_2-s) - \omega_\alpha(t_1-s)\big]\nabla u_X(s)\,ds$$
$$+ \int_{t_1-\delta/2}^{t_1}\big[\omega_\alpha(t_2-s) - \omega_\alpha(t_1-s)\big]\nabla u_X(s)\,ds + \int_{t_1}^{t_2}\omega_\alpha(t_2-s)\nabla u_X(s)\,ds,$$

and deduce from Theorem 3.3 that

$$\|\mathcal{I}^\alpha\nabla u_X(t_2) - \mathcal{I}^\alpha\nabla u_X(t_1)\| \le C\big(\|u_0\| + Mt_2^\eta\big)\big(I_1 + I_2 + I_3\big),$$

where

$$I_1 = \int_0^{t_1-\delta/2}\big[\omega_\alpha(t_1-s) - \omega_\alpha(t_2-s)\big]s^{-\alpha/2}\,ds,$$
$$I_2 = \int_{t_1-\delta/2}^{t_1}\big[\omega_\alpha(t_1-s) - \omega_\alpha(t_2-s)\big]s^{-\alpha/2}\,ds, \qquad I_3 = \int_{t_1}^{t_2}\omega_\alpha(t_2-s)s^{-\alpha/2}\,ds.$$

Since

$$I_1 + I_2 + I_3 \le C\big(\delta^{\alpha-2}(t_2-t_1) + \delta^{-\alpha/2}(t_2-t_1)^\alpha\big),$$

the proof is completed. ∎

The existence of the weak solution is proved in the next theorem. Furthermore, in Theorem 4.4, we show that the solution $u$ is continuous on the closed interval $[0,T]$ provided $u_0 \in \dot{H}^{\mu}(\Omega)$ for some $\mu > 0$.

**Theorem 3.4** *Assume that the coefficients satisfy* (3.1.4) *for* $m = 1$*, the source term satisfies* (3.2.2)*, and that the initial data* $u_0 \in L_2(\Omega)$*. Then, problem* (3.1.1)–(3.1.3) *has a weak solution* $u : [0,T] \to L_2(\Omega)$ *with the following properties.*

1. *The restriction* $u : (0,T] \to L_2(\Omega)$ *is continuous.*

2. *If* $0 < t \le T$*, then* $u(t) \in H_0^1(\Omega)$ *with* $\|u(t)\| + t^{\alpha/2}\|\nabla u(t)\| \le C\big(\|u_0\| + Mt^{\eta}\big)$*.*

**Proof.** Let $\psi_1$, $\psi_2$, $\psi_3$, … be a sequence of functions spanning a dense subspace of $H_0^1(\Omega)$. For each integer $n \ge 1$, let $X_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_n\}$ and for brevity denote the solution of (3.2.7) with $X = X_n$ by $u_n = u_X$, and likewise write $f_n = f_X$, so that

$$\langle u_n(t), v\rangle + \langle \kappa(\mathcal{I}^{\alpha}\nabla u_n)(t), \nabla v\rangle - \langle (B_1 u_n)(t), \nabla v\rangle = \langle f_n(t), v\rangle \qquad (3.3.1)$$

for $v \in X_n$ and $0 < t \le T$. From Theorem 3.3 and Lemma 3.7, the sequence of functions $u_n$ is bounded and equicontinuous in $C\big([\delta, T]; L_2(\Omega)\big)$ whenever $0 < \delta < T$. By choosing a sequence of values of $\delta$ tending to zero we can select a subsequence, and the resulting sequence is convergent since it is bounded and equicontinuous [[40], Theorem 1.14], denote this sequence by $u_n$, such that $u_n(t)$

converges in $L_2(\Omega)$ for $0 < t \le T$. We may therefore define

$$u(t) = \lim_{n\to\infty} u_n(t) \quad \text{for } 0 < t \le T,$$

and the resulting function satisfies property 1 because, given any fixed $\delta \in (0, T)$, the limit is uniform for $t \in [\delta, T]$ since it is a limit of a sequence of equicontinuous functions [[40], Theorem 1.11]. Similarly, the functions $\mathcal{I}^\alpha \nabla u_n$ are bounded and equicontinuous in $C\big([\delta, T]; L_2(\Omega)^d\big)$ so $\mathcal{I}^\alpha \nabla u : (0, T] \to L_2(\Omega)^d$ is continuous. In fact, it will follow from (3.3.2) below that $\|\mathcal{I}^\alpha \nabla u(t)\| \to 0$ as $t \to 0$, so $\mathcal{I}^\alpha \nabla u : [0, T] \to L_2(\Omega)^d$ is continuous.

By Theorem 3.3,

$$\|u_n(t)\| \le C\big(\|u_0\| + Mt^\eta\big) \quad \text{for } 0 < t \le T,$$

so by sending $n \to \infty$ we conclude that $\|u(t)\| \le C\big(\|u_0\| + Mt^\eta\big)$. Also, for $0 < t \le T$,

$$|\langle u_n(t), v\rangle| \le C\|u_n(t)\|_{H_0^1(\Omega)}\|v\|_{H^{-1}(\Omega)} \le Ct^{-\alpha/2}\big(\|u_0\| + Mt^\eta\big)\|v\|_{H^{-1}(\Omega)}$$

taking the limit as $n \to \infty$ it follows that

$$|\langle u(t), v\rangle| \le Ct^{-\alpha/2}\big(\|u_0\| + Mt^\eta\big)\|v\|_{H^{-1}(\Omega)}$$

for all $v \in L_2(\Omega)$, so $u(t) \in H_0^1(\Omega)$ with

$$\|u(t)\|_{H_0^1(\Omega)} \le C t^{-\alpha/2}(\|u_0\| + M t^\eta),$$

proving property 2. ∎

As a continuation of Theorem 3.4, in the next theorem we show some other properties of $u$.

**Theorem 3.5** *The function $u$ in Theorem 3.4 satisfies the following additional properties:*

1. *$\mathcal{I}^\alpha \nabla u$ and $B_1 u : [0,T] \longrightarrow L_2(\Omega)$ are continuous.*
2. *$\mathcal{I}^\alpha u(0) = 0$, $\mathcal{I}^\alpha \nabla u(0) = \vec{B}_1 u(0) = 0$ and $u(0) = u_0$.*
3. *$u(t)$ converges weakly to $u(0)$ as $t \longrightarrow 0$.*
4. *For $0 \le t \le T$ and $v \in H_0^1(\Omega)$, $u$ satisfies (3.1.5).*

**Proof.** As $\mathcal{I}^\alpha \nabla u(t)$ is bounded we conclude that $\mathcal{I}^\alpha \nabla u(t)$ is continuous with

$$\|\mathcal{I}^\alpha \nabla u(t)\| \le C \int_0^t (t-s)^{\alpha-1} s^{-\alpha/2} \big(\|u_0\| + M s^\eta\big)\, ds \le C(\|u_0\| + M t^\eta) t^{\alpha/2}; \quad (3.3.2)$$

likewise, for $n \ge 1$,

$$\|\mathcal{I}^\alpha u_n(t)\| \le C\big(\|u_0\| + M t^\eta\big) t^\alpha \quad \text{and} \quad \|\mathcal{I}^\alpha \nabla u_n(t)\| \le C\big(\|u_0\| + M t^\eta\big) t^{\alpha/2}. \quad (3.3.3)$$

Continuity of $B_1 u$ follow from (3.1.10) and (3.2.6), completing the proof of property 1, with

$$\begin{aligned} \|(\vec{B}_1 u)(t)\| &\le C\|(\mathcal{I}^\alpha u)(t)\| + C\int_0^t \big(\|(\mathcal{I}^\alpha u)(s)\| + \|u(s)\|\big)\,ds \\ &\le C\big(\|u_0\| + M\big)t^\alpha. \end{aligned} \tag{3.3.4}$$

Property 2 follows from Poincare's inequality (2.2.1), and the estimates (3.3.2) and (3.3.4).

If $0 \le \delta < t \le T$, then

$$\begin{aligned} \|(\mathcal{I}^\alpha u_n)(t) - (\mathcal{I}^\alpha u)(t)\| &\le \int_0^t \omega_\alpha(t-s)\|u_n(s) - u(s)\|\,ds \\ &\le C\int_0^\delta (t-s)^{\alpha-1}\big(\|u_0\| + Ms^\eta\big)\,ds + \int_\delta^t (t-s)^{\alpha-1}\|u_n(s) - u(s)\|\,ds \\ &\le C\delta^\alpha\big(\|u_0\| + M\delta^\eta\big) + \alpha^{-1}(t-\delta)^\alpha \max_{\delta\le s\le t}\|u_n(s) - u(s)\|, \end{aligned}$$

showing that $\mathcal{I}^\alpha u_n(t) \to \mathcal{I}^\alpha u(t)$ in $L_2(\Omega)$, uniformly for $t \in [\delta, T]$. In fact, the convergence is uniform for $t \in [0, T]$, owing to the estimate (3.3.3). Therefore, we see using (3.1.10) and (3.2.6) that, for $v \in H_0^1(\Omega)$,

$$\big\langle (B_1 u_n)(t), \nabla v\big\rangle \to \big\langle (B_1 u)(t), \nabla v\big\rangle.$$

Since $\langle f_n, \psi_j\rangle = \langle f, \psi_j\rangle$ for $j \le n$, we have

$$\lim_{n\to\infty}\langle f_n(t), \psi_j\rangle = \langle f(t), \psi_j\rangle \quad \text{for all } j \ge 1 \text{ and } 0 \le t \le T,$$

Thus, by sending $n \to \infty$ in (3.3.1), it follows that (3.2.7) holds for $v \in H_0^1(\Omega)$

and $0 < t \le T$. In light of (3.3.4) and (3.3.2), the variational equation (3.1.5) is satisfied when $t = 0$ if and only if $\langle u(0), v\rangle = \langle u_0, v\rangle$ for all $v \in H_0^1(\Omega)$, which is the case if and only if we define $u(0) = u_0$. Moreover, $\langle u(t), v\rangle \to \langle f(0), v\rangle = \langle u_0, v\rangle$ as $t \to 0$, for each $v \in H_0^1(\Omega)$, and hence by density for each $v \in L_2(\Omega)$, establishing properties 3 and 4. ∎

**Theorem 3.6** *The weak solution of the problem* (3.1.1)–(3.1.3) *is unique. More precisely, under the same assumptions as theorem 3.4, there is at most one function $u$ that satisfies* (3.1.5) *and is such that $u$ and $\mathcal{I}^\alpha u$ belong to $L_2\big((0,T); L_2(\Omega)\big)$, and $\mathcal{I}^\alpha \nabla u$ belongs to $L_2\big((0,T); L_2(\Omega)^d\big)$.*

**Proof.** Since the problem is linear, it suffices to show that if $u_0 = 0$ and $g(t) \equiv 0$ then $u(t) \equiv 0$. Thus, suppose that

$$\langle u(t), v\rangle + \big\langle \kappa(\mathcal{I}^\alpha \nabla u)(t), \nabla v\big\rangle - \big\langle (B_1 u)(t), \nabla v\big\rangle = 0$$

for $0 \le t \le T$ and $v \in H_0^1(\Omega)$. Proceeding as in the proof of (3.2.8), we have

$$\mathcal{Q}_1^\alpha(u,t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(\nabla u, t) \le \tfrac{1}{2}\mathcal{Q}^0(B_1 u, t) + \tfrac{1}{2}\mathcal{Q}_2^\alpha(u,t) \le C\mathcal{Q}_2^\alpha(u,t),$$

where in the final step we use (3.1.10), (3.1.11) and Lemma 2.4. Thus, by Lemma 2.2, the function $\mathsf{q}(t) = \mathcal{Q}_1^\alpha(u,t) + \mathcal{Q}_2^\alpha(\nabla u, t)$ satisfies

$$\mathsf{q}(t) \le C\mathcal{Q}_2^\alpha(u,t) \le C\mathcal{I}^\alpha q(t),$$

and hence $\mathsf{q}(t) = 0$ for $0 \leq t \leq T$ by Lemma 2.5. In particular, $\mathcal{Q}_1^{\alpha}(u, T) = 0$ and therefore if we put $u(t) = 0$ for $t > T$ then $\hat{u}(iy) = 0$ for $-\infty < y < \infty$ by (2.3.1), implying that $u(t) = 0$ for $0 \leq t \leq T$.

∎

## CHAPTER 4

# REGULARITY ANALYSIS

In the section 1 we show some technical lemmas. The regularity properties of $u$ is studied in section 2. Section 3 is devoted to show $H^2(\Omega)-$regularity properties of $u$.

## 4.1 Preliminaries

**Lemma 4.1** *Let $\mu > 0$ and $1 \leq q \leq m$. If, for $m - q + 1 \leq j \leq m$,*

$$\mathcal{M}^j \phi \in W_1^{j-(m-q)}(0,T) \quad \textit{with} \quad (\partial_t^k \mathcal{M}^j \phi)(0) = 0 \textit{ for } 0 \leq k \leq j - (m-q) - 1,$$

*then*

$$\partial_t^q \mathcal{M}^m \mathcal{I}^\mu \phi = \sum_{j=0}^{m-q} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-q-j} \mathcal{M}^j \phi + \sum_{j=m-q+1}^{m} \tilde{d}_j^{m,\mu} \mathcal{I}^\mu \partial_t^{j-(m-q)} \mathcal{M}^j \phi.$$

**Proof.** By Lemma 3.1,

$$\mathcal{M}^m \mathcal{I}^\mu = \sum_{j=0}^{m-q} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-j} \mathcal{M}^j + \sum_{j=m-q+1}^{m} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-j} \mathcal{M}^j.$$

If $0 \leq j \leq m-q$, then $m - q - j \geq 0$ so $\partial_t^q \mathcal{I}^{\mu+m-j} = \partial_t^q \mathcal{I}^q \mathcal{I}^{\mu+m-q-j} = \mathcal{I}^{\mu+m-q-j}$. Therefore,

$$\partial_t^q \sum_{j=0}^{m-q} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-j} \mathcal{M}^j \phi = \sum_{j=0}^{m-q} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-q-j} \mathcal{M}^j \phi \quad \text{for } \phi \in L_1(0,T).$$

If $m-q+1 \le j \le m$ then $j-(m-q) \ge 1$ so

$$\partial_t^q \mathcal{I}^{\mu+m-j} = \partial_t^{q-(m-j)} \partial_t^{m-j} \mathcal{I}^{m-j} \mathcal{I}^{\mu} = \partial_t^{j-(m-q)} \mathcal{I}^{\mu}$$

and thus

$$\partial_t^q \sum_{j=m-q+1}^{m} \tilde{d}_j^{m,\mu} \mathcal{I}^{\mu+m-j} \mathcal{M}^j \phi = \sum_{j=m-q+1}^{m} \tilde{d}_j^{m,\mu} \partial_t^{j-(m-q)} \mathcal{I}^{\mu} \mathcal{M}^j \phi.$$

By Lemma 3.1,

$$\partial_t^{j-(m-q)} \mathcal{I}^{\mu} \mathcal{M}^j \phi = \mathcal{I}^{\mu} \partial_t^{j-(m-q)} \mathcal{M}^j \phi + \sum_{k=0}^{j-(m-q)-1} (\partial_t^k \mathcal{M}^j \phi)(0)\, \omega_{\mu-k},$$

and our hypotheses on $\phi$ ensure that all terms in the sum over $k$ vanish. ∎

The next lemma will be used in the proof of Lemma 4.4.

**Lemma 4.2** *Let* $\psi \in W_\infty^{2m-1}\big((0,T); L_\infty(\Omega)^d\big)$ *for some* $m \ge 1$ *and let* $\mu \ge 0$. *Then,*

$$\mathcal{Q}^{0,m}(B_\psi^\mu \phi, t) \le C \sum_{j=0}^{m} \mathcal{Q}_2^{\mu,j}(\phi, t) \quad \text{for } 0 \le t \le T \text{ and } \phi \in C_\alpha^m.$$

**Proof.** We integrate by parts $m$ times to obtain

$$B_\psi^\mu \phi = \mathcal{I}^1(\psi \partial_t^{1-\mu} \phi) = \sum_{i=0}^{m-1} (-1)^i \psi^{(i)} \mathcal{I}^{\mu+i} \phi + (-1)^m \mathcal{I}^1\big(\psi^{(m)} \mathcal{I}^{\mu+m-1} \phi\big),$$

and so

$$B_{\psi}^{\mu,m}\phi = (\mathcal{M}^m B_{\psi}^{\mu}\phi)^{(m)} = \sum_{i=0}^{m}(-1)^i \mathcal{B}_i^m \phi, \tag{4.1.1}$$

where

$$\mathcal{B}_i^m\phi = \begin{cases} \partial_t^m \mathcal{M}^m \big(\psi^{(i)}\mathcal{I}^{\mu+i}\phi\big) & \text{for } 0 \le i \le m-1, \\ \partial_t^m \mathcal{M}^m \mathcal{I}^1 \big(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi\big) & \text{for } i = m. \end{cases}$$

If $0 \le i \le m-1$, then

$$\mathcal{B}_i^m\phi = \partial_t^m \big(\psi^{(i)}(\mathcal{M}^m\mathcal{I}^{\mu+i}\phi)\big) = \sum_{q=0}^{m}\binom{m}{q}\psi^{(i+m-q)}\partial_t^q \mathcal{M}^m \mathcal{I}^{\mu+i}\phi$$

so our assumption on $\psi$ implies that

$$\|(\mathcal{B}_i^m\phi)(t)\| \le C\sum_{q=0}^{m}\big\|\partial_t^q(\mathcal{M}^m\mathcal{I}^{\mu+i}\phi)(t)\big\|. \tag{4.1.2}$$

By Lemma 4.1,

$$\partial_t^q\mathcal{M}^m\mathcal{I}^{\mu+i}\phi = \sum_{j=0}^{m-q}\tilde{d}_j^{m,\mu+i}\mathcal{I}^{\mu+i+m-q-j}\mathcal{M}^j\phi + \sum_{j=m-q+1}^{m}\tilde{d}_j^{m,\mu+i}\mathcal{I}^{\mu+i}\partial_t^{j-(m-q)}\mathcal{M}^j\phi,$$

and by Lemma 3.1,

$$\mathcal{I}^{\mu}\mathcal{M}^j = \sum_{k=0}^{j}c_k^{j,\mu}\mathcal{M}^{j-k}\mathcal{I}^{\mu+k} \quad \text{for } \mu > 0 \text{ and } j \ge 0,$$

with

$$\partial_t^q\mathcal{M}^j = \sum_{r=0}^{q}a_r^{j,q}\mathcal{M}^{j-r}\partial_t^{q-r} = \sum_{r=0}^{q}\tilde{a}_r^{j,q}\mathcal{M}^{j-q+r}\partial_t^r \quad \text{for } 1 \le q \le j.$$

Thus, for $0 \leq j \leq m-q$,

$$\mathcal{I}^{\mu+i+m-q-j}\mathcal{M}^j\phi = \sum_{k=0}^{j} c_k^{j,\mu+i+m-q-j}\mathcal{M}^{j-k}\mathcal{I}^{\mu+i+m-q-j+k}\phi$$

and for $m-q+1 \leq j \leq m$,

$$\begin{aligned}
\mathcal{I}^{\mu+i}\partial_t^{j-(m-q)}\mathcal{M}^j\phi &= \sum_{r=0}^{j-(m-q)} \tilde{a}_r^{j,j-(m-q)}\mathcal{I}^{\mu+i}\mathcal{M}^{m-q}\mathcal{M}^r\partial_t^r\phi \\
&= \sum_{r=0}^{j-(m-q)} \tilde{a}_r^{j,j-(m-q)}\sum_{k=0}^{m-q} c_k^{m-q,\mu+i}\mathcal{M}^{m-q-k}\mathcal{I}^{\mu+i+k}\mathcal{M}^r\partial_t^r\phi,
\end{aligned}$$

so

$$\begin{aligned}
\left\|(\partial_t^q\mathcal{M}^m\mathcal{I}^{\mu+i}\phi)(t)\right\|^2 \leq\, & C\sum_{j=0}^{m-q}\left\|(\mathcal{I}^{\mu+i+m-q-j}\mathcal{M}^j\phi)(t)\right\|^2 \\
&+ C\sum_{j=m-q+1}^{m}\left\|(\mathcal{I}^{\mu+i}\partial_t^{j-(m-q)}\mathcal{M}^j\phi)(t)\right\|^2 \\
\leq\, & C\sum_{j=0}^{m-q}\sum_{k=0}^{j}\left\|(\mathcal{M}^{j-k}\mathcal{I}^{\mu+i+m-q-j+k}\phi)(t)\right\|^2 \\
&+ C\sum_{j=m-q+1}^{m}\sum_{r=0}^{j-(m-q)}\sum_{k=0}^{m-q}\left\|(\mathcal{M}^{m-q-k}\mathcal{I}^{\mu+i+k}\mathcal{M}^r\partial_t^r\phi)(t)\right\|^2.
\end{aligned}$$

Integrating in time, since $\mathcal{Q}^0(\mathcal{M}^j\mathcal{I}^\mu\phi,t) \leq t^{2j}\mathcal{Q}_2^\mu(\phi,t)$, we see that

$$\begin{aligned}
\mathcal{Q}^0\left(\partial_t^q\mathcal{M}^m\mathcal{I}^{\mu+i}\phi,t\right) \leq\, & C\sum_{j=0}^{m-q}\sum_{k=0}^{j} t^{2(j-k)}\mathcal{Q}_2^{\mu+i+m-q-j+k}(\phi,t) \\
&+ C\sum_{j=m-q+1}^{m}\sum_{r=0}^{j-(m-q)}\sum_{k=0}^{m-q} t^{2(m-q-k)}\mathcal{Q}_2^{\mu+i+k,r}(\phi,t)
\end{aligned}$$

and therefore, by Lemma 2.4,

$$\mathcal{Q}^0\big(\partial_t^q \mathcal{M}^m \mathcal{I}^{\mu+i}\phi, t\big) \le C t^{2(i+m-q)} \sum_{r=0}^{q} \mathcal{Q}_2^{\mu,r}(\phi,t). \tag{4.1.3}$$

Hence, recalling (4.1.2),

$$\mathcal{Q}^0(\mathcal{B}_i^m\phi, t) \le C\sum_{q=0}^{m} \mathcal{Q}^0\big(\partial_t^q \mathcal{M}^m \mathcal{I}^{\mu+i}\phi, t\big) \le C t^{2i} \sum_{r=0}^{m} \mathcal{Q}_2^{\mu,r}(\phi,t) \quad \text{for } 0 \le i \le m-1. \tag{4.1.4}$$

It remains to estimate $\mathcal{B}_m^m\phi = \partial_t^m \mathcal{M}^m \mathcal{I}^1(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi)$. Taking $q = m$ and $\mu = 1$ in Lemma 4.1 gives

$$\partial_t^m \mathcal{M}^m \mathcal{I}^1 = \tilde{d}_0^{m,1}\mathcal{I}^1 + \sum_{j=1}^{m} \tilde{d}_j^{m,1}\mathcal{I}^1\partial_t^j\mathcal{M}^j,$$

and so

$$\mathcal{B}_m^m\phi = \tilde{d}_0^{1,m}\mathcal{I}^1(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi) + \sum_{j=1}^{m} \tilde{d}_j^{1,m}\partial_t^{j-1}(\psi^{(m)}\mathcal{M}^j\mathcal{I}^{\mu+m-1}\phi).$$

Thus,

$$\mathcal{Q}^0(\mathcal{B}_m^m\phi, t) \le C\mathcal{Q}^0\big(\mathcal{I}^1(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi), t\big) + C\sum_{j=1}^{m}\sum_{q=0}^{j-1} \mathcal{Q}^0\big(\partial_t^q\mathcal{M}^j\mathcal{I}^{\mu+m-1}\phi, t\big),$$

and since

$$\left\|\left(\mathcal{I}^1(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi)(t)\right\|^2 \le \left(\int_0^t \|\psi^{(m)}(s)\|^2\,ds\right)\left(\int_0^t \|\mathcal{I}^{\mu+m-1}\phi)(s)\|^2\,ds\right)$$
$$\le Ct\mathcal{Q}_2^{\mu+m-1}(\phi,t)$$

we have, by Lemma 2.4,

$$\mathcal{Q}^0\left(\mathcal{I}^1(\psi^{(m)}\mathcal{I}^{\mu+m-1}\phi),t\right) \le Ct^2\mathcal{Q}_2^{\mu+m-1}(\phi,t) \le Ct^{2m}\mathcal{Q}_2^{\mu}(\phi,t).$$

Finally, using (4.1.3) with $m$ replaced by $j$ and with $i$ replaced by $m-1$,

$$\mathcal{Q}^0(\mathcal{B}_m^m\phi,t) \le Ct^{2m}\mathcal{Q}_2^{\mu}(\phi,t) + C\sum_{j=1}^{m}\sum_{q=0}^{j-1} t^{2(m-1+j-q)}\sum_{r=0}^{q}\mathcal{Q}_2^{\mu,r}(\phi,t)$$
$$\le Ct^{2m}\sum_{q=0}^{m-1}\mathcal{Q}_2^{\mu,r}(\phi,t).$$

The result now follows from (4.1.1) and (4.1.4). ∎

**Lemma 4.3** *If $m \ge 0$, $\psi \in W_\infty^m\big((0,T);L_\infty(\Omega)^d\big)$ and*

$$\mathcal{M}^k\phi \in W_1^k(0,T) \quad \textit{for } 0 \le k \le m+1,$$

*with*

$$(\partial_t^q\mathcal{M}^k\phi)(0) = 0 \quad \textit{for } 1 \le q \le k-1 \textit{ and } 1 \le k \le m+1,$$

*then*

$$t^{m+1}\big\|\partial_t^m \mathcal{I}^\mu(\psi\phi)(t)\big\| \le C \max_{0\le s\le t}\sum_{j=0}^{m}\|s^{\mu+1+j}\phi^{(j)}(s)\| \quad \textit{for } 0<t\le T.$$

**Proof.** By Lemma 3.1,

$$\big\|\mathcal{M}^{m+1}\partial_t^m \mathcal{I}^\mu(\psi\phi)\| = \left\|\sum_{j=0}^{m}\tilde{b}_j^{m+1,m}\partial_t^j\mathcal{M}^{1+j}\mathcal{I}^\mu(\psi\phi)\right\| \le C\sum_{j=0}^{m}\big\|\partial_t^j\mathcal{M}^{j+1}\mathcal{I}^\mu(\psi\phi)(t)\big\|, \tag{4.1.5}$$

and in turn,

$$\partial_t^j\mathcal{M}^{j+1}\mathcal{I}^\mu(\psi\phi) = \sum_{k=0}^{j+1}\tilde{d}_k^{j+1,\mu}\partial_t^j\mathcal{I}^{\mu+j+1-k}\mathcal{M}^k(\psi\phi).$$

Since $\partial_t^j\mathcal{I}^{\mu+j+1-k} = \partial_t^j(\partial_t\mathcal{I}^1)\mathcal{I}^{\mu+j+1-k} = \partial_t^k(\partial_t^{j+1-k}\mathcal{I}^{j+1-k})\mathcal{I}^{\mu+1} = \partial_t^k\mathcal{I}^{\mu+1}$

for $0\le k\le j+1$,

$$\begin{aligned}\|\partial_t^j\mathcal{M}^{j+1}\mathcal{I}^\mu(\psi\phi)(t)\| &\le C\sum_{k=0}^{j+1}\|\partial_t^j\mathcal{I}^{\mu+j+1-k}\mathcal{M}^k(\psi\phi)\| = C\sum_{k=0}^{j+1}\|\partial_t^k\mathcal{I}^{\mu+1}\mathcal{M}^k(\psi\phi)(t)\| \\ &= C\sum_{k=0}^{j+1}\|\mathcal{I}^{\mu+1}\partial_t^k\mathcal{M}^k(\psi\phi)(t)\|\end{aligned} \tag{4.1.6}$$

where, in the last step, we used the fact that $\partial_t^q\mathcal{M}^k(\psi\phi)(0)=0$ for $0\le q\le k-1$.

We have

$$\partial_t^k \mathcal{M}^k(\psi\phi) = \partial_t^k(\psi \mathcal{M}^k \phi) = \sum_{q=0}^{k} \binom{k}{q} \psi^{(k-q)} \partial_t^q \mathcal{M}^k \phi$$
$$= \sum_{q=0}^{k} \binom{k}{q} \psi^{(k-q)} \sum_{r=0}^{q} \tilde{a}_r^{k,q} \mathcal{M}^{k-(q-r)} \partial_t^r \phi,$$

and hence

$$\|\mathcal{I}^\mu \partial_t^k \mathcal{M}^k(\psi\phi)(t)\| \le C \sum_{q=0}^{k} \sum_{r=0}^{q} \|(\mathcal{I}^{\mu+1} \mathcal{M}^{k-q+r} \partial_t^r \phi)(t)\|$$
$$= C \sum_{q=0}^{k} \sum_{r=0}^{q} \int_0^t \omega_{\mu+1}(t-s) s^{k-q-\mu-1} \|s^{r+\mu+1} \phi^{(r)}(s)\| \, ds$$
$$\le C \sum_{r=0}^{k} \Big( \max_{0\le s\le t} \|s^{r+\mu+1} \phi^{(r)}(s)\| \Big) \sum_{q=r}^{k} (\omega_{\mu+1} * \omega_{k-q-\mu})(t).$$

The result now follows from (4.1.5), (4.1.6), and (2.1.1) ∎

## 4.2 Regularity in time

In this section, we estimate the time derivatives of $u$ and $\nabla u$. In Corollary 4.2.1 we show that if $g(t) \equiv 0$ then, with $m \ge 1$ such that (3.1.4) holds,

$$\|u^{(m)}(t)\| \le Ct^{-m} \|u_0\| \quad \text{and} \quad \|\nabla u^{(m)}(t)\| \le Ct^{-m-\alpha/2} \|u_0\| \quad \text{for } 0 < t \le T,$$

where $u^{(m)} := \partial_t^m u$. In contrast to classical parabolic PDEs, the fractional problem (3.1.1) exhibits only limited spatial smoothing for $t > 0$, because of the slow decay of the Mittag-Leffler function. In section 3.2, we estimate $u(t)$ in fractional

Sobolev norms. For example, Theorem 4.5 shows that if $u_0 \in \dot{H}^2(\Omega)$ and $g(t) \equiv 0$, and if $\kappa$ is Lipschitz and $\Omega$ is convex, then

$$\|u^{(m)}(t)\|_{H^2(\Omega)} \leq Ct^{-m}\|u_0\|_{H^2(\Omega)} \quad \text{for } 0 < t \leq T.$$

The above estimates is very important to perform the error analysis [38] of numerical methods for fractional problems of the form (3.1.1).

In this section we aim to estimate higher-order time derivatives of $u$ assuming appropriate bounds on the higher-order time derivatives of $g$, and the smoothness of the coefficients in (3.1.1) is required. Noting that, the *existence* of the higher-order derivatives of $u$ could be done using the same technique that used to establish the wellposedness of the weak solution in the previous chapter.

To show our results we will assume in addition to (3.1.4) that $\|g^{(j-1)}(t)\| \leq Ct^{\alpha-j}$ for $1 \leq j \leq m$. We introduce the following notations by extending them from section 2.3, put

$$B_\psi^{\mu,j}\phi = \partial_t^j \mathcal{M}^j \mathcal{I}^1(\psi \partial_t^{1-\mu}\phi) = (\mathcal{M}^j B_\psi^\mu \phi)^{(j)} \quad \text{and} \quad \mathcal{Q}_i^{\mu,j}(\phi, t) = \mathcal{Q}_i^\mu\big((\mathcal{M}^j\phi)^{(j)}, t\big)$$

for $0 \leq \mu \leq 1$, $j \in \{0, 1, 2, \ldots\}$, $0 \leq t \leq T$ and $i \in \{1, 2\}$, with $\mathcal{Q}^{0,j} = \mathcal{Q}_1^{0,j} = \mathcal{Q}_2^{0,j}$. The next result relies on Lemma 4.2 from section 2.3

**Lemma 4.4** *For $0 < t \leq T$ and for $m \geq 1$,*

$$\mathcal{Q}_1^{\alpha,m}(u, t) + \mathcal{Q}_2^{\alpha,m}(\nabla u, t) \leq Ct^\alpha \sum_{j=0}^{m} \mathcal{Q}^{0,j}(f, t),$$

*and*

$$\mathcal{Q}^{0,m}(u,t) + \mathcal{Q}_1^{\alpha,m}(\nabla u,t) \le C\sum_{j=0}^{m} \mathcal{Q}^{0,j}(f,t).$$

**Proof.** Since $(\mathcal{I}^\alpha \nabla u)(0) = 0$ by part 4 of Theorem 3.4,

$$\int_0^t \big\langle \kappa\nabla\partial_s^{1-\alpha}u(s), \nabla v\big\rangle\, ds = \left\langle \kappa\int_0^t (\mathcal{I}^\alpha\nabla u)'(s)\, ds, \nabla v\right\rangle = \big\langle \kappa(\mathcal{I}^\alpha\nabla u)(t), \nabla v\big\rangle,$$

and by Lemma 3.1,

$$\mathcal{M}^m\mathcal{I}^\alpha\nabla u = \mathcal{I}^\alpha\mathcal{M}^m\nabla u + \sum_{j=0}^{m-1}\tilde{d}_j^{m,\alpha}\mathcal{I}^{\alpha+m-j}\mathcal{M}^j\nabla u.$$

Thus, multiplying both sides of (3.1.5) by $t^m$ yields

$$\langle \mathcal{M}^m u, v\rangle + \langle \kappa\mathcal{I}^\alpha\mathcal{M}^m\nabla u, \nabla v\rangle + \sum_{j=1}^{m}\tilde{d}_j^{m,\alpha}\langle\kappa\mathcal{I}^{\alpha+m-j}\mathcal{M}^j\nabla u), \nabla v\rangle$$
$$= \langle\mathcal{M}^m B_F^\alpha u, \nabla v\rangle + \langle\mathcal{M}^m f, v\rangle$$

for $v \in H_0^1(\Omega)$. We have

$$\partial_t^m\mathcal{I}^{\alpha+m-j}\mathcal{M}^j\nabla u = \partial_t^j\partial_t^{m-j}\mathcal{I}^{m-j}\mathcal{I}^\alpha\mathcal{M}^j\nabla u = \partial_t^j\mathcal{I}^\alpha\mathcal{M}^j\nabla u = \mathcal{I}^\alpha\partial_t^j\mathcal{M}^j\nabla u,$$

where the final step follows by Lemma 3.1 because $\partial_t^i(\mathcal{M}^j u)(0) = 0$ for $0 \le i \le j-1 \le m-1$. Likewise, $\partial_t^m\mathcal{I}^\alpha\mathcal{M}^m\nabla u = \mathcal{I}^\alpha\partial_t^m\mathcal{M}^m\nabla u$ because $\partial_t^j(\mathcal{M}^m\nabla u)(0) = 0$

for $0 \le j \le m-1$, and therefore

$$\langle \partial_t^m \mathcal{M}^m u, v\rangle + \langle \kappa \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m \nabla u, \nabla v\rangle = \langle B_F^{\alpha,m} u, \nabla v\rangle$$
$$- \sum_{j=0}^{m-1} \tilde{d}_j^{m,\alpha} \langle \kappa \mathcal{I}^\alpha \partial_t^j \mathcal{M}^j \nabla u, \nabla v\rangle + \langle \partial_t^m \mathcal{M}^m f, v\rangle. \quad (4.2.1)$$

We let $\mathcal{E}(u) = 2\|B_{\vec{F}}^{\alpha,m} u\|^2$. By the Cauchy–Schwarz inequality,

$$\langle \partial_t^m \mathcal{M}^m u, v\rangle + \langle \kappa \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m \nabla u, \nabla v\rangle \le \mathcal{E}(u) + C \sum_{j=0}^{m-1} \|\mathcal{I}^\alpha \partial_t^j \mathcal{M}^j \nabla u\|^2$$
$$+ \tfrac{1}{2}\|\nabla v\|^2 + \tfrac{1}{2}\|v\|^2 + \langle \partial_t^m \mathcal{M}^m f, v\rangle.$$

Choosing $v = \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m u$ and integrating over the time interval $(0,t)$, we have

$$\mathcal{Q}_1^{\alpha,m}(u,t) + \tfrac{1}{2}\mathcal{Q}_2^{\alpha,m}(\nabla u, t) \le \int_0^t \mathcal{E}(u)\,ds + C \sum_{j=0}^{m-1} \mathcal{Q}_2^{\alpha,j}(\nabla u, t)$$
$$+ \mathcal{Q}_2^{\alpha,m}(u,t) + \int_0^t \langle \partial_t^m \mathcal{M}^m f, \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m u\rangle\,ds.$$

By the Cauchy-Schwarz inequality, and the inequality (2.3.8) with $\partial_t^m(\mathcal{M}^m u)$ in place of $\phi'$,

$$\int_0^t \langle \partial_t^m \mathcal{M}^m f, \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m u\rangle\,ds \le \int_0^t \|\partial_t^m \mathcal{M}^m f\|\,\|\mathcal{I}^\alpha \partial_t^m \mathcal{M}^m u\|\,ds$$
$$\le C\Big(\int_0^t (t-s)^\alpha \|\partial_t^m \mathcal{M}^m f\|^2\,ds\Big)^{1/2}\Big(\int_0^t (t-s)^{-\alpha}\|\mathcal{I}^\alpha \partial_t^m \mathcal{M}^m u\|^2\,ds\Big)^{1/2}$$
$$\le C\Big(t^\alpha \mathcal{Q}^{0,m}(f,t)\Big)^{1/2}\Big(\mathcal{I}^{1-\alpha}(\|\mathcal{I}^\alpha(\partial_t^m \mathcal{M}^m u)\|^2)(t)\Big)^{1/2}$$
$$\le C t^\alpha \mathcal{Q}^{0,m}(f,t) + \tfrac{1}{2}\mathcal{Q}_1^{\alpha,m}(u,t).$$

Thus, the function $\mathsf{q_m}(t) = \mathcal{Q}_1^{\alpha,m}(u,t) + \mathcal{Q}_2^{\alpha,m}(\nabla u,t)$ satisfies

$$\mathsf{q_m}(t) \le 2\int_0^t \mathcal{E}(u)\,ds + C\sum_{j=0}^{m-1} \mathcal{Q}_2^{\alpha,j}(\nabla u,t) + Ct^\alpha \mathcal{Q}^{0,m}(f,t).$$

By Lemma 4.2,

$$\mathcal{Q}^0(B_{\vec{F}}^{\alpha,m}u,t) \le C\sum_{j=0}^{m} \mathcal{Q}_2^{\alpha,j}(u,t) \tag{4.2.2}$$

By combining the above estimates,

$$\mathsf{q}_m(t) \le C\mathcal{Q}_2^{\alpha,m}(u,t) + C\sum_{j=0}^{m-1} \mathsf{q}_j(t) + Ct^\alpha \mathcal{Q}^{0,m}(f,t).$$

Consequently, we conclude (recursively) that

$$\mathsf{q}_m(t) \le C\sum_{j=0}^{m} \mathcal{Q}_2^{\alpha,j}(u,t) + Ct^\alpha \sum_{j=0}^{m} \mathcal{Q}^{0,j}(f,t),$$

so, by applying Lemma 2.2 with $\phi = (\mathcal{M}^j u)^{(j)}$,

$$\mathsf{q}_m(t) \le Ct^\alpha \sum_{j=0}^{m} \mathcal{Q}^{0,j}(f,t) + C\sum_{j=0}^{m} \mathcal{I}^\alpha \mathsf{q}_j(t)$$

Therefore, a repeated application of Lemma 2.5 yields the first desired estimate.

To show the second estimate, choose $v = \partial_t^m \mathcal{M}^m u$ in (4.2.1) and obtain

$$\|\partial_t^m \mathcal{M}^m u\|^2 + \langle \kappa \mathcal{I}^\alpha \partial_t^m \mathcal{M}^m \nabla u, \partial_t^m \mathcal{M}^m \nabla u\rangle = -\langle Eu, \partial_t^m \mathcal{M}^m u\rangle - \sum_{j=0}^{m-1} \tilde{d}_j^{m,\alpha} \langle \kappa \mathcal{I}^\alpha \partial_t^j \mathcal{M}^j \nabla u, \partial_t^m \mathcal{M}^m \nabla u\rangle + \langle \partial_t^m \mathcal{M}^m f, \partial_t^m \mathcal{M}^m u\rangle,$$

where $Eu = \nabla \cdot B_{\vec{F}}^{\alpha,m} u$. The first and the last terms on the right-hand side are bounded by

$$\|Eu\|^2 + \|\partial_t^m \mathcal{M}^m f\|^2 + \tfrac{1}{2}\|\partial_t^m \mathcal{M}^m u\|^2$$

so, after integrating in time and applying (2.3.2) (for a sufficiently *large* $\eta$),

$$\tfrac{1}{2}\mathcal{Q}^{0,m}(u,t) + \mathcal{Q}_1^{\alpha,m}(\nabla u, t) \le \int_0^t \|Eu(s)\|^2\,ds + \mathcal{Q}^{0,m}(f,t) + \tfrac{1}{2}\mathcal{Q}_1^{\alpha,m}(\nabla u,t) + C\sum_{j=0}^{m-1} \mathcal{Q}_1^{\alpha,j}(\nabla u, t).$$

Since $\nabla \cdot (\vec{F}\partial_t^{1-\alpha} u) = (\nabla \cdot \vec{F})\partial_t^{1-\alpha} u + \vec{F}\cdot \nabla \partial_t^{1-\alpha} u$, we see that

$$\nabla \cdot B_{\vec{F}}^{\alpha,m} u = \partial_t^m \mathcal{M}^m \mathcal{I}^1\big(\nabla \cdot (\vec{F}\partial_t^{1-\alpha} u)\big) = B_{\nabla\cdot\vec{F}}^{\alpha,m} u + B_{\vec{F}}^{\alpha,m} \nabla u,$$

and therefore, applying Lemma 4.2 followed by Lemma 2.4,

$$\int_0^t \|Eu(s)\|^2\,ds \;\le\; 4\Big(\mathcal{Q}^0(\nabla \cdot B_{\vec{F}}^{\alpha,m} u, t)\Big) \le\; C\sum_{j=0}^{m} \Big(\mathcal{Q}_2^{\alpha,j}(u,t) \;+\; \mathcal{Q}_2^{\alpha,j}(\nabla u, t)\Big).$$

Hence, the function $\mathsf{q}_m(t) = \mathcal{Q}^{0,m}(u,t) + \mathcal{Q}_1^{\alpha,m}(\nabla u,t)$ satisfies

$$\mathsf{q}_m(t) \le 2\mathcal{Q}^{0,m}(f,t) + C\sum_{j=0}^{m-1} \mathcal{Q}_1^{\alpha,j}(\nabla u,t) + C\sum_{j=0}^{m}\Big(\mathcal{Q}_2^{\alpha,j}(u,t) + \mathcal{Q}_2^{\alpha,j}(\nabla u,t)\Big),$$

and so, using (2.3.3) and (2.3.4), it follows that

$$\mathsf{q}_m(t) \le 2\mathcal{Q}^{0,m}(f,t) + C\sum_{j=0}^{m-1} \mathsf{q}_j(t) + C\Big(\mathcal{Q}_2^{\alpha,m}(u,t) + \mathcal{Q}_2^{\alpha,m}(\nabla u,t)\Big).$$

By Lemma 2.2 and (2.3.4),

$$\mathcal{Q}_2^{\alpha,m}(u,t) + \mathcal{Q}_2^{\alpha,m}(\nabla u,t) \le C\mathcal{I}^{\alpha}\mathsf{q}_m(s)\,ds,$$

and thus by Lemma 2.5,

$$\mathsf{q}_m(t) \le C\mathcal{Q}^{0,m}(f,t) + C\sum_{j=0}^{m-1} \mathsf{q}_j(t).$$

Applying this inequality recursively gives

$$\mathsf{q}_m(t) \le C\sum_{j=0}^{m} \mathcal{Q}^{0,j}(f,t),$$

which completes the proof. ∎

In the next theorem we estimate the fractional time derivatives of $u$ and $\nabla u$. Such estimates will help us in the study of properties of spatial regularity reflecting the presence of the time derivative of (3.1.1).

**Theorem 4.1** *For $m \geq 1$ and $0 < t \leq T$,*

$$\|(\partial_t^m u)(t)\|^2 + t^\alpha \|(\partial_t^m \nabla u)(t)\|^2 \leq Ct^{-1-2m} \sum_{j=0}^{m+1} \mathcal{Q}^{0,j}(f,t).$$

**Proof.** Since $\mathcal{M}\partial_t^m = \partial_t^m \mathcal{M} - m\partial_t^{m-1}$, we see using Lemma 3.1 (and setting $\tilde{b}_m^{m,m-1} = 0$) that

$$\mathcal{M}^{m+1}\partial_t^m = \mathcal{M}^m \partial_t^m \mathcal{M} - m\mathcal{M}^m \partial_t^{m-1} = \sum_{j=1}^{m+1} \big(\tilde{b}_{j-1}^{m,m} - m\tilde{b}_{j-1}^{m,m-1}\big)\partial_t^{j-1}\mathcal{M}^j,$$

and hence

$$\|(\mathcal{M}^{m+1}\partial_t^m u)(t)\|^2 \leq C \sum_{j=1}^{m+1} \|(\partial_t^{j-1}\mathcal{M}^j u)(t)\|^2 \tag{4.2.3}$$

Using Lemma 2.3 with $\phi = \partial_t^{j-1}\mathcal{M}^j u$ and the first bound in Lemma 4.4, we get

$$\|(\partial_t^{j-1}\mathcal{M}^j u)(t)\|^2 \leq Ct^{1-\alpha}\mathcal{Q}_1^\alpha(\partial_t^j \mathcal{M}^j u, t) \leq Ct \sum_{\ell=0}^{j} \mathcal{Q}^{0,\ell}(f,t)$$

and so

$$\|(\partial_t^m u)(t)\|^2 = t^{-2m-2}\|(\mathcal{M}^{m+1}\partial_t^m u)(t)\|^2 \leq Ct^{-1-2m} \sum_{j=0}^{m+1} \mathcal{Q}^{0,j}(f,t).$$

Applying the same argument to $\nabla u$ in place of $u$, and using the second bound in Lemma 4.4, the result follows. ∎

Next, we estimate fractional time derivatives of $u$ and $\nabla u$. These bounds will later help in our study of spatial regularity, reflecting the presence of the fractional time derivative in (3.1.1).

**Theorem 4.2** *For $m \geq 1$ and $0 < t \leq T$,*

$$\|(\partial_t^{m-\alpha} u)(t)\|^2 + t^\alpha \|(\partial_t^{m-\alpha} \nabla u)(t)\|^2 \leq C t^{-1-2(m-\alpha)} \sum_{j=0}^{m+1} \mathcal{Q}^{0,j}(f,t).$$

**Proof.** Using the inequality (4.2.3),

$$\big\|(\mathcal{M}^m \partial_t^{m-\alpha} u)(t)\big\|^2 = \big\|(\mathcal{M}^m \partial_t^{m-1} \partial_t^{1-\alpha} u)(t)\big\|^2 \leq C \sum_{j=1}^{m} \big\|(\partial_t^{j-1} \mathcal{M}^j \partial_t^{1-\alpha} u)(t)\big\|^2, \tag{4.2.4}$$

and using (3.1.9) and (2.3.6),

$$\begin{aligned} \mathcal{M}\partial_t^{1-\alpha} u = \mathcal{M}\partial_t \mathcal{I}^\alpha u = \mathcal{M}\big(\mathcal{I}^\alpha \partial_t u + u(0)\omega_\alpha\big) &= (\mathcal{I}^\alpha \mathcal{M} + \alpha \mathcal{I}^{\alpha+1})\partial_t u + u(0)\mathcal{M}\omega_\alpha \\ &= \mathcal{I}^\alpha \mathcal{M} u' + \alpha \mathcal{I}^\alpha\big(u - u(0)\big) + \alpha u(0)\omega_{1+\alpha} = \mathcal{I}^\alpha(\mathcal{M}u' + \alpha u). \end{aligned}$$

Thus, by Lemma 3.1,

$$\mathcal{M}^j \partial_t^{1-\alpha} u = \mathcal{M}^{j-1} \mathcal{I}^\alpha (\mathcal{M}u' + \alpha u) = \sum_{\ell=0}^{j-1} \tilde{d}_\ell^{j-1,\alpha} \mathcal{I}^{\alpha+j-1-\ell} \mathcal{M}^\ell (\mathcal{M}u' + \alpha u).$$

We have

$$\begin{aligned} \partial_t^{j-1} \mathcal{I}^{\alpha+j-1-\ell} \mathcal{M}^\ell (\mathcal{M}u' + \alpha u) &= \partial_t^\ell \big(\partial_t^{j-1-\ell} \mathcal{I}^{j-1-\ell}\big) \mathcal{I}^\alpha \mathcal{M}^\ell (\mathcal{M}u' + \alpha u) \\ &= \partial_t^\ell \mathcal{I}^\alpha \mathcal{M}^\ell (\mathcal{M}u' + \alpha u) = \mathcal{I}^\alpha \partial_t^\ell \mathcal{M}^\ell (\mathcal{M}u' + \alpha u), \end{aligned}$$

where we used the identity (2.3.6) and the fact that

$$\partial_t^i \mathcal{M}^\ell(\mathcal{M}u' + \alpha u)(0) = 0$$

for $0 \le i \le \ell - 1$. Hence,

$$\big\|(\partial_t^{j-1}\mathcal{M}^j\partial_t^{1-\alpha}u)(t)\big\| = \left\|\sum_{\ell=0}^{j-1}\tilde{d}_\ell^{j-1,\alpha}\mathcal{I}^\alpha\partial_t^\ell\mathcal{M}^\ell(\mathcal{M}u' + \alpha u)(t)\right\| \le C\sum_{\ell=0}^{j-1}\|\mathcal{I}^\alpha\phi_\ell(t)\|$$

where $\phi_\ell = \partial_t^\ell\mathcal{M}^\ell(\mathcal{M}u' + \alpha u)$. Using Lemma 3.1,

$$\phi_\ell = \sum_{i=0}^{\ell}\tilde{a}_i^{\ell,\ell}\mathcal{M}^i\partial_t^i(\mathcal{M}u' + \alpha u) = \sum_{i=0}^{\ell}\tilde{a}_i^{\ell,\ell}\mathcal{M}^i\big(\mathcal{M}\partial_t^i u' + i\partial_t^{i-1}u' + \alpha\partial_t^i u\big)$$

and so, by Theorem 4.1,

$$\|\phi_\ell(t)\|^2 \le C\sum_{r=0}^{\ell+1}\|(\mathcal{M}^r\partial_t^r u)(t)\|^2 \le C\sum_{r=0}^{\ell+1}t^{-1}\sum_{i=0}^{r+1}\mathcal{Q}^{0,i}(f,t) \le Ct^{-1}\sum_{r=0}^{\ell+2}\mathcal{Q}^{0,r}(f,t). \tag{4.2.5}$$

Since $\|\phi_\ell(t)\| \le C\omega_{1/2}(t)\psi_\ell(t)$ where $\psi_\ell(t) = \sqrt{\sum_{r=0}^{\ell+2}\mathcal{Q}^{0,r}(f,t)}$ is nondecreasing, we see that $\|\mathcal{I}^\alpha\phi_\ell(t)\| \le C\omega_{\alpha+1/2}(t)\psi_\ell(t)$. Therefore,

$$\begin{aligned}\big\|(\partial_t^{j-1}\mathcal{M}^j\partial_t^{1-\alpha}u)(t)\big\|^2 &\le C\sum_{\ell=0}^{j-1}\|\mathcal{I}^\alpha\phi_\ell(t)\|^2\\ &\le C\sum_{\ell=0}^{j-1}(t^{(\alpha+1/2)-1})^2\psi_\ell(t)^2 \le Ct^{2\alpha-1}\sum_{\ell=0}^{j+1}\mathcal{Q}^{0,\ell}(f,t).\end{aligned}$$

and the desired bound for $\|(\partial_t^{m-\alpha}u)(t)\|^2$ follows at once from (4.2.4).

Replacing $u$ with $\nabla u$ in the preceding argument, we have

$$\left\|(\mathcal{M}^m \partial_t^{m-\alpha} \nabla u)(t)\right\|^2 \le C \sum_{j=1}^{m} \left\|(\partial_t^{j-1} \mathcal{M}^j \partial_t^{1-\alpha} \nabla u)(t)\right\|^2 \le C \sum_{j=1}^{m} \sum_{\ell=0}^{j-1} \|\mathcal{I}^\alpha \phi_\ell(t)\|^2$$

where,

$$\phi_\ell = \partial_t^\ell \mathcal{M}^\ell (\mathcal{M} \nabla u' + \alpha \nabla u)$$

and hence $\|\phi_\ell(t)\| \le C\omega_{(1-\alpha)/2}(t)\psi_\ell(t)$. It follows that $\|\mathcal{I}^\alpha \phi_\ell\| \le C\omega_{(1+\alpha)/2}(t)\psi_\ell(t)$ and so $t^\alpha \left\|(\mathcal{M}^m \partial_t^{m-\alpha} \nabla u)(t)\right\|^2$ is bounded by

$$Ct^\alpha \sum_{\ell=0}^{m-1} \|\mathcal{I}^\alpha \phi_\ell(t)\|^2 \le Ct^\alpha \sum_{\ell=0}^{m-1} (t^{(1+\alpha)/2-1})^2 \psi_\ell(t)^2 \le Ct^{2\alpha-1} \sum_{\ell=0}^{m+1} \mathcal{Q}^{0,\ell}(f,t),$$

as required. ∎

To summarize the previous bounds we can easily conclude the following result.

**Corollary 4.2.1** *Let $m \ge 1$ and suppose that $g : (0,T] \to L_2(\Omega)$ is $C^m$ with*

$$\|g^{(j)}(t)\| \le Mt^{\eta-1-j} \quad \text{for } 0 \le j \le m \text{ and some } \eta > 0. \tag{4.2.6}$$

*Then*

$$\|(\partial_t^m u)(t)\| + t^{\alpha/2}\|(\partial_t^m \nabla u)(t)\| \le Ct^{-m}(\|u_0\| + Mt^\eta)$$

*and*

$$\|(\partial_t^{m-\alpha} u)(t)\| + t^{\alpha/2}\|(\partial_t^{m-\alpha} \nabla u)(t)\| \le Ct^{\alpha-m}(\|u_0\| + Mt^\eta).$$

**Proof.** Since $\|f^{(j)}(t)\| = \|g^{(j-1)}(t)\| \le Mt^{\eta-j}$ for $1 \le j \le m+1$, Lemma 3.1

implies that

$$\|(\mathcal{M}^j f)^{(j)}(t)\| \le CMt^\eta \text{ for } 1 \le j \le m+1, \quad \text{with } \|f(t)\| \le \|u_0\| + M\eta^{-1}t^\eta. \tag{4.2.7}$$

Thus,

$$\mathcal{Q}^{0,j}(f,t) \le CM^2 t^{2\eta+1} \text{ for } 1 \le j \le m+1, \quad \text{with } \mathcal{Q}^0(f,t) \le Ct(\|u_0\| + Mt^\eta)^2,$$

so

$$t^{-1-2m} \sum_{j=0}^{m+1} \mathcal{Q}^{0,j}(f,t) \le Ct^{-2m}(\|u_0\| + Mt^\eta)^2$$

and the result follows from Theorem 4.1 and Theorem 4.2. ∎

## 4.3 $H^2$-regularity in space

In this section we will investigate further the relation between the regularity of $u$ and that of the initial data $u_0$. As a result, Theorem 4.5 generalizes Corollary 4.2.1 which helps in studying the error analysis of a finite elements discretization of the fractional Fokker–Planck equation [29]. The fractional PDE (3.1.1) can be rewritten as

$$u' - \nabla \cdot (\kappa \partial_t^{1-\alpha} \nabla u) = \omega \quad \text{for } x \in \Omega \text{ and } 0 < t < T,$$

where

$$\omega = g - \nabla \cdot \big(F \partial_t^{1-\alpha} u\big).$$

Now we are able to apply known results for the fractional diffusion equation to establish the following bounds in the norm $\|v\|_\mu = \|A^{\mu/2} v\|$ of the fractional Sobolev space $\dot{H}^\mu(\Omega)$, where $A^{\mu/2}$ is defined via the spectral representation of $Av = -\nabla \cdot (\kappa \nabla v)$ using the Dirichlet eigenfunctions on $\Omega$ [37, 51]. Results of this section require $H^2$-regularity for the Poisson problem. To ensure this property we make the following assumptions [16, Theorems 2.2.2.3 and 3.2.1.2]

$$\kappa \text{ is Lipschitz on } \overline{\Omega} \tag{4.3.1}$$

We also require that $g$ satisfies (4.2.6). Next theorem result does not require any additional smoothness of $u_0$.

**Theorem 4.3** *Assume* (4.3.1) *and* (4.2.6). *If* $u_0 \in L_2(\Omega)$, *then*

$$t^m \|u^{(m)}(t)\|_\mu \le C\|u_0\| t^{-\mu\alpha/2} + CMt^{\eta-\mu\alpha/2} \quad \text{for } 0 \le \mu \le 2 \text{ and } 0 < t \le T.$$

**Proof.** We have [37, Theorems 4.1 and 4.2, and the inequality stated after Theorem 5.4]

$$t^m \|u^{(m)}(t)\|_\mu \le C t^{-\mu\alpha/2} \|u_0\| + C \sum_{j=0}^{m} \int_0^t (t-s)^{-\mu\alpha/2} s^j \|\omega^{(j)}(s)\| \, ds$$

for $m \geq 0$ and for $0 \leq \mu \leq 2$, with

$$\|\omega^{(j)}(s)\| \leq \|\partial_t^j g(s)\| + C \sum_{\ell=0}^{j} \Big( \|\partial_t^{\ell+1-\alpha} \nabla u(s)\| + \|\partial_t^{\ell} \nabla u(s)\| + \|\partial_t^{\ell+1-\alpha} u(s)\| + \|\partial_t^{\ell} u(s)\| \Big).$$

By Corollary 4.2.1,

$$\|\omega^{(j)}(s)\| \leq M s^{\eta-1-j} + C \sum_{\ell=0}^{j} \Big( s^{\alpha/2-\ell-1} + s^{-\alpha/2-\ell} + s^{\alpha-\ell-1} + s^{-\ell} \Big) (\|u_0\| + M s^{\eta})$$

so $s^j \|\omega^{(j)}(s)\| \leq C\|u_0\| s^{\alpha/2-1} + M s^{\eta-1}$ and hence

$$\begin{aligned} \int_0^t (t-s)^{-\mu\alpha/2} s^j \|\omega^{(j)}(s)\| \, ds &\leq C\|u_0\| (\omega_{1-\mu\alpha/2} * \omega_{\alpha/2})(t) + CM(\omega_{1-\mu\alpha/2} * \omega_{\eta})(t) \\ &\leq C\big(\|u_0\| t^{(1-\mu)\alpha/2} + M t^{\eta-\mu\alpha/2}\big), \end{aligned}$$

completing the proof. ∎

The first estimate in our next result shows that $u(t) \to u_0$ in the $L_2(\Omega)-$norm if we impose some additional spatial regularity on the initial data, namely if $u_0 \in \dot{H}^{\mu}(\Omega)$ for some $\mu > 0$. The second and third estimates extend the results of Corollary 4.2.1.

**Theorem 4.4** *Assume* (4.3.1) *and* (4.2.6)*. If* $0 \leq \mu \leq 2$ *and* $u_0 \in \dot{H}^{\mu}(\Omega)$*, then*

$$\|u(t) - u_0\| + t^{\alpha/2} \big\| \nabla \big(u(t) - u_0\big) \big\| \leq C \|u_0\|_{\mu} t^{\alpha\mu/2} + M t^{\eta},$$

*and, for* $m \geq 1$,

$$\|u^{(m)}(t)\| + t^{\alpha/2}\|\nabla u^{(m)}(t)\| \leq Ct^{-m}\big(\|u_0\|_\mu t^{\alpha\mu/2} + Mt^\eta\big)$$

*with*

$$\|\partial_t^{m-\alpha}u(t)\| + t^{\alpha/2}\|(\partial_t^{1-\alpha}\nabla u)(t)\| \leq Ct^{\alpha-m}\big(\|u_0\|_\mu t^{\alpha\mu/2} + Mt^\eta\big).$$

**Proof.** Introduce the solution operator $u(t) = \mathcal{U}(u_0, g, t)$. By linearity, $u = u_1 + u_2$ where $u_1(t) = \mathcal{U}(u_0, 0, t)$ and $u_2(t) = \mathcal{U}(0, g, t)$. In view of Corollary 4.2.1, it suffices to consider $u_1$. Let $w(t) = u_1(t) - u_0$ so that $w(0) = 0$, and suppose to begin with that $u_0 \in \dot{H}^2(\Omega)$. Using (3.2.7), we find that

$$\langle w(t), v\rangle + \big\langle \kappa(\mathcal{I}^\alpha \nabla w)(t), \nabla v\big\rangle - \big\langle (B_1 w)(t), \nabla v\big\rangle = \langle \rho(t), v\rangle,$$

where

$$\rho(t) = \mathcal{I}^\alpha \nabla \cdot (\kappa \nabla u_0) - \nabla \cdot B_1 u_0.$$

Since $(\mathcal{I}^\alpha u_0)'(t) = u_0 \omega_\alpha(t)$, and recalling the definitions (3.2.5), we have

$$\rho'(t) = \big(\nabla \cdot (\kappa \nabla u_0) - \nabla \cdot (F(t) u_0)\big)\omega_\alpha(t)$$

so $\|\rho^{(j+1)}(t)\| \leq C\|u_0\|_2 t^{\alpha-j}$. Therefore, by Corollary 4.2.1

$$\|w^{(m)}(t)\| + t^{\alpha/2}\|\nabla w^{(m)}(t)\| \leq Ct^{-m}\big(w(0) + \|u_0\|_2 t^\alpha\big) = C\|u_0\|_2 t^{\alpha-m},$$

which proves the result for integer-order time derivatives in the case $\mu = 2$. Similarly, for the fractional-order time derivatives,

$$\|\partial_t^{m-\alpha} w(t)\| + t^{\alpha/2}\|\partial_t^{m-\alpha}\nabla w(t)\| \le Ct^{\alpha-m}\big(w(0) + \|u_0\|_2 t^{\alpha}\big) = Ct^{2\alpha-m}\|u_0\|_2,$$

completing the proof for $\mu = 2$. Since Corollary 4.2.1 also implies the case $\mu = 0$, the result follows for $0 < \mu < 2$ by interpolation. ∎

**Theorem 4.5** *Assume* (4.3.1) *and* (4.2.6)*. If* $u_0 \in \dot{H}^2(\Omega)$*, then*

$$\|u^{(m)}(t)\|_2 \le Ct^{-m}\big(\|u_0\|_\mu t^{-(2-\mu)\alpha/2} + Mt^{\eta-\alpha}\big) \quad \textit{for } 0 < t \le T.$$

**Proof.** We know from Theorem 4.3 that

$$\|u^{(m)}(t)\|_2 \le Ct^{-\alpha-m}\big(\|u_0\| + Mt^{\eta}\big). \tag{4.3.2}$$

Thus, by linearity, we may assume that $g(t) \equiv 0$ and so $M = 0$. Integrating (3.1.1) in time, we see that

$$u - \nabla\cdot(\kappa\nabla\mathcal{I}^{\alpha}u) + \nabla\cdot B_1 u = u_0.$$

Applying the operator $\partial_t\mathcal{I}^{1-\alpha}$ to both sides,

$$-\nabla\cdot(\kappa\nabla u) = \rho \quad \text{where} \quad \rho = \partial_t\mathcal{I}^{1-\alpha}\big(u_0 - u - \nabla\cdot B_1 u\big).$$

Since $-\nabla \cdot (\kappa \nabla u^{(m)}) = \rho^{(m)}$ in $\Omega$, with $u^{(m)}(t) = 0$ on $\partial\Omega$ for $0 \le t \le T$, it follows by $H^2$-regularity for the Poisson problem that

$$\|u^{(m)}(t)\|_2 \le C\|\rho^{(m)}(t)\|. \tag{4.3.3}$$

Using the identity (2.3.6),

$$\begin{aligned}\rho &= \mathcal{I}^{1-\alpha}\partial_t\big(u_0 - u - \nabla \cdot B_1 u\big)\\ &= -\mathcal{I}^{1-\alpha}u' - \mathcal{I}^{1-\alpha}\big(\nabla \cdot (F\partial_t^{1-\alpha}u)\big).\end{aligned}$$

Lemma 4.3 and Theorem 4.4 imply that

$$\begin{aligned}t^{m+1}\|\partial_t^m \mathcal{I}^{1-\alpha}u'(t)\| &\le C \max_{0\le s\le t}\sum_{j=0}^{m} s^{1-\alpha+1+j}\|u^{(j+1)}(s)\|\\ &\le C \max_{0\le s\le t} s^{1-\alpha}\big(\|u_0\|_2 s^{\alpha}\big) = Ct\|u_0\|_2.\end{aligned}$$

Since

$$\nabla \cdot (F\partial_t^{1-\alpha}u) = (\nabla \cdot F)\partial_t^{1-\alpha}u + F \cdot \partial_t^{1-\alpha}\nabla u,$$

we see from Lemma 4.3 and Theorem 4.4 that

$$
\begin{aligned}
t^{m+1}\big\|\partial_t^m \mathcal{I}^{1-\alpha}\big(\nabla\cdot(F\partial_t^{1-\alpha}u)\big)(t)\big\| \\
&\le C\max_{0\le s\le t}\sum_{j=0}^{m} s^{1-\alpha+1+j}\Big(\|\partial_s^j(\partial_s^{1-\alpha}u)\|+\|\partial_s^j(\partial_s^{1-\alpha}\nabla u)\|\Big)\\
&= C\max_{0\le s\le t}\sum_{j=0}^{m} s^{1-\alpha+1+j}\Big(\|\partial_s^{j+1-\alpha}u\|+\|\partial_s^{j+1-\alpha}\nabla u\|\Big)\\
&\le C\max_{0\le s\le t} s^{1-\alpha}(s^{\alpha}+s^{\alpha/2})\big(\|u_0\|_2 s^{\alpha}\big)\le Ct^{1+\alpha/2}\|u_0\|_2.
\end{aligned}
$$

Thus,

$$t^{m+1}\|\omega^{(m)}(t)\|\le C\big(t+t^{1+\alpha/2}+t^{2-\alpha}\big)\|u_0\|_2,$$

showing that $t^m\|\omega^{(m)}(t)\|\le C\|u_0\|_2$ and therefore, by (4.3.2) and (4.3.3),

$$\|u^{(m)}(t)\|_2\le t^{-m}\|u_0\|_2,$$

and therefore, the proof is completed now. ∎

## CHAPTER 5

# CRANK-NICOLSON NUMERICAL SOLUTION

In the next section we propose a time discretization scheme that based on Crank-Nicolson method for the model problem (3.1.1). In section 2 we consider finite elements to discretize the space. In section 3 we prove the stability of the time discretization scheme. The error bound for the time discretization scheme is derived in section 4. In the last section we combine the time-stepping Crank-Nicolson scheme with the finite elements in space, this will define a fully discrete scheme, we also prove the existence and uniqueness of the solution for the fully discrete scheme.

## 5.1 An implicit Crank-Nicolson time-stepping scheme

We discretize in time the model problem (5.3.2). To do so, we let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and we use a time graded mesh with the following nodes $t_i = (i\,k)^\gamma$ for $0 \le i \le N$ with $\gamma \ge 1$ and $k = T^{1/\gamma}/N$, where $N$ is the number of subintervals. Denote by $k_n = t_n - t_{n-1}$ the length of the $n$th subinterval $I_n = (t_{n-1}, t_n)$, for $1 \le n \le N$. In our notation, we will often suppress the dependence on $x$ and think of $u = u(x,t)$ as a function of $t$ taking values in $L_2(\Omega)$. Integrating the fractional Fokker–Planck equation (1.1.1) over the $n$th time interval $I_n$ gives

$$u(t_n) - u(t_{n-1}) + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A} u \, dt = \int_{I_n} g(x,t) \, dt. \qquad (5.1.1)$$

where

$$\mathcal{A}u = -\nabla^2 u + \nabla \cdot (Fu)$$

We seek to compute $U^n(x) \approx u(x, t_n)$ for $n = 1, 2, \ldots, N$ by requiring that

$$U^n - U^{n-1} + \int_{I_n} \partial_t^{1-\alpha} \mathcal{A}\bar{U}\, dt = k_n \bar{g}^n \tag{5.1.2}$$

with $\bar{F}^n(x) = F(x, t_{n-\frac{1}{2}}),\ t_{n-\frac{1}{2}} = \frac{t_n + t_{n-1}}{2}$ and $\bar{U} = \frac{U^n + U^{n-1}}{2}$,

$\bar{g}^n \approx k_n^{-1} \int_{I_n} g(x,t)\, dt$.

The time stepping starts from the initial condition

$$U^0(x) = u_0(x) \quad \text{for } 0 \le x \le L, \tag{5.1.3}$$

and is subject to the boundary conditions $U^n(x) = 0$ for $x \in \partial\Omega$ where $1 \le n \le N$.

## 5.2 Stability of the numerical solution

In this section we show the stability of the semidiscrete approximate solution $U$ of (5.1.2) in the following theorem.

**Theorem 5.1** *Consider the implicit scheme* (5.1.2). *Assume that the driving force* $\vec{F} = \vec{F}(x)$ *satisfies that*

$$\nabla \cdot F \ge \frac{-2(\nabla u)^2}{u^2} \quad on \ \ \Omega$$

*then*

$$\|U^n\| \leq \|U^0\| + 2\sum_{j=1}^{n}\|\tilde{g}^j\|$$

**Proof.** Taking the inner product of (5.1.2) with $\bar{U}^n$,

$$\langle U^n - U^{n-1}, \bar{U}^n\rangle + \int_{I_n}\langle \partial_t^{1-\alpha}\mathcal{A}\bar{U}(t), \bar{U}(t)\rangle\, dt = \langle \bar{g}^n, \bar{U}^n\rangle$$

where

$$\mathcal{A}\bar{U} = -\nabla^2\bar{U} + \nabla\cdot(F\bar{U})$$

Now, using the given assumption on $\vec{F}$

$$\langle \mathcal{A}\bar{U}, \bar{U}\rangle = \langle -\nabla^2\bar{U} + \nabla\cdot(F\bar{U}), \bar{U}\rangle = \|\nabla\bar{U}\|^2 - \langle F\bar{U}, \nabla\bar{U}\rangle \geq 0. \qquad (5.2.1)$$

Let $U^{n^*} = \max_{0\leq n\leq N}\|U^n\|$. Summing the above equation from $n = 1$ to $n = n^*$ gives

$$\|U^{n^*}\|^2 - \|U^0\|^2 + 2\int_0^{t_{n^*}}\langle \partial_t^{1-\alpha}\mathcal{A}\bar{U}(t), \bar{U}(t)\rangle\, dt = \sum_{n=0}^{n^*}\langle \bar{g}^n, U^n + U^{n-1}\rangle.$$

Using (5.2.1) it follows that

$$\|U^{n^*}\|^2 \leq \|U^0\|^2 + 2\|U^{n^*}\|\sum_{n=0}^{n^*}\|\bar{g}^n\| \leq \|U^{n^*}\|\big(\|U^0\| + 2\sum_{n=0}^{n^*}\bar{\|}\bar{g}^n\|\big)$$

which implies the desired result. ▮

## 5.3 Error bound from the time dicretization

In this section we estimate the error $e^n = U^n - u(t_n)$ when $U^n$ is given by:

$$U^n - U^{n-1} + \int_{I_n} \partial_t^{1-\alpha}(\mathcal{A}\bar{U})\,dt = k_n \bar{g}^n \tag{5.3.1}$$

and $u$ is the solution of

$$u' + \left(\partial_t^{1-\alpha}\mathcal{A}u\right) = g, \tag{5.3.2}$$

Integrating (5.3.2) from $t = t_{n-1}$ to $t = t_n$ shows that the exact solution $u$ satisfies:

$$u(t_n) - u(t_{n-1}) + \int_{I_n} \left(\partial_t^{1-\alpha}\mathcal{A}u\right) dt = k_n \bar{g}^n.$$

Comparing this with (5.3.1), we observe that the error $e^n$ satisfies:

$$e^n - e^{n-1} + \int_{I_n} (\partial_t^{1-\alpha}\mathcal{A}\bar{e})\,dt = \eta^n \tag{5.3.3}$$

where

$$\eta^n = \int_{t_{n-1}}^{t_n} \left(\partial_t^{1-\alpha}\mathcal{A}(u - \bar{u})(t)\right) dt \tag{5.3.4}$$

since $e^0 = U^0 - u_0$, the stability result in Theorem 5.1 implies that

$$\|e^n\| \leq \|U^0 - u_0\| + 2\sum_{j=1}^{n} \|\eta^j\| \tag{5.3.5}$$

In next theorem we estimate the error from the time discretization. Some important results will be used from Mustapha [41].

**Theorem 5.2 (convergence Theorem)** *Let $u$ be the solution of the initial-value problem* (5.3.2) *and let $U^n$ be the solution of the discrete-time scheme* (5.3.1). *Assume that the initial data $u_0 \in H^2(\Omega)$ and Assume that*

$$t^{\alpha}\|u'(t)\| + t^{1+\alpha}\|u''(t)\| \le Ct^{\eta-1} \quad , \quad 0 < \eta < \alpha + 2, \quad 0 < t < T.$$

*Then, for $1 \le n \le N$, we have*

$$\|U_h^n - u(t_n)\| \le \|U_h^0 - u_0\| + Ch^2 + C \times \begin{cases} k^{\gamma\alpha} & \text{if } 1 \le \gamma < \frac{\alpha+1}{\alpha} \\ k^{\alpha+1}\max(1, \log(t_n/t_1)) & \text{if } \gamma = \frac{\alpha+1}{\alpha} \\ k^{\alpha+1} & \text{if } \gamma > \frac{\alpha+1}{\alpha} \end{cases}$$

**Proof.** By the achieved inequality in (5.3.5), the task reduces to estimate $\|\eta^j\|$. Thanks to Lemmas 4.1, and 4.2 in [41] where the following estimate was proved.

$$2\sum_{j=1}^{n}\|\eta^j\| \le C\Bigg(\int_0^{t_1} t^{\alpha}\|\mathcal{A}u'(t)\|\,dt + k_2^{\alpha+1}\|\mathcal{A}u'(t_2)\| + k_n^{\alpha+1}\|\mathcal{A}u'(t_n)\| + \sum_{j=2}^{n} k_j^{\alpha+1}\int_{t_{j-1}}^{t_j}\|\mathcal{A}u''(s)\|\,ds + \sum_{j=3}^{n} k_j^{\alpha}(k_j - k_{j-1})\|\mathcal{A}u'(t_{j-1})\|\Bigg) \quad (5.3.6)$$

From regularity properties of our solution $u$ (see Theorem 4.5) we have the following:

$$\|\mathcal{A}u'(t)\| + t\|\mathcal{A}u''(t)\| \le C(t^{-1} + t^{\eta-\alpha-1}). \quad (5.3.7)$$

Substitute the above estimates in (5.3.6) we get :

$$\int_0^{t_1} t^\alpha \|\mathcal{A}u'(t)\|\, dt \le \int_0^{t_1} (t^{\alpha-1} + t^{\eta-1})\, dt \le C(t_1^\alpha + t_1^\eta) \le Ct_1^\alpha \le Ck^{\gamma\alpha}.$$

Since $k_j \le \gamma k t_j^{1-1/\gamma}$ for $j \ge 1$,

$$\begin{aligned} k_j^{\alpha+1}\|\mathcal{A}u'(t_j)\| &\le Ck_j^{\alpha+1}(t_j^{-1} + t_j^{\eta-\alpha-1}) \le Ck^{\alpha+1}t_j^{-(\alpha+1)/\gamma}(t_j^\alpha + t_j^\eta) \\ &\le Ck^{\alpha+1}t_j^{\alpha-(\alpha+1)/\gamma} \end{aligned}$$

Using $k_j^{\alpha+1} \ge k_{j-1}^{1+\alpha}$ and

$$\begin{aligned} \|\mathcal{A}u'(t_{j-1})\| &\le \|\mathcal{A}u'(t_j) - \mathcal{A}u'(t_{j-1})\| + \|\mathcal{A}u'(t_j)\| \\ &= \int_{t_{j-1}}^{t_j} \|\mathcal{A}u''(s)\|\, ds + \|\mathcal{A}u'(t_j)\| \end{aligned}$$

we get the following estimate

$$\begin{aligned} k_j^\alpha(k_j - k_{j-1})\|\mathcal{A}u'(t_{j-1})\| &\le (k_j^{\alpha+1} - k_{j-1}^{\alpha+1})\|\mathcal{A}u'(t_{j-1})\| \\ &\le k_j^{\alpha+1}\int_{t_{j-1}}^{t_j} \|\mathcal{A}u''(s)\|\, ds \\ &\quad + k_j^{\alpha+1}\|\mathcal{A}u'(t_j)\| - k_{j-1}^{\alpha+1}\|\mathcal{A}u'(t_{j-1})\|. \end{aligned}$$

This implies

$$\sum_{j=2}^{n} k_j^{\alpha}(k_j - k_{j-1})\|\mathcal{A}u'(t_{j-1})\| \le \sum_{j=2}^{n} k_j^{\alpha+1} \int_{t_{j-1}}^{t_j} \|\mathcal{A}u''(s)\|\,ds$$
$$+ k_n^{\alpha+1}\|\mathcal{A}u'(t_n)\| - k_2^{\alpha+1}\|\mathcal{A}u'(t_2)\|.$$

By (5.3.7),

$$\int_{t_{j-1}}^{t_j} \|\mathcal{A}u''(s)\|\,ds \le \int_{t_{j-1}}^{t_j} (s^{-2} + s^{\eta-\alpha-2})\,ds = k_j(t_j^{-2} + t_j^{\eta-\alpha-2}) \qquad \text{for } j \ge 2.$$

Therefore,

$$\sum_{j=2}^{n} k_j^{\alpha+1} \int_{t_{j-1}}^{t_j} \|\mathcal{A}u''(s)\|\,ds$$
$$\le \sum_{j=2}^{n} k_j^{\alpha+1} k_j (t_j^{-2} + t_j^{\eta-\alpha-2}) \le C \sum_{j=2}^{n} k^{\alpha+1} k_j (t_j^{\alpha-(\alpha+1)/\gamma-1} + t_j^{\eta-(\alpha+1)/\gamma-1})$$
$$\le Ck^{\alpha+1} \int_{t_1}^{t_n} (t^{\alpha-(\alpha+1)/\gamma-1})\,dt \le Ck^{\alpha+1} \times \begin{cases} t_1^{\alpha-(1+\alpha)/\gamma} & \text{if } \alpha - \frac{\alpha+1}{\gamma} < 0 \\ \max(1, \log(t_n/t_1)) & \text{if } \alpha = \frac{\alpha+1}{\gamma} \\ t_n^{\alpha-(\alpha+1)/\gamma} & \text{if } \alpha - \frac{\alpha+1}{\gamma} > 0 \end{cases}$$

Regarding the term $k_n^{\alpha+1}\|\mathcal{A}u'(t_n)\|$, again by the regularity property in (5.3.7)

$$
\begin{aligned}
k_n^{\alpha+1}\|\mathcal{A}u'(t_n)\| &\le k_n^{\alpha+1}(t_n^{-1}+t_n^{\eta-\alpha-1})\\
&\le Ck^{\alpha+1}t_n^{\alpha-(1+\alpha)/\gamma}\\
&\le C\times\begin{cases} k^{1+\alpha}(nk)^{\alpha\gamma-(1+\alpha)} \le k^{\alpha\gamma} & \text{if } \alpha-\frac{\alpha+1}{\gamma}<0\\[2ex] k^{\alpha+1} & \text{if } \alpha-\frac{\alpha+1}{\gamma}\ge 0\end{cases}
\end{aligned}
$$

Inserting the above contribution in (5.3.6) will complete the proof. ∎

## 5.4 Well-posedness of the fullydiscrete solution

In this section we show the existence and uniqueness of the solution of the fully discrete scheme (5.4.1). Our discrete-time solution $U^n \in H_0^1(\Omega)$ of (5.1.2) satisfies

$$\langle U^n - U^{n-1}, v\rangle + \int_{I_n}\langle \partial_t^{1-\alpha}\nabla\bar{U}, \nabla v\rangle\, dt - \int_{I_n}\langle \bar{F}^n\partial_t^{1-\alpha}\bar{U}, \nabla v\rangle\, dt = \int_{I_n}\langle g, v\rangle\, dt$$

for all $v \in H_0^1(\Omega)$.

For the spatial discretization via the standard Galerkin finite elements method, let $\mathcal{T}_h$ be a family of regular triangulations (made of simplexes $K$) of the domain $\overline{\Omega}$ and let $h = \max_{K\in\mathcal{T}_h}(\text{diam}K)$, where $h_K$ denotes the diameter of the elements $K$. Let $S_h \subset H_0^1(\Omega)$ denote the usual space of continuous, piecewise-linear functions on $\mathcal{T}_h$ that vanish on $\partial\Omega$.

We therefore seek a fully-discrete solution $U_h^n \in S_h$ given by

$$\langle U_h^n - U_h^{n-1}, v\rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \nabla \bar{U}_h, \nabla v\rangle \, dt - \int_{I_n} \langle \bar{F}^n \partial_t^{1-\alpha} \bar{U}_h, \nabla v\rangle \, dt = \int_{I_n} \langle g, v\rangle \, dt, \tag{5.4.1}$$

for all $v \in S_h$

**Theorem 5.3** *For $k$ sufficiently small, the solution $U_h^n$ of the fully discrete scheme in (5.4.1) exists and is unique.*

**Proof.** We assume that (5.4.1) has two solutions $U_h^n$, and $W_h^n$. Then, $Q_h^n := U_h^n - W_h^n$ satisfies

$$\langle Q_h^n - Q_h^{n-1}, v\rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \nabla \bar{Q}_h, \nabla v\rangle \, dt - \int_{I_n} \langle \bar{F}^n \partial_t^{1-\alpha} \bar{Q}_h, \nabla v\rangle \, dt = 0. \tag{5.4.2}$$

Therefore, after integrating, we get:

$$\langle Q_h^1, v\rangle + \frac{1}{2}\omega_{11} \langle \nabla Q_h^1, \nabla v\rangle - \frac{1}{2}\omega_{11} \langle \bar{F} Q_h^1, \nabla v\rangle = 0 \qquad \text{for } n = 1,$$

where we used the fact that $Q_h^0 = 0$.

Choose $v = Q_h^1$, then apply Cauchy-Shwarz inequality and $\epsilon$ inequality,

$$\|Q_h^1\|^2 + \frac{1}{2} k^\alpha \|\nabla Q_h^1\|^2 \le \frac{1}{2\epsilon} C k^\alpha \|Q_h^1\| + \frac{1}{2} k^\alpha \epsilon \|\nabla Q_h^1\|^2$$
$$\implies (1 - \frac{1}{2\epsilon} C k^\alpha) \|Q_h^1\|^2 + \frac{1}{2} k^\alpha (1 - \epsilon) \|\nabla Q_h^1\|^2 \le 0.$$

Therefore, for $k$ sufficiently small, $\|Q_h^1\|^2 \le 0$. This implies that $Q_h^1 = 0$ .

In a similar fashion, we show that $Q_h^2 = 0$ using $Q_h^1 = 0$. Recursively, we can show that $Q_h^n = 0 \quad \text{for } n \geq 1$.

∎

## CHAPTER 6

# $L1$ APPROXIMATION SCHEME

This chapter is devoted to discuss the time-stepping $L1$ numerical method combined with the finite elements in space. As mentioned earlier, the time-stepping $L1$ scheme has some advantages over the Crank-Nicolson scheme in terms of the convergence rates. In section 1, we define the computational scheme. Then, the error estimates are established in section 2.

## 6.1 $L1$ finite elements scheme

Recall that our time-fractional Fokker-Planck equation,

$$\partial_t u(x,t) - \nabla \cdot (\partial_t^{1-\alpha} \kappa_\alpha(x) \nabla u(x,t)) + \nabla \cdot (F \partial_t^{1-\alpha} u(x,t)) = g(x,t), \qquad (6.1.1)$$

with initial condition $u(x,0) = v(x)$, and subject to homogeneous boundary Dirichlet boundary conditions. The diffusivity coefficient $0 < c_0 \leq \kappa_\alpha(x) \leq c_1$ on $\Omega$ for some positive constants $c_1$ and $c_2$.

For the error analysis part, we assume that

$$\|u'(t)\|_{H^2(\Omega)} + t\|u''(t)\|_{H^2(\Omega)} + t^2\|u'''(t)\|_{H^2(\Omega)} \leq Ct^{\sigma-1}, \qquad \text{for some } \sigma > 0. \quad (6.1.2)$$

To define our schemes we introduce the following notations:

$$\partial v(t) = \partial v^n = \frac{v^n - v^{n-1}}{k_n} \quad \text{for } t_{n-1} < t < t_n,$$

and the piecewise-linear interpolation function

$$\check{v}(t) = v^{n-1} + (t - t_{n-1})\partial v^n \quad \text{for } t_{n-1} < t < t_n. \tag{6.1.3}$$

Integrating the weak formulation problem in (3.1.1) over the $n$th time interval $I_n$ gives

$$\langle u(t_n) - u(t_{n-1}), v\rangle + \int_{I_n} \langle \mathcal{A}(\partial_t^{1-\alpha} u(t)), v\rangle \, dt + \int_{I_n} \langle \nabla \cdot (F\partial_t^{1-\alpha} u(t)), v\rangle \, dt = \langle \bar{g}^n, v\rangle \tag{6.1.4}$$

for all $v \in H_0^1(\Omega)$, with $\bar{g}^n = \int_{I_n} g(t)\, dt$. For the fully discrete computational solution, we seek $u_h^n \in S_h$ (see section 4.2 for the definition of $S_h$) approximates $u(t_n)$ such that, for $1 \le n \le N$,

$$k_n\langle \partial u_h^n, v_h\rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \mathcal{A}(\check{u}_h(t), v_h)\rangle \, dt + \int_{I_n} \langle \nabla \cdot (F\partial_t^{1-\alpha} \check{u}_h(t)), v_h\rangle \, dt = \langle \bar{g}^n, v_h\rangle \tag{6.1.5}$$

for all $v_h \in S_h$, with $u_h^0 = R_h v$, where $R_h : H_0^1(\Omega) \to S_h$ is the Ritz projection defined by

$$A(R_h w, \phi) = A(w, \phi), \quad \text{for all } \phi \in S_h.$$

## 6.2 Error analysis

For the error analysis, we follow the approach in [Mustapha, [44]]. We start by the following decomposition

$$u_h^n - u(t_n) = (u_h^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) = \theta^n + \rho^n, \quad \text{for} \ \ 1 \le n \le N.$$

From the Ritz projector approximation property,

$$\|\rho^n\| \le Ch^2 \|u(t_n)\|_{H^2(\Omega)} \quad \text{for } 0 \le n \le N. \tag{6.2.1}$$

Thus, the main task now is to bound the term $\theta^n$. A preliminary estimate will be derived in the next lemma. For convenience, we introduce the following notations:

$$\eta_1^n = \int_{I_n} \partial_t^{1-\alpha} \mathcal{A}(u - \check{u})(t)\, dt \quad \text{and} \quad \eta_2^n = \rho(t_{n-1}) - \rho(t_n)\,.$$

Furthermore, for $t \in I_n$, let $\eta_1(t) = \eta_1^n$ and let $\eta_2(t) = \eta_2^n$.

**Lemma 6.1** *For $1 \le n \le N$, we have*

$$\|\theta^n\|^2 \le Ct_n \sum_{j=1}^{n} \Big( \frac{1}{k_j} (\|\eta_1^j\|^2 + \|\eta_2^j\|^2) \Big)$$

**Proof.** By comparing (6.1.4) with (6.1.5), we get

$$k_n \langle \partial\theta^n, v_h \rangle + \int_{I_n} \mathcal{A}\big(\partial_t^{1-\alpha}\check{\theta}(t), v_h\big)\, dt + \int_{I_n} (\nabla(F\partial_t^{1-\alpha}\check{\theta}(t)), v_h)\, dt = \langle \eta_1^n + \eta_2^n, v_h \rangle,$$

for all $v_h \in V_h$. Choosing $v_h = \partial\theta^n = \partial_t\check{\theta}(t)$ for $t \in I_n$, we reach

$$k_n\|\partial\theta^n\|^2 + \int_{I_n} \mathcal{A}\Big(\partial_t^{1-\alpha}\check{\theta}(t)dt, \partial_t\check{\theta}(t)\Big)\,dt - \int_{I_n} (F\partial_t^{1-\alpha}\check{\theta}(t), \nabla\partial_t\check{\theta}(t))\,dt = \langle \eta_1^n + \eta_2^n, \partial\theta^n\rangle. \tag{6.2.2}$$

Summing over $n$, we observe that

$$\sum_{j=1}^{n} k_j\|\partial\theta^j\|^2 + \Theta^n = \sum_{j=1}^{n}\langle \eta_1^j + \eta_2^j, \partial\theta^j\rangle + \int_0^{t_n} (F\partial_t^{1-\alpha}\check{\theta}(t), \nabla\partial_t\check{\theta}(t))\,dt \tag{6.2.3}$$

where

$$\Theta^n = \int_0^{t_n} \mathcal{A}\Big(\partial_t^{1-\alpha}\check{\theta}(t)dt, \partial_t\check{\theta}(t)\Big)\,dt.$$

We bound the last term in (6.2.3) as follows:

$$\begin{aligned}
\Big|\int_0^{t_n}\int_\Omega F\partial_t^{1-\alpha}\check{\theta}(t)\nabla\partial_t\check{\theta}(t)\,dx\,dt\Big| &= \Big|\int_\Omega F\int_0^{t_n} \mathcal{I}^\alpha\partial_t\check{\theta}(t)\nabla\partial_t\check{\theta}(t)\,dt\,dx\Big| \\
&\le \int_\Omega |F|\Big|\int_0^{t_n} \mathcal{I}^\alpha\partial_t\check{\theta}(t)\nabla\partial_t\check{\theta}(t)\,dt\Big|\,dx \\
&\le C\int_\Omega \Big|\int_0^{t_n} \mathcal{I}^\alpha\partial_t\check{\theta}(t)\nabla\partial_t\check{\theta}(t)\,dt\Big|\,dx \\
\le C\int_\Omega\left(\frac{1}{2\epsilon}\int_0^{t_n} \mathcal{I}^\alpha\partial_t\check{\theta}(t)\partial_t\check{\theta}(t)\,dt + \frac{\epsilon}{2}\int_0^{t_n}\right. & \left.\mathcal{I}^\alpha\nabla\partial_t\check{\theta}(t)\nabla\partial_t\check{\theta}(t)\,dt\right)dx \\
= C_\epsilon\int_0^{t_n}\big(\mathcal{I}^\alpha\partial_t\check{\theta}(t), \partial_t\check{\theta}(t)\big)\,dt + \epsilon & \int_0^{t_n}\big(\mathcal{I}^\alpha\nabla\partial_t\check{\theta}(t), \nabla\partial_t\check{\theta}(t)\big)\,dt \\
= C_\epsilon\int_0^{t_n}\big(\mathcal{I}^\alpha\partial_t\check{\theta}(t), \partial_t\check{\theta}(t)\big)\,dt & + \epsilon\int_0^{t_n}\big(\mathcal{I}^\alpha\nabla\partial_t\check{\theta}(t), \nabla\partial_t\check{\theta}(t)\big)\,dt.
\end{aligned} \tag{6.2.4}$$

Choose $\epsilon = \frac{1}{2}$ and then inserting (6.2.4) in (6.2.3) yield

$$\sum_{j=1}^{n} k_j\|\partial\theta^j\|^2 + \frac{1}{2}\Theta^n = \sum_{j=1}^{n}\langle \eta_1^j + \eta_2^j, \partial\theta^j\rangle + C\int_0^{t_n}\big(\mathcal{I}^\alpha\check{\theta}(t), \partial_t\check{\theta}(t)\big)\,dt \tag{6.2.5}$$

An application of the Cauchy-Schwarz inequality gives

$$\sum_{j=1}^{n}\langle \eta_1^j+\eta_2^j,\partial\theta^j\rangle \le \sum_{j=1}^{n}(\|\eta_1^j\|+\|\eta_2^j\|)\|\partial\theta^j\| \le \sum_{j=1}^{n}\Big(\frac{1}{2k_j}(\|\eta_1^j\|^2+\|\eta_2^j\|^2)+\frac{k_j}{2}\|\partial\theta^j\|^2\Big). \tag{6.2.6}$$

Using Cauchy-Schwarz inequality, $\epsilon-$inequality followed by Lemma 2.2,

$$\begin{aligned}
\int_0^{t_n}\big(\mathcal{I}^\alpha\partial_t\check{\theta}(t),\partial_t\check{\theta}(t)\big)\,dt &\le \int_0^{t_n}\|\mathcal{I}^\alpha\partial_t\check{\theta}(t)\|\|\partial_t\check{\theta}(t)\|\,dt \\
&\le C_\epsilon\int_0^{t_n}\|\mathcal{I}^\alpha\partial_t\check{\theta}(t)\|^2\,dt+\epsilon\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt \\
&\le C_\epsilon t_n^\alpha\sum_{j=1}^{n}\int_{I_j}\omega_\alpha(t_n-s)\int_0^s\|\partial_t\check{\theta}(q)\|^2\,dq\,ds+\epsilon\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt \\
&\le C_\epsilon t_n^\alpha\sum_{j=1}^{n}\int_{I_j}\omega_\alpha(t_n-s)\int_0^{t_j}\|\partial_t\check{\theta}(q)\|^2\,dq\,ds+\epsilon\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt \\
&\le C_\epsilon t_n^\alpha\sum_{j=1}^{n}z^j\int_{I_j}\omega_\alpha(t_n-s)\,ds+\epsilon\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt
\end{aligned} \tag{6.2.7}$$

where $z^j=\int_0^{t_j}\|\partial_t\check{\theta}(t)\|^2\,dt$.

Using

$$\sum_{j=1}^{n}k_j\|\partial\theta^j\|^2=\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt$$

and then inserting (6.2.6) and (6.2.7) in (6.2.3), give

$$\begin{aligned}
&\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt+\Theta^n\le \\
&\quad\sum_{j=1}^{n}\Big(\frac{1}{k_j}(\|\eta_1^j\|^2+\|\eta_2^j\|^2)\Big)+C_\epsilon t_n^\alpha\sum_{j=1}^{n}z^j\int_{I_j}\omega_\alpha(t_n-s)\,ds+\epsilon\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt \\
&\quad\Longrightarrow\int_0^{t_n}\|\partial_t\check{\theta}(t)\|^2\,dt\le\sum_{j=1}^{n}\Big(\frac{1}{k_j}(\|\eta_1^j\|^2+\|\eta_2^j\|^2)\Big)+C_\epsilon t_n^\alpha\sum_{j=1}^{n}z^j\int_{I_j}\omega_\alpha(t_n-s)\,ds
\end{aligned} \tag{6.2.8}$$

Therefore,

$$z^n \leq C \sum_{j=1}^{n} \Big( \frac{1}{k_j} (\|\eta_1^j\|^2 + \|\eta_2^j\|^2) \Big) + C_\epsilon t_n^\alpha \sum_{j=1}^{n} z^j \int_{I_j} \omega_\alpha(t_n - s)\, ds.$$

By the Gronwall inequality in Lemma 2.6, we get:

$$z^n \leq C E_\alpha(C'_\epsilon t_n^{2\alpha}) \sum_{j=1}^{n} \Big( \frac{1}{k_j} (\|\eta_1^j\|^2 + \|\eta_2^j\|^2) \Big).$$

Combine the above contribution with the following inequality

$$\|\theta^n\|^2 \leq t_n \sum_{j=1}^{n} k_j \|\partial \theta^j\|^2,$$

will complete the proof. ∎

In the next theorem, the error bound for the fully discrete scheme is derived.

**Theorem 6.1 (Convergence Theorem)** *Let $u$ be the solution of problem* (6.1.1). *Let $u_h^n \in S_h$ be the solution of the fully discrete scheme* (6.1.5). *Assume the regularity properties in* (6.1.2) *with $\frac{1}{2} < \sigma < 3$ hold true. Then*

$$\|u_h^n - u(t_n)\| \leq Ch^2 + C \times \begin{cases} k^{\gamma(\alpha+\sigma-1/2)} & \text{if } 1 \leq \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^2 \max(1, \sqrt{\log(t_n/t_2)}) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^2 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

**Proof.** From [Mustapha, [44]] we have

$$\|\eta_1^j\|^2 \leq C\Big(k^{2\gamma(\sigma+\alpha)} + k^{2+\gamma(\alpha+\sigma)} k_j t_j^{\alpha+\sigma-1-2/\gamma} + k^4 k_j^2 t_j^{2(\alpha+\sigma-1-2/\gamma)}\Big).$$

Therefore,

$$\frac{1}{k_j}\|\eta_1^j\|^2 \leq C\big(\frac{1}{k_j}k^{2\gamma(\sigma+\alpha)} + k^{2+\gamma(\alpha+\sigma)}t_j^{\alpha+\sigma-1-2/\gamma} + k^4 k_j t_j^{2(\alpha+\sigma-1-2/\gamma)}\big).$$

Using

$$k_j = t_j - t_{j-1} = k^\gamma(j^\gamma - (j-1)^\gamma) = \gamma k^\gamma \int_{j-1}^{j} t^{\gamma-1}\,dt \geq \gamma k^\gamma (j-1)^{\gamma-1},$$

implies that

$$\frac{1}{k_j} \leq C_\gamma k^{-\gamma}(j-1)^{1-\gamma}.$$

$$\begin{aligned}
\sum_{j=1}^{n}\Big(\frac{1}{k_j}\|\eta_1^j\|^2\Big) =& \frac{1}{k_1}\|\eta_1^1\|^2 + \sum_{j=2}^{n}\Big(\frac{1}{k_j}\|\eta_1^j\|^2\Big) \\
&\leq \frac{1}{k_1}\|\eta_1^1\|^2 + \sum_{j=2}^{n} Ck^{\gamma(2(\sigma+\alpha)-1)}(j-1)^{1-\gamma} \\
&+ C\sum_{j=2}^{n} k^{2+\gamma(\alpha+\sigma)}t_j^{\alpha+\sigma-1-2/\gamma} + C\sum_{j=2}^{n} k^4 k_j t_j^{2(\alpha+\sigma-1-2/\gamma)} \\
&\leq Ck^{\gamma(2(\alpha+\sigma)-1))} + \big(Ck^{\gamma(2(\sigma+\alpha)-1)}\big)\sum_{j=2}^{n}(j-1)^{1-\gamma} \\
&+ Ck^{\gamma(2(\alpha+\sigma)-1)}\sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2} + C\sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)}\,dt.
\end{aligned}$$

Now,

$$\sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)}\,dt = k^4 \int_{t_2}^{t_n} t^{2(\alpha+\sigma-1-2/\gamma)}\,dt$$

$$= Ck^4 \times \begin{cases} t_2^{2(\alpha+\sigma)-1-4/\gamma} & \text{if } 2(\alpha+\sigma)-2-4/\gamma < -1 \\ \log(t_n/t_2) & \text{if } 2(\alpha+\sigma)-2-4/\gamma = -1 \\ t_n^{2(\alpha+\sigma)-1-4/\gamma} & \text{if } 2(\alpha+\sigma)-2-4/\gamma > -1 \end{cases}$$

$$= Ck^4 \times \begin{cases} t_2^{2(\alpha+\sigma)-1-4/\gamma} & \text{if } 1 \le \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ \log(t_n/t_2) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ t_n^{2(\alpha+\sigma)-1-4/\gamma} & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

$$\le C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if } 1 \le \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^4 \log(t_n/t_2) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^4 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

For the series in the term

$$k^{\gamma(2(\alpha+\sigma)-1)} \sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2}$$

it converges only if $2+\gamma-\gamma(\alpha+\sigma) > 1$ which implies

$$\begin{cases} \text{if } \alpha+\sigma < 1 \Longrightarrow \gamma > 1/(\alpha+\sigma-1) \quad \text{which is true for any } \gamma \\ \text{if } \alpha+\sigma > 1 \Longrightarrow 1 \le \gamma < 1/(\alpha+\sigma-1) \end{cases}$$

Therefore we can combine the above results as follows

$$k^{\gamma(2(\alpha+\sigma)-1)} \sum_{j=2}^{n} j^{\gamma(\alpha+\sigma)-\gamma-2} + C \sum_{j=2}^{n} k^4 \int_{I_j} t^{2(\alpha+\sigma-1-2/\gamma)} \, dt$$
$$\leq C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if } 1 \leq \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^4 \log(t_n/t_2) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^4 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

For the case $(\alpha + \sigma) < 1/2$ it is contained by the first case since $\gamma > \frac{2}{\alpha+\sigma-1/2}$ become the right hand side is negative and $\gamma \geq 1$. The first series on the left side is convergent by the integral test. Therefore,

$$\sum_{j=1}^{n} \Big( \frac{1}{k_j} \|\eta_1^j\|^2 \Big) \leq C \times \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if } 1 \leq \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^4 \log(t_n/t_2) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^4 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}$$

Using (6.2.1), followed by regularity assumption (6.1.2) one can conclude that

$$\|\eta_2^j\|^2 \leq Ch^4 \Big( \int_{I_j} \|u'(t)\|_2 \, dt \Big)^2 \leq Ch^4 \Big( \int_{I_j} t^{\sigma-1} \, dt \Big)^2 \leq Ch^4 k_j^2 t_j^{2(\sigma-1)}$$

Hence,

$$\sum_{j=1}^{n}\left(\frac{1}{k_j}(\|\eta_2^j\|^2)\right) \leq Ch^4\sum_{j=1}^{n}\frac{1}{k_j}k_j^2 t_j^{2(\sigma-1)}$$
$$\leq Ch^4\sum_{j=1}^{n}k_j t_j^{2\sigma-2} \leq Ch^4\int_{t_1}^{t_n} t^{(2\sigma-1)-1}\,dt$$
$$\leq Ch^4 \times \begin{cases} \log(t_n/t_1) & \text{if } \sigma = 1/2 \\ t_n^{2\sigma-1} & \text{if } \sigma > 1/2 \end{cases}$$

Combining the above estimates leads to:

$$\|\theta^n\|^2 \leq t_n\sum_{j=1}^{n}\left(\frac{1}{k_j}\big(\|\eta_1^j\|^2 + \eta_2^j\|^2\big)\right) \leq$$
$$Ct_n\left(h^4 k^{\gamma(2\sigma-1)} + \begin{cases} k^{\gamma(2(\alpha+\sigma)-1)} & \text{if } 1 \leq \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^4\log(t_n/t_2) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^4 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}\right).$$

consequently,

$$\|\theta^n\| \leq Ch^2k^{\gamma(\sigma-1/2)}+C\times\begin{cases} k^{\gamma(\alpha+\sigma-1/2)} & \text{if } 1 \leq \gamma < 2/\big(\alpha+\sigma-1/2\big) \\ k^2\max(1, \sqrt{\log(t_n/t_2)}) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2\big) \\ k^2 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}.$$

Combining the above estimates,

$$
\begin{aligned}
\|u_h^n - u(t_n)\| = \|\theta^n + \rho^n\| &\le Ch^2 + \|\theta^n\| + \|\rho^n\| \\
&\le Ch^2 + C \times \begin{cases} k^{\gamma(\alpha+\sigma-1/2)} & \text{if } 1 \le \gamma < 2/\big(\alpha+\sigma-1/2) \\ k^2 \max(1, \sqrt{\log(t_n/t_2)}) & \text{if } \gamma = 2/\big(\alpha+\sigma-1/2) \\ k^2 & \text{if } \gamma > 2/(\alpha+\sigma-1/2) \end{cases}
\end{aligned}
$$

for $1 \le n \le N$. $\blacksquare$

## CHAPTER 7

# IMPLEMENTATION AND NUMERICAL EXPERIMENTS

In section 1 we discuss the implementation of the Crank-Nicolson finite elements scheme in one dimension. The implementation of the $L1$ approximation scheme is discussed in section 2. The last section contained numerical experiments that confirm our theoretical convergence results for both numerical schemes. Some figures and numerical tables will be included.

## 7.1 Implementations of the Crank-Nicolson finite element scheme

Recall that, our fully-discrete solution $U_h^n \in S_h$ is given by

$$\langle U_h^n - U_h^{n-1}, v\rangle + \int_{I_n} \langle \partial_t^{1-\alpha} \nabla \bar{U}_h, \nabla v\rangle \, dt - \int_{I_n} \langle \bar{F}^n \partial_t^{1-\alpha} \bar{U}_h, \nabla v\rangle \, dt = \int_{I_n} \langle g, v\rangle \, dt$$

for all $v \in S_h$ and for $1 \le n \le N$, with $U_h^0 = R_h u_0$. Explicitly, let $\phi_p \in S_h$ denote the $p$th nodal basis function, so that $\phi_p(x_q) = \delta_{pq}$. So,

$$U_h^n(x) = \sum_{p=1}^{P-1} U_p^n \phi_p(x) \quad \text{where } U_p^n = U_h^n(x_p) \approx U^n(x_p) \approx u(x_p, t_n).$$

Define the $(P-1) \times (P-1)$ tridiagonal matrices $\mathbf{M}$ and $\mathbf{B^n}$ with entries

$$\mathbf{M_{pq}} = \langle \phi_q, \phi_p\rangle \qquad \text{and} \quad \mathbf{B^n_{pq}} = \langle \phi_{qx}, \phi_{px}\rangle - \langle \bar{F}^n \phi_q, \phi_{px}\rangle,$$

and define $(P-1)$-dimensional column vectors $U^n$ and $\mathbf{G^n}$ with components $\mathbf{U^n_p}$ and $G^n_p = \int_{I_n} \langle g, \phi_p \rangle \, dt$. We find that

$$\begin{aligned}
&\mathbf{M U^n} - \mathbf{M} U^{n-1} + \frac{1}{2}\omega_{nn}\mathbf{B^n U^n} + \frac{1}{2}\sum_{j=1}^{n-1}\omega_{nj}\mathbf{B^n U^j} + \frac{1}{2}\omega_{nn}\mathbf{B^n U^{n-1}} \\
&+ \frac{1}{2}\sum_{j=1}^{n-1}\omega_{nj}\mathbf{B^n U^{j-1}} + \sum_{j=1}^{n}\omega_{nj}\mathbf{B^n U^j} - \sum_{j=1}^{n-1}\omega_{n-1,j}\mathbf{B^n} U^j - \sum_{j=1}^{n-1}\omega_{n-1,j}\mathbf{B^n U^{j-1}} = \mathbf{G^n},
\end{aligned} \tag{7.1.1}$$

where

$$\omega_{nj} = \int_{I_j} \omega_\alpha(t_n - s)\, ds = \omega_{1+\alpha}(t_n - t_{j-1}) - \omega_{1+\alpha}(t_n - t_j) \quad \text{for } n \geq 2.$$

Therefore, at the $n$th time step, we must solve the following linear system

$$\begin{aligned}
\big(\mathbf{M} + \frac{1}{2}\omega_{nn}\mathbf{B^n}\big)\mathbf{U^n} = \big(\mathbf{M} - \frac{1}{2}\omega_{nn}\mathbf{B^n}\big)\mathbf{U^{n-1}} + \mathbf{G^n} - \frac{1}{2}\sum_{j=1}^{n-1}\big(\omega_{nj} - \omega_{n-1,j}\big)\mathbf{B^n U^j} \\
- \frac{1}{2}\sum_{j=1}^{n-1}\big(\omega_{nj} - \omega_{n-1,j}\big)\mathbf{B^n U^{j-1}}
\end{aligned} \tag{7.1.2}$$

with

$$\mathbf{M} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & 0 & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} & 0 \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \phi_{p-2,p-3} & \phi_{p-2,p-2} & \phi_{p-2,p-1} \\ 0 & \cdots & 0 & \phi_{p-1,p-2} & \phi_{p-1,p-1} \end{bmatrix},$$

$$\mathbf{B^n} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & 0 & 0 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \psi_{p-2,p-3} & \psi_{p-2,p-2} & \vdots \\ 0 & \cdots & 0 & \psi_{p-1,p-2} & \psi_{p-1,p-1} \end{bmatrix}$$

$$- \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & 0 & 0 \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \xi_{p-2,p-3} & \xi_{p-2,p-2} & \vdots \\ 0 & \cdots & 0 & \xi_{p-1,p-2} & \xi_{p-1,p-1} \end{bmatrix}$$

where $\phi_{ij} = \langle \phi_i, \phi_j \rangle$, $\psi_{ij} = \langle \partial_x(\phi_i), \partial_x(\phi_j) \rangle$, $\xi_{ij} = \langle F^n \phi_i, \partial_x \phi_j \rangle$,

$g_i = \int_{t_{n-1}}^{t_n} \langle g(t), \phi_i \rangle, \quad 0 \leq i, j \leq p-1$

$$\mathbf{U^n} = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{P-1}^n \end{bmatrix} \quad , \quad \mathbf{G^n} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{p-1} \end{bmatrix}$$

## 7.2 Implementation of $L1$ approximation scheme

For the fully discrete computational solution, we seek $u_h^n \in S_h$ approximates $u(t_n)$ such that, for $1 \le n \le N$,

$$k_n\langle \partial u_h^n, v_h\rangle + \int_{I_n} \langle \partial_t^{1-\alpha}\mathcal{A}(\check{u}_h(t), v_h)\rangle\, dt + \int_{I_n} \langle \nabla\cdot(F\partial_t^{1-\alpha}\check{u}_h(t)), v_h\rangle\, dt = \langle \bar{g}^n, v_h\rangle \quad (7.2.1)$$

for all $v_h \in S_h$ with $u_h^0 = R_h v$. Following the notations of the previous section, the fully discrete scheme (7.2.1) can be written in the matrix form as:

$$\begin{aligned}\big(\mathbf{M} + \omega_{2+\alpha}(k_n)\mathbf{B^n}\big)\mathbf{U^n} &= \big(\mathbf{M} - \omega_{2+\alpha}(k_n)\mathbf{B^n}\big)\mathbf{U^{n-1}} + \mathbf{G^n} \\ &\quad - \sum_{j=1}^{n-1}\big(\omega_{nj} - \omega_{n-1,j}\big)\mathbf{B^n}\mathbf{U^j} - \sum_{j=1}^{n-1}\big(\hat{\omega}_{nj} - \hat{\omega}_{n-1,j}\big)\mathbf{B^n}\Big(\frac{U^i - U^{j-1}}{k_j}\Big) \quad (7.2.2)\end{aligned}$$

where

$$\hat{\omega}_{nj} = \omega_{2+\alpha}(t_n - t_{j-1}) - \omega_{2+\alpha}(t_n - t_j) - k_j\omega_{1+\alpha}(t_n - t_j)$$

## 7.3 Numerical convergence

The convergence of both numerical methods (Crank-Nicolson and $L1$) will be tested on sample example below. Choose

$$F(x,t) = x + \sin t, \quad T = 1, \quad L = \pi, \quad \kappa_\alpha = \mu_\alpha = 1,$$

where the source term $g$ is chosen so that the exact solution $u(x,t) = [1+\omega_{1+\alpha}(t)]\sin x$. In this example the solution $u$ satisfies the following regularity properties:

$$t^{\alpha}\|u'(t)\| + t^{1+\alpha}\|u''(t)\| \leq Ct^{2\alpha-1}.$$

This valid for $\sigma = 2\alpha$. Hence, from the error analysis in chapter 4 we expect the convergence rate of the Crank-Nicolson finite elements scheme to be of order $O(k^{2\alpha\gamma})$ for $1 \leq \gamma < \frac{1+\alpha}{2\alpha}$ and $O(k^{1+\alpha})$ for $\gamma > \frac{1+\alpha}{2\alpha}$.

Whereas for the $L1$ approximation scheme, the required regularity assumption is

$$\|u'(t)\| + t^{2}\|u''(t)\| \leq t^{\sigma-1}$$

This is valid for $\sigma = \alpha$. Hence we expect $O(k^{\gamma(2\alpha-1/2)})$ rates of convergence for $1 \leq \gamma < 2/(2\alpha-0.5)$ and $O(k^{2})$ for $\gamma > 2/(2\alpha-0.5)$. The numerical results in Table 6.1 show a better convergence rate.

| $\alpha = 0.3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 1$ | | | | $\gamma = 2$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 6.87e-02 | | 1.72e-02 | | 1.52e-02 | | 4.46e-03 | |
| 40 | 4.96e-02 | 0.47 | 1.31e-02 | 0.39 | 7.22e-03 | 1.07 | 2.17e-03 | 1.03 |
| 80 | 3.53e-02 | 0.49 | 9.73e-03 | 0.43 | 3.34e-03 | 1.11 | 1.03e-03 | 1.08 |
| 160 | 2.48e-02 | 0.51 | 7.04e-03 | 0.47 | 1.52e-03 | 1.14 | 4.78e-04 | 1.11 |
| 320 | 1.73e-02 | 0.52 | 5.02e-03 | 0.49 | 6.79e-04 | 1.16 | 2.17e-04 | 1.14 |
| 640 | 1.2e-02 | 0.53 | 3.54e-03 | 0.51 | 3.01e-04 | 1.17 | 9.72e-05 | 1.16 |
| Theory | | 0.6 | | 0.1 | | 1.2 | | 0.2 |

| $\alpha = 0.3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 3$ | | | | $\gamma = 3.3$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 9.279e-03 | | 9.08e-04 | | 1.204e-02 | | 5.5e-04 | |
| 40 | 4.067e-03 | 1.19 | 2.85e-04 | 1.67 | 4.493e-03 | 1.18 | 1.54e-04 | 1.83 |
| 80 | 1.74e-03 | 1.21 | 8.63e-05 | 1.72 | 1.95e-03 | 1.2 | 4.33e-05 | 1.83 |
| 160 | 7.51e-04 | 1.22 | 2.56e-05 | 1.75 | 8.4e-04 | 1.21 | 1.2e-05 | 1.85 |
| 320 | 3.19e-04 | 1.23 | 7.42e-06 | 1.78 | 3.55e-04 | 1.23 | 3.18e-06 | 1.92 |
| 640 | 1.34e-04 | 1.25 | 2.16e-06 | 1.78 | 1.5e-04 | 1.24 | 8.2e-07 | 1.96 |
| Theory | | 1.3 | | 0.3 | | 1.3 | | 2 |

Table 7.1: Errors and convergence rates for different mesh grading $\gamma$ with $\alpha = 0.3$.

We observe better order for $L1$ scheme. The errors and convergence rates for Crank-Nicolson and $L1$ improved when the mesh is graded. We observe that the numerical results of Crank-Nicolson are as expected in Theorems (6.1, 5.2). However, the numerical results of the $L1$ scheme shows that the theoretical results are pessimistic.

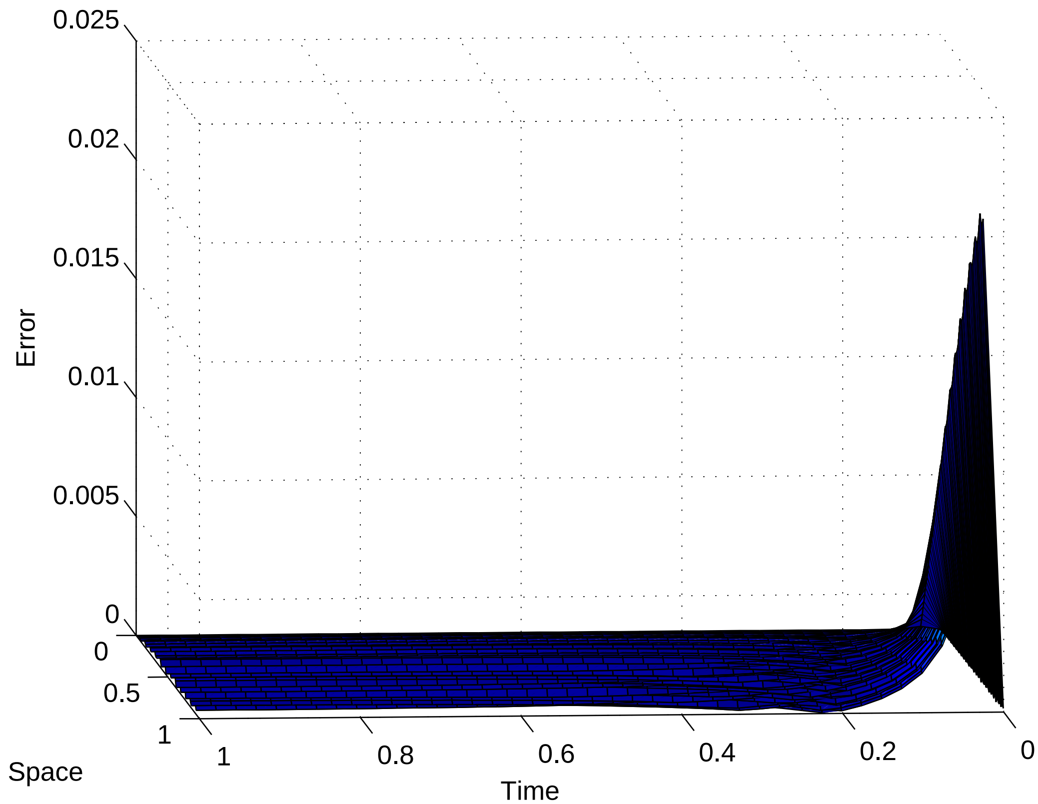


Figure 7.1: Surface error for $\alpha = 0.3$ and $\gamma = 1$ in the spatial domain $[0, \pi]$ using $L1$ scheme

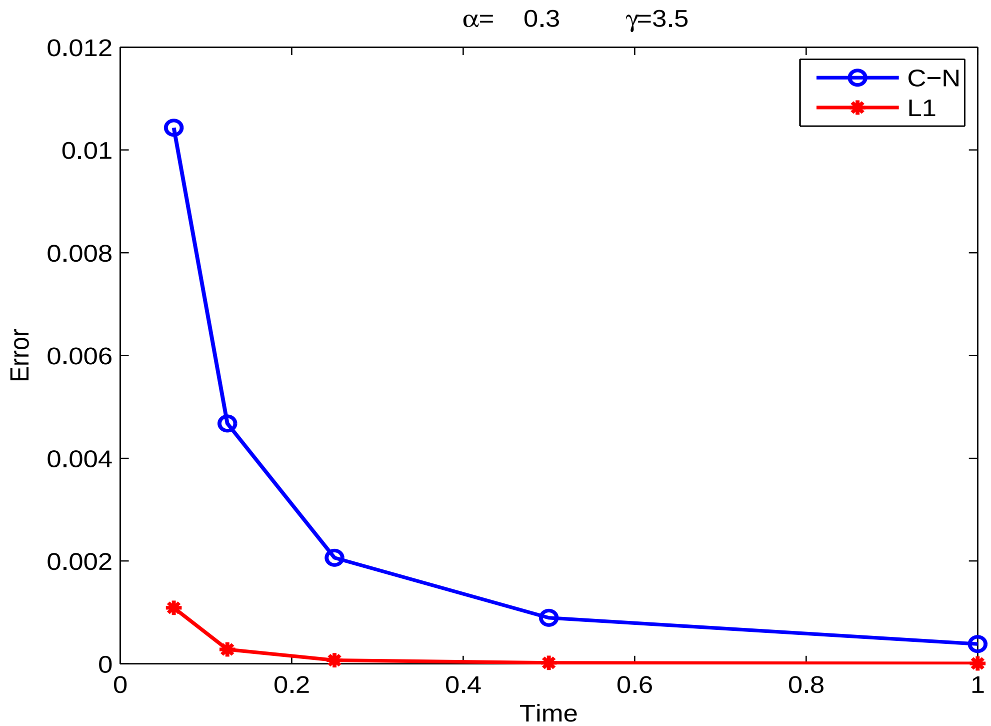


Figure 7.2: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

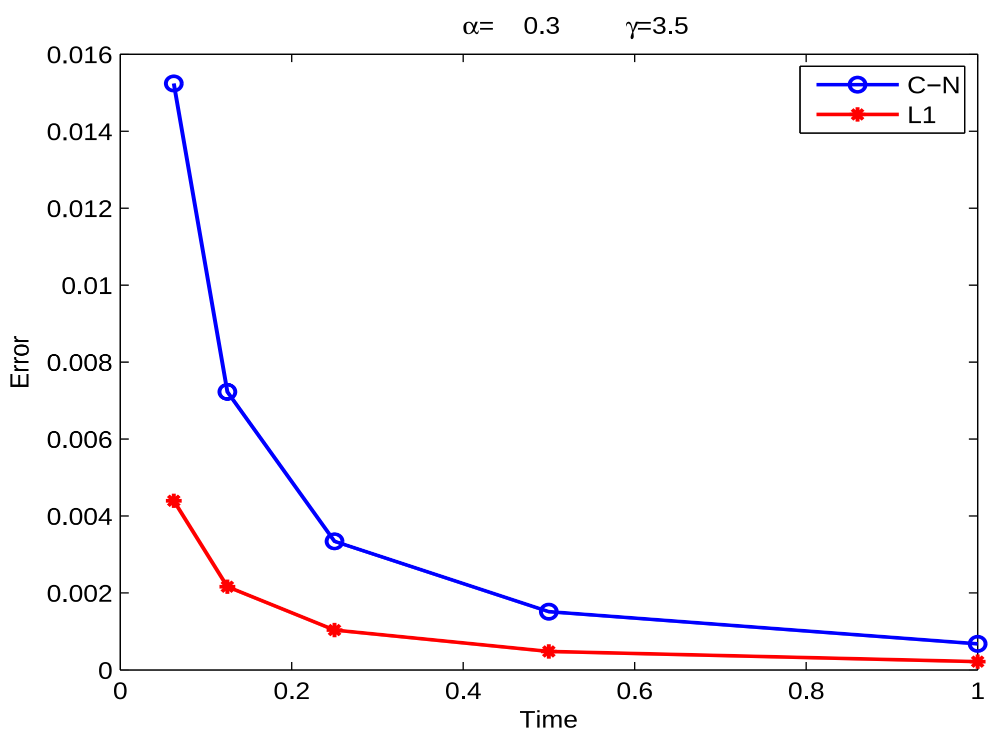


Figure 7.3: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

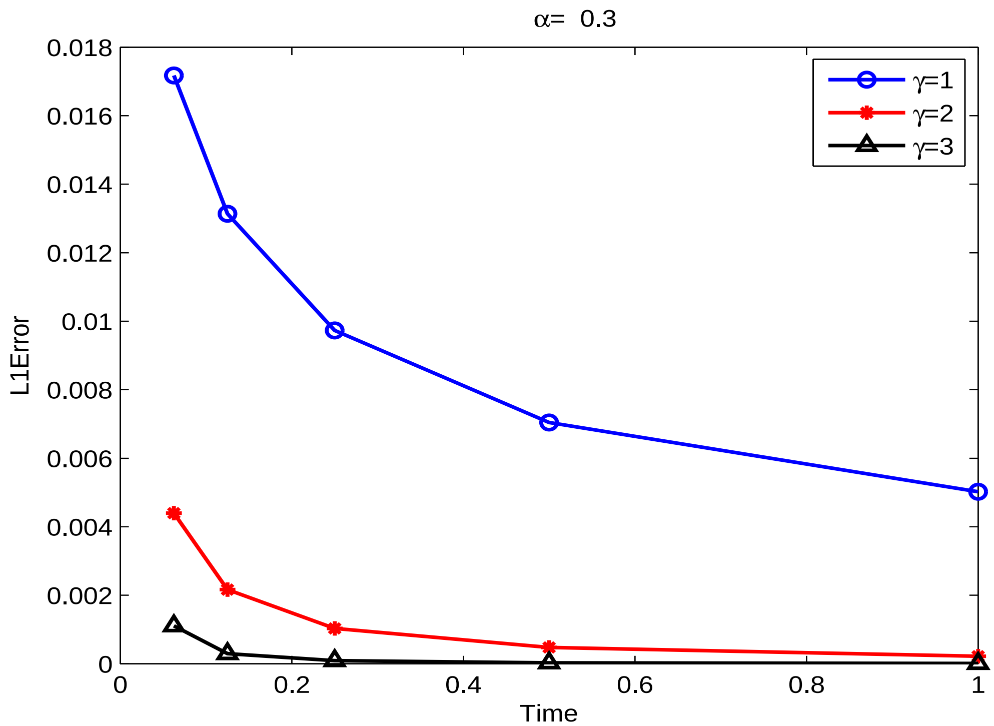


Figure 7.4: Error in the spatial domain $[0, \pi]$ for different time meshes using $L1$ scheme.

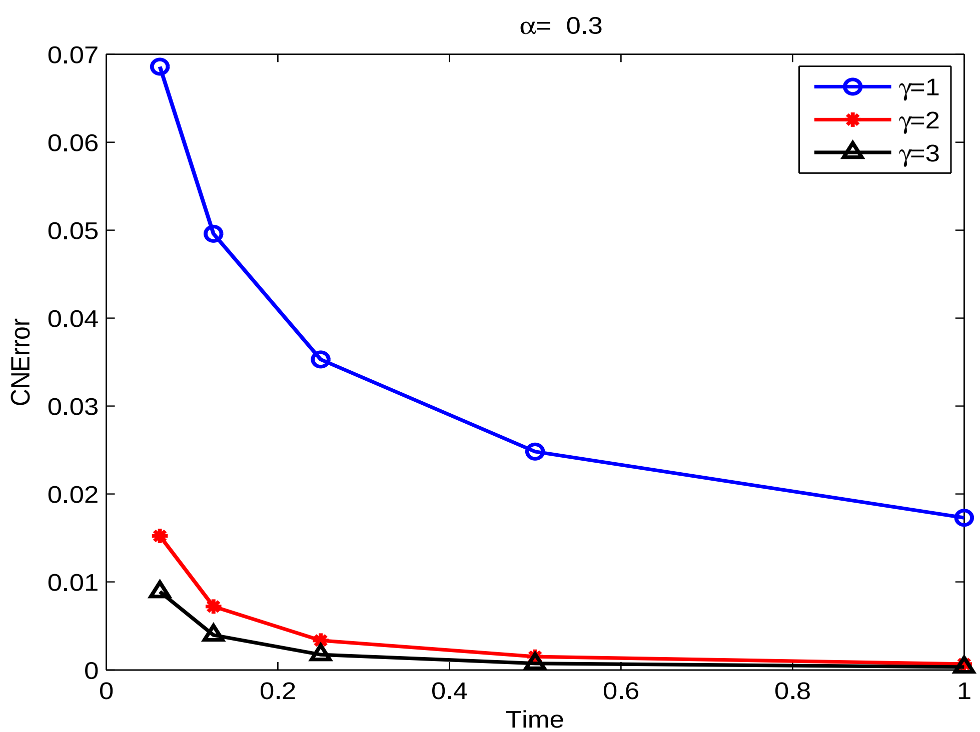


Figure 7.5: Error in the spatial domain $[0, \pi]$ for different time meshes using Crank-Nicolson scheme.

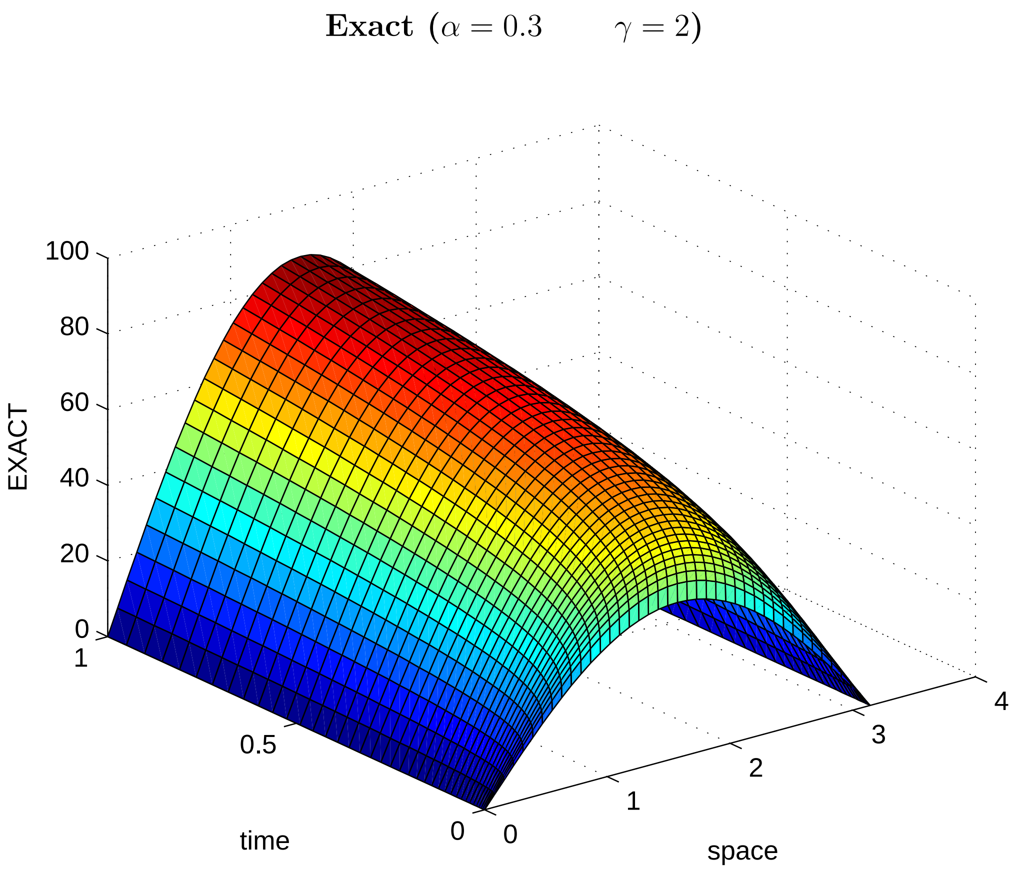


Figure 7.6: Exact solution in the spatial domain $[0, \pi]$.

**Numerical** ($\alpha = 0.3 \qquad \gamma = 2$)

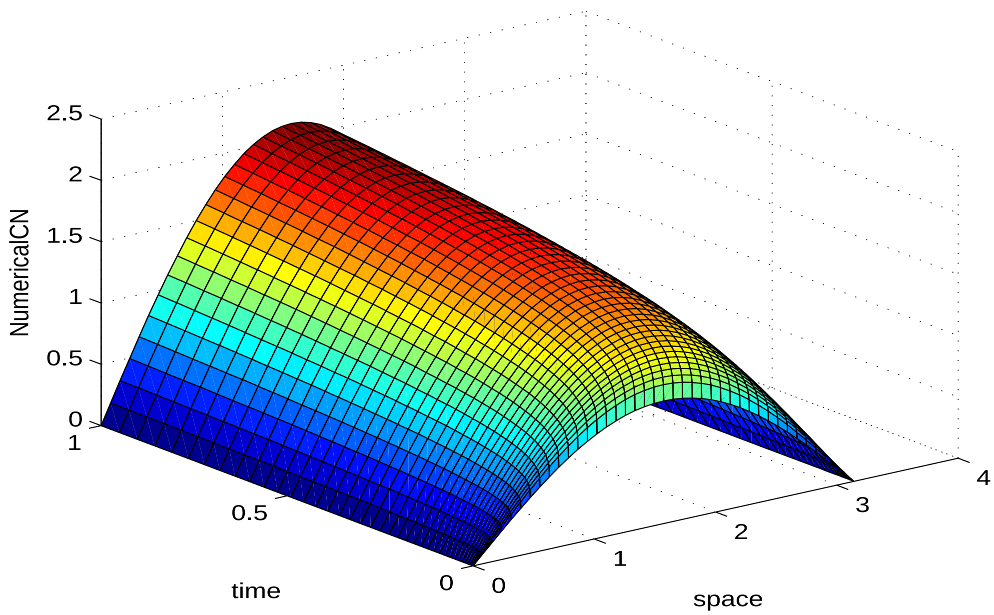


Figure 7.7: Numerical solution in the spatial domain $[0, \pi]$ using Crank-Nicolson scheme.

**Numerical** ($\alpha = 0.3 \qquad \gamma = 2$)

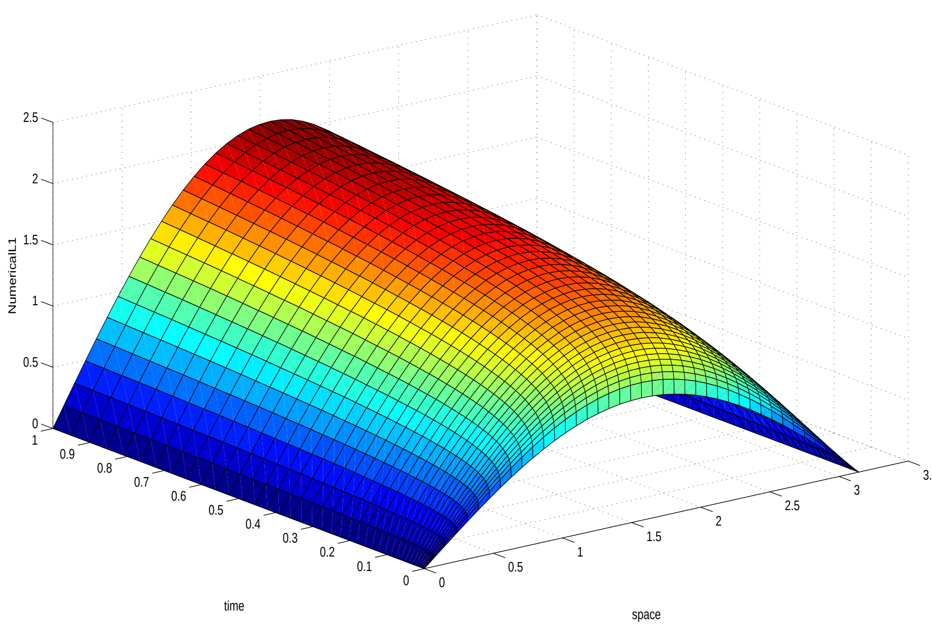


Figure 7.8: Numerical solution in the spatial domain $[0, \pi]$ using $L1$ scheme.

| $\alpha = 0.5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 1$ | | | | $\gamma = 1.5$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 1.81e-02 | | 6.32e-03 | | 4.81e-03 | | 1.85e-03 | |
| 40 | 9.96e-03 | 0.86 | 3.65e-03 | 0.79 | 1.834e-03 | 1.39 | 7.2e-04 | 1.36 |
| 80 | 5.38e-03 | 0.89 | 2.04e-03 | 0.84 | 6.86e-04 | 1.42 | 2.76e-04 | 1.38 |
| 160 | 2.84e-03 | 0.92 | 1.1e-03 | 0.89 | 2.5e-04 | 1.45 | 1.02e-04 | 1.44 |
| 320 | 1.29e-03 | 0.94 | 5.88e-04 | 0.9 | 9.02e-05 | 1.47 | 3.71e-05 | 1.46 |
| 640 | 7.66e-04 | 0.95 | 3.07e-04 | 0.93 | 3.23e-05 | 1.48 | 1.33e-05 | 1.48 |
| Theory | | 1 | | 0.5 | | 1.5 | | 0.75 |

| $\alpha = 0.5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 2$ | | | | $\gamma = 3$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 4.55e-03 | | 4.74e-04 | | 7.29e-03 | | 2.66e-04 | |
| 40 | 1.77e-03 | 1.37 | 1.33e-04 | 1.83 | 2.97e-03 | 1.3 | 8.96e-05 | 1.57 |
| 80 | 6.63e-04 | 1.41 | 3.7e-05 | 1.84 | 1.18e-03 | 1.33 | 2.9e-05 | 1.63 |
| 160 | 2.49e-04 | 1.41 | 1.02e-05 | 1.85 | 4.53e-04 | 1.38 | 8.02e-06 | 1.86 |
| 320 | 9.18e-05 | 1.43 | 2.71e-06 | 1.91 | 1.71e-04 | 1.41 | 2.12e-06 | 1.92 |
| 640 | 3.37e-05 | 1.45 | 7.05e-07 | 1.94 | 6.33e-05 | 1.43 | 5.4e-07 | 1.97 |
| Theory | | 1.5 | | 1 | | 1.5 | | 1.5 |

Table 7.2: Errors and convergence rates for different mesh grading $\gamma$ with $\alpha = 0.5$.

We observe better order for $L1$ scheme. The errors and convergence rates for Crank-Nicolson and $L1$ improved when the mesh is graded. We observe that the numerical results Crank-Nicolson scheme are as expected in Theorems (6.1, 5.2). However, the numerical results of the $L1$ scheme shows that the theoretical results are pessimistic.

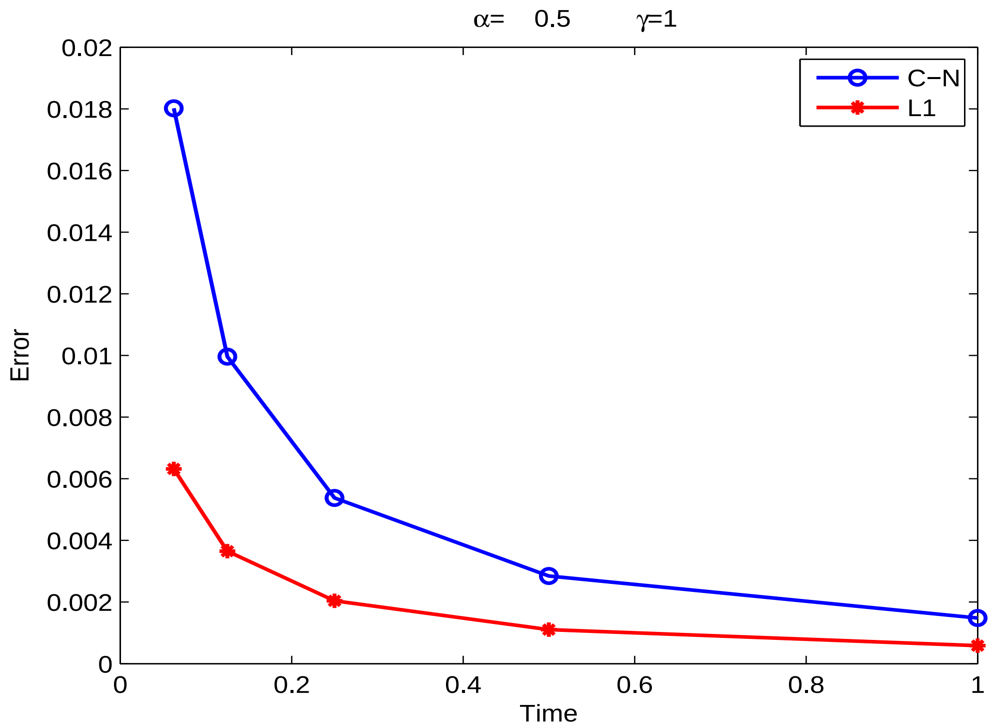


Figure 7.9: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

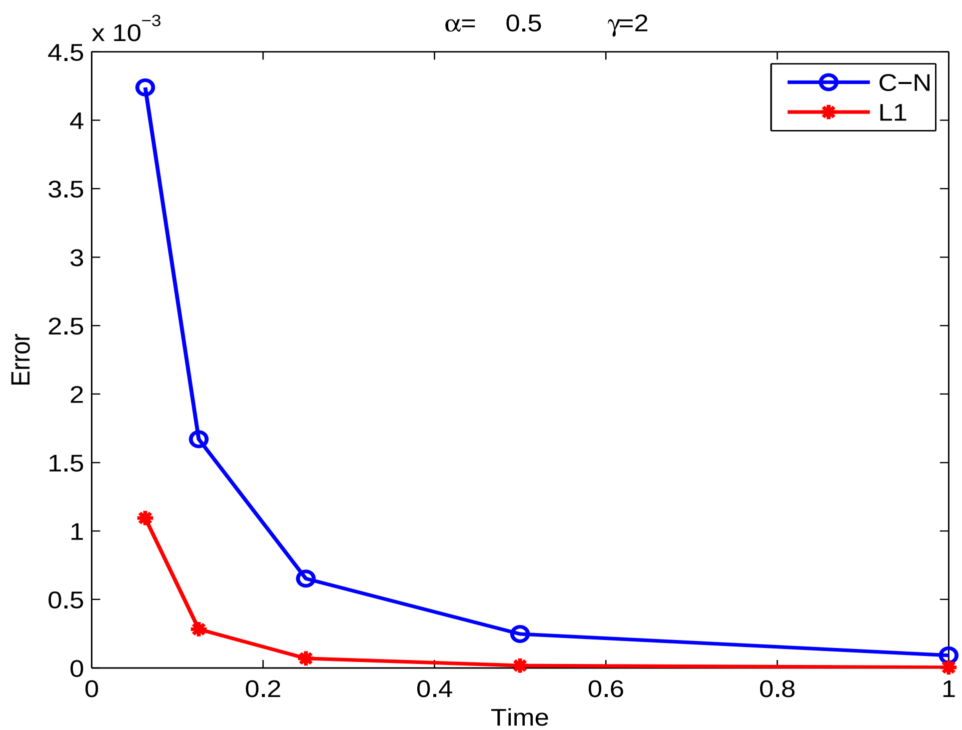


Figure 7.10: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

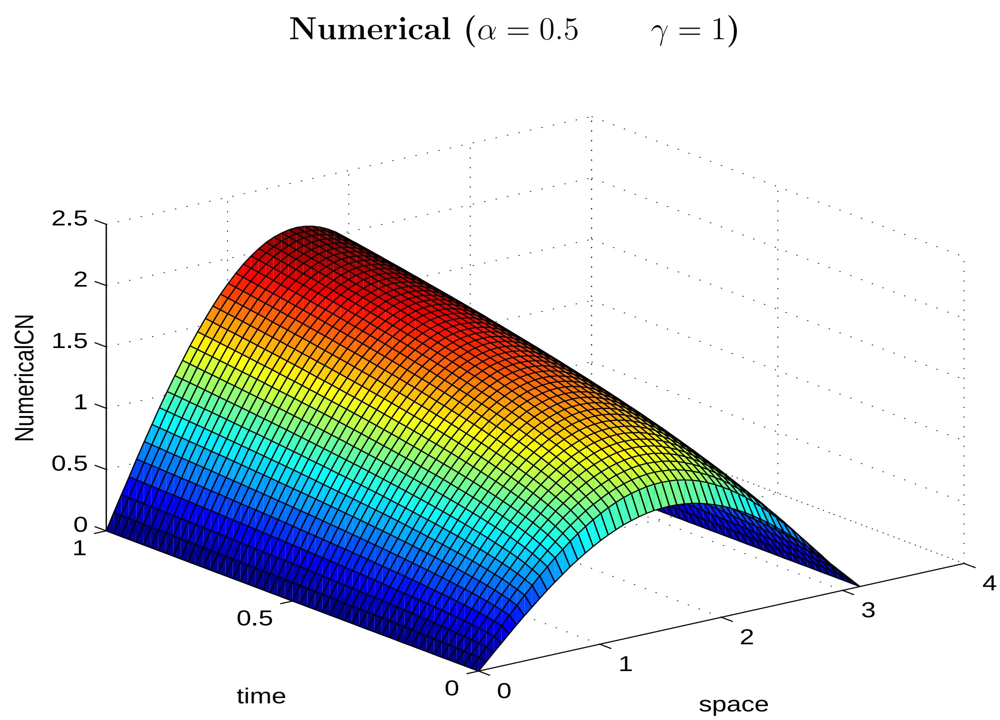


Figure 7.11: Numerical in the spatial domain $[0, \pi]$ using Crank-Nicolson scheme.

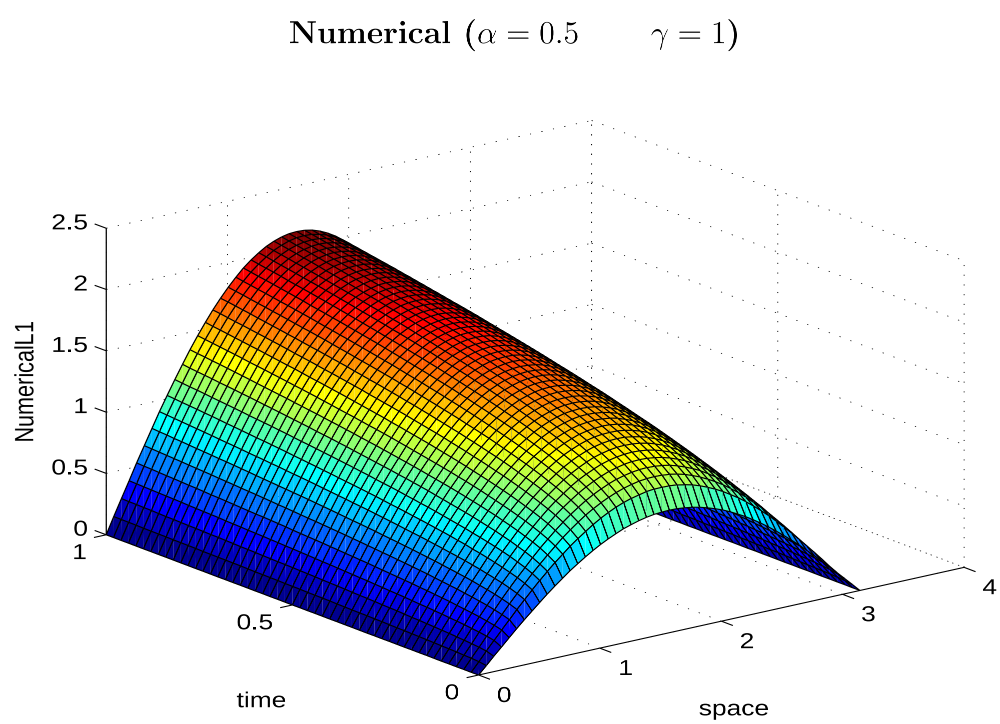


Figure 7.12: Error in the spatial domain $[0, \pi]$ using $L1$ scheme.

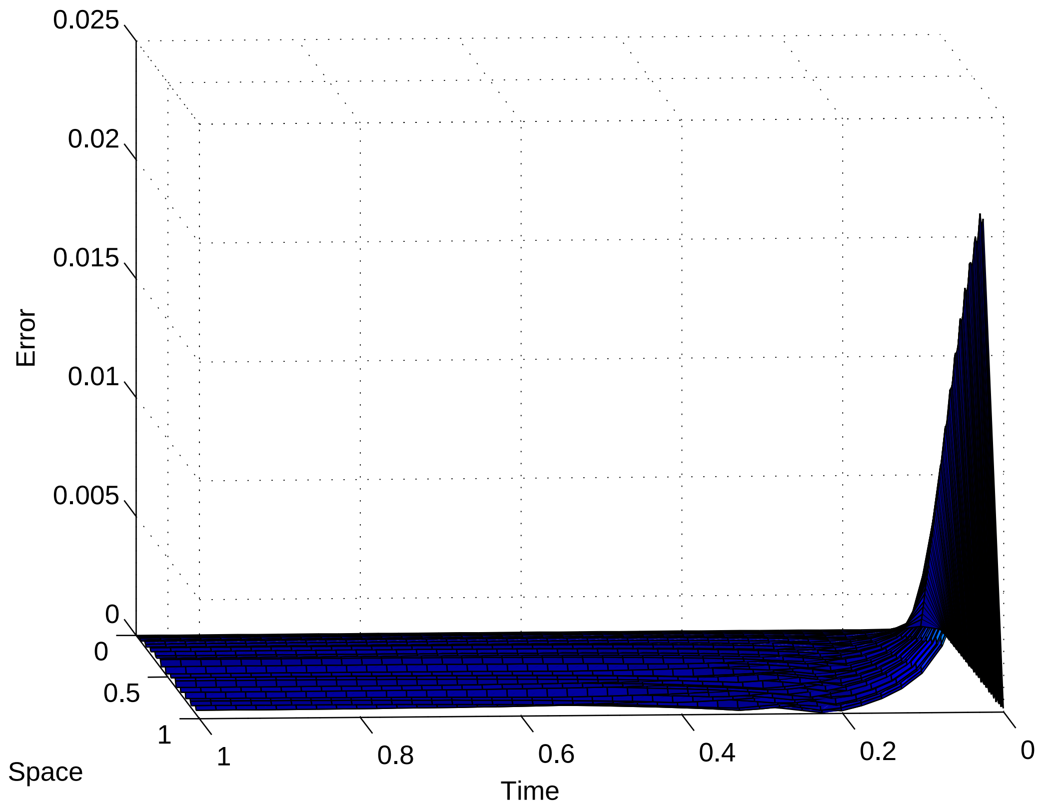


Figure 7.13: Error for $\alpha = 0.5$ and $\gamma = 2$ in the spatial domain $[0, \pi]$ using $L1$ scheme.

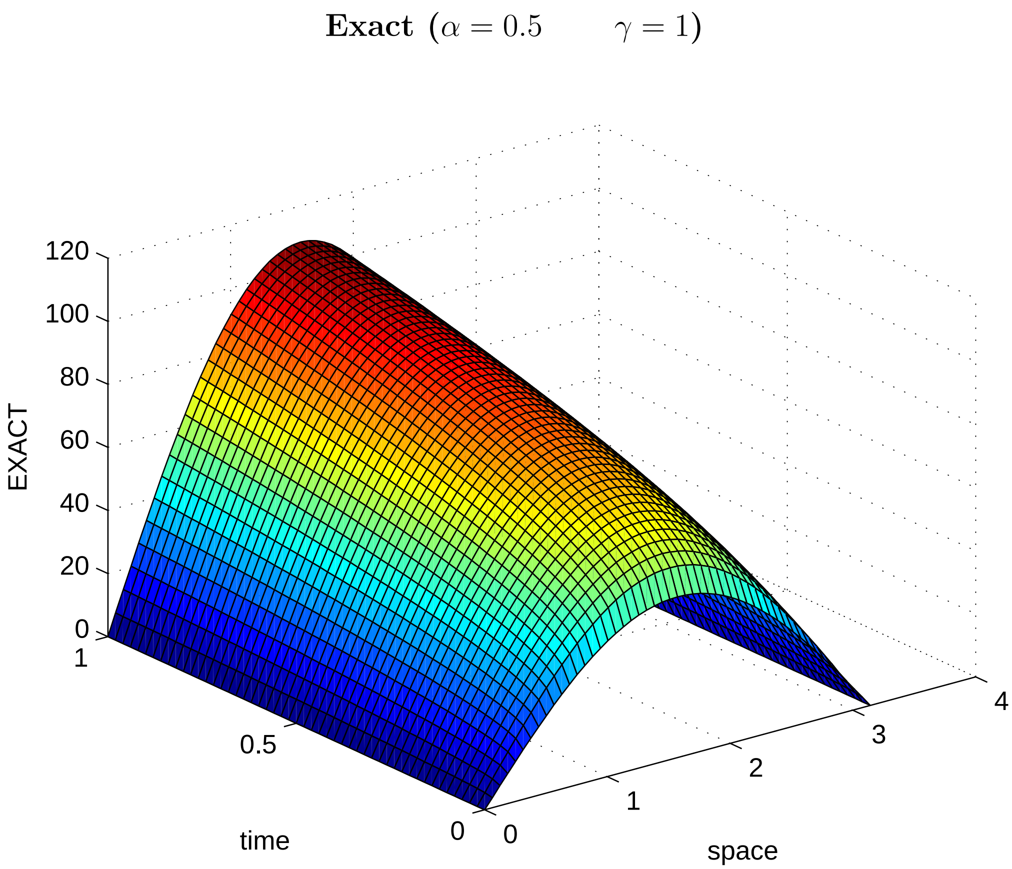


Figure 7.14: Exact solution in the spatial domain $[0, \pi]$.

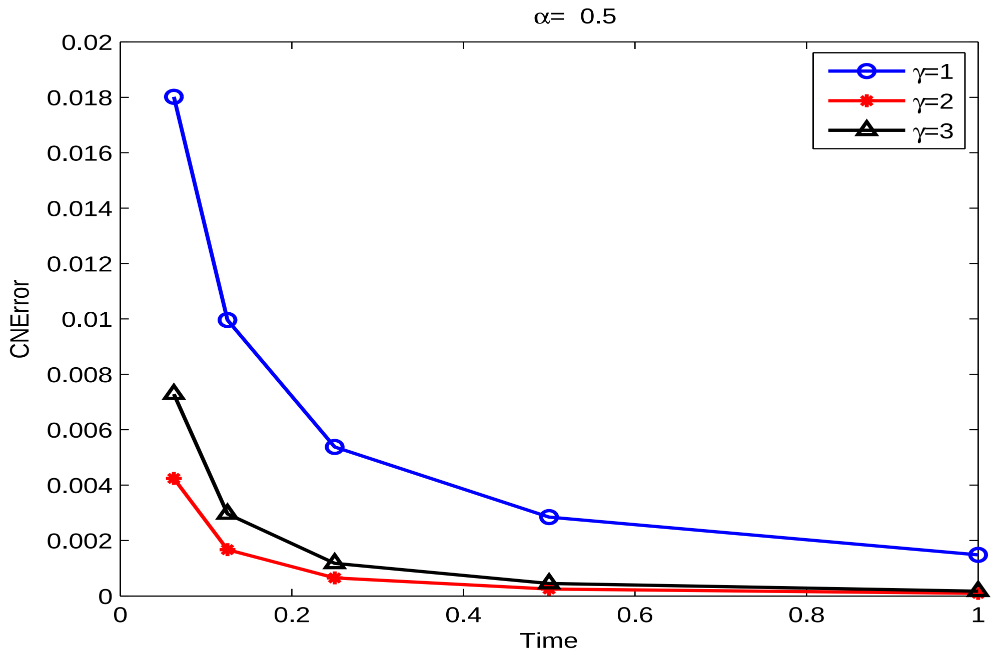


Figure 7.15: Error in the spatial domain $[0, \pi]$ for different time meshes using Crank-Nicolson scheme.

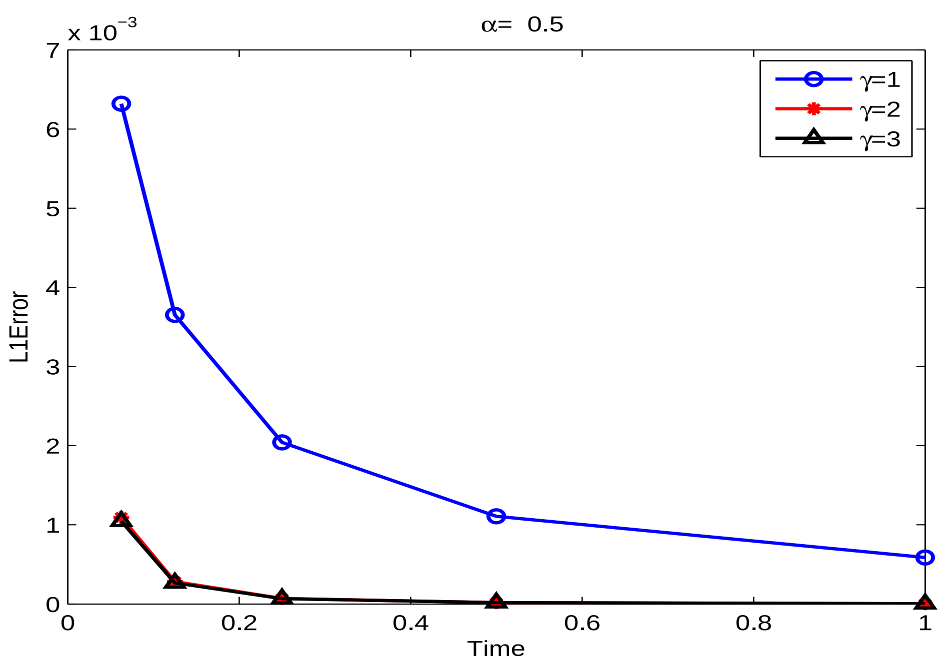


Figure 7.16: Error in the spatial domain $[0, \pi]$ for different time meshes using $L1$ scheme.

| $\alpha = 0.7$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 1$ | | | | $\gamma = 1.5$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 3.3e-03 | | 1.38e-03 | | 1.57e-03 | | 1.08e-03 | |
| 40 | 1.35e-03 | 1.29 | 5.631e-04 | 1.3 | 5e-04 | 1.65 | 2.8e-04 | 1.95 |
| 80 | 5.48e-04 | 1.31 | 2.34e-04 | 1.27 | 1.69e-04 | 1.56 | 7.1e-05 | 1.98 |
| 160 | 2.16e-04 | 1.34 | 9.39e-05 | 1.32 | 5.54e-05 | 1.61 | 1.78e-05 | 1.99 |
| 320 | 8.436e-05 | 1.36 | 3.71e-05 | 1.34 | 1.8e-05 | 1.62 | 4.45e-06 | 2 |
| 640 | 3.26e-05 | 1.37 | 1.45e-05 | 1.36 | 5.76e-06 | 1.64 | 1.11e-06 | 2 |
| Theory | | 1.4 | | 0.9 | | 1.7 | | 1.35 |

| $\alpha = 0.7$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\gamma = 2$ | | | | $\gamma = 2.5$ | | | |
| | $Crank - Nicolson$ | | $L1$ | | $Crank - Nicolson$ | | $L1$ | |
| | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ | $M.E.$ | $O.C.$ |
| 20 | 2.35e-03 | | 1.05e-03 | | 3.27e-03 | | 1.03e-03 | |
| 40 | 8.1e-04 | 1.54 | 2.72e-04 | 1.95 | 1.17e-03 | 1.48 | 2.64e-04 | 1.96 |
| 80 | 2.75e-04 | 1.56 | 6.9e-05 | 1.97 | 4.13e-04 | 1.5 | 6.71e-05 | 1.97 |
| 160 | 9.2e-05 | 1.58 | 1.73e-05 | 1.99 | 1.4e-04 | 1.56 | 1.68e-05 | 1.99 |
| 320 | 3.01e-05 | 1.61 | 4.35e-06 | 1.99 | 4.6e-05 | 1.6 | 4.21e-06 | 1.99 |
| 640 | 9.7e-06 | 1.63 | 1.08e-06 | 2 | 1.48e-05 | 1.63 | 1.05e-06 | 2 |
| Theory | | 1.7 | | 1.8 | | 1.7 | | 2 |

Table 7.3: Errors and convergence rates for different mesh grading $\gamma$ with $\alpha = 0.7$.

We observe better order for $L1$ scheme. The errors and convergence rates for Crank-Nicolson and $L1$ improved when the mesh is graded. We observe that the numerical results of Crank-Nicolson schemes are as expected in Theorems (6.1, 5.2). However, the numerical results of the $L1$ scheme shows that the theoretical results are pessimistic.

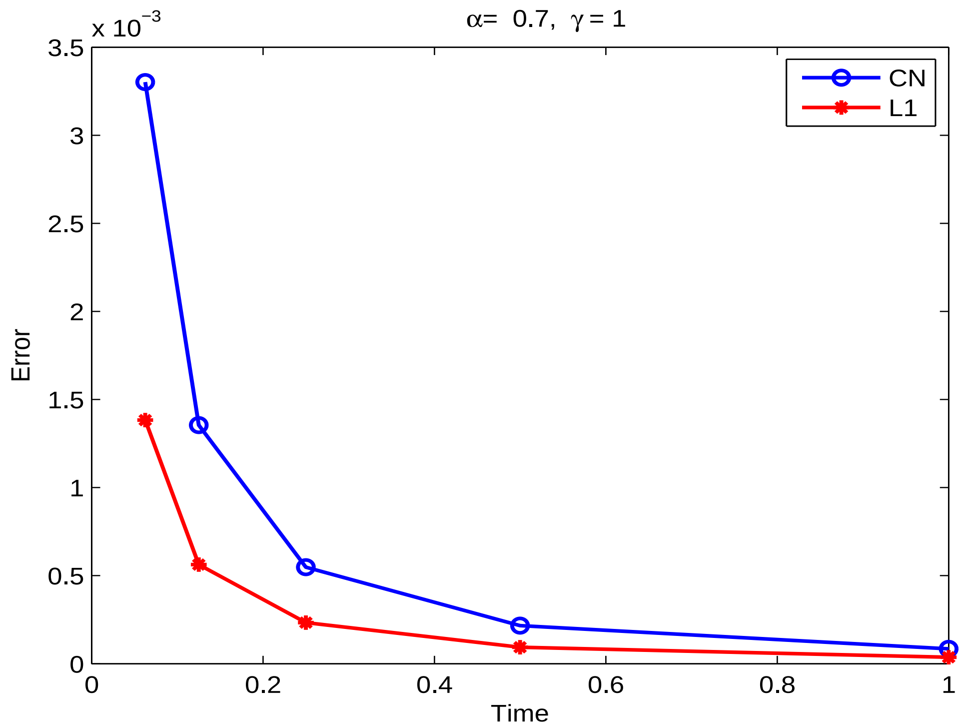


Figure 7.17: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

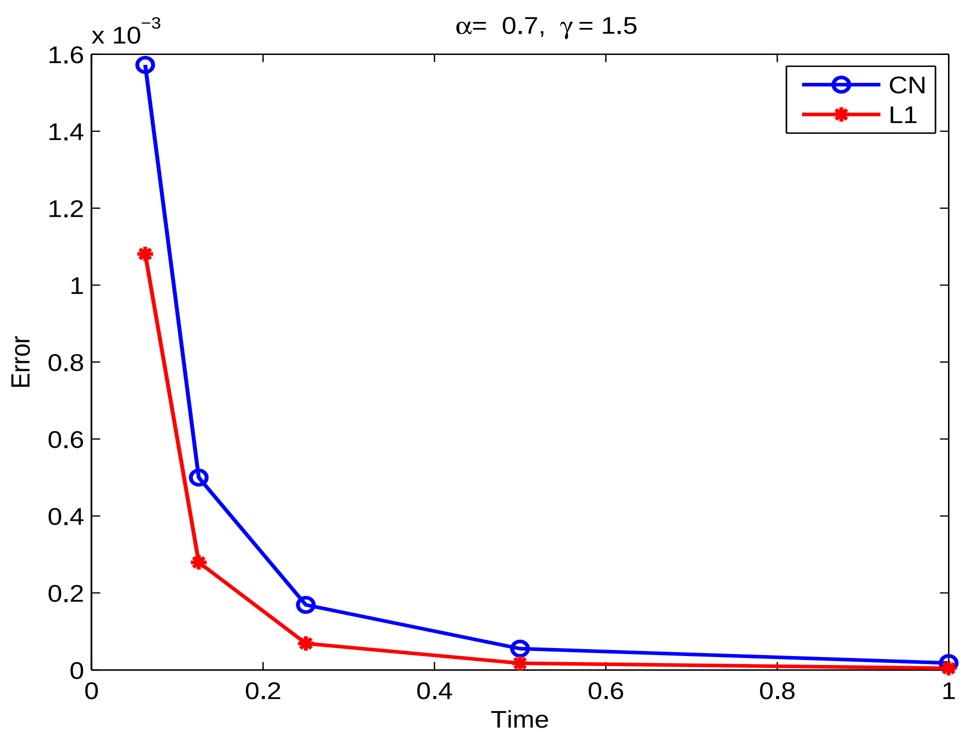


Figure 7.18: Error in the spatial domain $[0, \pi]$ using both $L1$ and Crank-Nicolson schemes.

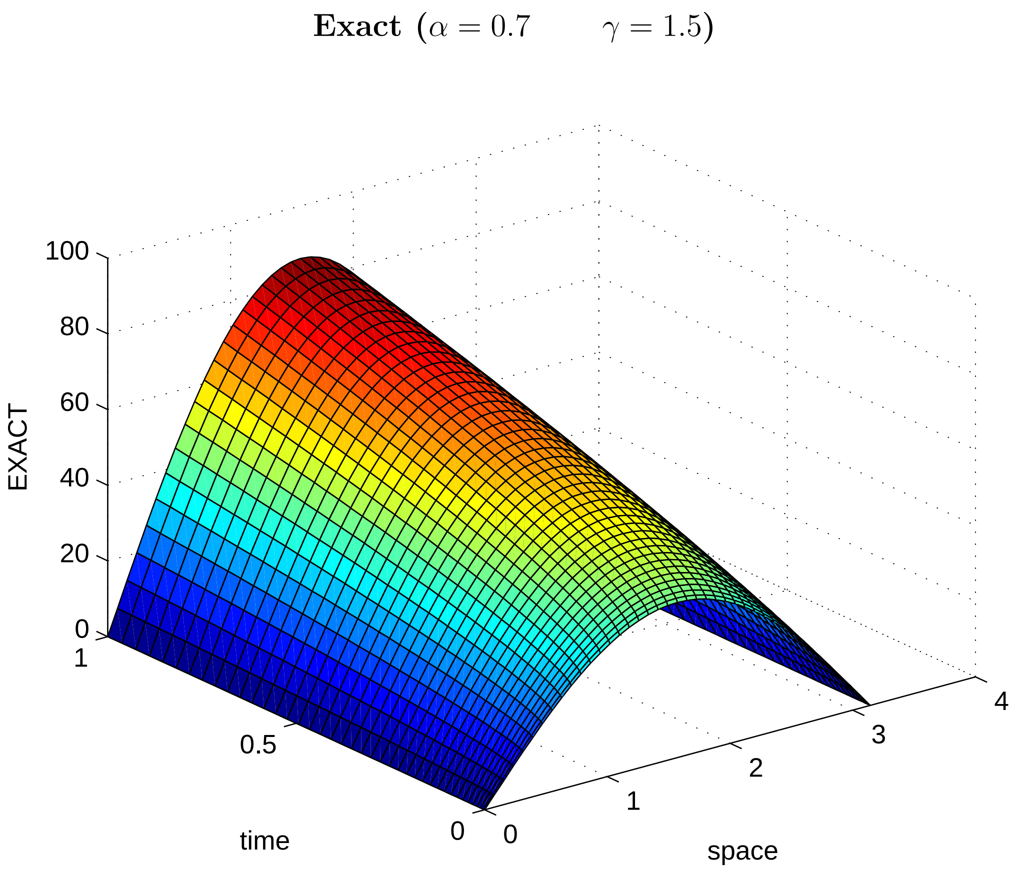


Figure 7.19: Exact solution in the spatial domain $[0, \pi]$.

**Numerical** $(\alpha = 0.7 \qquad \gamma = 1.5)$

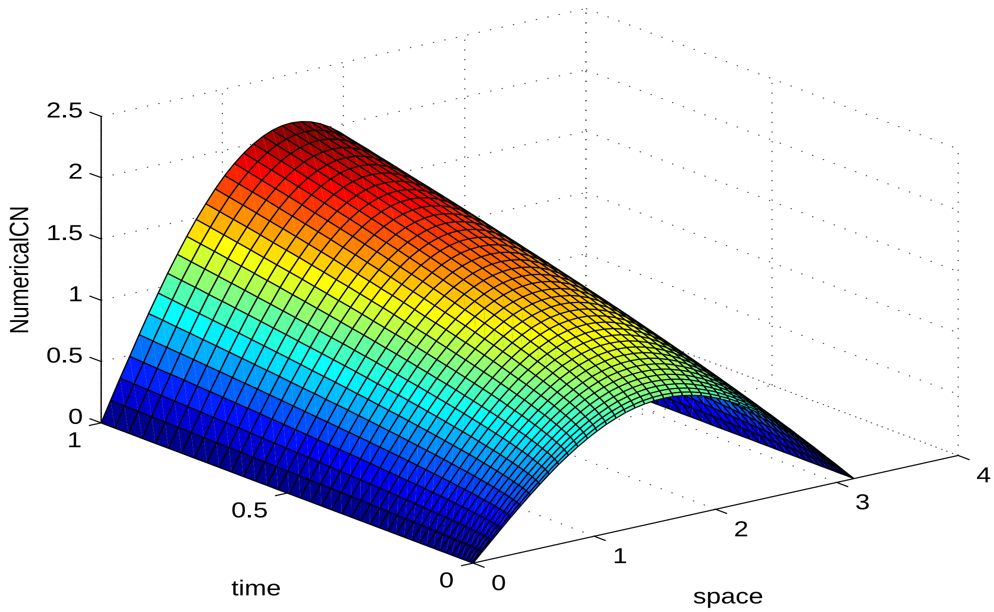


Figure 7.20: Error in the spatial domain $[0, \pi]$ using Crank-Nicolson scheme.

**Numerical** $(\ \alpha = 0.7 \qquad \gamma = 1.5)$

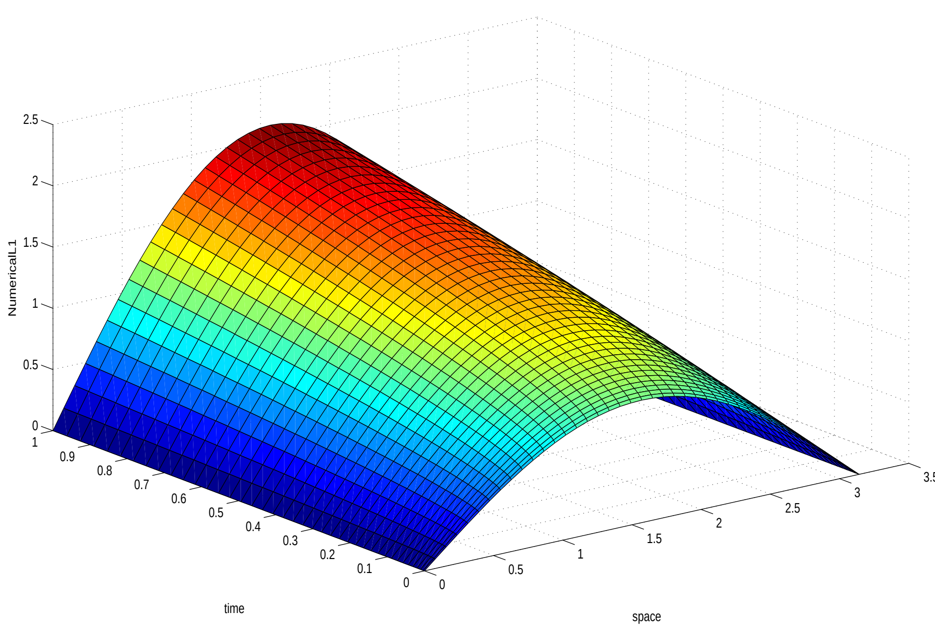


Figure 7.21: Error in the spatial domain $[0, \pi]$ using $L1$ scheme.

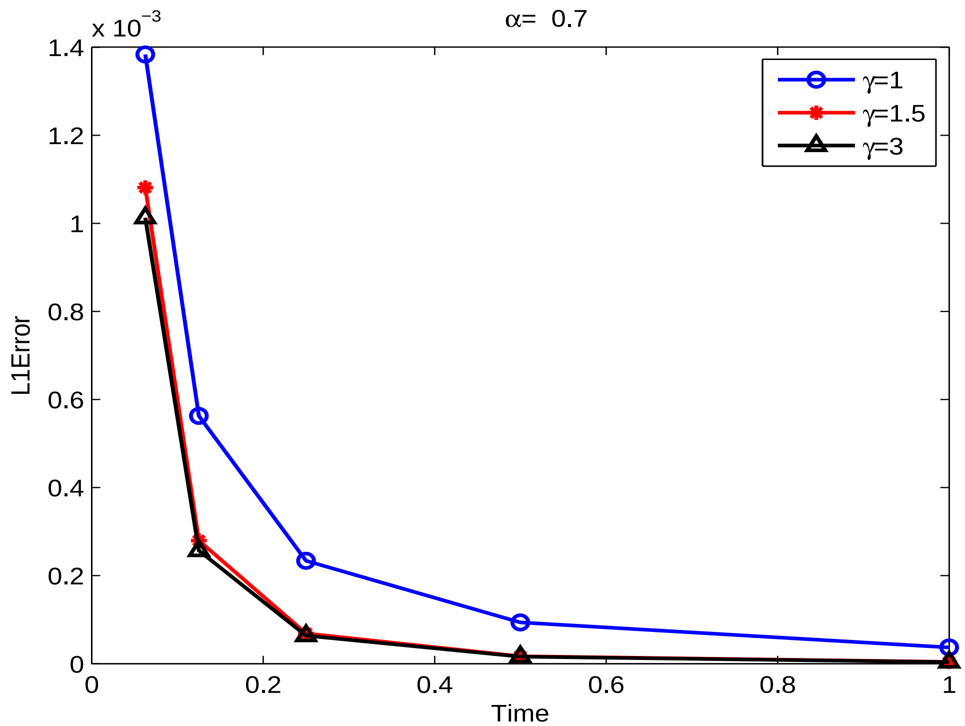


Figure 7.22: Error in the spatial domain $[0, \pi]$ for different time meshes using $L1$ scheme.

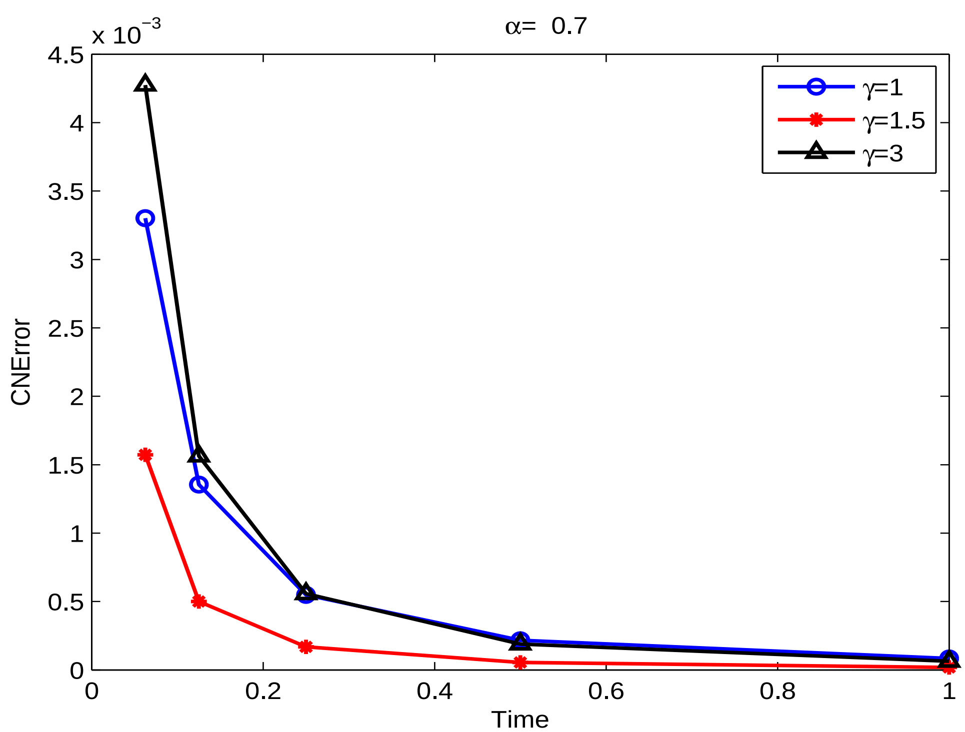


Figure 7.23: Error in the spatial domain $[0, \pi]$ for different time meshes using Crank-Nicolson scheme.

## 7.4 Conclusions and future work

We established the existence and uniqueness of the weak solution for the general form of the model problem (1.1.1) in the case of space-time dependent driving forcing via Galerkin method. Furthermore the behavior of the time derivatives of the weak solution was studied, proving estimates that play an important role in the error analysis of the numerical schemes. For the numerical solution of the model problem (1.1.1), an implicit Crank-Nicolson scheme to discretize in time was proposed such a scheme is formally second-order accurate. However due to the presence of a weakly singular kernel and the fractional derivative operator $\partial_t^{1-\alpha}$, we only proved an $O(k^{1+\alpha})$ convergence for $0 < \alpha < 1$ in the case of non-uniform time meshes, where $k$ denotes the maximum time step. A fully discrete scheme that combined finite elements in space with Crank-Nicolson in time was proposed, and the existence and uniqueness of the solution of the fully discrete scheme was proved. We introduced another numerical scheme based on $L1$ approximation in time and finite elements in space and we performed the error analysis for the fully discrete scheme. We got results better than the first method, we got an order of $O(k^2)$ convergence rate in the case of non-uniform time meshes.

In comparison of the previous work regarding the convergence rate we find that our results is better than the work done by Le et al. [29] in their work they proved an $O(k^\alpha)$ order of convergence. However in our numerical methods we got $O(k^{\alpha+1})$ using Crank-Nicolson method and $O(k^2)$ using $L1$ approximation scheme.

For the future work we will investigate the numerical solution of the time-fractional Fokker-Planck equation in the case of non-smooth initial data.

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