WAREHOUSE SELECTION AND INVENTORY OPTIMIZATION

BY

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In dedication to my parents, who have been my source of inspiration and gave me strength during this work, who continually provide their moral, spiritual, emotional, and financial support.

To my brothers, mentors, friends, and classmates whose words of encouragement, and advice always eased my journey toward the MS degree.

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ABSTRACT

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Warehouse capacitated inventory optimization problems are rarely addressed in the literature. This is because capacity constraints imposed by warehouses can always be lifted by leasing or owning more warehouses, and also because warehousing expenses are considered negligible compared to the whole problem. Hence, problems that deal with inventory optimization and warehouses are usually neglected in the literature due to the lack of real–life applications. However, in land-scarce regions, land acquisition and upkeep are becoming more and more expensive, mainly due to population growth. This growth is anticipated to make warehousing a major problem, where leasing warehouses is the only viable option for small business owners to survive. Awarding longer leasing contracts with cheaper rates is the main rivalry tactic between warehouses. This is where this proposed work comes to the benefit of the business owners, by aiding them in selecting the optimal ordering and warehousing plan. Ultimately, this helps in competing business environment. This work will introduce a new capacitated inventory optimization problem called the Warehouse Selection and Inventory Optimization (WSIO) problem. The work includes developing mathematical models for both the deterministic and stochastic demand cases, developing exact and heuristic solution methods to solve the WSIO problem, suggesting ideas to speed up the solution process, and finally presenting interesting insights and observation about the WSIO problem through experimental work.

ملخص الرسالة

الاسم الكامل: إبراهيم تركي التركي

عنوان الرسالة: تحقيق الأمثلية في إدارة المخزون واختيار المستودعات

التخصص: هندسة النظم الصناعية

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نادرا ما يتطرق الباحثون في مجال تحقيق الأمثلية إلى المسائل الرياضية المتعلقة بإدارة المخزون المقيدة بمساحة تخزين ثابتة متوفرة من قبل مجموعة مستودعات. السبب في ذلك يعود إلى كون تلك القيود قابلة للثني عن طريق التوسع بمساحة التخزين المتوفرة عن طريق استئجار مستودعات إضافية، وكذلك لرخص تكاليف إيجار المستودعات مقارنة بالمصاريف التشغيلية الأخرى. لتلك الأسباب ولقلة التطبيقات الحياتية والحاجة العملية لبحث علمي في هذا المجال، تكاد تضمحل الأعمال الأكاديمية في مجال تحقيق الأمثلية في إدارة المخزون المقيدة بمساحة تخزين المستودعات. غير أن في البلدان المكتظة بالسكان والشحيحة بالمساحات السكنية والتجارية، تجد أن التكاليف المتعلقة بامتلاك وإبقاء مساحات شاسعة للتخزين في ارتفاع مستمر، ويعود السبب في ذلك لجهود الجهات التنظيمية هناك لفك الاحتكار القائم على تلك المساحات ولمواجهة أزمات الإسكان والتوسع العمراني. في مثل هذا المناخ، قد تقيد شركات الأعمال بالذات صغيرة الحجم منها من امتلاك مستودعات تخزين خاصة بها ، واللجوء على الاعتماد على طرف ثالث موفراَ لخدمة التخزين كبديل حيث أن الإستراتيجية التنافسية المتوقعة بين موفري خدمة التخزين لاجتذاب شركات الأعمال هي مكافأة العقود الطويلة برسوم إيجار أرخص. هنا يكمن هدف هذا العمل بمساعدة شركات الأعمال في تحقيق الأمثلية في إدارة المخزون واختيار المستودعات لتحقيق خطة تعود في النهاية بأفضل الأرباح. هذا العمل سيطرح مسألة رياضية جديدة بمسمى " تحقيق الأمثلية فإدارة المخزون واختيار المستودعات "، وسيقوم بتوفير نموذجين رياضيين إحداهما للمسائل الحتمية والأخر للمسائل العشوائية، وهذا يتضمن طرح طرق مختلفة لحل النموذجين سواء بطرق حتمية أو تقريبية مع نصوصها البرمجية، مع مناقشة طرق مختلفة لتسريع حل النموذجين برمجيا، وحل عدة أمثلة عددية مع مناقشة نتائج حلها.

CHAPTER 1

INTRODUCTION

In this current age, most corporations operate in highly populated land-scarce regions, where they tend to suffer from an escalating financial hemorrhaging when expanding horizontally. Especially for the corporations that invested in business areas that require large warehouses, like in retail and logistics. This escalation is due to the continuous increase in land prices, and government taxation efforts to reduce land seizure. Therefore, corporations tend to escape this managerial and financial nightmare by outsourcing their storage needs instead of acquiring new warehouses or expanding the existing ones. Primarily, outsourcing comes in the form where corporations lease warehouses from a third-party company at a mutually agreed price for a certain lease duration. Typically, the third-party companies are able to turn a profit — as oppose to corporations not invested in storage services — due to various reasons, some are technological, and others are managerial. For instance, a heavy capital investment in advanced automated storage and retrieving technology will allow for a full utilization of space, reduced upkeep expenses, and eventually a profit. Another possible reason is having a policy to sublet a single large warehouse to several corporations with small storage requirements. Regardless of the mechanism the storage service providers use to turn a profit, the main concern here is "how corporations can optimally utilize the warehouse services?". Now, lease contracts are mostly dependent on the provided storage space, the service provider pricing, and their leasing policy (whether they reward longer contracts with lower leasing rates or not). For a retailer seeking to expand, the emergence of many leasing options will urge them to reconsider their whole supply chain business plan, especially their ordering policy. In a nutshell, the new business plan must be able to provide answers to the simple questions: Should we expand? Should we lease warehouses? Which warehouses to lease? When to lease them and for how long? How much to order for each product? These questions can pose a challenge for any decision maker, especially if:

- the storage size for each product is different,
- the warehouses have different sizes and lease policies,
- the demand for each product is different from one period to another and possibly is uncertain, and
- the storage service providers reward longer contracts.

The decision problem with the above issues is referred to in this paper as the Warehouse Selection and Inventory Optimization (WSIO) problem. The WSIO problem with the above-described complexity is beyond the scope of the classical Economic Ordering Quantity (EOQ) models and requires further investigation. This work finds its motivation from the expected population growth levels, and land–scarcity in many highly–populated regions across the globe, such as in Honk–Kong, Tokyo, and New York. This growth is anticipated to create a climate where leasing warehouses is the only viable option for small business owners to survive, and where awarding longer leasing contracts with cheaper rates is the main rivalry tactic among the warehouse owners to the attract business owners. This is where this work comes to the benefit for the business owners to aid them in selecting the optimal plan, and ultimately survive in the competitive business world.

This thesis's objective is to introduce the WSIO problem for multi–periods inventory problems with multi–products that have time dependent selling prices, develop mathematical models for both the deterministic demand case and stochastic demand case, suggest exact and develop heuristic solution methods to solve the WSIO problem, suggest ideas to speed up the solution time, and finally draw interesting insights and observation about the WSIO problem through experimental work.

The rest of this work is organized as follows: In Chapter 2, a concise literature review is presented. In Chapter 3, a deterministic mathematical model is developed, and ideas to speed up the solution process are suggested. In Chapter 4, a stochastic model is developed. In Chapter 5, exact and heuristic solution methods to solve WISO problem are developed and presented. In Chapter 6, experimental performance of the proposed solution methods is illustrated through solving several numerical examples on the WSIO problem. In Chapter 7, a conclusion and a summary about this work is presented.

CHAPTER 2

LITERATURE REVIEW

In this chapter, a review is presented on the literature that focus on the connectivity between inventory optimization, warehousing, and demand uncertainty. This overlap area of research was found to be better represented by the following three categories: First category, research work done on Hartley's two warehouse model. Second category, research work done on inventory optimization and warehousing that is unrelated to Hartley's model. Third category, research that involves introducing demand uncertainty to capacitated inventory optimization problems. Hence, this chapter is divided into five sections: The first three sections each will discuss one category from the three categories above, the fourth section will discuss the gap found in the literature, and the fifth section will present the WSIO problem statement and its role in filling the literature gap.

2.1. Hartley's Two–Warehouse Model

The first to relate the Economic Ordering Quantity (EOQ) model to warehouses was Hartley in 1976 [1]. He conferred a simple two-warehouse model, one warehouse he referred to as Owned Warehouse (OW), and the other one as Rented Warehouse (RW). His work paved the way for many researchers interested in warehouse inventory optimization problems. Here are few examples: Sarma in 1983 [2] expanded Hartley's original model by considering transference cost between the two warehouses and proposed different reordering rules. In 1987 [3], Sarma provided further expansion to Hartley's model by accommodating for deterioration effect on the two warehouses. In 1992, Chaudhuri and Goswami [4] extended Sarma's models to include demand that varies linearly with time. Later, numerous other researchers followed by developing similar two-warehouse inventory models where each considered different factors such as shortages, deteriorating items, stock level-dependent demand, inflation rate, time value of money, finite production, and finite time horizon, (see for example Pakkala and Achary [5] , Maiti and Bhunia [6], Kar, Bhunia and Maiti [7], Yang [8], Zhou and Yang [9], Hsieh, Dye and Ouyang [10]). Furthermore, beside extending Hartley's model, there has been other type of papers inspired by the two-warehouse model. For instance, Chung, Her, and Lin [11] work where they converted Salameh and Jaber's [12] single warehouse model with imperfect products to a two-warehouse model. Also, Lee, M and Elsayed, E. [13] work where they provided their own NLP formulation and solution procedure for a two-warehouse problem with warehouses that operates under a dedicated storage policy, and a full-turnover-based storage policy. The latest work extensions to the two-warehouse problem was Moncer's [14] and Sana's [15]. The former extended previous work to accommodate for the two options: long fixed contract, and flexible contract, and the latter provided a formulation for the two-warehouse problem when demand is uncertain following the newsvendor concept.

In the above section, published papers on Hartley's two warehouse model was reviewed. The review unveils that Hartley's two warehouse model's still an active research area, and is the dominant area when looking up for the topics in inventory optimization and warehousing.

2.2. Capacitated Inventory Optimization Problems

Although, Hartley's model appears to be the most popular work on warehouse inventory optimization, there have been other contributions to this field that are not based on Hartley's model. For instance, Zhang, Zhu, and Hu [16] provided a different approach on the warehouse optimization problem. They proposed a simple mathematical model that provides warehouse owners looking to expand their capacity with the optimal decision between expanding current warehouses or leasing new warehouses. Another example is Jucker, Kropp, and Carlson [17] work where they considered leased warehouses in the classical plant-region allocation problem. They considered a company manufacturing a single product, and planning to increase production volume by building a plant to supply new districts, each to be supplied by a local warehouse. Local warehouses are rented in a way that no fixed costs are associated with the warehouses. The goal is to select the plant and warehouse capacities which maximizes the net profit, with no stock-outs due to insufficient plant capacity. Also, the plant installation cost is nonlinear w.r.t. its capacity, and the warehouses lease costs are linear w.r.t its capacities. They also tackled uncertainty using expectations to replace stochastic variables or parameters. Moreover, Ng et al. [18] provided a closed form solution to a capacitated EOQ problem, where the decision variables are the batch sizes for each period, and the warehouse capacity size. In their model, it is assumed that warehousing costs dominate non-warehousing costs. Goh et al. [19] provided a closed form solution to a problem slightly similar to the one proposed in our work, and an iterative algorithmic solution for a complicated variation of the problem. The problem described in their paper requires solving simultaneously an inventory problem, and a warehouse sizing problem, where the demand is known, and constant. In addition, the warehouses are leased for a fixed time period equal to the demand rate time unit, with a lease cost that is a step function of the warehouse size. The closed form solution was for a single product, where the algorithm was for the multi-products. Mousavi et al. [20] gave a model for a multi-product multi-period inventory control problem under an all-unit discount policy and inflation, all constrained by a limited capacity and a dedicated budget. The problem was solved using a particle-swarm based algorithm.

In the above section, published papers on capacitated inventory optimization was reviewed, while excluding all work relevant to Hartley's model. The review unveils that once Hartley's two warehouse model is cast aside, the remaining research areas are (A) Warehouse–Plants allocation problems, (B) Warehouse capacity design problems, (C) Automation in the warehouse area, and (D) Buy or expand decision problems. This shows the lack of research on capacitated inventory optimization problems. Specially, in the variety in of handling warehouse capacity.

2.3. Stochastic Capacitated Inventory Optimization Problems

Although, most capacitated EOQ models in the literature assume all parameters are known for certain, there are some researchers who investigated the capacitated EOQ problems when one or more parameters are not known for certain. In 1988, Rosenblatt and Roll [21] tackled uncertainty using simulation, where an (s, Q) inventory policy and a random storing strategy were assumed, s being the reordering inventory level and Q the ordering quantity.

In particular, the warehouse capacity essential to uphold a certain service–level was discovered to be directly related to the reorder quantity, and the average everyday demands, and inversely related to the number of products, reorder points and the inconsistency in the day-to-day demand. A multiplicative regression model shows that the last two factors have only a negligible influence. Sungur [22], Sungur, Ordonez, and Dessouky [23] tackled a real-life allocation problem where vehicles with limited capacities are allocated to the destinations with uncertain demand. Ordonez and Zhao [24] examined the robust capacity growth problem of network flows under travel time and demand uncertainty. Atamtürk and Zhang [25] studied the design problem and network flow under demand uncertainty, with applications to location–transportation and lot-sizing problems. Paolo [26] provided a mathematical model for a multi-item time capacitated multi-period lot-sizing problem with uncertain demand. He investigated different existing heuristics to solve the scenarios-tree based model and discussed their efficiency and effectiveness. In addition to the above, comprehensive review papers that deals with capacitated inventory problems were taken into consideration while reviewing this topic. Among them is the review done by Gabrel, Murat, and Thiele [27] which covers all the recent inventory optimization problems focused on uncertainty and robustness. Also, there is the most recent review done by Díaz-Madroñero, Peidro, and Mula [28] that covers the recent development in tactical optimization models for the integrated production, warehousing and transport routing planning decisions.

In the above section, published papers on stochastic capacitated inventory optimization were reviewed. The review unveils that there is lack of research on stochastic capacitated inventory optimization problems, especially in the recent years.

2.4. Gap Analysis

The literature review presented in Sections 2.1, 2.2, and 2.3 reveals a gap in the literature that concerns capacitated inventory optimization problems, especially problems that involves warehouses. The gap does not exist due to a lack of research done on capacitated inventory optimization, but due to the majority of this research being extensions to previous work. This led most research about capacitated inventory optimization and warehousing to be extensions to Hartley's two–warehouse model, or the warehouses allocation model, or the warehouse capacity inventory–based design model. Of course, there have been other research efforts about warehouses, but most of them are irrelevant to our work, and hence, they were dismissed from our literature review. For example, facility layout optimization problems, and research about automating warehouses. Hence, although the literature is rich with published papers about capacitated inventory optimization problems, the variety of the original models these papers are based upon is limited. In Hartley's model, the capacity is presented as a fixed starting resource that is extendable by a fixed amount through an option called *Rented–Warehouse*, and the problem mainly revolves around the question, should the *Rented–Warehouse* option be selected? Of course, this involves other decisions like how much to order? when to order? and the other usual inventory optimization questions. In the allocation problem, the capacity is also presented as a fixed resource, but now the question is how to allocate this resource to maximize the decision maker's goal. In the warehouse–design problem, the capacity is no longer a fixed resource, but rather is a first–stage decision that cannot be altered at later stages, hence, the varieties of future decisions will be limited by that first stage decision. To illustrate the lack of original models, observe how the capacity of the warehouses is represented in these three research areas that dominate the topic inventory optimization and warehousing. In all three problems, the capacity is represented as a fixed resource or as a first stage decision, which certainly does not accommodate for all the real–life warehouses capacitated inventory optimization problems. The gap also doubles in size when considering the variety of existing uncapacitated inventory optimization problems, and the different assumptions they can have (e.g. fixed demand, and continuous demand).

This work aims to reduce the above gap by introducing the WSIO problem, where capacity is represented in a unique way, and assumptions are made so that a family of real–life problems that have not been addressed before in the literature can be addressed by our work.

2.5. Problem Statement

Nowadays, corporations tend to favor leasing warehouses over owning and maintaining warehouses. Specially, corporations that are located in highly–populated land–scarce regions. The reasoning for this is twofold: First, leasing warehouses offer more flexibility and mobility. The corporations can liquefy their assets, modify their supply chain business plans, expand or shrink their operations, or even simply switch warehouses, all at much faster pace. Second, leasing warehouses is mostly favored because it is a risk-averse strategy. That is, it shields corporations from any potential financial risks associated with investing upfront on owning a warehouse, or any other post–ramifications like increased taxation on owned lands, or sudden drops in warehouses salvage value. To illustrate, contemplate the following example: a small retailer invests his capital in owning a large warehouse, and starts to import products that are new to the region and have high demand rates. He starts to make profit, but suddenly his market share drops significantly because other retailers notice the trend and join in. Suddenly his business plan becomes unprofitable due to the competition. Now, because the retailer invested all his capital upfront in owning the warehouse, his ability to endure loss is weaker and his options are limited. This is a single example among many other examples where leasing a warehouse for a small business owner is certainly a better strategy than owning one. However, with a leasing strategy in mind, more options are available, and hence more questions are to be answered. The WSIO problem tackles these questions mathematically and when solved offer an optimal solution to these questions.

The WSIO problem assumes that a corporation is seeking to maximize its total profit by selling multiple products. However, the total demand for each product is different from one period to another, and possibly stochastic (independent or correlated). Each product has different ordering cost, holding cost, purchasing cost, lost sale cost (opportunity cost), storage space requirements, selling price, and selling price depreciation rate with time. Furthermore, the corporation needs to choose from several available warehousing options at different time periods. Each leasing option is characterized by its warehouse capacity and its reward policy for longer contracts. The warehouses can be leased for any duration of demand periods, or for a minimum duration of multiple demands periods. The latter suggests warehousing is more of a strategic decision compared to reordering.

Now, for any corporation seeking to maximize its total profit, it needs to answer the following questions:

(1) How much to order for each product? When to make the order?

(2) Which warehouses to lease? When to lease them? For how long to lease?

The WSIO problem distinguishes itself apart from the capacitated EOQ models in the literature in different ways. First, most warehouse inventory optimization problems are based on Hartley's two-warehouses model, while WSIO is not. Clearly, it has a different purpose, structure, and set of variables and parameters. Second, most capacitated EOQ models that are not based on Hartley's model, either assume capacity is given at the beginning (allocation problems) or to be decided at the start, and then is fixed for the rest of the planning horizon (design problems). Third, the questions answered by the WSIO problem, makes it an inventory optimization problem, warehouse selection problem, and interestingly a scheduling problem as well. Only few inventory problems fall all at once under these three categories. Fourth, rarity of real-life existence of the WSIO problem in the past and presumably till today, which suggests that it has never been an active area of research. However, with population growth accelerating in land-scarce regions like in Tokyo and Honk Hong, it is possible that in the not so far future this work will be part of an active area of research.

CHAPTER 3

THE DETERMINSTIC MODEL

In this chapter, a deterministic mathematical model will be developed for the WSIO problem described in Chapters 2. First, a deterministic model will be developed for the problem, where the demand is assumed to be known for certain. Second, different techniques to improve the model's solution time are proposed and analyzed.

3.1. Motivation

In the literature, any attempt to construct a stochastic model for any problem, starts first by composing a deterministic model for that problem. This practice eases building the stochastic model and allows later for comparative verification and testing. In the proposed deterministic model, it is assumed that the products demands are known for certain beforehand, for the entire planning horizon. Whereas, the stochastic model assumes that the demand is not known for certain, and that it may be represented by a probability distribution. Although, any other parameter could be stochastic such as lead time, or prices reduction rate with time, demand uncertainty was only considered due to it being the most relevant to the WSIO problem. While, some would argue that replacing stochastic parameters by their expectation is a valid approach to avoid overcomplications brought by stochastic models, many would argue that this approach is merely solving for a single scenario among many more that would be left unconsidered resulting in a dishonest solution. This argument inspired the development of two decision tools: (1) Expected Value for Perfect Information (EVPI), and (2) Value for Stochastic Solution (VSS) [29]. The two tools are used primarily to test the effectiveness of a stochastic model solution against a solution found using expectation in a deterministic model. EVPI estimates the monetary worth of obtaining perfect information, if investments to eliminate uncertainty are under consideration. On the other hand, VSS estimates the worth of solving the stochastic model as oppose to solving the deterministic expectation model. Further details about the two decision tools will be provided in Chapter 6. In general, both tools provide very interesting insights about the uncertainty in the WSIO problem, and both measures require the deterministic and stochastic models. Hence, there are many key incentives to pursue developing the deterministic model first.

3.2. Model Development

In this section, the steps toward obtaining the deterministic model are listed and explained in detail. First, the problem is described in mathematical notations, followed by listing the assumptions made for this work. Second, the deterministic model is developed.

Table 1 Available Warehousing Options Summary

		Lease Period						
		1	2	3	4	…	j	……
Warehouse Capacity	r_1	h_{11}	h_{12}	h_{13}	h_{14}	…	h_{1j}	……
	r_2	h_{21}	h_{22}	h_{23}	h_{24}	…	h_{2j}	……
	…	…	…	…	…	…	…	……
	r_w	h_{w1}	h_{w2}	h_{w3}	h_{w4}	…	h_{wj}	……

The WSIO problem requires that all available warehousing options, including the corporation's owned warehouses are known beforehand. For example, in Table 1 each row corresponds to a warehousing option with r_i refers to the warehouse capacity, and h_{ij} refers to warehouse type *i* lease cost for *j* periods. To illustrate, assume a corporation is leasing warehouse *i* = 1 for 4 periods, then they will be leasing a warehouse with capacity r_1 for 4 periods and paying in return for this service h_{14} price unit. Now, $h_{i,j+1} \div (j+1)$ could be equal to $h_{i,j} \div j$, or it could be less. In the latter case, the service provider for warehouse *i* is deploying a reward policy, where longer contracts are rewarded with cheaper rates. Observe that the lower rate could be offered at each lease period or could be offered after every certain number of periods. There is a total of *w* warehousing options, and unbounded possible leasing duration unless bounded by the planning horizon length *M*.

Typically, the discrete demands for each product over the planning horizon is represented in a table similar to Table 2.

Table 2 Total Demand for Each Product Over the Planning Horizon

			Demand Periods							
			1	2	3	4	…	k		M
Products	1	R_1	D_{11}	D_{12}	D_{13}	D_{14}	…	D_{1k}	…	D_{1M}
	2	R_2	D_{21}	D_{22}	D_{23}	D_{24}	…	D_{2k}	…	D_{2M}
	…	⋯	…	…	…	…	…	…	…	…
	g	R_g	D_{g1}	D_{g2}	D_{g3}	D_{g4}	…	D_{gk}	…	D_{gM}
	…	⋯	…	…	…	…	…	…	…	…
	v	R_v	D_{v1}	D_{v2}	D_{v3}	D_{v4}	…	D_{vk}	…	D_{vM}

In Table 2, each row corresponds to a different product with storage size equal to R_g, and D_{gk} referring to product g demand at period k. There is a total of v products and M demand periods to plan for. Now, a key step in developing any model is first to list all the assumptions made about the actual problem. Following is a list of all the assumptions made during the development of the deterministic model:

1. Demand is known for certain but not constant (changes from one period to another.)
2. Ordering cost is constant and known for certain.
3. Holding cost consist of two parts:
   a) holding cost for the leased warehouses, which varies based on their capacities and their lease duration (leasing option) and is independent on number of units stored.
   b) holding cost per unit per unit time on stored inventory.
4. Demand periods are equal in duration.
5. Warehouses are leased for a duration that is a multiple of a demand period duration.
6. Lead time is equal to a discrete number of demands periods, and can be equal to zero.
7. Number of available warehouses are enough to store all the inventory.
8. Products' unit selling price declines linearly with time at the rate b_g per period.
9. The warehouses with the least remaining duration are consumed first.
10. All the products share the available warehouses.
11. Lost sales are permitted.

Note that the above assumptions are shared between the deterministic model, and the stochastic model, except for the first assumption. That is, demand in the stochastic model is not known for certain. Hence expenses to dispose excess inventory at the last period are considered. Further assumptions for the stochastic model will be revealed later in Chapter

4. Now, the notations used for developing the deterministic model are listed and defined as follows:

Notations & Parameters:

g : Index used to refer to a product, $g = 1, \dots v$

i : Index used to refer to a warehousing option, $i = 1, \dots W$

j : Index used to refer to lease duration, $j = 1, \dots M$

k : Index used to refer to a demand period, $k = 1, \dots M$. Aliases: *l, m, t, n*.

A : Set containing the periods numbers at which leasing is permitted.

K_{gk} : Ordering cost of product g at period k, and $\widehat{K}_k$ is order placement cost.

r_i : Storage capacity for a warehousing option type i in storage unit (SU).

h_{ij} : Warehousing option i 's lease cost for a lease duration of j periods.

$\hat{h}_g$: Holding cost per unit per unit time for product g.

η_g : Lost sale penalty per unit for product g.

D_{gk} : Demand for product g at period k.

C_g : Purchase cost per unit for product g.

P_g : Initial selling price per unit for product g.

b_g : Per unit decline in selling price for product g after one period.

R_g : Storage size in (SU) for product g.

lt_g : Lead time duration for product g, from the suppliers to the warehouses.

<u>Decision Variables:</u>

X_{ijk} : Number of warehouses to lease from option *i* having a capacity r_i and is leased for *j* periods including and starting from period k.

q_{gk} : Quantity ordered of product g at period *k*.

I_{gk} : Planned inventory of product g at period *k*, where I_{g0} is the starting inventory.

u_{gk} : Planned lost sales of product g at period *k*.

δ_{gk} : Binary variable that takes the value 1, when product g is ordered at period *k*, otherwise 0.

ξ_k : Binary variable that takes the value 1, when an order is placed at period *k*, otherwise 0.

Next, the objective function sought for optimization is formulated using the above notations. The objective function is to maximize total profit over the planning horizon *M*, where total profit is defined as follows:

Total Profit = Total Revenue – Holding Cost – Ordering Cost – Purchase Cost

Now, each part is calculated as follows:

$$\text{Total revenue} = \sum_{g=1}^{v} \sum_{k=1}^{M} q_{gk} \cdot \bar{P}_{gk} \tag{3.1}$$

where $\bar{P}_{gk}$ is the average selling price for order q_{gk}. However, to compute $\bar{P}_{gk}$ for each q_{gk}, the periods at which q_{gk} is consumed must be known beforehand, which is not an attainable information when this model is extended for demand uncertainty. Hence, the following expression is used instead:

$$\text{Total revenue} = \sum_{g=1}^{v}\sum_{k=1}^{M} P_g \cdot \left(D_{gk} - u_{gk}\right) - \sum_{g=1}^{v}\sum_{k=1}^{M} I_{gk} \cdot b_g \tag{3.2}$$

In Equation (3.2), the total revenue is calculated by first computing the total revenue from sold products $\left(D_{gk} - u_{gk}\right)$, assuming that they are all sold at their initial selling price P_g, and then subtracting by the total loss caused by the price reduction b_g. This mathematical representation is valid and can be proved as follows: First assume that q_{kl} refers to an order made at period *k* and consumed at period *l* for product *g* = 1. Now, $I_{1k} = \sum_{n=1}^{k}\sum_{l=k+1}^{M} q_{nl}$, then $\sum_{k=1}^{M} I_{1k} \cdot b = \sum_{k=1}^{M}\sum_{n=1}^{k}\sum_{l=k+1}^{M} q_{nl} \cdot b$. Consider the following:

$$\sum_{k=1}^{M} I_{1k} = \sum_{k=1}^{M}\sum_{n=1}^{k}\sum_{l=k+1}^{M} q_{nl} \tag{3.3}$$

$$= \sum_{n=1}^{M}\sum_{k=n}^{M}\sum_{l=k+1}^{M} q_{nl} \tag{3.4}$$

$$= \sum_{n=1}^{M}\left(\sum_{l=n+1}^{M} q_{nl} + \sum_{l=n+2}^{M} q_{nl} + \cdots + \sum_{l=M-1}^{M} q_{nl} + \sum_{l=M}^{M} q_{nl}\right) \tag{3.5}$$

$$= \sum_{n=1}^{M}\left([M-n]q_{nM} + [(M-1)-n]q_{n,M-1} + \cdots + [(n+1)-n]q_{n,n+1}\right) \tag{3.6}$$

$$= \sum_{n=1}^{M}\sum_{l=n+1}^{M}(l-n)q_{nl} \tag{3.7}$$

$$= \sum_{k=1}^{M}\sum_{l=k+1}^{M}(l-k)q_{kl} \tag{3.8}$$

$$\therefore \sum_{k=1}^{M} I_{1k} \cdot b = \sum_{k=1}^{M}\sum_{l=k+1}^{M}(l-k)q_{kl} \cdot b \tag{3.9}$$

where, the right-hand side refers to the total reduction caused by b in the revenue generated by all the orders q_{kl}, and hence Equation (3.2) is valid.

Holding cost = $\sum_{k \in A} \sum_{i=1}^{W} \sum_{j=1}^{M-k+1} (X_{ijk} \cdot h_{ij}) + \sum_{g=1}^{v} \sum_{k=1}^{M} I_{gk} \cdot \hat{h}_g$ (3.10)

Ordering cost = $\sum_{k=1}^{M} \sum_{g=1}^{v} \delta_{gk} \cdot K_{gk} + \xi_k \cdot \widehat{K}_k$ (3.11)

Purchasing cost = $\sum_{g=1}^{v} \sum_{k=1}^{M} q_{gk} \cdot C_{gk}$ (3.12)

Now, since all the components for the objective function are computed, the equation for the total profit can be represented by:

Total profit =

$$\sum_{g=1}^{v} \sum_{k=1}^{M} P_g \cdot (D_{gk} - u_{gk}) - \sum_{k \in A} \sum_{i=1}^{W} \sum_{j=1}^{M-k+1} (X_{ijk} \cdot h_{ij}) - \sum_{g=1}^{v} \sum_{k=1}^{M} u_{gk} \cdot \eta_g$$
$$- \sum_{g=1}^{v} \sum_{k=1}^{M} I_{gk} \cdot (\hat{h}_g + b_g) - \sum_{k=1}^{M} \sum_{g=1}^{v} \delta_{gk} \cdot K_{gk} - \sum_{k=1}^{M} \xi_k \cdot \widehat{K}_k$$
$$- \sum_{g=1}^{v} \sum_{k=1}^{M} q_{g,k} \cdot C_g$$

(3.13)

In addition to the objective function, following are the constraints governing the problem logic:

$$q_{g,k} \leq \delta_{gk} \sum_{l=k+lt(g)}^{M} D_{gl} \qquad \forall \ k, g \qquad (3.14)$$

$$\delta_{gk} \leq \xi_k \qquad \forall\ k, g \qquad (3.15)$$

$$I_{g1} = q_{g,1} + I_{g0} - D_{g1} + u_{g1} \qquad \forall\ g \qquad (3.16)$$

$$I_{gk} = I_{g,k-1} - D_{gk} + u_{gk} \qquad \forall\ g, 2 \leq k \leq 1 + lt(g) \qquad (3.17)$$

$$I_{gk} = q_{g,k-lt(g)} + I_{g,k-1} - D_{gk} + u_{gk} \qquad \forall\ g, k > 1 + lt(g) \qquad (3.18)$$

$$\sum_{m \in A \cap \{t | t \leq k\}} \sum_{j=k-m+1}^{M} \sum_{i=1}^{W} r_i X_{ijm} \geq \sum_{g=1}^{v} R_g \left(I_{gk} + D_{gk} - u_{gk}\right) \qquad \forall k \qquad (3.19)$$

Constraints (3.14) imply that if $q_{g,k} > 0$, then $\delta_{gk} = 1$. Similarly, Constraints (3.15) imply that if $\delta_{gk} = 1$, then $\xi_k = 1$. Constraints (3.16), (3.17) and (3.18) are the inventory flow balance constraints, where constraint (3.16) is for the first period where lead time is ignored to avoid unavoidable lost sales. In Constraints (3.17), lead time is considered but the index k here only spans the periods between the first period and the period at which the first order has arrived. In constraints (3.18), lead time is also considered but k spans the periods that comes after the arrival of the first order affected by lead time. Finally, constraints (3.19) are the capacity constraints, where the left-hand side is all the space available at period k by warehouses leased at periods A, and the right-hand side is all the capacity needed at period k, which is rendered by inventory carried to the next periods and products sold at period k.

Regarding the lead time $lt(g)$, it is only considered when it is large enough that it can be rounded to a multiple of demand periods. Otherwise, lead time is ignored and later is reflected on the optimal solution. Furthermore, note that if b_g is relatively large, tighter upper bounds on each $q_{g,k}$ in Constraints (3.14) must be considered. This is to avoid $q_{g,k}$

spanning a number of demand periods such that the total reduction in price caused by b_g exceeds the unit selling price P_g.

Following is the complete mathematical model for the WSIO problem:

$$Max\ Z = \sum_{g=1}^{v}\sum_{k=1}^{M} P_g \cdot (D_{gk} - u_{gk}) - \sum_{k\in A}\sum_{i=1}^{W}\sum_{j=1}^{M-k+1} (X_{ijk} \cdot h_{ij}) - \sum_{g=1}^{v}\sum_{k=1}^{M} u_{gk} \cdot \eta_g$$

$$- \sum_{g=1}^{v}\sum_{k=1}^{M} I_{gk} \cdot (\hat{h}_g + b_g) - \sum_{k=1}^{M}\sum_{g=1}^{v} \delta_{gk} \cdot K_{gk} - \sum_{k=1}^{M} \xi_k \cdot \hat{K}_k$$

$$- \sum_{g=1}^{v}\sum_{k=1}^{M} q_{g,k} \cdot C_g \qquad (3.13)$$

Subject to:

$$q_{g,k} \le \delta_{gk} \sum_{l=k+lt(g)}^{M} D_{gl} \qquad \forall\ k, g \qquad (3.14)$$

$$\delta_{gk} \le \xi_k \qquad \forall\ k, g \qquad (3.15)$$

$$I_{g1} = q_{g,1} + I_{g0} - D_{g1} + u_{g1} \qquad \forall\ g \qquad (3.16)$$

$$I_{gk} = I_{g,k-1} - D_{gk} + u_{gk} \qquad \forall\ g, 2 \le k \le 1 + lt(g) \qquad (3.17)$$

$$I_{gk} = q_{g,k-lt(g)} + I_{g,k-1} - D_{gk} + u_{gk} \qquad \forall\ g, k > 1 + lt(g) \qquad (3.18)$$

$$\sum_{m\in A\cap\{t|t \le k\}} \sum_{j=k-m+1}^{M} \sum_{i=1}^{W} r_i X_{ijm} \ge \sum_{g=1}^{v} R_g \left(I_{gk} + D_{gk} - u_{gk}\right) \qquad \forall k \qquad (3.19)$$

$$\delta_{gk} \in \{0,1\}\ ,\ I_{gk} \ge 0\ ,\ q_{gk} \ge 0\ ,\ X_{ijk} \in \mathbb{Z}_{+}, \xi_k \in \{0,1\}$$

Note that although the parameters K_{gk}, $\widehat{K}_k$, b_g, η_g, and $\hat{h}_g$ are assumed to be given, following are suggestions on estimating their monetary values: First, the parameters $\widehat{K}_k$ and K_{gk}, which represent respectively the cost incurred by placing an order at period *k*, and the additional cost incurred by ordering product g at period *k*. To illustrate, if an order is placed for products 1 and 2 at period 3, the objective function would be penalized with the dollar amount $\widehat{K}_3 + K_{13} + K_{23}$ solely for that order. Hence, to better estimate $\widehat{K}_k$ and K_{gk} values for each period, the order placement total cost for different orders of different products combinations for each period are collected, and then the cost effect of ordering each product is isolated. For instance, if placing an order for product 1 and 2 at period 3 is equal to \$100 and placing an order for product 1 at period 3 is equal to \$80, then a good estimate for K_{23} is \$20. Second, the parameter $\hat{h}_g$ which represent the cost for holding a single unit of product *g* as an inventory for a single period. This parameter can be estimated as the unit opportunity cost for not investing the dollar amount went in purchasing a unit product g in the bank. Hence, $\hat{h}_g$ could be estimated by $e \times C_g$, where *e* is the bank rate of return. Third, the parameter η_g which refer to the cost incurred from losing a single sale of product g. This parameter can be estimated as the opportunity cost for missing on potential profit, or can estimated as the monetary cost endured when the business owner reputation is negatively affected by losing a single sale of product g. Fourth, the parameter b_g which refer to the price reduction rate with time for product g. This parameter can be estimated for each product by analyzing the effect of time on the selling price through previously recorded sales.

3.3. Model Validation

In operations research, model validation is defined as the process of ensuring that a mathematical model is correctly the intended real–life problem. It is an important process that is to be undertaken whenever a new mathematical model is introduced. This is mainly accomplished by either comparing the new model results with the results of an older valid model, or by comparing the model results against real–life data. Success in the latter approach indicates an existence of empirical evidence on model's validity. In the WSIO model case, both approaches are not possible. This is due to the lack of previous similar models, and to the inaccessibility to real–life data. Hence, instead of the above two approaches, the WSIO model was validated through an extensive testing procedure for the model's rational behavior. This was accomplished by testing the model against many different problems that have optimal solutions, which can be anticipated beforehand. For instant, assigning high values for K_{gk}, and not allowing lost sales would push the model to yield a solution where orders are only placed at the first period. Similarly, assigning high values for $\widehat{K}_k$ would push the orders to be more aligned. This approach was repeated many times over many parameters, and was successful in unveiling modeling errors that were ultimately fixed. Furthermore, another reason that supports the WSIO model validity is the model ease of readability. The model can be easily read and logically understood. For instance, the objective function is simply the sum of the product of each dollar–unit parameter by its corresponding decision variable, except for the term $I_{gk} \cdot (\hat{h}_g + b_g)$ which was mathematically proven earlier to be valid. This is similarly true to the rest of the model. Hence, the model is assumed valid.

3.4. Efforts to Improve the Model's Solution Time

In this section, efforts are made to improve the deterministic model efficiency through introducing additional constraints that uphold necessary for solution optimality conditions. These constraints generate valid cuts that may reduce the feasible region. Typically, this approach is hypothesized to reduce the solution time to reach the optimal solution. The incentive to pursue these efforts is due to the heavy reliance of the problem on integer variables.

The rest of this section will be organized as follows: First, the problem's optimality conditions are introduced, followed by their equivalent mathematical expressions. Then, new mathematical models are proposed that incorporate the conditions. Then, a complete study among the models is illustrated to find the best model.

First set of optimality conditions to explore is concerned with the relationship between inventory and lost sales during each period. In optimal solutions, if lost sales happen to occur at a certain period, then inventory passed down to next period must be equal to zero, and vice versa. In simple terms, for a given k and g, if $u_{g,k}$ is greater than zero, then $I_{g,k}$ must be equal to zero, and vice versa. The reason why this is an optimality condition is because any situation where both $u_{g,k}$ and $I_{g,k}$ are greater than zero for a given g and k would mean that demand was deliberately not satisfied although inventory did exist. Intuitively, this is certainly not an optimal situation, since it promotes deliberate rejection of sales in exchange for more holding cost, and more reduction in the product selling price.

Following are the optimality conditions. Note that $\hat{\delta}_{gk}$ is a binary decision variable dedicated only for Constraints (3.20 – 3.21):

$$u_{g,k} \leq \hat{\delta}_{gk} D_{gk} \qquad \forall\ k, g \qquad (3.20)$$

$$I_{g,k} \leq (1 - \hat{\delta}_{gk}) \sum_{l=1}^{M} D_{gl} \qquad \forall\ k, g \qquad (3.21)$$

Second set of optimality conditions to explore is inspired by the dynamic lot sizing algorithms such as the Wagelmans–Hoesel–Kolen (WHK) algorithm. Basically, all dynamic lot sizing algorithms are based on three mathematically proven [30] optimality conditions:

A. The order quantity for any period must only be equal to the sum of a number of future periods demands. Hence, it cannot be fractions of demands.

B. If a demand is being satisfied from a different period than its own period, then no order can be made at that demand period.

C. If a demand is satisfied from a certain period call it *k*, then all previous demands starting from period *k* demand, up to the satisfied demand must all be satisfied from the same order made at period *k*.

Following are the optimality conditions. Note that y_{gkl} is a binary decision variable:

$$y_{gkl} \geq y_{gk,l+1} \qquad \forall g, k, k \leq l \leq M-1, l-k \geq lt(g) \qquad (3.22)$$

$$y_{g1l} \geq y_{g1,l+1} \qquad \forall g, l \qquad (3.23)$$

$$y_{gmn} \leq \left(1 - y_{gkl}\right) \quad \forall g, k \leq M-1, k+1 \leq l \leq M, k+1 \leq m \leq l, m \leq n,\ m + lt(g) < l$$

(3.24)

$$y_{gkn} \leq \left(1 - y_{gkl}\right) \qquad \forall g, k > 1, l \geq lt(g) + k,\ k \leq n < k + lt(g) \tag{3.25}$$

$$q_{gk} = \sum_{l=k+lt(d)}^{M} y_{gkl}\, D_{gl} \qquad \forall g, k > 1 \tag{3.26}$$

$$q_{g1} = \sum_{l=k}^{M} y_{g1l}\, D_{gl} \qquad \forall g \tag{3.27}$$

The above Constraints (3.26) and (3.27) would mean that q_{gk} must be exactly equal to a certain sum of future demands, which is a correct optimality condition if there was no restriction on capacity. In the existence of capacity constraints, q_{gk} is bounded by the capacity limitations, and hence it resorts to the second-best solution that is a fraction of the demand periods q_{gk} spans. This is accomplished by replacing (3.26) and (3.27) with the following constraints:

$$\sum_{n=k+1}^{n=l-lt(g)} \delta_{gn} \leq (l - lt(g) - k)(1 - y_{gkl}) \qquad \forall g, l > k \tag{3.28}$$

$$\sum_{l=k+lt(g)}^{M} y_{gkl} \leq \delta_{gk} \cdot M \qquad \forall g, k \tag{3.29}$$

Furthermore, another variation that is worth exploring is to replace constraint (3.29) with the following constraint:

$$y_{gkl} \leq \delta_{gk} \qquad \forall g, l \geq k + lt(g) \tag{3.30}$$

Now, these optimality conditions will be tested for their intended purpose. This is accomplished by first listing the different models that can be assembled by the different optimality conditions from constraints (3.20 - 3.30). Table 3 shows the different models

(A, B, ... H.) to be tested, and which constraints they have as additional constraints to the deterministic model mentioned in section 3.2.

Table 3 Summary of which Optimality Conditions are Included in which Model

		CONSTRAINTS										
		3.20	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30
MODELS	A											
	B	✓	✓									
	C			✓	✓	✓	✓	✓	✓			
	D	✓	✓	✓	✓	✓	✓	✓	✓			
	E			✓	✓	✓	✓	✓		✓	✓	
	F	✓	✓	✓	✓	✓	✓	✓		✓	✓	
	G			✓	✓	✓	✓	✓		✓		✓
	H	✓	✓	✓	✓	✓	✓	✓		✓		✓

The seven problem sizes considered for the test are 18×10, 9×10, 6×10, 3×10, 3×20, 3×30, and 3×60, where 3×10 means a problem with 3 products and 10 periods. For each problem size, 100 random instances were generated. Then, all the proposed models (A – H) were executed to solve the 100 random instances for each problem size. Then, the average solution time and objective function value is recorded. During this test, the same computer was used (Windows 10 pro 64bit operating system, with a processor Interl(R) Core(TM) i5-7600 CPU @ 3.5GHz, and an installed memory (RAM) equal to 8.00 GB). Similarly, the same CPLEX solver was used for all the problems. Also, all problems were solved in series, no parallel execution was allowed. Table 4 shows a summary of the test results.

Table 4 Deterministic Model Variants Performance Average Test Results

		PROBLEM SIZE						
		18×10	9×10	6×10	3×10	3×20	3×30	3×60
MODELS	A	0.785	0.74	0.74	0.73	0.73	0.80	0.93
	B	0.85	0.75	0.76	0.79	0.74	0.82	0.99
	C	1.56	1.03	0.93	0.85	3.11	26.19	103.00
	D	1.10	0.91	0.89	0.82	1.36	3.52	91.00
	E	0.80	0.75	0.76	0.74	0.87	1.24	18.00
	F	0.89	0.75	0.76	0.72	0.85	1.24	18.00
	G	0.804	0.72	0.72	0.73	0.81	1.23	+100
	H	0.85	0.72	0.72	0.73	0.83	1.28	+100

The data displayed in each cell in Table 4 is the average solving time for the hundred instances for a given model and a given problem size. During the test, outliers were not dismissed since each model is tested against the same data instance in each size category. The result shows model A as the most efficient among its peers in all problem sizes; except for the sizes 9×10 and 6×10, where the models G and H appear to be slightly better than model A; and size 3×10, where model F appear to be slightly better than the other models. Therefore, the test was extended for models A, G, and H to accommodate for size 100×10, and the result was 0.99, 2.588, and 3.22 seconds, respectively. Hence, despite the anomaly in sizes 9×10, 6×10, and 3×10, it is safe to conclude that model A is the most efficient among its peers, and that adding inventory optimality conditions as constraints is not beneficial to the solution time. A possible reason is the heavy reliance on the optimality conditions new binary variables. This reasoning is supported by the contrast in solution times between long planning horizon time problems, and short planning horizon time problems. In long planning horizon time problems, the model is required to introduce more integer variables, which is found to have a drastic impact on solution time as seen in Table 4. Furthermore, another purpose for this test is to check for the validity of the models. This was accomplished by recording all the objective function values for all 7×100 problems for each model, and then comparing them for discrepancies. The test showed no discrepancies in the objective function values for all the models.

CHAPTER 4

THE STOCHASTIC MODEL

In this chapter, a stochastic mathematical model will be developed for the WSIO problem. First, a brief background about stochastic programming is provided. Second, the steps toward developing the WSIO stochastic model are shown. Third, the approach used to validate the stochastic model is demonstrated.

4.1. Background

Stochastic programming simply refers to mathematical programming that deals with parameters that are not known for certain. Although, deterministic programming is more popular in the literature, most real-life applications are actually inhabited with uncertainty. This uncertainty comes in the form where some (or all) of the problem parameters are not known for certain, but their probability distributions are known or at least can be estimated. For instance, in most financial models, the *return-on-investment* data is provided by a set of different possibilities with different probabilities (or sometimes referred to as risk levels). Stochastic programming was developed for such problems, by finding a solution that is feasible for all possible scenarios (or almost all), while optimizing an objective function. The objective function consists of two parts: the first part deals with decisions made when uncertainty still unrealized, and the second part deals with decisions made when uncertainty is realized. The following mathematical model is the standard form for any stochastic programming problem:

$$Min\ Z = C^T x + \mathcal{L}(x)$$

Subject to:

$$Ax = b,$$

$$x \geq 0,$$

$$A \in R^{m \cdot n}, b \in R^m, C \in R^n, x \in R^n$$

where $\mathcal{L}(x)$ is the second part mentioned earlier, and is equal to the following expression:

$$\mathcal{L}(x) = E_\xi[\, Q\big(x, \xi(\omega)\big)\,]$$

where $Q\big(x, \xi(\omega)\big)$ is a function that maps x (the decisions made prior to uncertainty realization) and $\xi(\omega)$ (the scenario realized) to the best possible value when all recourse decisions are optimized. Hence, $Q\big(x, \xi(\omega)\big)$ is itself another optimization program, and is expressed by:

$$Q\big(x, \xi(\omega)\big) = \underset{y}{Min}\ \{q(\omega)^T y \mid Wy = h(\omega) - T(\omega)x, y \geq 0\}$$

$$W \in R^{m \cdot n}, y \in R^n, h \in R^m, T \in R^m$$

Note that $\xi(\omega)$ is a vector such that $\xi^T(\omega) = (q(\omega)^T, h(\omega), T_1(\omega), \dots\dots, T_{m_2}(\omega))$, and hence, for each realization of ω, there is a probably different optimal recourse actions y. Also, in certain problems $\mathcal{L}(x)$ can be computed in terms of x and added to the original problem, which is the case with the newsvendor problem. However, some problems are very complicated, and such approach can become very challenging or even inapplicable. Therefore, such problems must be represented by what is called *the extensive form*, which is simply an explicit mathematical programming approach that aims toward optimizing the expectation of the objective function over all scenarios, while upholding all the constraints for all the scenarios, and it goes as follow:

$$Min\ Z = C^T x + \sum_{\omega} p \cdot q(\omega)^T y$$

Subject to:

$$Ax = b,$$

$$Wy = h(\omega) - T(\omega)x \quad \forall\ \omega,$$

$$x \geq 0, y \geq 0,$$

In the stochastic WSIO problem, the extensive form is used to represent the problem mathematically. This is because when demand at each period for each product in uncertain, $\mathcal{L}(x)$ does not exist in a simple closed-form function of x.

4.2. Model Development

Prior to developing the stochastic model for the WSIO problem, it a good practice to depict uncertainty by a tree diagram. Eventually, this will serve as the basis for the stochastic model. For simplicity, assume there is a single product with three planning periods, and four possible demands at each period. The scenario tree for this problem is depicted in

Figure 4. Scenario trees are very useful when trying to mathematically model a stochastic problem. They provide a graphical representation that allows for observing the different possible scenarios, and when decisions are due. Most importantly, they provide a structure that can be utilized to translate the problem into a mathematical model similar to the deterministic model.

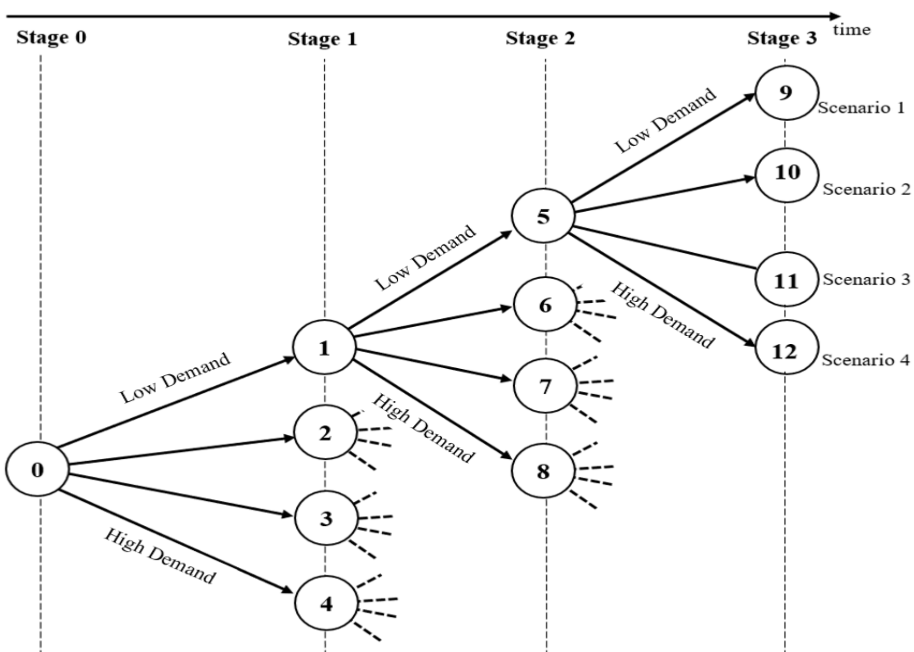

Figure 1 Scenario Tree for a Single Product with 3 Demand Outcomes over 3 Periods

For instance, in Figure 4, the stages above the tree represent the time at which uncertainty is reveled, and sometimes refer to the time at which decisions can be made. Stages for the WSIO problem represent the demand periods, where Stage 0 refers to the beginning of a planning horizon where no uncertainty has been realized yet, but decisions concerning future periods are to be made. Stage i ($1 \leq i \leq M$) refers to a point in time where demand for period i has been realized, and consequently decisions concerning future periods are to be revised, and recourse (corrective) actions regarding previous decisions are to be made. The nodes at each stage represent the possible different incidents at which some uncertainty is realized. For example, at Node 1, it is realized that the demand for the first period is low, while the arcs coming out Node 1 represent the different possibilities for the demand in Period 2, given the demand in first period was low. Note that Node 0 is called the *root node* where no uncertainty has been realized yet, and all *make-now* decisions are due. Furthermore, the nodes at the end are called the *terminal nodes*, and their count number is equal to the number of all possible scenarios to the problem. Now, to harness the benefits of the scenario tree structure and convert it into a mathematical model, let us first introduce the following notations (modified from other scenario-tree based stochastic models) [26]:

- $n \in N$ is a node of the scenario tree, N is the set of all nodes, and O is the set of all terminating nodes, where $|O|$ is the number of all possible scenarios to the problem.
- $T(n)$ is the time period for node n; for instance, $T(1) = 1$ and $T(12) = 3$.
- $a(n)$ is the immediate predecessor for node n, $n \neq 1$; for instance, $a(6) = 1$.
- $\Omega(n, t)$ is the unique ancestor of node n at stage t; for instance, $\Omega(12,1) = 1$.
- $D_g^{[n]}$ is the demand for product g at node n.
- $p^{[n]}$ is the unconditional probability for node n, where $\sum_{n\in\{n|T(n)=t\}} p^{[n]} = 1$.
- $\delta_g^{[n]} \in \{0,1\}$ is the order variable for product g at node n. Similarly, $\xi^{[n]}$ is defined.
- $q_{g\tau}^{[n]}$ is the order quantity for product g made at node n to meet the demand in time period τ.
- $I_g^{[n]}$ is the leftover inventory of product g at node n passed for use to immediate successor nodes since it was not consumed at n. Also, $I_g^{[0]}$ is the starting inventory, and $I_g^{[a(0)]} = 0$.
- $u_g^{[n]}$ is the lost sales of product g at node n. Note that $u_g^{[0]} = 0$.

- γ_g cost of getting rid of excess inventory at the end of the planning horizon.
- $A = \{n | T(n) \in$ set of stages at which warehouse selection is allowed$\}$, also $\{0\}$ always is in A to avoid unavoidable lost sales.

The objective function derived previously for the deterministic model can be used here. This is because while developing the deterministic model, the goal to eventually have a stochastic model was considered. This greatly helped into an easy transition toward introducing uncertainty in demand. For instance, since the revenue in our deterministic model is computed by the expression $\sum_{g=1}^{v}\sum_{k=1}^{M} P_g \cdot (D_{gk} - u_{gk})$ then using the simple expression $\sum_{g=1}^{v}\sum_{n\in N} p^{[n]} \cdot P_g \cdot (D_g^{[n]} - u_g^{[n]})$ can accommodate for the demand uncertainty. However, if our deterministic model would have used a different expression to compute revenue, say $\sum_{g=1}^{v}\sum_{k=1}^{M} P_g \cdot q_{gk}$, then simply using the expression $\sum_{g=1}^{v}\sum_{n\in N} p^{[n]} \cdot P_g \cdot q_g^{[n]}$ to accommodate for demand uncertainty would be misleading, since not all $q_g^{[n]}$ are necessary sold due to the demand uncertainty. Therefore, the stochastic model can be derived from our previously developed deterministic model, without the need to make any major alterations except for considering the additional expenses caused by getting rid of excess inventory at the end of planning horizon, since it is now a possibility. This additional expense can be represented by the expression $\sum_{g=1}^{v}\sum_{n\in O} p^{[n]} \cdot I_g^{[n]} \cdot \gamma_g$. Note that γ_g refers to the per unit cost to get rid of excess unsold inventory at the end of the planning horizon, where γ_g could refer to the per unit cost of dismantle service. Beside this alteration, any model from the previous chapter can be extended toward demand uncertainty using the notations defined earlier in this chapter. Following is the WSIO stochastic model extended from model A:

$$
\begin{aligned}
Max\ Z = &\sum_{g=1}^{v}\sum_{n\in N} p^{[n]}\cdot P_g\cdot (D_g^{[n]}-u_g^{[n]}) \\
&-\sum_{n\in A}\sum_{i=1}^{W}\sum_{j=1}^{M-k+1} p^{[n]}\cdot (\,X_{ij}^{[n]}\cdot h_{ij}) - \sum_{g=1}^{v}\sum_{n\in N} p^{[n]}\cdot u_g^{[n]}\cdot \eta_g \\
&-\sum_{g=1}^{v}\sum_{n\in N\setminus O} p^{[n]}\cdot I_g^{[n]}\cdot (\hat{h}_g+b_g) - \sum_{g=1}^{v}\sum_{n\in N} p^{[n]}\cdot \delta_g^{[n]}\cdot K_g{}^{[n]} \\
&-\sum_{n\in N} p^{[n]}\cdot \xi^{[n]}\cdot \widehat{K}^{[n]} - \sum_{g=1}^{v}\sum_{n\in N} p^{[n]}\cdot q_g^{[n]}\cdot C_g - \sum_{g=1}^{v}\sum_{n\in O} p^{[n]}\cdot I_g^{[n]}\cdot \gamma_g
\end{aligned}
\tag{4.1}
$$

Subject to:

$$q_g^{[n]} \le \delta_g^{[n]}\cdot \max_{i\in\Sigma(n)} D_g^{[i]}\cdot (M+1-T(n)) \qquad \forall\ n,g \tag{4.2}$$

$$\delta_g^{[n]} \le \xi^{[n]} \qquad \forall\ n,g \tag{4.3}$$

$$I_g^{[0]} = q_g^{[0]} + I_g^{[a(0)]} \qquad \forall\ g \tag{4.4}$$

$$I_g^{[n]} = I_g^{[a(n)]} - D_g^{[n]} + u_g^{[n]} \qquad \forall\ g, n\in\{i \mid 1\le T(i)\le 1+lt(g)\} \tag{4.5}$$

$$I_g^{[n]} = q_g^{[\Omega(n,T(n)-lt(g))]} + I_g^{[a(n)]} - D_g^{[n]} + u_g^{[n]} \qquad \forall\ g, n\in\{i \mid T(i) > 1+lt(g)\} \tag{4.6}$$

$$\textstyle\sum_{k\in A\cap\bigcup_{t\le T(n)}\Omega(n,t)} \sum_{j=T(n)-T(k)+1}^{M} \sum_{i=1}^{W} r_i X_{ij}^{[k]} \ge\ \sum_{g=1}^{v} R_g\ \left(I_g^{[n]} + D_g^{[n]} - u_g^{[n]}\right)\ \ \forall n \tag{4.7}$$

$$\delta_g^{[n]} \in \{0,1\},\ \xi^{[n]} \in \{0,1\},\ I_g^{[n]} \ge 0\ ,\ q_g^{[n]} \ge 0\ ,\ X_{ij}^{[n]} \in\ \mathbb{Z}_+,\ u_g^{[n]} \ge 0,$$

Note that Z here refers to the expected total profit given our decisions about $q_g^{[n]}$, and $X_{ij}^{[n]}$. The constraints here hold the same logic from the deterministic model except that their number is multiplied to accommodate for each possible scenario in accordance to the scenario tree. All the stochastic model constraints and objective function are similar to the deterministic model but were extended to accommodate for all the possible scenarios. For instance, Constraint (4.5) $I_g^{[n]} = I_g^{[a(n)]} - D_g^{[n]} + u_g^{[n]}$ is Constraint (3.15) extended to accommodate for the uncertainty in products demands. $I_g^{[n]}$ is the inventory at node *n*, $D_g^{[n]}$ is the demand for product *g* at node *n*, $u_g^{[n]}$ is the lost sales for product *g* at node *n*, $I_g^{[a(n)]}$ is the inventory at the parent node (immediate ancestor of node *n*) for node *n*. The rest of the model was extended similarly to accommodate for demand uncertainty. Note that solving this model will yield a solution that is 100% feasible against all scenarios.

4.3. Model Validation

The stochastic model presented in Section 4.2 was verified through comparative testing against the deterministic model. Basically, the stochastic model was fed with different discrete probability distributions for several demand parameters, D_{gk}, in a way that tricks the model to think there are multiple scenarios, when in reality there is just a single scenario. For example, assume that the product demand for the first period is uncertain with a discrete probability distribution that have outcomes with values equal to each other (see Table 5 for example), Thus, there is actually a single possible outcome, but the model will assume that there are multiple outcomes and will solve accordingly. Now, if the stochastic model is valid, it is supposed to yield the same solution as the deterministic model's solution. This test was repeated 100 times using different numbers (problems parameters, and probability distribution functions), and in all of them, the stochastic model solution was identical to the deterministic model solution. This is a strong indicator that the stochastic model is valid.

Table 5 Example for a PDF Used During the Validation Process

P(n)	$D_1^{[n]}$
0.1	$D_1^{[1]} = 100$
0.3	$D_1^{[2]} = 100$
0.6	$D_1^{[3]} = 100$

CHAPTER 5

SOLUTION METHODS

In this chapter, the methods used to solve the deterministic and stochastic WSIO models are demonstrated. First, the exact methods used to solve the WISO problem are discussed and explained, with a main focus on the stochastic model. Second, a novel heuristic to solve the deterministic WSIO problem is proposed, which is intended for problems that are too large for the common Mixed Integer Programming (MIP) algorithms.

5.1. Exact Methods

The term *Exact Methods* refers to the family of well-established algorithms that guarantee convergence to the optimal solution. In this section, the exact methods used to solve the WISO problem are demonstrated, both for the deterministic model, and the stochastic model. First, the exact method and software code used for the deterministic model are outlined. Second, the exact method and software code used for the stochastic model are demonstrated and discussed.

For the deterministic model, CPLEX solver was used as an exact method. This is because the deterministic model is a Mixed Integer Linear Programming (MILP) problem. Hence, investments in developing and programming exact methods that reaches optimality were not considered. Since there exist mathematically well-established algorithms that solves

MILP problems such as CPLEX solver. However, developing a heuristic method was considered in this work, since the WSIO problem relays heavily on integer decision variables, and like any MILP problem, if its model size exceeds a certain threshold, it can make any MIP algorithm inefficient, or simply impractical depending on the machine solving the problem. The heuristic is discussed in the next section.

For the stochastic model, a different approach and code was used to generate and solve the model as oppose to the deterministic model. In stochastic programming, a persisting issue facing researchers and has been an active research area for so long is how to manage the enormous size of the stochastic model, and the ramifications such size can have on the solution efficiency. This explosion in size (compared to the problem deterministic model) is mostly anticipated, since solving a stochastic model means seeking optimization over all possible scenarios. Whereas, solving a deterministic model means finding the optimal solution only for a single scenario. This issue does not only affect efficiency (the time to reach the optimal solution). but could also cause most optimization software packages to crash or reject solving the model since the number of variables, and constraints can become intractable. Therefore, this issue started to attract many researchers to develop remedies to make stochastic problems more manageable. The most famous remedy is known as Slyke and Wets's *L-shaped method* [31], which primarily exploit the explicit-form dual structure, while using either Wolfe decomposition (inner linearization) [32] of the dual or a Benders decomposition (outer linearization) [33] of the primal. In a nutshell, the method starts with an unconstrained variable referring to the recourse function in the objective function, and the first-stage objective function and constraints. Next, a solution is generated and checked for feasibility over all scenarios, where if it is feasible, the solution proceeds to next step, otherwise a feasibility cut is added to the original model and previous steps are repeated. Then, using the first stage feasible solution, all second-stage scenarios are solved separately for the optimal recourse action, and an optimality cut is generated using the simplex multipliers of each scenario. This process is repeated until a certain criterion is reached (for the explicit details, please refer to [34]). The *L-shaped* method is mostly described as a *divide-and-conquer* method where the single large size stochastic model is replaced with many more small size models, trading a problem enormous size for more iterations. Since then, many more extensions and improvements were added to the *L-shaped method* making it a vital tool in solving any scenario-tree based stochastic problem. However, since the *L-shaped method's* first introduction, computing power has grown exponentially promoting many engineers, and researchers to avoid the complications associated with the *L-shaped method* and simply generate the explicit form using any modeling software and later solving it using any appropriate solver. Fortunately, the modeling software package GAMS (General Algebraic Modeling System) has recently released a new tool known as EMP (Extended Mathematical Programming) [34], which can combine the former option efficiency, and the latter option convince. EMP primarily allows for reformulating any given model into an equivalent model where mathematical programming is more established. For instance, EMP can generate the KKT conditions for a given NLP model, which allows for reformulating the model to an MCP (Mixed Complementarity Problems) model. In our case, this powerful tool is capable of generating the deterministic equivalent model for any stochastic model efficiently, and then solving it using any appropriate algorithm. To achieve this, GAMS require three main code segments:

(1) The core model, which is the stochastic model presented as a deterministic model by replacing the stochastic variables with their expectations.

(2) EMP annotations, where the random variables and their probability distributions are defined, and the constraints and decision variables are allocated to their corresponding stages.

(3) A directory, which records the decision variables' values for each scenario, and allows the software to run more efficiently by storing the model structure and prevent recreating the model each time a scenario is optimized.

The core model is written in GAMS language. The EMP annotations will always start with the stochastic variables' (parameters) definitions, followed by a specific allocation of which variables (both decision variables, and stochastic) and constraints belong to which stage. Finally, the dictionary, which will contain a set that maps each scenario decision–variable to its corresponding variable in the original deterministic model. For explicit sample codes, and further read about the matter please refer to reference [34]. Furthermore, usually SP models in GAMS require two solvers, EMP tend to use either the solver DE or JAMS to generate the deterministic equivalent model, then a secondary sub-solver is used to solve the generated model (e.g. lingo or Cplex). In addition to the simplicity offered by EMP, this tool also allows for other optimization goals beside optimizing the expected objective function. For instance, it allows for CVaR optimization (conditional value at risk), worst-case scenario optimization and chance constraints. Hence, the EMP tool was chosen to solve the WSIO problem, due to its convince and efficiency. Note that a separate GAMS code beside the code used for solving, was used first to generate the EMP annotations (since there were 10 stages, and many variables and constraints, writing the annotations manually is impractical). Once the annotations are generated, they are added manually to the GAMS code used to solve the stochastic WSIO problem.

5.2. Heuristic Methods

Heuristic methods refer to a family of algorithms that seek to find solutions among all possible solutions, but the optimal solution is not guaranteed to be among them. These algorithms usually find near optimal solutions, and they find them fast and easy compared to exact algorithms. Since the WSIO problem relies heavily on integer variables, and exact methods can become inefficient for large intractable problems, there is certainly a need to develop a heuristic for large problems, where optimality is traded for a faster solution time. In this section, a heuristic is developed for the deterministic WSIO problem.

In abstract, the heuristic developed in this section consist of five major phases: (1) Generating and selecting the best feasible solutions to the WSIO problem, while assuming capacity restriction is not a constraint, (2) Computing the capacity requirements for each period for each solution, (3) Selecting heuristically the least expensive warehouses to satisfy the capacity requirements for each solution, (4) Reducing the number of leased warehouses for each solution, and consequently reducing the ordered quantities, (5) Phase four is repeated until no improvement is observed in the objective function (total profit). Finally, the solution with the best total profit is selected. The heuristic is tentatively named *The Reduced Capacity Heuristic* and is abstractly depicted in Figure 2.

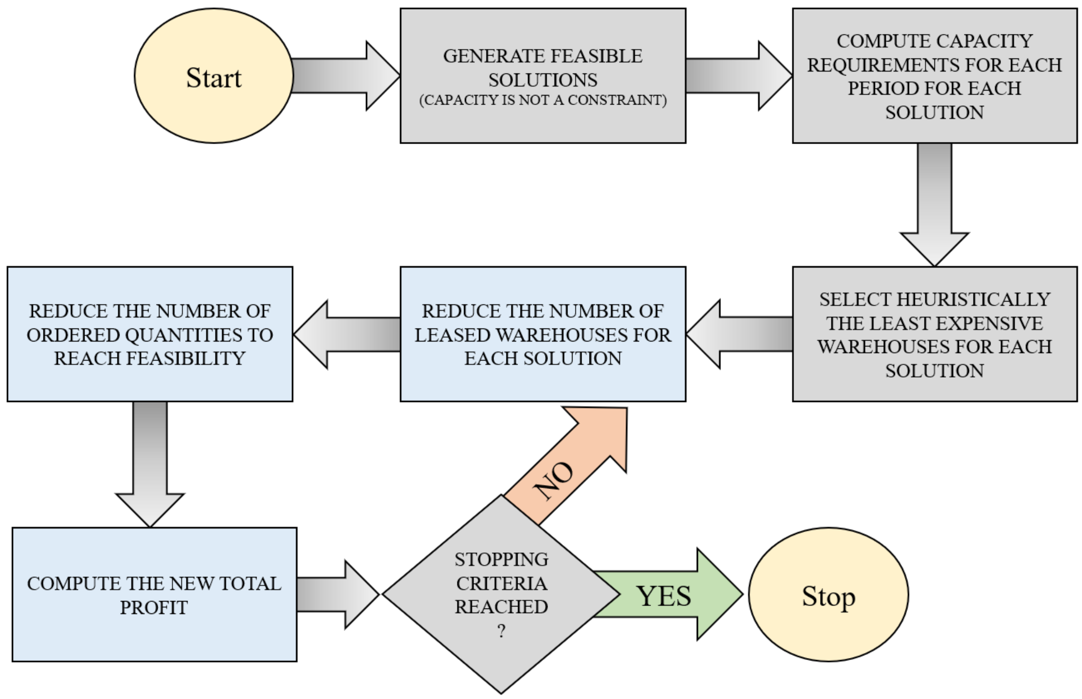

Figure 2 Flowchart for The Reduced Capacity Heuristic

In details, the explicit heuristic algorithm is as follows:

Step 1. Generate feasible solutions to the WSIO problem, while assuming capacity is not a constraint. This is accomplished by various means, such as solving the uncapacitated deterministic model H using the genetics algorithm, or through applying the silver-meal heuristic, or any other similar easy to apply heuristics. Then, let each solution be referred to by S_c^n where c = 1, 2 ... and n = 1.

Step 2. Compute the total capacity needed at each period for each solution, $TC_k^{(c)}$. This is accomplished by computing $TC_k^{(c)} = \sum_{g=1}^{v} R_g \left(I_{gk} + D_{gk} - u_{gk}\right)$ for each period k for each solution S_c^n.

Step 3. Select the warehouses, $X_{ijA}^{(c)}$, for each solution S_c^n as follows:

a. Compute each period total capacity requirements, $TC_k^{(c)}$, and once they are computed, all $TC_k^{(c)}$ values are ordered by their period number *k* in a single row grid (start with the smallest *k* value in the left and ascend to the largest *k* value in the right) ,call it *the capacity requirements grid*. To illustrate, the following is a capacity requirement grid:

Period	1	2	3	4	5	6	7	8	9
Capacity Needed	3000	9000	8000	6000	4000	5000	3000	4000	4500

b. Now, remove from *the capacity requirements grid* the minimum number of periods (columns) to maintain a steadily declining $TC_k^{(c)}$, call it *the reduced capacity requirements grid.* To illustrate, the grid would be as follows for the capacity requirement grid:

Period	2	3	4	6	9
Capacity Needed	9000	8000	6000	5000	4500

Figure 3 shows the difference between planning for the actual capacity requirements grid, and the *reduced capacity requirements grid*. Clearly, selecting which warehouses to choose is easier for the reduced grid as oppose to the actual grid, since the reduced grid has a downward stairs shape, and satisfying the reduced grid satisfies the actual capacity grid.

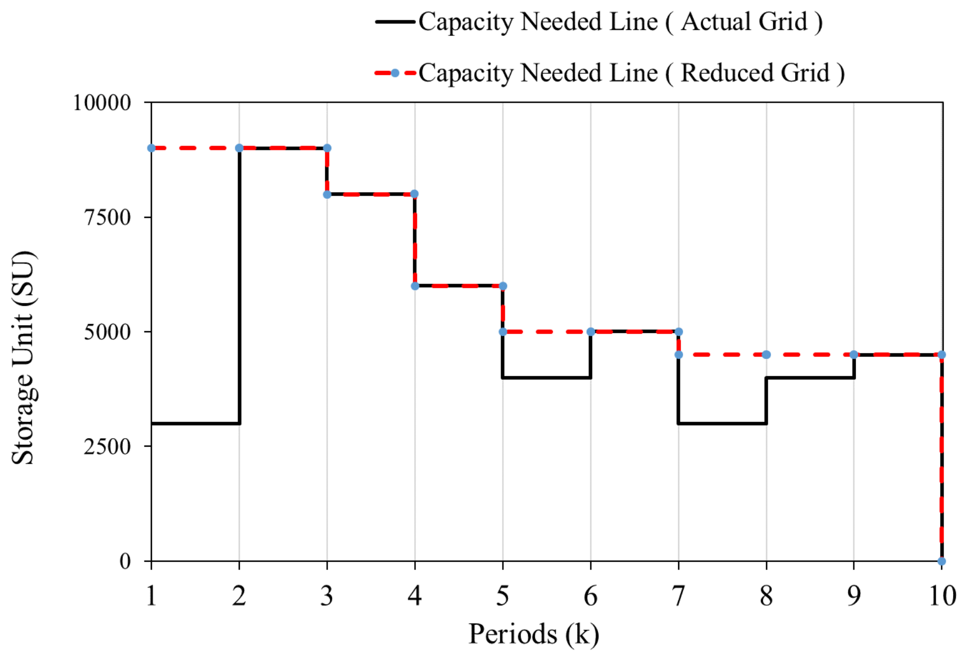

Figure 3 Difference between Actual and Reduced Grid Capacities

c. Now, starting from the end period in the *reduce capacity requirements grid*, find the least expensive warehousing plan that satisfy its $TC_k^{(c)}$. This is accomplished by first dividing the last $TC_k^{(c)}$ over all r_i, and rounding up to the nearest integer, this will give the number of warehouses need of $X_{ijA}^{(c)}$ to satisfy the last $TC_k^{(c)}$with a warehouse capacity r_i, call it $WN_i = \lceil TC_k^{(c)}/r_i \rceil$ Then, compute the leasing cost for the different possible plans to satisfy that $TC_k^{(c)}$ from all the periods, and pick the least expensive. The leasing cost for each plan is computed by multiplying WN_i by its corresponding h_{ij}. To illustrate, assume that the last $TC_k^{(c)} = 4500$ (from the previous grid), and assume A={1,8}. Then, the different possible plans and their prices would be: Plan (1) $X_{i91}^{(c)} = WN_i \times h_{i9}$, compute this for all *i* and pick the least expensive, this will be the option to satisfy the last $TC_k^{(c)}$ from warehouses leased at the first period, where plan (2) $X_{i71}^{(c)} = WN_i \times h_{i8}$ and $X_{i28}^{(c)} = WN_i \times h_{i2}$, pick the least expensive *i* for both, and add their costs together. Then, compare Plan (1) with Plan (2), and pick the least expensive, add it to its corresponding S_c^n.

d. Deduct the newly provided capacity value from all $TC_k^{(c)}$in *the reduced capacity requirements grid*. Now, the last period will have $TC_k^{(c)} = 0$, hence, remove it and update the reduced grid. To illustrate, the updated reduced grid for the earlier example will be as follows, assuming the newly provided capacity equal to $TC_9^{(c)}$ = 4500:

Period	2	3	4	6
Capacity Needed	5500	3500	1500	500

Furthermore, the updated plot for Figure 3 is in Figure 4.

e. Repeat the process starting from point (c), until all $TC_k^{(c)} = 0$ for all *k*.

<u>Step 4.</u>Compute the total profit for each S_c^n, and assign them in the variables Z_c^n, where *n* = 1. If a certain solution S_c^n has a total profit Z_c^n that meet the termination criteria (minimum targeted profit), then terminate the algorithm and S_c^n is the solution. Otherwise procced to the next step to improve new feasible solutions S_c^n for all *c*.

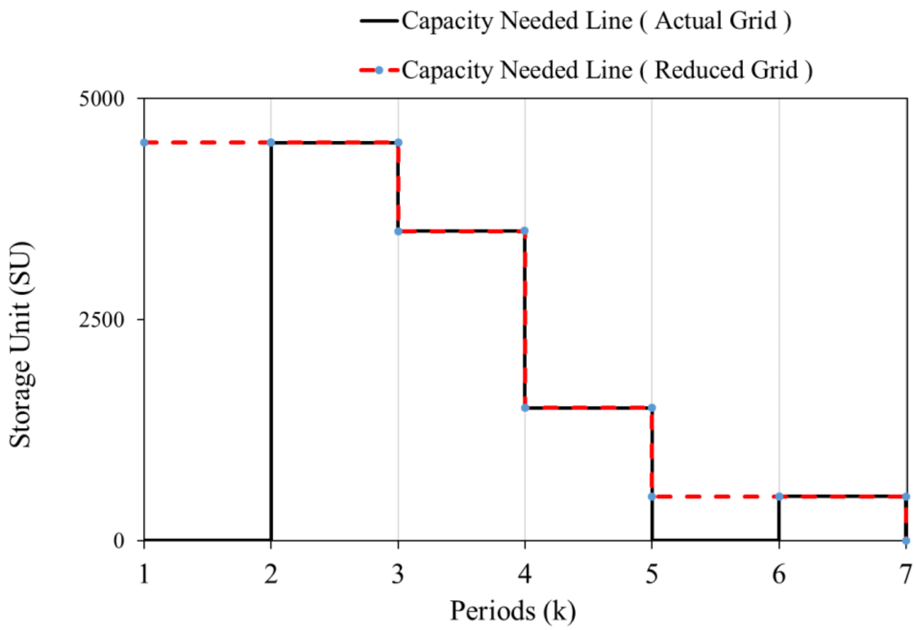

Figure 4 Change between Actual and Reduced Grid Capacities

Step 5. Assign a priority index ($\mathrm{PI_g}$) to each product using the following formula:

$$\mathrm{PI_g} = \frac{(P_g - C_g + \eta_g) \cdot \bar{d}_g}{K_g + (h_g + b_g) \cdot \bar{d}_g + R_g \cdot \bar{d}_g} \quad (5.1)$$

The lower $\mathrm{PI_g}$, the better is to cause lost sales in product g ordering plan. This formula uses a heuristic sense by having the reasons to keep a product order at the denominator, and the reasons to eliminate the order at the numerator.

Step 6. Let $n = n + 1$. Identify the product with the lowest $\mathrm{PI_g}$, pick the order $Q_{gk}^{(c)}$ in its ordering plan that spans the most demand periods. Cause lost sales $u_{gk}^{(c)}$ is the last period spanned by $Q_{gk}^{(c)}$ by an amount that is enough to reduce one from the most expensive $X_{ijA}^{(c)}$ that covers $Q_{gk}^{(c)}$, while maintaining solution feasibility. If the reduced

$Q_{gk}^{(c)}$ satisfies the two points: (I) $Q_{gk} \leq \frac{k_g + \ddot{D}_{gk} \cdot \eta_g}{P_g - C_g + \eta_g}$ and (II) happens only to exists for a single demand period, eliminate that order, hence $u_{gk}^{(c)} = Q_{gk}^{(c)}$, this is because satisfying these two conditions means that $Q_{gk}^{(c)}$ is so small that $Q_{gk}^{(c)}$ existence does not justify the ordering cost k_g. Note that $\ddot{D}_{gk}$ is not the demand from the original problem, but it is the satisfied demand by the reduced $Q_{gk}^{(c)}$, $\ddot{D}_{gk} = D_{gk} - u_{gk}^{(c)}$.

Step 7. Compute the new Z_c^n, if there is improvement in Z_c^n over Z_c^{n-1}, update S_c^n and go to step 6, otherwise let $S_c^n = S_c^{n-1}$ and repeat step 6 by choosing a different $Q_{gk}^{(c)}$. If all $Q_{gk}^{(c)}$ are exhausted, terminate the algorithm and choose the best S_c^n with the highest Z_c^n as the best solution yielded.

This heuristic is guaranteed to converge to a set of solutions and terminate, since step 1 to step 5 are non-iterative, and the rest of the steps are governed by α%; where an improvement in total profit means continuous reduction in $X_{ijA}^{(c)}$, which will lead to zero capacity (which means that the cost of leasing warehouses is too overwhelming), and no improvement in total profit will lead α% to reach zero. This heuristic seeks to generate feasible solutions, and then improve on them through an iterative process of capacity reduction and order plans adjustments. Note that the heuristic algorithm was written with an intent to improve all starting solutions, S_c^1, through a parallel process. However, a series improving process is also possible by looping the heuristic over *c* for each starting solution S_c^1. A numerical example on this heuristic demonstrated above is presented in the next chapter.

5.3. Presolving Techniques for the WSIO Problem

In this section, a list of pre–solving techniques are suggested to improve the solution time for the WSIO problem, mainly by either reducing the problem size or setting upper limits on the decision variables. These techniques can be used in conjunction with many exiting pre–solving techniques for MIP problems [35]

Dominated Solutions: One very effective pre–solving technique is to lookup for dominated solutions and remove them before solving the model. The term dominated solutions refer to the set of solutions that is guaranteed not to be among the optimal solutions. In the WSIO problem, dominated solutions are identified through the parameter h_{ij}, which refers to the cost of leasing a warehouse with a capacity of r_i for *j* periods. Now, if there was to exist a warehouse, say i_1, and another warehouse, say i_2, and $r_{i2} = n \cdot r_{i1}$, and *n* is a positive integer, and $h_{i2,j} > n \cdot h_{i1,j}$ for a given *j*. Then, $X_{i2,jk}$ for that given *j* is assigned a zero or removed from the model before solving. This is because any solution with a non–zero $X_{i2,jk}$ is dominated by the same solution but with $X_{i1,jk}$ set to fulfill the capacity secured by $X_{i2,jk}$, and $X_{i2,jk}$ is set to zero. This is a very effective technique that is guaranteed to maintain optimality and reduces solution time. (since it will reduce the number of integer variables, and hence reduce the model size

Upper Bounds: Furthermore, another very effective pre–solving technique is to set an upper bound on each X_{ijk}. This is accomplished by assuming that the optimal solution is a solution that will depend mainly on the variable X_{ijk} given the indices i, j, and *k* to supply for the capacity needed starting from period *k* up to period $k + j$. Hence, the upper bound for each X_{ijk} would be equal to $\lceil max[TC_k \ , ... \ , TC_{k+j}] \,/\, r_i \rceil$. Following constraint depict a mechanism to identify upper bounds:

$$X_{ijk} \leq \lceil \max[TC_k \ , ... \ , TC_{k+j}] \,/\, r_i \rceil \qquad \forall\, i, j, k \tag{5.2}$$

Correlated Demand: Also, a very effective pre–solving technique that concern the stochastic model is to reduce the number of stochastic variables by identifying a correlation between the demands of the different products. For instance, assume there are three products with demands for 10 periods, and each demand could be high or low at each period for each product. The total number of possible scenarios would be 8^{10}. Now, assume a correlation is found between the products' demands, such that there is a dominant product where if its demand is high, the others have low demands, and if its low, the others have high demands. Now, the stochastic variable $D_g^{[i]}$ can be replaced by a regular variable D_g , and the uncertainty is represented by a binary variable that once realized, a value is assigned to the dominant D_g , and consequently the demands for the other products. This is will reduce the number of scenarios from 8^{10} to 2^{10} (as illustrated in Figure 5), hence, a considerable amount of solution time is saved. In brief, there is a reasonable incentive to lookup for possible demand correlation between the products prior to solving, which will considerably reduce the number of stochastic variables, the scenario–tree size, and eventually reduce solution time.

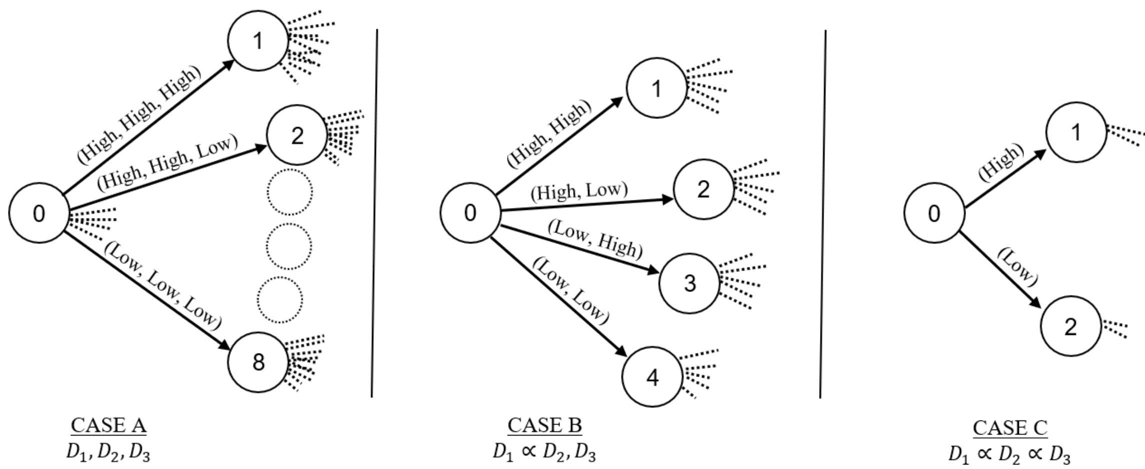

Figure 5 Depiction of The Scenario Trees when Demand Correlation is Considered

CHAPTER 6

EXPERIMENTAL RESULTS

In this chapter, experimental results for the work presented in the previous chapters are provided in the form of numerical examples. First, a numerical example on the deterministic WSIO problem is presented. Second, the same numerical example but with demand uncertainty is solved, and the result is comparatively analyzed and discussed. Next, the same numerical example is re–solved but using the heuristic presented in Chapter 4. This is followed by a comparative analysis in efficiency and effectiveness between the heuristic's performance and the exact methods' performance.

6.1. Numerical Example I

In this section, a numerical example is presented on the deterministic model derived earlier in a previous chapter. Following are the problem parameters:

Table 6 and Table 7 shows the products' demand grid and warehouse leasing prices.

P_1 = \$60/unit, b_1 = \$10/unit, C_1 = \$10/unit, $\hat{h}_1$ = \$0.01/(unit·period), R_1 = 1.0 SU.

P_2 = \$80/unit, b_2 = \$15/unit, C_2 = \$20/unit, $\hat{h}_2$ = \$0.02/(unit·period), R_1 = 3.0 SU.

P_3 = \$100/unit, b_3 = \$20/unit, C_3 = \$30/unit, $\hat{h}_2$ = \$0.03/(unit·period), R_3 = 5.0 SU.

η_1 = \$10/unit, η_2 = \$15/unit, η_3 = \$20/unit, lt_1 = 1 period, lt_2 = 2 period, lt_3 = 3 period.

K_0 = \$2500/order, K_1 = \$5000/order, K_2 = \$6000/order, K_3 = \$8000 /order, A = {1, 4, 8}.

Table 6 Demand Requirements Assumed for the Test Problems

		Demand Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	200	300	400	150	1000	800	400	300	190	200
	2	300	800	1000	4000	700	400	300	50	120	10
	3	1000	1000	1000	100	10	900	700	200	0	100

Table 7 Available Warehousing Options Assumed for the Test Problems

		Leasing Periods, and their cost									
		1	2	3	4	5	6	7	8	9	10
Warehouse Type	1(20)	100	190	270	340	400	450	490	520	540	550
	2(40)	190	370	540	700	850	990	1120	1240	1350	1450
	3(60)	280	550	810	1060	1300	1530	1750	1960	2160	2350
	4(80)	360	710	1050	1380	1700	2010	2310	2600	2880	3150
	5(100)	450	890	1320	1740	2150	2550	2940	3320	3690	4050

The problem solution using CPLEX as the solver is as follows:

The maximum possible total profit is $469,076.852 and the optimal solution has the following products' ordering, and warehouse leasing schemes:

Table 8 Solution Products' Ordering Plan for Numerical Example I

		Ordering Plan									
		1	2	3	4	5	6	7	8	9	10
Products	1	500	550		1000	800	700		390		
	2	2100	4000	700	400	480					
	3	2588		900	700	300					

Above is the solution's products ordering plan, and below is the warehouses leasing plan

$$X_{1,6,1} = 43 \text{ , } X_{1,7,1} = 165 \text{ , } X_{1,8,1} = 53 \text{ , } X_{1,9,1} = 27 \text{ , } X_{1,10,1} = 37 \text{ , } X_{4,1,4} = 67,$$

$$X_{4,2,1} = 88 \text{ , } X_{5,1,1} = 61 \text{ , } X_{5,1,4} = 3 \text{ , } X_{5,2,1} = 1$$

Figure 5 shows the solution's total capacity needed at each period in black solid line, and the total capacity provided by the solution's set of leased warehouses in red dashed line. Also, the yellow boxes refer to the points in time at which leasing warehouses is permitted. It can be inferred from the plot that the solution is feasible, since the storage capacity provided at each period is higher or equal to the storage capacity needed. Also, it can be inferred that the solution is rational, since the gap between the storage capacity need and provided is kept at minimum.

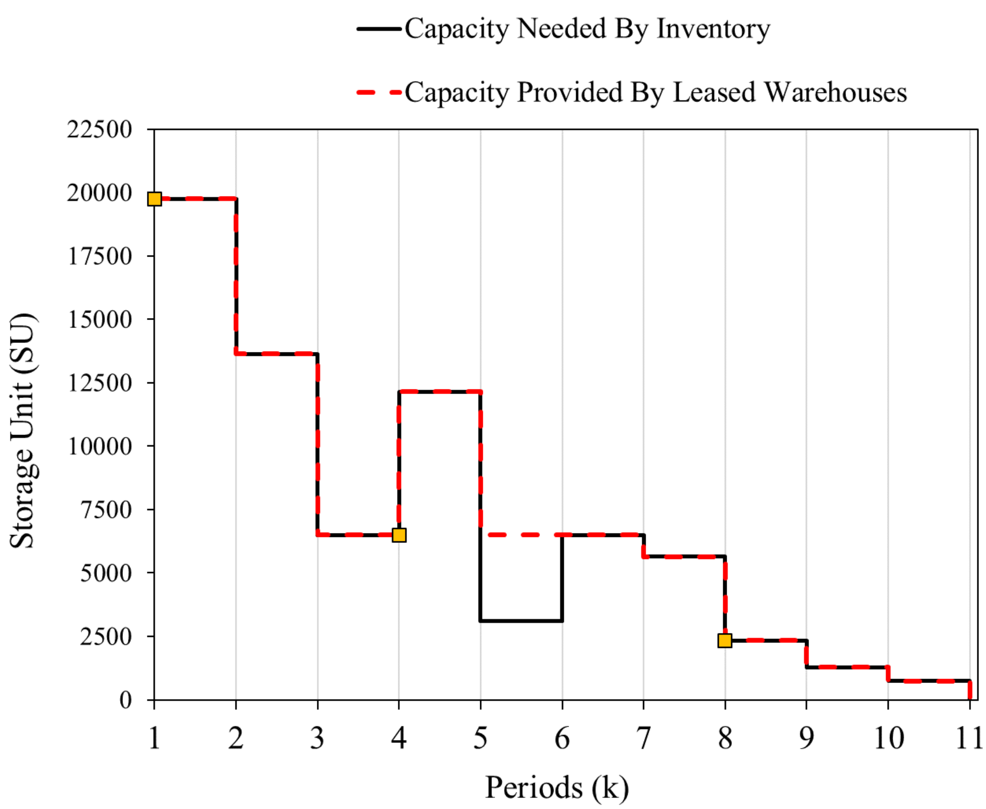

Figure 6 Storage Capacity Needed, and Capacity Provided at Each Period

The result for this deterministic numerical example will be used to analyze the stochastic model in the next section.

6.2. Numerical Example II

In this section, a numerical example is presented on the stochastic model derived earlier in a previous chapter. The problem used in this section is the same one used by the previous deterministic numerical example, except for the assumption of demand uncertainty, which is reflected by the following probability mass function for products' demands at each period k, and the additional parameter $\gamma_g = 0.01$:

Table 9 Probability Mass Function (PMF) For Products' Demands at Each Period *k*

k	D_1^{I}	D_2^{I}	D_3^{I}	$\mathrm{P}(k, D_g^{\mathrm{I}})$	D_1^{II}	D_2^{II}	D_3^{II}	$\mathrm{P}(k, D_g^{\mathrm{II}})$
1	200	300	1000	0.5	200	300	1000	0.5
2	330	700	1000	0.9	30	1700	1000	0.1
3	700	100	1000	0.55	100	2100	1000	0.45
4	157	1500	100	0.65	137	8550	100	0.35
5	2000	500	10	0.20	750	750	10	0.80
6	2100	10	900	0.35	100	610	900	0.65
7	1100	90	700	0.30	100	390	700	0.70
8	330	30	200	0.80	180	130	200	0.20
9	250	60	0	0.70	50	260	0	0.30
10	500	4	100	0.25	100	12	100	0.75

Notice that the expression $\mathrm{P}(k, D_g^{\mathrm{I}}) \times D_g^{\mathrm{I}} + \mathrm{P}(k, D_g^{\mathrm{II}}) \times D_g^{\mathrm{II}}$ is equal D_{gk} from the deterministic numerical example for any g and k. Hence, the demand grid in the deterministic numerical example is basically the expectation demand grid, not the actual demand. This choice of numbers is to study the difference between the practice of replacing demand uncertainty with their expectation, and the practice of representing demand uncertainty through the extensive stochastic form, at least for this numerical example.

The problem was solved using EMP-CPLEX as the solver, and the solution is as follows: The maximum possible expected total profit is \$218,523.23. The solution was found to be the best solution to behave against 512 possible demand scenarios given their probability of occurring and is feasible for all 512 scenarios. The solution is represented as 512 products' ordering and warehouse leasing schemes, which each differ in their reaction to the demand realized at each period. For instance, all planning schemes will have the following part in their solution, which is made at time zero, and when no products demands have been realized yet: $q_{1,1} = 530$, $q_{2,1} = 1100$, $q_{3,1} = 2922$, $X_{1,7,1} = 163$, $X_{1,8,1} = 47$, $X_{1,9,1} = 14$, $X_{1,10,1} = 65$, $X_{4,2,1} = 82$, $X_{5,1,1} = 61$. However, as the products' demands start to unveil at each period, the complete solution starts to form up at each period as a reflex to the realized demand. The complete solution is in Appendix A. To illustrate, following is a comparison for the highest possible total profit scenario (the best-case scenario), and the least possible total profit scenario (the worst-case scenario):

- The best-case scenario (S18, Probability = 0.005, profit = $350,059.457):

Table 10 Best Case-Scenario's Products Demand Grid (Numerical Example II)

		Demand Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	200	330	700	157	2000	100	1100	330	250	100
	2	300	700	100	1500	500	610	90	30	60	12
	3	1000	1000	1000	100	10	900	700	200	0	100

Table 11 Best Case-Scenario's Products Ordering Plan (Numerical Example II)

		Ordering Plan									
		1	2	3	4	5	6	7	8	9	10
Products	1	530	857		1910	410	470	680			
	2	1100	1674	669		187					
	3	2922		900	700	300					

The warehouses leasing scheme:

$$X_{1,7,1} = 163, X_{1,8,1} = 47, X_{1,9,1} = 14, X_{1,10,1} = 65, X_{4,2,1} = 82, X_{5,1,1} = 61, X_{4,3,4} = 2$$

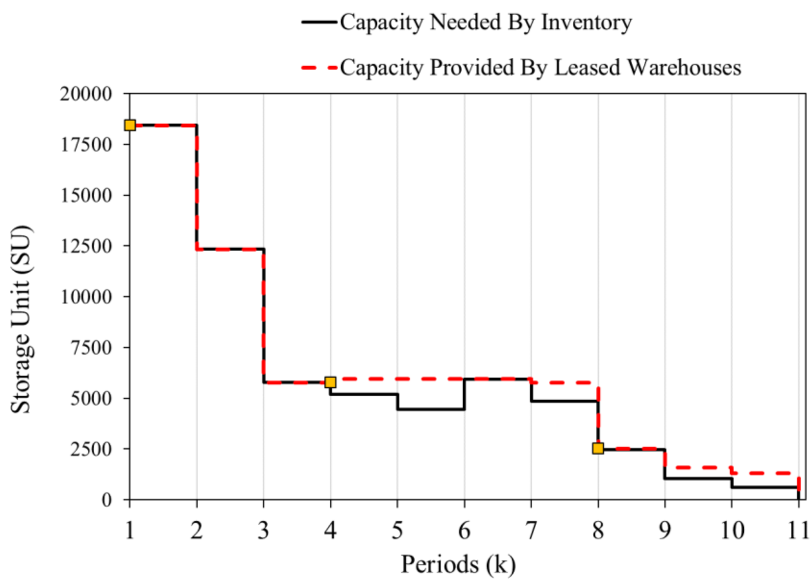

Figure 7 Plot for the Best-Case Scenario Stochastic Solution

- The worst-case scenario (S512[1], Probability = 2.58×10^{-4}, profit = \$14,351.94):

Table 12 Worst Case-Scenario's Products Demand Grid (Numerical Example II)

		Demand Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	200	30	100	137	750	100	100	180	50	100
	2	300	1700	2100	8550	750	610	390	130	260	12
	3	1000	1000	1000	100	10	900	700	200	0	100

Table 13 Worst Case-Scenario's Products Ordering Plan (Numerical Example II)

		Ordering Plan									
		1	2	3	4	5	6	7	8	9	10
Products	1	530	557		1330						
	2	1100	1674	666		393					
	3	2922		900	700	300					

[1] S512 refers to scenario number 512 in the solution report

The warehouses leasing scheme:

$$X_{1,7,1} = 163, X_{1,8,1} = 47, X_{1,9,1} = 14, X_{1,10,1} = 65, X_{4,2,1} = 82, X_{5,1,1} = 61$$

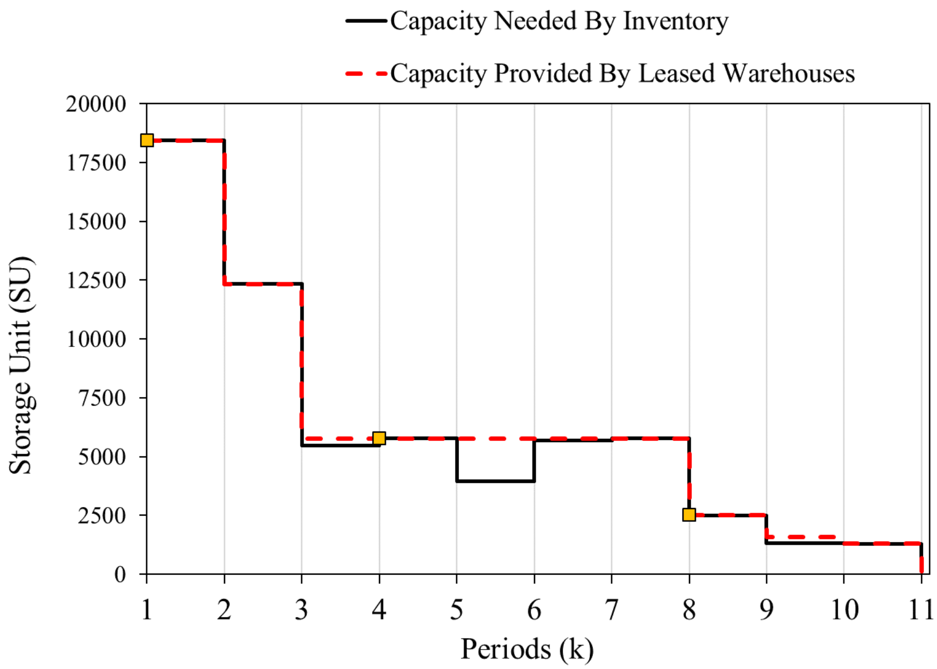

Figure 8 Plot for the Worst-Case Scenario Stochastic Solution

Notice the difference between the two above cases, both started with the same ordering and warehouse selection plan, but as the products' demands being realized, the plan starts to change. In the best-case scenario, the ordering and warehouse selection plan better succeeds in matching the products' anticipated demands as oppose to the worst-case scenario, where the plan fails to match the anticipated demands, resulting in high level of lost sales. This is attributed to the high probability the best-case scenario has in comparison to the probability of the worst-case scenario. Hence, the model was swayed to give more emphasis in optimization on scenario 18 (the best-case scenario) and other similar scenarios with high probability of occurring, on the account of scenarios with low probability of occurring (e.g. scenario 512, the worst-case scenario). Furthermore, notice the difference between the deterministic numerical example solution, and the stochastic numerical example solution. Although, they share the same problem parameters, and the fact that the deterministic problem's products demands are actually the expectation of the stochastic problem's products uncertain demands, they have completely two different solutions and expected profit. This is because attempting to remedy demand uncertainty through demand expectation is a misleading approach. It simply makes the optimization process exclusive only for a single scenario that is the expectation, while neglecting the totality of all the 512 possible scenarios. Actually, feeding the deterministic model solution to the stochastic model as fixed parameters resulted in the solution being infeasible to all 512 scenarios due to the high variation in demand uncertainty. The infeasibility is caused by the violation of the products' capacity requirement constraints by all 512 scenarios. This is because in some periods, actual demand can be lower than the expectation causing more inventory to be carried out to future periods, while the capacity of the leased warehouses cannot accommodate for the additional inventory, since they are tailored for the capacity requirements of the expected demand. To illustrate, observe Figure 8, where the deterministic solution was used in a stochastic environment, and assuming scenario 18 (the best case scenario) happened to take place. Notice the infeasibility highlighted in red, which is described by the capacity–needed line exceeding the capacity–provided line for around six periods. Similarly, all 512 scenarios will have the same issue with the deterministic model solution. This observation is very important because it reveals a very effective pre–solution procedure that could reduce the solution time considerably. The procedure suggested is to eliminate before solving any solution with X_{ijk} that cannot provide storage capacity to the maximum possible demand in Table 9. The solutions removed are all infeasible solutions, since feasibility is mainly affected by the capacity provided at each period. This procedure will surely improve the solution time, since the number of integer variables will drop. Now, since deterministic solution is 100% infeasible as shown earlier, and computing a real value for the decision–making tool VSS (value of stochastic solution, mentioned in previous chapters) is impossible. However, if infeasibility is avoided by penalizing the additional inventory by a negligible amount, say $0.001 per unit per time period, then a rough estimate of the VSS tool can be computed.

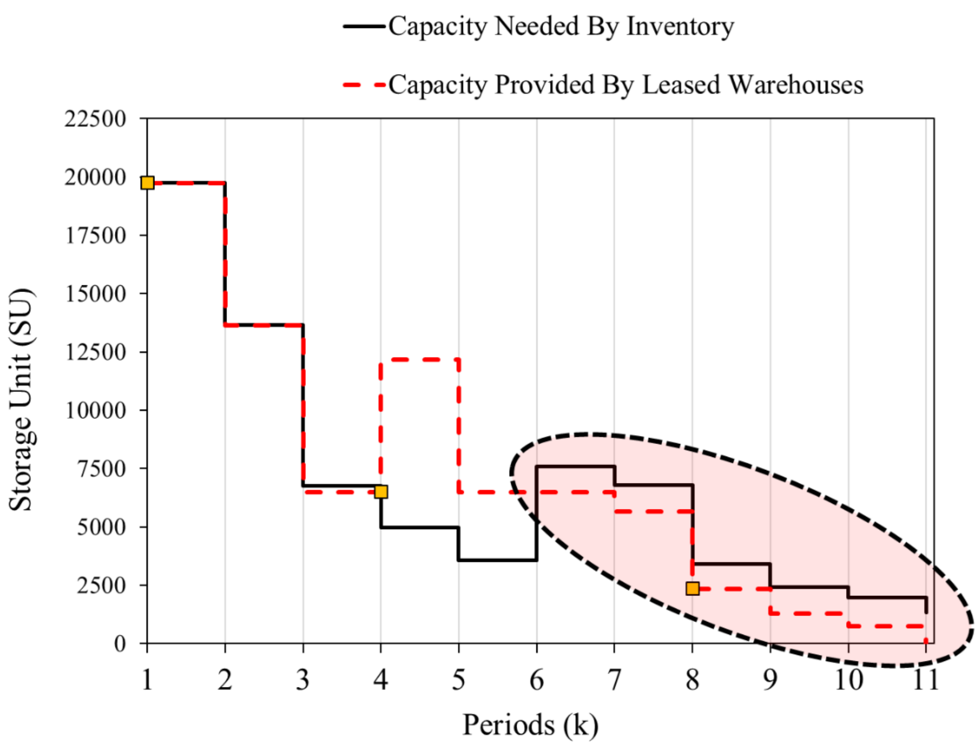

Figure 9 Infeasibility of the Deterministic Solution used in the Stochastic Problem

First, the mathematical expression for VSS is written as follows:

$$\text{VSS} = Z^{s} - Z^{D} \tag{6.1}$$

where, Z^s is the stochastic model's optimal objective function value, and Z^d is the stochastic model's objective function value, given the solution is the deterministic expectation model's optimal solution. Z^s is already computed, and it is equal to \$218,523.23, while Z^d is computed by adding a dummy positive variable to the right–hand side of the capacity constraint, and then is used to penalize the objective function by \$0.001 per infeasible unit, and Z^d becomes equal to – \$32939.12. Hence, the monetary value for investing in solving the stochastic model is roughly VSS = $Z^s - Z^D =$ \$251,462.35. The second key measurement indicator is EVPI, which simply attempts to give a monetary value for collecting perfect information that reduce demand uncertainty toward complete certainty. This is accomplished by evaluating the following mathematical expression for EVPI:

$$\text{EVPI} = \sum_n P(n) \cdot \tilde{Z}_n^d - Z^s \tag{6.2}$$

where $\tilde{Z}_n^d$ is the optimal solution objective function for scenario *n* given that all demand uncertainty is realized at Stage 1. This is simply accomplished by introducing a dummy stage as the first stage with dummy variables, while changing all the previous EMP annotations stage numbers to two. This will tell the model that all the information will be reveled and all the non–dummy decisions will have to be made together after the dummy stage. Z^s is already computed, and it is equal to \$218,523.23, while $\sum_n P(n) \cdot \tilde{Z}_n^d$ is equal to \$454,310.92. Hence, EVPI $= \sum_n P(n) \cdot \tilde{Z}_n^d - Z^s = \$235{,}787.7$, which is the monetary value for obtaining perfect information at the beginning of the planning horizon. This information is very valuable when deciding whether to invest in seeking perfect information or settle with the existing information.

VSS and EVPI are key measures that are consistently reintroduced or redeveloped mathematically in the literature whenever a new stochastic problem is introduced. The above calculations are a guide toward recomputing these two key measurements to any stochastic WSIO problem.

Now, inspired by both the VSS and EVPI decision–making tool, we introduce in this work another new decision–making tool called the Expected Value for Reduce Lead Time (EVRLT), which is a very relevant decision–making tool to the WSIO problem. EVRLT seeks to measure the monetary value for investing in reducing the lead time for a certain product, which will result in improving the reaction time toward demand realization, and ultimately better solutions with higher total profit. Following is the mathematical expression for EVRLT:

$$\text{EVRLT}(\,g, u\,) \;= Z^{s}_{g,u} - Z^{s} \tag{6.3}$$

where $Z^{s}_{g,u}$ is the stochastic model's optimal objective function value, when the lead time for product g is reduced by u. Both Tables 14 and 15 show the summary computations of $Z^{s}_{g,u}$, and EVRLT on the stochastic numerical example presented earlier in this section.

Table 14 $Z_{g,u}$ Values for g, and u For Numerical Example II

		Lead Time Reduction (u)			
	$Z^{s}_{g,u}$	0	1	2	3
Product (g)	1	$218,523.23	$254,000.97	NA	NA
	2	$218,523.23	$276,997.28	$458,2 65.46	NA
	3	$218,523.23	$218,527.00	$296,771.41	$330,550.81

Table 15 EVRLT (*g*, *u*) Values for *g*, and *u* for Numerical Example II

		Lead Time Reduction (*u*)			
	EVRLT	0	1	2	3
Product (*g*)	1	$0.00	$35,477.74	NA	NA
	2	$0.00	$58,474.05	$239,742.23	NA
	3	$0.00	$3.77	$78,248.18	$112,027.58

Table 15 provides very useful information for the decision–maker concerning which product should be invested in reducing its lead time, and how much is expected in return on that investment. For instance, if the investment amount of reducing the lead time for any product by one period is the same, then clearly from Table 15, Product 2 should be made top priority, while Product 3 should be avoided entirely. Furthermore, although Table 15 shows the EVRLT values for each product separately, EVRLT can be extended to study the effect of reducing the lead time for multiple products simultaneously.

6.3. Numerical Example III

In this section, a numerical example is presented on the heuristic algorithm developed in the preceding chapter. The problem used in this section is similar to the one used in the previously discussed deterministic numerical example. This is to allow for a comparative analysis between the heuristic solution, and the actual optimal solution.

Now, the first step in the *reduced capacity heuristic* is to generate many feasible solutions to the uncapacitated WSIO problem, preferably variant model E (without the capacity constraint). This is because model E contain all the optimality conditions as constraints. The generated solutions from variant model E have better quality compared to solutions generated from other variant models. Now, assume one of the possible generated solutions is the following solution, call it S_1^1:

		Ordering Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	500	550		1000	800	700		390		
	2	2100	4000	700	400	480					
	3	3000		900	700	300					

Although, the heuristic algorithm uses multiple starting solutions (seeds), and then improves them toward feasibility and optimality, and ultimately chooses the best among them, the above solution is the only solution that will be used during this numerical example (c = 1). This is because the main goal for this numerical example is to show how the heuristic algorithm will be applied step by step toward each single starting solution. Hence, the steps will be shown are for S_1^1, the above single starting solution only.

Now, the step that comes after generating the starting solutions is to compute their capacity requirements grids. Following is the grid for the above starting solution, S_1^1:

1	2	3	**4**	5	6	7	**8**	9	10
21800	15700	8550	12150	3100	6500	5640	2340	1280	730

Next step is to compute the reduced capacity requirements grid, which goes as follow:

1	2	**4**	6	7	**8**	9	10
21800	15700	12150	6500	5640	2340	1280	730

Next step is to start with the end $TC_k^{(1)}$, and compare between all the possible different warehousing plans to satisfy this capacity requirement from all the periods. Then, pick the least expensive and add it to the solution. Following is all the possible warehousing plans to satisfy $TC_{10}^{(1)} = 730$:

Plan (1): $X_{1,10,1}^{(1)} = 37$ with a cost equal to $20,075.

Plan (2): $X_{4,3,1}^{(1)} = 10$ and $X_{1,7,4}^{(1)} = 37$ with a cost equal to $27,466.

Plan (3): $X_{1,7,1}^{(1)} = 37$ and $X_{4,3,8}^{(1)} = 37$ with a cost equal to $27,466.

Plan (4): $X_{4,3,1}^{(1)} = 10$, $X_{1,4,4}^{(1)} = 37$ and $X_{4,3,8}^{(1)} = 37$ with a cost equal to $31,572.

The plans are generated as follows: observe plan (1), the last index in $X_{1,10,1}^{(1)}$ refers to the first leasing period, the second index refers to the lease duration required to reach the targeted period, and the first index refers to the warehouse size, and it was set to be equal to one, because $WN_1 \times h_{1,10}$ was the least expensive compared to the other $WN_i \times h_{i,10}$, and finally the 37 is actually WN_1. For plan (2), the same process is repeated but with a different chosen path that is leasing at Period 1, and then at Period 4. Similarly, the same process with a similar variation choosing the path was done to compute Plan (3) and (4). Now, notice that Plan (1) is the least expensive plan, hence, add the term $X_{1,10,1}^{(1)} = 37$ to the solution S_1^1. Now, the newly provided capacity by $X_{1,10,1}^{(1)} = 37$ is subtracted from all the capacity requirements $TC_k^{(1)}$ in the reduced grid, and $TC_{10}^{(1)}$ is taken out. Following is the updated grid:

1	2	4	6	7	8	9
21060	14960	11410	5760	4900	1600	540

Similarly, we compute the different warehousing plans to satisfy the last period total capacity requirements, $TC_9^{(1)} = 540$. The plans are as follow:

Plan (1) $X_{1,9,1}^{(1)} = 27$, with a cost equal to \$14,580.

Plan (2) $X_{4,3,1}^{(1)} = 7$ and $X_{1,6,4}^{(1)} = 27$, with a cost equal to \$19,238.

Plan (3) $X_{1,7,1}^{(1)} = 27$ and $X_{4,2,8}^{(1)} = 7$, with a cost equal to \$18,023.

Plan (4) $X_{4,3,1}^{(1)} = 7$ and $X_{1,4,4}^{(1)} = 27$, and $X_{4,2,8}^{(1)} = 7$, with a cost equal to \$21,061.

Since plan (1) is the least expensive, the term $X_{1,9,1}^{(1)} = 27$ is added to the solution S_1^1. Also, the newly provided capacity by $X_{1,9,1}^{(1)} = 27$ is subtracted from all the capacity requirements $TC_k^{(1)}$ in the reduced grid, and $TC_9^{(1)}$ is taken out. This process is repeated until all $TC_k^{(1)} = 0$. Following are the remaining solution details.

The updated *reduced capacity requirements grid* is as follows:

1	2	4	6	7	8
20520	14420	10870	5220	4360	1060

The possible plans to satisfy $TC_8^{(1)} = 1060$:

Plan (1) $X_{1,8,1}^{(1)} = 53$, \$27,560. (✔)

Plan (2) $X_{4,3,1}^{(1)} = 14$ and $X_{1,5,4}^{(1)} = 53$, \$34,393.

Plan (3) $X_{1,7,1}^{(1)} = 53$ and $X_{4,1,8}^{(1)} = 14$, \$30,740.

Plan (4) $X_{4,3,1}^{(1)} = 14$ and $X_{1,4,4}^{(1)} = 53$ and $X_{4,1,8}^{(1)} = 14$, \$36,703.

The updated *reduced capacity requirements grid* is as follows:

1	2	4	6	7
19460	13360	9810	4160	3300

The possible plans to satisfy $TC_7^{(1)} = 3300$:

Plan (1) $X_{1,7,1}^{(1)} = 165$, \$80,850. (✔)

Plan (2) $X_{4,3,1}^{(1)} = 42$ and $X_{1,4,4}^{(1)} = 165$, \$99,413.

The updated *reduced capacity requirements grid* is as follows:

1	2	4	6
16160	10060	6510	860

The possible plans to satisfy $TC_6^{(1)} = 860$:

Plan (1) $X_{1,6,1}^{(1)} = 43$, \$19,350. (✔)

Plan (2) $X_{4,3,1}^{(1)} = 11$ and $X_{4,3,4}^{(1)} = 11$, \$22,576.

The updated *reduced capacity requirements grid* is as follows:

1	2	4
15300	9200	5650

The possible plans to satisfy $TC_4^{(1)} = 5650$:

Plan (1) $X_{1,4,1}^{(1)} = 283$, \$96,050. (✔)

Plan (2) $X_{4,3,1}^{(1)} = 71$ and $X_{4,1,4}^{(1)} = 71$, \$99,581.

The updated *reduced capacity requirements grid* is as follows:

1	2
9650	3550

The possible plans to satisfy $TC_2^{(1)} = 3550$:

Plan (1) $X_{4,2,1}^{(1)} = 45$, \$31,506. (✔)

The updated *reduced capacity requirements grid* is as follows:

1
6100

The possible plans to satisfy $TC_1^{(1)} = 6100$:

Plan (1) $X_{4,1,1}^{(1)} = 77$, \$27,450. (✔)

Eventually, the solution S_1^1 becomes:

$$X_{1,10,1}^{(1)} = 37, X_{1,9,1}^{(1)} = 27, X_{1,8,1}^{(1)} = 53, \; X_{1,7,1}^{(1)} = 165, \; X_{1,6,1}^{(1)} = 43, \; X_{1,4,1}^{(1)} = 283,$$
$$X_{4,2,1}^{(1)} = 45, \; X_{4,1,1}^{(1)} = 77,$$

Table 16 Heuristic Solution's Products Ordering Plan for Numerical Example III

		Ordering Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	500	550		1000	800	700		390		
	2	2100	4000	700	400	480					
	3	3000		900	700	300					

Now, the total profit for S_1^1 is $Z_1^1 = \$450{,}052.1$. Notice that Z_1^1 is relatively very close to Z^* (the optimal from the deterministic numerical example) with an absolute gap difference equal to $Z^* - Z_1^1 = \$19{,}024.75$, and a relative gap equal to $Z^* - Z_1^1/Z^* = 4\%$. This is an indication that this hurestic has good potentials to yield more suboptimal solutions, once given good uncapacitated starting solutions. In the algorithm, it is suggested that S_1^1 can be checked for further possible improvements in its Z_1^1 by causing lost sales in the least important product at the product order that spans the most demand periods. This is accomplished by first updating n = 1 to n = 2, and computing the new S_1^2 and Z_1^2 as follows:

[1] First, compute the priority index for each product:

$$\mathrm{PI}_1^1 = 2\ ,\ \mathrm{PI}_2^1 = 3,\ \mathrm{PI}_3^1 = 1.$$

[2] Then, Identify the order that spans the most demand periods for the least important product, that becomes $\mathrm{Q}_{31}^{(1)} = 3000$.

		Demand Periods									
		1	2	3	4	5	6	7	8	9	10
Products	1	500	550		1000	800	700		390		
	2	2100	4000	700	400	480					
	3	3000		900	700	300					
Capacity Needed		21800	15700	8550	12150	3100	6500	5640	2340	1280	730
Available Capacity		21920	15760	12160	12160	6500	6500	5640	2340	1280	740

[3] Then, cause lost sales in $\mathrm{Q}_{31}^{(1)}$ so it becomes $\mathrm{Q}_{31}^{(1)} = 2984$ which save enough space to eliminate a single warehouse from $X_{421}^{(1)} = 45$, so it becomes $X_{421}^{(1)} = 44$. The changes in $\mathrm{Q}_{31}^{(1)} = 2984$, and $X_{421}^{(1)} = 44$ makes the new solution S_1^2.

[4] Then, compute the total profit for S_1^2 that is $Z_1^2 = \$449{,}963.06$. Hence, this direction negatively affects the total profit. Hence, we return to the solution S_1^1 and try a different similar change to a different order, and then to a different product.

[5] All products orders were checked is accordance to the steps shown above, no resulting S_1^n provided a better total profit than S_1^1's total profit.

CHAPTER 7

CONCLUSIONS

In this thesis, the literature concerning the topic "*capacitated inventory optimization and warehousing*" was extensively reviewed, and consequently a gap was successfully identified. The gap revealed that although the literature is rich with published papers about capacitated inventory optimization, the bulk of these papers are actually extensions to few well–known inventory optimization problems. Hence, the variety of the original models these papers are based upon is limited, there is a need to introduce new models, which can accommodate for other real–life inventory problems. The objective of this thesis is to introduce a new capacitated inventory optimization and warehousing problem. The problem introduced is named as the Warehouses Selection and Inventory Optimization (WSIO) problem. The WSIO problem is unique in the way capacity is viewed as a resource, along with the assumptions made for this problem. In the WSIO problem, business owners are seeking to find the optimal ordering plan for multiple products, along with the optimal warehousing plan. It is assumed that a variety of warehousing options exist, which mainly differ in storage space, and in their reward system for longer contracts. The thesis has successfully introduced the problem, developed both a deterministic and a stochastic model for the problem, suggested exact methods to solve both models, developed and tested possible modeling techniques to improve the exact methods' efficiency, developed heuristic methods to solve the models, and finally drew important insights about the WSIO problem through performing experiments and analyzing the results. The motivation behind this accomplishment is twofold: First, the anticipated need for this work in the not so far future in highly–populated land–scarce regions like Honk–Kong and Tokyo. Second, the lack of past research on this area as reveled by the literature review. Furthermore, this work is considered novel in not only introducing the WSIO problem, but in the other accomplishments that accompanied this introduction. This involved developing a variety of novel mathematical formulations for several existing optimality conditions of the inventory problem dynamic model. Then, developing an elaborate test to examine whether adding these newly fashioned mathematical formulations would improve solution time or not, which was initially hypothesized that they would improve solution time, but instead they were proven to worsen solution time. Furthermore, this work is novel in developing a GAMS code for the stochastic model using the newly added EMP tool, which was first introduced in late 2017. Also, the work is novel in developing a heuristic approach to the WSIO problem that was shown to have good potentials but require further testing, along with a new decision tool called the Expected Value for Reduced Lead Time (EVRLT). In addition to the previous accomplishments, this work provides experimental data that can be citied, and used for other researches, such as the data concerning the solution performance of the optimality conditions' models, and the data concerning the difference between using the deterministic model solution and the stochastic model solution. Most importantly, this work might be able to shed the light on an uncharted or forgotten territory in the capacitated inventory optimization research area.

The future work for this thesis is to develop a software program for the reduced capacity heuristic and have it tested for performance against the exact MIP solution methods. In addition, looking up for other real–life application that can utilize the work presented by this thesis, not necessarily inventory and warehousing applications.

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Appendix A

Numerical Example II Solution Report

The solution report consists of over than 190 pages. Hence, it was enclosed on a CD accompanying this document.

Vitae

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