A DISCONTINUOUS PETROV-GALERKIN METHOD FOR TIME-FRACTIONAL DIFFUSION EQUATIONS

BY

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I dedicate my Dissertation work to my loving mother, my wife, my sons, my daughter and my relatives.

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THESIS ABSTRACT

NAME: BASHEER SALEH MOHAMMAD ABDALLAH

TITLE OF STUDY: **A discontinuous Petrov-Galerkin method for time-fractional diffusion equations**

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In this thesis, we propose and analyze a discontinuous Petrov-Galerkin (DPG) method for the numerical solution of time-fractional diffusion problems. We prove the existence and uniqueness of the approximate solution, and derive error estimates. We employ a non-uniform mesh based on concentrating the cells near the singularity. This leads to improve the accuracy of the approximate solution. Then we combine the Petrov-Galerkin scheme with respect to time with a standard finite element discretization in space and obtain a fully discrete scheme, for which we present suboptimal error analysis. Finally, we compare the theoretical results with the results of numerical computations.

ملخص الرسالة

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عنوان الرسالة: طريقة بيتروف جالركن المنفصلة لحل المعادلة التفاضلية الجزئية الكسرية عدديا.

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في هذه الرسالة ، فإننا نقترح ونحلل طريقة بيتروف جالركن المنفصلة لحل المعادلة التفاضلية الجزئية الكسرية عدديا. سوف نثبت أن لهذه المعادلات حل وحيد. وأيضا سنشتق الخطأ في القياس بين الحل التقريبي والحل المضبوط. علاوة على ذلك سوف نوظف تجزئة غير منتظمة تقوم على تركيز خلايا قرب التفرد. وهذا يؤدي إلى تحسين دقة الحل التقريبي. ثم نجمع بين طريقة بيتروف للوقت وطريقة مشهورة الا وهي طريقة العناصر المحددة للفضاء للحصول على نظام منفصل تماما، والذي يقدم تحليل الخطأ الامثل. أخيرا سوف نقارن بين النتائج النظرية والنتائج العددية.

CHAPTER 1

Introduction

1.1 Fractional Diffusion Equations

Fractional diffusion equations are partial differential equations that involve fractional derivatives in space, time, or both. Fractional derivatives provide an exellent instrument for the description of memory and hereditary of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical derivatives, in which such effects are in fact neglected.

The advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, fluid flow, dynamical processes in self-similar and porous structures, electrical networks, probability and statistics, viscoelasticity, chemical physics, optics and signal processing, and in many other fields. Fractional derivatives also appear in the theory of control of dynamical systems when the controlled system is described by a fractional differential equation, see

[20, 28, 29, 37] and the references therein.

Fractional diffusion equations, in particular anomalous diffusion, have been observed in various physical fields such as tracer transfer in underground water and dispersive transports in amorphous semiconductors, see [6, 28, 49] and the references therein.

1.2 Mathematical Problem

Our work is concerned with the following problem

$$\begin{aligned}
&{}^{c}D^{1-\alpha}u(x,t) + Au(x,t) = f(x,t), && \text{in } \Omega \times (0,T], \qquad (1.1)\\
&u = 0, && \text{on } \partial\Omega \times (0,T],\\
&u|_{t=0} = g(x), && \text{on } \Omega,
\end{aligned}$$

where A is an operator that has a complete orthonormal eigensystems. For example $A = -\Delta$ is the laplacian operator. Ω is a convex bounded polyhedral domain of $\mathbb{R}^n$ with $n = 1, 2, 3$ and $0 < \alpha < 1$. The functions f and g are given.

Several authors have proposed a variety of low-order numerical methods for model problem (1.1). For example, Zhao *et al.* [48] considered the initial-boundary value problem for the time fractional differential equation (1.1) for $n = 1$ with Neumann boundary conditions in the domain $\Omega = (0, L)$. They studied the system by combining an order reduction approach and L_1 discretization of the fractional derivative considered by Oldham and Spanier [36], and constructed a box-type scheme. Also they proved that the global convergence order in maximum norm is $O(t^{2-\alpha} + h^2)$.

Jin *et al.* [18] considered the initial boundary value problem for the homogeneous time-fractional diffusion equation (1.1). They studied the Galerkin Finite Element Method (FEM) and lumped mass Galerkin FEM, by using piecewise linear functions. They established optimal error with respect to the regularity error estimates, including the cases of smooth and nonsmooth initial data, i.e., $v \in H^2(\Omega) \bigcap H_0^1(\Omega)$ and $v \in L_2(\Omega)$, respectively. For the lumped mass method, the optimal L_2-norm error estimate is valid only under an additional assumption on the mesh, which in two dimensions is known to be satisfied for symmetric meshes.

Sweilam *et al.* [44] developed a Crank-Nicolson finite difference method for solving the initial boundary value problem for the time-fractional diffusion equation (1.1) and studied the stability of the discrete solution. Moreover, they proved that the global convergence order in maximum norm is $O(k + h^2)$.

Cui [11] considered the initial boundary value problem for the time fractional diffusion equation (1.1) for $n = 2$ with $A = a\frac{\partial^2}{\partial x^2}u(x, y, t) + b\frac{\partial^2}{\partial y^2}u(x, y, t)$, where a and b are positive constants. He solved this fractional diffusion equation by means of High-order compact finite difference method with operator-splitting technique (The Caputo derivative is evaluated by the L_1 approximation, and the second order derivatives with respect to the space variables are approximated by the compact finite differences to obtain fully discrete implicit schemes) and he studied the stability of the discrete solution. Moreover, he proved that the global convergence order in maximum norm is $O(k^{\min(1+\alpha,2-\alpha)} + h^4)$.

Mainardi *et al.* [24] considered the initial boundary value problem for the time fractional diffusion equation (1.1) for $n = 1$. By using the Fourier transform (in space) and the Laplace transform (in time), the fundamental solutions (Green functions) are shown to be high transcendental functions of the Wright-type that can be interpreted as spatial probability density functions evolving in time with similarity properties.

Lin and Xu [45] considered the initial boundary value problem for the time fractional diffusion equation (1.1) for $n = 1$. The proposed method is based on a finite difference scheme in time and Legendre spectral methods in space. Stability and convergence of the method are established. They proved that the full discretization is unconditionally stable, and they proved the convergence of order $O(k^{2-\alpha} + N^{-m})$ where k, N and m are the time step size, polynomial degree and regularity of the exact solution, respectively.

Joaquin and Santos [38] considered the initial boundary value problem for the time fractional diffusion equation (1.1) for $n = 1$. The stability analysis was carried out by means of a kind of fractional von Neumann method. Partial convergence analyses (truncation error of order $O(k + h^2)$) were provided assuming that u is sufficiently regular.

Zhang and Sun [47] considered a standard initial boundary value problem for the time fractional diffusion equation (1.1) for $n = 2$. The proposed method is based on central difference approximation for the spatial discretization, and, for the time stepping, two new alternating direction implicit (ADI) schemes based on the L_1 approximation and backward Euler method. They proved the convergence of order $O(k^{\min(2+\alpha,2-\alpha)} + h^2)$ (for the first scheme) and $O(k^{\min(1+\alpha,2-\alpha)} + h^2)$ (for the second scheme) assuming that u is smooth.

Furthermore, various numerical methods have been applied for the fractional subdiffusion problem

$$u'(x,t) + D^{1-\alpha}Au(x,t) = f(x,t) \quad \text{in } \Omega \times (0,T],\, 0 < \alpha < 1. \tag{1.2}$$

This equation is closely related to, but different from (1.1). For example, Mclean and Thomee [27] developed a numerical method based on spatial finite element discretization and Laplace transformation with quadrature in time for (1.2) with a homogeneous Dirichlet boundary data. They proved that the maximum-norm error estimate is of order $O(t^{-1-\alpha}h^2 l_h^2), l_h = |\ln h|$.

Mustapha and Mclean[26] employed a piecewise-constant, discontinuous Galerkin method for the time discretization of a sub-diffusion equation. They proved a priori error bound of order k under realistic assumptions on the regularity of the solution. They also showed that a spatial discretization using continuous, piecewise-linear finite elements leads to an additional error term of order $h^2 \max(1, \log k^{-1})$.

Mustapha and Mclean [33] used a piecewise-linear, discontinuous Galerkin method for the time discretization of a fractional diffusion equation. Their analysis showed that, for a time interval (0,T) and a spatial domain Ω, the error in $L_\infty((0,T);L_2(\Omega))$ is of order $k^{2+\alpha}$. They employed a non-uniform mesh based on concentrating the cells near the singularity. In the limiting case, $\alpha = 0$, they recovered the known $O(k^2)$ convergence for the classical diffusion (heat) equation. They also considered a fully-discrete scheme that employs standard (continuous) piecewise-linear finite elements in space, and showed that the additional error is of order $h^2 \log(1/k)$. Mustapha [31] studied a semidiscrete in time and fully discrete schemes, Crank-Nicolson in time and finite elements in space, and derived error bounds for smooth initial data.

Yuste and Acedo [46] proposed an explicit finite difference (FD) method for solving fractional diffusion equation (1.2) for $n = 1$. An $O(k+h^2)$ truncation error was shown assuming that u is sufficiently smooth at $t = 0$. Chen *et al.* [7]considered problem (1.2) for $n = 2$ and solved it by using the Grunwald–Letnikov expansion for time and finite difference method for space and he proved the convergence of order $O(k + h^2)$.

Cui [9] proposed high-order compact finite difference scheme (After approximating the second-order derivative with respect to space by the compact finite difference, they used the Grunwald-Letnikov discretization of the Riemann-Liouville derivative to obtain a fully discrete implicit scheme) and he proved the method has accuracy of four in the spatial grid size and one in the fractional time step, provided u is sufficiently smooth.

Mustapha [31] studied an implicit finite-difference Crank-Nicolson method in time combined with spatial piecewise-linear finite elements(FEs) scheme for solving fractional diffusion equation (1.2) . Convergence of order $O(h^2+k^{2+\alpha})$ was proven. A time-space FD scheme was studied recently in [32] where convergence of order $O(h^2+k^{2+\alpha})$ was achieved.

A compact ADI scheme was studied recently in [10]. This method is used to split the original problem into two separate one-dimensional problems. The local truncation error was analyzed and the stability was discussed by the Fourier method.

1.3 Discontinuous Petrov Galerkin Method

Petrov Galerkin (PG) is a class of finite elements (FEs) introduced initially by Hulme [17] for systems of nonlinear first-order ordinary differential equations. The PG method allows the trial and the test function spaces to be different. If the test space is discontinuous, (PG) is called Discontinuous Petrov Galerkin (DPG). In the last fifteen years, DPG has been applied to different class of problems. For instance, DPG was used to solve a dvective- diffusive problem [4], transport equation [13], optimal test function [14], the elliptic problem [3] $-\mathrm{div}(v\nabla u) = f$ in $\Omega \subset \mathbb{R}^2$ with $u = g$ in $\partial\Omega$, where $\partial\Omega$ is the boundary of the set Ω and the functions f, g and v are given. DPG method was used for sloving linear volterra integro differential equations [23, 30] and fredholm equation of the second kind [8, 19].

In [13] optimal error estimates for the aproximate solution were proven in both the element size h and polynomial degree p. In [3] the discontinuous nature of the test functions at the element interfaces allows to introduce new boundary unknowns that on one hand enforce the weak continuity of the trial functions, and on the other avoid the need to define a priori algorithmic fluxes as in standard discontinuous Galerkin methods.

In [8, 19] one of the advantages of the Petrov-Galerkin method is that it allows us to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree. In [23, 30] the DPG method can produce numerical solutions with optimal/superconvergence rates in the involved spaces. Practically, the number of unknowns in the DPG method is fewer than the number of unknowns in the discontinuous Galerkin method with respect to time.

1.4 Our work

In this thesis we propose and analyze a time-stepping discontinuous Petrov-Galerkin method combined with a standard continous, conforming finite element method in space for the numerical solution of time-fractional diffusion model (1.1). We prove the existence, uniqueness and stability of approximate solutions, and derive error estimates. The time mesh is graded appropriately near $t = 0$ to compensate the singular (temporal) behavior of the exact solution near $t = 0$ caused by the weakly singular kernel, but the spatial mesh is quasi-uniform.

In the $L_\infty(0, T, L_2(\Omega))$-norm((0; T) is the time domain and Ω is the spatial domain), for sufficiently graded time meshes, a global convergence of order $k^{m+\alpha/2} + h^{r+1}$ is shown, where k is the maximum time step, h is the maximum diameter of the spatial finite elements, and m and r are the degrees of approximate solutions in time and spatial variables, respectively. Numerical experiments indicate that our theoretical error bound is pessimistic. We observe that the error is of order $k^{m+1} + h^{r+1}$, that is, optimal in both variables.

1.5 Thesis Outline

In the next chapter, we will introduce some basic notations, definitions and theorems that will be used throughout the dissertation.

In the third chapter, we will consider (1.1) with $\mathcal{A} = -\Delta$,

$$\begin{aligned}
&{}^{c}D^{1-\alpha}u(x,t) - \Delta u(x,t) = f(x,t), && \text{in } \Omega \times (0,T], \qquad (1.3)\\
&u = 0, && \text{on } \partial\Omega \times (0,T],\\
&u|_{t=0} = g(x), && \text{on } \Omega.
\end{aligned}$$

In chapter three, we use the positivity, coercivity and continuity properties of the fractional time derivative operator that we showed in chapter two. This will be the key to establish the existence, uniqueness and stability of the approximate solution of problem (1.3). In Section 3.1 we show some stability properties of the continuous solution of problem (1.3). In Section 3.2, we introduce the time-stepping DPG method. In Section 3.3, we prove our main results regarding the well-posedness of the approximate solution. In Sections 3.5, we work on the error analysis of the method and convergence of order k^1 will be proved, where k is the maximum time step. In Section 3.6, we present the implementation of DPG scheme of (1.3).

In chapter four we extend the discontinuous petrov-galerkin method to high order. The techniques that we used in chapter three are not valid to high order to prove the stability of the approximate solution of problem (1.3) and deriving the error estimate.

In this chapter, we will use techniques which are different from the techniques that we used in chapter three to prove the stability and the order of convergence and derive error estimate, which are completely explicit in the local step sizes, the local polynomial degrees, and the local regularity of the analytical solution. We show that DPG schemes based on appropriate refined time-steps (towards $t = 0$) achieve convergence of order $k^{m+\alpha/2}$, where m is the degree of approximate solutions in time.

In Chapter five we show the stability, uniqueness and the existence of the approximate solution in space for our problem (1.3). Also we will combine the Petrov-Galerkin scheme with respect to time with a standard finite elements discretization in space and obtain a fully discrete scheme of our problem (1.3) and we work on the error analysis of the method and show the convergence of order $k^{m+\alpha/2} + h^{r+1}$, where m and r are the degrees of approximate solutions in time and spatial variables, respectively. Finally we implement the fully discrete discontinuous petrov galerkin finite element (DPGFE) scheme defined on rectangular polygons.

In chapter six, we present a series of numerical examples and compare the theoretical results with numerical results.

CHAPTER 2

Preliminaries

In this chapter we introduce some basic notations, definitions and theorems that will be used throughout the dissertation.

2.1 Spaces

Definition 2.1 ([20]) We denote by $L_2(\Omega)$ the space of all Lebesgue real-valued measurable functions v defined on a bounded, convex domain $\Omega \subseteq \mathbb{R}^n$ for which $\|v\| < \infty$, where

$$\|v\| := \|v\|_{L_2} = \left(\int_\Omega v^2(x)\, dx \right)^{1/2}.$$

The space $L_2(\Omega)$ can be equipped with the inner product

$$\langle v, w \rangle := \int_\Omega v(x)\, w(x)\, dx.$$

Definition 2.2 ([21, 43]) Given an integer $k \geq 1$, the distributional derivative of order k of $f \in L_{1,loc}$ is the linear functional,

$$D^m v(\phi) = (-1)^m \int_\Omega v(x) D^m \phi(x)\, dx, \qquad \forall \phi \in C_c^\infty(\Omega),\ |m| \leq r.$$

If there exists a locally integrable function g such that $D^m v(\phi) = g(\phi)$, namely

$$\int_\Omega g(x)\phi(x)\, dx = (-1)^m \int_\Omega v(x) D^m \phi(x)\, dx, \qquad \forall \phi \in C_c^\infty(\Omega),$$

then we say that g is the weak derivative of order k of f, where C_c^∞ denotes the space of continuous functions with compact support, having continuous derivatives of every order and $m = (m_1, \cdots, m_n)$ is a n-vector where the m_i are non-negative integers, $|m| = \sum_{i=1}^n m_i$ and $D^m \phi = \frac{\partial^{|m|}\phi}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}$.

Definition 2.3 ([21]) We define $\mathbb{H}^r(\Omega)\,, r \geq 0$, to be the space of all functions whose weak derivatives of order $\leq r$ belong to $L_2(\Omega)$, i.e.,

$$\mathbb{H}^r(\Omega) = \{v \in L_2(\Omega) : D^m v \in L_2(\Omega) \ \text{ for } \quad |m| \leq r\}.$$

The space $\mathbb{H}^r(\Omega)$ can be equipped with the norm and seminorm

$$\|v\|_r := \|v\|_{\mathbb{H}^r} = \left(\sum_{|m| \leq r} \|D^m v\|^2 \right)^{1/2},$$

$$|v|_r := |v|_{\mathbb{H}^r} = \left(\sum_{|m| = r} \|D^m v\|^2 \right)^{1/2},$$

respectively.

where $\mathbb{R}$ is the set of real numbers and $D^m v$ is the weak derivative of order k of v.

Also we introduce the space,

$$H_0^1 := \{v \in \mathbb{H}^1 : v|_{\partial\Omega} = 0\},$$

where $\partial\Omega$ is the boundary of the set Ω.

Remark 2.4 If v is a sufficiently smooth function, say $v \in C^k(\Omega)$, then its weak derivative $D^m v$ of order $|m| \leq k$ coincides with the corresponding partial derivative (in the classical pointwise sense); $D^m v := \frac{\partial^{|m|} v}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}$.

Definition 2.5 ([15]) A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \epsilon$ for any finite collection $\{[a_i, b_i] : 1 \leq i \leq N\}$ of non-overlapping subintervals $[ai, bi]$ of $[a, b]$ with $\sum_{i=1}^{N} |b_i - a_i| < \delta$.

2.2 Fractional Calculus

In this section, we will define Riemann–Liouville fractional integral and derivative and Caputo derivative, follows by some properties.

Definition 2.6 ([20]) *(i)* The Riemann–Liouville fractional integral operator I^α is defined by

$$\mathrm{I}^\alpha v(t) := \begin{cases} \int_0^t \omega_\alpha(t-s)v(s)\,ds, & \alpha > 0, \\ \\ v(t), & \alpha = 0, \end{cases}$$

where $\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(\alpha)$ is the standard gamma function.

(ii) The Riemann–Liouville fractional derivative D^α, $\alpha > 0$, is defined by

$$\mathrm{D}^\alpha v(t) := \mathrm{D}^n \mathrm{I}^{\mathrm{n}-\alpha} v(t) = \frac{d^n}{dt^n} \int_0^t \omega_{n-\alpha}(t-s)v(s)\,ds \quad \text{where} \ \ n = [\alpha] + 1,$$

if it exists, where $[\alpha]$ is the greatest integer of α.

(iii) The Caputo fractional derivative ${}^cD^\alpha$, $\alpha > 0$, is defined by

$${}^cD^\alpha v(t) := \mathrm{I}^{\mathrm{n}-\alpha} \mathrm{D}^n v(t) = \int_0^t \omega_{n-\alpha}(t-s)\,\frac{d^n v(s)}{ds^n}\,ds, \quad \text{where} \ \ n = -[-\alpha],$$

if it exists.

Properties 2.7 *(i)* *(*[22]**)** **Assume that** $0 < \alpha \leq 1$, $f \in L_2(0,T)$**, then we have**

$$\|I^\alpha f\|^2_{L_2(0,T)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} I^\alpha \, \|f\|_{L_2(0,T)}.$$

(ii) ([16, 20]) Let $\alpha > 0$, $n = [\alpha] + 1$ and $v \in AC[0,T]$, then D^α exists almost everywhere and has the representation

$$\mathrm{D}^\alpha v(t) := \frac{v(0)\, t^{-\alpha}}{\Gamma(1-\alpha)} + {}^c D^\alpha v(t).$$

(iii) ([20], Semi Group Inequality) If $\alpha > 0$ and $\beta > 0$, then

$$I^{\alpha+\beta} v = I^\alpha I^\beta v,$$

is satisfied at almost every point in $[0, T]$ for $v \in L_p(0, T)$, $1 \le p \le \infty$.

2.3 Classical Inequalities

In this section, we display some inequalities that we will use in the next chapters.

Inequalities 2.8 *(i)* *(***[43], Cauchy-Schwarz Inequality)** If $v, w \in L_2(0, T)$, then $v\, w \in L_1(0, T)$ and

$$|\langle v, w\rangle| \le \|v\| \|w\|.$$

(ii) ([21], Poincare's Inequality) If Ω is a bounded domain in $\mathbb{R}^n$, then there exists a constant C such that

$$\|v\|_1 \le C \|\nabla v\|, \quad \forall v \in \mathbb{H}_0^1.$$

(iii) (Geometric Arithmetic Mean Inequality) If $a, b \in \mathbb{R}$, then for any $\epsilon > 0$,

$$ab \leq \frac{\epsilon\, a^2}{2} + \frac{b^2}{2\,\epsilon}.$$

Lemma 2.9 ([40], Discrete Gronwall's Inequality) Let $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ be sequences of non-negative numbers with $b_1 \leq b_2 \leq, ..., \leq b_n$. Assume that, for $C \geq 0$ and weights $(k_1, ..., k_{n-1}) \in \mathbb{R}_+^{n-1}$, $a_1 \leq b_1$, $a_j \leq b_j + C\sum_{i=1}^{j-1}(k_i\, a_i)$, $j = 2, ..., n$. Then

$$a_j \leq b_j \, \exp\left(C \sum_{i=1}^{j-1} k_i\right), \quad j = 2, ..., n,$$

where $\mathbb{R}_+$ is the set of positive real numbers.

Lemma 2.10 ([40],Lemma 3.1, Inverse Inequality) If ϕ is a polynomial of degree r on the interval $I = (a, b)$, then we have

$$\|\phi\|_{L_\infty(I)}^2 \leq C \log(r+1) \int_a^b |\phi'(t)|^2 (t-a) dt + C|\phi(b)|^2.$$

We end this section with the Generalized Minkowski Inequality.

Lemma 2.11 ([39]:page 9, Generalized Minkowski Inequality) For I_1=$[a, b]$, I_2=$[c, d]$ and $1 \leq p \leq \infty$. Let $f(x, y)$ be a measurable function on

$I_1 \times I_2$, then

$$\left(\int_{I_1} \left| \int_{I_2} f(x,y) dy \right|^p dx \right)^{1/p} \leq \int_{I_2} \left(\int_{I_1} |f(x,y)|^p \, dx \right)^{1/p} dy.$$

2.4 Additional Results

Definition 2.12 ([5]) Let $\mathbf{A} = (\mathbf{a_{ij}})$ be an $n \times m$ matrix and $\mathbf{B}$ be a $p \times q$ matrix, then the kronecker product of $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$, is a $np \times mq$ matrix defined by

$$\mathbf{A} \otimes \mathbf{B} := \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{bmatrix}.$$

Definition 2.13 ([1, 42]) A set $\{x_j\}_{j=1}^{\infty}$ in a Hilbert space V is orthonormal if $\langle x_i, x_j \rangle = \delta_{i,j}$, where $\delta_{i,j}$ defined by

$$\delta_{i,j} = \begin{cases} 1, & \text{for } \; i = j, \\ 0, & \text{for } \; i \neq j. \end{cases}$$

Definition 2.14 ([1, 42]) A set $\{x_j\}_{j=1}^{\infty}$ is said to be a basis for a Hilbert space V if every y in V can be expressed uniquely in the form $y = \sum_{i=1}^{\infty} c_i \, x_i$.

Remark 2.15 ([1, 42]) A set $\{x_j\}_{j=1}^{\infty}$ is said to be a complete orthonormal set for a Hilbert space V or orthonormal basis for a Hilbert space V if it is orthonormal in V and is a basis for V.

Theorem 2.16 ([15],[20]:page 2, Fundamental Theorem of Calculus) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is differentiable almost everywhere with integrable derivative such that $f(t) = \int_a^t f'(x)\, dx + f(a)$ holds if and only if f is absolutely continuous.

Lemma 2.17 ([12], Bramble Hilbert Lemma) Let Ω be a convex domain of diameter h and let $g \in \mathbb{H}^m,\ m \in N$. Then there exists a polynomial P of degree $m-1$ for which $|g - P|_k \leq Ch^{m-k}|g|_m,\ k = 0, 1, ..., m-1$.

Theorem 2.18 ([1], Plancherel's Theorem) *If $f \in L_2(\mathbb{R}^n)$ and $g \in L_2(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \widehat{f}(y)\overline{\widehat{g}(y)}dy$$

where $\hat{f}$ and $\hat{g}$ are the fourier transforms of f and g defined by

$$\widehat{f}(y) = \int_{-\infty}^{\infty} e^{-iyt} f(t)dt,$$

$$\widehat{g}(y) = \int_{-\infty}^{\infty} e^{-iyt} g(t)dt,$$

respectively.

Remark 2.19 ([21], Green's Formula) Let $u \in C^2$ and $v \in C^1$, then

$$\int_{\Omega} \nabla u\, \nabla v dx = \int_{\Gamma} \frac{\partial u}{\partial n}\, v ds - \int_{\Omega} \Delta u\, v dx,$$

where $\frac{\partial u}{\partial n} = n.\nabla u$ is the exterior normal derivative of u on Γ.

Theorem 2.20 ([1, 2]) Let n be a positive integer , and let A be a square matrix of order n. Then the following are equivalent statements about the linear system $Ax = b$

(i) For each right side b, the system $Ax = b$ has exactly one solution x

(ii) For each right side b, the system $Ax = b$ has at least one solution x

(iii) The homogeneous form of $Ax = 0$ has exactly one solution

$$x_1 = x_2 = ... = x_n = 0$$

(iv) $\det(A) \neq 0$.

2.5 Fractional Inequalities

In this section, we show some important fractional inequalities that we will use through the dissertation.

Lemma 2.21 If $0 < \alpha < 1$, $u \in AC[0,T]$ and ${}^cD^{1-\frac{\alpha}{2}}u \in L_2(0,T)$, then we have

$$|u(t)|^2 \leq 2|u(0)|^2 + 2\frac{T^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})} \int_0^T |{}^cD^{1-\frac{\alpha}{2}}u(s)|^2\, ds. \qquad (2.1)$$

for $0 < t \leq T$.

Proof. By applying fundamental theorem of calculus (2.16), Cauchy-Schwarz inequality (2.8)(i) and using semigroup property (2.7)(iii), we find

$$\begin{aligned}
|u(t)| &\leq |u(0)| + |(\mathrm{I}\mathrm{D})u(t)| \\
&= |u(0)| + |(\mathrm{I}^{1-\frac{\alpha}{2}}\mathrm{I}^{\frac{\alpha}{2}}\mathrm{D})u(t)| \\
&\leq |u(0)| + \int_0^t \frac{(t-s)^{-\frac{\alpha}{2}}}{\Gamma(1-\frac{\alpha}{2})}\, |{}^c\mathrm{D}^{1-\frac{\alpha}{2}}u(s)|\, ds \\
&\leq |u(0)| + \left(\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma^2(1-\frac{\alpha}{2})} ds\right)^{\frac{1}{2}} \left(\int_0^t |{}^c\mathrm{D}^{1-\frac{\alpha}{2}}u(s)|^2\, ds\right)^{\frac{1}{2}} \\
&= |u(0)| + \left(\frac{t^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\right)^{\frac{1}{2}} \left(\int_0^t |{}^c\mathrm{D}^{1-\frac{\alpha}{2}}u(s)|^2\, ds\right)^{\frac{1}{2}} \\
&\leq |u(0)| + \left(\frac{T^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\right)^{\frac{1}{2}} \left(\int_0^T |{}^c\mathrm{D}^{1-\frac{\alpha}{2}}u(s)|^2\, ds\right)^{\frac{1}{2}}. \qquad (2.2)
\end{aligned}$$

Squaring both sides of the inequality (2.2) and using the geometric-arithmetic mean inequality (2.8)(iii), then we obtain the desired result.

∎

Remark 2.22 For the rest of the thesis, we consider a partition of the interval $[0, T]$ given by the points:

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and let $k_n = t_n - t_{n-1}$, $v^n = v(t_n)$ and $\nabla v^n = \nabla v(t_n), 0 \leq n \leq N$.

Definition 2.23 We denote by $C^1(\bigcup_{n=1}^N I_n)$ is the space of functions $v : \bigcup_{n=1}^N I_n \to \mathbb{R}$ such that the restriction $v'|_{I_n}$ extends to a continuous function on the closed interval $[t_{n-1}, t_n], 1 \leq n \leq N$.

Lemma 2.24 Let $v,\, w \in C[0,t_n] \cap C^1(\bigcup_{j=1}^{n} I_j),\ 1 \le n \le N$. Then for the memory term $\int_0^{t_n} {}^c\!D^{1-\alpha} v(\tau)\, v'(\tau)\, d\tau$, we have

(i) If

$$\max_{j=0}^{n} |v^j| = 0 \quad \text{and} \quad \int_0^{t_n} v'(\tau)\, {}^c\!D^{1-\alpha} v(\tau)\, d\tau = 0,$$

then $v \equiv 0$ on $(0, t_n)$.

(ii) The memory term satisfies the coercivity property:

$$\int_0^{t_n} {}^c\!D^{1-\alpha} v(\tau)\, v'(\tau)\, d\tau \ge c_\alpha \int_0^{t_n} |{}^c\!D^{1-\frac{\alpha}{2}} v(\tau)|^2\, d\tau, \quad c_\alpha = \cos(\alpha\pi/2).$$

(iii) The memory term satisfies the continuity property:

$$\left| \int_0^{t_n} {}^c\!D^{1-\alpha} w(\tau)\, v'(\tau)\, d\tau \right| \le \frac{1}{c_\alpha^2} \int_0^{t_n} {}^c\!D^{1-\alpha} v(\tau)\, v'(\tau) d\tau \int_0^{t_n} {}^c\!D^{1-\alpha} w(\tau)\, w'(\tau)\, d\tau\,.$$

Proof. To prove this Lemma, we will follow the derivation used to obtain Lemma 3.1 in [35]. The proof of (i) is based on using Fourier transform. To that end, we extend v by zero outside the interval $(0, t_n]$ and we extend $\omega_\alpha(\tau) = \tau^{\alpha-1}$ by zero for $\tau \le 0$, i.e, let

$$\tilde{v}(\tau) = \begin{cases} v(\tau) & \text{if } \ 0 < \tau \le t_n \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{w}_\alpha(\tau) = \begin{cases} w_\alpha(\tau) & \text{if } \ \tau > 0 \\ 0 & \text{if } \ \tau \le 0. \end{cases}$$

Let $\widehat{v}$ denotes the fourier transform of v,

$$\widehat{v}(y) = \int_{-\infty}^{\infty} e^{-iy\tau} v(\tau) d\tau.$$

By using Plancherel's theorem (2.18), the fact $\overline{\widehat{\tilde{v}'}(y)} = \widehat{\tilde{v}'}(-y)$, v is a real-valued function, and $\widehat{\omega}_\alpha(y) = (iy)^{-\alpha}$, we find that

$$\begin{aligned}
\int_0^{t_n} v'(\tau)\, {}^c\!D^{1-\alpha} v(\tau)\, d\tau &= \int_{-\infty}^{\infty} \overline{\tilde{v}'(\tau)}\, {}^c\!D^{1-\alpha} \tilde{v}(\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} \overline{\tilde{v}'(\tau)} \int_{-\infty}^{\infty} \tilde{\omega}_\alpha(\tau - s) \tilde{v}'(s) ds\, d\tau \\
&= \frac{1}{2\pi} Re \int_{-\infty}^{\infty} \overline{\widehat{\tilde{v}'}(y)}\, \widehat{\tilde{\omega}_\alpha}(y) \widehat{\tilde{v}'}(y)\, dy \\
&= \frac{1}{2\pi} Re \int_{-\infty}^{\infty} \widehat{\tilde{v}'}(-y)\, (iy)^{-\alpha} \widehat{\tilde{v}'}(y)\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} Re(iy)^{-\alpha} |\widehat{\tilde{v}'}(y)|^2\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} Re(\exp^{-\alpha(\ln|y| + i\pi/2)}) |\widehat{\tilde{v}'}(y)|^2\, dy \\
&= \frac{\cos(\alpha\pi/2)}{2\pi} \int_{-\infty}^{\infty} |y|^{-\alpha} |\widehat{\tilde{v}'}(y)|^2\, dy \qquad (2.3) \\
&= \frac{\cos(\alpha\pi/2)}{\pi} \int_0^{\infty} y^{-\alpha} |\widehat{\tilde{v}'}(y)|^2\, dy
\end{aligned}$$

Since we have $\int_0^{t_n} v'(\tau)\, {}^c\!D^{1-\alpha} v(\tau)\, d\tau = 0$ and $y^{-\alpha} \cos(\alpha\pi/2) > 0$ for $y \in (0, \infty)$ and $0 < \alpha < 1$, it follows that $\widehat{\tilde{v}'} = 0$ almost everywhere in $(0, \infty)$. This leads to $\tilde{v}' = 0$ almost everywhere in $(0, \infty)$. This implies $\tilde{v} = constant$ almost everywhere in $(0, \infty)$. Using this and because v vanishes outside the interval $[0, t_n]$ then we have $v = constant$. Using the first equation in part (i) of this lemma, we obtain

$v \equiv 0$ on $[0, t_n]$ and hence, the proof of (i) is completed.

To prove (ii) we write $|y|^{-\alpha} = |(iy)^{-\alpha/2}|^2$, using (2.3) and Plancherel's theorem again, we have

$$
\begin{aligned}
\int_0^{t_n} v'(\tau)\, {}^cD^{1-\alpha} v(\tau)\, d\tau &= \frac{\cos(\alpha\pi/2)}{2\pi} \int_{-\infty}^{\infty} |(iy)^{-\alpha/2} \widehat{\tilde{v}'}(y)|^2 \, dy \\
&= \frac{\cos(\alpha\pi/2)}{2\pi} \int_{-\infty}^{\infty} |\widehat{\omega}_{\alpha/2}(y) \widehat{\tilde{v}'}(y)|^2 \, dy \\
&= \frac{\cos(\alpha\pi/2)}{2\pi} \int_{-\infty}^{\infty} |\widehat{{}^cD^{1-\frac{\alpha}{2}} \tilde{v}}(y)|^2 \, dy \\
&= \cos(\alpha\pi/2) \int_{-\infty}^{\infty} |{}^cD^{1-\frac{\alpha}{2}} \tilde{v}(t)|^2 \, dt \\
&= \cos(\alpha\pi/2) \Bigg(\int_{-\infty}^{0} \big|\mathcal{I}_1(t) + \mathcal{I}_2(t)\big|^2 \\
&+ \int_0^{\infty} |\mathcal{I}_1(t) + \mathcal{I}_3(t) + \mathcal{I}_4(t)|^2 \, dt \Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_1(\tau) &= \int_{-\infty}^{0} \tilde{w}_{\alpha/2}(\tau - s) \tilde{v}'(s) ds, \\
\mathcal{I}_2(\tau) &= \int_{0}^{\infty} \tilde{w}_{\alpha/2}(\tau - s) \tilde{v}'(s) ds, \\
\mathcal{I}_3(\tau) &= \int_{0}^{t_n} \tilde{w}_{\alpha/2}(\tau - s) \tilde{v}'(s) ds,
\end{aligned}
$$

and

$$
\mathcal{I}_4(\tau) = \int_{t_n}^{\infty} \tilde{w}_{\alpha/2}(\tau - s) \tilde{v}'(s) ds.
$$

We notice that $\mathcal{I}_1(\tau) = 0$ since $\tilde{v}'(s) = 0$ on $(-\infty, 0)$, $\forall \tau \in (-\infty, 0)$. Also, $\mathcal{I}_2(\tau) = 0$ since $\tilde{w}_{\alpha/2} = 0$ on $(0, \infty)$, $\forall \tau \in (-\infty, 0)$. Moreover, $\mathcal{I}_3(\tau) =$

$\int_0^{t_n} w_{\alpha/2}(\tau - s)v'(s)ds$ since $\tilde{w}_{\alpha/2} = w_{\alpha/2}$, and $\tilde{v}(s) = v(s)$ on $(0, t_n)$, $\forall \tau \in (0, \infty)$. Finally $\mathcal{I}_4(\tau) = 0$ since $\tilde{w}_{\alpha/2} = 0$ on (t_n, ∞), $\forall \tau \in (0, t_n]$ and $\tilde{v}(s) = 0$ on (t, ∞), $\forall \tau \in (t_n, \infty)$. This leads to

$$\begin{aligned}\int_0^{t_n} v'(\tau)\, {}^c\!D^{1-\alpha} v(\tau)\, d\tau &= \cos(\alpha\pi/2) \int_0^{\infty} |{}^c\!D^{1-\frac{\alpha}{2}} v(\tau)|^2\, d\tau \\ &\geq \cos(\alpha\pi/2) \int_0^{t_n} |{}^c\!D^{1-\frac{\alpha}{2}} v(\tau)|^2\, d\tau \qquad (2.4)\end{aligned}$$

This proves part (ii).

To prove the inequality in (iii), from (2.3), Plancherel's theorem and Cauchy Schwarz inequality 2.8(i), we have

$$\begin{aligned}&\left| \int_0^{t_n} v'(\tau)\, {}^c\!D^{1-\alpha} w(\tau)\, d\tau \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \overline{\widehat{\tilde{v}'(y)}}\ (iy)^{-\alpha} \widehat{\tilde{w}}'(y) \right| dy \\ &\leq \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} |(iy)^{-\alpha/2}\, \widehat{\tilde{v}}'(y)|^2\, dy \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |(iy)^{-\alpha/2}\, \widehat{\tilde{w}}'(y)|^2\, dy \right)^{\frac{1}{2}} \\ &= \frac{1}{\pi} \left(\int_0^{\infty} y^{-\alpha} |\widehat{\tilde{v}}'(y)|^2\, dy \right)^{\frac{1}{2}} \left(\int_0^{\infty} y^{-\alpha} |\widehat{\tilde{w}}'(y)|^2\, dy \right)^{\frac{1}{2}} \\ &= c_\alpha \left(\int_0^{t_n} v'(\tau)\, {}^c\mathrm{D}^{1-\alpha} v(\tau) d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_n} w'(\tau)\, {}^c\!D^{1-\alpha} w(\tau)\, d\tau \right)^{\frac{1}{2}}.\end{aligned}$$

∎

The proof is completed now.

Lemma 2.25 Let $v,\ w \in C([0, t_n]; L_2(\Omega)) \cap C^1(\bigcup_{j=1}^n I_j; L_2(\Omega))$, $1 \leq n \leq N$. Then for the memory term $\int_0^{t_n} \langle v'(\tau), {}^c\!D^{1-\alpha} v(\tau) \rangle\, d\tau$, we have

(i) If $\max_{j=0}^n \|v^j\| = 0$ and $\int_0^{t_n} \langle v'(\tau), {}^c\!D^{1-\alpha} v(\tau) \rangle\, d\tau = 0$, then $v \equiv 0$ on

$(0, t_n)$.

(ii) The memory term satisfies the coercivity property:

$$\int_0^{t_n} \langle v'(\tau), {}^c\!D^{1-\alpha} v(\tau)\rangle \, d\tau \geq c_\alpha \int_0^{t_n} \|{}^c\!D^{1-\frac{\alpha}{2}} v(\tau)\|^2 d\tau \text{ with } c_\alpha = \cos(\alpha\pi/2).$$

(iii) The memory term satisfies the continuity property:

$$\left| \int_0^{t_n} \langle v'(\tau), {}^c\!D^{1-\alpha} w(\tau)\rangle \, d\tau \right| \leq \frac{\epsilon}{2\, c_\alpha^2} \int_0^{t_n} \langle v'(\tau), {}^c\!D^{1-\alpha} v(\tau)\rangle \, d\tau$$
$$+ \frac{1}{2\,\epsilon} \int_0^{t_n} \langle w'(\tau), {}^c\!D^{1-\alpha} w(\tau)\rangle \, d\tau \quad \text{for any} \quad \epsilon > 0.$$

Proof. Let $\{\Phi_m\}_{m=0}^{\infty}$ be the complete set of eigenfunctions of the operator $-\Delta$ subject to homogeneous dirichlet boundary conditions. Then we can write

$$v = \sum_{m=0}^{\infty} v_m \Phi_m \quad \text{and} \quad w = \sum_{n=0}^{\infty} w_n \Phi_n,$$

where $v_m = \langle v, \Phi_m\rangle$ and $w_m = \langle w, \Phi_m\rangle$.

Using the orthonormality property of the eignfunctions $\{\Phi_m\}_{m=0}^{\infty}$ and Lemma 2.24 we obtain the first and second part of this lemma. To prove the third part of this lemma we follow the same derivation in the first and second part of this lemma and the geometric-arithmetic mean inequality (2.8)(iii) we obtain the desired results.

∎

CHAPTER 3

Piecewise Linear, Discontinuous Petrov Galerkin Method

We start this chapter by showing some stability properties of the continuous solution of problem (1.3). We introduce a time-stepping DPG method for (1.3) and show the existence, uniqueness and stability of the approximate solution. Then, we conduct the error analysis of the method and prove convergence of order k^1, where k is the maximum time step. Finally, we present the implementation of DPG scheme to problem (1.3).

3.1 Stability of The Continuous Solution

A stability property of the solution for problem (1.3) will be proved in the next theorem. More precisely, we find an upper bound of $\|u(t)\|_1$ that depends on the initial data and the source function.

Theorem 3.1 We assume that $f \in H^1([0,T];L_2(\Omega))$, and $u(0) \in H_0^1(\Omega)$, then $u \in L_\infty((0,T);H_0^1(\Omega))$ and

$$\|u(t)\|_1^2 \leq C_1 \left(\|\nabla u(0)\|^2 + \|f(0)\|^2 + \int_0^t \|f'(s)\|^2 \, ds \right), \tag{3.1}$$

where C_1 depends on Ω and T.

Proof. By taking the inner product of (1.3) with u', using Green's formula 2.19 and integrating over the interval $[0,t]$, we obtain

$$\int_0^t \langle {}^c\!D^{1-\alpha} u(\tau), u'(\tau) \rangle \, d\tau + \int_0^t \langle \nabla u(\tau), \nabla u'(\tau) \rangle \, d\tau = \int_0^t \langle f(\tau), u'(\tau) \rangle \, d\tau. \tag{3.2}$$

Using part (ii) of Lemma 2.25 and the equality

$$\int_0^t \langle \nabla u(\tau), \nabla u'(\tau) \rangle \, d\tau = \frac{1}{2} \int_0^t \frac{d}{d\tau} \|\nabla u(\tau)\|^2 \, d\tau = \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\|\nabla u(0)\|^2,$$

we notice that

$$\|\nabla u(t)\|^2 - \|\nabla u(0)\|^2 \leq 2 \left| \int_0^t \langle f(\tau), u'(\tau) \rangle \, d\tau \right|. \tag{3.3}$$

To bound the right hand side of the inequality (3.3), we integrate by parts to obtain

$$\int_0^t \langle f(\tau), u'(\tau) \rangle \, d\tau = \langle f(t), u(t) \rangle - \langle f(0), u(0) \rangle - \int_0^t \langle f'(\tau), u(\tau) \rangle \, d\tau, \tag{3.4}$$

and using Cauchy-Schwarz inequality 2.8(i), Poincare's inequality 2.8(ii) and the geometric-arithmetic mean inequality 2.8(iii), yield

$$\begin{aligned}\left|\int_0^t \langle f(\tau), u'(\tau)\rangle \, d\tau\right| &\leq C\|f(t)\|\|\nabla u(t)\| + \|f(0)\|\|u(0)\| + C\int_0^t \|f'(\tau)\|\|\nabla u(\tau)\| \, d\tau \\ &\leq C\|f(t)\|^2 + \|f(0)\|\|u(0)\| + C\int_0^t \|f'(\tau)\|^2 \, d\tau \\ &\qquad + \frac{1}{4}\|\nabla u(t)\|^2 + \frac{1}{4}\int_0^t \|\nabla u(\tau)\|^2 \, d\tau,\end{aligned} \tag{3.5}$$

and consequently, we notice from (3.3) that

$$\begin{aligned}&\|\nabla u(t)\|^2 \\ &\leq 2\|\nabla u(0)\|^2 + 4\|f(0)\|\|u(0)\| + 4C\|f(t)\|^2 + 4C\int_0^t \|f'(s)\|^2 \, ds + \int_0^t \|\nabla u(s)\|^2 \, ds,\end{aligned}$$

for any $0 < t < T$. By writing

$$f(t) = f(0) + \int_0^t f'(s) \, ds,$$

we obtain

$$\begin{aligned}&\|\nabla u(t)\|^2 \\ &\leq 2\|\nabla u(0)\|^2 + (2+2C)\|f(0)\|^2 + 2\|u(0)\|^2 + 4C(2T+1)\int_0^t \|f'(s)\|^2 \, ds + \int_0^t \|\nabla u(s)\|^2 \, ds.\end{aligned}$$

Hence, using the standard Gronwall's inequality, we obtain

$$\|\nabla u(t)\|^2 \leq 2\|\nabla u(0)\|^2 + (2+2C)\|f(0)\|^2 + 2\|u(0)\|^2 + 4C(2T+1)\int_0^t \|f'(s)\|^2\,ds.$$

Therefore, again an application of Poincare's inequality 2.8(ii) we obtain (3.1), where $C_1 = 2C(4T+2)+6$. ∎

3.2 Time-Stepping DPG Method

To describe the time-stepping discontinuous Petrov-Galerkin (DPG) method, we introduce a (possibly nonuniform) time partition of the interval $[0,T]$ given by the points:

$$0 = t_0 < t_1 < \cdots < t_N = T\,.$$

We set $k_n = t_n - t_{n-1}$. The maximum step-size is defined as $k = \max_{1\leq n\leq N} k_n$. The supremum of the function u is defined as $\|u(t)\|_{I_n} = \sup_{t\in I_n}\|u(t)\|$. We assume that for a fixed parameter $\gamma \geq 1$, there holds

$$c_\gamma k^\gamma \leq k_1 \leq C_\gamma k^\gamma, \tag{3.6}$$

and

$$c_\gamma k {t_n}^{1-\frac{1}{\gamma}} \leq k_n \leq C_\gamma k {t_n}^{1-\frac{1}{\gamma}} \quad \text{and} \quad t_n \leq C_\gamma t_{n-1} \quad \text{for} \quad 2 \leq n \leq N. \tag{3.7}$$

For instance, these properties hold if

$$t_n = \left(\frac{n}{N}\right)^{\gamma} T \qquad \text{for} \qquad 0 \le n \le N. \tag{3.8}$$

Next, for any Sobolev space H, and for a fixed $m \ge 1$, we introduce the trial space

$$\mathcal{W}_m(H) = \{v \in C([0,T]; H) : v|_{I_n} \in P_m(H), 1 \le n \le N\}$$

and the test space

$$\mathcal{T}_m(H) = \{v \in L_2((0,T); H) : v|_{I_n} \in P_{m-1}(H), 1 \le n \le N\},$$

where $P_m(H)$ denotes the space of polynomials of degree $\le m$ in the time variable t, with coefficients in H. For $m = 1$ we set $\mathcal{W}(H) := \mathcal{W}_m(H)$ and $\mathcal{T}(H) := \mathcal{T}_m(H)$.

Now, we are ready to define our numerical scheme. Following [23, 30], we define the DPG approximation $U \in \mathcal{W}(H_0^1)$ of the solution u of problem (1.3) is now defined as follows: Find $U \in \mathcal{W}(H_0^1)$ such that

$$\begin{aligned} G_N(U, X) &= \int_0^T \langle f(t), X(t)\rangle \, dt \qquad \text{for all } X \in \mathcal{T}(H_0^1), \\ U(0) &= g(x), \end{aligned} \tag{3.9}$$

where G_N is the global bilinear form defined by

$$G_N(U, X) := \int_0^T \langle {}^{c}D^{1-\alpha} U(t), X(t)\rangle \, dt + \int_0^T \langle \nabla U(t), \nabla X(t)\rangle \, dt.$$

We notice that the approximate solution $U(x,t)$ is piecewise linear in time with coefficients in $H_0^1(\Omega)$.

3.3 Existence and Uniqueness

In this section, we will show the existence and uniqueness of the DPG solution.

Theorem 3.2 The discrete solution U of (3.9) exists and is unique.

Proof. Since the operator $-\Delta$ possesses a complete orthonormal eigensystem $\{\lambda_j, \Phi_j\}_{j=1}^{\infty}$, problem (3.9) can be reduced to a finite linear algebraic equation on each subinterval I_n. To see this, let P_1 be the space of piecewise linear function in time. If we now take $X(t) = \Phi_j$ on I_n and zero elsewhere in (3.9), then we find that

$$\int_{t_{n-1}}^{t_n} \Big({}^{c}D^{1-\alpha} U_j(t) + \lambda_j U_j(t) \Big)\, dt = \int_{t_{n-1}}^{t_n} f_j(t)\, dt \tag{3.10}$$

for all $j = 1, 2, 3, \ldots$ with $U_j = \langle U, \Phi_j \rangle \in P_1$ and $f_j = \langle f, \Phi_j \rangle$. By using theorem 2.20 we note that the finite dimensionality of problem (3.10) on each subinterval I_n the existence of the scalar function U_j follows from its uniqueness. Since the DPG solution U is constructed step by step, it is enough to show the uniqueness on the first time interval $[0, t_1]$. That is, it is enough to consider $n = 1$ in (3.10) (for $n \geq 2$ the proof is completely analogous). To this end, let U_{j_1} and U_{j_2} be two DPG solutions on I_1. By linearity, the difference $V_j := (U_{j_1} - U_{j_2})|_{I_1}$ then satisfies:

$$\int_0^{t_1} \left({}^c\!D^{1-\alpha} V_j(t) + \lambda_j |V_j(t)|^2 \right) dt = 0, \qquad j = 1, 2, \cdots \tag{3.11}$$

with $V_j^0 = 0$. Using lemma 2.25 (ii) and (3.11) we conclude that

$$\int_0^{t_1} {}^c\!D^{1-\alpha} V_j(t)\, dt = 0$$

Since $V_j^0 = 0$, $V_j^1 = 0$ and using Lemma 2.25 (i), we obtain $V_j \equiv 0$ on $[0, t_1]$. This completes the proof. ∎

3.4 Stability of the Approximate Solution

In this section we show the stability of the approximate solution U of (3.9) in the following theorem.

Theorem 3.3 Let $f \in AC([0,T]; H^1(\Omega))$ and $g \in H^1(\Omega)$, then for $1 \le n \le N$, the DPG solution U of (3.9) satisfies

$$c_\alpha \int_0^{t_n} \|{}^c\!D^{1-\frac{\alpha}{2}} U(t)\|^2 dt + \|\nabla U^n\|^2 \le C_1 G_n(f, g),$$

where $c_\alpha = \cos(\alpha\pi/2)$, and

$$G_n(f,g) = \|f(t_n)\|^2 + \|f(0)\|\|g\| + (1 + k_1)\|\nabla g\|^2 + \int_0^{t_n} \|f'(t)\|^2\, dt\,.$$

Proof. In (3.9) we choose $X|_{(0,t_n)} = U'$ and zero elsewhere,

$$\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha} U(t), U'(t)\rangle \, dt + \int_0^{t_n} \langle \nabla U(t), \nabla U'(t)\rangle \, dt = \int_0^{t_n} \langle f(t), U'(t)\rangle \, dt,$$

$1 \leq n \leq N$. Following the derivation used to obtain (3.5), we have

$$\begin{aligned} \left| \int_0^{t_n} \langle f(t), U'(t)\rangle \, dt \right| \leq C\|f(t_n)\|^2 + \|f(0)\|\|g\| + C\int_0^{t_n} \|f'(t)\|^2 \, dt \\ + \frac{1}{4}\|\nabla U^n\|^2 + \frac{1}{4}\int_0^{t_n} \|\nabla U(t)\|^2 \, dt. \end{aligned} \tag{3.12}$$

By using the equality

$$\int_0^{t_n} \langle \nabla U(t), \nabla U'(t)\rangle \, dt = \frac{1}{2}\|\nabla U^n\|^2 - \frac{1}{2}\|\nabla g\|^2,$$

we obtain

$$\begin{aligned} \|\nabla U^n\|^2 + 4\int_0^{t_n} \langle {}^cD^{1-\alpha} U(t), U'(t)\rangle \, dt \\ \leq 2\|\nabla U^0\|^2 + 4C\|f(t_n)\|^2 + 4\|f(0)\|\|g\| + 4C\int_0^{t_n} \|f'(t)\|^2 \, dt \\ + \int_0^{t_n} \|\nabla U(t)\|^2 \, dt. \end{aligned}$$

Also from the coercivity property in Lemma 2.25 we have

$$\int_0^{t_n} \langle {}^cD^{1-\alpha} U(t), U'(t)\rangle \, dt \geq c_\alpha \int_0^{t_n} \|{}^cD^{1-\frac{\alpha}{2}} U(t)\|^2 \, dt. \tag{3.13}$$

Thus,

$$\begin{aligned}
&\|\nabla U^n\|^2 + 4c_\alpha \int_0^{t_n} \|{}^cD^{1-\frac{\alpha}{2}}U(t)\|^2 dt \\
&\le 2\|\nabla U^0\|^2 + 4C\|f(t_n)\|^2 + 4\|f(0)\|\|g\| + 4C\int_0^{t_n} \|f'(t)\|^2\, dt \\
&\quad + \int_0^{t_n} \|\nabla U(t)\|^2\, dt. \qquad (3.14)
\end{aligned}$$

However, $U|_{I_j}$ is a linear polynomial (in time), so $\|U\|_{I_j} \le \max\{\|U^{j-1}\|, \|U^j\|\}$ and hence,

$$\begin{aligned}
\int_0^{t_n} \|\nabla U(t)\|^2\, dt &\le \sum_{j=1}^{n} k_j \Big(\|\nabla U^{j-1}\|^2 + \|\nabla U^j\|^2 \Big) \\
&= \sum_{j=1}^{n-1} (k_j + k_{j+1}) \|\nabla U^j\|^2 + k_1 \|\nabla U^0\|^2 + k_n \|\nabla U^n\|^2 \\
&\le C \sum_{j=1}^{n-1} k_j \|\nabla U^j\|^2 + k_1\|\nabla U^0\|^2 + k_n\|\nabla U^n\|^2, 1 \le n \le N,
\end{aligned} \qquad (3.15)$$

where in the last inequality we shifted the summation indices in the first term and used the mesh assumption, $k_{j+1} \le C_\gamma k_j$ for $j \ge 1$.

Insert (3.15) in (3.14) and using the assumption that k_n is sufficiently small, then for $1 \le n \le N$, we obtain

$$\begin{aligned}
&c_\alpha \int_0^{t_n} \|{}^cD^{1-\frac{\alpha}{2}}U(t)\|^2 dt + (1-k_n)\|\nabla U^n\|^2 \\
&\qquad \le (6+8C)G_n(f,g) + C_\gamma \sum_{j=1}^{n-1} k_j \|\nabla U^j\|^2 . \qquad (3.16)
\end{aligned}$$

Multiplying both sides of (3.16) by $(1-k_n)^{-1}$, then we have

$$4c_\alpha(1-k_n)^{-1}\int_0^{t_n}\|{}^cD^{1-\frac{\alpha}{2}}U(t)\|^2dt+\|\nabla U^n\|^2$$
$$\leq(6+8C)(1-k_n)^{-1}G_n(f,g)+C_\gamma(1-k_n)^{-1}\sum_{j=1}^{n-1}k_j\|\nabla U^j\|^2\,. \quad (3.17)$$

From (3.17), we have

$$\|\nabla U^n\|^2\leq(6+8C)(1-k_n)^{-1}G_n(f,g)+C_\gamma(1-k_n)^{-1}\sum_{j=1}^{n-1}k_j\|\nabla U^j\|^2$$
$$\leq(6+8C)(1-k)^{-1}G_n(f,g)+C_\gamma(1-k)^{-1}\sum_{j=1}^{n-1}k_j\|\nabla U^j\|^2\,. \quad (3.18)$$

Thus, by using Gronwall's inequality 2.9 ($a_n=\|\nabla U^n\|^2$ and $b_n=(6+8C)(1-k)^{-1}G_n(f,g)$, we notice that a_n and b_n are non-negative and b_n is increasing) we have

$$\|\nabla U^n\|^2\leq(6+8C)(1-k)^{-1}G_n(f,g)\exp\left(C_\gamma(1-k)^{-1}\sum_{i=1}^{n-1}k_i\right)$$
$$\leq(6+8C)(1-k)^{-1}G_n(f,g)\exp(C_\gamma k(1-k)^{-1}(n-1))$$
$$\leq(6+8C)(1-k)^{-1}G_n(f,g)\exp(C_\gamma(1-k)^{-1}k(n-1)). \quad (3.19)$$

Inserting (3.19) in (3.17) yields

$$
\begin{aligned}
&4c_\alpha(1-k_n)^{-1}\int_0^{t_n}\|{}^cD^{1-\frac{\alpha}{2}}U(t)\|^2dt+\|\nabla U^n\|^2\\
&\le(6+8C)(1-k)^{-2}G_n(f,g)+k(1-k)^{-2}(n-1)\exp(C_\gamma k(1-k)^{-1}(n-1))G_{n-1}(f,g)\\
&\le(6+8C)(1-k)^{-2}G_n(f,g)+k(1-k)^{-2}(n-1)\exp(C_\gamma k(1-k)^{-1}(n-1))G_n(f,g)\\
&\le C_1G_n(f,g),\ \ 1\le n\le N,
\end{aligned}
$$

where $C_1=\max\{(6+8C)(1-k)^{-2},k(1-k)^{-2}(n-1)\exp(C_\gamma k(1-k)^{-1}(n-1))\}$.

This completes the proof. ∎

3.5 Error Estimate

This section is devoted to the derivation of error estimates for the time-stepping DPG method. To do so, we introduce piecewise linear interpolants that have been frequently used in the analysis of discontinuous Galerkin time- stepping methods and finite element methods for one dimensional reaction-diffusion problems. For any function $u\in C([0,T];L_2(\Omega))$ we define the piecewise linear interpolant $\Pi u:[0,T]\to L_2(\Omega)$ by setting

$$\Pi u(t_n)=u(t_n),\quad n=0,1,\dots,N\,.$$

Explicitly, we have

$$\Pi u(t)=u^n+\frac{u^n-u^{n-1}}{k_n}(t-t_n),\tag{3.20}$$

and

$$(\Pi u)'(t) = \frac{u^n - u^{n-1}}{k_n}. \tag{3.21}$$

The interpolation error admits the following integral representations

$$\begin{aligned}
\Pi u(t) - u(t) &= u^n - u(t) + \frac{u^n - u^{n-1}}{k_n}(t - t_n) \\
&= \int_t^{t_n} u'(s)ds + \frac{t - t_n}{k_n} \int_{t_{n-1}}^{t_n} u'(s)ds \\
&= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_t^{t_n} [u'(s) - u'(q)]\, ds\, dq \\
&= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_t^{t_n} \int_q^s u''(\ell) d\ell\, ds\, dq,
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
(\Pi u)'(t) - u'(t) &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} u'(s)ds - u'(t) \\
&= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} u'(s)ds - \frac{1}{k_n} \int_{t_{n-1}}^{t_n} u'(t)ds \\
&= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_t^s u''(\ell) d\ell\, ds.
\end{aligned} \tag{3.23}$$

In the next theorem, we derive the error estimates and some useful properties of the interpolation operator Π.

Theorem 3.4 For $2 \leq n \leq N$, then the following properties hold.

(i) For any $u|_{I_n} \in \mathbb{H}^2(I_n; L_2(\Omega))$,

$$\|(u - \Pi u)\|_{I_n}^2 + k_n^2 \|(u - \Pi u)'\|_{I_n}^2 \leq C k_n^3 \int_{t_{n-1}}^{t_n} \|u''(t)\|^2\, dt.$$

(ii) For any $u|_{I_n} \in \mathbb{H}^2(I_n; L_2(\Omega))$,

$$\int_{t_{n-1}}^{t_n} \|(u - \Pi u)(t)\|^2 dt + k_n^2 \int_{t_{n-1}}^{t_n} \|(u - \Pi u)'(t)\|^2 dt \le C k_n^4 \int_{t_{n-1}}^{t_n} \|u^{''}(t)\|^2 \, dt.$$

(iii) For any $u|_{I_n} \in \mathbb{H}^1(I_n; L_2(\Omega))$,

$$\int_{t_{n-1}}^{t_n} \|(\Pi u)'(t)\|^2 dt \le \int_{t_{n-1}}^{t_n} \|u'(t)\|^2 \, dt \,.$$

Proof. To estimate the first term in part (i), we use the explicit representation of Πu, (3.22) and Cauchy-Schwarz inequality 2.8(i) to obtain

$$\begin{aligned}
\|(\Pi u - u)\|_{I_n}^2 &= \sup_{t \in I_n} \|(\Pi u - u)(t)\|^2 \\
&\le \int_\Omega \left(\frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} |u''(\ell, x)| d\ell \, ds \, dq \, dt \right)^2 dx \\
&\le \int_\Omega \left(\frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} |u''(\ell, x)| d\ell \, ds \, dq \, dt \right)^2 dx \\
&\le k_n^3 \int_\Omega \int_{t_{n-1}}^{t_n} (u''(t, x))^2 \, dt \, dx \\
&= k_n^3 \int_{t_{n-1}}^{t_n} \int_\Omega (u''(t, x))^2 \, dx \, dt \\
&= k_n^3 \int_{t_{n-1}}^{t_n} \|u''(t)\|^2 \, dt, \quad n \ge 2 \,.
\end{aligned} \tag{3.24}$$

To bound the second term in part (i) we follow the same steps that we used to show the first term and using (3.23), then we have

$$
\begin{aligned}
\|(\Pi u - u)'\|_{I_n}^2 &\leq \int_\Omega \left(\int_{t_{n-1}}^{t_n} |u''(t,x)|\, dt \right)^2 dx \\
&\leq \int_\Omega \int_{t_{n-1}}^{t_n} 1\, dt \int_{t_{n-1}}^{t_n} (u''(t,x))^2\, dt\, dx \\
&= k_n \int_{t_{n-1}}^{t_n} \int_\Omega (u''(t,x))^2\, dx\, dt \\
&= k_n \int_{t_{n-1}}^{t_n} \|u''(t)\|^2\, dt \quad \text{for } n \geq 2\,. \qquad (3.25)
\end{aligned}
$$

For the first term in part (ii), from part (i) we have the bound

$$
\int_{t_{n-1}}^{t_n} \|(\Pi u - u)(t)\|^2\, dt \leq \|(\Pi u - u)\|_{I_n}^2 \int_{t_{n-1}}^{t_n} dt \leq k_n^4 \int_{t_{n-1}}^{t_n} \|u''(t)\|^2\, dt \qquad (3.26)
$$

Similarly, from part(i) we have the bound

$$
\int_{t_{n-1}}^{t_n} \|(\Pi u - u)'(t)\|^2\, dt \leq \|(\Pi u - u)'\|_{I_n}^2 \int_{t_{n-1}}^{t_n} dt \leq k_n^2 \int_{t_{n-1}}^{t_n} \|u''(t)\|^2\, dt. \qquad (3.27)
$$

By combining (3.26) and (3.27) we obtain (ii).

Finally, to show (iii), we use (3.23) and Cauchy-Schwarz inequality 2.8(i),

$$
\|(\Pi u)'\|_{I_n}^2 = \left(\frac{1}{k_n} \int_{t_{n-1}}^{t_n} \|u'(s)\|\, ds \right)^2 \leq \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \|u'(s)\|^2\, ds. \qquad (3.28)
$$

Thus,

$$
\int_{I_n} \|(\Pi u)'(t)\|^2\, dt \leq \|(\Pi u)'\|_{I_n}^2 \int_{t_{n-1}}^{t_n} dt \leq \int_{t_{n-1}}^{t_n} \|u'(s)\|^2\, ds
$$

∎

Now, we decompose the error $U - u$ into two terms,

$$U - u = (U - \Pi u) + (\Pi u - u) = \theta + \eta \tag{3.29}$$

where $\theta = U - \Pi u$ and $\eta = \Pi u - u$. Accordingly, we have the following bound.

Theorem 3.5 For $1 \le n \le N$, we have

$$\begin{aligned} c_\alpha \int_0^{t_n} \|{}^c D^{1-\frac{\alpha}{2}} \theta(t)\|^2 dt + \|\nabla \theta^n\|^2 \\ \le C\left(\|\nabla \theta^0\|^2 + \frac{\omega_{\alpha+1}(t_n)}{c_\alpha^2} \int_0^{t_n} \|\eta'(t)\|^2 dt + \sum_{j=1}^n k_j \|\nabla \eta'\|_{I_j}^2 \right). \end{aligned} \tag{3.30}$$

Proof. First, since the solution u of problem (1.3) satisfies

$$G_N(u, X) = \int_0^T \langle f(t), X(t) \rangle \, dt \quad \text{for all } X \in \mathcal{T},$$

$U - u$ satisfies the orthogonality condition

$$G_N(U - u, X) = 0 \quad \text{for all } X \in \mathcal{T}. \tag{3.31}$$

Hence,

$$G_N(\theta, X) = -G_N(\eta, X) = -\int_0^{t_N} \langle {}^c D^{1-\alpha} \eta, X \rangle \, dt - \int_0^{t_N} \langle \nabla \eta, \nabla X \rangle \, dt \tag{3.32}$$

By choosing $X|_{(0,t_n)} = \theta'$ and zero elsewhere, and using the equality:

$\langle \nabla\theta(t), \nabla\theta'(t)\rangle = \frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2$, we obtain

$$2\int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt + \|\nabla\theta^n\|^2 - \|\nabla\theta^0\|^2$$
$$= -2\int_0^{t_n} \langle {}^cD^{1-\alpha}\eta, \theta'\rangle\, dt - 2\int_0^{t_n} \langle \nabla\eta(t), \nabla\theta'(t)\rangle\, dt$$

From the continuity result in Lemma 2.24, we have

$$2\left|\int_0^{t_n} \langle {}^cD^{1-\alpha}\eta(t), \theta'(t)\rangle\, dt\right|$$
$$\leq \frac{1}{c_\alpha^2}\int_0^{t_n} \langle\, {}^cD^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt + \int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt.$$

By simplifying and employing the coercivity property in Lemma 2.25, we obtain

$$c_\alpha \int_0^{t_n} \|{}^cD^{1-\frac{\alpha}{2}}\theta(t)\|^2 dt \,+\, \|\theta^n\|^2 \,-\, \|\theta^0\|^2 \;\leq\; \frac{1}{c_\alpha^2}\int_0^{t_n} \langle {}^cD^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt \,+\, \mathcal{I}_n, \tag{3.33}$$

where

$$\mathcal{I}_n = -2\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \langle \nabla\eta(t), \nabla\theta'(t)\rangle\, dt.$$

Next, we want to estimate the first term on the right-hand side of (3.33). To do so, we recall the inequality in 2.7 (i): for $g \in L_2(0,T)$, there holds

$$\begin{aligned}\int_0^T \left(\int_0^t \omega_\alpha(t-s)\, g(s)\, ds\right)^2 dt &\leq \omega_{\alpha+1}(T)\int_0^T \omega_\alpha(T-t)\int_0^t g^2(s)\, ds\, dt \\ &\leq \omega_{\alpha+1}^2(T)\int_0^T g^2(t)\, dt \quad \text{for} \quad 0<\alpha<1\,.\end{aligned} \tag{3.34}$$

From Generalized Minkowski Inequality 2.11 and inequality (3.34), we have

$$\int_0^{t_n} \|{}^c\!D^{1-\alpha}\eta(t)\|^2 dt \le \omega_{\alpha+1}^2(t_n) \int_0^{t_n} \|\eta'(t)\|^2 dt. \tag{3.35}$$

Using the Cauchy-Schwarz inequality 2.8(i) again and inequality (3.35) we obtain

$$\begin{aligned}\left(\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt\right)^2 &\le \left(\int_0^{t_n} \|\eta'(t)\|\, \|{}^c\!D^{1-\alpha}\eta(t)\| dt\right)^2 \\ &\le \int_0^{t_n} \|\eta'(t)\|^2 dt \int_0^{t_n} \|{}^c\!D^{1-\alpha}\eta(t)\|^2 dt \\ &\le \left(\omega_{\alpha+1}(t_n) \int_0^{t_n} \|\eta'(t)\|^2 dt\right)^2 . \end{aligned} \tag{3.36}$$

To bound $\mathcal{I}_n$ on the right hand side of (3.33), we integrate by parts, use the property $\eta^j = 0$, $0 \le j \le N$, and the inequality

$$2\langle \nabla\eta(t),\, \nabla\theta'(t)\rangle \le 2\|\nabla\eta'(t)\|\, \|\nabla\theta(t)\| \le \|\nabla\eta'(t)\|^2 + \|\nabla\theta(t)\|^2,$$

$$|\mathcal{I}_n| \le 2\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \langle \nabla\eta(t), \nabla\theta'(t)\rangle\, dt \le \sum_{j=1}^{n} k_j \Big(\|\nabla\eta'\|_{I_j}^2 + \|\nabla\theta\|_{I_j}^2 \Big). \tag{3.37}$$

Since, $\nabla\theta|_{I_j}$ is linear in time then $\nabla\theta|_{I_j} \le \max\{|\nabla\theta^{j-1}|, |\nabla\theta^j|\}$. Thus,

$$\begin{aligned}\sum_{j=1}^{n} k_j \|\nabla\theta\|_{I_j}^2 &\le \sum_{j=1}^{n} k_j \|\nabla\theta^{j-1}\|^2 + \sum_{j=1}^{n} k_j \|\nabla\theta^j\|^2 \\ &\le C\sum_{j=1}^{n-1} k_j \|\nabla\theta^j\|^2 + k_1\|\nabla\theta^0\|^2 + k_n\|\nabla\theta^n\|^2. \end{aligned} \tag{3.38}$$

Following the steps that used in the proof of theorem 3.3 we obtain (3.30). ▌

In the next theorem, we derive the error estimate for the DPG solution, giving rise to suboptimal algebraic rates of convergence. Following [26], we assume that the solution u of (1.3) satisfies the finite regularity assumptions:

$$\|\nabla u'(t)\| + t\|\nabla u''(t)\| \leq C_2\, t^{\sigma-1}, \tag{3.39}$$

for some positive constants C_2 and σ.

Theorem 3.6 Let the solution u of problem (1.3) satisfies (3.39). Let $U \in \mathcal{W}$ be the DPG approximation defined by (3.9). Then we have the error estimate

$$\begin{aligned}
&\|(U-u)\|^2_{L_\infty((0,T);L_2(\Omega))} \\
&\leq C\bigg(\frac{\omega_{\alpha+1}(T)}{c_\alpha^2}\int_0^T \|\eta'(t)\|^2 dt + \sum_{j=1}^N k_j\|\nabla\eta'\|^2_{I_j}\bigg) + 2\|\eta\|^2_{L_\infty((0,T);L_2(\Omega))}.
\end{aligned} \tag{3.40}$$

Proof. From (3.29) and theorem 3.5, we find

$$\begin{aligned}
\|(U-u)\|^2_{L_\infty((0,T);L_2(\Omega))} &\leq 2\|\theta\|^2_{L_\infty((0,T);L_2(\Omega))} + 2\|\eta\|^2_{L_\infty((0,T);L_2(\Omega))} \\
\leq C\bigg(\frac{\omega_{\alpha+1}(T)}{c_\alpha^2}\int_0^T \|\eta'(t)\|^2 dt &+ \sum_{j=1}^N k_j\|\nabla\eta'\|^2_{I_j}\bigg) + 2\|\eta\|^2_{L_\infty((0,T);L_2(\Omega))}\,.
\end{aligned}$$

∎

Theorem 3.7 Let the solution u of problem (1.3) satisfies (3.39) with $1 < 2\sigma < 2$. Let $U \in \mathcal{W}(H_0^1)$ be the DPG approximation defined by (3.9) with the partition satisfying (3.6) and (3.7). Then we have the error estimate:

$$\|(U-u)\|^2_{L_\infty((0,t_n);L_2(\Omega))} \leq C\, k^2, \quad \text{for} \;\; \gamma > \frac{2}{2\sigma - 1}$$

where C is a constant that depends only on C_2, T, α, γ and σ.

Proof. To estimate the term $\sum_{j=1}^{n} k_j\|\nabla\eta'\|_{I_j}^2$ in (3.40) we use (3.24), (3.39), (3.7) and the assumption $\gamma > 2/(2\sigma-1)$, then for $n \geq 2$, we have

$$k_j\|\nabla\eta'\|_{I_j}^2 \leq k_j^2 \int_{t_{j-1}}^{t_j} \|\nabla u''(t)\|^2\, dt \leq Ck_j^2 \int_{t_{j-1}}^{t_j} t^{2\sigma-4} dt \leq Ck^2 \int_{t_{j-1}}^{t_j} t^{2\sigma-\frac{2}{\gamma}-2}\, dt.$$

So,

$$\sum_{j=2}^{n} k_j\|\nabla\eta'\|_{I_j}^2 \leq Ck^2 \int_{t_1}^{t_n} t^{2\sigma-\frac{2}{\gamma}-2}\, dt \leq C\, k^2. \tag{3.41}$$

Similarly,

$$\int_{t_{j-1}}^{t_j} \|\eta'(t)\|^2 dt \leq Ck^2 \int_{t_{j-1}}^{t_j} t^{2\sigma-\frac{2}{\gamma}-2}\, dt.$$

So,

$$\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \|\eta'(t)\|^2 dt \leq Ck^2 \int_{t_1}^{t_n} t^{2\sigma-\frac{2}{\gamma}-2}\, dt \leq C\, k^2. \tag{3.42}$$

Similarly,

$$\|\eta\|_{I_j}^2 \leq k_j^3 \int_{t_{j-1}}^{t_j} \|\nabla u''(t)\|^2\, dt \leq Ck_j^2 \int_{t_{j-1}}^{t_j} t^{2\sigma-3} dt \leq Ck^2 \int_{t_{j-1}}^{t_j} t^{2\sigma-\frac{2}{\gamma}}\, dt.$$

So,

$$\sum_{j=2}^{n} \|\eta\|_{I_j}^2 \le Ck^2 \int_{t_1}^{t_n} t^{2\sigma - \frac{2}{\gamma}}\, dt \le C\, k^2. \tag{3.43}$$

Now, from theorem 3.4, (3.39) and (3.6) with $2\sigma > 1$, we have

$$\begin{aligned} k_1 \|\nabla \eta'\|_{I_1}^2 &\le k_1^{-1} \left(\int_0^{t_1} (\|\nabla u'(s)\| + \|\nabla u'(t)\|) ds \right)^2 \\ &\le C k_1^{-1} \left(\int_0^{t_1} (s^{\sigma-1} + t^{\sigma-1}) ds \right)^2 \\ &= C(1/\sigma + 1)^2 {t_1}^{2\sigma-1} \le C k^{\gamma(2\sigma-1)} \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^{t_1} \|\eta'(t)\|^2 dt &\le \int_0^{t_1} \left(\int_0^{t_1} (\|\nabla u'(s)\| + \|\nabla u'(t)\|) ds \right)^2 dt \\ &= C(1/\sigma + 1)^2 {t_1}^{2\sigma-1} \le C k^{\gamma(2\sigma-1)}. \end{aligned}$$

and

$$\|\eta\|_{I_1}^2 \le k_1^{-1} \left(\int_0^{t_1} \int_0^{t_1} (\|u'(s)\| + \|u'(q)\|) ds\, dq \right)^2 \le C\, k^{2\gamma\,\sigma}.$$

This completes the proof.

$\blacksquare$

3.6 Implementations of the Numerical Schemes

In this section we discuss the implementation of the time-stepping DPG scheme defined by (3.9). We start by defining the hat basis functions $\phi_1, \phi_2, ..., \phi_N$ of the trial space $\mathcal{W}$ as follows: for $j = 1, \dots, N-1$,

$$\phi_j(t) = \begin{cases} \frac{t-t_{j-1}}{k_j}, & t \in I_j \\ \frac{t_{j+1}-t}{k_{j+1}}, & t \in I_{j+1} \\ 0, & \text{elsewhere} \end{cases} .$$

$$\phi_0(t) = \begin{cases} \frac{t_1-t}{k_1}, & t \in I_1 \\ 0, & \text{elsewhere} . \end{cases}$$

and

$$\phi_N(t) = \begin{cases} \frac{t-t_{N-1}}{k_N}, & t \in I_N \\ 0, & \text{elsewhere} . \end{cases}$$

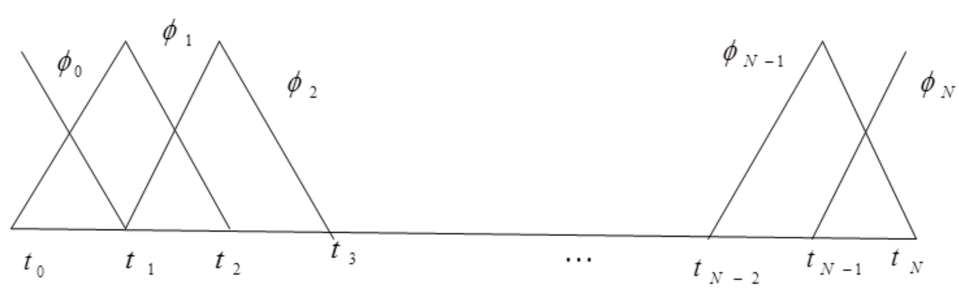

Figure 3.1: The hat basis functions $\phi_1, \phi_2, ..., \phi_N$ of the trial space $\mathcal{W}$.

To proceed in our implementation, we reformulate the DPG scheme (3.9) locally and obtain

$$\begin{aligned}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}\omega_\alpha(t-s)\langle U',X\rangle ds\,dt+\int_{t_{n-1}}^{t_n}\langle\nabla U,\nabla X\rangle dt\\ =\int_{t_{n-1}}^{t_n}\langle f(t),X\rangle dt-\int_{t_{n-1}}^{t_n}\int_{0}^{t_{n-1}}\omega_\alpha(t-s)\langle U',X\rangle ds\,dt,\end{aligned}\quad(3.44)$$

for all $X\in H_0^1(\Omega)$, $n=1,\cdots,N$. Let

$$U(x,t)=\sum_{i=0}^{N}a_i(x)\phi_i(t),$$

with $a_0=U(0,x)$. Then from (3.44), we have

$$\begin{aligned}\frac{1}{k_n}\langle a_n-a_{n-1},X\rangle\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}\omega_\alpha(t-s)\,ds\,dt\\ +\frac{1}{k_n}\int_{t_{n-1}}^{t_n}\langle(t_n-t)\nabla a_{n-1}+(t-t_{n-1})\nabla a_n,\nabla X\rangle dt\\ =\int_{t_{n-1}}^{t_n}\langle f(t),X\rangle dt-\sum_{i=1}^{n-1}\frac{1}{k_i}\langle a_i-a_{i-1},X\rangle\int_{t_{n-1}}^{t_n}\int_{t_{i-1}}^{t_i}\omega_\alpha(t-s)\,ds\,dt,\end{aligned}\quad(3.45)$$

for $n=1,\cdots,N$. By integrating (3.45), we obtain

$$\begin{aligned}\frac{\omega_{\alpha+2}(k_n)}{k_n}\langle a_n-a_{n-1},X\rangle+\frac{k_n}{2}\langle\nabla(a_n+a_{n-1}),\nabla X\rangle\\ =\int_{t_{n-1}}^{t_n}\langle f(t),X\rangle dt-\sum_{i=1}^{n-1}\omega^{n,i}\langle a_i-a_{i-1},X\rangle,\end{aligned}\quad(3.46)$$

where

$$
\begin{aligned}
\omega^{n,i} &= \frac{1}{k_i}\int_{t_{n-1}}^{t_n}\int_{t_{i-1}}^{t_i}\omega_\alpha(t-s)\,ds\,dt \\
&= \frac{1}{k_i}\left[\omega_{\alpha+2}(t_{n-1}-t_j)-\omega_{\alpha+2}(t_N-t_j)+\omega_{\alpha+2}(t_n-t_{j-1})-\omega_{\alpha+2}(t_{n-1}-t_{j-1})\right].
\end{aligned}
\tag{3.47}
$$

So, we arrive at the following system

$$
2\mathbf{B}\langle\mathbf{a},X\rangle+\Gamma(\alpha+2)\mathbf{D}\langle\nabla\mathbf{a},\nabla X\rangle=2\Gamma(\alpha+2)\mathbf{F} \tag{3.48}
$$

where the bi-diagonal matrices $\mathbf{B}$ and $\mathbf{D}$ are

$$
\mathbf{B}=\begin{bmatrix}
k_1^\alpha & 0 & 0 & \cdots & 0 \\
-k_2^\alpha & k_2^\alpha & 0 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -k_{N-1}^\alpha & k_{N-1}^\alpha & 0 \\
0 & \cdots & 0 & -k_N^\alpha & k_N^\alpha
\end{bmatrix},\quad
\mathbf{D}=\begin{bmatrix}
k_1 & 0 & 0 & \cdots & 0 \\
k_2 & k_2 & 0 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & k_{N-1} & k_{N-1} & 0 \\
0 & \cdots & 0 & k_N & k_N
\end{bmatrix},
$$

and the vectors $\mathbf{a}$ and $\mathbf{F}$ are

$$
\mathbf{a}=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix},\quad
\mathbf{F}=\begin{bmatrix}
\int_0^{t_1}\langle f(t),X\rangle dt+\frac{\omega_{\alpha+2}(k_1)}{k_1}\langle U^0,X\rangle-\frac{k_1}{2}\langle\nabla U^0,\nabla X\rangle \\
\int_{t_1}^{t_2}\langle f(t),X\rangle dt-\omega^{1,j}\langle a_j-a_{j-1},X\rangle \\
\vdots \\
\int_{t_{N-1}}^{t_N}\langle f(t),X\rangle dt-\sum_{j=1}^{N-1}\omega^{N,j}\langle a_j-a_{j-1},X\rangle
\end{bmatrix}.
$$

We notice that (3.48) is the result of discretizing time to find **a**. However, we will combine this system with a standard finite elements discretization in space to obtain a fully discrete system.

CHAPTER 4

High Order, Discontinuous Petrov Galerkin Method

In this chapter, we define a time-stepping high order DPG method for the time fractional diffusion problem (1.3). The existence, uniqueness and stability of the approximate solution will be shown. Then, we present the error analysis of the method and convergence of order $k^{m+\alpha/2}$ will be proved, where m is the degree of the approximate solution in time.

4.1 Discontinuous Petrov-Galerkin Scheme

In this section, we define the high order DPG approximate solution for problem (1.3).

The time-stepping DPG approximation of the solution u of problem (1.3) is now defined as follows: Find $U \in \mathcal{W}_m(H_0^1)$ such that

$$\begin{aligned} G_N(U, X) &= \int_0^{t_N} \langle f(t), X(t) \rangle \, dt, \qquad \text{for all } X \in \mathcal{T}_m(H_0^1)\,, \\ U(0) &= g(x), \end{aligned} \tag{4.1}$$

where G_N is the global bilinear form defined by

$$G_N(U, X) := \int_0^{t_N} \langle {}^c D^{1-\alpha} U(t), X(t) \rangle \, dt + \int_0^{t_N} \langle \nabla U(t), \nabla X(t) \rangle \, dt\,.$$

For later use, since the solution u of problem (1.3) satisfies

$$G_N(u, X) = \int_0^{t_N} \langle f(t), X(t) \rangle \, dt \qquad \text{for all } X \in \mathcal{T}_m(H_0^1)\,,$$

then $U - u$ satisfies the orthogonality condition

$$G_N(U - u, X) = 0 \qquad \text{for all } X \in \mathcal{T}_m(H_0^1)\,. \tag{4.2}$$

The orthogonality condition (4.2), we will use it to derive error estimate.

4.2 Well-posedness of The DPG Scheme

The existence, uniqueness and stability of the discrete DPG solution U will be shown in this section. We will use the properties of the memory term that we showed in chapter 2.

We start by proving the existence and uniqueness of the DPG solution.

Theorem 4.1 The discrete solution U of (4.1) exists and is unique.

Proof. Since the operator $-\Delta$ possesses a complete orthonormal eigensystem $\{\lambda_j, \Phi_j\}_{j\geq 1}$, problem (4.1) can be reduced to a finite linear system of algebraic equations on each subinterval I_n. To see this, let P_m be the space of polynomial of degree $\leq m$ in time. If we now take $X(x,t) = \Phi_j(x)w(t)$ on I_n and zero elsewhere in (4.1), then

$$\int_{t_{n-1}}^{t_n} \Big({}^c\mathrm{D}^{1-\alpha} U_j w(t) + \lambda_j U_j w(t) \Big)\, dt = \int_{t_{n-1}}^{t_n} f_j\, w(t)\, dt \tag{4.3}$$

for all $j = 1, 2, 3, \ldots$ and $w \in P_{m-1}$, with $U_j = \langle U, \Phi_j \rangle \in P_m$ and $f_j = \langle f, \Phi_j \rangle$. Because of the finite dimensionality of system (4.3) ($m \times m$ equations), the existence of the scalar function U_j follows from its uniqueness. Since the DPG solution is constructed element by element, it is enough to show the uniqueness on the first time interval $[0, t_1]$. That is, it is enough to consider $n = 1$ in (4.3) (for $n \geq 2$ the proof is completely analogous). To this end, let U_{j_1} and U_{j_2} be two DPG solutions on I_1. By linearity, the difference $V_j := (U_{j_1} - U_{j_2})|_{I_1}$ then satisfies:

$$\int_0^{t_1} \Big({}^c\mathrm{D}^{1-\alpha} V_j(t) w(t) + \lambda_j V_j(t) w(t) \Big)\, dt = 0 \qquad \forall\, w \in P_{m-1},\ \forall\, j \geq 1, \tag{4.4}$$

with $V_j^0 = 0$. Choosing $w = V_j'$ yields

$$\int_0^{t_1} V_j'(t)\, {}^c\mathrm{D}^{1-\alpha} V_j(t)\ dt + \frac{\lambda_j}{2} \int_0^{t_1} \frac{d}{dt} |V_j(t)|^2\, dt = 0.$$

Integrating then we have

$$\int_0^{t_1} V_j'(t)^c\mathrm{D}^{1-\alpha}V_j(t)\ dt + \frac{\lambda_j}{2}(|V_j^1|^2 - |V_j^0|^2) = 0.$$

Since $V_j^0 = 0$ then we obtain

$$\int_0^{t_1} V_j'(t)^c\mathrm{D}^{1-\alpha}V_j(t)\ dt + \frac{\lambda_j}{2}|V_j^1|^2 = 0.$$

Using lemma 2.25 (ii), we conclude that

$$|V_j^1| = 0 \quad \text{and} \quad \int_0^{t_1} V_j'(t)\,{}^c\mathrm{D}^{1-\alpha}V_j(t)\,dt = 0.$$

Thus, from Lemma 2.25 (i), we obtain $V_j \equiv 0$ on $[0, t_1]$. This completes the proof.

∎

Next, we show the stability of the DPG scheme.

Theorem 4.2 Assume that $g \in H_0^1(\Omega)$, and $f \in H^1([0,T]; L_2(\Omega))$. Then, for $1 \le n \le N$, the DPG solution U defined by (4.1) satisfies the following stability property:

$$\int_0^{t_n} \langle {}^cD^{1-\alpha}U(t), U'(t)\rangle\,dt + \|\nabla U^n\|^2 \le \|\nabla g\|^2 + \frac{1}{c_\alpha^2}\int_0^{t_n} \langle f, \mathrm{D}^\alpha f\rangle\,dt, \tag{4.5}$$

where $c_\alpha = \cos(\alpha\pi/2)$.

Proof. In (4.1) choose $X|_{(0,t_n)} = U'$ and zero elsewhere. Noticing that $f(t) =$

${}^{c}D^{1-\alpha}(\mathrm{I}^{1-\alpha} f)(t)$, we obtain

$$\int_0^{t_n} \langle {}^{c}D^{1-\alpha}U(t), U'(t)\rangle\, dt + \int_0^{t_n} \langle \nabla U(t), \nabla U'(t)\rangle\, dt$$
$$= \int_0^{t_n} \langle {}^{c}D^{1-\alpha}(\mathrm{I}^{1-\alpha} f)(t), U'(t)\rangle\, dt, \quad 1 \le n \le N\,. \tag{4.6}$$

Thus, from the continuity property (iii) of Lemma 2.25,

$$\begin{aligned} 2\left|\int_0^{t_n} \langle {}^{c}D^{1-\alpha}(\mathrm{I}^{1-\alpha} f)(t), U'(t)\rangle\, dt\right| &\le \frac{1}{c_\alpha^2} \int_0^{t_n} \langle {}^{c}D^{1-\alpha}(\mathrm{I}^{1-\alpha} f)(t), (\mathrm{I}^{1-\alpha} f)'(t)\rangle\, dt \\ &\quad + \int_0^{t_n} \langle {}^{c}D^{1-\alpha}U(t), U'(t)\rangle\, dt \\ &= \frac{1}{c_\alpha^2} \int_0^{t_n} \langle f(t), \mathrm{D}^{\alpha} f(t)\rangle + \int_0^{t_n} \langle {}^{c}D^{1-\alpha}U(t), U'(t)\rangle\, dt. \end{aligned}$$

By inserting this in (4.6) and using the equality

$$2\int_0^{t_n} \langle \nabla U(t), \nabla U'(t)\rangle\, dt = \int_0^{t_n} \frac{d}{dt}\|\nabla U(t)\|^2\, dt = \|\nabla U^n\|^2 - \|\nabla U^0\|^2,$$

we obtain (4.5). ∎

4.3 Error Estimate

The error estimates for the time-stepping high order DPG scheme will be derived in this section. For this purpose, we introduce a projection operator that has been used frequently in the analysis of discontinuous Galerkin time-stepping methods as well as finite element methods for one dimensional reaction-diffusion problems.

For any $u \in C([0,T];L_2(\Omega))$, the projection operator $(m \geq 2)$ $\Pi u : [0,T] \to L_2(\Omega)$: for $2 \leq n \leq N$, is defined by

$$\Pi u(t_n) = u(t_n) \text{ and } \Pi u(t_{n-1}) = u(t_{n-1}) \text{ and } \int_{t_{n-1}}^{t_n} \langle \Pi u - u,\, v\rangle\, dt = 0 \quad \text{for all } v \in P_{m-2}.$$

For instance, we define piecewise quadratic projection $(m = 2)$ $\Pi u : I_n \to H_0^1(\Omega)$ by setting

$$\Pi u(t_n) = u(t_n), \quad \Pi u(t_{n-1}) = u(t_{n-1})$$

and

$$\int_{I_n} (\Pi u - u)(t)\, dt = 0, \quad n = 1, \cdots, N\,.$$

Explicitly,

$$\begin{aligned}\Pi u(t) = {} & \frac{(t-t_n)^2}{k_n^2}\left(3u^n + 3u^{n-1} - \frac{6}{k_n}\int_{I_n} u(t)\, dt\right) \\ & + \frac{(t-t_n)}{k_n}\left(4u^n + 2u^{n-1} - \frac{6}{k_n}\int_{I_n} u(t)\, dt\right) + u^n. \end{aligned} \tag{4.7}$$

If further, $u \in \mathbb{H}^3(I_n;L_2(\Omega))$, then using integration by parts, we find that the projection error admits the integral representations in terms of u'''

$$
\begin{aligned}
&\Pi u(t) - u(t) \\
&\quad = \frac{(t-t_n)^2}{k_n^2}\bigg(3u^n + 3u^{n-1} - 6u^n + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) \\
&\qquad + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\bigg) + \frac{(t-t_n)}{k_n}\bigg(4u^n + 2u^{n-1} - 6u^n \\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\bigg) + \int_t^{t_n} u'(s)\, ds \\
&\quad = \frac{(t-t_n)^2}{k_n^2}\bigg(-3\int_{t_{n-1}}^{t_n} u'(s)\, ds + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) \\
&\qquad + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\bigg) + \frac{(t-t_n)}{k_n}\bigg(-2\int_{t_{n-1}}^{t_n} u'(s)\, ds + 3k_n\, u'(t_n) \\
&\qquad - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\bigg) - (t-t_n)\, u'(t) \\
&\qquad + \frac{(t-t_n)^2}{2}\, u''(t) + \frac{1}{2}\int_{t_{n-1}}^{t_n} (s-t_n)^2\, u'''(s)\, ds
\end{aligned}
$$

Through further using integration by parts,

$$
\begin{aligned}
&\Pi u(t) - u(t) \\
&\quad = \frac{(t-t_n)^2}{k_n^2}\Bigg(-3k_n\, u'(t_n) + \frac{3}{2}\, k_n^2\, u''(t_n) - \frac{3}{2}\int_{t_{n-1}}^{t_n} (t-t_n)^2 u'''(t)\, dt \\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\Bigg) \\
&\qquad + \frac{(t-t_n)}{k_n}\Bigg(-2k_n\, u'(t_n) + k_n^2\, u''(t_n) - \int_{t_{n-1}}^{t_n} (t-t_n)^2 u'''(t)\, dt \\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt\Bigg) \\
&\qquad - (t-t_n)\, u'(t) + \frac{(t-t_n)^2}{2}\, u''(t) + \frac{1}{2}\int_t^{t_n} (s-t_n)^2\, u'''(s)\, ds \\
&\quad = \frac{(t-t_n)^2}{2}\, u''(t_n) + \frac{(t-t_n)^2}{2}\, u''(t) + (t-t_n)\int_t^{t_n} u''(s)\, ds \\
&\qquad - \frac{3(t-t_n)^2}{4k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\, dt + \frac{(t-t_n)^2}{k_n^3}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt \\
&\qquad + \frac{(t-t_n)}{k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^3\, u'''(t)\, dt + \frac{1}{2}\int_t^{t_n} (s-t_n)^2\, u'''(s)\, ds \\
&\quad = \frac{(t-t_n)^2}{2}\int_{t_n}^{t} u'''(t)\, dt - \frac{3(t-t_n)^2}{4k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\, dt \\
&\qquad + \frac{(t-t_n)^2}{k_n^3}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\, dt \\
&\qquad + \frac{(t-t_n)}{k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^3\, u'''(t)\, dt + \frac{1}{2}\int_t^{t_n} (s-t_n)^2\, u'''(s)\, ds \qquad (4.8)
\end{aligned}
$$

From (4.8), using integration by parts we have

$$
\begin{aligned}
&\Pi u'(t) - u'(t)\\
&\quad= \frac{2(t-t_n)}{k_n^2}\Bigg(3u^n + 3u^{n-1} - 6u^n + 3k_n\, u'(t_n) - k_n^2\, u''(t_n)\\
&\qquad + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg) + \frac{1}{k_n}\Bigg(4u^n + 2u^{n-1} - 6u^n\\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg) - u'(t)\\
&\quad= \frac{2(t-t_n)}{k_n^2}\Bigg(-3\int_{t_{n-1}}^{t_n} u'(s)\,ds + 3k_n\, u'(t_n) - k_n^2\, u''(t_n)\\
&\qquad + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg) + \frac{1}{k_n}\Bigg(-2\int_{t_{n-1}}^{t_n} u'(s)\,ds\\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg) - u'(t)\\
&\quad= \frac{2(t-t_n)}{k_n^2}\Bigg(-3k_n\, u'(t_n) + \frac{3}{2}\,k_n^2\, u''(t_n) - \frac{3}{2}\int_{t_{n-1}}^{t_n} (t-t_n)^2 u'''(t)\,dt\\
&\qquad + 3k_n\, u'(t_n) - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg)\\
&\qquad + \frac{1}{k_n}\Bigg(-2k_n\, u'(t_n) + k_n^2\, u''(t_n) - \int_{t_{n-1}}^{t_n} (t-t_n)^2 u'''(t)\,dt + 3k_n\, u'(t_n)\\
&\qquad - k_n^2\, u''(t_n) + \frac{6}{k_n}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt\Bigg) - u'(t)\\
&\quad= (t-t_n)\, u''(t) + \int_t^{t_n} u''(s)\,ds + \frac{3(t-t_n)}{2k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\,dt\\
&\qquad + \frac{12(t-t_n)}{k_n^3}\int_{I_n} (t-t_{n-1})^3\, u'''(t)\,dt - \frac{1}{k_n}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\,dt\\
&\qquad + \frac{1}{k_n^2}\int_{t_{n-1}}^{t_n} (s-t_n)^2\, u'''(s)\,ds\\
&\quad= \int_t^{t_n} (t_n - t)\, u'''(s)\,ds + \frac{3(t-t_n)}{2k_n^2}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\,dt\\
&\qquad + \frac{12(t-t_n)}{k_n^3}\int_{t_{n-1}}^{t_n} (t-t_{n-1})^3\, u'''(t)\,dt - \frac{1}{k_n}\int_{t_{n-1}}^{t_n} (t-t_n)^2\, u'''(t)\,dt\\
&\qquad + \frac{1}{k_n^2}\int_{t_{n-1}}^{t_n} (s-t_n)^2\, u'''(s)\,ds.
\end{aligned}
\tag{4.9}
$$

So, for $n \geq 2, \Pi u|_{I_n} \in P_m(L_2(\Omega))$ is the Raviart-Thomas projection operator of u. However, due to the singular behaviour of u at $t = 0$ in the model problem (1.3), we consider $\Pi u|_{I_1} \in P_1(L_2(\Omega))$(linear polynomial in the time variable), interpolates u at the endpoints t_0 and t_1.

In the next theorem, we show the error estimates of the projection operator Π and derive some useful bounds of the error estimate.

Theorem 4.3 For $2 \leq n \leq N$ and $m \geq 2$, we have

(i) For any $u|_{I_n} \in \mathbb{H}^1(I_n; L_2(\Omega))$, there holds

$$\int_{t_{n-1}}^{t_n} \|(\Pi u)'(t)\|^2 dt \leq 2 \int_{t_{n-1}}^{t_n} \|u'(t)\|^2 \, dt \, .$$

(ii) For any $u|_{I_n} \in \mathbb{H}^{m+1}(I_n; L_2(\Omega))$, there holds

$$\int_{t_{n-1}}^{t_n} \|(u - \Pi u)(t)\|^2 dt + k_n^2 \int_{t_{n-1}}^{t_n} \|(u - \Pi u)'(t)\|^2 dt$$
$$\leq C k_n^{2m+2} \int_{t_{n-1}}^{t_n} \|u^{(m+1)}(t)\|^2 \, dt.$$

(iii) For any $u|_{I_n} \in \mathbb{H}^{m+1}(I_n; L_2(\Omega))$, there holds

$$\|(u - \Pi u)\|_{I_n}^2 + k_n^2 \|(u - \Pi u)'\|_{I_n}^2 \leq C k_n^{2m+1} \int_{t_{n-1}}^{t_n} \|u^{(m+1)}(t)\|^2 \, dt.$$

Proof. From the definition of Πu, integration by parts yields

$$\int_{t_{n-1}}^{t_n} \langle (\Pi u - u)'(t),\, v(t)\rangle \, dt = -\int_{t_{n-1}}^{t_n} \langle (\Pi u - u)(t),\, v'(t)\rangle \, dt = 0 \quad \text{for all} \;\; v \in P_{m-1}\,. \tag{4.10}$$

So, choosing $v(t) = (\Pi u)'(t)$, we notice that

$$\int_{t_{n-1}}^{t_n} \|(\Pi u)'(t)\|^2 \, dt = \int_{t_{n-1}}^{t_n} \langle u'(t), (\Pi u)'(t)\rangle \, dt \leq \int_{t_{n-1}}^{t_n} \|(\Pi u)'(t)\| \, \|u'(t)\| \, dt\,.$$

Therefore, we use the inequality $2\|(\Pi u)'(t)\| \, \|u'(t)\| \leq \|(\Pi u)'(t)\|^2 + \|u'(t)\|^2$ to obtain part(i). To estimate the second term in (ii) we use (4.10), then we have

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \|(u - \Pi u)'(t)\|^2 dt &= \int_{t_{n-1}}^{t_n} \langle (u - \Pi u)'(t), (u - \Pi u)'(t)\rangle dt \\
&= \int_{t_{n-1}}^{t_n} \langle (u - \Pi u)'(t), (u' - (\Pi u)')(t)\rangle dt \\
&= \int_{t_{n-1}}^{t_n} \langle (u - \Pi u)'(t), u'(t)\rangle dt - \int_{t_{n-1}}^{t_n} \langle (u - \Pi u)'(t), (\Pi u)'(t)\rangle dt \\
&= \int_{t_{n-1}}^{t_n} \langle (u - \Pi u)'(t), u'(t)\rangle dt
\end{aligned}$$

By using Bramble-Hilbert Lemma 2.17, there exists $\phi \in P_{m-1}$ such that

$$\|(u' - \phi)(t)\| \leq C k_n^m \|u^{(m+1)}(t)\|$$

and using Cauchy-Schwarz inequality 2.8(i), then we have

$$
\begin{aligned}
\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(t)\|^2 dt &= \int_{t_{n-1}}^{t_n} \langle (u-\Pi u)'(t), (u'-\phi)(t)\rangle dt \\
&\leq \int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(t)\| \|(u'-\phi)(t)\| dt \\
&\leq \left(\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} \|(u'-\phi)(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\leq Ck_n^m \left(\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} \|u^{(m+1)}(t)\|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

So,

$$
\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(t)\|^2 dt \leq Ck_n^{2m} \int_{t_{n-1}}^{t_n} \|u^{(m+1)}(t)\|^2 dt.
$$

Next, we want to estimate the first term in part(ii) $\int_{t_{n-1}}^{t_n} \|(u-\Pi u)(t)\|^2 dt$, we apply fundamental theorem of calculus 2.16, Cauchy-Schwarz inequality 2.8 (i) and Generalized Minkowski inequality 2.11, then we find

$$
\begin{aligned}
\int_{t_{n-1}}^{t_n} \|(u-\Pi u)(t)\|^2 dt &= \int_{t_{n-1}}^{t_n} \langle (u-\Pi u)(t), (u-\Pi u)(t)\rangle dt \\
&= \int_{t_{n-1}}^{t_n} \left\langle \int_{t_{n-1}}^{t} (u-\Pi u)'(q)\, dq, \int_{t_{n-1}}^{t} (u-\Pi u)'(q)\, dq \right\rangle dt \\
&\leq \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \|(u-\Pi u)'(q)\|\, dq \int_{t_{n-1}}^{t} \|(u-\Pi u)'(q)\|\, dq\, dt \\
&= \int_{t_{n-1}}^{t_n} \left(\int_{t_{n-1}}^{t} \|(u-\Pi u)'(q)\|\, dq\right)^2 dt \\
&\leq \int_{t_{n-1}}^{t_n} (t-t_{n-1}) \int_{t_{n-1}}^{t} \|(u-\Pi u)'(q)\|^2 dq\, dt. \qquad (4.11)
\end{aligned}
$$

So,

$$\int_{t_{n-1}}^{t_n} \|(u-\Pi u)(t)\|^2 dt \le C k_n^{2m+2} \int_{t_{n-1}}^{t_n} \|u^{m+1}\|^2 dt. \tag{4.12}$$

To estimate $\|(u-\Pi u)\|_{I_n}$ in part(iii), we use fundamental theorem of calculus 2.16 and Cauchy-Schwarz inequality 2.8, then we get

$$\begin{aligned}\|(u-\Pi u)\|_{I_n}^2 &\le \big(\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(q)\| dq\big)^2 \\ &\le (t_n - t_{n-1}) \left(\int_{t_{n-1}}^{t_n} \|(u-\Pi u)'(q)\|^2 dq\right). \end{aligned} \tag{4.13}$$

this leads

$$\|(u-\Pi u)\|_{I_n}^2 \le C k_n^{2m+1} \int_{t_n-1}^{t_n} \|u^{(m+1)}(t)\|^2 dt. \tag{4.14}$$

To estimate $\|(u-\Pi u)'\|_{I_n}$, we have two cases:

Case(1) for $m \ge 3$ we decompose it as:

$$\|(u-\Pi u)'\|_{I_n} \le \|(u-\tilde{\Pi} u)'\|_{I_n} + \|(\tilde{\Pi} u - \Pi u)'\|_{I_n},$$

where $\tilde{\Pi} u|_{I_n} \in P_m(L_2(\Omega))$ will be defined such that

$$\|(u-\tilde{\Pi} u)\|_{I_n}^2 + k_n^2 \|(u-\tilde{\Pi} u)'\|_{I_n}^2 \le C k_n^{2m+1} \int_{t_n-1}^{t_n} \|u^{m+1}(t)\|^2 dt.$$

For instance, one may choose $\tilde{\Pi} u$ on I_n as follows: $\tilde{\Pi} u(t_n) = u(t_n)$, $\tilde{\Pi} u(t_{n-1}) = u(t_{n-1})$, $(\tilde{\Pi} u)'(t_n) = u'(t_n)$ and $\tilde{\Pi} u(\xi_{n,l}) = u(\xi_{n,l})$, $l = 1, \ldots, m-2$, where $\xi_{n,l} = t_{n-1} + k_n \xi_l$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2}$ are the $(m-2)$-point Gauss-Legendre quadrature on the interval $(0,1)$. By using inverse and triangle inequalities, we have $\|(\tilde{\Pi u}-\Pi u)'\|_{I_n} \le Ck_n^{-1}(\|(\tilde{\Pi}u-u)\|_{I_n}+\|(u-\Pi u)\|_{I_n})$. So,

$$
\begin{aligned}
\|(u-\Pi u)'\|_{I_n}^2 &\le C\|(u-\tilde{\Pi}u)'\|_{I_n}^2 + Ck_n^{-2}(\|(\tilde{\Pi}u-u)\|_{I_n}^2+\|(u-\Pi u)\|_{I_n}^2) \\
&\le Ck_n^{2m-1}\int_{t_n-1}^{t_n}\|u^{m+1}(t)\|^2 dt.
\end{aligned}
$$

Case(2) for $m=2$ we use the explicit representation of Πu, (4.9) and using Cauchy-Schwarz inequality 2.8 (i) to obtain

$$
\begin{aligned}
\|(\Pi u-u)'\|_{I_n}^2 \le &\int_\Omega\Big(\int_{t_{n-1}}^{t_n}(t_n-t)\,|u'''(t,x)|\,dt+\frac{9}{4k_n^2}\int_{t_{n-1}}^{t_n}(t-t_n)^2\,|u'''(t,x)|\,dt \\
&+\frac{144}{k_n^4}\int_{t_{n-1}}^{t_n}(t-t_{n-1})^3\,|u'''(t,x)|\,dt+\frac{1}{k_n^2}\int_{t_{n-1}}^{t_n}(t-t_n)^2\,|u'''(t,x)|\,dt \\
&+\frac{1}{k_n^2}\int_{t_{n-1}}^{t_n}(t-t_n)^2\,|u'''(t,x)|\,dt\Big)^2\,dx \\
&\le \int_\Omega\Big((\frac{k_n^3}{3}\int_{t_{n-1}}^{t_n}(u'''(t,x))^2\,dt)^{1/2}+(\frac{9k_n^3}{20}\int_{t_{n-1}}^{t_n}(u'''(t,x))^2\,dt)^{1/2} \\
&+(\frac{144k_n^3}{7}\int_{t_{n-1}}^{t_n}(u'''(t,x))^2\,dt)^{1/2}+(\frac{k_n^3}{5}\int_{t_{n-1}}^{t_n}(u'''(t,x))^2\,dt)^{1/2} \\
&+(\frac{k_n^3}{5}\int_{t_{n-1}}^{t_n}(u'''(t,x))^2\,dt)^{1/2}\Big)^2\,dx \\
&= Ck_n^3\int_{t_{n-1}}^{t_n}\int_\Omega(u''(t,x))^2\,dx\,dt \\
&= Ck_n^3\int_{t_{n-1}}^{t_n}\|u'''(t)\|^2\,dt
\end{aligned}
$$

and thus, the proof of the third bound is completed now. ∎

Now, we decompose the error $U-u$ as before (chapter three) into two terms,

$$
U-u=(U-\Pi u)+(\Pi u-u)=\theta+\eta \tag{4.15}
$$

where $\theta = U - \Pi u, \eta = \Pi u - u$. The main task reduces to bound θ. In the next theorem we derive an interesting upper bound of θ that depends on η.

Theorem 4.4 For $1 \le n \le N$, we have

$$\begin{aligned}
&\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt + \|\nabla\theta^n\|^2 \\
&\le \frac{2}{c_\alpha^2}\left(\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt + \int_0^{t_n} \langle \Delta\eta, {}^c\,\mathrm{D}^{\alpha}\Delta\eta\rangle\, dt\right). \qquad (4.16)
\end{aligned}$$

Proof. First, the orthogonality property (4.2) and the decomposition (4.15) imply

$$\begin{aligned}
G_N(\theta, X) &= -G_N(\eta, X) \\
&= -\int_0^{t_N} \langle {}^cD^{1-\alpha}\eta(t), X(t)\rangle \; - \int_0^{t_N} \langle \nabla\eta(t), \nabla X(t)\rangle\, dt\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^{t_N} \langle {}^cD^{1-\alpha}\theta(t), X(t)\rangle\, dt + \int_0^{t_N} \langle \nabla\theta(t), \nabla X(t)\rangle\, dt \\
&= -\int_0^{t_N} \langle {}^c\mathrm{D}^{1-\alpha}\eta(t), X(t)\rangle\, dt - \int_0^{t_N} \langle \nabla\eta(t), \nabla X(t)\rangle\, dt\,. \qquad (4.17)
\end{aligned}$$

By choosing $X|_{(0,t_n)} = \theta'$ and zero elsewhere, and with the aid of the equality:

$$2\int_0^{t_n} \langle \nabla\theta(t), \nabla\theta'(t)\rangle\, dt = \|\nabla\theta^n\|^2 - \|\nabla\theta^0\|^2$$

and the fact that $\theta^0 = 0$, we obtain

$$2\int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt + \|\nabla\theta^n\|^2 = -2\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\eta(t), \theta'(t)\rangle\, dt$$
$$+2\int_0^{t_n} \langle \Delta\eta(t), \theta'(t)\rangle\, dt. \qquad (4.18)$$

Now, using the continuity property, Lemma 2.25 (*iii*), and Cauchy-Schwarz inequality 2.8 (iii), we notice that

$$2\left|\int_0^{t_n} \langle {}^cD^{1-\alpha}\eta(t), \theta'(t)\rangle\, dt\right|$$
$$\leq \frac{2}{c_\alpha^2}\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt + \frac{1}{2}\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt\,,$$

and in addition, we use the identity $\Delta\eta = {}^cD^{1-\alpha}(\mathrm{I}^{1-\alpha}\Delta\eta)$ and observe

$$2\left|\int_0^{t_n} \langle \Delta\eta(t), \theta'(t)\rangle\, dt\right| \leq \frac{2}{c_\alpha^2}\int_0^{t_n} \langle {}^cD^{1-\alpha}(\mathrm{I}^{1-\alpha}\Delta\eta)(t), (\mathrm{I}^{1-\alpha}\Delta\eta)'(t)\rangle\, dt$$
$$+\frac{1}{2}\int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt$$
$$= \frac{2}{c_\alpha^2}\int_0^{t_n} \langle \Delta\eta(t), {}^cD^{\alpha}\Delta\eta(t)\rangle\, dt + \frac{1}{2}\int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle\, dt\,,$$

where in the second inequality we used properties 2.7 (iii)

$$D^\alpha\Delta\eta(t) = (\mathrm{I}^{1-\alpha}\Delta\eta)'(\mathrm{t}) = \omega_{1-\alpha}(\mathrm{t})\,\Delta\eta(0) + {}^c\mathrm{D}^\alpha\Delta\eta(\mathrm{t}) = {}^c\mathrm{D}^\alpha\Delta\eta(\mathrm{t}). \qquad (4.19)$$

Inserting the above inequalities in (4.18) yields

$$\int_0^{t_n} \langle {}^cD^{1-\alpha}\theta(t), \theta'(t)\rangle \, dt + \|\nabla\theta^n\|^2 \le \frac{2}{c_\alpha^2}\left(\int_0^{t_n} \langle {}^cD^{1-\alpha}\eta(t), \eta'(t)\rangle \, dt + \int_0^{t_n} \langle \Delta\eta(t), {}^cD^{\alpha}\Delta\eta(t)\rangle \, dt\right). \tag{4.20}$$

∎

Before proceeding in our error analysis, we want to remind the reader that the continuous solution u of the time fractional model problem (1.3) is not regular. More precisely, u has a singular behaviour near $t = 0$ where typically, u satisfies the finite regularity assumptions:

$$\|u^{(q)}(t)\| \le c_q \, t^{\sigma-q} \quad \text{for} \quad t > 0 \ \text{ with } \ 1 \le q \le m+1, \tag{4.21}$$

for some positive constants c_q and σ. Using this, we can show another class of regularity properties that are needed in our forthcoming error analysis given by:

$$\|\Delta u^{(q)}(t)\| \le d_q \, t^{\delta-q-1} \quad \text{for} \quad t > 0 \ \text{ with } \ 1 \le q \le m+1, \tag{4.22}$$

for some positive constant d_q, with $(1-\alpha)/2 < \sigma < 1$ and $\delta > 1$. For $\sigma > 1$ all proof can be easily modified. The proof of the above regularity properties follows from the regularity analysis in [25, 27].

Because the exact solution u is not sufficiently smooth near $t = 0$, the global error in U fails to be $O(k^{m+1})$ accurate in time if we use a uniform time step k.Indeed, for problems of the form (1.3) and based on the above regularity assumptions, by using any class of finite difference or finite element methods over a uniform time mesh, we often expect to observe a global convergence of order $O(k^\sigma)$ where typically $\sigma < 1$. To this end, to capture the singular behaviour of u near $t = 0$, following see [25, 26, 33, 34], we employ a family of non-uniform meshes that concentrate the time levels near $t = 0$. Using the above regularity and time mesh assumptions, we estimate the first and second terms on the right-hand side of (4.16) in the next two lemmas.

Lemma 4.5 Assume that u satisfies the regularity assumptions (4.21) with $2 > 2\sigma > 1-\alpha$, and the time mesh satisfies the conditions (3.6) and (3.7). Then, there exists a constant C depends on c_q, σ, α, γ, m and T such that, for $1 \le n \le N$

$$\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt \le C\, k^{2m+\alpha} \quad \text{for} \quad \gamma > \max\left\{\frac{2m+\alpha}{\alpha+2\sigma-1}, \frac{m+1}{\sigma}\right\}.$$

Proof. We start our proof by splitting $\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt$ as follows:

$$\begin{aligned}\int_0^{t_n} &\langle {}^c\!D^{1-\alpha}\eta(t), \eta'(t)\rangle\, dt \\ &= \int_{I_1} \langle \mathcal{A}_1^\alpha(t), \eta'(t)\rangle\, dt + \sum_{j=2}^{n} \int_{I_j} \langle \mathcal{A}_2^\alpha(t) + \mathcal{A}_{3,j}^\alpha(t) + \mathcal{A}_{4,j}^\alpha(t), \eta'(t)\rangle\, dt, \quad (4.23)\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{A}_1^\alpha(t) &= \int_0^t \omega_\alpha(t-s)\,\eta'(s)\,ds\,,\\
\mathcal{A}_2^\alpha(t) &= \int_0^{t_1} \omega_\alpha(t-s)\,\eta'(s)\,ds\,,\\
\mathcal{A}_{3,j}^\alpha(t) &= \int_{t_1}^{t_{j-1}} \omega_\alpha(t-s)\,\eta'(s)\,ds\,,\\
\mathcal{A}_{4,j}^\alpha(t) &= \int_{t_{j-1}}^{t} \omega_\alpha(t-s)\,\eta'(s)\,ds.
\end{aligned}
\tag{4.24}
$$

For $t \in I_1$, from (3.23) and the regularity properties, (4.21) (recall that $\sigma < 1$), we observe

$$
\begin{aligned}
\|\eta'(t)\| = \frac{1}{t_1}\int_0^{t_1} \|u'(s)\|\,ds + \|u'(t)\| &= \frac{C}{t_1}\int_0^{t_1} s^{\sigma-1}\,ds + C\,t^{\sigma-1}\\
&\le C(t_1^{\sigma-1} + t^{\sigma-1}) \le Ct^{\sigma-1}.
\end{aligned}
\tag{4.25}
$$

Hence, using the Cauchy-Schwarz inequality (2.8)(i), we have

$$
\begin{aligned}
\int_{I_1} |\langle \mathcal{A}_1^\alpha(t), \eta'(t)\rangle|\,dt &\le \int_{I_1} \|\eta'(t)\| \int_0^t \omega_\alpha(t-s)\,\|\eta'(s)\|\,ds\,dt\\
&\le C\int_{I_1} t^{\sigma-1}\int_0^t (t-s)^{\alpha-1}\,s^{\sigma-1}\,ds\,dt\\
&\le C\int_{I_1} t^{\sigma-1}\,t^{\alpha+\sigma-1}\,dt \le Ct_1^{\alpha+2\sigma-1}.
\end{aligned}
\tag{4.26}
$$

To estimate the term involving $\mathcal{A}_2^\alpha(t)$, we use the above bound (4.25), the projection errors in Theorem 4.3, and then, integrating follows by the use of the inequality:$(t_j - s)^\alpha - (t_{j-1} - s)^\alpha \leq k_j^\alpha$, yield

$$\begin{aligned}
\int_{I_j} |\langle \mathcal{A}_2^\alpha(t), \eta'(t)\rangle|\, dt &\leq \int_{I_j} \|\eta'(t)\| \int_0^{t_1} \omega_\alpha(t-s)\, \|\eta'(s)\|\, ds\, dt \\
&\leq C\, \|\eta'\|_{I_j} \int_{I_1}\int_{I_j} (t-s)^{\alpha-1}\, s^{\sigma-1}\, dt\, ds \\
&\leq C\, k_j^{m-1/2} \left(\int_{I_j} \|u^{(m+1)}(t)\|^2\, dt \right)^{\frac{1}{2}} k_j^\alpha\, t_1^\sigma.
\end{aligned}$$

Thus, using the time mesh assumption (3.7), the regularity properties and (4.21) we obtain

$$\begin{aligned}
\int_{I_j} |\langle \mathcal{A}_2^\alpha(t), \eta'(t)\rangle|\, dt &\leq C\, k_j^{m+\alpha-1}\, k_j^{1/2} \left(\int_{I_j} t^{2\sigma-2(m+1)}\, dt \right)^{\frac{1}{2}} t_1^\sigma \\
&\leq C\, k^{m+\alpha-1}\, t_j^{m+\alpha-1-(m+\alpha-1)/\gamma}\, k_j^{1/2} \left(\int_{I_j} t^{2\sigma-2(m+1)}\, dt \right)^{\frac{1}{2}} t_1^\sigma \\
&\leq C\, k^{m+\alpha-1}\, t_{j-1}^{m+\alpha-1-(m+\alpha-1)/\gamma}\, k_j^{1/2} \left(\int_{I_j} t^{2\sigma-2(m+1)}\, dt \right)^{\frac{1}{2}} t_1^\sigma \\
&\leq C\, k^{m+\alpha-1}\, k_j^{1/2} \left(\int_{I_j} t^{2\sigma+2\alpha-4-2(m+\alpha-1)/\gamma}\, dt \right)^{\frac{1}{2}} t_1^\sigma \\
&\leq C\, k^{m+\alpha-1}\, k_j\, t_{j-1}^{\sigma+\alpha-2-(m+\alpha-1)/\gamma}\, t_1^\sigma \\
&\leq C\, k^{m+\alpha-1}\, k_j\, t_j^{\sigma+\alpha-2-(m+\alpha-1)/\gamma}\, t_1^\sigma \\
&\leq C\, k^{m+\alpha-1} \int_{I_j} t^{\sigma+\alpha-2-(m+\alpha-1)/\gamma}\, dt \;\; t_1^\sigma\, .
\end{aligned}$$

Hence, summing over j, integrating and using the assumption $\gamma > (m+1)/\sigma$, we achieve

$$\sum_{j=2}^{n}\int_{I_j}|\langle\mathcal{A}_2^\alpha(t),\eta'(t)\rangle|\,dt \le C\,t_1^\sigma\,k^{m+\alpha-1}\int_{t_1}^{t_n} t^{\sigma+\alpha-2-(m+\alpha-1)/\gamma}\,dt$$
$$\le C\max\{k^{2m+\alpha},t_1^{\alpha+2\sigma-1}\}\,. \tag{4.27}$$

To estimate the term involving $\mathcal{A}_{3,j}^\alpha(t)$, we integrate by parts to obtain

$$\mathcal{A}_{3,j}^\alpha(t) = -\int_{t_1}^{t_{j-1}}\omega_{\alpha-1}(t-s)\eta(s)\,ds.$$

Following the arguments we used above yield

$$\begin{aligned}\int_{I_j}\langle\mathcal{A}_{3,j}^\alpha(t),\eta'(t)\rangle\,dt &\le \|\eta'\|_{I_j}\int_{I_j}\int_{t_1}^{t_{j-1}}\omega_{\alpha-1}(t-s)\,\|\eta(s)\|\,ds\,dt\\ &= \|\eta'\|_{I_j}\int_{t_1}^{t_{j-1}}[\omega_\alpha(t_{j-1}-s)-\omega_\alpha(t_j-s)]\,\|\eta(s)\|\,ds\\ &= \|\eta'\|_{I_j}\sum_{i=2}^{j-1}\|\eta\|_{I_i}\int_{I_i}[\omega_\alpha(t_{j-1}-s)-\omega_\alpha(t_j-s)]\,ds\,.\end{aligned}$$

Changing the order of summations give

$$\begin{aligned}&\sum_{j=2}^{n}\int_{I_j}|\langle\mathcal{A}_{3,j}^\alpha(t),\eta'(t)\rangle|\,dt\\ &\qquad\le\sum_{i=2}^{n}\max_{j=i+1}^{n}\|\eta'\|_{I_j}\,\|\eta\|_{I_i}\int_{I_i}\sum_{j=i+1}^{n-1}[\omega_\alpha(t_{j-1}-s)-\omega_\alpha(t_j-s)]\,ds\\ &\qquad\le\sum_{i=2}^{n}\|\eta\|_{I_i}\,\max_{j=i+1}^{n}\|\eta'\|_{I_j}\int_{I_i}\omega_\alpha(t_i-s)\,ds\\ &\qquad\le C\sum_{i=2}^{n}k_i^{\alpha+1}\,\max_{j=i}^{n}\|\eta'\|_{I_j}^2\,.\end{aligned} \tag{4.28}$$

But, by following similar arguments in the estimation of the term

$\int_{I_j} \langle \mathcal{A}_{3,j}^{\alpha}(t), \eta'(t)\rangle \, dt$, we have

$$
\begin{aligned}
& k_i^{\alpha+1} \ \max_{j=i}^{n} \|\eta'\|_{I_j}^2 \\
& \le C\, k_i^{\alpha+1} \ \max_{j=i}^{n} k_j^{2m-1} \left(\int_{I_i} t^{2\sigma-2(m+1)} \, dt \right) \\
& \le C\, k_i^{\alpha+1} \ \max_{j=i}^{n} \, k^{2m}\, k_j^{-1}\, t_j^{2m-2m/\gamma} \left(\int_{I_j} t^{2\sigma-2(m+1)} \, dt \right) \\
& \le C\, k_i^{\alpha+1} \ \max_{j=i}^{n} \, k^{2m}\, k_j^{-1}\, t_{j-1}^{2m-2m/\gamma} \left(\int_{I_j} t^{2\sigma-2(m+1)} \, dt \right) \\
& \le C\, k_i^{\alpha+1} \ \max_{j=i}^{n} \, k^{2m}\, k_j^{-1} \ \left(\int_{I_j} t^{2\sigma-2-2m/\gamma} \, dt \right) \\
& \le C\, k_i^{\alpha+1} \ \max_{j=i}^{n} \, k^{2m}\, t_{j-1}^{2\sigma-2-2m/\gamma} \\
& \le C\, k_i^{\alpha+1} \ \ k^{2m}\, t_i^{2\sigma-2-2m/\gamma} \\
& \le C\, k^{2m+\alpha}\, k_i^{1} \ t_i^{2\sigma+\alpha-2-(2m+\alpha)/\gamma}.
\end{aligned}
$$

Hence, summing over j, integrating and using $2\sigma + \alpha - 1 > (2m+\alpha)/\gamma$ (since $\gamma > (2m+\alpha)/(2\sigma+\alpha-1)$), then we find

$$
\begin{aligned}
\sum_{j=2}^{n} \int_{I_j} |\langle \mathcal{A}_{3,j}^{\alpha}(t), \eta'(t)\rangle| \, dt &\le C\, k^{2m+\alpha} \int_{t_1}^{t_n} t^{2\sigma+\alpha-2-(2m+\alpha)/\gamma} \, dt \\
&\le C\, k^{2m+\alpha} t_n^{2\sigma+\alpha-1-(2m+\alpha)/\gamma}.
\end{aligned} \tag{4.29}
$$

Finally, to estimate the term involving $\mathcal{A}_4^\alpha(t)$, we use similar arguments and get

$$
\begin{aligned}
\int_{I_j} |\langle \mathcal{A}_{4,j}^\alpha(t), \eta'(t)\rangle|\, dt &\le \|\eta'\|_{I_j}^2 \int_{I_j}\int_{t_{j-1}}^t \omega_\alpha(t-s)\, ds\, dt \\
&\le C\, k_j^{2m-1} \int_{I_j} \|u^{(m+1)}(t)\|^2\, dt\, \omega_{\alpha+2}(k_j) \\
&\le C\, k^{2m+\alpha}\, t_j^{2m+\alpha-(2m+\alpha)/\gamma} \int_{I_j} t^{2\sigma-2(m+1)}\, dt \\
&\le C\, k^{2m+\alpha}\, t_{j-1}^{2m+\alpha-(2m+\alpha)/\gamma} \int_{I_j} t^{2\sigma-2(m+1)}\, dt \\
&\le C\, k^{2m+\alpha} \int_{I_j} t^{2\sigma+\alpha-2-(2m+\alpha)/\gamma}\, dt.
\end{aligned}
$$

Summing over j and using $2\sigma+\alpha-1 > (2m+\alpha)/\gamma$ (since $\gamma > (2m+\alpha)/(2\sigma+\alpha-1)$) , we obtain

$$
\begin{aligned}
\sum_{j=2}^n \int_{I_j} |\langle \mathcal{A}_{4,j}^\alpha(t), \eta'(t)\rangle|\, dt &\le C\, k^{2m+\alpha} \int_{t_1}^{t_n} t^{2\sigma+\alpha-2-(2m+\alpha)/\gamma}\, dt \\
&\le C\, k^{2m+\alpha}\, t_n^{2\sigma+\alpha-1-(2m+\alpha)/\gamma}\,.
\end{aligned}
\tag{4.30}
$$

Finally, combine (4.23), (4.26), (4.27), (4.29), and (4.30) to obtain the desired estimate. ∎

Lemma 4.6 Assume that u satisfies the regularity assumptions (4.22) and the time mesh satisfies the conditions (3.6) and (3.7). Then, we have

$$
\int_0^{t_n} \langle {}^cD^\alpha \Delta\eta(t), \Delta\eta(t)\rangle\, dt \le C\, k^{2m+\alpha}, \quad \gamma > \max\left\{\frac{m+\alpha}{\delta-1}, \frac{2m+\alpha}{2\delta-\alpha-1}\right\},
$$

where the constant C depends on d_q,δ, σ, α, γ, m and T.

Proof. As in the proof of Lemma (4.5), first we split $\int_0^{t_n}\langle {}^cD^\alpha \Delta\eta(t), \Delta\eta(t)\rangle\, dt$ as follows:

$$\int_0^{t_n}\langle {}^cD^\alpha \Delta\eta(t), \Delta\eta(t)\rangle\, dt = \int_{I_1}\langle \Delta\mathcal{A}_1^{1-\alpha}(t), \Delta\eta(t)\rangle\, dt \\ + \sum_{j=2}^{n}\int_{I_j}\langle \Delta(\mathcal{A}_2^{1-\alpha}(t) + \mathcal{A}_{3,j}^{1-\alpha}(t) + \mathcal{A}_{4,j}^{1-\alpha}(t)), \Delta\eta(t)\rangle\, dt, \tag{4.31}$$

where $\mathcal{A}_1^{1-\alpha}(t)$, $\mathcal{A}_2^{1-\alpha}(t)$, $\mathcal{A}_{3,j}^{1-\alpha}(t)$ and $\mathcal{A}_{4,j}^{1-\alpha}(t)$ defined in (4.24) be replaced α by $1-\alpha$. To bound the first term we use $\Delta\mathcal{A}_1^{1-\alpha}(t) = \frac{\partial}{\partial t}\int_0^t \omega_{1-\alpha}(t-s)\,\Delta\eta(s)\,ds$, and thus, integrating by parts, we obtain

$$\left|\int_{I_1}\langle \Delta\mathcal{A}_1^{1-\alpha}(t), \Delta\eta(t)\rangle\, dt\right| = \left|\int_{I_1}\left\langle \Delta\eta'(t), \int_0^t \omega_{1-\alpha}(t-s)\,\Delta\eta(s)\,ds\right\rangle dt\right| \\ = \left|\int_{I_1}\left\langle \Delta\eta'(t), \int_0^t \omega_{2-\alpha}(t-s)\,\Delta\eta'(s)\,ds\right\rangle dt\right|.$$

Now, following the derivation in (4.26) and using the regularity property (4.22) instead of (4.21), we obtain

$$\left|\int_{I_1}\langle \Delta\mathcal{A}_1^{1-\alpha}(t), \Delta\eta(t)\rangle\, dt\right| \le \int_{I_1}\|\Delta\eta'(t)\|\int_0^t \omega_{2-\alpha}(t-s)\,\|\Delta\eta'(s)\|\,ds\,dt \\ \le C\int_{I_1} t^{\delta-2}\int_0^t (t-s)^{1-\alpha}\, s^{\delta-2}\,ds\,dt \\ \le Ck_1^{2\delta-\alpha-1}. \tag{4.32}$$

To estimate the term that contains $\mathcal{A}_2^{1-\alpha}$, we use the same steps that we used to obtain (4.27) and using $\delta > 1$, we have

$$
\begin{aligned}
\left| \int_{I_j} \langle \Delta \mathcal{A}_2^{1-\alpha}(t), \Delta\eta(t) \rangle \, dt \right| &= \left| \int_{I_j} \left\langle \Delta\eta'(t), \int_0^{t_1} \omega_{2-\alpha}(t-s)\, \Delta\eta'(s)\, ds \right\rangle dt \right| \\
&\le \int_{I_j} \|\Delta\eta'(t)\| \int_0^{t_1} \omega_{2-\alpha}(t-s)\, \|\Delta\eta'(s)\|\, ds\, dt \\
&\le C \int_{I_j} \|\Delta\eta'(t)\| \int_0^{t_1} \omega_{2-\alpha}(t-s)\, s^{\delta-2}\, ds\, dt \\
&\le C\, k_j^{m-1/2} \left(\int_{I_j} \|\Delta u^{(m+1)}(t)\|^2\, dt \right)^{\frac{1}{2}} t_1^{\delta-1}\, t_j^{2-\alpha}\, k_j \\
&= C\, t_j^{1-\alpha}\, k_j^{m+1/2} \left(\int_{I_j} \|\Delta u^{(m+1)}(t)\|^2\, dt \right)^{\frac{1}{2}} t_1^{\delta-1} \\
&\le C\, t_j^{1-\alpha}\, k_j^{m}\, k_j^{1/2} \left(\int_{I_j} t^{2\delta-2m-4}\, dt \right)^{\frac{1}{2}} t_1^{\delta-1} \\
&\le C\, t_j^{1-\alpha}\, k^{m}\, k_j^{1/2} \left(\int_{I_j} t^{2\sigma+2\alpha-4-2m/\gamma}\, dt \right)^{\frac{1}{2}} t_1^{\delta-1} \\
&\le C\, t_j^{1-\alpha}\, k^m\, k_j\, t_{j-1}^{\delta-2-m/\gamma}\, t_1^{\delta-1} \\
&\le C\, t_j^{1-\alpha}\, k^m\, k_j\, t_j^{\delta-2-m/\gamma}\, t_1^{\delta-1} \\
&= C\, k^m\, k_j\, t_j^{\sigma-1-m/\gamma}\, t_1^{\delta-1} \\
&\le C\, k^m\, t_1^{\delta-1} \int_{I_j} t^{\sigma-1-m/\gamma}\, dt.
\end{aligned}
$$

Hence, summing over j, integrating and using $\gamma > \frac{m+\alpha}{\delta-1}$, we achieve

$$
\begin{aligned}
\sum_{j=2}^{n} \int_{I_j} \langle \Delta \mathcal{A}_2^{1-\alpha}(t), \Delta\eta(t) \rangle | \, dt &\le C\, k^m\, t_1^{\delta-1} \int_{t_1}^{t_n} t^{\sigma-1-m/\gamma} \\
&\le C\, \max\{k^{2m+\alpha}, t_1^{2\delta-\alpha-1}\}\,.
\end{aligned}
$$

Following the derivation in (4.28), and using Theorem 4.3 and (4.22),

$$\sum_{j=2}^{n}\int_{I_j}|\langle\Delta\mathcal{A}_{3,j}^{1-\alpha}(t),\Delta\eta(t)\rangle|\,dt \le C\sum_{i=2}^{n} k_i^{1-\alpha}\,\|\Delta\eta\|_{I_i}\ \max_{j=i+1}^{n}\|\Delta\eta\|_{I_j}$$

where

$$\begin{aligned}
&k_i^{1-\alpha}\,\|\Delta\eta\|_{I_i}\ \max_{j=i+1}^{n}\|\Delta\eta\|_{I_j}\\
&\le C\,k_i^{1-\alpha}\,k_i^{m+1}\|\Delta u^{(m+1)}(t)\|_{I_i}\max_{j=i+1}^{n}\ k_j^{m+1}\,\|\Delta u^{(m+1)}(t)\|_{I_j}\\
&\le C\,k_i\ \max_{j=i}^{n}\ k_j^{2m+2-\alpha}\ t_{j-1}^{2\delta-2m-4}\\
&\le C\,k_i\ \max_{j=i}^{n}\ k_j^{2m+\alpha}\,k_j^{2-2\alpha}\,t_j^{2\delta-2m-4}\\
&\le C\,k_i\ \max_{j=i}^{n}\ k_j^{2m+\alpha}\ t_j^{2\delta-2\alpha-2m-2}\\
&\le C\,k_i\ \max_{j=i}^{n}\ k^{2m+\alpha}\ t_j^{2\delta-\alpha-2-(2m+\alpha)/\gamma}\\
&\le C\,k_i\ k^{2m+\alpha}\ \max\{t_j^{2\delta-\alpha-2-(2m+\alpha)/\gamma},1\}\\
&\le C\,k^{2m+\alpha}\left(\int_{I_j} t^{2\delta-\alpha-2-(2m+\alpha)/\gamma}\,dt+k_i\right)
\end{aligned}$$

Thus, we employ the assumption $\gamma>(2m+\alpha)/(2\delta-\alpha-1)$

$$\sum_{j=2}^{n}\int_{I_j}|\langle\Delta\mathcal{A}_{3,j}^{1-\alpha}(t),\Delta\eta(t)\rangle|\,dt \le C\,k^{2m+\alpha}(t_n^{2\delta-\alpha-(2m+\alpha)/\gamma}+1)\le C\,k^{2m+\alpha}. \quad (4.33)$$

To estimate $\sum_{j=2}^{n}\int_{I_j}|\langle\Delta\mathcal{A}_{4,j}^{1-\alpha}(t),\Delta\eta(t)\rangle|\,dt$, we use the preceding arguments and obtain

$$
\begin{aligned}
&\sum_{j=2}^{n} \int_{I_j} |\langle \Delta\mathcal{A}_{4,j}^{1-\alpha}(t), \Delta\eta(t)\rangle|\, dt \\
&\qquad \le \sum_{j=2}^{n} \|\Delta\eta\|_{I_j}^2 \int_{I_j} \int_{t_{j-1}}^{t} \omega_{1-\alpha}(t-s)\, ds\, dt \\
&\qquad \le C \sum_{j=2}^{n} k_j^{2m+1} \int_{I_j} t^{2\delta-2m-4}\, dt\, k_j^{1-\alpha} \\
&\qquad \le C \sum_{j=2}^{n} k_j^{2m+\alpha}\, k_j^{2-2\alpha} \int_{I_j} t^{2\delta-2m-4}\, dt \\
&\qquad \le C\, k^{2m+\alpha} \sum_{j=2}^{n} t_j^{2m+\alpha-(2m+\alpha)/\gamma}\, t_j^{2-2\alpha} \int_{I_j} t^{2\delta-2m-4}\, dt \\
&\qquad \le C\, k^{2m+\alpha} \sum_{j=2}^{n} \int_{I_j} t^{2\delta-\alpha-2-(2m+\alpha)/\gamma}\, dt \\
&\qquad = C\, k^{2m+\alpha} \int_{I_1}^{t_n} t^{2\delta-\alpha-2-(2m+\alpha)/\gamma}\, dt \\
&\qquad \le C\, k^{2m+\alpha}\, {t_n}^{2\delta-\alpha-1-(2m+\alpha)/\gamma}
\end{aligned}
$$

Finally, combine this with (4.32), (4.33), (4.33), and (4.34) yield the desired estimate. ∎

In the next theorem, we derive the main error results for the DPG solution giving rise to suboptimal algebraic rates of convergence in time (short by order $1-\alpha/2$ from being optimal) if the solution u of (1.3) is sufficiently regular. However, our numerical results illustrate an optimal convergence rate.

Theorem 4.7 Let the solution u of problem (1.3) satisfy (4.21) and (4.22). Let $U \in \mathcal{W}_m(H_0^1)$ be the DPG approximation defined by (4.1). Assume that the time mesh satisfies assumptions (3.6) and (3.7). Then for $\gamma >$

$\max\left\{\frac{2m+\alpha}{\alpha+2\sigma-1}, \frac{2m+\alpha}{2\delta-\alpha-1}, \frac{m+\alpha}{\delta-1}, \frac{m+1}{\sigma}\right\}$, we have the error estimate:

$$\|U-u\|^2_{L_\infty((0,T);L_2(\Omega))} \le C\, k^{2m+\alpha},$$

where C is a constant that depends only on d_1, d_2, δ, σ, α, γ, m and T.

Proof. Setting $e := U-u$, recalling that $e=\theta+\eta$, we observe

$$\|e\|^2_{L_\infty((0,T);L_2(\Omega))} \le C\left(\|\theta\|^2_{L_\infty((0,T);L_2(\Omega))} + \|\eta\|^2_{L_\infty((0,T);L_2(\Omega))}\right) .$$

By using lemma 2.1, theorem 2.25, theorem 4.4, theorem 4.5 and theorem 4.6, we find that

$$\|\theta\|^2_{L_\infty((0,T);L_2(\Omega))} \le C\, k^{2m+\alpha}.$$

From Theorem 4.3, the regularity assumption (4.21) and the inequality

$$\gamma > (2m+\alpha)/2\sigma,$$

we obtain

$$\begin{aligned}
\|\eta\|_{I_n}^2 &\le Ck_n^{2m+1}\int_{t_{n-1}}^{t_n}\|u^{(m+1)}(t)\|^2\,dt\\
&\le Ck_n^{2m+1}\int_{t_{n-1}}^{t_n} t^{2\sigma-2m-2}\,dt\\
&\le Ck_n^{2m+\alpha}\,k_n^{1-\alpha}\int_{t_{n-1}}^{t_n} t^{2\sigma-\alpha-2m-1}\,dt\\
&\le Ck_n^{2m+\alpha}\,t_n^{1-\alpha}\int_{t_{n-1}}^{t_n} t^{2\sigma-\alpha-2m-1}\,dt\\
&\le Ck_n^{2m+\alpha}\,t_{n-1}^{1-\alpha}\int_{t_{n-1}}^{t_n} t^{2\sigma-\alpha-2m-1}\,dt\\
&\le Ck_n^{2m+\alpha}\int_{t_{n-1}}^{t_n} t^{2\sigma-\alpha-2m-1}\,dt\\
&\le Ck^{2m+\alpha}\int_{t_{n-1}}^{t_n} t^{2\sigma-1-(2m+\alpha)/\gamma}\,dt\\
&\le Ck^{2m+\alpha}\,{t_n}^{2\sigma-(2m+\alpha)/\gamma},
\end{aligned} \tag{4.34}$$

for $2 \le n \le N$. However, for $n = 1$, by using (3.22) and the mesh assumption $\gamma > (m+1)/\sigma$, we observe

$$\|\eta\|_{I_1}^2 \le k_1^{-1}\left(\int_0^{t_1}\int_0^{t_1}(\|u'(s)\| + \|u'(q)\|)ds\,dq\right)^2 \le C\,k^{2\gamma\sigma} \le k^{2m+\alpha}.$$

So,

$$\|\eta\|_{L_\infty((0,T);L_2(\Omega))}^2 = \max_{1\le n\le N}\|\eta\|_{I_n}^2 \le C\,k^{2m+\alpha},$$

This proves the assertion.

∎

4.4 Implementations of The Numerical Schemes

This section is devoted to discuss the implementation of the time-stepping DPG scheme defined by (4.1). Let $\{\phi_n^j(t)\}_{j=0}^{m-1} = \{\frac{(t-t_{n-1})^{j+1}}{(j+1)k_n^j}\}_{j=0}^{m-1}$ be the set of basis functions of the dimentional trial space $\mathcal{W}_m$ over the subinterval (t_{n-1}, t_n). So,

$$U(x,t)|_{I_n} = U(x,t_{n-1}) + \sum_{j=0}^{m-1} a_n^j(x)\phi_n^j(t)$$

Using this in (4.1) with $X = \frac{(t-t_{n-1})^r}{k_n^r}\chi$, we obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1}\langle a_n^{j+1}, X\rangle \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \omega_\alpha(t-s)\,\frac{(s-t_{n-1})^j}{k_n^j}\frac{(t-t_{n-1})^r}{k_n^r}ds\,dt \\
&\qquad\qquad + \sum_{j=0}^{m-1}\langle \nabla a_n^{j+1}, \nabla\chi\rangle \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{j+r+1}}{(j+1)\,k_n^{j+r}}dt \\
&= \int_{t_{n-1}}^{t_n}\langle f(t), \frac{(t-t_{n-1})^r}{k_n^r}\chi\rangle dt - \langle \nabla U(t_{n-1}), \nabla\chi\rangle \int_{t_{n-1}}^{t_n}\frac{(t-t_{n-1})^r}{k_n^r}dt \\
&\qquad\qquad - \int_{t_{n-1}}^{t_n}\int_0^{t_{n-1}} \omega_\alpha(t-s)\langle U', X\rangle ds\,dt,
\end{aligned}$$

for all $\chi \in H_0^1(\Omega)$ and for $r = 0, 1, ..., m-1$ and for $n = 1, \cdots, N$. Integrating,

$$\begin{aligned}
&{k_n}^{\alpha+1}\sum_{j=0}^{m-1}\langle a_n^{j+1}, \chi\rangle \frac{\Gamma(j+1)}{(j+\alpha+r+1)\Gamma(j+\alpha+1)} + k_n^2\sum_{j=0}^{m-1}\langle \nabla a_n^{j+1}, \nabla\chi\rangle \frac{1}{(j+1)(j+r+2)} \\
&= \int_{t_{n-1}}^{t_n}\frac{(t-t_{n-1})^r}{k_n^r}\big(\langle f(t), \chi\rangle - \int_0^{t_{n-1}} \omega_\alpha(t-s)\langle U', \chi\rangle ds\,\big)dt - \frac{k_n}{r+1}\langle \nabla U(t_n), \nabla\chi\rangle.
\end{aligned} \tag{4.35}$$

Then

$$
\begin{aligned}
&{k_n}^{\alpha+1}\sum_{j=0}^{m-1}\langle a_n^{j+1},\chi\rangle\frac{\Gamma(j+1)}{(j+\alpha+r+1)\Gamma(j+\alpha+1)}+k_n^2\sum_{j=0}^{m-1}\langle\nabla a_n^{j+1},\nabla\chi\rangle\frac{1}{(j+1)(j+r+2)}\\
&=\int_{t_{n-1}}^{t_n}\frac{(t-t_{n-1})^r}{k_n^r}\Big(\langle f(t),\chi\rangle-\sum_{j=0}^{m-1}\sum_{d=1}^{n-1}\langle a_d^{j+1},\chi\rangle\int_{t_{d-1}}^{t_d}\omega_\alpha(t-s)\frac{(s-t_{d-1})^j}{k_d^j}ds\Big)dt\\
&-\frac{k_n}{r+1}\langle\nabla U(t_{n-1}),\nabla\chi\rangle. \qquad (4.36)
\end{aligned}
$$

So, we arrive at the following system

$$
k_n^{\alpha+1}\mathbf{B_1}\langle\mathbf{a_n},X\rangle+k_n^2\mathbf{D_1}\langle\nabla\mathbf{a_n},\nabla X\rangle=\mathbf{F_n} \qquad (4.37)
$$

where the matrices $\mathbf{B_1}$ and $\mathbf{D_1}$ are

$$
\mathbf{B_1}=\begin{bmatrix}
\frac{\Gamma(1)}{(1+\alpha)\Gamma(1+\alpha)} & \frac{\Gamma(2)}{(2+\alpha)\Gamma(2+\alpha)} & \cdots & \frac{\Gamma(m)}{(m+\alpha)\Gamma(m+\alpha)}\\
\frac{\Gamma(1)}{(2+\alpha)\Gamma(1+\alpha)} & \frac{\Gamma(2)}{(3+\alpha)\Gamma(2+\alpha)} & \cdots & \frac{\Gamma(m)}{(m+\alpha+1)\Gamma(m+\alpha)}\\
\vdots & \vdots & & \vdots\\
\frac{\Gamma(1)}{(m+\alpha)\Gamma(1+\alpha)} & \frac{\Gamma(2)}{(m+\alpha+1)\Gamma(2+\alpha)} & \cdots & \frac{\Gamma(m)}{(2m+\alpha-1)\Gamma(m+\alpha)}
\end{bmatrix},
$$

$$
\mathbf{D_1}=\begin{bmatrix}
\frac{1}{2} & \frac{1}{(2)(3)} & \cdots & \frac{1}{(m)(m+1)}\\
\frac{1}{3} & \frac{1}{(2)(4)} & \cdots & \frac{1}{(m)(m+2)}\\
\vdots & \vdots & & \vdots\\
\frac{1}{(m+1)} & \frac{1}{(2)(m+2)} & \cdots & \frac{1}{(m)(2m)}
\end{bmatrix},
$$

and the vectors $\mathbf{a_n}$ and $\mathbf{F_n}$ are

$$\mathbf{a_n} = \begin{bmatrix} a_n^1 \\ a_n^2 \\ \vdots \\ a_n^m \end{bmatrix},$$

$$\mathbf{F_n} = \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \chi \rangle dt \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})}{k_n} \langle f(t), \chi \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{m-1}}{k_n^{m-1}} \langle f(t), \chi \rangle \end{bmatrix} - \begin{bmatrix} k_n \langle \nabla U(t_{n-1}), \nabla \chi \rangle \\ \frac{k_n}{2} \langle \nabla U(t_{n-1}), \nabla \chi \rangle \\ \vdots \\ \frac{k_n}{m} \langle \nabla U(t_{n-1}), \nabla \chi \rangle \end{bmatrix}$$
$$- \begin{bmatrix} \int_{t_{n-1}}^{t_n} \sum_{j=0}^{m} \sum_{d=1}^{n-1} \langle a_d^j, \chi \rangle \int_{t_{d-1}}^{t_d} \omega_\alpha(t-s) \frac{(s-t_{d-1})^j}{k_d^j} ds\, dt \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^2}{k_n^2} \sum_{j=0}^{m} \sum_{d=1}^{n-1} \langle a_d^j, \chi \rangle \int_{t_{d-1}}^{t_d} \omega_\alpha(t-s) \frac{(s-t_{d-1})^j}{k_d^j} ds\, dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{m-1}}{k_n^{m-1}} \sum_{j=0}^{m} \sum_{d=1}^{n-1} \langle a_d^j, \chi \rangle \int_{t_{d-1}}^{t_d} \omega_\alpha(t-s) \frac{(s-t_{d-1})^j}{k_d^j} ds\, dt \end{bmatrix}.$$

The system (4.37) is the result of discretizing time to find $\mathbf{a_n}$ on the subinterval I_n for $1 \le n \le N$. However, in chapter five we will combine the system (4.37) with a standard finite elements discretization in space to obtain a fully discrete system.

CHAPTER 5

Fully Discrete Scheme

In this chapter, we show the stability, the uniqueness and the existence of the approximate solution in space for our problem (1.3). Also we will combine the Petrov-Galerkin scheme with respect to time with a standard finite element discretization in space and obtain a fully discrete scheme of our problem (1.3). In addition, we analyze the error analysis of the method and show the convergence of order $k^{m+\alpha/2} + h^{r+1}$, where m and r are the degrees of approximate solutions in time and spatial variables, respectively. Finally we derive the implementation of the fully discrete discontinuous petrov galerkin finite element (DPGFE) scheme defined on rectangular polygons.

5.1 Error for the spatial discretization

Let $S_h \subseteq H_0^1(\Omega)$ denote the space of continuous, piecewise polynomials of degree $\leq r (r \geq 1)$ with respect to a quasi-uniform partition of Ω into triangular or quadrilateral (or tetrahedral etc.) finite elements, with maximum diameter h.

Let the operator $R_h : H_0^1(\Omega) \to S_h$ denote the Ritz projection defined by

$$\langle \nabla(R_h v - v), \nabla \chi \rangle = 0 \quad \text{for all} \quad \chi \in S_h, \tag{5.1}$$

and has the approximation property

$$\|R_h v - v\| \leq C\, h^{\min\{s,r\}+1} \|v\|_{s+1}^2 \quad \text{for} \quad v \in H^{s+1} \cap H_0^1\,, \quad r \geq 1\,, \quad s \geq 0. \tag{5.2}$$

The solution of the continuous problem (1.3) satisfies

$$\langle {}^c D^{1-\alpha} u(t), X \rangle - \langle \Delta u(t), X \rangle = \langle f(t), X \rangle, \quad \forall X \in H_0^1(\Omega). \tag{5.3}$$

So, we define the spatially semi-discrete solution $u_h : [0, T] \to S_h$ by

$$\langle {}^c D^{1-\alpha} u_h(t), X \rangle + \langle \nabla u_h(t), \nabla X \rangle = \langle f(t), X \rangle \, dt, \quad \forall X \in S_h. \tag{5.4}$$

with $u_h(0) = R_h g$.

Definition 5.1 *[41]* Let $\Omega = (a, b) \subset \mathbb{R}$ be a bounded interval. Amesh T on Ω is a partition of Ω into $M(T)$ open, disjoint subintervals Ω_j^T, $T = \{\Omega_j^T\}_{j=1}^{M(T)}$, $\{\Omega_j^T\} = (x_{j-1}^T, x_j^T)$, $a = x_0^T < x_1^T < \cdots < x_M^T$. The points $x_j \in \bar{\Omega}$ are nodal points (nodes) and the Ω_j are the elements of the mesh T. Set $h_j^T := x_j^T - x_{j-1}^T$.

Definition 5.2 *[41]* A family of meshes $\{T_j\}$ is a quasi-uniform in Ω if there exists constants c_1, $c_2 > 0$ independent of j such that

$$c_1 \leq \frac{\max_{1\leq i\leq M(T_j)} h_i}{\min_{1\leq i\leq M(T_j)} h_i} \leq c_2.$$

Theorem 5.3 The spatially semi-discrete solution satisfies

$$\|u_h(t)\|_1^2 \leq C_1 \left(\|\nabla u_h(0)\|^2 + \|f(0)\|^2 + \int_0^t \|f'(s)\|^2 \, ds \right), \tag{5.5}$$

where C_1 depends on Ω and T.

Proof. By using $\langle \nabla u_h'(t), \nabla u_h(t) \rangle = d/dt \|u_h(t)\|$ and choosing $X = u_h'$ in (5.4) then we have

$$\int_0^t \langle {}^c D^{1-\alpha} u_h(\tau), u_h'(\tau) \rangle \, d\tau + \int_0^t \langle \nabla u_h(\tau), \nabla u_h'(\tau) \rangle \, d\tau = \int_0^t \langle f(\tau), u_h'(\tau) \rangle \, d\tau.$$

Following the same steps that we used in the proof of Theorem 3.1 we obtain (5.5). ∎

The uniqueness and existence of the semi-discrete solution u_h of problem (5.4) will be proved in the next theorem.

Theorem 5.4 For each fixed t the finite element solution u_h exists and is unique.

Proof. Let

$$u_h(x,t) = \sum_{i=1}^{M_h} c_i(t) \, \phi_i(x),$$

where $\phi_i(x)$ are the basis of S_h and M_h is the number of the interior nodes.

By subsituting u_h in (5.4) and choosing $X = \phi_j(x)$, we obtain

$$\sum_{i=1}^{M_h} {}^c\!D^{1-\alpha} c_i(t) \langle \phi_i, \phi_j \rangle + \sum_{i=1}^{M_h} c_i(t) \langle \nabla\phi_i, \nabla\phi_j \rangle$$
$$= \langle f(t), \phi_j \rangle, \quad j = 1, ..., M_h.$$

Then we arrive at the following system $(B_1 {}^c\!D^{1-\alpha} + B_2) C(t) = b$ where

$$B_1 := \begin{bmatrix} \langle \phi_1, \phi_1 \rangle & \cdots & \langle \phi_1, \phi_{M_h} \rangle \\ \vdots & & \\ \langle \phi_{M_h}, \phi_1 \rangle & \cdots & \langle \phi_{M_h}, \phi_{M_h} \rangle \end{bmatrix}, \ C(t) := \begin{bmatrix} c_1(t) \\ \vdots \\ c_{M_h}(t) \end{bmatrix}$$

and

$$B_2 := \begin{bmatrix} \langle \nabla\phi_1, \nabla\phi_1 \rangle & \cdots & \langle \nabla\phi_1, \nabla\phi_{M_h} \rangle \\ \vdots & & \\ \langle \nabla\phi_{M_h}, \nabla\phi_1 \rangle & \cdots & \langle \nabla\phi_{M_h}, \nabla\phi_{M_h} \rangle \end{bmatrix}, \ b := \begin{bmatrix} \langle f(t), \phi_1 \rangle \\ \vdots \\ \langle f(t), \phi_{M_h} \rangle \end{bmatrix}$$

Because of the finite dimensionality of system (5.4) ($M_h \times M_h$ systems), the existence of the scalar function u_h follows from its uniqueness. To show this, let u_{h1} and u_{h2} be two solutions for (5.4) on I_1. By linearity, the difference $v_h := (u_{h1} - u_{h2})|_{I_1}$ then satisfies:

$$\langle {}^c\!D^{1-\alpha} v_h(t), X(t) \rangle + \langle \nabla v_h(t), \nabla X(t) \rangle = 0.$$

Choosing $X = v_h'(t)$ and integrating from 0 to t_1, and following the derivation that used in theorem 3.2, we obtain

$$v_h \equiv 0.$$

This completes the proof. ∎

Now, we decompose $u_h - u$ as

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \phi + \xi, \tag{5.6}$$

where $\phi = u_h - R_h u$ and $\xi = R_h u - u$.

Theorem 5.5 *For $0 \le t \le T$, the solution of the spatially semidiscrete problem* (5.3) *satisfies*

$$\|u_h(t) - u(t)\| \\ \le C\left(\|\nabla(R_h u - u)^0\|^2 + \frac{\omega_{\alpha+1}(t)}{c_\alpha^2} h^{2r+2} \int_0^t \|u'(s)\|_{r+1}^2 ds\right). \tag{5.7}$$

Proof. From (5.3) and (5.4), we observe that for $X \in S_h$

$$\begin{aligned}
&\langle {}^c D^{1-\alpha} \phi(t), X\rangle + \langle \nabla\phi(t), \nabla X\rangle \\
&= \langle f(t), X\rangle - \langle {}^c \mathrm{D}^{1-\alpha} R_h u(t), X\rangle - \langle \nabla R_h u(t), \nabla X\rangle \\
&= -\langle {}^c D^{1-\alpha} \xi(t), X\rangle - \langle \nabla\xi(t), \nabla X\rangle
\end{aligned}$$

By following the steps of Theorem 3.5 we obtain

$$c_\alpha \int_0^t \|{}^cD^{1-\frac{\alpha}{2}}\phi(s)\|^2\, ds + \|\nabla\phi(t)\|^2 \le C\bigg(\|\nabla\phi^0\|^2 + \frac{\omega_{\alpha+1}(t)}{c_\alpha^2}\int_0^t \|\xi'(s)\|^2\, ds\bigg).$$

From (5.2), we have

$$\begin{aligned} c_\alpha \int_0^t \|{}^cD^{1-\frac{\alpha}{2}}\phi(t)\|^2 dt + \|\nabla\phi(t)\|^2 \\ \le C\bigg(\|\nabla(u_h - R_h u)^0\|^2 + \frac{\omega_{\alpha+1}(t)}{c_\alpha^2} h^{2r+2}\int_0^t \|u'(s)\|_{r+1}^2 ds\bigg). \end{aligned} \tag{5.8}$$

Using (5.8) and Lemma 2.21 we have

$$\|\phi(t)\| \le C\bigg(\|\nabla(R_h u - u)^0\|^2 + \frac{\omega_{\alpha+1}(t)}{c_\alpha^2} h^{2r+2}\int_0^t \|u'(s)\|_{r+1}^2 ds\bigg),$$

and using (5.6) and (5.2) we obtain (5.7). ∎

5.2 Error Estimate for high order in time and spatial variable in space

In this section we define high order DPG finite dimensional trial and test space of U_h, then we derive error estimates for high order DPG-FE scheme.

The DPG-FE approximation of the solution u of problem (1.3) is now defined as follows: Find $U_h \in \mathcal{W}_m(S_h)$ such that

$$
\begin{aligned}
G_N(U_h, X) &= \int_0^T \langle f(t), X(t)\rangle \, dt \quad \text{ for all } X \in \mathcal{T}_m(S_h)\,, \\
U_h(0) &= R_h g\,.
\end{aligned}
\tag{5.9}
$$

where the global bilinear form

$$
G_N(U_h, X) := \int_0^T \langle {}^c\!D^{1-\alpha} U_h(t), X(t)\rangle \, dt + \int_0^T \langle \nabla U_h(t), \nabla X(t)\rangle \, dt\,.
$$

We are ready now to prove the existence and uniqueness of the DPG-FE solution.

Theorem 5.6 *The discrete solution U_h of* (5.9) *exists and is unique.*

Proof. Because of the finite dimensionality of problem (5.9) on each sub-domain $\Omega \times I_n$, the existence of the approximate solution U_h follows from its uniqueness. To see this, we take $X \equiv 0$ outside $\Omega \times I_n$ in (5.9), then we find that

$$
\int_{t_{n-1}}^{t_n} \Big(\langle {}^c\!D^{1-\alpha} U_h, X\rangle + \langle \nabla U_h, \nabla X\rangle \Big) \, dt = \int_{t_{n-1}}^{t_n} \langle f, U_h' \rangle \, dt \quad \text{with} \;\; U_h^0 = R_h g\,. \tag{5.10}
$$

Since the U_h is constructed element by element (in time), it is enough to show the uniqueness on the first sub-domain $\Omega \times I_1$. That is, it is enough to consider $n = 1$ in (5.10) (for $n \geq 2$ the proof is completely analogous). To this end, let $U_{h,1}$ and $U_{h,2}$ be two solutions of (5.10) on $\Omega \times I_1$. By linearity, the difference

$V_h := (U_{h,1} - U_{h,2})$ on $\Omega \times I_1$ then satisfies:

$$\int_0^{t_1} \langle {}^cD^{1-\alpha} V_h, X\rangle \, dt + \int_0^{t_1} \langle \nabla V_h, \nabla X\rangle \, dt = 0 \qquad \forall\, w \in P_{m-1},\ \forall\, j \geq 1 \tag{5.11}$$

with $V_h^0 = 0$. Choosing $X = V_h'$ Choosing $X = V_h'$, using lemma 2.25 (i), (ii) and following the derivation that we used in the proof of theorem 3.2, we obtain

$$\int_0^{t_1} \langle {}^cD^{1-\alpha} V_h, V_h'\rangle \, dt + \frac{1}{2}\int_0^{t_1} \frac{d}{dt}\|\nabla V_h(t)\|^2 \, dt = 0.$$

Integrating, then using $V_h^0 = 0$ and Lemma 2.25 (ii), we conclude that

$$\|\nabla V_h^1\|^2 = \|\nabla V_h^0\|^2 = 0 \ \text{ and } \ \int_0^{t_1} \langle {}^cD^{1-\alpha} V_h, V_h'\rangle \, dt = 0.$$

Therefore, $\|V_h^1\| = \|V_h^0\| = 0$ and consequently, an application of Lemma 2.25 (i) yields $V_h \equiv 0$ on $\Omega \times [0, t_1]$. This completes the proof. ∎

Now, to derive fully error estimate, we decompose the error $U_h - u$ into three terms:

$$U_h - u = \zeta + \Pi\xi + \eta \tag{5.12}$$

with

$$\zeta = U_h - \Pi R_h u, \quad \xi = R_h u - u \quad \text{and} \quad \eta = \Pi u - u,$$

The main task reduces to bound ζ. In the next theorem we derive an interesting upper bound of ζ that depends on η and ξ.

Theorem 5.7 *For $1 \le n \le N$, we have*

$$\int_{t_n-1}^{t_n} \langle {}^c\mathcal{D}^{1-\alpha}\zeta, \zeta' \rangle \, dt + \|\nabla \zeta^n\|^2 \le \frac{4}{c_\alpha^2} \bigg(\int_{t_n-1}^{t_n} \langle {}^c\mathcal{D}^{1-\alpha}\eta(t), \eta'(t) \rangle \, dt$$
$$+ \int_{t_n-1}^{t_n} \langle \Delta\eta(t), {}^c\mathcal{D}^{\alpha}\Delta\eta(t) \rangle \, dt + \omega_{\alpha+1}^2(t_n) \int_0^{t_n} \|\xi'(t)\|^2 \, dt \bigg). \quad (5.13)$$

Proof. First, the orthogonality property, (4.2) and the decomposition (5.12) imply

$$\begin{aligned} G_N(\zeta, X) &= -G_N(\Pi\xi + \eta, X) \\ &= -\int_0^{t_N} \langle {}^c\mathcal{D}^{1-\alpha}(\Pi\xi + \eta)(t), X(t) \rangle \, dt - \int_0^{t_N} \langle \nabla(\Pi\xi + \eta)(t), \nabla X(t) \rangle \, dt \,. \end{aligned}$$

But, the operators Π and R_h are commute ($\Pi \, R_h = R_h \, \Pi$) and so, from the definition of Ritz projector, we have $\langle \nabla \Pi\xi, \nabla X \rangle = \langle \nabla(R_h(\Pi u) - \Pi u), \nabla X \rangle = 0$. Hence,

$$\int_0^{t_N} \langle {}^c\mathcal{D}^{1-\alpha}\zeta(t), X(t) \rangle \, dt + \int_0^{t_N} \langle \nabla\zeta(t), \nabla X(t) \rangle \, dt$$
$$= -\int_0^{t_N} \langle {}^c\mathrm{D}^{1-\alpha}(\Pi\xi + \eta), X \rangle \, dt - \int_0^{t_N} \langle \nabla\eta, \nabla X \rangle \, dt \,. \quad (5.14)$$

By choosing $X|_{(0,t_n)} = \zeta'$ and zero elsewhere with the aid of the equality:

$$2\langle \nabla\zeta, \nabla\zeta' \rangle_{t_n} = \|\nabla\zeta^n\|^2 - \|\nabla\zeta^0\|^2$$

and the fact that $\zeta^0 = 0$, we obtain

$$2\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\zeta, \zeta'\rangle\, dt + \|\nabla\zeta^n\|^2 = -2\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}(\Pi\xi + \eta), \zeta'\rangle\, dt + 2\int_0^{t_n} \langle \Delta\eta, \zeta'\rangle\, dt \tag{5.15}$$

Now, using the continuity property, Lemma 2.25 (iii), and the ϵ inequality, we notice that

$$\begin{aligned}
2\left|\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\eta, \zeta'\rangle\, dt\right| &\le \frac{4}{c_\alpha^2}\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\eta, \eta'\rangle\, dt + \frac{1}{4}\int_0^{t_n} \langle {}^c\mathrm{D}^{1-\alpha}\zeta, \zeta'\rangle\, dt,\\
2\left|\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\Pi\xi, \zeta'\rangle\, dt\right| &\le \frac{4}{c_\alpha^2}\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\Pi\xi, (\Pi\xi)'\rangle\, dt + \frac{1}{4}\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\zeta, \zeta'\rangle\, dt,
\end{aligned}$$

and following the derivation used in theorem 4.4 we obtain

$$\begin{aligned}
&\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\zeta, \zeta'\rangle\, dt + \|\nabla\zeta^n\|^2\\
&\le \frac{4}{c_\alpha^2}\left(\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\eta, \eta'\rangle\, dt + \int_0^{t_n} \langle \Delta\eta, {}^c\!D^{\alpha}\Delta\eta\rangle\, dt + \int_0^{t_n} \langle {}^c\!D^{1-\alpha}\Pi\xi, (\Pi\xi)'\rangle\, dt\right).
\end{aligned} \tag{5.16}$$

To complete the proof, we need to estimate the third term on the right-hand side of (3.35). To do so, we use the same derivation that we used in theorem 3.5

$$\begin{aligned}
|\int_0^{t_n} \langle {}^c\!D^{1-\alpha}\Pi\xi, \Pi\xi'\rangle\, dt| &\le \int_0^{t_n} \|{}^c\!D^{1-\alpha}\Pi\xi\|\,\|\Pi\xi'\|\, dt\\
&\le \|{}^c\!D^{1-\alpha}\Pi\xi\|_{L_2(0,t_n;\Omega)}\|(\Pi\xi)'\|_{L_2(0,t_n;\Omega)}\\
&\le \omega_{\alpha+1}^2(t_n)\|(\Pi\xi)'\|_{L_2(0,t_n;\Omega)}^2 \le \omega_{\alpha+1}^2(t_n)\|\xi'\|_{L_2(0,t_n;\Omega)}^2.
\end{aligned}$$

Finally, we insert this estimate in (5.16) and the proof is completed now. ∎

In the next theorem, we derive main error results for the DPG-FE solution giving rise to suboptimal algebraic rates of convergence in time (short by order $1-\alpha/2$ from being optimal), and optimal convergence rates in the spatial discretization provided the the solution u of (1.3) is sufficiently regular. However, our numerical results illustrate an optimal convergence rate.

Theorem 5.8 Let the solution u of problem (1.3) satisfy (4.21) and (4.22). In addition we assume that $u(0)$, $u(t_1) \in H^{r+1}(\Omega)$ and $u_t \in L_2((t_1,T);H^{r+1}(\Omega))$. Let $U_h \in \mathcal{W}_m(S_h)$ be the DPG approximation defined by (5.9). Assume that the time mesh satisfies assumptions (3.6) and (3.7), then for $\gamma \geq \max\left\{\frac{2m+\alpha}{2\sigma+\alpha-1}, \frac{2m+\alpha}{2\delta-\alpha-1}, \frac{m+\alpha}{\delta-1}, \frac{m+1}{\sigma}\right\}$, we have the error estimate:

$$\begin{aligned}
&\|U_h - u\|_{L_\infty((0,T);L_2(\Omega))}\\
&\leq C\left(k^{m+\alpha/2} + h^{r+1}(\|u(0)\|_{r+1} + \|u(t_1)\|_{r+1} + \int_{t_1}^{T} \|u'(t)\|_{r+1}^2\,dt)\right) \text{ for } 1 \leq n \leq N,
\end{aligned}$$

where C is a constant that depends on d_1, d_2, σ, α, γ, m and T.

Proof. Setting $e := U_h - u$, recalling that $e = \zeta + \Pi\xi + \eta$, we observe

$$\|e\|^2_{L_\infty((0,T);L_2(\Omega))} \leq C\left(\|\zeta\|^2_{L_\infty((0,T);L_2(\Omega))} + \|\Pi\xi\|^2_{L_\infty((0,T);L_2(\Omega))} + \|\eta\|^2_{L_\infty((0,T);L_2(\Omega))}\right) .$$

To estimate the term $\|\Pi\xi\|_{I_n}$. For $n=1$, $\Pi\xi$ is a linear polynomial in time and so, $\|\Pi\xi\|_{I_1} \leq \max\|\Pi\xi(0)\|, \|\Pi\xi(t_1)\| = \max\|\xi(0)\|, \|\xi(t_1)\|$ and thus, from the Ritz projection approximation error (5.1), we have

$$\|\Pi\xi\|_{I_1} \leq Ch^{r+1}(\|u(0)\| + \|u(t_1)\|)$$

Now, for $n \geq 2$ we use Inverse inequality(2.10) and Theorem 4.3 and the interpolation properties of the projection operator Π, and obtain

$$\begin{aligned}
\|\Pi\xi\|_{I_n}^2 &\leq C\left(\int_{I_n} \|\Pi\xi'(t)\|^2(t-t_{n-1})dt + \|(\Pi\xi)^n\|^2\right) \\
&\leq Ck_n\left(\int_{I_n} \|\xi'(t)\|^2 dt\right)
\end{aligned}$$

after noting from the approximation property of the Ritz projection that

$$\int_{I_n} \|\xi'(t)\|^2\, dt \leq C\, h^{2r+2} \int_0^{t_n} \|u'(t)\|_{s+1}^2\, dt,$$

To estimate the first term $\|\zeta\|_{I_n}$, we use lemma 2.21, lemma 2.25 and then lemma 4.5 , lemma 4.6 and theorem 5.7, we find that

$$\|\zeta\|_{L_\infty((0,T);L_2(\Omega))}^2 \leq C\, k^{2m+\alpha} + C\, h^{2r+2} \int_0^{t_n} \|u'\|_{r+1}^2\, dt,$$

From (4.34), we obtain that

$$\|\eta\|_{L_\infty((0,T);L_2(\Omega))}^2 \leq C\, k^{2m+\alpha}$$

the assertion follows. $\blacksquare$

5.3 Implementations of DPGFE scheme

5.3.1 Case1: Piecewise linear in time and one dimension in space

This section is devoted to implement the fully discrete DPGFE scheme defined by (5.9). Let $a_n = \sum_{i=1}^{M_h} a_{n,i}\psi_i$, where ψ_i are basis of S_h for all $n = 1, 2, .., N$ and choose $X = \psi_j$ for all $j = 1, 2, .., M_h$. We subsitute (replace $a_n = \sum_{i=1}^{M_h} a_{n,i}\psi_i$, $X = \psi_j$) in (3.46) and integrating, we get

$$\frac{\omega_{\alpha+2}(k_n)}{k_n}\left\langle \sum_{i=1}^{M_h} a_{n,i}\psi_i - \sum_{i=1}^{M_h} a_{n-1,i}\psi_i, \psi_j \right\rangle + \frac{k_n}{2}\left\langle \nabla \sum_{i=1}^{M_h} a_{n,i}\psi_i + \nabla \sum_{i=1}^{M_h} a_{n-1,i}\psi_i, \nabla\psi_j \right\rangle$$
$$= \int_{t_{n-1}}^{t_n} \langle f(t), \psi_j\rangle dt - \sum_{l=1}^{n-1} \omega^{n,i}\left\langle \sum_{i=1}^{M_h} a_{l,i}\Psi_i - \sum_{i=1}^{M_h} a_{l-1,i}\psi_i, \psi_j \right\rangle$$

then

$$\frac{\omega_{\alpha+2}(k_n)}{k_n}\mathbf{A}\begin{bmatrix} a_{n,1}-a_{n-1,1} \\ \vdots \\ a_{nM_h}-a_{n-1,M_h} \end{bmatrix}+\frac{k_n}{2}\mathbf{E}\begin{bmatrix} a_{n,1}+a_{n-1,1} \\ \vdots \\ a_{nM_h}+a_{n-1,M_h} \end{bmatrix}$$

$$=\begin{bmatrix} \int_{t_{n-1}}^{t_n}\langle f(t),\psi_1\rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n}\langle f(t),\psi_{M_h}\rangle dt \end{bmatrix}-\sum_{l=1}^{n-1}\omega^{n,i}\mathbf{A}\begin{bmatrix} a_{l,1}-a_{l-1,1} \\ \vdots \\ a_{l,M_h}-a_{l-1,M_h} \end{bmatrix}$$

So, we arrive at the following system

$$\Big(2\mathbf{A}_1+\Gamma(\alpha+2)\mathbf{E_1}\Big)\mathbf{Y}=\mathbf{2\Gamma}(\alpha+\mathbf{2})\mathbf{G}$$

$$\mathbf{A}=[\langle\psi_i,\psi_j\rangle]_{M_h\times M_h},\,\mathbf{E}=[\langle\nabla\psi_i,\nabla\psi_j\rangle]_{M_h\times M_h}$$

$$\mathbf{A}_1=\mathbf{B}\otimes\mathbf{A}=\begin{bmatrix} k_1^\alpha\mathbf{A} & 0 & 0 & \cdots & 0 \\ -k_2^\alpha\mathbf{A} & k_2^\alpha\mathbf{A} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{N-1}^\alpha\mathbf{A} & k_{N-1}^\alpha\mathbf{A} & 0 \\ 0 & \cdots & 0 & -k_N^\alpha\mathbf{A} & k_N^\alpha\mathbf{A} \end{bmatrix},$$

$$\mathbf{E}_1 = \mathbf{D} \otimes \mathbf{E} = \begin{bmatrix} k_1\mathbf{E} & 0 & 0 & \cdots & 0 \\ k_2\mathbf{E} & k_2\mathbf{E} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & k_{N-1}\mathbf{E} & k_{N-1}\mathbf{E} & 0 \\ 0 & \cdots & 0 & k_N\mathbf{E} & k_N\mathbf{E} \end{bmatrix}$$

$$Y = \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{1,M_h} \\ a_{2,1} \\ \vdots \\ a_{2,M_h} \\ \vdots \\ an,1 \\ \vdots \\ a_{n,M_h} \end{bmatrix}, \mathbf{G} = \begin{bmatrix} G^1 \\ G^2 \\ \vdots \\ G^N \end{bmatrix}$$

$$G^1 = \begin{bmatrix} \int_{t_0}^{t_1} \langle f(t), \psi_1 \rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1} \langle U^0, \psi_1 \rangle - \frac{k_1}{2} \langle \nabla U^0, \nabla \psi_1 \rangle \\ \vdots \\ \int_{t_0}^{t_1} \langle f(t), \psi_{M_h} \rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1} \langle U^0, \psi_{M_h} \rangle - \frac{k_1}{2} \langle \nabla U^0, \nabla \psi_{M_h} \rangle \end{bmatrix},$$

$$G^n = \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \Psi_1 \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \Psi_{M_h} \rangle dt \end{bmatrix} - \sum_{j=1}^{n-1} \omega^{n,j} \mathbf{A} \begin{bmatrix} \langle a_{j,1} - a_{j-1,1}, \Psi_1 \rangle \\ \vdots \\ \langle a_{j,M_h} - a_{j-1,M_h}, \Psi_{M_h} \rangle \end{bmatrix}$$

for $n = 2, \ldots, N$.

where $\omega^{n,j}$ is defined in (3.47).

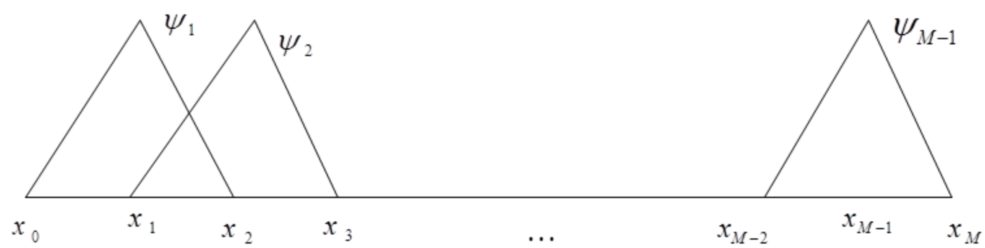

Figure 5.1: The figure that shows the basis of S_h.

5.3.2 Case2: Piecewise linear in time and two dimension in space

In this section, we will implement the fully discrete DPGFE scheme defined by (5.9) on rectangular polygons. To do this, let $a_n(x, y) = \sum_{j=1}^{M_h} \sum_{i=1}^{M_h} a_{n,i,j} \psi_i(x)\psi_j(y)$, where $\psi_i(x), \psi_i(y)$ are basis for all $n = 1, 2, .., N$ and choose $X = \psi_d(x)\psi_m(y)$ for all $d, m = 1, 2, .., M_h$. We subsitute (replace

$a_n(x,y) = \sum_{j=1}^{M_h}\sum_{i=1}^{M_h} a_{n,i,j}\psi_i(x)\psi_j(y)$, $X = \psi_d(x)\psi_m(y)$) in (3.46), we obtain

$$\begin{aligned}
&\frac{\omega_{\alpha+2}(k_n)}{k_n}\Big\langle \sum_{j=1}^{M_h}\sum_{i=1}^{M_h}(a_{n,i,j} - a_{n-1,i,j})\psi_i\psi_j, \psi_d\psi_m\Big\rangle_{xy} \\
&+ \frac{k_n}{2}\Big\langle \nabla\sum_{j=1}^{M_h}\sum_{i=1}^{M_h} a_{n,i,j}\psi_i\psi_j + \nabla\sum_{j=1}^{M_h}\sum_{i=1}^{M_h} a_{n-1,i,j}\psi_i\psi_j, \nabla\psi_d\psi_m\Big\rangle_{xy} \\
&= \int_{t_{n-1}}^{t_n}\langle f(t), \psi_i\psi_j\rangle_{xy}dt - \sum_{l=1}^{n-1}\omega^{n,i}\Big\langle \sum_{j=1}^{M_h}\sum_{i=1}^{M_h} a_{n,i,j}\psi_i\psi_j - \sum_{j=1}^{M_h}\sum_{i=1}^{M_h} a_{n,i,j}\psi_i(x)\psi_j(y), \psi_d\psi_m\Big\rangle_{xy}
\end{aligned} \tag{5.17}$$

where

$$\langle\psi_i\psi_j, \psi_d\psi_m\rangle_{xy} := \langle\psi_i, \psi_d\rangle\langle\psi_j, \psi_m\rangle$$

So, we can write (5.17) in the form

$$\frac{\omega_{\alpha+2}(k_n)}{k_n}\mathbf{A}\otimes\mathbf{A}\begin{bmatrix} a_{n,1,1} - a_{n-1,1,1} \\ \vdots \\ a_{n,1,M_h} - a_{n-1,1,M_h} \\ ,..., \\ a_{n,M_h,1} - a_{n-1,M_h,1} \\ \vdots \\ a_{n,M_h,M_h} - a_{n-1,M_h,M_h} \end{bmatrix} + \frac{k_n}{2}(\mathbf{E}\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{E})\begin{bmatrix} a_{n,1,1} + a_{n-1,1,1} \\ \vdots \\ a_{n,1,M_h} + a_{n-1,1,M_h} \\ ,..., \\ a_{n,M_h,1} - a_{n-1,M_h,1} \\ \vdots \\ a_{n,M_h,M_h} + a_{n-1,M_h,M_h} \end{bmatrix}$$

$$
= \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1\psi_1 \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1\psi_{M_h} \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h}\psi_1 \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h}\psi_{M_h} \rangle dt \end{bmatrix} - \sum_{l=1}^{n-1} \omega^{n,l} \mathbf{A} \otimes \mathbf{A} \begin{bmatrix} a_{l,1,1} - a_{l-1,1,1} \\ \vdots \\ a_{l,1,M_h} - a_{l-1,1,M_h} \\ \vdots \\ a_{l,M_h,1} - a_{l-1,M_h,1} \\ \vdots \\ a_{l,M_h,M_h} - a_{l-1,M_h,M_h} \end{bmatrix}
$$

So, we arrive at the following system

$$
\left(2\mathbf{B}^{\star} + \mathbf{\Gamma}(\alpha + \mathbf{2})\mathbf{D}^{\star}\right)\mathbf{V} = \mathbf{2}\mathbf{\Gamma}(\alpha + \mathbf{2})\mathbf{Q}
$$

where

$$
V = \left[a_{1,1,1} \cdots a_{1,1,M_h} \cdots a_{1,M_h,1} \cdots, a_{1,M_h M_h} \cdots a_{1,1,1} \cdots a_{n,1,M_h} \cdots, a_{n,M_h,1} \cdots a_{n,M_h,M_h}\right]^T
$$

$$
\mathbf{B}^{\star} = \mathbf{B} \otimes (\mathbf{A} \otimes \mathbf{A}), \mathbf{D}^{\star} = \mathbf{D} \otimes (\mathbf{E} \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{E})
$$

$$
\mathbf{Q} = \begin{bmatrix} Q^1 & Q^2 & \dots & Q^N \end{bmatrix}^T,
$$

$$
Q^1 = \begin{bmatrix} \int_{t_0}^{t_1} \langle f(t), \psi_1\psi_1\rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1}\langle U^0, \psi_1\psi_1\rangle - \frac{k_1}{2}\langle \nabla U^0, \nabla\psi_1\nabla\psi_1\rangle \\ \vdots \\ \int_{t_0}^{t_1} \langle f(t), \psi_1\psi_{M_h}\rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1}\langle U^0, \psi_1\psi_{M_h}\rangle - \frac{k_1}{2}\langle \nabla U^0, \nabla\Psi_1\nabla\psi_{M_h}\rangle \\ \vdots \\ \int_{t_0}^{t_1} \langle f(t), \psi_{M_h}\psi_1\rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1}\langle U^0, \psi_{M_h}\psi_1\rangle - \frac{k_1}{2}\langle \nabla U^0, \nabla\psi_{M_h}\nabla\psi_1\rangle \\ \vdots \\ \int_{t_0}^{t_1} \langle f(t), \psi_{M_h}\psi_{M_h}\rangle dt + \frac{\omega_{\alpha+2}(k_1)}{k_1}\langle U^0, \psi_{M_h}\psi_{M_h}\rangle - \frac{k_1}{2}\langle \nabla U^0, \nabla\psi_{M_h}\nabla\psi_{M_h}\rangle \end{bmatrix}
$$

$$
Q^n = \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1\psi_1\rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1\psi_{M_h}\rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h}\psi_1\rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h}\psi_{M_h}\rangle dt \end{bmatrix} - \sum_{j=1}^{n-1} \mathbf{A} \otimes \mathbf{A} \begin{bmatrix} a_{l,1,1} - a_{l-1,1,1} \\ \vdots \\ a_{l,1,M_h} - a_{l-1,1,M_h} \\ \vdots \\ a_{l,M_h,1} - a_{l-1,M_h,1} \\ \vdots \\ a_{l,M_h,M_h} - a_{l-1,M_h,M_h} \end{bmatrix}
$$

for $n = 2, \ldots, N.$

5.3.3 Case3: High order in time and one dimension in space

This section is devoted to implement the fully discrete DPGFE scheme defined by (5.9). Let $a_n^j = \sum_{l=1}^{M_h} a_{n,l}^j \Psi_l$, where ψ_l are basis for all $l = 1, 2, .., n$ and choose $\chi = \psi_{l^*}$ for all $l^* = 1, 2, .., M_h$. We subsitute (replace $a_n^j = \sum_{l=1}^{M_h} a_{n,l}^j \psi_l$, $\chi = \psi_{l^*}$) in (4.37) and integrating, we obtain

$$
\begin{aligned}
& k_n^{\alpha+1} \sum_{j=0}^{m-1} \Big\langle \sum_{l=1}^{M_h} a_{n,l}^j \psi_l, \psi_{l^*} \Big\rangle \frac{\Gamma(j+1)}{(j+\alpha+r+1)\Gamma(j+\alpha+1)} \\
& + k_n^2 \sum_{j=0}^{m-1} \Big\langle \sum_{l=1}^{M_h} a_{n,l}^j \nabla \Psi_l, \nabla \psi_{l^*} \Big\rangle \frac{1}{(j+1)(j+r+2)} \\
& = \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^r}{k_n^r} \big(\langle f(t), \psi_{l^*} \rangle - \frac{k_n}{r+1} \langle \nabla U(t_{n-1}), \nabla \psi_{l^*} \rangle \\
& - \sum_{j=0}^{m-1} \sum_{d=1}^{n-1} \sum_{l=1}^{M_h} \langle a_{d,l}^j \psi_l, \psi_{l^*} \rangle \int_{t_{d-1}}^{t_d} \omega_\alpha(t-s) \frac{(s-t_{d-1})^j}{k_d^j} ds \big) dt,
\end{aligned}
$$

for $l^* = 1, 2, ..., M_h$ and for $r = 0, 1, ..., m-1$ and for $j = 0, 2, ..., m-1$.

We introduce the following notations

$$
\mathbf{A_1} = [\langle \psi_i, \psi_j \rangle]_{M_h \times M_h}, \; \mathbf{E_1} = [\langle \nabla \psi_i, \nabla \psi_j \rangle]_{M_h \times M_h}
$$

$$
\mathbf{B_2} = \mathbf{B_1} \otimes \mathbf{A_1}\,, \qquad \mathbf{D_2} = \mathbf{D_1} \otimes \mathbf{E_1},
$$

where the two matrices $\mathbf{B_1}$ and $\mathbf{D_1}$ are defined in chapter 4 section (4.4) and

$$
w^{n,d,s,j} = \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{s-1}}{k_n^{s-1}} \sum_{d=1}^{n-1} \int_{t_{d-1}}^{t_d} \omega_\alpha(t-s) \frac{(s-t_{d-1})^j}{k_d^j} ds\, dt.
$$

So, we arrive at the following system

$$\left(k_n^{\alpha+1}\mathbf{B_2}+\mathbf{k_n^2D_2}\right)\mathbf{Y}=\mathbf{G_n}, \tag{5.18}$$

where

$$Y=\begin{bmatrix} a_{n,1}^1 \\ \vdots \\ a_{n,M_h}^1 \\ \vdots \\ a_{n,M_h}^m \\ \vdots \\ a_{n,M_h}^m \end{bmatrix},$$

$$
\mathbf{G_n} = \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1 \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h} \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{m-1}}{k_n^{m-1}} \langle f(t), \psi_1 \rangle dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^{m-1}}{k_n^{m-1}} \langle f(t), \psi_{M_h} \rangle dt \end{bmatrix} - \begin{bmatrix} k_n \left\langle U(t_{n-1}, \psi_1 \right\rangle \\ \vdots \\ k_n \left\langle U(t_{n-1}, \psi_{M_h} \right\rangle \\ \vdots \\ \frac{k_n}{m} \langle U(t_{n-1}), \psi_1 \rangle \\ \vdots \\ \frac{k_n}{m} \langle U(t_{n-1}, \psi_{M_h} \rangle \end{bmatrix}
$$

$$
- \begin{bmatrix} w^{n,d,0,0} & w^{n,d,0,1} & \cdots & w^{n,d,0,m-1} \\ w^{n,d,1,0} & w^{n,d,1,1} & \cdots & w^{n,d,1,m-1} \\ \vdots & \vdots & & \vdots \\ w^{n,d,m-1,0} & w^{n,d,m-1,1} & \cdots & w^{n,d,m-1,m-1} \end{bmatrix} \otimes \mathbf{A_1} \begin{bmatrix} a_{d,1}^1 \\ \vdots \\ a_{d,1}^m \\ \vdots \\ a_{d,M_h}^1 \\ \vdots \\ a_{d,M_h}^m \end{bmatrix}.
$$

The system (5.18) obtained after combining the DPG scheme in time with FEM in space to find a.

CHAPTER 6

Numerical Results

In this chapter, we present a series of numerical examples and compare the theoretical results with the numerical results.

6.1 Composite p-point Gauss Quadrature Rule

In this section, we will introduce a quadrature rule that we use to compute integrals. Let g be a function defined on [0,1], consider a mesh $0 = x_0 < x_1 < x_2 < ... < x_M = 1$ and let $h_i = x_i - x_{i-1}$. The composite p-point Gauss quadrature rule for approximating $\int_0^1 g(x)dx$ is defined by

$$\int_0^1 g(x)dx = \sum_{i=1}^{M} \sum_{l=1}^{p} w_l h_i g(x_{i,l}),$$

where $x_{i,l} = x_{i-1} + h_i \xi_l$, $w_1, w_2, ..., w_p$ are the weights and $\xi_1, \xi_2, ..., \xi_p$ are the nodes.

To determine the p nodes (ξ_i) and p weights (w_i) we need to solve the $2p$

nonlinear equations

$$w_1\xi_1^r + w_2\xi_2^r + ... + w_p\xi_p^r = \frac{1}{r+1}, \;\; r = 0, 1, ..., 2p-1.$$

The r^{th} equation is obtained by the requirement that the rule

$$w_1\xi_1^r + w_2\xi_2^r + ... + w_p\xi_p^r = \int_0^1 p(x)dx$$

be exact for $p(x) = x^r$, $r = 0, 1, ..., 2p-1$.

For instance, if $p = 2$ the weights are w_1, w_2 and the nodes are ξ_1, ξ_2. To determine these, we require it to be exact for the four monomials

$$p(x) = 1, x, x^2, x^3$$

This leads to the four equations

$$1 = w_1 + w_2$$

$$1/2 = w_1\xi_1 + w_2\xi_2$$

$$1/3 = w_1\xi_1^2 + w_2\xi_2^2$$

$$1/4 = w_1\xi_1^3 + w_2\xi_2^3$$

To evaluate L_∞ error in time, we introduce the finer grid

$$\mathcal{G}^q = \{\, t_{j-1} + nk_j/q \,:\, 1 \le j \le N,\ 0 \le n \le q\}, \tag{6.1}$$

where N is the number of time mesh subintervals. Thus, for large values of q, the error measure $|||v|||_q := \max_{t\in\mathcal{G}^q} \|v(t)\|$ approximates the norm $\|v\|_{L_\infty(0,T;L_2(\Omega))}$. To compute spatial L_2-norm, we apply a composite 2-point Gauss quadrature rule on each interval of the uniform spatial mesh.

We compute the order of convergence with respect to the change in the number of subintervals N in the spatial mesh by using the following formula:

$$\frac{\log(\mathrm{error(N(i-1))/error(N(i))}}{\log(N(i)/N(i-1))},$$

where $error(N(i))$ is the error measure $|||v|||_q$.

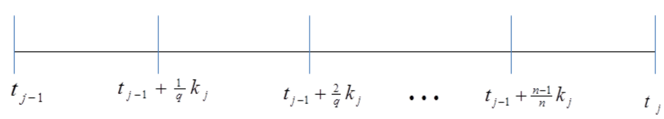

Figure 6.1: This figure shows the set G^q on the interval $[t_{j-1}, t_j]$.

6.2 Examples

Example 1(one dimension in space)

We let $\Omega = (0,1)$, $\triangle u = u_{xx}$, $T = 1$ and $u_0 = 0$.

(a) Smooth solution

We assume that $u = u(x,t)$ satisfies (1.3). We choose $f(x,t)$ such that $u = t^{1+\alpha}\sin(\pi x)$ is the exact solution of (1.3). Thus

$$f(x,t) = (\pi^2 t^{1+\alpha} + \Gamma(2+\alpha))\sin(\pi x).$$

It can be seen that the regularity conditions in (4.21) and (4.22) hold for $\sigma = 1+\alpha$ and $\delta = 2+\alpha$. To test the accuracy of of the DPG-FE scheme (5.9) (with degree m in the time variable and r in the spatial variable) on the non-uniform mesh in (3.6)–(3.7) for various choices of $\gamma \geq 1$, h (the spatial step-size) will be chosen such that the temporal errors are dominant. Thus, from Theorem 5.8, we expect to observe convergence of order $O(k^{m+\alpha/2})$ for $\gamma > \max\left\{\frac{2m+\alpha}{3\alpha+1}, \frac{m+1}{\alpha+1}\right\}$. However, the numerical results in Tables 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8 and 6.9 illustrate more optimistic convergence rates compared and also demonstrated that the grading mesh parameter γ is slightly relaxed. We observe a uniform global error bounded by $Ck^{\min\{\gamma(1+\alpha),m+1\}}$ for $\gamma \geq 1$, which is optimal for $\gamma \geq (m+1)/(\alpha+1)$.

The results are also displayed graphically in Figures 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8 and 6.9, where we show the errors against the number of time subintervals N, in the semilogarithmic scale.

N	$\gamma=1$		$\gamma=1.4$		$\gamma=1.8$	
20	4.12e-03		1.9607e-03		2.29e-03	
40	1.5e-03	1.47	4.90e-04	1.99	5.76e-04	1.99
80	6.99e-04	1.09	1.53e-04	1.68	1.48e-04	1.95
160	3.16e-04	1.15	4.89e-05	1.64	3.69e-05	2.00
320	1.43e-04	1.15	1.57e-05	1.63	8.99e-06	2.04

Table 6.1: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.2$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.2\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1} .

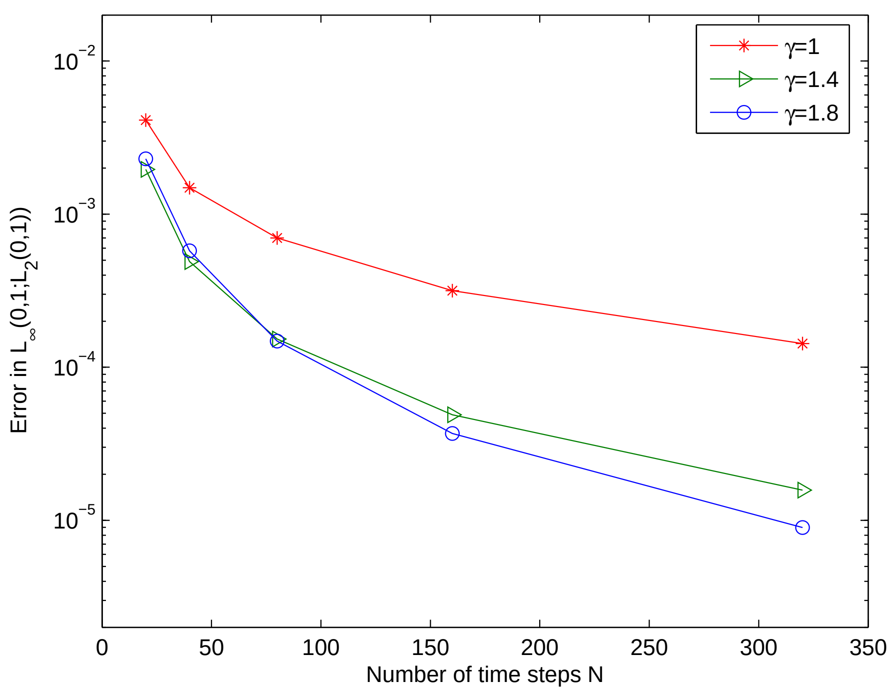

Figure 6.2: Graph of error in Table 6.1 .

N	$\gamma = 1$		$\gamma = 1.2$		$\gamma = 1.4$	
20	6.76e-03		3.69e-03		3.22e-03	
40	2.29e-03	1.56	1.01e-03	1.88	7.77e-04	2.05
80	8.22e-04	1.48	2.93e-04	1.78	2.03e-04	1.94
160	2.77e-04	1.57	7.96e-05	1.88	5.06e-05	2.00
320	9.51e-05	1.55	2.21e-05	1.85	1.26e-05	2.01

Table 6.2: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.5$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.5\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1} .

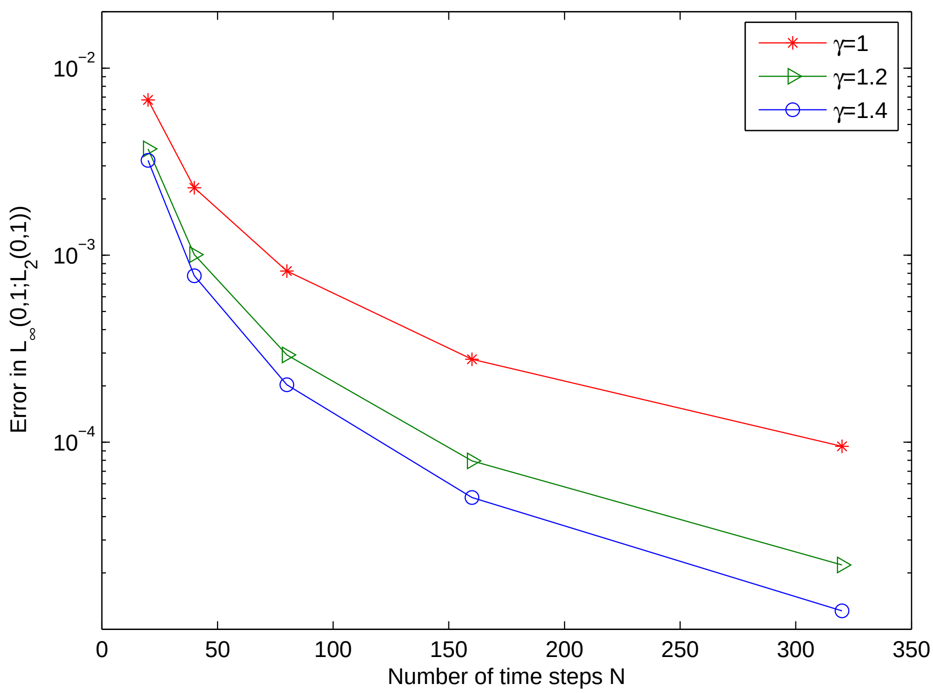

Figure 6.3: Graph of error in Table 6.2 .

N	$\gamma = 1$		$\gamma = 1.1$		$\gamma = 1.2$	
20	6.45e-03		4.6201e-03		4.15e-03	
40	1.96e-03	1.72	1.2473e-03	1.89	8.75e-04	2.24
80	6.356e-04	1.62	3.6122e-04	1.79	2.25e-04	1.96
160	1.94e-04	1.72	9.7623e-05	1.89	5.57e-05	2.01
320	5.98e-05	1.69	2.6826e-05	1.86	1.39e-05	2.00

Table 6.3: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.7$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.7\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

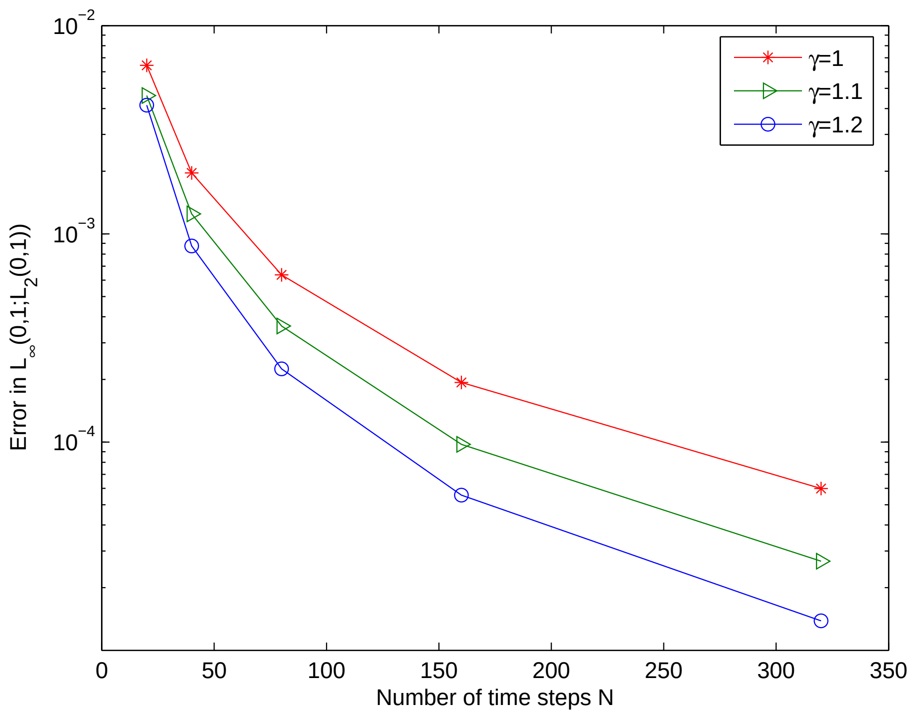

Figure 6.4: Graph of error in Table 6.3 .

N	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2.5$	
10	5.43e-04		1.23e-04		6.61e-05	
20	1.95e-04	1.48	2.88e-05	2.10	5.43e-06	3.60
40	7.59e-05	1.36	7.22e-06	1.99	5.22e-07	3.38
80	3.05e-05	1.3	1.84e-06	1.97	5.91e-08	3.14
160	1.25e-05	1.29	4.73e-07	1.96	6.95e-09	3.09

Table 6.4: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ (that is, the DPG time stepping solution is piecewise quadratic) and $\alpha = 0.3$. We observe convergence of order $k^{(1+\alpha)\gamma} (= k^{1.3\gamma})$ for $1 \leq \gamma < (m+1)/(1+\alpha)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

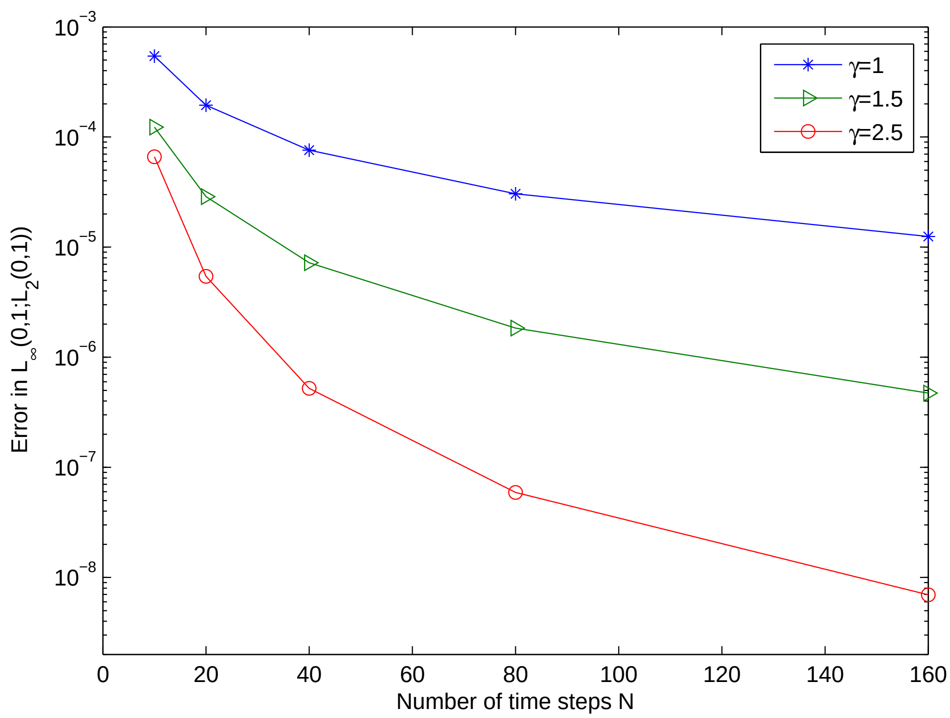

Figure 6.5: Graph of error in Table 6.4 .

N	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
10	5.57e-04		1.12e-04		1.04e-04	
20	1.68e-04	1.73	1.7116e-05	2.7055	9.18e-06	3.49
40	5.29e-05	1.67	2.8873e-06	2.5675	8.73e-07	3.39
80	1.71e-05	1.63	5.3985e-07	2.4191	8.85e-08	3.30
160	5.59e-06	1.61	1.0643e-07	2.3426	9.39e-09	3.24

Table 6.5: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ and $\alpha = 0.5$. We observe convergence of order $k^{(1+\alpha)\gamma} (= k^{1.5\gamma})$ for $1 \leq \gamma < (m+1)/(1+\alpha)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

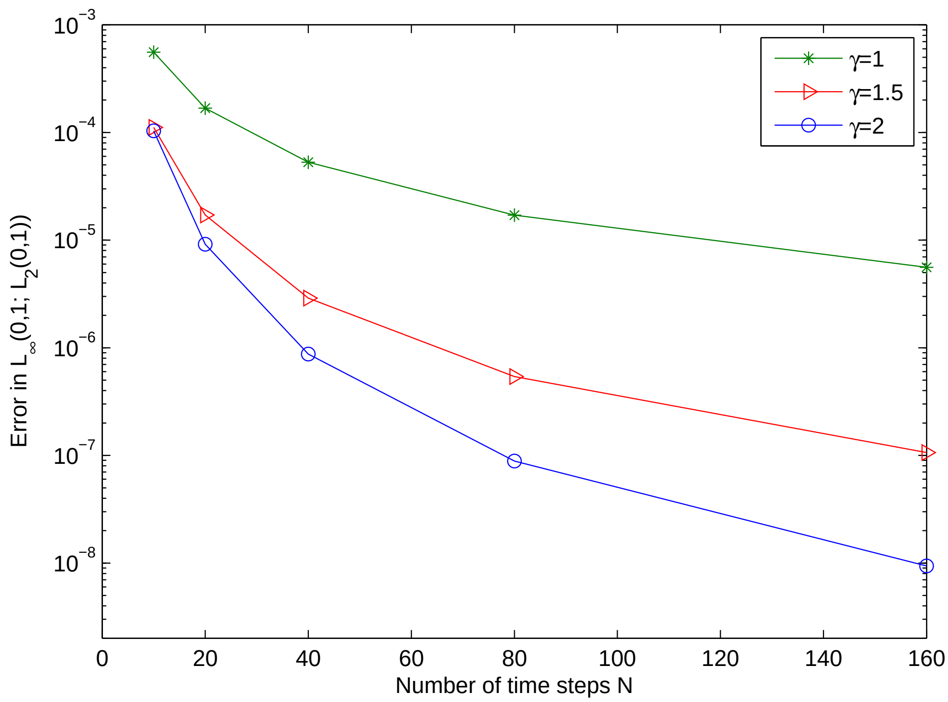

Figure 6.6: Graph of error in Table 6.5 .

N	$\gamma = 1$		$\gamma = 1.4$		$\gamma = 1.8$	
10	3.28e-04		1.29e-04		1.51e-04	
20	8.91e-05	1.8820	1.93e-05	2.74	1.80e-05	3.07
40	2.57e-05	1.7909	3.14e-06	2.62	1.90e-06	3.25
80	7.59e-06	1.7602	5.33e-07	2.56	2.01e-07	3.24
160	2.26e-06	1.7460	9.29e-08	2.52	2.17e-08	3.21

Table 6.6: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ and $\alpha = 0.7$. We observe convergence of order $k^{(1+\alpha)\gamma}(= k^{1.7\gamma})$ for $1 \leq \gamma < (m+1)/(1+\alpha)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

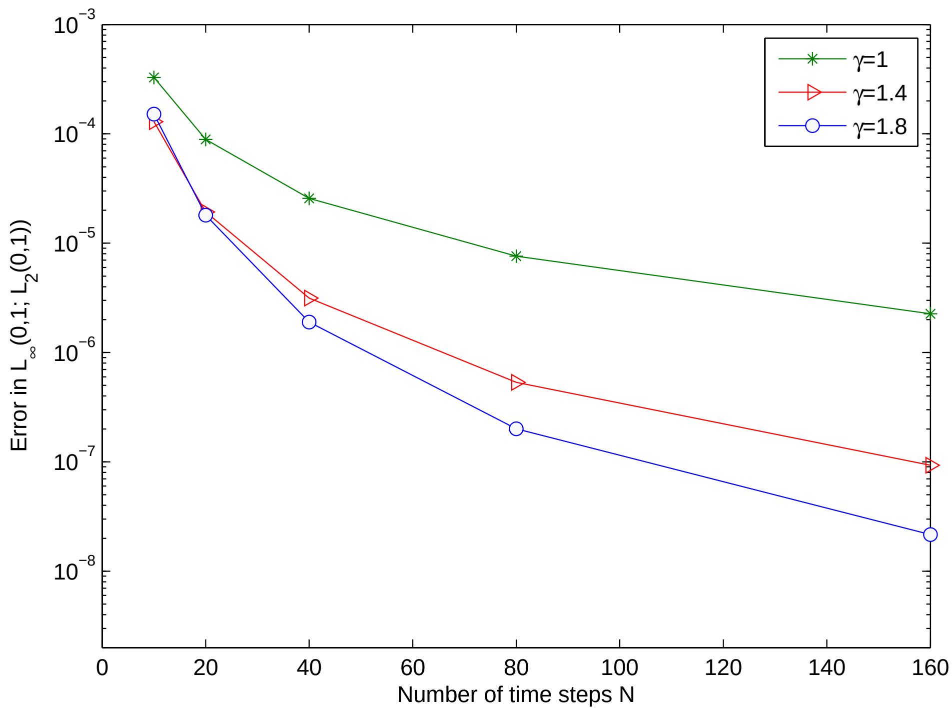

Figure 6.7: Graph of error in Table 6.6 .

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3.5$	
6	3.87e-04		5.46e-05		2.49e-05	
10	1.89e-04	1.40	1.34e-05	2.73	2.09e-06	4.85
14	1.20e-04	1.33	5.59e-06	2.61	5.04e-07	4.22
18	8.69e-05	1.31	2.94e-06	2.56	1.80e-07	4.09
22	6.72e-05	1.28	1.77e-06	2.53	7.83e-08	4.15

Table 6.7: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ (that is, the DPG time stepping solution is piecewise cubic) and $\alpha = 0.2$. We observe convergence of order $k^{(1+\alpha)\gamma}(= k^{1.2\gamma})$ for $1 \leq \gamma < (m+1)/(1+\alpha)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

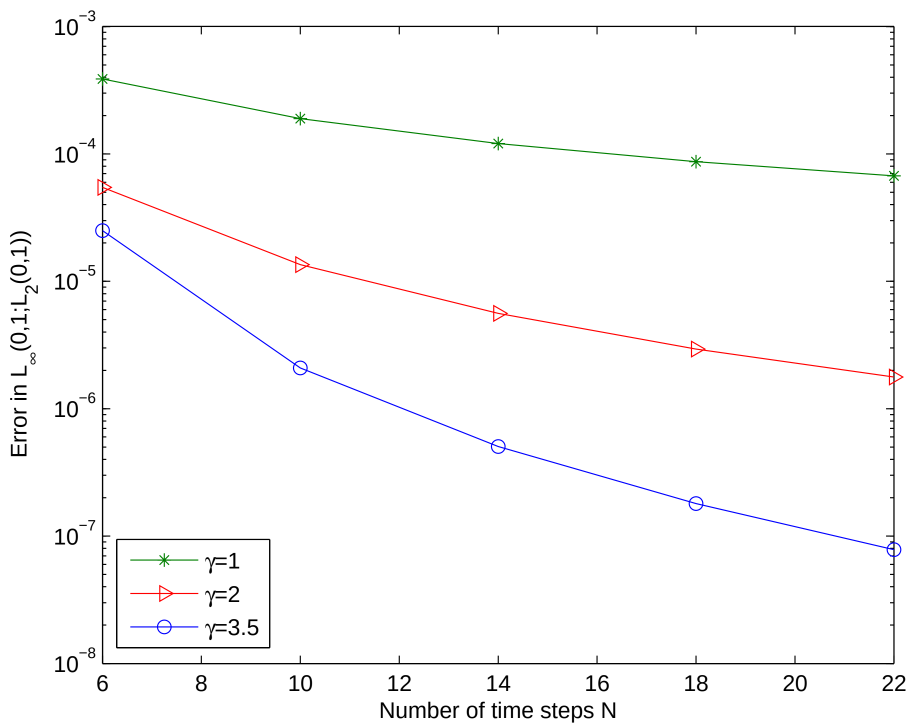

Figure 6.8: Graph of error in Table 6.7 .

N	$\gamma=1$		$\gamma=2$		$\gamma=3$	
6	4.18e-04		3.02e-05		2.78e-05	
10	1.60e-04	1.87	4.47e-06	3.74	2.68e-06	4.57
14	8.81e-05	1.78	1.37e-06	3.50	6.14e-07	4.38
18	5.68e-05	1.74	5.87e-07	3.38	2.08e-07	4.29
22	4.02e-05	1.72	3.02e-07	3.31	8.94e-08	4.22

Table 6.8: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ and $\alpha = 0.5$. We observe convergence of order $k^{(1+\alpha)\gamma}(= k^{1.5\gamma})$ for $1 \le \gamma < (m+1)/(1+\alpha)$ and for $\gamma \ge (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

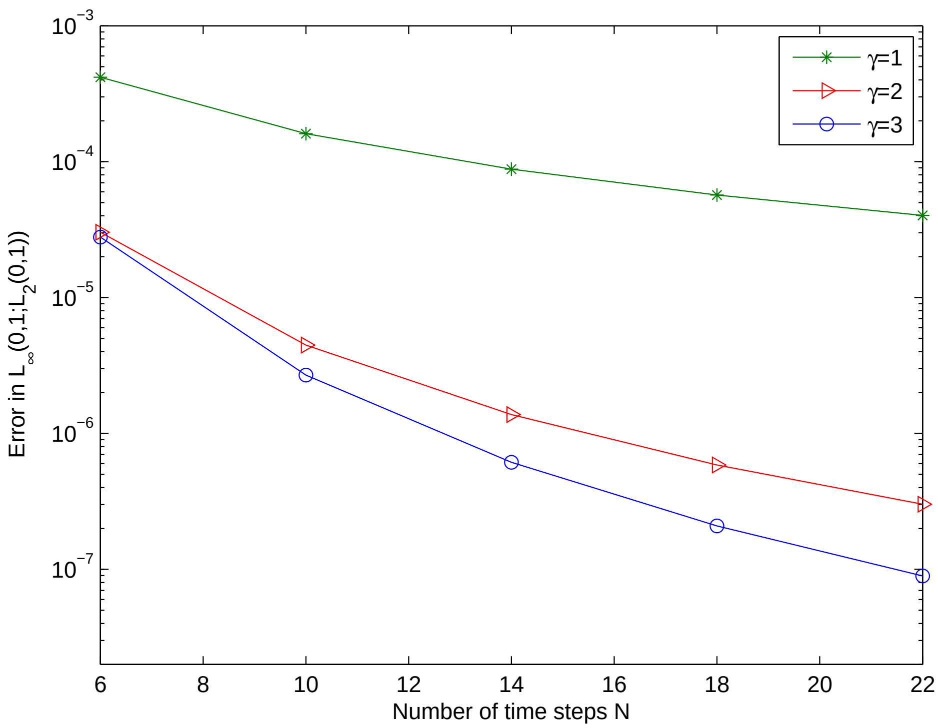

Figure 6.9: Graph of error in Table 6.8 .

N	$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2.5$	
6	2.36e-04		5.88e-05		1.25e-05	
10	8.56e-05	1.99	1.27e-05	2.99	1.14e-06	4.69
14	4.53e-05	1.89	4.89e-06	2.84	2.61e-07	4.38
18	2.85e-05	1.85	2.43e-06	2.78	9.14e-08	4.18
22	1.97e-05	1.83	1.40e-06	2.75	3.98e-08	4.14

Table 6.9: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ and $\alpha = 0.7$. We observe convergence of order $k^{(1+\alpha)\gamma}(= k^{1.7\gamma})$ for $1 \leq \gamma < (m+1)/(1+\alpha)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

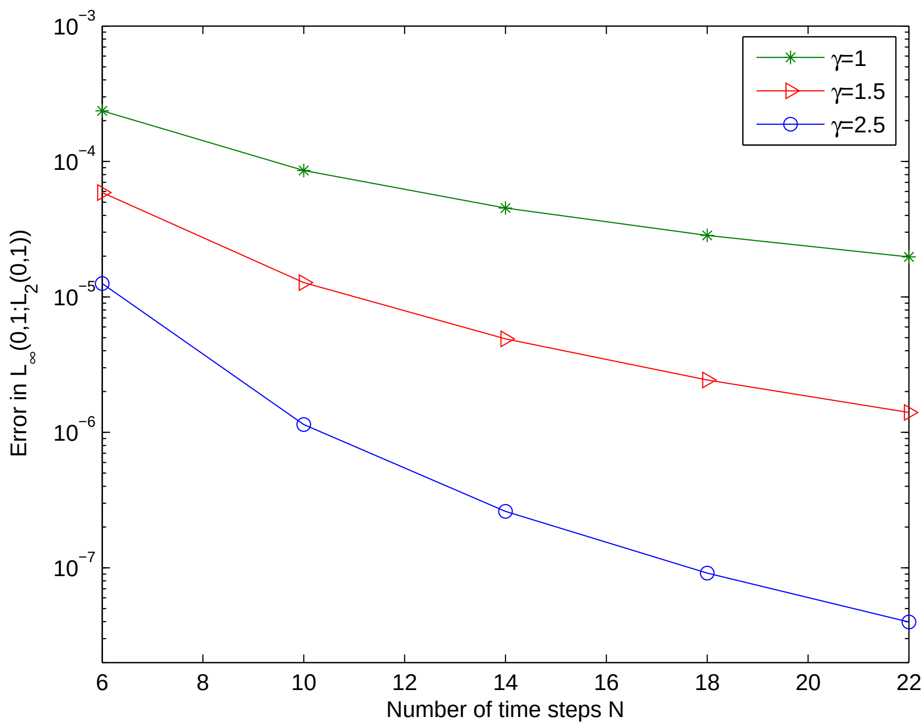

Figure 6.10: Graph of error in Table 6.9 .

Next, we test the performance of the spatial finite elements discretizaton of (order degree r) of the scheme (5.9). We use a uniform spatial mesh consists of Nx subintervals and each is of width h. The time step-size k and the degree of the time-stepping DPG discretization are chosen such that the spatial errors is dominating. Hence, we expect from Theorem 5.8 to see convergence of order $O(h^{r+1})$. We illustrate these results in Tables 6.10 and 6.11 for $r = 1, 2, 3$.

N_x	$r = 1$		$r = 2$		$r = 3$	
10	5.70e-03		1.29e-04		2.38e-06	
15	2.52e-03	2.16	3.80e-05	3.25	4.68e-07	4.34
20	1.43e-03	2.08	1.51e-05	3.38	1.58e-07	3.97
25	9.20e-04	2.07	8.25e-06	2.85	6.23e-08	4.38
30	5.69e-04	2.02	3.95e-06	3.08	4.02e-08	4.01

Table 6.10: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $\alpha = 0.2$. We observe convergence of order h^{r+1} for $r = 1, 2, 3$.

N_x	$r = 1$		$r = 2$		$r = 3$	
10	5.64e-03		1.77e-04		2.29e-06	
15	2.51e-03	2.16	5.67e-05	3.03	4.55e-07	4.32
20	1.42e-03	2.08	2.41e-05	3.15	1.50e-07	4.07
25	9.17e-04	2.06	1.26e-05	3.00	6.01e-08	4.29
30	5.68e-04	2.01	6.02e-06	3.12	2.28e-08	4.05

Table 6.11: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $\alpha = 0.7$. We observe convergence of order h^{r+1} for $r = 1, 2, 3$.

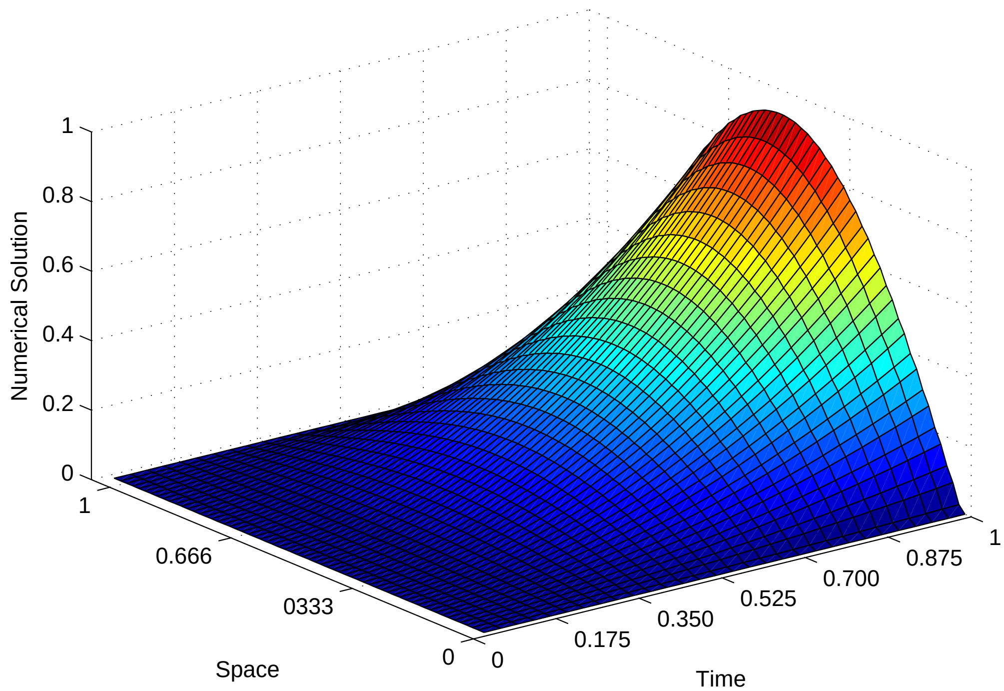

Figure 6.11: This figure that shows the numerical solution for $\alpha = 0.5$ and $\gamma = 1.4$.

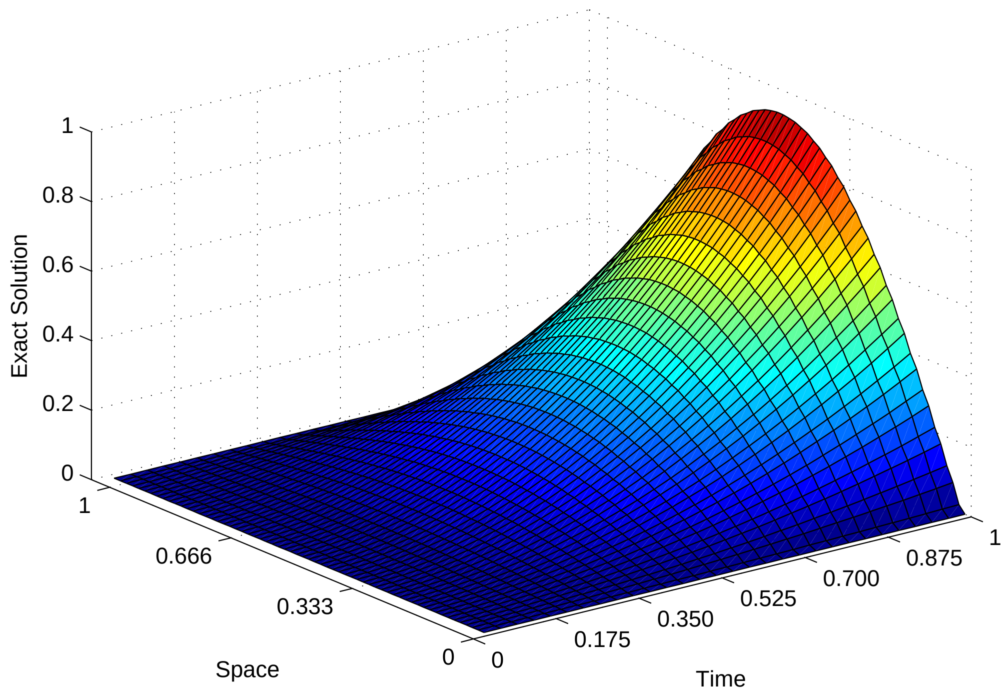

Figure 6.12: This figure that shows the exact solution for $\alpha = 0.5$ and $\gamma = 1.4$.

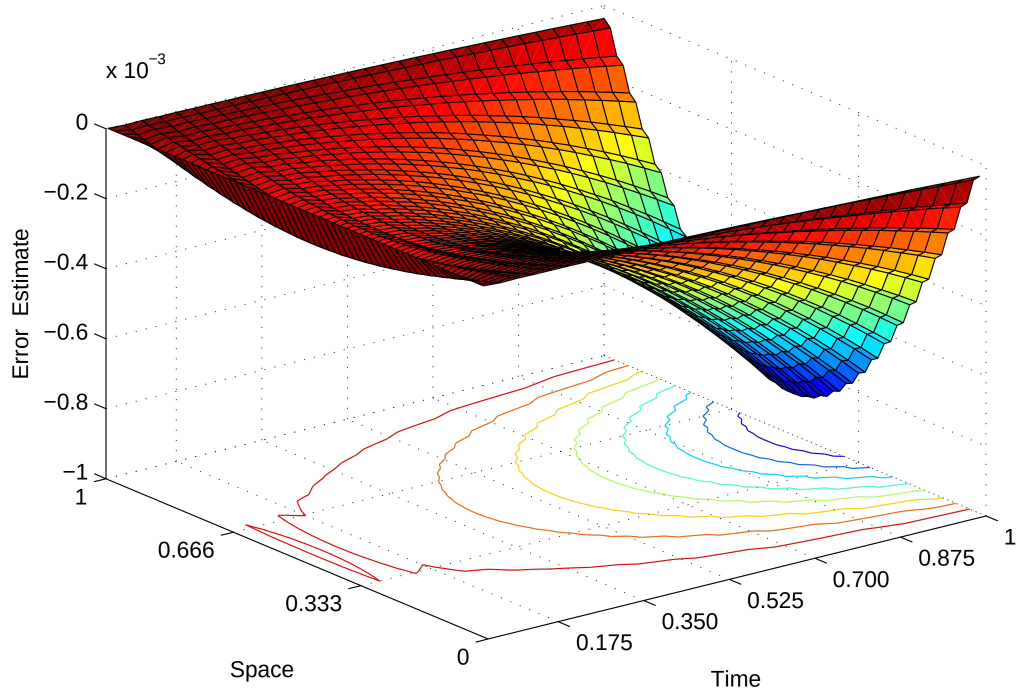

Figure 6.13: This figure that shows the error estimate for $\alpha = 0.5$ and $\gamma = 1.4$.

(b) Nonsmooth solution

We choose the initial datum such that the exact solution is:

$$u(x,t) = t^{1-\alpha}\sin(\pi x). \tag{6.2}$$

Thus $f(x,t) = (\pi^2 t^{1-\alpha} + \Gamma(2-\alpha))\sin(\pi x)$. It can be seen that the regularity conditions in (4.21) and (4.22) hold for $\sigma = 1-\alpha$ and $\delta = 2-\alpha$. In this part we test the accuracy of the time-stepping DPG when the solution of (1.3) in (6.2) is less smooth than part (a). We From Theorem 5.8, we expect to observe convergence of order $O(k^{m+\alpha/2})$ for $\gamma > (2m+\alpha)/(1-\alpha)$. However, the numerical results in Tables 6.12, 6.13, 6.14, 6.15, 6.16, 6.17, 6.18, 6.19, 6.20 and 6.21 illustrate more optimistic convergence rates compared and also demonstrated that the grading mesh parameter γ is slightly relaxed. We observe a uniform global error bounded by $Ck^{\min\{\gamma(1-\alpha),m+1\}}$ for $\gamma \geq 1$, which is optimal for $\gamma \geq (m+1)/(1-\alpha)$.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
20	4.21e-03		4.26e-04		4.44e-04	
40	2.52e-03	7.41e-01	1.42e-04	1.58	1.14e-04	1.96
80	1.49e-03	7.58e-01	4.69e-05	1.59	2.88e-05	1.98
160	8.74e-04	7.72e-01	1.56e-05	1.60	7.22e-06	2.00
320	5.08e-04	7.82e-01	5.12e-06	1.60	1.70e-06	2.08

Table 6.12: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.2$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.8\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

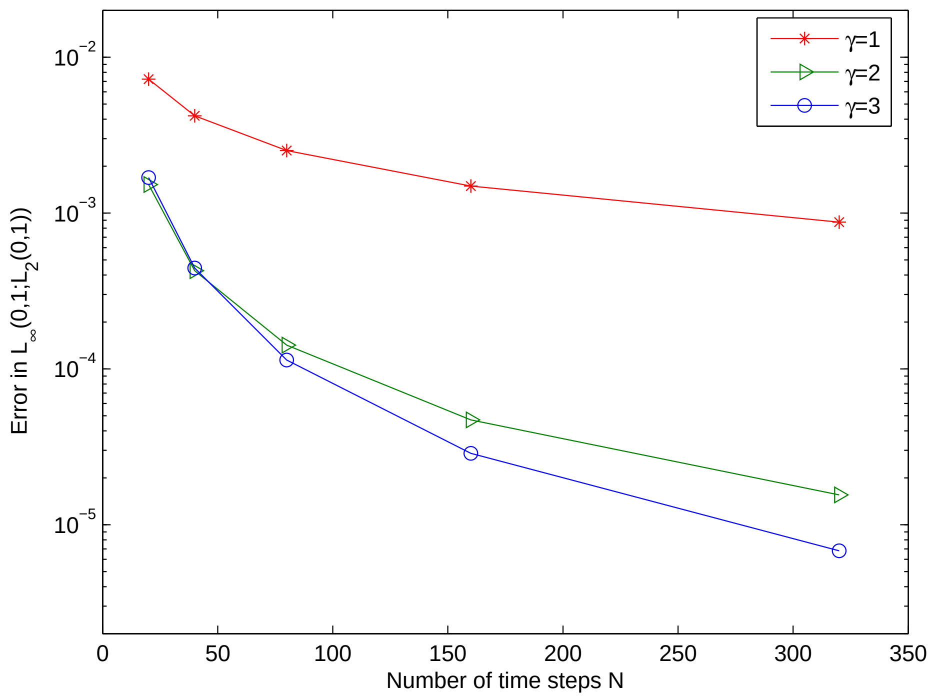

Figure 6.14: Graph of error in Table 6.12.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
20	9.33e-03		1.48e-03		1.52e-03	
40	5.62e-03	7.31e-01	4.50e-04	1.72	3.91e-04	1.96
80	3.53e-03	6.71e-01	1.72e-04	1.39	9.89e-05	1.98
160	2.20e-03	6.78e-01	6.52e-05	1.39	2.48e-05	2.01
320	1.38e-03	6.85e-01	2.47e-05	1.40	5.70e-06	2.12

Table 6.13: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.3$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.7\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

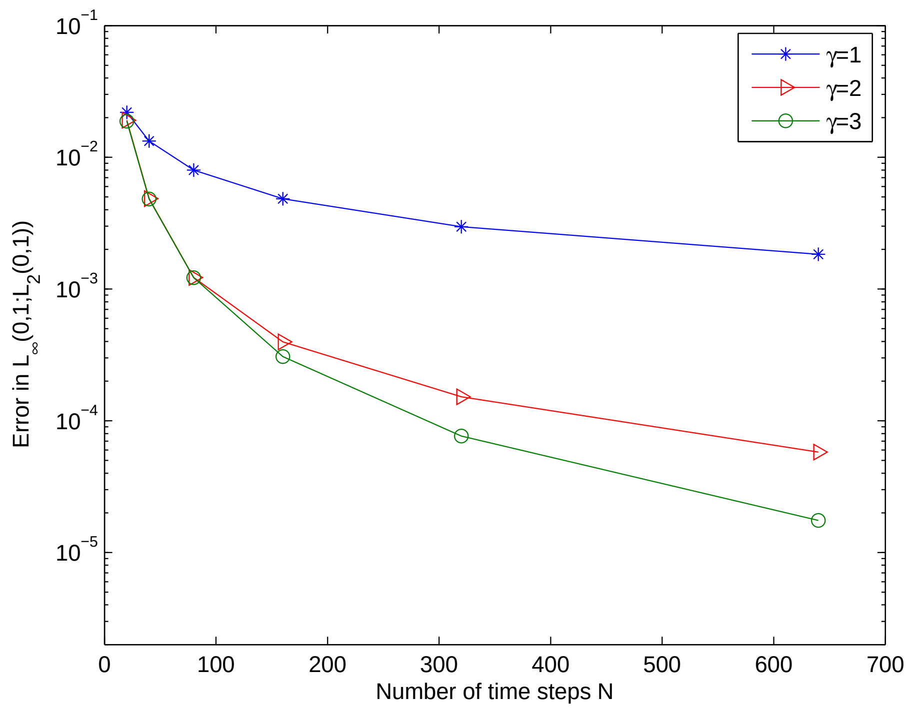

Figure 6.15: Graph of error in Table 6.13.

N	$\gamma = 1$		$\gamma = 2.5$		$\gamma = 4$	
20	4.25e-02		3.47e-03		1.68e-03	
40	2.87e-02	5.66e-01	1.48e-03	1.23	4.28e-04	1.97
80	1.93e-02	5.76e-01	6.23e-04	1.24	1.08e-04	1.98
160	1.29e-02	5.77e-01	2.62e-04	1.25	2.71e-05	2.00
320	8.69e-03	5.73e-01	1.10e-04	1.25	6.33e-06	2.09

Table 6.14: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.5$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.5\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

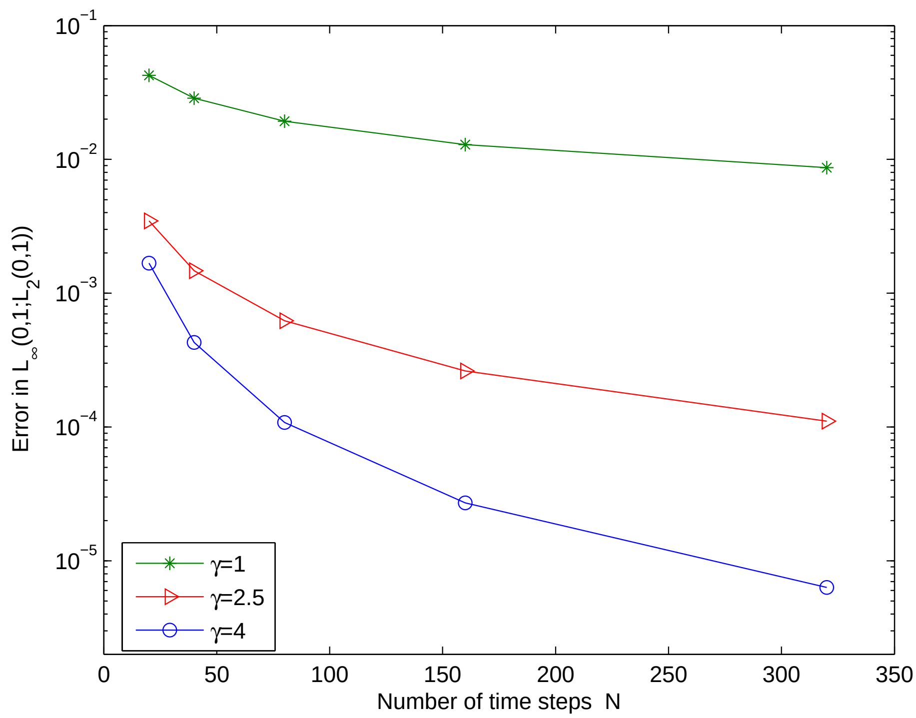

Figure 6.16: Graph of error in Table 6.14.

N	$\gamma = 1$		$\gamma = 4$		$\gamma = 7$	
10	1.82e-01		2.20e-02		8.64e-03	
20	1.48e-01	2.94e-01	9.14e-03	1.27	2.26e-03	1.93
40	1.19e-01	3.12e-01	3.84e-03	1.25	5.78e-04	1.97
80	9.59e-02	3.18e-01	1.64e-03	1.22	1.45e-04	1.98
160	7.68e-02	3.20e-01	7.07e-04	1.21	3.81e-05	1.93

Table 6.15: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 1$ and $\alpha = 0.7$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.3\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

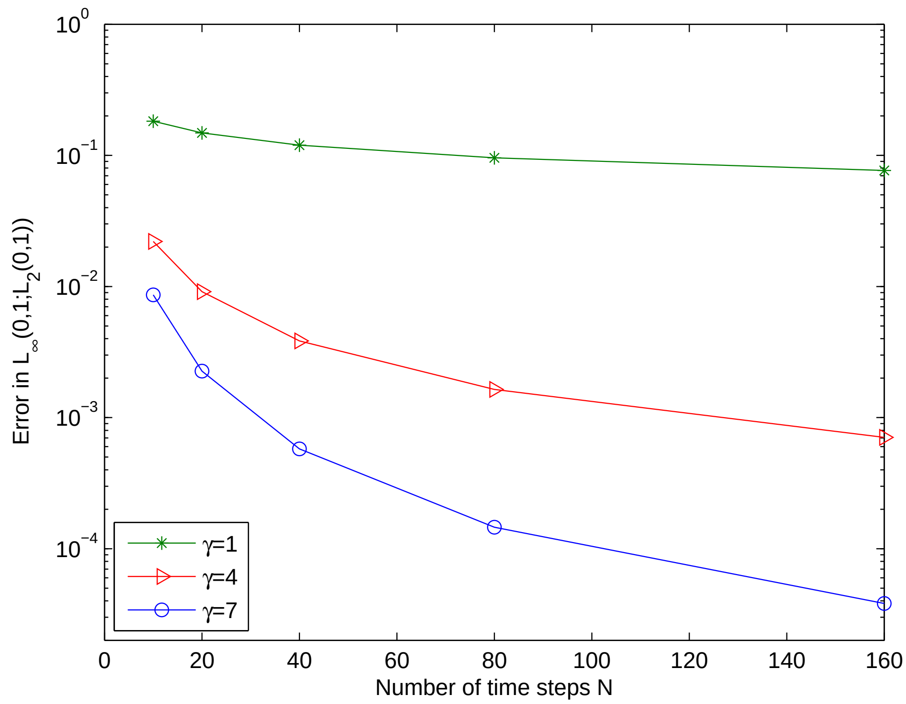

Figure 6.17: Graph of error in Table 6.15.

N	$\gamma = 1$		$\gamma = 2.5$		$\gamma = 4.5$	
10	6.25e-03		6.85e-04		4.54e-04	
18	4.12e-03	7.09e-01	2.27e-04	1.87	5.65e-05	3.54
26	3.19e-03	6.93e-01	1.16e-04	1.82	1.55e-05	3.51
34	2.65e-03	6.88e-01	7.16e-05	1.81	6.26e-06	3.39
42	2.29e-03	6.86e-01	4.91e-05	1.79	3.17e-06	3.22

Table 6.16: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ and $\alpha = 0.3$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.7\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

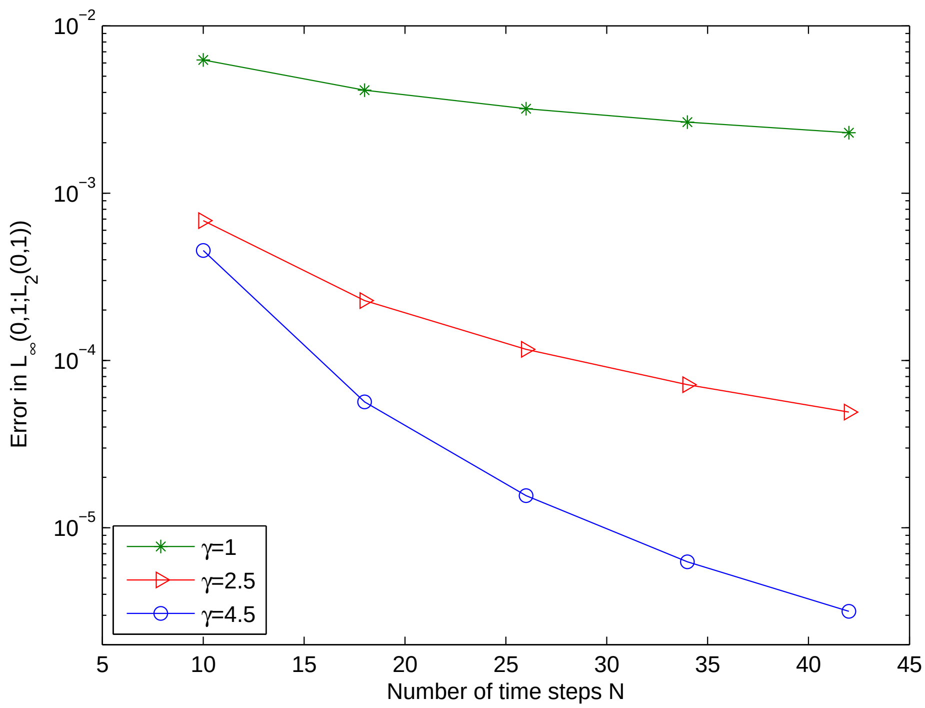

Figure 6.18: Graph of error in Table 6.16 .

N	$\gamma = 1$		$\gamma = 4$		$\gamma = 6$	
5	5.56e-02		1.38e-02		2.31e-02	
10	3.43e-02	7.01e-01	1.36e-03	3.34	2.98e-03	2.95
15	2.62e-02	6.65e-01	5.00e-04	2.47	7.99e-04	3.24
20	2.16e-02	6.53e-01	2.72e-04	2.10	2.97e-04	3.44
25	1.87e-02	6.46e-01	1.71e-04	2.08	1.42e-04	3.28

Table 6.17: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ and $\alpha = 0.5$. We observe convergence of order $k^{(1-\alpha)\gamma} (= k^{0.5\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

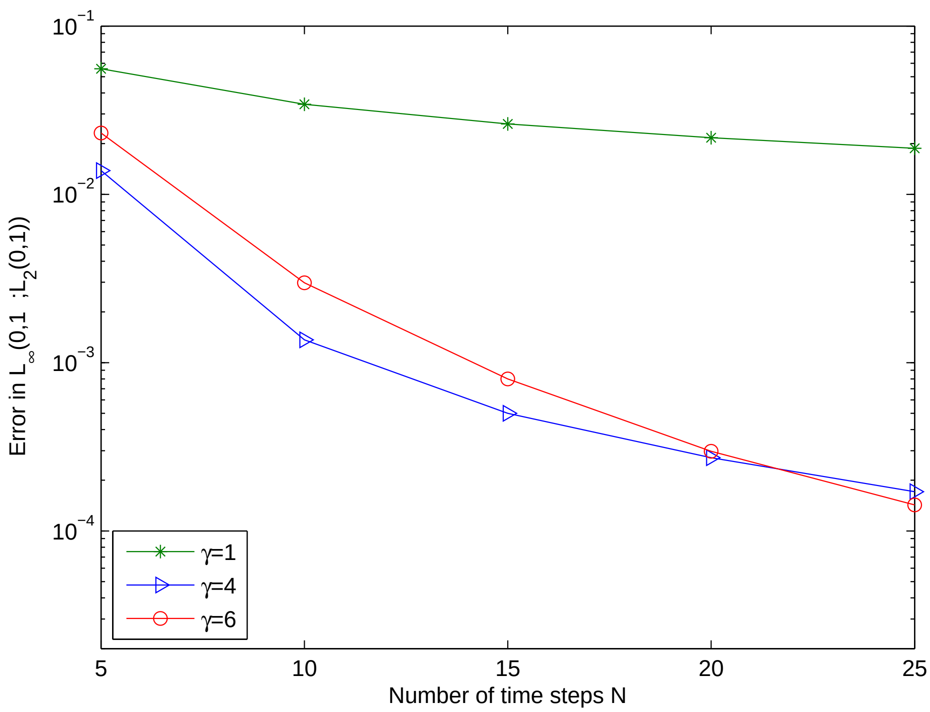

Figure 6.19: Graph of error in Table 6.17.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
10	1.31e-01		6.32e-02		3.01e-02	
20	1.02e-01	3.53e-01	3.83e-02	7.25e-01	1.42e-02	1.09
30	8.90e-02	3.44e-01	2.88e-02	7.04e-01	9.34e-03	1.03
40	8.07e-02	3.41e-01	2.35e-02	6.95e-01	6.99e-03	1.01
50	7.48e-02	3.39e-01	2.02e-02	6.88e-01	5.61e-03	9.90e-01

Table 6.18: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 2$ and $\alpha = 0.7$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.3\gamma})$ for $1 \leq \gamma \leq (m+1)/(1-\alpha)$.

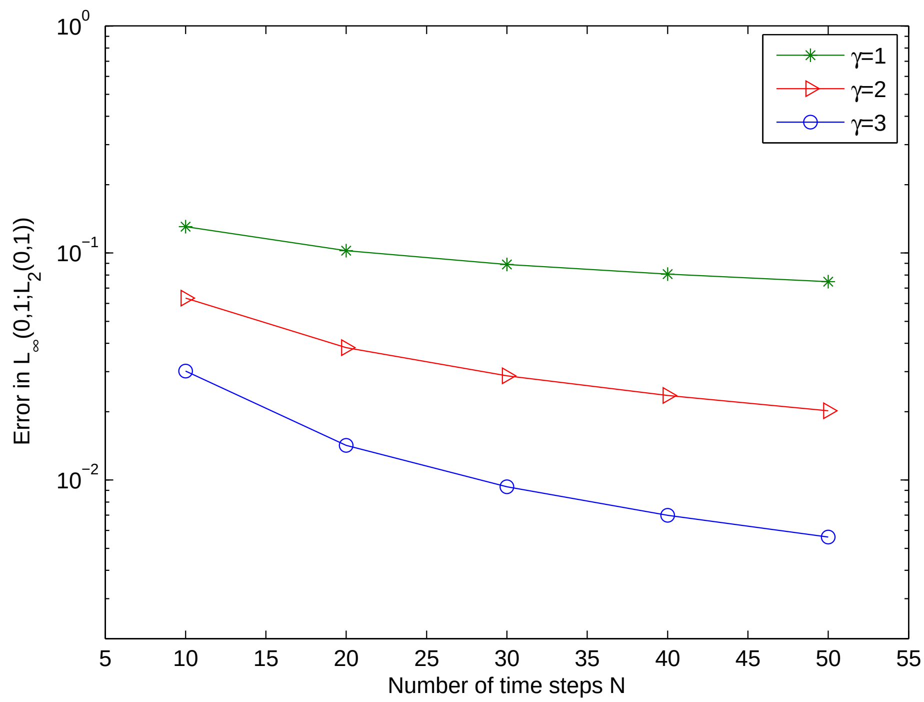

Figure 6.20: Graph of error in Table 6.18 .

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
4	8.30e-03		3.28e-03		1.50e-03	
8	3.92e-03	1.08	9.82e-04	1.74	2.46e-04	2.61
12	2.85e-03	7.88e-01	5.16e-04	1.58	9.49e-05	2.36
16	2.28e-03	7.66e-01	3.32e-04	1.52	4.94e-05	2.27
20	1.93e-03	7.55e-01	2.37e-04	1.49	3.01e-05	2.23

Table 6.19: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ (that is, the DPG time stepping solution is piecewise cubic) and $\alpha = 0.3$. We observe convergence of order $k^{(1-\alpha)\gamma} (= k^{0.7\gamma})$ for $1 \le \gamma \le (m+1)/(1-\alpha)$.

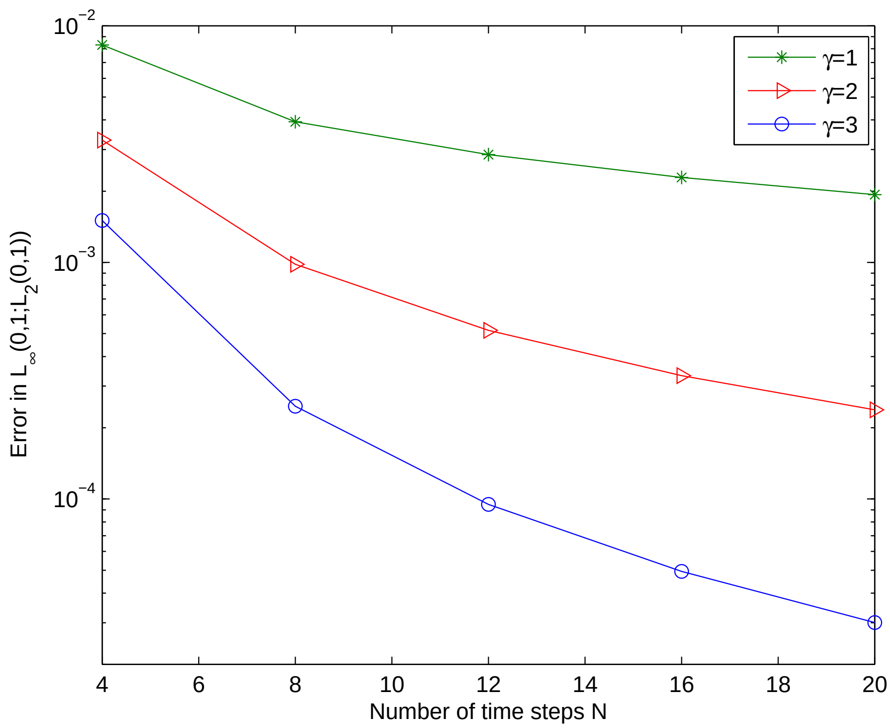

Figure 6.21: Graph of error in Table 6.19 .

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
6	4.87e-02		1.83e-02		7.87e-03	
10	3.43e-02	6.92e-01	8.86e-03	1.42	2.65e-03	2.13
14	2.74e-02	6.67e-01	5.67e-03	1.32	1.54e-03	1.61
18	2.32e-02	6.56e-01	4.15e-03	1.25	1.03e-03	1.58
22	2.04e-02	6.49e-01	3.37e-03	1.02	7.56e-04	1.56

Table 6.20: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ and $\alpha = 0.5$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.5\gamma})$ for $1 \leq \gamma \leq (m+1)/(1-\alpha)$.

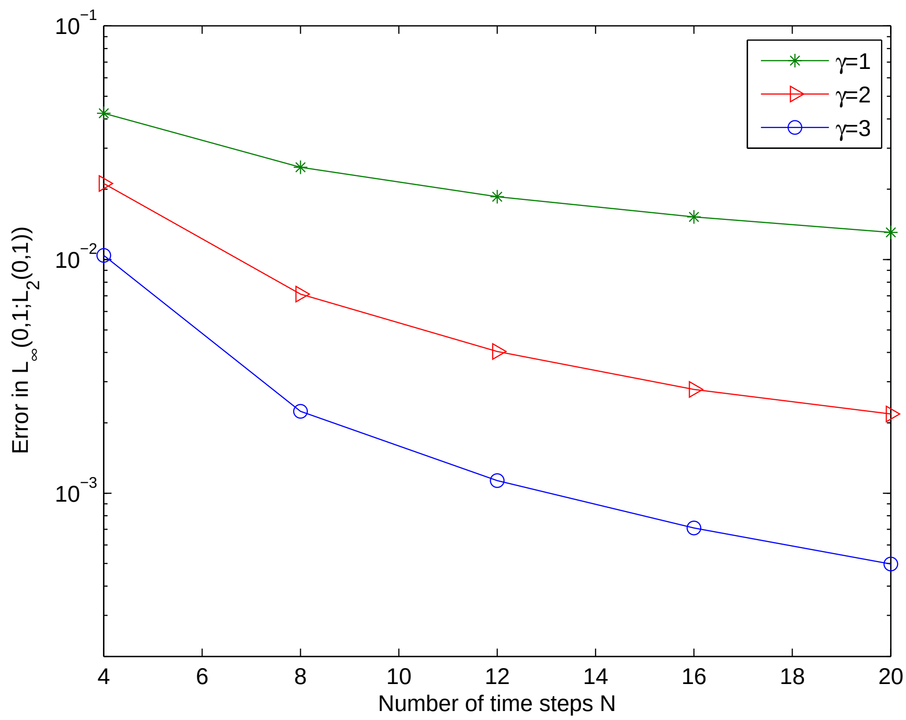

Figure 6.22: Graph of error in Table 6.20.

N	$\gamma=1$		$\gamma=2$		$\gamma=3$	
4	1.47e-01		1.02e-01		7.07e-02	
8	1.11e-01	4.04e-01	5.77e-02	8.27e-01	2.95e-02	1.26
12	9.58e-02	3.72e-01	4.23e-02	7.66e-01	1.84e-02	1.15
16	8.63e-02	3.62e-01	3.41e-02	7.45e-01	1.34e-02	1.11
20	7.97e-02	3.57e-01	2.90e-02	7.33e-01	1.06e-02	1.07

Table 6.21: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $m = 3$ and $\alpha = 0.7$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.3\gamma})$ for $1 \le \gamma \le (m+1)/(1-\alpha)$.

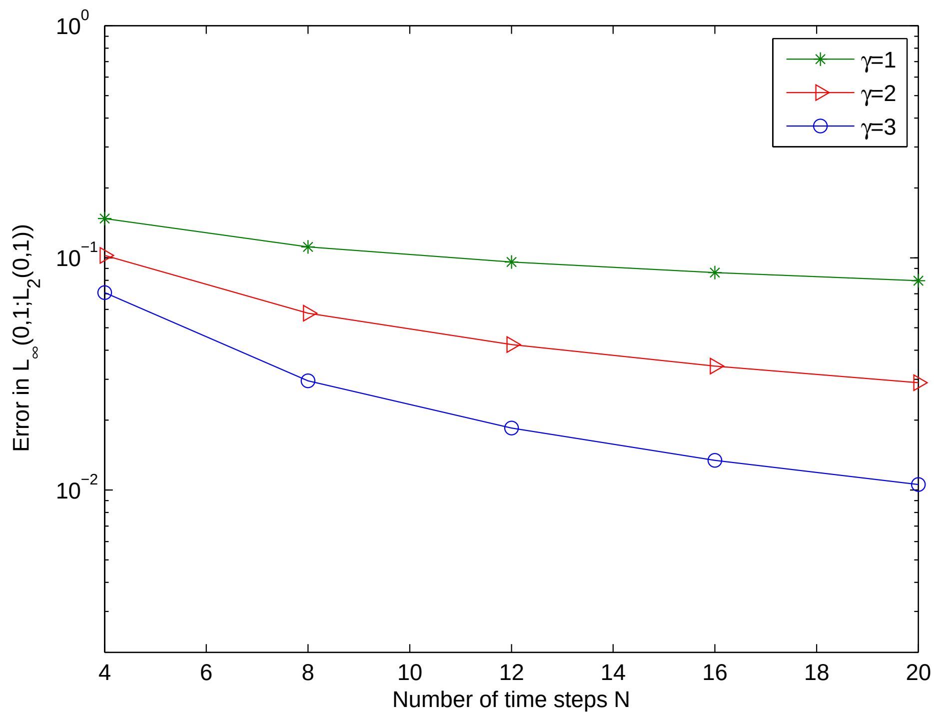

Figure 6.23: Graph of error in Table 6.21.

Next, we test the performance of the spatial finite elements discretizaton of (order degree r) of the scheme (5.9). We use a uniform spatial mesh consists of Nx subintervals and each is of width h. The time step-size k and the degree of the time-stepping DPG discretization are chosen such that the spatial errors is dominating. Hence, we expect from Theorem 5.8 to see convergence of order $O(h^{r+1})$. We illustrate these results in Tables 6.22 for $r = 1, 2$.

N_x	$r=1$		$r=2$	
6	1.65e-02		1.21e-03	
10	5.92e-03	2.26	1.71e-04	3.11
14	3.02e-03	2.17	5.49e-05	2.97
18	1.83e-03	2.13	2.44e-05	2.93
22	1.09e-03	2.18	1.12e-05	3.59

Table 6.22: The errors $|||U_h - u|||_{10}$ for different time mesh gradings with $\alpha = 0.3$. We observe convergence of order h^{r+1} for $r = 1, 2$.

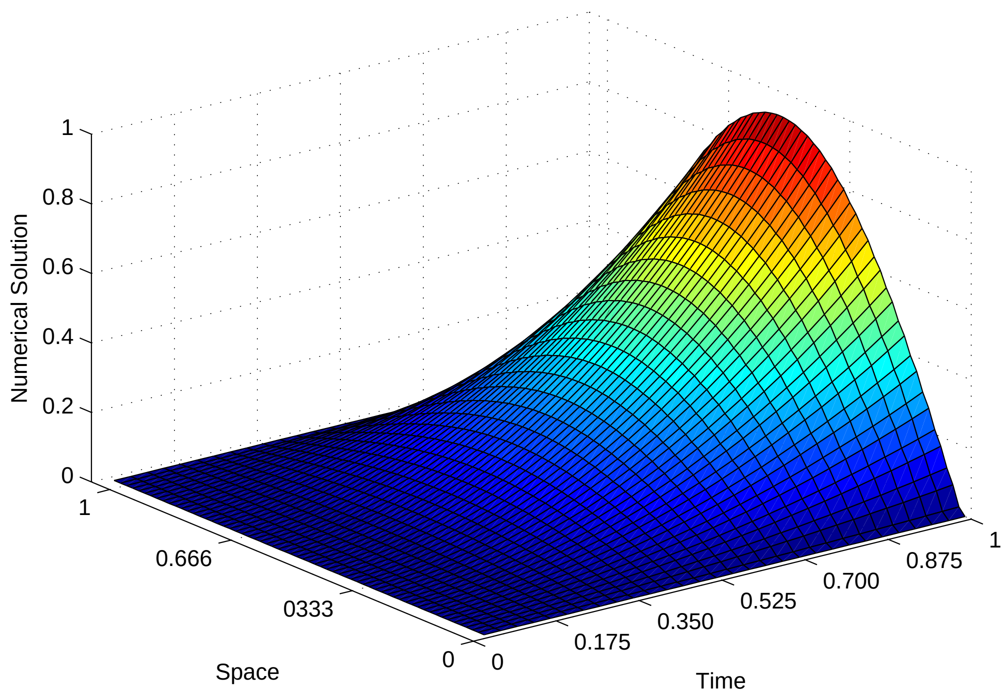

Figure 6.24: This figure that shows the numerical solution for $\alpha = 0.3$ and $\gamma = 3$.

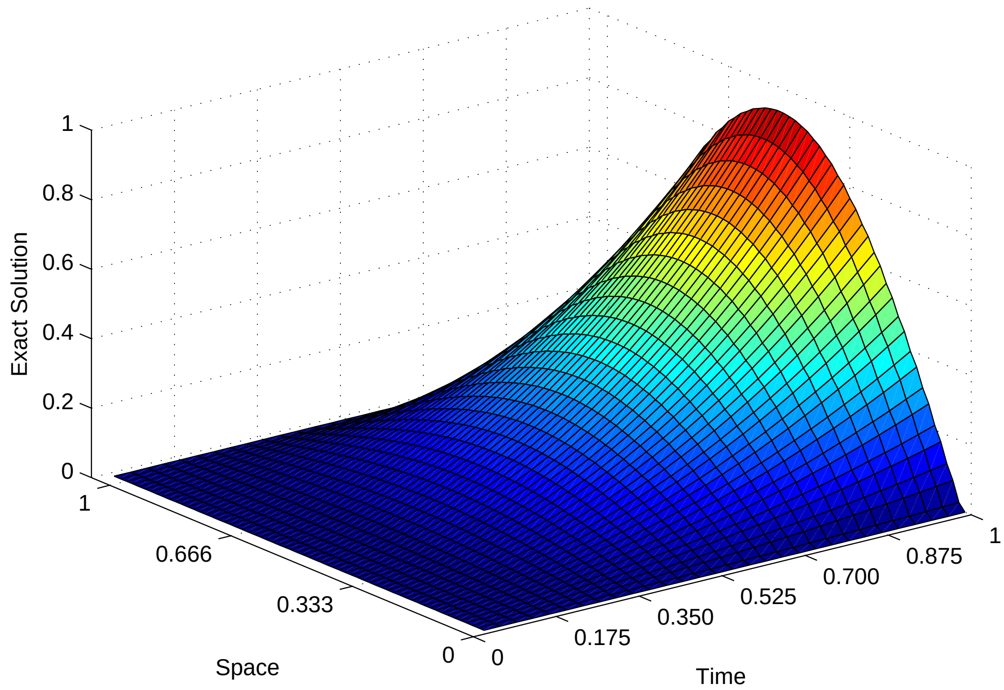

Figure 6.25: This figure that shows the exact solution for $\alpha = 0.3$ and $\gamma = 3$.

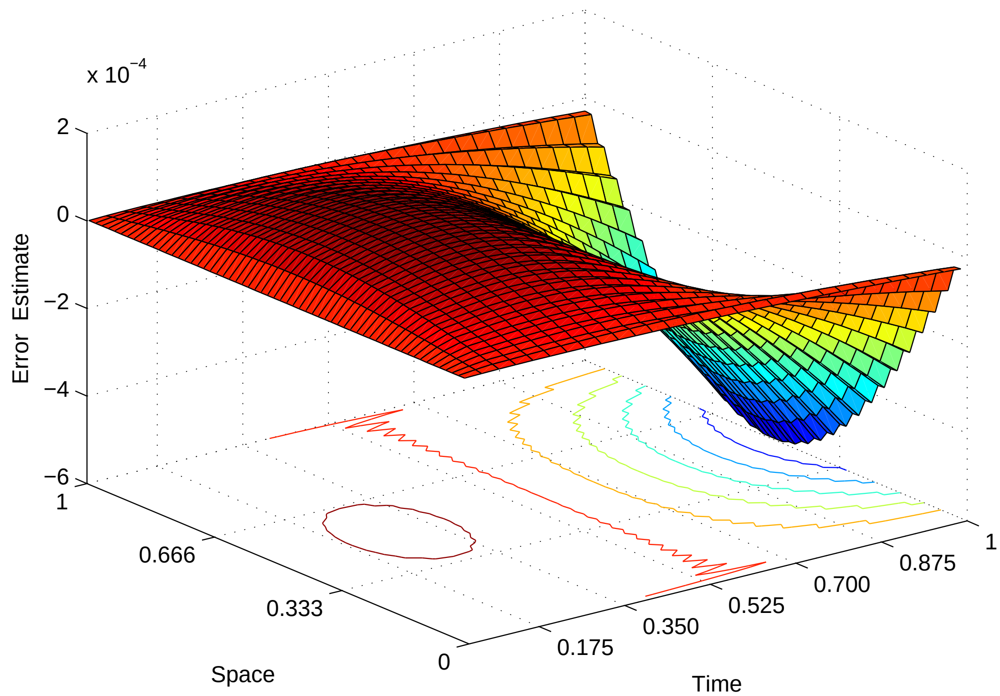

Figure 6.26: This figure that shows the error estimate for $\alpha = 0.3$ and $\gamma = 3$.

Example 2 (two dimension in space)

We let $\Omega = (0,1) \times (0,1)$, $\triangle u = u_{xx} + u_{yy}$, $T = 1$ and $u_0 = 0$.

(a) Smooth solution

We choose the initial datum such that the exact solution is:

$$u(x,y,t) = t^{1+\alpha}\sin(\pi x)\sin(\pi y) \tag{6.3}$$

Thus $f(x,y,t) = (\pi^2 t^{1+\alpha} + \Gamma(2+\alpha))\sin(\pi x)\sin(\pi y)$. As part (a) in example1, the theoretical results in Theorem 5.8 show the convergence of order $O(k^{m+\alpha/2})$ for $\gamma \geq \max\left\{\frac{2m+\alpha}{3\alpha+1}, \frac{m+1}{\alpha+1}\right\}$.. However, the numerical results in Tables 6.23, 6.24 and 6.25 illustrate more optimistic convergence rates compared and also demonstrated that the grading mesh parameter γ is slightly relaxed. We observe the uniform global error bounded by $Ck^{\min\{\gamma(1+\alpha),m+1\}}$ for $\gamma \geq 1$, which is optimal for $\gamma \geq (m+1)/(\alpha+1)$.

N	$\gamma = 1$		$\gamma = 1.4$		$\gamma = 1.8$	
10	1.58e-01		8.11e-02		4.16e-02	
20	8.05e-02	9.74e-01	3.15e-02	1.36	1.23e-02	1.75
40	3.49e-02	1.20e+00	9.79e-03	1.69	2.75e-03	2.17
80	1.58e-02	1.14e+00	3.24e-03	1.59	6.64e-04	2.05
160	6.87e-03	1.20e+00	1.01e-03	1.68	1.48e-04	2.16

Table 6.23: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.2$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.2\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1} .

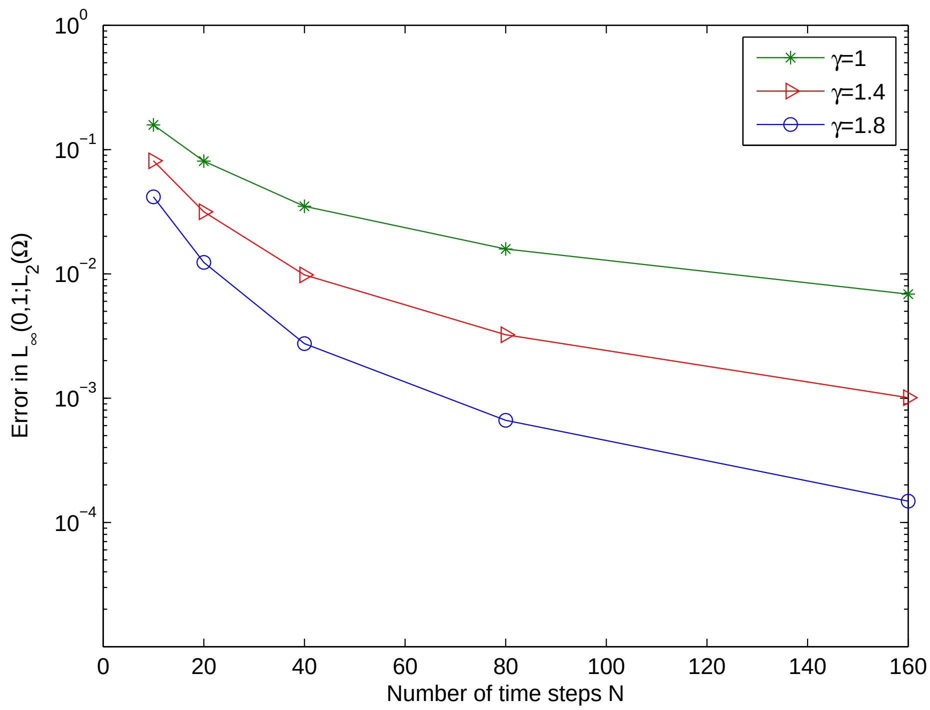

Figure 6.27: Graph of error in Table6.23 .

N	$\gamma = 1$		$\gamma = 1.2$		$\gamma = 1.4$	
10	1.09e-01		7.1560e-02		4.72e-02	
20	4.68e-02	1.21	2.61e-02	1.46	1.45e-02	1.70
40	1.65e-02	1.50	7.46e-03	1.81	3.38e-03	2.10
80	6.15e-03	1.42	2.28e-03	1.71	8.47e-04	1.99
160	2.17e-03	1.50	6.54e-04	1.80	1.97e-04	2.11

Table 6.24: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.5$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.5\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ for and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}

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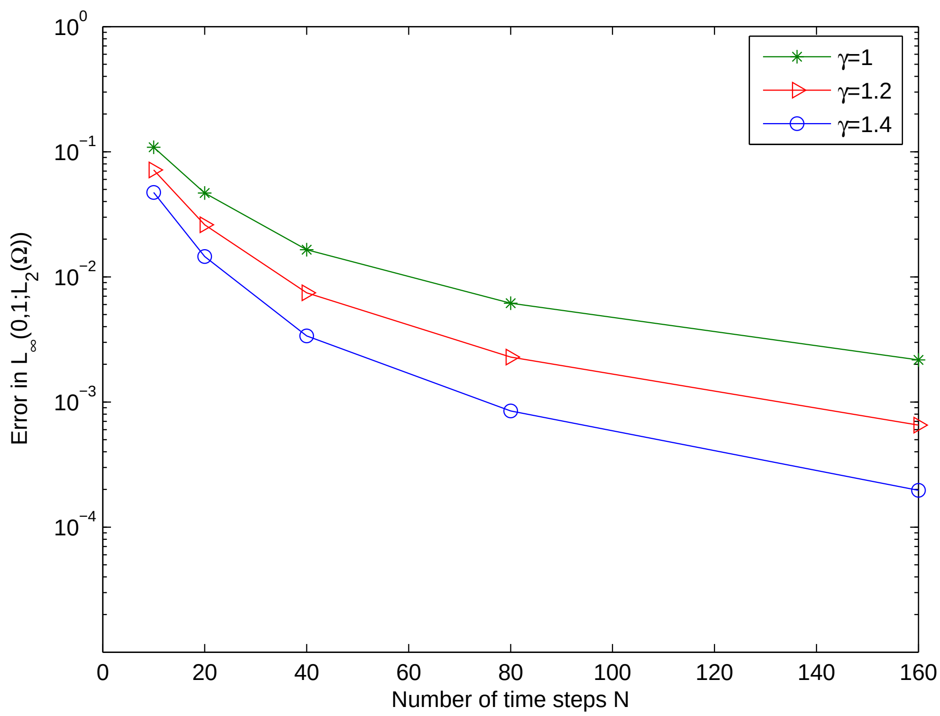

Figure 6.28: Graph of error in Table 6.24 .

N	$\gamma=1$		$\gamma=1.1$		$\gamma=1.2$	
10	8.42e-02		6.65e-02		5.25e-02	
20	3.25e-02	1.36	2.33e-02	1.51	1.67e-02	1.65
40	9.98e-03	1.70	6.37e-03	1.87	4.07e-03	2.04
80	3.27e-03	1.61	1.86e-03	1.77	1.06e-03	1.93
160	1.00e-03	1.70	5.09e-04	1.87	2.58e-04	2.04

Table 6.25: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.7$. We observe convergence of order $k^{(\alpha+1)\gamma}(= k^{1.7\gamma})$ for $1 \leq \gamma < (m+1)/(\alpha+1)$ and for $\gamma \geq (m+1)/(\alpha+1)$ the order of convergence is k^{m+1}.

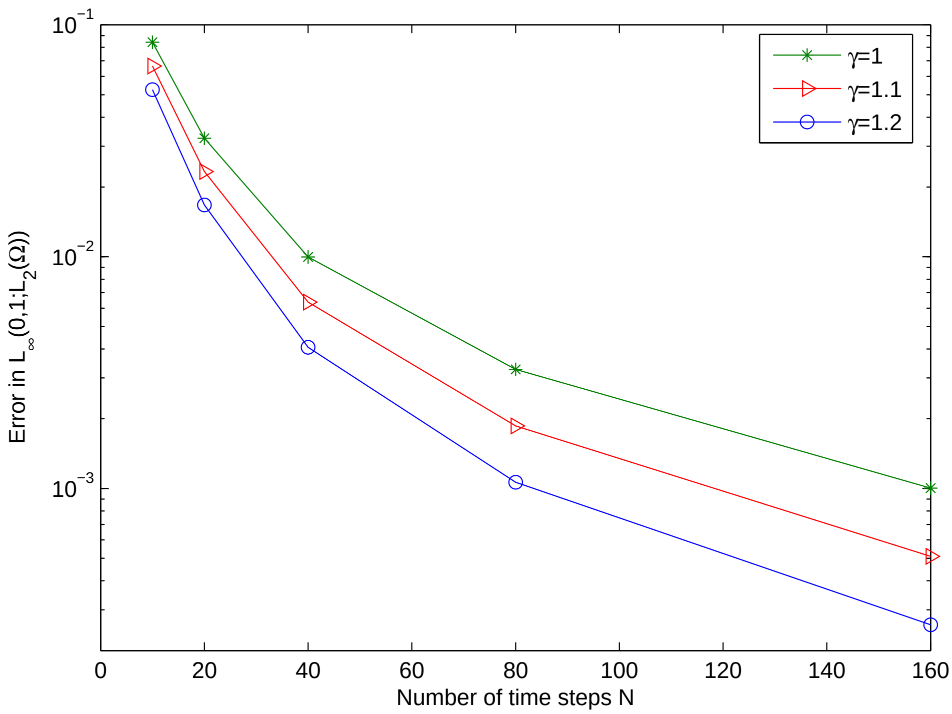

Figure 6.29: Graph of error in Table 6.25 .

(b)(Nonsmooth solution)

We choose the initial datum such that the exact solution is:

$$u(x,y,t) = t^{1-\alpha}\sin(\pi x)\sin(\pi y) \tag{6.4}$$

Thus

$$f(x,y,t) = (\pi^2 t^{1-\alpha} + \Gamma(2-\alpha)) sin(\pi x)\sin(\pi y)$$

Just as part (b) in example1, the theoretical results in Theorem 5.8 prove that convergence of order $O(k^{m+\alpha/2})$ for $\gamma \leq (2m+\alpha)/(1-\alpha)$. However, the numerical results in Tables 6.26, 6.27 and 6.28 illustrate more optimistic convergence rates compared and also demonstrated that the grading mesh parameter γ is slightly relaxed. We observe a uniform global error bounded by $Ck^{\min\{\gamma(1-\alpha),m+1\}}$ for $\gamma \geq 1$, which is optimal for $\gamma \geq (m+1)/(1-\alpha)$.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
10	1.55e-01		3.14e-02		6.31e-03	
20	9.57e-02	6.97e-01	1.19e-02	1.39	1.47e-03	2.10
40	5.92e-02	6.94e-01	4.54e-03	1.39	3.44e-04	2.10
80	3.66e-02	6.94e-01	1.72e-03	1.40	8.03e-05	2.10
160	2.26e-02	6.95e-01	6.53e-04	1.40	1.87e-05	2.10

Table 6.26: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.3$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.7\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

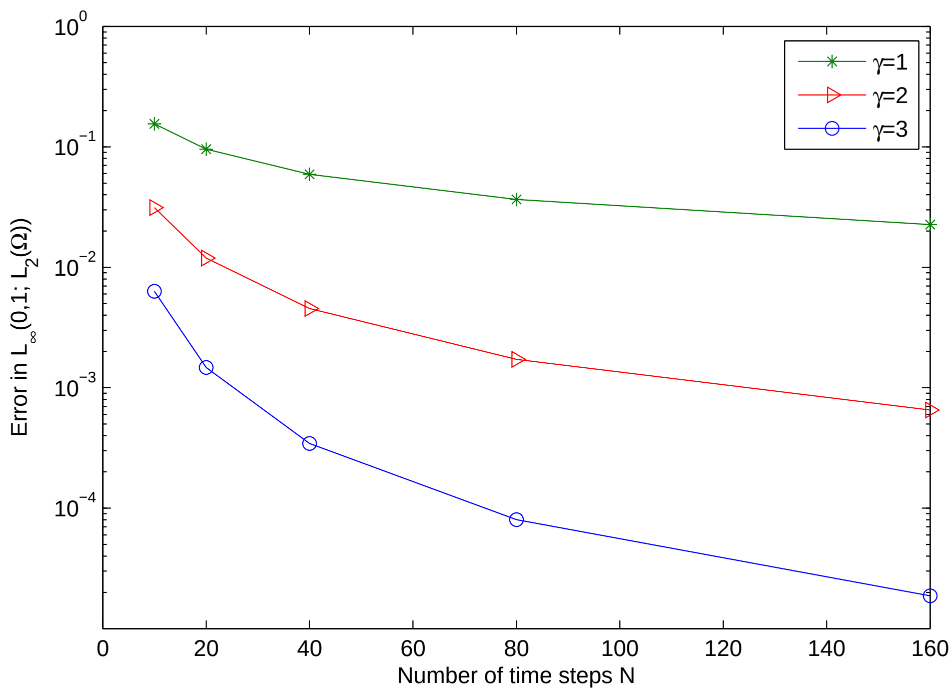

Figure 6.30: Graph of error in Table 6.26 .

N	$\gamma = 1$		$\gamma = 2.5$		$\gamma = 4$	
10	3.32e-01		1.00e-01		3.02e-02	
20	2.35e-01	4.99e-01	4.24e-02	1.24	7.64e-03	1.99
40	1.66e-01	4.97e-01	1.79e-02	1.24	1.92e-03	1.99
80	1.18e-01	4.96e-01	7.59e-03	1.24	4.79e-04	1.99
160	8.36e-02	4.95e-01	3.20e-03	1.25	1.19e-04	1.99

Table 6.27: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.5$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.5\gamma})$ for $1 \leq \gamma < (m+1)/(1-\alpha)$ and for $\gamma \geq (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

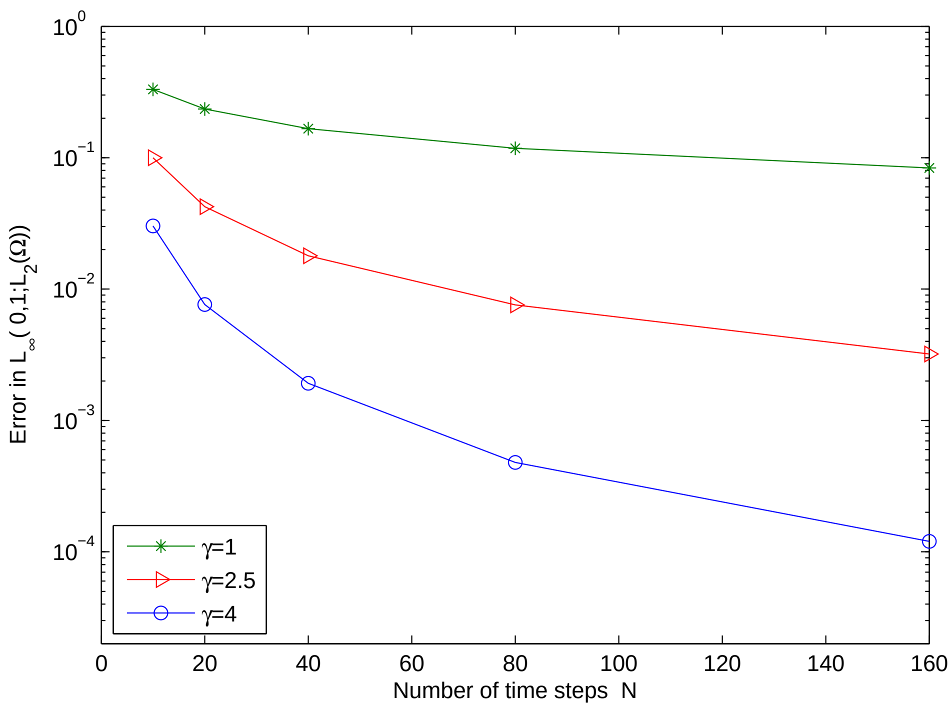

Figure 6.31: Graph of error in Table 6.27 .

N	$\gamma = 1$		$\gamma = 4$		$\gamma = 7$	
10	4.67e-01		1.35e-01		3.89e-02	
20	3.94e-01	2.44e-01	6.90e-02	9.65e-01	1.21e-02	1.69
40	3.20e-01	2.99e-01	3.02e-02	1.19	2.84e-03	2.09
80	2.63e-01	2.83e-01	1.38e-02	1.13	7.16e-04	1.99
160	2.14e-01	2.99e-01	6.04e-03	1.19	1.67e-04	2.09

Table 6.28: The errors $|||U_h - u|||_7$ for different time mesh gradings with $m = 1$ and $\alpha = 0.7$. We observe convergence of order $k^{(1-\alpha)\gamma}(= k^{0.3\gamma})$ for $1 \le \gamma < (m+1)/(1-\alpha)$ and for $\gamma \ge (m+1)/(1-\alpha)$ the order of convergence is k^{m+1}.

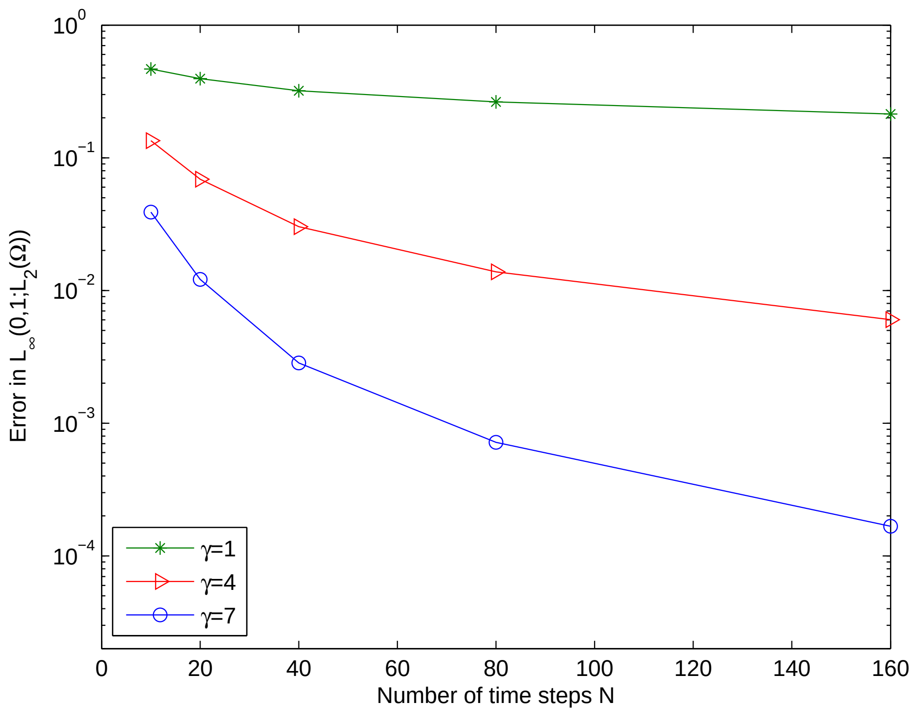

Figure 6.32: Graph of error in Table 6.28.

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2. *Education*

University	Date of graduate	Degree Obtained
King Fahd University of Petroleum and Minerals (Saudi Arabia)	24 /2 /2014	Ph.D. in Mathematics.
AlQuds University(Palestine)	26 /1 /2006	M.Sc. in Mathematics.
AlQuds University(Palestine)	15/9/1999	B.Sc. in Mathematics.

3. *Awards and Scholarships*

Year	Award or Scholarship
2010 – 2014	Teaching and Research Assistantship (KFUPM, Saudi Arabia)

4. *Teaching*

University	Period of Service	Position
KFUPM, Dhahran, KSA	2010 – 2014	Lecturer B
AlQuds Open University	2006 – 2010	Lecturer (part time)
Beit Iksa Secondary School	1999 – 2010	Teacher

5. Course *Taught*

- **King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia,** Calculus I, Calculus II, Statistics 319.
- **AlQuds Open University, Palestine** .
  - Linear Algebra.
  - Methods of solving Differential Equations.
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6. *Seminars*

Date	Title	Location
11/07/2004	Indirect Method for Solving first Order linear system of Difference Equations	Math. department Seminar, ALQuds University
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