MODELS OF FRACTIONAL DIFFUSION FOR RADIAL-COMPOSITE RESERVOIRS

BY

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Dedication

To my dear parents, sister and uncle for their endless love and support

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All praise is to ALLAH Almighty who made it possible for me to accomplish this research work successfully.

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NOMENCLATURE

a_{ij} = terms in the system of equation of radial geometry

A = arbitrary constant in the system of equations for radial composite system

b_{ij} = terms in the system of equation of radial geometry

B = arbitrary constant in the system of equations for radial composite system

B_1 = Formation volume factor of oil in region1of radial composite system, RB/STB

B_2 = Formation volume factor of oil in region2 of radial composite system, RB/STB

c_{ij} = terms in the system of equation of radial geometry

c_t = total compressibility, psi^{-1}

C = Wellbore storage constant, bbl/psi, or arbitrary constant in the system of equations for radial composite system

D = arbitrary constant in the system of equations for radial composite system

F = storativity ratio for a two region reservoir $=\left(\dot{k}/\varphi\mu c_t\right)_1 \Big/ \left(\dot{k}/\varphi\mu c_t\right)_2$

h = reservoir thickness, ft

I_j	=	modified Bessel function of first kind of order j
k	=	reservoir permeability, md
$\dot{k}$	=	memory affected reservoir permeability, $md/\sec^{\alpha}$
$\dot{k}_1$	=	modified reservoir permeability for region 1, $md/\sec^{\alpha}$
$\dot{k}_2$	=	modified reservoir permeability for region 2, $md/\sec^{\alpha}$
K_j	=	modified Bessel function of second kind of order j
M	=	mobility ratio for a two-region reservoir= $=\left(\dot{k}_1/\mu_1 B_1\right)/\left(\dot{k}_2/\mu_2 B_2\right)$
p	=	pressure, psi
$\dot{p}$	=	memory effected pressure, $psi/\sec^{\alpha}$
p_D	=	dimensionless pressure drop
$\hat{p}_D$	=	dimensionless pressure drop in Laplace space
p_i	=	initial reservoir pressure, psi
p_{wf}	=	bottom hole flowing pressure, psi
$\hat{p}_{wD}$	=	dimensionless wellbore pressure in Laplace space
q	=	injection or production rate, STB/D

r = radius or radial distance for radial reservoir, ft

r_D = dimensionless distance in radial geometry $= \frac{r}{r_w}$

r_{eD} = dimensionless distance to outer boundary for radial geometry $= \frac{r_e}{r_w}$

r_f = discontinuity radius for a two region, radial composite reservoir, ft

R_D = dimensionless discontinuity radius for a two region reservoir $= \frac{r_f}{r_w}$

s = skin factor

S_1 = arbitrary constant in the system of equations for radial composite system

S_2 = arbitrary constant in the system of equations for radial composite system

S_3 = arbitrary constant in the system of equations for radial composite system

S_4 = arbitrary constant in the system of equations for radial composite system

S_5 = arbitrary constant in the system of equations for radial composite system

S_6 = arbitrary constant in the system of equations for radial composite system

t = time, hrs

t_D = dimensionless time $= 0.00002637\left(k/\varphi\mu c_t\right)_1\left(t/r_w^2\right)$

V_i = Constant in Stehfest Algorithm

z = Laplace parameter

Greek Symbols

α = memory parameter

μ = viscosity, *cp*

φ = porosity

$\dot{\eta}_1$ = hydraulic diffusivity constant for region 1

$\dot{\eta}_2$ = hydraulic diffusivity constant for region 2

Δp_s = pressure drop due to skin, psia

Δp_w = wellbore pressure drop, psia

ABSTRACT

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Accurate understanding of the physics of fluid flow in a porous media is still questionable in the field of hydrogeology, geo-mechanics, and soil mechanics and indeed in the recovery of oil and gas. This topic has been researched for several decades for the said fields and the outcomes of these studies provide the basis for future predictions and decisions. The main objective of this research is to improve the physics of fluid flow in a porous media (petroleum reservoirs) under anomalous diffusion where classical Darcy's law fails to describe the process.

The first equation describing fluid flow through porous media was first developed by Henry Darcy in 1856. The equation was developed to calculate flow rate of water through sand beds. The law forms the basis of estimating conductivity of sand beds which is also known as permeability. Darcy's Law is analogous to Fick's Law in diffusion theory that is used to describe the normal diffusion in porous media. However there are several cases where the fluid flow paths are complex and diffusion occur is not normal. Several experimental observations are evident in literature that shows non-Fickian dispersion process in heterogeneous porous media where classical Darcy's Law fails to describe the process adequately. Many authors consider the use of fractional derivative as a mean to

describe the anomalous diffusion process that requires some modification in conventional Darcy's law. In this research we propose the use of memory formalisms on pressure gradient term to modify Darcy's Law. Fractional order derivatives are used to represent the memory formalisms.

In this study, we consider a two region radial composite reservoir that mimics a number of reservoir situations. Modified Darcy's law is used to derive diffusivity equation and its solutions are obtained in Laplace space. The pressure behavior for a two region composite system is modeled after incorporating the memory parameter (α) and the effect of changing memory parameter on bottom hole pressure and pressure distribution over time is analyzed. Results show that bottom hole pressure is affected by memory parameter also and a larger pressure drop occurs as the value of α increases. Also pressure derivatives curves deviate from each other in radial flow regime in both region 1 and 2. This will affect the calculation of permeability values from graphical analysis. Finally, parameters are estimated using non-linear regression (Levenberg-Marquardt algorithm) considering both normal and fractional diffusion.

ملخص الرسالة

الاسم الكامل: زعيـم حـسـن خـان

عنوان الرسالة: صياغـة الانـتـشار الـجـزئي في الـمكـامـكن الـشـعاعـية الـمركبة

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الفهم الفيزيائي الدقيق لتدفق الموائع خلال الأوساط المسامية في المكامن النفطيه يعترية الكثير من الغموض من جوانب عدة مثل الهيدروجيولوجيا و الميكانيكا الجيولوجية و ميكانيكا التربة كما هو الواقع أثناء عمليات استخلاص النفط والغاز. خلال العقود السابقه نال هذا المجال اهتمام الكثير من الباحثين في عدة حقول نفطية لإستنباط الأسس السليمة للتنبوئات والقرارات المستقبليه لتلك الحقول. لذا فإن الهدف الرئيسي لهذه الدراسة هو إضْفاء فهماً دقيقاً لفيزيائية تدفق الموائع في الأوساط المسامية (المكامن النفطيه) خلال عمليات الانتشار الغير منتظم لـ تلك الموائع، حيث لايمكن تطبيق قانون Darcy في مثل هذه الحالات.

في سـنة 1856 وُضعت معادلة لحساب معدل تدفق المياه من خلال طبقات الرمل والذي يشكل المبدأ الأولي لتحديد مدى قابلية الصخور لتمرير السائل أو مايسمى النفاذية. يعتبر قانون Darcy مماثل لقانون Fick من الناحية النظرية في وصف انتشار الموائع خلال الأوساط المسامية. ولكن هناك العديد من الحالات التي يكون فيها مسارات تدفق السوائل معقدة وانتشارها ليس منتظما. العديد من التجارب العلمية التي أُجريت في هذا المجال تظهر أنّ عملية انتشار الموائع لاتخضع لقانون Fick في المكامن الغير متجانسة حيت يقصر أيضا قانون Darcy في وصف عملية الانشار بشكل كاف. لقد عَمَد الكثير من الكتاب إلى استخدام المشتقات الجزئية كوسيلة لوصف عملية الانتشار الشاذة (الغير منتظمة) والتي تتطلب إجراء بعض التعديلات في القانون الأساسي لِ Darcy. لذا فإننا من خلال هذا البحث سنعمل على استخدام تدرجات الضغوط المخزنه بالذاكره من أجل تعديل قانون Darcy باستعمال المشتقات الجزئية لتمثيل نُظم الذاكره المناطة.

في هذه الدارسة سوف نأخذ بعين الاعتبار وجود منطقين في المكمن الشعاعي وهي الحالة التي تُحاكي العديد من المكامن النفطية المركبة كما سنستخدم قانون Darcy المعدّل لايجاد حل لمعادلة الانتشار بطريقة Laplace space . وعلى غرار سلوك الضغط في المكمن المركب من منطقتين والمنوط دراسته في هذا البحث فان النظام الناتج سوف يُصاغ بعد دمج عوامل الذاكره (α) وتأثير تغيير تلك العوامل على توزيع الضغط في قعر البئر البترولي. تُظهر نتائج هذه الدراسة أنّ الضغط السفلي للبئر يتأثر بمعامل الذاكرة كما أنه يُحدث انخفاظ عالي في الضغوط بزيادة المعامل (α). كما لوحظ أيضاً انحراف منحيات مشتقات الضغوط عن بعضها البعض في المنطقتين 1 و 2 في النظام الشعاعي المركب والذي سيؤدي بدوره إلى التأثير على حسابات النفاذيات بواسطة المخطاطات. أخيراً تم تقدير هذه العوامل باستخدام العلاقات الغير خطية (Levenberg-Marquardt algorithm) باعتبار الانتشار الطبيعي والجزئي.

CHAPTER 1

INTRODUCTION

The flow behavior of fluids through a porous media has been of interest not only in petroleum industry but in the field of chemical engineering, hydrogeology, agricultural, soil mechanics etc. Due to the complex nature of porous media various authors were attracted to tackle the problems related to this topic and formulated different relations for studying diffusion of fluids in porous media. However all the authors based their relations on classical Darcy's law and provided solutions for different interesting cases (Barry and Sposito, 1989; Kabala and Sposito, 1991; Neuman and Orr, 1993; Indelman and Abramovich, 1994; Steefel and Lasaga, 1994; Dewers and Ortoleva, 1994; Hu and Cushman, 1991; Cristakos et al., 1995; Cushman and Moroni, 2001).

The classical theory of propagation of pressure and fluids is based on Darcy's Law which states the proportionality between the flux and pressure gradient. Darcy's Law forms the scientific basis of permeability of the reservoir rock. The law is comparable to Fick's Law in diffusion so Darcy's law is a constitutive equation in defining normal diffusion through porous media in a petroleum reservoir. On the other hand if diffusion process is not normal i.e. anomalous diffusion, Darcy's law fails to describe the physics of the process adequately and requires some modification. Some authors like Caputo relates anomalous diffusion process to memory i.e. diffusion process will depend upon previous value of pressure and flow of fluids. Hence some of the fluids behaviors in rock possess properties that cannot be modeled with classical propagation theory (Bell and Nur, 1978; Roeloffs, 1998) and mathematical representation of these flow behaviors is still inadequate and requires generalization of existing flow equations.

In this research, our focus will be on the flow behavior in radial composite reservoirs. Two or more regions with different fluid or rock properties combined together to form a composite reservoir. This consideration has been of interest in well testing where numerous reservoir situations mimic a composite system. A composite reservoir model helps in analyzing transient pressure data from acidization and injection processes, in reservoirs where rock or fluid property differs and geothermal reservoirs with thermal discontinuities (Ambastha, 1995). An oil reservoir with aquifer is considered to be an example of naturally composite reservoir whereas steam injection, insitu combustion, polymer flooding and CO_2 miscible flooding create artificial composite reservoirs (Issaka, 1996). In simple words, composite reservoirs can be modeled more adequately in all reservoir cases where two regions of different either rock or fluid properties exist.

Linear composite reservoirs may be created due to geological factors, such as faulting, facies changes or pinch outs. It is possible that these boundaries will resist the flow across them, and be partially communicating (Ambastha, 1987). In case of vertically fractured wells, elliptical geometry is more appropriate to model the effects of steam injection (Obut and Ertekin, 1987; Stanislav et al., 1987; Stanislav et al., 1992).

1.1 Statement of the Problem

It is evident from the literature that classical Darcy law has been used mostly for describing fluid flow through porous media. This law states that flux is directly proportional to the pressure gradient. However there are several cases where the fluid flow paths are complex and diffusion occur is not normal. Several experimental observations are evident in literature that shows non-Fickian dispersion process in heterogeneous porous media where classical Darcy's Law fails to describe the process adequately. Many authors consider the use of fractional derivative as a mean to describe the anomalous diffusion process that requires some modification in conventional Darcy's law. Caputo relates anomalous diffusion to the memory formalisms, this means that system is affected by memory term i.e. the diffusion of fluids will depend on the previous value of pressure and flow of fluid. So in order to have a better representation of these processes, classical equations should be modified.

One way to modify Darcy law is the use of fractional derivatives on pressure gradient term to represent memory formalisms. These derivatives are integro-differential operators that are being used in modeling transport and describing anomalous diffusion. It has been applied to model transport of passive tracers in turbulence. So the use of fractional derivatives to model memory proves out to be very useful in recent studies.

1.2 Research Objectives

It includes the following,

1. To use modified classical equation of fluid flow through porous media i.e. Darcy's Law in the diffusivity equation to account for anomalous diffusion.

$$q = \frac{\dot{k}h}{141.2\mu B}\left(r\frac{\partial \dot{p}}{\partial r} \right) \qquad (1.1)$$

where $\dot{k}$ is the modified Darcy's Permeability and $\dot{p}$ is the memory affected pressure that is defined as:

$$\dot{p} = \frac{\partial^{\alpha} p}{\partial t^{\alpha}} \tag{1.2}$$

2. The partial differential equations and boundary conditions for a two region radial composite system will be modified to obtain new solutions. All the cases will assume constant rate inner boundary condition with well bore storage and skin. Three outer boundary conditions will be considered. These are infinite acting, no-flow and constant pressure.

3. The analytical solutions obtained for two region radial composite system will be presented in Laplace space.

4. After incorporation of memory formalism parameter α in constitutive equations, a sensitivity study will be performed for different values of α. The Effect of memory parameter α on bottom hole flowing pressure, pressure drop and Bourdet Pressure derivative will be analyzed.

5. A pressure transient analysis problem will be analyzed, reservoir and wellbore parameters including memory parameter α will be calculated using Levenberg-Marquardt method i.e. Non Linear Regression.

CHAPTER 2

LITERATURE REVIEW

2.1 Fluid Flow through Porous Media

Fluid flow through porous media is the topic of interest in many fields. Its application is far but not limited to hydrogeology, soil engineering and chemical engineering. One of the most important application is the extraction of oil and gas from petroleum reservoirs, a resource on which world heavily depends. As long as the contrast between world oil supply and demand will increase, new methods will be required to make an efficient use of this resource. The subject of fluid flow through porous media combines fluid dynamics, thermodynamics, applied mathematics, chemistry and geology. The wide scope of this subject and involvement of difficult physical processes make the relevant equations perplexing.

The understanding of the physics behind movement of fluids in a porous media is still questionable and a challenging task. Also movement of fluids in a porous media is not possible to be visualized directly under certain cases. Many authors (Biot, 1941, 1956a, 1956b, 1973; Biot and Willis, 1957; McNamee and Gibson, 1960; Bell and Nur, 1978) derived different form of useful equations for diffusion of fluid and their solutions in many interesting cases. However most of the authors mentioned assumed empirically derived Darcy's Law and formulated their equations of diffusion based on it.

The classical equation describing the flow of fluid through porous media relating pressure gradient and fluid flux was formulated by Henry Darcy in 1856. This law was developed as a result of experiments on flow of water through sands. According to this law, flux is directly proportional to the pressure gradient. Although Darcy's Law (an expression of conservation of momentum) was determined experimentally, it has since been derived from the Navier-Stokes equations while considering it to be homogeneous. Darcy’s Law has many analogies; it is comparable to Ohm's Law for the Conduction of Electricity, Fourier's expression for the conduction of heat or Fick's law in diffusion theory (Hubbert, 1956). This law forms the scientific basis of permeability of the medium that remains constant with time in case of Darcy’s flow.

However it has been observed that some flow behavior does not follow the Darcy’s law trend while moving through the porous media. In fact these behaviors contradict with the classic theory of diffusion of pressure and fluids in the porous media. These phenomena might cause the permeability of the system to change such as fluid may carry solid particles that caused pore plugging or chemical reaction with other minerals can change the permeability of the system. It has been experimentally proved that when a fluid flows through a porous medium the permeability of the matrix may be locally variable in time (Caputo, 2000; Iaffaldano et al., 2006; Cloot and Botha, 2006) for the several reasons mentioned above.

It has been observed that modern diffusion equation fails to describe the behavior of subterranean water in flow through porous media. However most of the research has been done while considering the diffusion of flux rather than the flux of the fluid (Christakos et al., 1995; Mainardi et al., 1996). The main difficulty arises in computing the flux with constant pressure at the boundary because of mathematical computations. So the diffusion of flux requires more attention and a different approach.

In order to describe the flow behavior of fluids, one needs the modification of Darcy's law by introducing general memory formalisms terms on the flow and pressure gradient as well. Diffusion equation will also require some modification; so memory formalism was introduced as rheology in the fluid. These memory formalisms are defined as fractional derivatives (Caputo, 2006).

2.2 Anomalous Diffusion

The concept of diffusion is used in variety of sciences: physics, transport phenomena, biological sciences etc. Normal diffusion can be modeled with the help of classical Fick's law which states that diffusion flux is directly proportional to the negative concentration gradient. Not in all diffusion processes, particles distribute themselves randomly and uniformly. In some cases, particles exhibit complex motion and their trajectories produce complex objects (Afananasiev et al., 1991). In this case probability distribution of the particles can no longer be approximated by Gaussian distribution so cannot be modeled by classical diffusion equation based on Fick's law. Several authors described the complex situations that can be described by the use of fractional derivatives (Compte, 1996; Benson et al., 2000; Benson et al., 2001; Del-Castillo-Negrete et al., 2003; Meerschaert, 2002; Metzler and Klafter, 2000).

The common perception about diffusion is that particles move randomly in the space. However, if a particle is headed in one direction than there is a probability that it will continue in its direction for some time until this probability goes to zero (Taylor, 1921).

This type of diffusion can be called as anomalous diffusion in which particles move coherently for a longer period of time until they disperse. Anomalous diffusion can no longer be described by the classical diffusion equations as these processes are characterized by non-Gaussian probability distribution functions. However fractional diffusion equations provide an adequate means of describing anomalous transport.

Reservoirs containing natural fractures possess complex geometries so the elementary particles moving along the fractures and porous medium will perform complex motion, this structure can be considered as fractal. Fractional diffusion equation for the modeling of fractal geometry was formulated while considering a comb like structure of the medium, in this case fractional temporal derivatives were used to model sub diffusion (slow diffusion) process (Nigmatullin, 1984). Also in a latter study, fractional advection diffusion equation was developed analytically for fractured aquifer considering a double porosity model (Barenblatt et al., 1990) whereas the order of the fractional advection equation depends upon the fractal dimensions of the porous medium. Various approaches for studying diffusion in fractal geometries have been used extensively in recent years (Havlin, 2002; Uchaikin, 2008; Sibatov and Uchaikin, 2009). Modeling of diffusion for normal fractals is done by considering Fick's Law with spatially variable diffusivity; also it has the same form of governing conventional partial differential equations and proves to be a better representation of anomalous diffusion (Fomin et al., 2011).

Many experimental results have proved the presence of anomalous diffusion process especially in heterogeneous medium where concentration of solutes on average scale causes non linearity between second moment and time. Many authors use fractional derivatives model for diffusion of solutes in heterogeneous porous media. Erochenkova and Lima use partial differential equations whose coefficients can be represented by random processes to model diffusion (Erochenkova & Lima, 2001).

2.3 Fractional Derivatives and Memory Formalisms

Fractional calculus is the field of mathematics which deals with the derivatives and integrals to non-integer orders. The fundamentals of fractional calculus were developed by Leibniz (1695), Liouville (1834), Riemann (1892) and others. Oliver Heaviside in 1890s provides the basis for applying fractional calculus in the engineering field. Researchers have widely used fractional order derivatives to model various physical phenomena in the last several decades. In order to generalize the systems of differential equations, fractional calculus plays an important role.

Since the appearance of fractional calculus, different authors suggested various definitions of fractional derivatives and integrals. Some of the most popular definitions are as under:

2.3.1 Riemann-Liouville Definition

The popular definition of fractional calculus is this which shows Rieman integral of order α:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{2.1} $$

where α is:

$(n-1 \le \alpha < n)$.

2.3.2 M. Caputo Definition

The second popular definition is:

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_{a}^{t} \frac{f^{n}(\tau)\, d\tau}{(t-\tau)^{\alpha+n-1}} \tag{2.2} $$

and α is defined as:

$(n - 1 \leq \alpha < n)$.

2.3.3 Grünwald-Letnikove Definition

This is another joined definition which is sometimes useful:

$$ {}_{a}D_{t}^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^{j} \binom{\alpha}{j} f(t - jh) \tag{2.3} $$

2.3.4 Hadamard fractional integral

J Hadamard proposes the following definition:

$$ {}_{a}D_{t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(\log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \tag{2.4} $$

$for\, t > a$.

Fractional derivatives have been used previously in study of electric transmission lines (Heaviside, 1892), to describe ultrasonic wave propagation physics in human cancellous bone (Sebaa et al., 2006). A new technique for the modeling of speech signal was developed based on fractional integration (Khaled Assaleh and Wajdi Ahmad, 2007). Fractional derivatives in time can provide improve description of behavior of sound waves in rigid porous materials (Fellah and Depollier, 2002). Also fractional derivatives are useful in modeling of different viscoelastic materials that exhibit complex elastic moduli (Soczkiewicz, 2002).

Fractional order time derivative and space derivative are somewhat different in describing the physics of the flow. This concept is well defined and presented by Caputo (Caputo, 2002). If modeling of local perturbation is concerned then fractional order time derivative will be useful, however if variations in an infinite medium is to be captured then fractional order space derivatives are appropriate i.e. flow will be related to memory by recalling the path of pressure gradient from the beginning of the flow.

Qinghe Wang and Dengke Tong used fractional calculus in seepage mechanics for development of a three-dimensional relaxation model of Non-Newtonian viscoelastic fluid. The exact solution for the model in an infinite acting reservoir is obtained using Laplace transform, Fourier sine and cosine integral transform. In this study, Stehfest algorithm and Gauss-Lauguerre numerical inversion techniques are used to find out the pressure transient trends in real space. Fluid characteristics of Non-Newtonian fluid are found as strong function of the order of fractional derivative. Also these effects are observed at the initial stage of pressure distribution curved which then merges into a single curve at the end (Wang and Tong, 2009).

The fluid flow in a fractal reservoir is somewhat similar to the diffusion in a disordered medium or anomalous diffusion due to complex structures. A mathematical model for pressure transient analysis of fractal reservoirs is proposed and solved using Green's function method (Park, et al., 1998). In this research, effective diffusion coefficient is developed that represents the memory of the system. The solutions were interpreted for many cases and it was concluded that additional pressure drop occurs due to delay from diffusion. However the solutions were limited and cannot separate the two fractal dimensions d_f and d_w. Park suggested a modified constant rate that is applicable to whole spatio-temporal ranges without wellbore storage and skin (Park et al., 2000).

To include the effect of wellbore storage and skin in presence of memory, new mathematical procedures and a generalized form of bottom hole pressure for fractal reservoir was formulated (Park, et al., 2001). In this paper, a new solution is derived and analyzed for bottom-hole pressure distribution which permits the wellbore storage and skin effects for fractal reservoirs. After that, a general mathematical formula is proposed for the analysis of pressure behavior in the case of three-dimensional anisotropic fractally fractured reservoirs. This formula is motivated from the fact that many fractured reservoirs show spatial anisotropy, i.e., spatial asymmetry. In the research, the solutions are obtained on the basis of fractional diffusion theory. Also with sensitivity analysis, it is observed that at early time less pressure drop occurs for larger dynamic fractal dimension.

Mishra use fractional derivative in pressure transient analysis of fractal reservoirs with phase redistribution (Mishra. A. S., 2010). In this research, it was observed that due to slow down of diffusion, the bottom hole pressure is less affected by the formation for same wellbore storage compared to that of conventional reservoir. The results of this study are helpful in characterizing the fractal reservoirs and other properties such as wellbore storage and skin are compared with Chang and Yortsos methods. The mathematical solution presented here is similar one recommended for pressure transient analysis of transient data from naturally fractured reservoir.

2.4 Composite reservoirs

In different scenarios, composite reservoir systems may exist naturally or artificially. These reservoirs are composed of two or more regions of different rock or fluid properties. One example of naturally occurring of these reservoirs is the reservoir with peripheral water encroachment from an aquifer. Steam flooding, in situ combustion, CO_2 miscible flooding and other enhanced oil recovery processes create an artificial composite system. Acid stimulation can result in change of permeability near wellbore is an example of composite system. A finite thickness skin region was treated as composite system (Wattenbarger and Ramey, 1970). Solutions for a composite reservoir considering finite skin i.e. damage around a well bore is also presented (Olarewaju and Lee, 1987a) where a damage portion was considered as inner region and rest of the reservoir as outer region. Reservoirs undergoing thermal recovery processes are perfect idealization for composite reservoirs. In this case, inner region is steam swept region where as unswept oil region is considered as outer region.

Several authors studied the transient pressure behavior while considering the radial composite system (Guerrero, 1961; Carter, 1966; Bixel and Van Pollen, 1967; Eggenschwiler et al., 1980; Olarewaju and Lee, 1987b; Ambastha and Ramey, 1989). Most of the authors presented their solutions for a two region composite reservoirs that consist of an inner and outer region separated by a sharp interface. However in real scenarios, assumption about sharp interface is not accurate. In order to present solutions more adequately, a three region composite reservoir (Onyekonwu and Ramey, 1986; Barua and Horne, 1987; and Ambastha and Ramey, 1992) and multi regions composite reservoir (Nanba and Home, 1989; Abbaszadeh-Dehghani and Kamal, 1989; Bratvold and Horne, 1990) models are considered.

Figure 2.1 is a graphical representation of a two-region, radial composite reservoir. These inner and outer regions are separated by a discontinuity and R is the radius (or distance) from the center of the wellbore to the discontinuity. The regions themselves are homogeneous i.e. uniform in fluid and rock properties but different from each other.

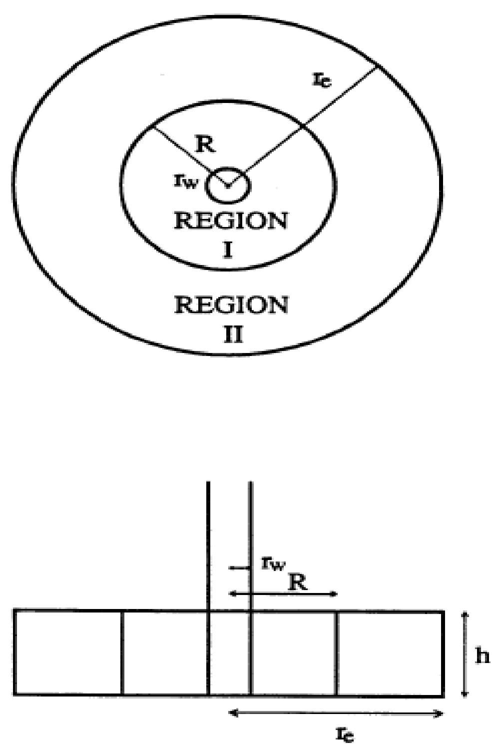

Figure 2.1: Two Region Radial Composite System
(Courtesy Issaka, 1995)

2.5 Well Test Analysis

Proper characterization of petroleum reservoirs is essential for prediction of accurate reservoir performance. Well testing is a part of formation evaluation that has a greater ability to find in situ reservoir conditions. In early era of well testing most of the analysis was done using straight line analysis proposed by Theis (1935). The semilog analysis became popular in 1960's and 1970's. Miller et al (1950) and Horner (1951) used middle time data while Muskat (1937), Horner (1951), Matthews et al (1954) and Jones (1962) used late time data for straight line analysis. Type curve analysis was introduced by Ramey in 1970 however significant advances in this area were done by Gringarten (1979) and Gringarten and Bourdet (1980). Derivative analysis by Bourdet (1983) opened a new way of analyzing complex reservoir and well behaviors. In recent times, deconvolution technique significantly improves the interpretation of well test data, providing accurate information about reservoir and well parameters.

From several decades, non-linear regression technique is being used to obtain the reservoir parameters. Nonlinear regression is also known as automated type curve matching. In this technique, the objective is to minimize the sum of squares of the difference between the observed pressure data and the model pressures. However this technique has disadvantage of getting trapped in local minima which is usually in the vicinity of initial guess.

CHAPTER 3

MATHEMATICAL BACKGROUND

3.1 Laplace Transforms

Laplace transform is one of the most common types of transforms that is being widely used in physics and other engineering disciplines. Laplace transform of a piecewise continuous function $f(t)$ is denoted by $\mathcal{L}[f(t)]$and defined as,

$$\mathcal{L}\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt \tag{3.1}$$

Laplace transform is used to solve linear ordinary differential equations. It has a variety of applications in various areas of science and engineering like electrical engineering, physics, control engineering, optics, signal processing, well test analysis and mathematics etc. It transforms input and outputs that are in time domain to the input and outputs that are in frequency domain. Laplace transformable functions must satisfy the Dirichlet Conditions (A. D. Poularikas, 2000):

i. $f(t)$ must be piecewise continuous.

ii. $f(t)$ must be exponential order which means that $f(t)$ must remain less than Ce^{-at} as t approaches 1 where C is a positive constant and a is a real positive number.

Laplace transform has various properties but some of the most important properties that are used in this research are as follows:

1. Linearity

The Laplace transform of the linear sum of two Laplace transformable functions $f(t)$ $+g(t)$ is given by,

$$\mathcal{L}\left[f(t)+g(t)\right]=F(z)+G(z) \tag{3.2}$$

2. Differentiation

If the function $f(t)$ is continuous and $f'(t)$ is piecewise continuous then

$$\mathcal{L}\left[f^{'}(t)\right]=zF(z)-f(0) \tag{3.3}$$

In general,

$$\mathcal{L}\left[f^{(n)}(t)\right]=z^{n}F(z)-z^{n-1}f^{n-1}(0)-z^{n-2}f^{n-2}(0)\ldots\ldots-f(0), \tag{3.4}$$

For fractional Derivatives (special case),

$$\mathcal{L}\left[\frac{d^{\gamma}f}{dt^{\gamma}}\right]=z^{\gamma}F(z)-z^{\gamma-1}f(0) \tag{3.5}$$

Laplace transformation is an effective mathematical tool that can solve very complex engineering problems with ease especially in area of control and stability.

3.2 Application of Laplace Transforms in Well Test Analysis

Fluid flow problems in porous media were originally solved using Fourier-Bessel series. After the work of Van Everdingen and Hurst in 1949, Laplace transformation was recognized as the powerful tool for solving complex problems in less time. Van Everdingen and Hurst in 1949 presented solutions for the radial diffusivity equation that governs fluid flow through porous media. They developed two sets of solutions for the diffusivity equation i.e. constant terminal pressure and constant terminal rate solutions. Since then some original solutions were obtained in Laplace that were not possible with previous methods.

The use of Laplace transformation in pressure transient analysis is evident and has advantage over other techniques. Solutions can be obtained in Laplace space easily however numerical inversion is required to convert solutions to real time domain. Stehfest algorithm is mostly used today for the inversion of Laplace space to real time.

3.3 Stehfest Algorithm

Laplace transformation is applied for finding out solutions of diffusivity equations subjected to various boundary conditions. These solutions require inversion from Laplace space to real space. Stehfest algorithm due to its simplicity is used for the numerical inversion of Laplace transform. The algorithm was developed in 1960’s by Gaver-Stehfest. Due to its simplicity and convergence, the algorithm is widely used in petroleum engineering especially in pressure transient analysis.

If a Laplace transform F(s) is available then according to Stehfest its approximate inversion is calculated as:

$$F\left(t\right)=\frac{ln2}{T}\sum_{i=1}^{N}V_i P\left(\frac{ln2}{T}i\right) \quad (3.6)$$

where N is an even integar and its value usually lies between 4 & 20.

3.4 Non Linear Regression

The use of nonlinear regression algorithms in well test analysis for estimating reservoir and well bore parameters was introduced by Rosa and Horne (1983). Following are the some advantages of nonlinear regression over old techniques that make it to use widely today in well test analysis for parameters estimation:

1. Nonlinear regression can interpret uninterpretable tests i.e. it can be applied for any possible reservoir models by generating the corresponding pressure transient solution.

2. Nonlinear regression can analyze multirate or variable rate tests. For these types of variable rate tests, pressure response is calculated for a constant rate production drawdown test based on the reservoir model. After getting the solution, superposition principle is applied to compute the pressure response for an arbitrary flow rate history.

3. The method avoids inconsistent interpretations hence the results are free from human bias.

4. Nonlinear regression provides confidence estimates on answers in conjunction with statistical inference.

3.5 Levenberg-Marquardt Algorithm

In this work, the Levenberg-Marquardt (LM) algorithm is used for nonlinear regression that finds out the minimum of the objective function that is expressed as the sum of squares of non-linear real-valued functions (Levenberg K. 1944). This technique is considered as a standard for non-linear least-squares problems (Mittelmann, H.D. 2004). The Hessian matrix H for standard Newton inverse analysis method can be defined as the second derivative of the objective function. So, the Hessian of the objective function can be written as,

$$H\left(\vec{\alpha}\right) = S^T C_D^{-1} S + C_M^{-1} + \nabla S^T C_D^{-1}\left(\vec{d}_{cal} - \vec{d}_{meas}\right) \tag{3.7}$$

where ΔS is the second derivative matrix and it can be given as,

$$\nabla S = \frac{\partial S}{\partial \vec{\alpha}^T} = \frac{\partial^2 \vec{d}_{cal}}{\partial \vec{\alpha}\, \partial \vec{\alpha}^T} \tag{3.8}$$

In above equation (3.5), the Hessian matrix should be positive-definite at each iteration to meet the convergence; because when it is positive-definite the Newton approach yields a downhill direction and meet the quadratic convergence in the neighborhood of the actual solution $\vec{\alpha}$. If Hessian matrix is close to singular i.e. not positive-definite then convergence or optimum solution may not reached.

$$0 < \vec{s}^T H \vec{s}^T < -\vec{g}^T \vec{s} \tag{3.9}$$

In order to solve Equation (3.10) for $\vec{s}_{\kappa}$, we need to compute gradient and Hessian at each iteration, κ. The gradient for the objective function defined in equation (3.6) can be represented as,

$$\vec{g}(\vec{\alpha}) = S^T C_D^{-1}\left(\vec{d}_{cal} - \vec{d}_{meas}\right) + C_M^{-1}\left(\vec{\alpha} - \vec{\alpha}_{pri}\right) \tag{3.10}$$

where S is the sensitivity matrix and is given as,

$$S = \frac{\partial \vec{d}_{cal}}{\partial \vec{\alpha}} \tag{3.11}$$

The calculation of exact Hessian is computationally expensive and takes very long time. The Levenberg-Marquardt method (Levenberg, 1944; Marquardt, 1963) approximates Hessian matrix to be equal to the diagonal matrix $(\dot{\lambda} I)$. So the equation becomes,

$$H_{LM}(\vec{\alpha}) = S^T C_D^{-1} S + C_M^{-1} + \dot{\lambda} I \tag{3.12}$$

where $\dot{\lambda}$ is a scalar quantity that is multiplied with the identity matrix I of the Hessian which makes it to be always positive definite. This diagonal perturbation will shift every eigenvalue of the Gauss-Newton Hessian by the value of $\dot{\lambda}$. Any eigenvalue that is negative or too close to zero, becomes positive, using this diagonal perturbation. This also improves the condition number of matrix. This perturbation is not limited to Gauss-Newton Hessian, but can even be applied to exact Hessian if it is close to singular. The Levenberg-Marquardt method is a combination of the Gauss–Newton algorithm and the method of steepest descent. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method that slows the convergence rate and sometimes cannot reach to the minimum point. When the current solution is close to the correct solution, it becomes a Gauss-Newton method.

3.6 Analytical Solutions to Radial Composite Reservoir

This section comprises of the analytical solutions of two region radial composite reservoir that was developed by Ambastha (1988) and modified later by Issaka (1996). The solutions are developed in Laplace space firstly without the inclusion of wellbore storage and skin. Following are the assumptions on which equations are derived:

- Constant production rate at wellbore is considered.
- The formation consists of two discontinuous regions with homogenous and isotropic properties on each side of the discontinuity.
- Laminar flow of a single phase fluid with slightly but constant compressibility occurs in each region.
- Gravity and capillarity effects are negligible.

For a two region radial composite reservoir, the classical governing flow equations in field units can be written as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_1}{\partial r}\right)=3792.2\left(\frac{\varphi\mu c_t}{\dot{k}}\right)_1\frac{\partial p_1}{\partial t}\quad\left(0<r\le r_f\right), \tag{3.13}$$

and,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_2}{\partial r}\right)=3792.2\left(\frac{\varphi\mu c_t}{\dot{k}}\right)_2\frac{\partial p_2}{\partial t}\quad\left(r_f\le r\le r_e\ or\ \infty\right) \tag{3.14}$$

p_1 and p_2 are the pressures in region 1 and 2 respectively. φ is porosity, μ is viscosity in centipoise, c_t is the total compressibility and $\dot{k}$ is the modified permeability. Equation 3.13 represents the diffusivity equation for the first region that extends from the wellbore radius r_w to r_f (distance to discontinuity). On the other hand, Equation 3.14 is for the second region that starts from r_f and ends depending upon the where the external boundary is located.

After converting the above equations into dimensionless form, the classical equations are as follows:

$$r_D^2 \frac{\partial^2 p_{D1}}{\partial r_D^2} + r_D \frac{\partial p_{D1}}{\partial r_D} = z r_D^2 p_{D1} \qquad \left(0 < r \le r_f\right), \tag{3.15}$$

and,

$$r_D^2 \frac{\partial^2 p_{D2}}{\partial r_D^2} + r_D \frac{\partial p_{D2}}{\partial r_D} = F z r_D^2 p_{D2} \qquad \left(r_f < r \le r_e \; or \; \infty\right) \tag{3.16}$$

Following are the initial and boundary conditions for classical diffusion equations for a two region radial composite reservoir:

3.6.1 Initial Conditions

Before any production/injection from the well takes place the whole reservoir is assumed to be at the uniform initial reservoir pressure, p_i. In dimensionless form, initial conditions for two regions can be written as,

$$p_{D1}(r_D, t_D = 0) = 0, \tag{3.17}$$

and,

$$p_{D2}(r_D, t_D = 0) = 0 \tag{3.18}$$

3.6.2 Inner Boundary Condition

Since constant rate production is assumed at well bore so Darcy's law will be applicable in this case and can be written as:

$$\left.\frac{\partial p_{D1}}{\partial r_D}\right|_{r_D=1} = -1 \tag{3.19}$$

Since well bore storage and skin effects are not taken into account, so:

$$p_{wD} = p_{D1} \quad (r_D = 1) \tag{3.20}$$

3.6.3 Interface Conditions

A very thin film discontinuity is considered at the interface so pressure and flow rate at the interface (R_D) will be continuous:

1. Equal Pressure

$$p_{D1}\left(r_D = R_D, t\right) = p_{D2}\left(r_D = R_D, t\right) \tag{3.21}$$

2. Equal Flux

$$\left.\frac{\partial p_{D2}}{\partial r_D}\right|_{r_D=R_D} = M \left.\frac{\partial p_{D1}}{\partial r_D}\right|_{r_D=R_D} \tag{3.22}$$

where M is the mobility ratio between region 1 and 2 and is defined as:

$$M = \frac{(k/\mu B)_1}{(k/\mu B)_2}$$

3.6.4 Outer Boundary Condition

Three different boundary conditions are normally encountered in solving problems of diffusivity that are as follows:

1. Infinite Acting Reservoir Boundary

In this case well is assumed to be located at the center of porous medium of infinite radial extent. Also for finite reservoir, this condition is valid as it means that pressure disturbance generated at the well bore has not seen the outer boundary. In dimensional form, it can be written as follows:

$$p_{D2}\left(r_D \to \infty, t\right) = 0 \qquad (3.23)$$

2. No-flow Reservoir Boundary

For this condition, well is located at the center of a cylindrical reservoir of radius r_e with No-flow reservoir outer boundary. The condition is widely applicable for volumetric reservoir. The condition from Darcy's Law is as under:

$$\left(\frac{\partial p_{D2}}{\partial r_D}\right)_{r_D = r_{eD}} = 0 \qquad (3.24)$$

3. Constant Pressure Outer Boundary

The well is assumed to be located at the center of a cylindrical reservoir of radius r_e and constant pressure is maintained at the outer boundary. This condition takes the form:

$$p_{D2}\left(r_D \to r_{eD}, t\right) = 0 \tag{3.25}$$

The dimensionless variables used are as follow:

$$p_{D1} = \frac{1}{141.2q}\left(\frac{kh}{B\mu}\right)_1 (p_i - p_1)\ , \tag{3.26}$$

$$p_{D2} = \frac{1}{141.2q}\left(\frac{kh}{B\mu}\right)_2 (p_i - p_2)\,, \tag{3.27}$$

$$p_{wD1} = \frac{1}{141.2q}\left(\frac{kh}{B\mu}\right)_1 (p_i - p_{wf})\,, \tag{3.28}$$

$$t_D = 0.00002637\left(\frac{k}{\varphi\mu c_t}\right)_1 \frac{t}{r_w^2}\,, \tag{3.29}$$

$$F = \left(k / \varphi\mu c_t\right)_1 \Big/ \left(k / \varphi\mu c_t\right)_2\ , \tag{3.30}$$

$$M = \left(k / \mu B\right)_1 \Big/ \left(k / \mu B\right)_2\ , \tag{3.31}$$

$$r_D = \frac{r}{r_w}\,, \tag{3.32}$$

$$r_{Df} = \frac{r_f}{r_w}\,, \tag{3.33}$$

and,

$$r_{eD} = \frac{r_e}{r_w}. \tag{3.34}$$

The solutions to Equations 3.9 & 3.10 subjected to various boundary conditions from Equations 3.11 to 3.19 are as under:

$$\hat{p}_{D1} = AI_0\left(r_D\sqrt{z}\right) + BK_0\left(r_D\sqrt{z}\right) \quad \left(0 < r \le r_f\right), \tag{3.35}$$

and,

$$\hat{p}_{D2} = CI_0\left(r_D\sqrt{Fz}\right) + DK_0\left(r_D\sqrt{Fz}\right) \quad \left(r_{Df} < r \le r_e \; or \; \infty\right) \tag{3.36}$$

In above equations, z is the Laplace parameter that is equivalent to time parameter in real space. Dimensionless pressure at the wellbore can be written from Equation 3.20 as,

$$\hat{p}_{wD1} = AI_0\left(\sqrt{z}\right) + BK_0\left(\sqrt{z}\right) \quad \left(r_D = 1\right) \tag{3.37}$$

The constants A, B, C and D are found using boundary conditions from Equations 3.19 to 3.25. Three different sets of constants are obtained for three different outer boundary conditions. These constants subjected to different boundary conditions are as under:

3.6.5 Constants for Infinite Acting Reservoir Boundary

The constant *A*, *B*, C and D for infinite acting reservoir boundary are as under:

$$A = \frac{S_2}{z^{\frac{3}{2}}\left(S_1K_1\left(\sqrt{z}\right) - S_2I_1\left(\sqrt{z}\right)\right)}, \tag{3.38}$$

$$B = \frac{S_2}{z^{\frac{3}{2}}\left(S_1 K_1\left(\sqrt{z}\right) - S_2 I_1\left(\sqrt{z}\right)\right)}, \tag{3.39}$$

$$C = 0, \tag{3.40}$$

and,

$$D = \frac{I_0\left(r_{Df}\sqrt{z}\right)S_1 + K_0\left(r_{Df}\sqrt{z}\right)S_2}{z^{\frac{3}{2}}\left(S_1 K_1\left(\sqrt{z}\right) - S_2 I_1\left(\sqrt{z}\right)\right)} \tag{3.41}$$

whereas S_1 and S_2 are,

$$S_1 = MK_0\left(r_{Df}\sqrt{Fz}\right)I_1\left(r_{Df}\sqrt{z}\right) + \sqrt{F}K_1\left(r_{Df}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{z}\right), \tag{3.42}$$

and,

$$S_2 = MK_0\left(r_{Df}\sqrt{Fz}\right)K_1\left(r_{Df}\sqrt{z}\right) - \sqrt{F}K_1\left(r_{Df}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{z}\right). \tag{3.43}$$

3.6.6 Constants for No-flow Reservoir Boundary

The constant for No-flow reservoir external boundary are:

$$A = \frac{S_2}{z^{\frac{3}{2}}\left(S_1 K_1\left(\sqrt{z}\right) - S_2 I_1\left(\sqrt{z}\right)\right)}, \tag{3.44}$$

$$B = \frac{S_2}{z^{\frac{3}{2}}\left(S_1 K_1\left(\sqrt{z}\right) - S_2 I_1\left(\sqrt{z}\right)\right)}, \tag{3.45}$$

$$C = \frac{K_1\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{z}\right)S_2 + K_1\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{z}\right)S_1}{z^{\frac{3}{2}}S_3\left(S_1 K_1\left(\sqrt{z}\right) - S_2 I_1\left(\sqrt{z}\right)\right)}, \tag{3.46}$$

and,

$$D = \frac{I_1\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{z}\right)S_2 + I_1\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{z}\right)S_1}{z^{\frac{3}{2}}S_3\left(S_1K_1\left(\sqrt{z}\right) - S_2I_1\left(\sqrt{z}\right)\right)} \quad (3.47)$$

S_1, S_2, S_3 and S_4 are defined as,

$$S_1 = MI_1\left(r_{Df}\sqrt{z}\right)S_4 - \sqrt{F}I_0\left(r_{Df}\sqrt{z}\right)S_3, \quad (3.48)$$

$$S_2 = MK_1\left(r_{Df}\sqrt{z}\right)S_4 + \sqrt{F}K_0\left(r_{Df}\sqrt{z}\right)S_3, \quad (3.49)$$

$$S_3 = K_1\left(r_{eD}\sqrt{Fz}\right)I_1\left(r_{Df}\sqrt{Fz}\right) - I_1\left(r_{eD}\sqrt{Fz}\right)K_1\left(r_{Df}\sqrt{Fz}\right), \quad (3.50)$$

and,

$$S_4 = K_1\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{Fz}\right) + I_1\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{Fz}\right). \quad (3.51)$$

3.6.7 Constants for Constant Pressure Reservoir Boundary

The constants under this condition are:

$$A = \frac{S_2}{z^{\frac{3}{2}}\left(S_1K_1\left(\sqrt{z}\right) - S_2I_1\left(\sqrt{z}\right)\right)}, \quad (3.52)$$

$$B = \frac{S_2}{z^{\frac{3}{2}}\left(S_1K_1\left(\sqrt{z}\right) - S_2I_1\left(\sqrt{z}\right)\right)}, \quad (3.53)$$

$$C = \frac{K_0\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{z}\right)S_2 + K_0\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{z}\right)S_1}{z^{\frac{3}{2}}S_3\left(S_2I_1\left(\sqrt{z}\right) - S_1K_1\left(\sqrt{z}\right)\right)}, \quad (3.54)$$

and,

$$D = \frac{I_0\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{z}\right)S_2 + I_0\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{z}\right)S_1}{z^{\frac{3}{2}}S_3\left(S_1K_1\left(\sqrt{z}\right) - S_2I_1\left(\sqrt{z}\right)\right)} \tag{3.55}$$

where,

$$S_1 = MS_4I_1\left(r_{Df}\sqrt{z}\right) + \sqrt{F}S_3I_0\left(r_{Df}\sqrt{z}\right), \tag{3.56}$$

$$S_2 = MS_4K_1\left(r_{Df}\sqrt{z}\right) - \sqrt{F}S_3K_0\left(r_{Df}\sqrt{z}\right), \tag{3.57}$$

$$S_3 = K_0\left(r_{eD}\sqrt{Fz}\right)I_1\left(r_{Df}\sqrt{Fz}\right) + I_0\left(r_{eD}\sqrt{Fz}\right)K_1\left(r_{Df}\sqrt{Fz}\right), \tag{3.58}$$

and,

$$S_4 = I_0\left(r_{eD}\sqrt{Fz}\right)K_0\left(r_{Df}\sqrt{Fz}\right) - K_0\left(r_{eD}\sqrt{Fz}\right)I_0\left(r_{Df}\sqrt{Fz}\right). \tag{3.59}$$

Dimensionless wellbore pressure without skin and wellbore storage is calculated from Equation 3.37. Stehfest Algorithm (1970) is used to obtain real well bore pressure from Laplace space.

3.6.8 Well bore Storage and Skin Effect

The solutions presented so far are without wellbore storage and skin effects. It is practical to add well bore storage and skin effects in final calculations to mimic the true picture of pressure distribution with respect to time or distance. In order to include well bore storage and skin into the solution, Van Everdingen and Hurst 1949 proposed a solution and solved the problem as convolution integral. So the dimensionless wellbore pressure with skin and wellbore storage effect can be written as:

$$\hat{p}_{wD}(z) = \frac{z\hat{p}_D + s}{z\left\{1 + C_D z\left(z\hat{p}_D + s\right)\right\}} \tag{3.60}$$

In above equation, $\hat{p}_D$ is the dimensionless well bore pressure in Laplace space without wellbore storage and skin effects.

CHAPTER 4

ANALYTICAL SOLUTIONS TO THE FRACTIONAL DIFFUSION EQUATION IN RADIAL COMPOSITE RESERVOIRS

4.1 Classical Darcy's Law

The basic constitutive equation that governs the flow of fluids through porous media is Darcy's law. The French civil engineer Henry Darcy formulated the famous law in 1856 on the basis of his experiments on vertical water filtration through sand beds. Darcy (1856) found out the relationship that could best describe his experimental data as,

$$q = C\frac{\Delta h}{L} \tag{4.1}$$

In the above equation that yields from Darcy's experiments, q is the volumetric flow rate, L is the length of the sand pack, Δh is the difference between heights h_1 and h_2, the heights above the standard datum of the water in the manometers and represents hydraulic heads at points 1 and 2.

However in describing the fluid flow in a petroleum reservoir it is more convenient to represent Darcy Law differentially in radial form with field units. The said form of the Darcy's Law can be written as,

$$q = \frac{kh}{141.2\mu B}\left(r\frac{\partial p}{\partial r}\right) \tag{4.2}$$

In Equation 4.2, q is in STB/D, h and r are in ft, μ is in centipoise, B is formation volume factor in RB/STB, p is in psia.

4.2 Modified Darcy's Law

In this study, Darcy's Law is modified and fractional derivatives are used to represent memory formalism parameter. Modified Darcy's Law can be written as,

$$q = \frac{\dot{k}h}{141.2\mu B}\left(r\frac{\partial \dot{p}}{\partial r}\right) \tag{4.3}$$

where,

$$\dot{p} = \frac{\partial^{\alpha} p}{\partial t^{\alpha}} \tag{4.4}$$

$\dot{k}$ is the modified Darcy's permeability and its units are $md/\sec^{\alpha}$ and α represents the memory parameter in Equation 4.4.

4.3 Models of Fractional Diffusion

In this research, Ambastha (1988) solution for two region radial composite reservoir is modified and fractional derivatives are introduced for the first time for composite reservoir. First the solution is obtained without wellbore storage and skin. Following are some of the assumptions on which the equations are derived:

- Constant production rate at wellbore is considered.

- The formation consists of two discontinuous regions with homogenous and isotropic properties on each side of the discontinuity.
- The front is of infinitesimal thickness and is considered stationary throughout the test period.
- Laminar flow of a single phase fluid with slightly compressible fluid.
- Gravity and capillarity effects are negligible.

By using Equation 4.3 i.e. modified Darcy's Law, the modified diffusivity equations describing fluid flow through porous media in a two region radial composite reservoir are given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \dot{p}_1}{\partial r}\right)=3792.2\left(\frac{\varphi\mu c_t}{\dot{k}}\right)_1\frac{\partial p_1}{\partial t} \quad \left(0<r\le r_f\right), \tag{4.5}$$

and,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \dot{p}_2}{\partial r}\right)=3792.2\left(\frac{\varphi\mu c_t}{\dot{k}}\right)_2\frac{\partial p_2}{\partial t} \quad \left(r_f\le r\le r_e\right) \tag{4.6}$$

where,

$$\dot{p}_1=\frac{\partial^\alpha p_1}{\partial t^\alpha}, \tag{4.7}$$

and,

$$\dot{p}_2=\frac{\partial^\alpha p_2}{\partial t^\alpha}. \tag{4.8}$$

It will be easier to incorporate boundary conditions in the form of pressure drop rather than just pressure. So modified diffusivity equations in terms of pressure drop are as under:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta \dot{p}_1}{\partial r}\right)=3792.2\left(\frac{\varphi \mu c_t}{\dot{k}}\right)_1\frac{\partial \Delta p_1}{\partial t} \quad \left(0<r\leq r_f\right), \tag{4.9}$$

and,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta \dot{p}_2}{\partial r}\right)=3792.2\left(\frac{\varphi \mu c_t}{\dot{k}}\right)_2\frac{\partial \Delta p_2}{\partial t} \quad \left(r_f\leq r\leq r_e\right). \tag{4.10}$$

where,

$$\Delta p = p_i - p(r,t)$$

Let,

$$\dot{\eta}_1=\frac{1}{3792.2}\left(\frac{\dot{k}}{\varphi \mu c_t}\right)_1, \tag{4.11}$$

and,

$$\dot{\eta}_2=\frac{1}{3792.2}\left(\frac{\dot{k}}{\varphi \mu c_t}\right)_2. \tag{4.12}$$

Since Equations 4.9 & 4.10 are second order in space and first order in time so generally it is required to have two boundary conditions and one initial condition. For radial flow, we usually specify boundary conditions at the wellbore (Inner boundary condition) and at the external radius of the reservoir (Outer boundary condition, considering the reservoir is circular). As we have two region radial composite reservoir so two interface conditions will also be present. These conditions are:

4.3.1 Initial Conditions

The conditions which are specified at time t=0 are termed as initial conditions. Usually in petroleum reservoirs it is reasonable to assume a uniform initial pressure in the reservoir.

For a two region composite, we assume that initial pressure in both the region is the same and is equal to p_i. So pressure drop at time t=0 can be defined as,

$$\Delta p_1\left(r, t=0\right)=0, \tag{4.13}$$

and,

$$\Delta p_2\left(r, t=0\right)=0 \tag{4.14}$$

4.3.2 Inner Boundary Condition

Generally the two types of inner boundary conditions are constant rate and constant pressure. A well is typically produced either at one of two conditions and it may have the effects of well bore storage. In this research, we restrict our solutions to the most commonly encountered inner boundary condition i.e. a well producing/injecting at constant rate to solve the diffusivity equations. The inner boundary for a well producing/injecting at constant rate is given by,

$$q=\frac{\dot{k}h}{141.2\mu B}\left(r\frac{\partial\Delta\dot{p}_1}{\partial r}\right)_{r=r_w}-\frac{24C}{B}\frac{d\Delta p_{wf}}{dt} \tag{4.15}$$

In this study, q is taken as negative for production well whereas for injection well it is positive. The effect of skin that cause an additional pressure drop near the wellbore due to impaired permeability must be added to calculate actual pressure from diffusivity equation. Skin effect is always present in an oil/gas reservoir usually caused by drilling and completion procedures. The skin effect in terms of pressure drop is as follows,

$$s\left(r\frac{\partial\Delta\dot{p}_1}{\partial r}\right)_{r=r_w}=\Delta p_{r=r_w}-\Delta p_{wf} \tag{4.16}$$

4.3.3 Interface Conditions

In most of the cases, it is reasonable to assume two regions of different but uniform and isotropic properties. The two regions are separated by a sharp interface present between them and there is significant contrast in mobility and storativity of the two regions. However it is necessary to have continuity between pressure and flow rates as the fluid moves from region 1 to region 2. The interface conditions are expressed as,

1. Equal Pressure at interface

$$\Delta p_1\left(r=r_f,t\right)=\Delta p_2\left(r=r_f,t\right) \tag{4.17}$$

2. Equal Flow at interface

$$\left(\frac{\partial \Delta \dot{p}_2}{\partial r}\right)_{r=r_f}=M\left(\frac{\partial \Delta \dot{p}_1}{\partial r}\right)_{r=r_f} \tag{4.18}$$

where M is the mobility ratio between region 1 and region 2.

$$M=\frac{\left(\frac{\dot{k}}{\mu B}\right)_1}{\left(\frac{\dot{k}}{\mu B}\right)_2} \tag{4.19}$$

4.3.4 Outer Boundary Conditions

Three cases are generally considered for outer boundary conditions of a reservoir. One is infinite acting; it means that pressure disturbance created at the wellbore is not felt at the outer boundary for practical distances from the wellbore at any time during the well test. Second case is of a closed reservoir, an example of no-flow boundary reservoir is the volumetric reservoir. Third outer boundary condition is of constant pressure reservoir, reservoirs with very strong water drive are the examples of this case.

Case 1: Infinite Acting Reservoir Boundary

Pressure becomes equal to the initial reservoir pressure as the radius becomes very large for all time. Eventually pressure drop will be zero. It can be written as,

$$\Delta p_2\left(r \to \infty, t\right) = 0 \tag{4.20}$$

Case 2: Closed Reservoir

Considering a cylindrical reservoir with external radius as re, for all time greater than zero from Darcy's Law we can write the boundary condition as,

$$\left(\frac{\partial \Delta p_2}{\partial r}\right)_{r=r_e} = 0 \tag{4.21}$$

Case 3: Constant Pressure Reservoir Boundary

For a reservoir with strong water drive with outer radius re, constant pressure condition can be written as,

$$\Delta p_2\left(r = r_e, t\right) = 0 \tag{4.22}$$

4.3.5 Laplace transformation of Initial and Boundary Conditions

Laplace transformation of Initial and Boundary Conditions from Equation 4.13 to Equation 4.22 are as follows,

4.3.5.1 Initial Conditions

After Laplace transform, initial conditions for the two regions can be written as,

$$\Delta\hat{p}_1\left(r,t=0\right)=0, \tag{4.23}$$

and,

$$\Delta\hat{p}_2\left(r,t=0\right)=0. \tag{4.24}$$

4.3.5.2 Inner Boundary Condition

Rearranging Equation 4.15 to separate out pressure gradient terms from other and is given as,

$$\left(r\frac{\partial\Delta\dot{p}_1}{\partial r}\right)_{r=r_w}=141.2\left(\frac{\mu B}{\dot{k}h}\right)_1\left[q+\frac{24C}{B_1}\frac{d\Delta p_{wf}}{dt}\right]$$

Applying Laplace transforms and substituting initial condition from Equation 4.23 in to the above equation,

$$\left(r\frac{d}{dr}\left[z^{\alpha}\Delta\hat{p}_1-z^{\alpha-1}\Delta\hat{p}_1\left(r,t=0\right)\right]\right)_{r=r_w}=\frac{\dot{q}}{z}+\dot{C}\left[z\Delta\hat{p}_{wf}-\Delta\hat{p}_{wf}\left(r,t=0\right)\right],$$

Further simplification of the above equation will give,

$$\left(r\frac{d\Delta\hat{p}_1}{dr}\right)_{r=r_w}=\frac{\dot{q}}{z^{1+\alpha}}+\dot{C}\left[z^{1-\alpha}\Delta\hat{p}_{wf}\right] \tag{4.25}$$

where,

$$\dot{q}=141.2q\left(\frac{\mu B}{\dot{k}h}\right)_1,$$

and,

$$\dot{C} = 3388.8C\left(\frac{\mu}{\dot{k}h}\right)_1$$

In Laplace form, Equation 4.16 can be written as:

$$\Delta\hat{p}_{wf} = \Delta\hat{p}_{r=r_w} - s\left(r\frac{d}{dr}\left[z^{\alpha}\Delta\hat{p}_1 - z^{\alpha-1}\Delta\hat{p}_1\left(r,t=0\right)\right]\right)_{r=r_w} \quad ,$$

Substituting initial condition from Equation 4.23 and further simplification of the above equation will give,

$$\Delta\hat{p}_{wf} = \Delta\hat{p}_{r=r_w} - sz^{\alpha}\left(r\frac{d\Delta\hat{p}_1}{dr}\right)_{r=r_w} \tag{4.26}$$

4.3.5.3 Interface Conditions

Laplace transform of interface conditions are as follows:

1. Equal Pressure at interface

Laplace form of Equation 4.17 can be written as,

$$\Delta\hat{p}_1\left(r=r_f,t\right) = \Delta\hat{p}_2\left(r=r_f,t\right) \tag{4.27}$$

2. Equal Flow at interface

Equation 4.18 in Laplace form is as follows:

$$\left(\frac{d\Delta\hat{p}_2}{dr}\right)_{r=r_f} = M\left(\frac{d\Delta\hat{p}_1}{dr}\right)_{r=r_f} \tag{4.28}$$

4.3.5.4 Outer Boundary Conditions

After transforming into Laplace space, three outer boundary conditions can be written as:

Case 1: Infinite Acting Reservoir Boundary

Equation 4.20 in Laplace space is as under:

$$\Delta\hat{p}_2\left(r \to \infty, t\right) = 0 \tag{4.29}$$

Case 2: Closed Reservoir

Laplace transformation of Equation 4.21 will give,

$$\left(\frac{d\Delta\hat{p}_2}{dr}\right)_{r=r_e} = 0 \tag{4.30}$$

Case 3: Constant Pressure Reservoir Boundary

Constant pressure reservoir outer boundary in Laplace space is as under:

$$\Delta\hat{p}_2\left(r = r_e, t\right) = 0 \tag{4.31}$$

Taking Laplace transform of Equations 4.9 & 4.10 and applying initial conditions from Equations 4.23 & 4.24,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left[z^{\alpha}\Delta\hat{p}_1 - z^{\alpha-1}\Delta\hat{p}_1\left(r, t=0\right)\right]\right] = \frac{1}{\dot{\eta}_1}\left[z\Delta\hat{p}_1 - \Delta\hat{p}_1\left(r, t=0\right)\right],$$

Simplifying above equation,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Delta\hat{p}_1}{dr}\right] = \frac{z^{1-\alpha}}{\dot{\eta}_1}\Delta\hat{p}_1,$$

finally we have,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Delta\hat{p}_1}{dr}\right] - \gamma_1\Delta\hat{p}_1 = 0 \qquad \left(0 \le r \le r_f\right) \ . \tag{4.32}$$

Similarly for second region of radial composite system,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Delta\hat{p}_2}{dr}\right]-\gamma_2\Delta\hat{p}_2=0 \quad \left(r_f\le r\le r_e\right) \tag{4.33}$$

where,

$$\gamma_1=\frac{z^{1-\alpha}}{\dot{\eta}_1},$$

and,

$$\gamma_2=\frac{z^{1-\alpha}}{\dot{\eta}_2}$$

Laplace Solutions for Equations 4.32 and 4.33 in terms of Bessel functions are as follows:

$$\Delta\hat{p}_1=AI_o\left(r\sqrt{\gamma_1}\right)+BK_o\left(r\sqrt{\gamma_1}\right) \quad \left(0<r\le r_f\right), \tag{4.34}$$

and,

$$\Delta\hat{p}_2=CI_o\left(r\sqrt{\gamma_2}\right)+DK_o\left(r\sqrt{\gamma_2}\right) \quad \left(r_f\le r\le r_e\right) \tag{4.35}$$

The constants A, B, C & D are subjected to outer boundary conditions and will vary as the outer boundary conditions are changed.

Taking derivative of Equation 4.26 and incorporating inner boundary condition from Equation 4.17 and solving for $\Delta\hat{p}_{wf}$,

$$Ar_w\sqrt{\gamma_1}I_1\left(r_w\sqrt{\gamma_1}\right)-Br_w\sqrt{\gamma_1}K_1\left(r_w\sqrt{\gamma_1}\right)=\frac{\dot{q}}{z^{1+\alpha}}+\dot{C}\left[z^{1-\alpha}\Delta\hat{p}_{wf}\right],$$

Rearrangement of the above equation will give,

$$\Delta\hat{p}_{wf} = \frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}}\left[AI_1\left(r_w\sqrt{\gamma_1}\right) - BK_1\left(r_w\sqrt{\gamma_1}\right)\right] - \frac{\dot{q}}{\dot{C}z^2} \tag{4.36}$$

Taking derivative of Equation 4.26 and incorporating inner boundary condition from Equation 4.18 and solving for $\Delta\hat{p}_{wf}$,

$$\begin{aligned}\Delta\hat{p}_{wf} &= A\left[I_o\left(r_w\sqrt{\gamma_1}\right) - sz^{\alpha}r_w\sqrt{\gamma_1}I_1\left(r_w\sqrt{\gamma_1}\right)\right] + \\ &B\left[K_o\left(r_w\sqrt{\gamma_1}\right) + sz^{\alpha}r_w\sqrt{\gamma_1}K_1\left(r_w\sqrt{\gamma_1}\right)\right]\end{aligned} \tag{4.37}$$

4.3.6 Solutions for Infinite Acting Reservoir Boundary Case

Using Equation 4.29 in Equation 4.35 to incorporate Infinite acting outer boundary condition,

$CI_0(\infty) + DK_0(\infty) = 0$,

and,

$$C = 0 \tag{4.38}$$

Using Equation 4.27 for equal pressure condition,

$$AI_o\left(r_f\sqrt{\gamma_1}\right) + BK_o\left(r_f\sqrt{\gamma_1}\right) = CI_o\left(r_f\sqrt{\gamma_2}\right) + DK_o\left(r_f\sqrt{\gamma_2}\right) \tag{4.39}$$

Putting value of C from Equation 4.38 in Equation 4.39,

$$AI_o\left(r_f\sqrt{\gamma_1}\right) + BK_o\left(r_f\sqrt{\gamma_1}\right) = DK_o\left(r_f\sqrt{\gamma_2}\right) \tag{4.40}$$

Using Equation (4.28) for equal flux condition,

$$\sqrt{\gamma_2}\left[CI_1\left(r_f\sqrt{\gamma_2}\right) - DK_1\left(r_f\sqrt{\gamma_2}\right)\right] = M\sqrt{\gamma_1}\left[AI_1\left(r_f\sqrt{\gamma_1}\right) - BK_1\left(r_f\sqrt{\gamma_1}\right)\right] \tag{4.41}$$

Substituting Equation 4.30 into Equation 4.33,

$$M\left[AI_1\left(r_f\sqrt{\gamma_1}\right)-BK_1\left(r_f\sqrt{\gamma_1}\right)\right]=-\sqrt{F}\left[K_1\left(r_f\sqrt{\gamma_2}\right)\right]D \tag{4.42}$$

Rearranging Equation 4.36, 4.37, 4.40 and 4.42 as,

$$a_{11}\Delta\hat{p}_{wf}+a_{12}A+a_{13}B+a_{14}D=a_{15}, \tag{4.43}$$

$$a_{21}\Delta\hat{p}_{wf}+a_{22}A+a_{23}B+a_{24}D=a_{25}, \tag{4.44}$$

$$a_{31}\Delta\hat{p}_{wf}+a_{32}A+a_{33}B+a_{34}D=a_{35}, \tag{4.45}$$

and,

$$a_{41}\Delta\hat{p}_{wf}+a_{42}A+a_{43}B+a_{44}D=a_{45} \tag{4.46}$$

Coefficients Descriptions

The coefficient under infinite acting reservoir boundary are defined as,

$$a_{11}=1, \tag{4.47}$$

$$a_{12}=-\frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}}I_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.48}$$

$$a_{13}=\frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}}K_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.49}$$

$$a_{14}=0, \tag{4.50}$$

$$a_{15}=-\frac{\dot{q}}{\dot{C}z^2}, \tag{4.51}$$

$$a_{21}=1, \tag{4.52}$$

$$a_{22}=-\left[I_o\left(r_w\sqrt{\gamma_1}\right)-sz^{\alpha}r_w\sqrt{\gamma_1}I_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.53}$$

$$a_{23}=-\left[K_o\left(r_w\sqrt{\gamma_1}\right)+sz^{\alpha}r_w\sqrt{\gamma_1}K_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.54}$$

$$a_{24}=0, \tag{4.55}$$

$$a_{25}=0, \tag{4.56}$$

$$a_{31} = 0\,, \tag{4.57}$$

$$a_{32} = I_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.58}$$

$$a_{33} = K_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.59}$$

$$a_{34} = -K_o\left(r_f\sqrt{\gamma_2}\right), \tag{4.60}$$

$$a_{35} = 0\,, \tag{4.61}$$

$$a_{41} = 0\,, \tag{4.62}$$

$$a_{42} = MI_1\left(r_f\sqrt{\gamma_1}\right), \tag{4.63}$$

$$a_{43} = -MK_1\left(r_f\sqrt{\gamma_1}\right), \tag{4.64}$$

$$a_{44} = \sqrt{F}K_1\left(r_f\sqrt{\gamma_2}\right), \tag{4.65}$$

and,

$$a_{45} = 0 \tag{4.66}$$

Solving Equations 4.43 to 4.46 will yield value of required pressure drop that will contain the memory effect.

4.3.7 Constants for Closed Reservoir Case

Taking derivative of Equation 4.35 and subtituting it in Equation 4.30 to incorporate No-flow outer boundary condition,

$$C\sqrt{\gamma_2}I_1\left(r_e\sqrt{\gamma_2}\right) - D\sqrt{\gamma_2}K_1\left(r_e\sqrt{\gamma_2}\right) = 0\,,$$

Simplification will give,

$$C = \frac{K_1\left(r_e\sqrt{\gamma_2}\right)}{I_1\left(r_e\sqrt{\gamma_2}\right)}D \tag{4.67}$$

Using Equation 4.67 in Equation 4.39 and 4.41,

$$AI_o\left(r_f\sqrt{\gamma_1}\right)+ BK_o\left(r_f\sqrt{\gamma_1}\right)=$$
$$\left[\frac{K_1\left(r_e\sqrt{\gamma_2}\right)I_o\left(r_f\sqrt{\gamma_2}\right)+ I_1\left(r_e\sqrt{\gamma_2}\right)K_o\left(r_f\sqrt{\gamma_2}\right)}{I_1\left(r_e\sqrt{\gamma_2}\right)}\right]D, \tag{4.68}$$

and,

$$M\left[AI_1\left(r_f\sqrt{\gamma_1}\right)- BK_1\left(r_f\sqrt{\gamma_1}\right)\right]=$$
$$\sqrt{F}\left[\frac{K_1\left(r_e\sqrt{\gamma_2}\right)I_1\left(r_f\sqrt{\gamma_2}\right)- I_1\left(r_e\sqrt{\gamma_2}\right)K_1\left(r_f\sqrt{\gamma_2}\right)}{I_1\left(r_e\sqrt{\gamma_2}\right)}\right]D. \tag{4.69}$$

Rearranging Equation 4.36, 4.37, 4.68 and 4.69 as,

$$b_{11}\Delta\hat{p}_{wf} + b_{12}A + b_{13}B + b_{14}D = b_{15}, \tag{4.70}$$

$$b_{21}\Delta\hat{p}_{wf} + b_{22}A + b_{23}B + b_{24}D = b_{25}, \tag{4.71}$$

$$b_{31}\Delta\hat{p}_{wf} + b_{32}A + b_{33}B + b_{34}D = b_{35}, \tag{4.72}$$

and,

$$b_{41}\Delta\hat{p}_{wf} + b_{42}A + b_{43}B + b_{44}D = b_{45} \tag{4.73}$$

Coefficients Descriptions

The coefficients description under No-flow reservoir outer boundary is as follows:

$$b_{11} = 1, \tag{4.74}$$

$$b_{12} = -\frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}} I_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.75}$$

$$b_{13} = \frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}} K_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.76}$$

$$b_{14} = 0, \tag{4.77}$$

$$b_{15} = -\frac{\dot{q}}{\dot{C}z^2}, \tag{4.78}$$

$$b_{21} = 1, \tag{4.79}$$

$$b_{22} = -\left[I_o\left(r_w\sqrt{\gamma_1}\right) - sz^{\alpha} r_w \sqrt{\gamma_1} I_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.80}$$

$$b_{23} = -\left[K_o\left(r_w\sqrt{\gamma_1}\right) + sz^{\alpha} r_w \sqrt{\gamma_1} K_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.81}$$

$$b_{24} = 0, \tag{4.82}$$

$$b_{25} = 0, \tag{4.83}$$

$$b_{31} = 0, \tag{4.84}$$

$$b_{32} = I_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.85}$$

$$b_{33} = K_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.86}$$

$$b_{34} = -\left[\frac{K_1\left(r_e\sqrt{\gamma_2}\right) I_o\left(r_f\sqrt{\gamma_2}\right) + I_1\left(r_e\sqrt{\gamma_2}\right) K_o\left(r_f\sqrt{\gamma_2}\right)}{I_1\left(r_e\sqrt{\gamma_2}\right)}\right], \tag{4.87}$$

$$b_{35} = 0, \tag{4.88}$$

$$b_{41} = 0, \tag{4.89}$$

$$b_{42} = MI_1\left(r_f\sqrt{\gamma_1}\right), \tag{4.90}$$

$$b_{43} = -MK_1\left(r_f\sqrt{\gamma_1}\right), \tag{4.91}$$

$$b_{44} = -\sqrt{F}\left[\frac{K_1\left(r_e\sqrt{\gamma_2}\right) I_1\left(r_f\sqrt{\gamma_2}\right) - I_1\left(r_e\sqrt{\gamma_2}\right) K_1\left(r_f\sqrt{\gamma_2}\right)}{I_1\left(r_e\sqrt{\gamma_2}\right)}\right], \tag{4.92}$$

and,

$$b_{45} = 0 \tag{4.93}$$

Solving Equations 4.70 to 4.73 will yield value of required pressure drop that will contain the memory effect.

4.3.8 Constants for Constant Pressure Boundary Case

Using Equation 4.31 and substitute it in Equation 4.35 to incorporate Constant Pressure Outer boundary condition,

$$CI_0\left(r_e\sqrt{\gamma_2}\right)+DK_0\left(r_e\sqrt{\gamma_2}\right)=0\,,$$

which further simplifies to,

$$C=-\frac{K_0\left(r_e\sqrt{\gamma_2}\right)}{I_0\left(r_e\sqrt{\gamma_2}\right)}D \tag{4.94}$$

Substituting the value of Equation 4.86 in Equation 4.39 and 4.41,

$$AI_o\left(r_f\sqrt{\gamma_1}\right)+BK_o\left(r_f\sqrt{\gamma_1}\right)= \left[\frac{I_0\left(r_e\sqrt{\gamma_2}\right)K_o\left(r_f\sqrt{\gamma_2}\right)-K_0\left(r_e\sqrt{\gamma_2}\right)I_o\left(r_f\sqrt{\gamma_2}\right)}{I_0\left(r_e\sqrt{\gamma_2}\right)}\right]D\,, \tag{4.95}$$

and,

$$M\left[AI_1\left(r_f\sqrt{\gamma_1}\right)-BK_1\left(r_f\sqrt{\gamma_1}\right)\right]= -\sqrt{F}\left[\frac{K_0\left(r_e\sqrt{\gamma_2}\right)I_1\left(r_f\sqrt{\gamma_2}\right)+I_0\left(r_e\sqrt{\gamma_2}\right)K_1\left(r_f\sqrt{\gamma_2}\right)}{I_0\left(r_e\sqrt{\gamma_2}\right)}\right] \tag{4.96}$$

Rearranging Equation 4.36, 4.37, 4.95 and 4.96 as,

$$c_{11}\Delta\hat{p}_{wf}+c_{12}A+c_{13}B+c_{14}D=c_{15}\,, \tag{4.97}$$

$$c_{21}\Delta\hat{p}_{wf}+c_{22}A+c_{23}B+c_{24}D=c_{25}\,, \tag{4.98}$$

$$c_{31}\Delta\hat{p}_{wf}+c_{32}A+c_{33}B+c_{34}D=c_{35}\,, \tag{4.99}$$

and,

$$c_{41}\Delta\hat{p}_{wf} + c_{42}A + c_{43}B + c_{44}D = c_{45} \tag{4.100}$$

Coefficients Descriptions

For Constant pressure reservoir outer boundary, coefficients are as follows:

$$c_{11} = 1, \tag{4.101}$$

$$c_{12} = -\frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}} I_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.102}$$

$$c_{13} = \frac{r_w\sqrt{\gamma_1}}{\dot{C}z^{1-\alpha}} K_1\left(r_w\sqrt{\gamma_1}\right), \tag{4.103}$$

$$c_{14} = 0, \tag{4.104}$$

$$c_{15} = -\frac{\dot{q}}{\dot{C}z^2}, \tag{4.105}$$

$$c_{21} = 1, \tag{4.106}$$

$$c_{22} = -\left[I_o\left(r_w\sqrt{\gamma_1}\right) - sz^{\alpha} r_w\sqrt{\gamma_1} I_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.107}$$

$$c_{23} = -\left[K_o\left(r_w\sqrt{\gamma_1}\right) + sz^{\alpha} r_w\sqrt{\gamma_1} K_1\left(r_w\sqrt{\gamma_1}\right)\right], \tag{4.108}$$

$$c_{24} = 0, \tag{4.109}$$

$$c_{25} = 0, \tag{4.110}$$

$$c_{31} = 0, \tag{4.111}$$

$$c_{32} = I_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.112}$$

$$c_{33} = K_o\left(r_f\sqrt{\gamma_1}\right), \tag{4.113}$$

$$c_{34} = -\left[\frac{I_0\left(r_e\sqrt{\gamma_2}\right)K_o\left(r_f\sqrt{\gamma_2}\right) - K_0\left(r_e\sqrt{\gamma_2}\right)I_o\left(r_f\sqrt{\gamma_2}\right)}{I_0\left(r_e\sqrt{\gamma_2}\right)}\right], \tag{4.114}$$

$$c_{35} = 0, \tag{4.115}$$

$$c_{41} = 0 \text{ ,} \tag{4.116}$$

$$c_{42} = MI_1\left(r_f \sqrt{\gamma_1}\right), \tag{4.117}$$

$$c_{43} = -MK_1\left(r_f \sqrt{\gamma_1}\right), \tag{4.118}$$

$$c_{44} = \sqrt{F}\left[\frac{K_0\left(r_e\sqrt{\gamma_2}\right)I_1\left(r_f\sqrt{\gamma_2}\right) + I_0\left(r_e\sqrt{\gamma_2}\right)K_1\left(r_f\sqrt{\gamma_2}\right)}{I_0\left(r_e\sqrt{\gamma_2}\right)}\right], \tag{4.119}$$

and,

$$c_{45} = 0 \tag{4.120}$$

Solving Equations 4.97 to 4.100 will yield value of required pressure drop that will contain the memory effect.

4.4 Pressure response and its Derivatives under fractional diffusion

Before going to inverse analysis, the basic step is the identification of the reservoir recognition of the reservoir model, because without defining the model, the corresponding reservoir and wellbore parameters cannot be estimated.

Pressure derivative plots that were first proposed by Bourdet et al. (1983a) have become a standard procedure for model identification. The pressure derivative plot provides a simultaneous presentation of the following two sets of plots.

$$\log\left(P_D\right) Vs \log\left(t_D\right)$$

$$\log\left(P_D{}'\right) Vs \log\left(t_D\right)$$

In our case we plotted $\left(\Delta p\right)$ & $\left(\Delta p'\right)$ instead of P_D & $P_D^{'}$ respectively. We are now applying this concept of memory formalisms for the generation of pressure and pressure derivative curves suggested by Bourdet (1983). Three examples are considered for the generation of pressure and derivative curves. These examples mainly vary in terms of their outer boundary conditions and specific reservoir and well bore properties. Derivative $(\Delta p')$ in this case is defined as:

$$\Delta p' = t\frac{d\Delta p}{dt}$$

4.5 Model Validation

The models equations provided in literature by Ambastha (1988) are compared with fractional diffusion models providing alpha=0. The model validations for different outer boundary conditions under different reservoir and wellbore parameters are as under:

4.5.1 Model Verification for Infinite Acting reservoir Boundary

Table 4.1: Reservoir and Wellbore parameters for infinite acting outer boundary condition

Parameters	**Value**
$\dot{k}_1$	500
μ_1	0.8
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
c_{t2}	1e-5
φ_2	0.25
r_f	400
s	2
C	0.005
α	0

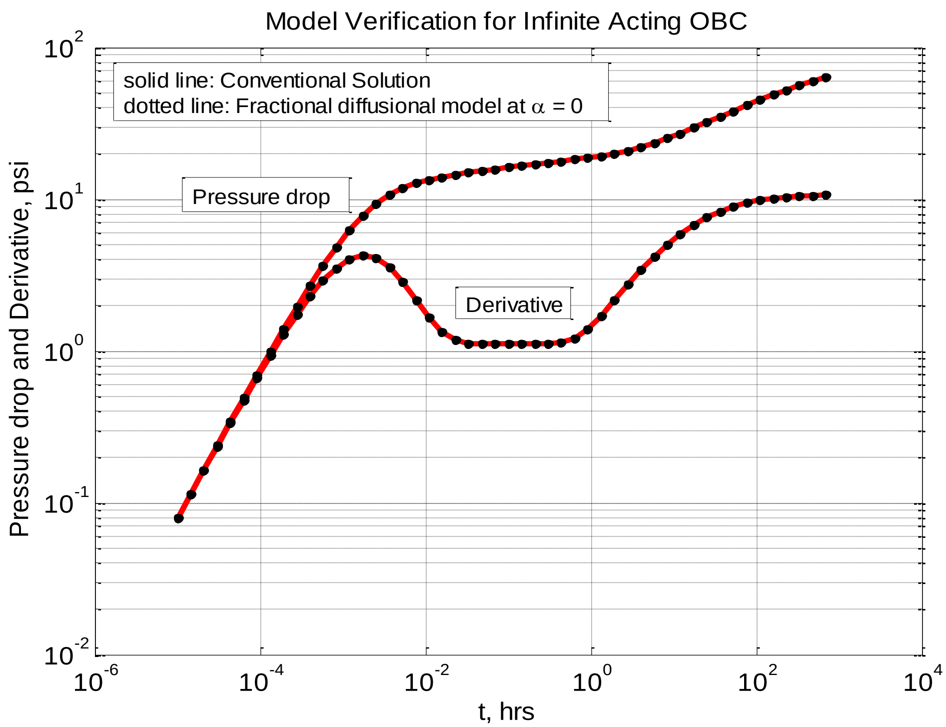

Figure 4.1: Matched Pressure drop and derivative versus time for infinite acting outer boundary condition

4.5.2 Model Verification for No-flow reservoir Boundary

Table 4.2: Reservoir and Wellbore parameters for no-flow outer boundary condition

Parameters	**Value**
$\dot{k}_1$	500
μ_1	0.8
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0

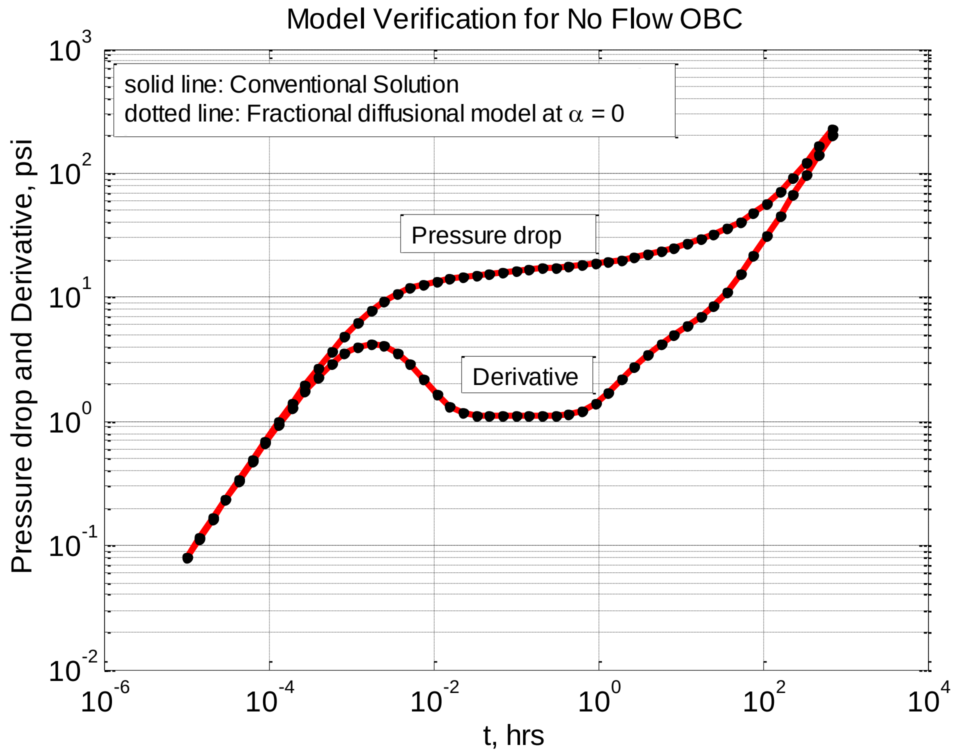

Figure 4.2: Matched Pressure drop and derivative versus time for no-flow outer boundary condition

4.5.3 Model Verification for Constant Pressure reservoir Boundary

Table 4.3: Reservoir and Wellbore parameters for constant pressure outer boundary condition

Parameters	Value
$\dot{k}_1$	500
μ_1	0.8
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0

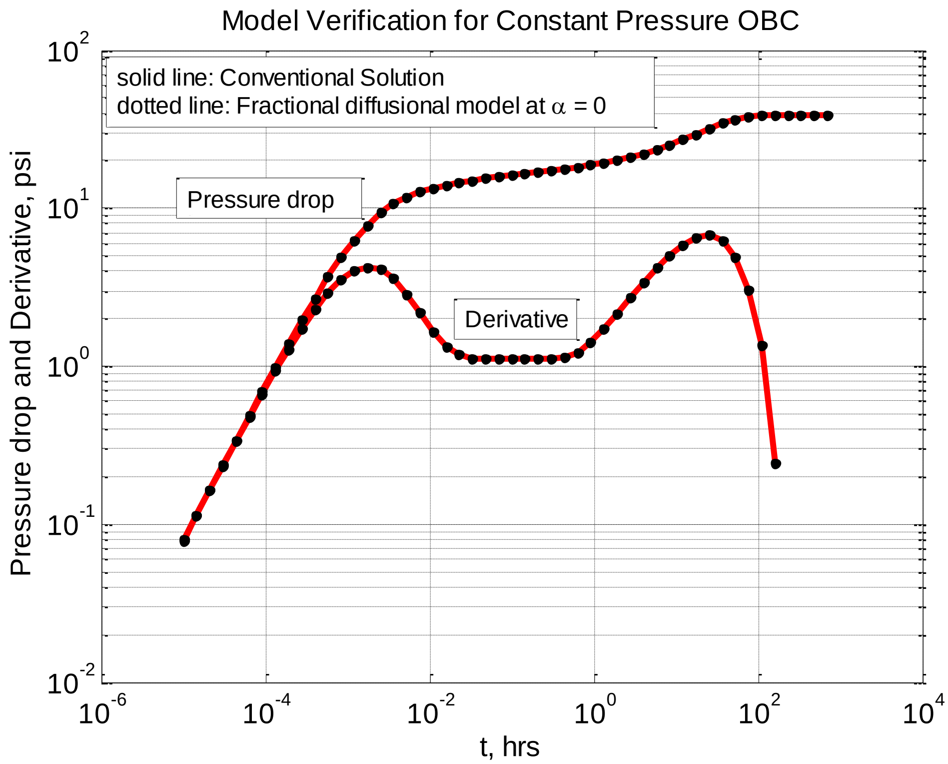

Figure 4.3: Matched Pressure drop and derivative versus time for constant pressure outer boundary condition

4.6 Sensitivity analysis

In this section, we perform sensitivity analysis of memory parameter for favorable and unfavorable mobility ratios:

4.6.1 Example 1: Two Region Radial Composite Reservoir with infinite acting outer boundary condition

In this example, we generate pressure and pressure derivatives curves for a two region radial composite reservoir with constant rate inner boundary condition and infinite acting outer boundary condition. We considered two scenarios of favorable and unfavorable mobility ratios i.e. when $M > 1$ (favorable mobility ratio) and $M < 1$ (unfavorable mobility ratio). Wellbore storage and skin effects are also considered. These curves are plotted considering the fractional diffusion models and sensitivity of memory parameter

α is also being analyzed. The reservoir is initially at a pressure of 6000 psia and its thickness is 100 ft. A well with radius of 0.5 ft under constant production of 800 STB/D is considered. A logarithmic function is used to make time steps starting with time of about 10^{-6} and ending with about 1000 hrs with 1500 data points. Stehfest algorithm is used to convert the pressure data from Laplace space to real space and value of 'N' even integar is taken to be 6. Typical reservoir and well bore properties used for generation of pressure and pressure derivatives curves for two mobility ratios are shown in Table 4.4 and 4.5.

It is evident from the Figure 4.4 and Figure 4.7 that bottom hole flowing pressure is influenced by the memory parameter α. Due to large time scale, effect of memory parameter is not visible for early time on Cartesian plot. On the other hand, semi log plot in Figure 4.5 and 4.8 of bottom hole flowing pressure versus time shows that with increasing value of memory parameter α, p_{wf} is becoming higher i.e. additional pressure drop may be due to anomalous diffusion. This effect can be compared with results discussed by Park (Park et al., 2001) in which they described the memory of the system from effective diffusion coefficient, their results show that additional pressure drops should occur because of the delay from diffusion. Figure 4.6 shows the pressure drop and pressure derivative plots for mobility ratio greater than one where as Figure 4.9 is for mobility ratio less than one, these distributions are plotted to see the effect of memory parameters. In both figures, wellbore storage period is easily identifiable by unit slope line. At the end of well bore storage, pressure derivative is showing radial flow regime for region one for α equals to zero, however it is noticed that as the value of α is increasing; the deviation from horizontal line is becoming more. Each of the four lines for different values of α crosses each other before the start of the radial homogeneous system. The pressure derivative curve shows higher pressure drop due to value of mobility ratio greater than one whereas it is opposite in Figure 4.9 (Ambastha, 1988; Issaka, 1996). It is observed that the pressure drop is less for larger values of α, then the pressure drop curves merges to a single one at one point. They become separated out once again at later stage; however at the later stage pressure drop in the reservoir is more for higher values of α for both cases of mobility ratio. The pressure derivative curves in the Figure 4.6 and Figure 4.9 are much more sensitive to the memory parameter, as there are strong separations between the radial flow lines for different values of α. This depicts the values of α to be very small because in practical field data not that much deviation in pressure derivative line is expected from the horizontal line.

Table 4.4: Reservoir and Wellbore parameters for two region radial composite reservoir with infinite acting outer boundary condition (M > 1)

Parameters	**Value**
$\dot{k}_1$	500
μ_1	0.8
c_{t1}	1e-5
B_1	1.2
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

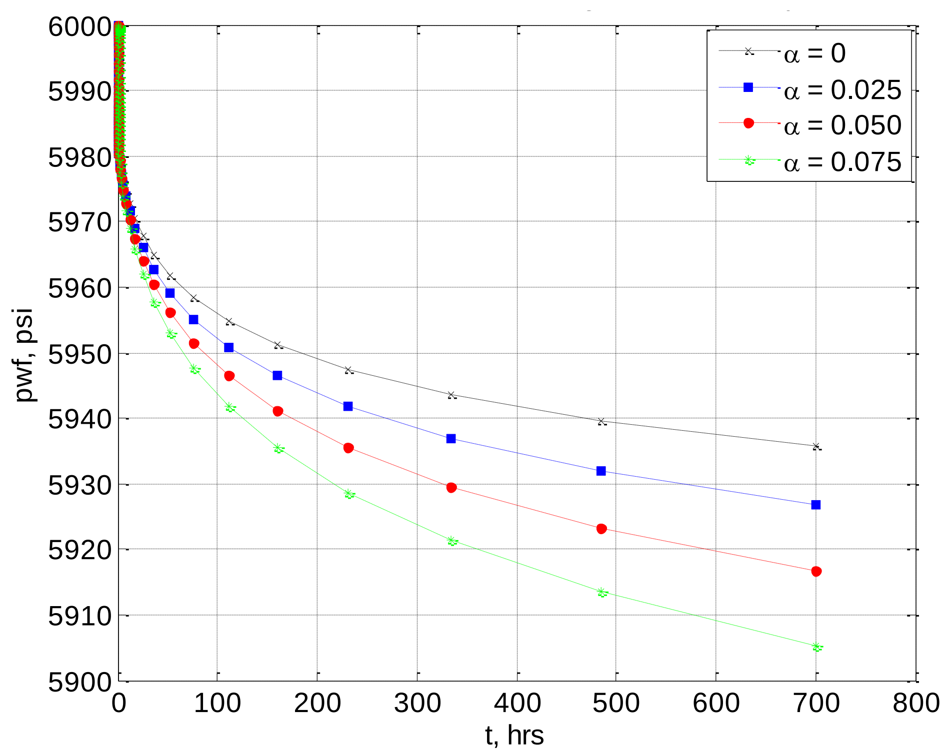

Figure 4.4: Bottom hole pressures versus time for M > 1 for infinite acting outer boundary

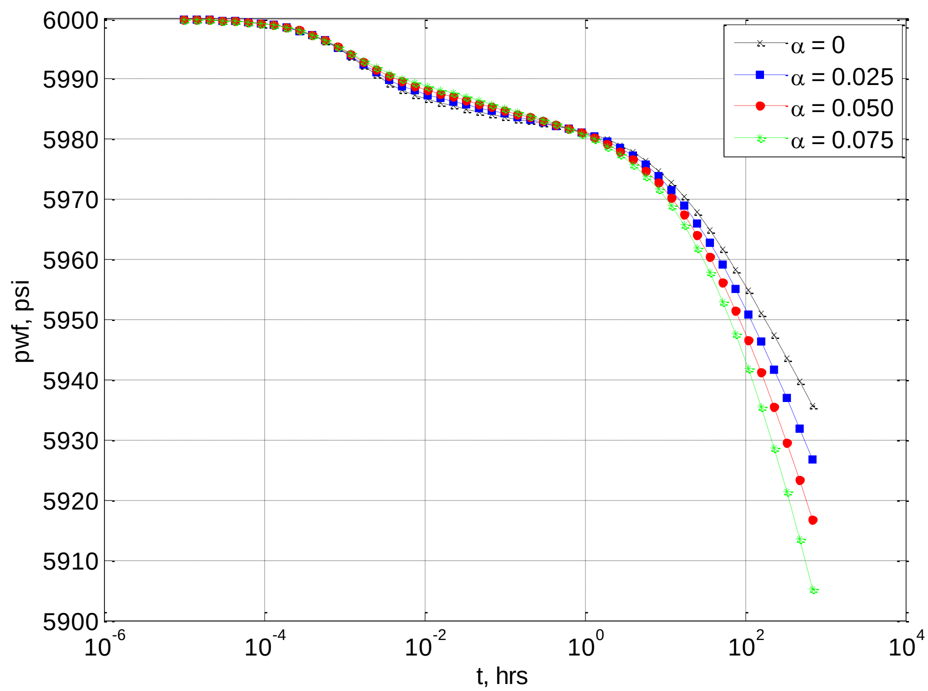

Figure 4.5: Semilog plot of bottom hole pressures versus time for M > 1 for infinite acting outer boundary

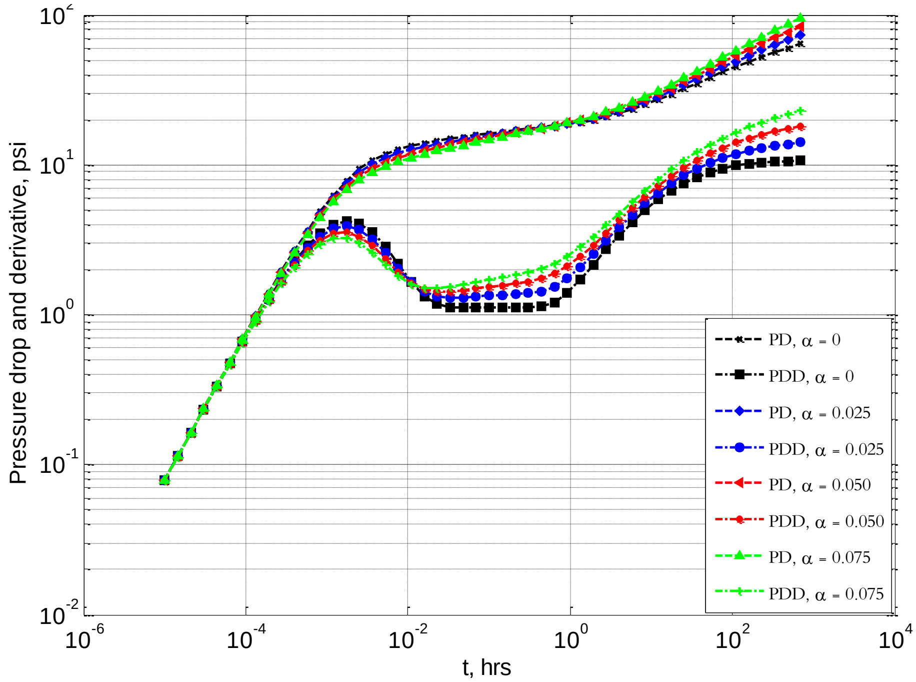

Figure 4.6: Pressure drop and pressure derivative versus time for M > 1 for infinite acting outer boundary

Table 4.5: Reservoir and Wellbore parameters for two region radial composite reservoir with infinite acting outer boundary condition (M < 1)

Parameters	**Value**
$\dot{k}_1$	50
μ_1	0.8
B_1	1.2
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	200
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

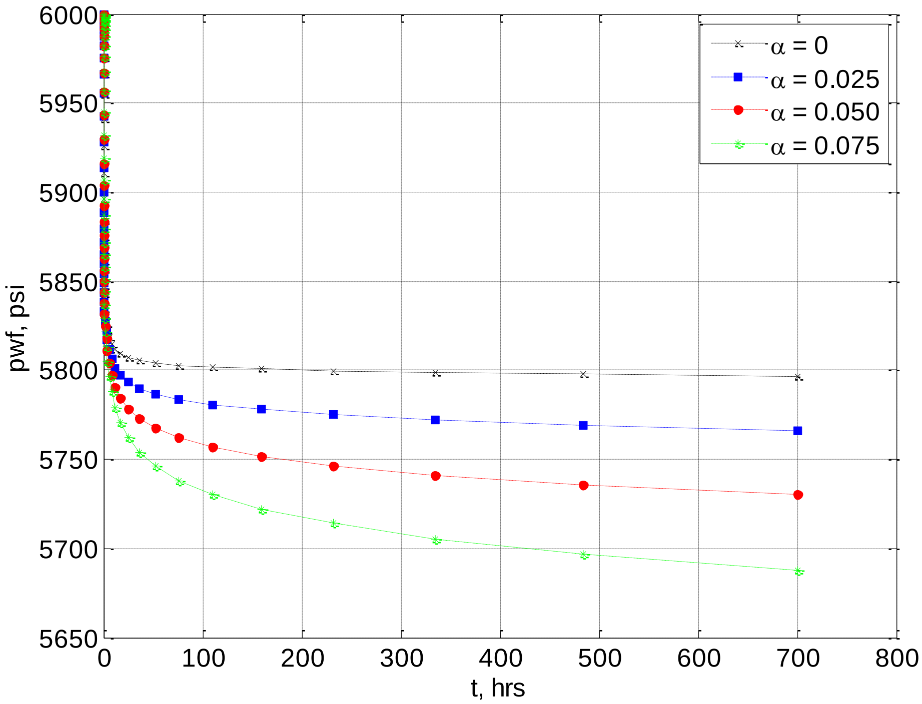

Figure 4.7: Bottom hole pressures versus time for M < 1 for infinite acting outer boundary

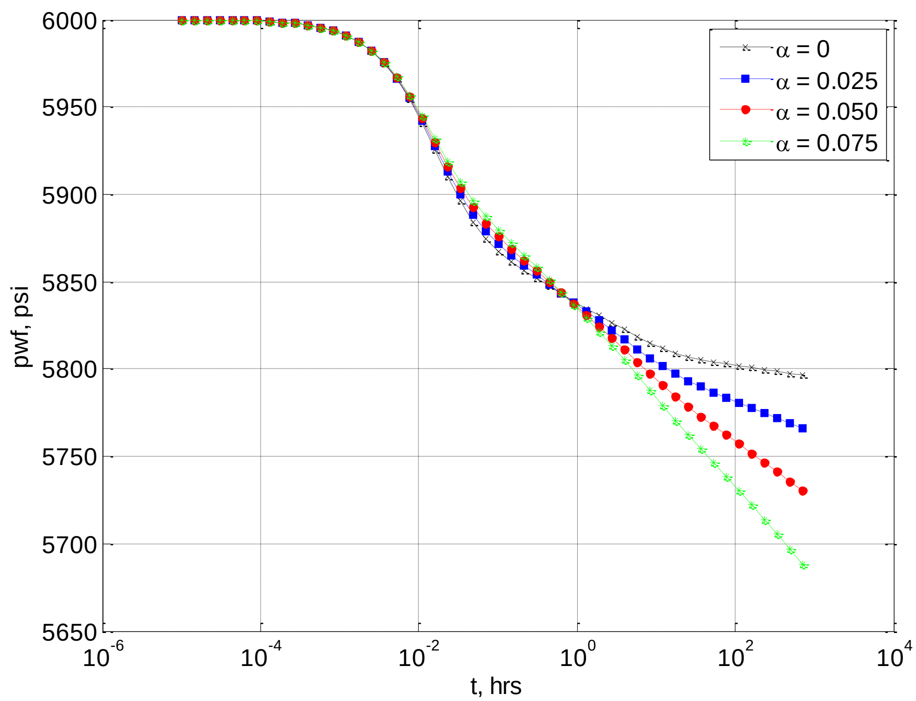

Figure 4.8: Semilog plot of bottom hole pressures versus time for M < 1 for infinite acting outer boundary

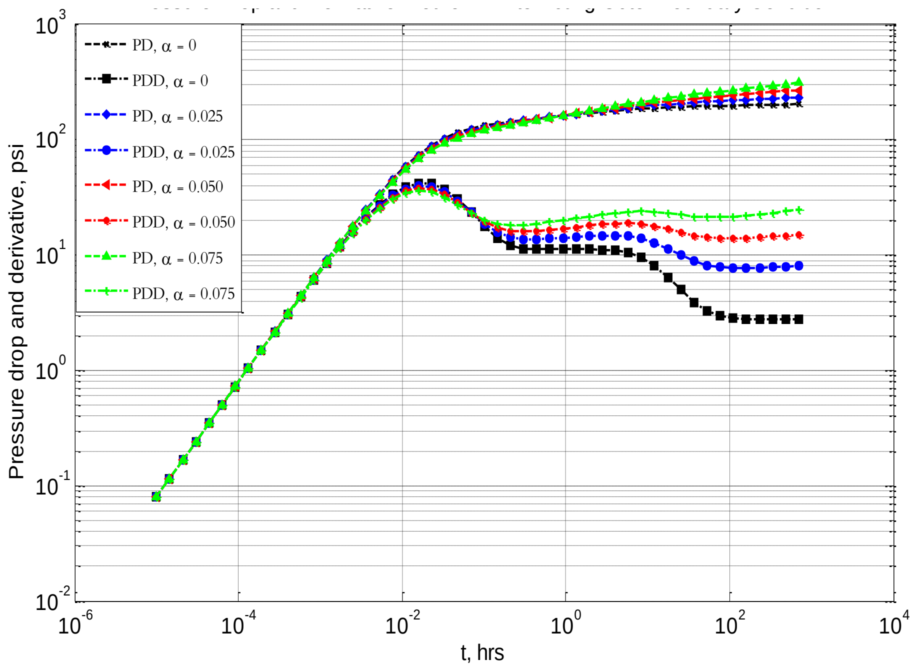

Figure 4.9: Pressure drop and pressure derivative for M < 1 for infinite acting outer boundary

4.6.2 Example 2: Two Region Radial Composite Reservoir with No-flow outer boundary condition

In this example, a two region radial composite reservoir with constant rate inner boundary condition and no-flow condition at the outer boundary is considered for the developing the graphical model of pressure and pressure derivatives. Memory effects are incorporated in terms of fractional derivative form. Wellbore storage and skin effects are also considered. These curves are plotted considering the fractional diffusion represented by memory parameter α. Figure 4.10, 4.11 & 4.12 shows the sensitivity of memory parameter α on pressure and pressure derivative plots for mobility ratio greater than one whereas Figures 4.13, 4.14 & 4.15 are for mobility ratio less than one, four different values of α are used for sensitivity analysis. The reservoir is initially at a pressure of 6000 psia and its thickness is 100 ft. A well with radius of 0.5 ft under constant production of 800 STB/D is considered. A logarithmic function is used to make time steps starting with time of about 10^{-6} and ending with about 1000 hrs with 1500 data points. Typical reservoir and well bore properties used for generation of pressure and pressure derivatives curves are shown in Table 4.2.

The pressure distributions are obtained from the numerical inversion of the solutions using Stehfest Algorithm and even integer ‘N’ is equal to 6 for calculations. It is observed that bottom hole flowing pressure is influenced by the memory parameter α from Figure 4.10 and 4.13. Semi log plot of bottom hole flowing pressure versus time in Figure 4.11 and Figure 4.14 shows that with increasing value of memory parameter α, p_{wf} is becoming higher i.e. slower pressure drop may be due to slow diffusion. This effect was somewhat described by Park (Park et al., 2001) in which they presented the justification of the additional pressure drop due to delay in diffusion. It is noticed that all pressure lines for four values of α plotted in Figure 4.11 and 4.14 merge together. Comparable effect was described by (Wang. Q. & Tong. D., 2009) on the flow analysis of viscoelastic fluid with fractional order derivative in horizontal well, after that pressure drops faster for larger value of α.

Figure 4.12 and 4.15 shows the pressure drop and pressure derivative plots for mobility ratio greater than one and less than one respectively. These distributions are plotted to see the effect of memory parameters. At the end of well bore storage, pressure derivative is showing radial flow regime for region one for α equals to zero, however it is noticed that as the value of α is increasing; the deviation from horizontal line is becoming more. Each of the four lines for different values of α crosses each other before the start of the radial homogeneous system. The pressure derivative curve shows higher pressure drop due to the value of the mobility ratio which is greater than one, this is because the high permeability value in region 1 whereas it is opposite in Figure 4.9 (Ambastha, 1988; Issaka, 1996). It is observed that the pressure drop is less for large values of α, then the pressure drop curves merge to a single one at one point. They become separated out once again at later stage; however at the later stage pressure drop in the reservoir is more for higher values of α for both cases of mobility ratio. Pressure derivative in the Figure 4.12 and 4.15 is much more sensitive to the memory parameter, as there are strong separations between the radial flow lines for different values of α. However all the lines of pressure drop and pressure merges to the unit slope line as the effect of outer boundary is reached. It is also noticed that for all the cases pseudo steady state is reaching, it can be observed in Figure 4.10 and Figure 4.13 as the slope of bottom hole pressure lines are almost constant that means these models can be used to calculate drainage volume and it will be affected much by changing α values.

Table 4.6: Reservoir and Wellbore parameters for two region radial composite reservoir with no-flow outer boundary condition

Parameters	**Value**
$\dot{k}_1$	500
μ_1	0.8
B_1	1.2
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

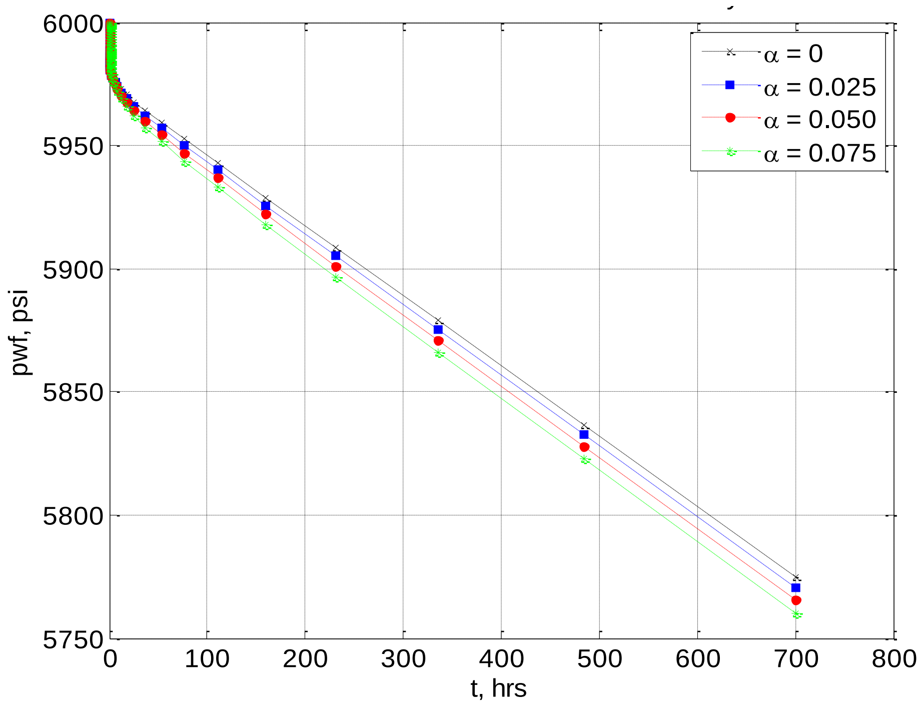

Figure 4.10: Bottom hole pressures versus for M > 1 for no flow outer boundary

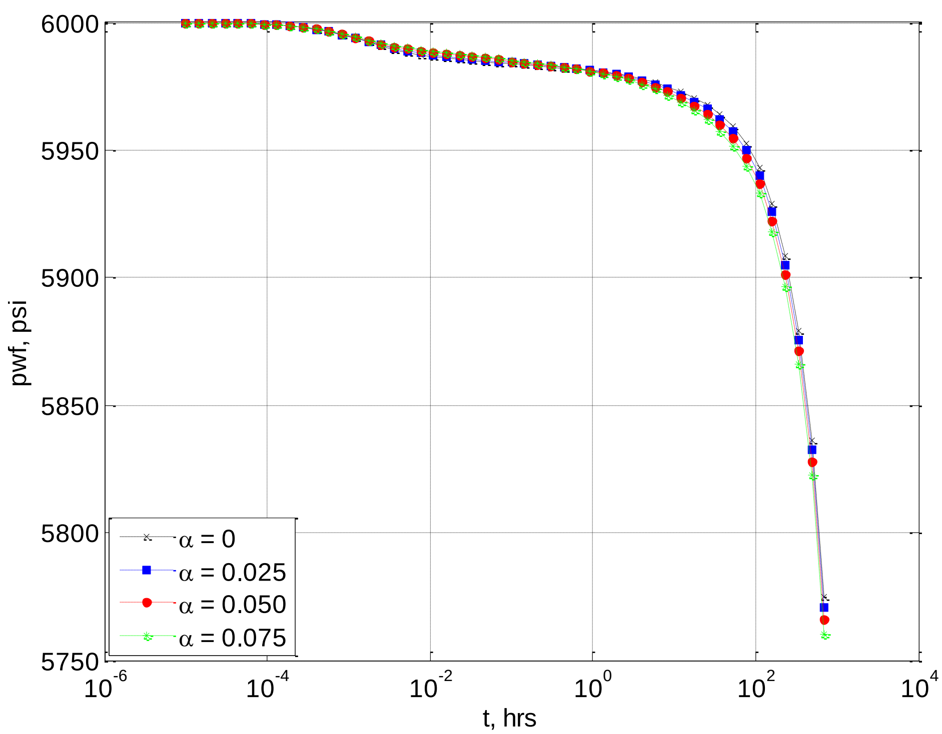

Figure 4.11: Semilog plot of bottom hole pressures versus time for M > 1 for no-flow outer boundary

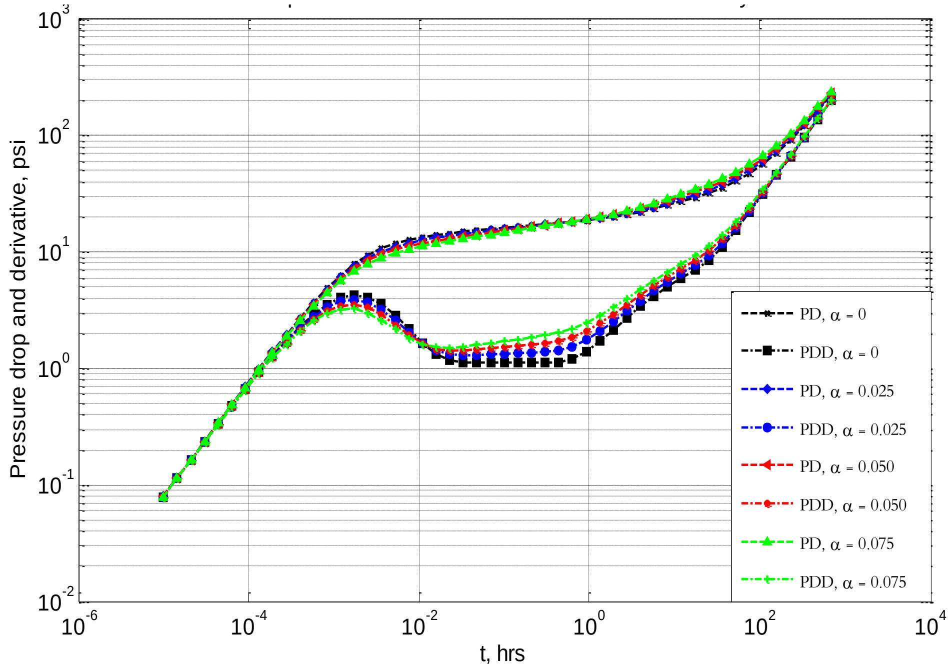

Figure 4.12: Pressure drop and pressure derivative versus time for M > 1 for no-flow outer boundary condition

Table 4.7: Reservoir and Wellbore parameters for two region radial composite reservoir with no-flow outer boundary (M < 1)

Parameters	**Value**
$\dot{k}_1$	50
μ_1	0.8
B_1	1.2
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	200
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

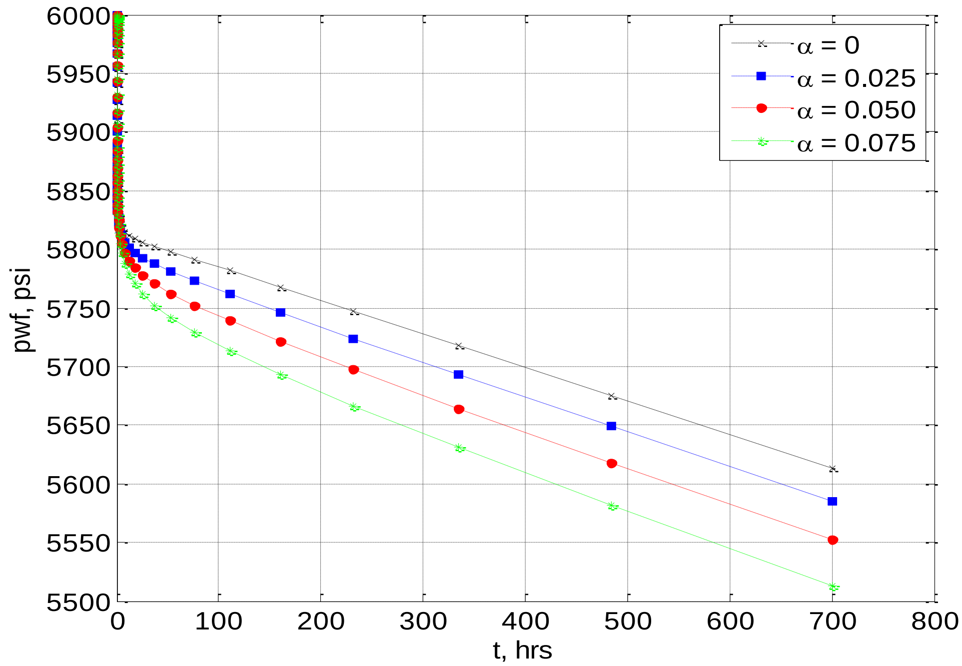

Figure 4.13: Bottom hole pressures versus time for M < 1 for no-flow outer boundary

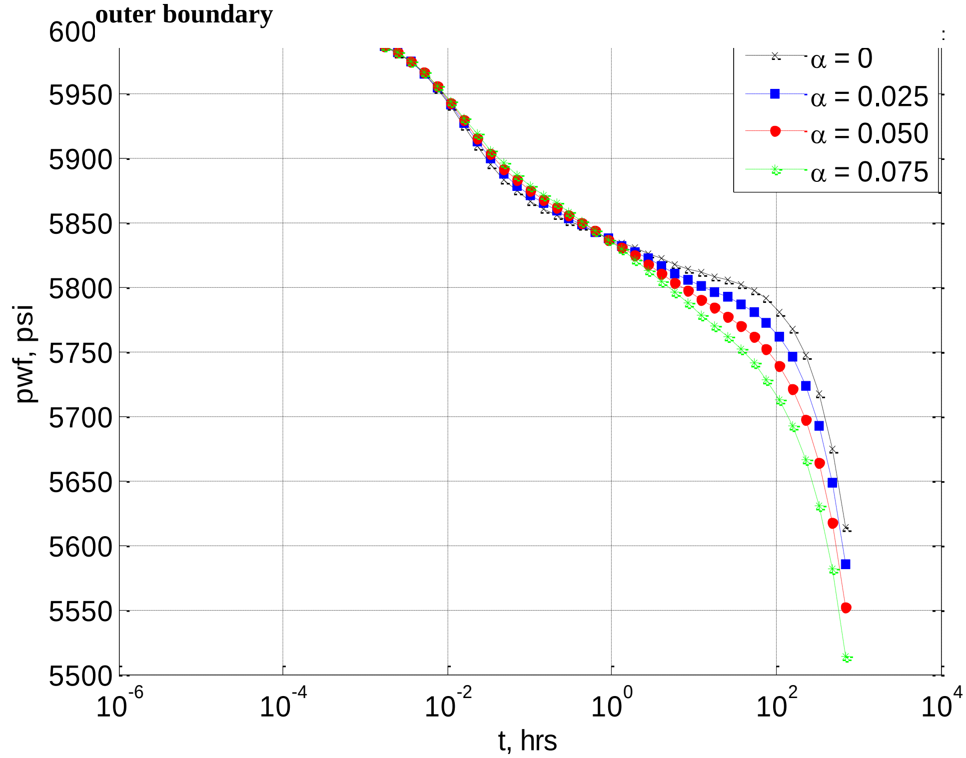

Figure 4.14: Semilog plot of bottom hole pressures versus time for M < 1 for no-flow outer boundary

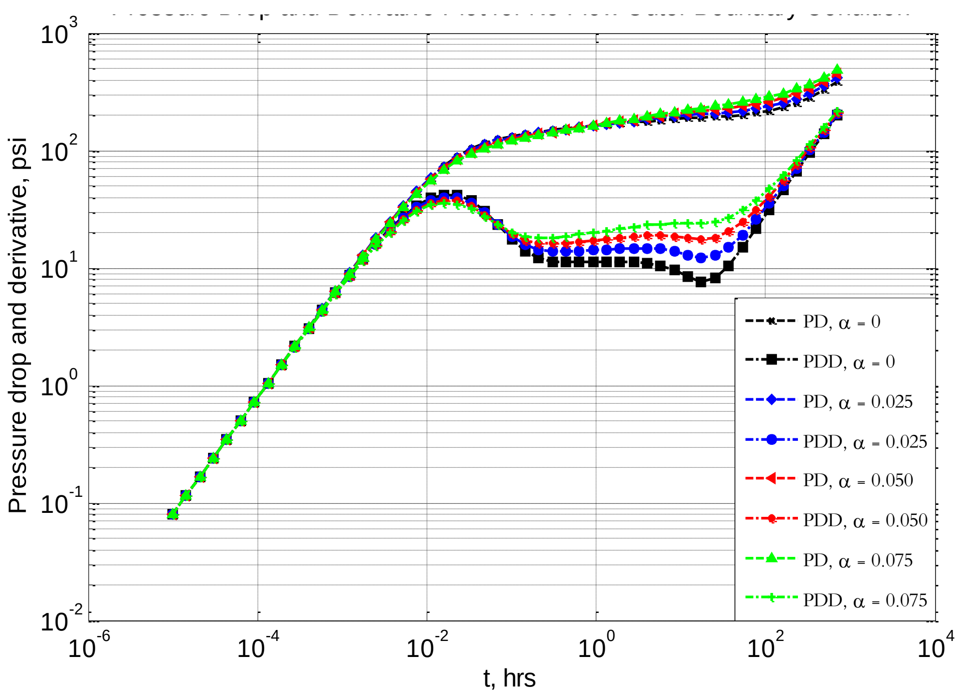

Figure 4.15: Pressure drop and pressure derivative versus time for M < 1 for no-flow outer boundary condition

4.6.3 Example 3: Two Region Radial Composite Reservoir with constant pressure outer boundary condition

This example considers a reservoir with two different regions but uniform in reservoir properties. A well with radius of 0.5 ft is considered in the center of the reservoir with constant rate production of 800 STB/D. At the outer boundary of the reservoir, constant pressure case is taken that is common due to the presence of strong aquifer. Two cases are presented i.e. for favorable and unfavorable mobility ratios. Bottom hole flowing pressure, pressure drop and pressure derivatives curves for first case are being plotted as shown in Figures 4.16, 4.17 and 4.18 whereas for the second case the plots are shown in Figures 4.19, 4.20 and 4.21. Wellbore storage and skin effects are also considered. These curves are plotted considering the fractional diffusion and sensitivity of memory parameter α is also being analyzed. The reservoir is initially at a pressure of 6000 psia and its thickness is 100 ft. A logarithmic function is used to make time steps starting with time of about 10^{-6} and ending with about 1000 hrs with 1500 data points. Typical reservoir and well bore properties used for generation of pressure and pressure derivatives curves for the two cases are shown in Table 4.8 and 4.9.

The pressure distributions are obtained from the numerical inversion using Stehfest Algorithm. It is evident from the Figure 4.16 and 4.19 that bottom hole flowing pressure is influenced by the memory parameter α. Semi log plot of bottom hole flowing pressure versus time gives a more clear view which shows that with increasing value of memory parameter α, p_{wf} is becoming higher i.e. slower pressure drop may be due to slow diffusion (Park et al., 2001). However eventually all lines for four values of α plotted in Figure 4.17 and Figure 4.20, the pressure lines merge together, after that pressure drops faster for the larger value of α. Figure 4.18 and Figure 4.21 shows the pressure drop and pressure derivative plots on log-log scale for both the cases, these distributions are plotted to see the effect of memory parameter. After the end of the well bore storage, radial homogeneous system is easily identifiable for the case α equals to zero from horizontal line having zero slope. It is noticed that as the value of α increases, pressure derivatives lines becomes deviated from horizontal line and deviation is more for large α values. The pressure derivative curve in Figure 4.18 shows higher pressure drop due to the value of the mobility ratio which is greater than one, this is because the high permeability value in region 1 whereas it is opposite in Figure 4.21 (Ambastha, 1988; Issaka, 1996). At initial stage pressure drop is slower for larger values of α, then pressure drop curves merges to a single one at one point. It can be seen at late time that pressure drop in the reservoir is much more for larger values of α. Pressure derivative curves in Figures 4.18 and 4.21 are much more sensitive to the memory parameter, as there are strong separations between the radial flow lines for different values of α. At late times, when boundary effect is reached the pressure derivative curve with α value equals to zero drops down and pressure drop curve flattens due to constant pressure at the boundary, theoretically larger time will be required for α values greater than zero to see the same effect.

Table 4.8: Reservoir and Wellbore parameters for two region radial composite reservoir with constant pressure OBC

Parameters	**Value**
$\dot{k}_1$	500
μ_1	0.8
B_1	1.2
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	50
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

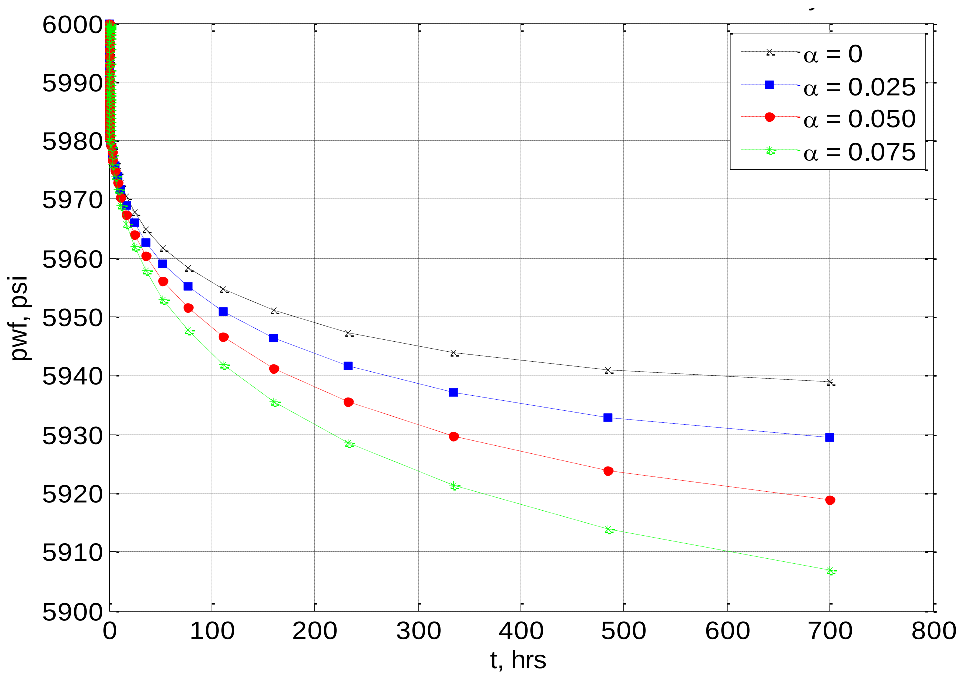

Figure 4.16: Bottom hole pressures versus for M > 1 for constant pressure outer boundary (Cartesian Plot)

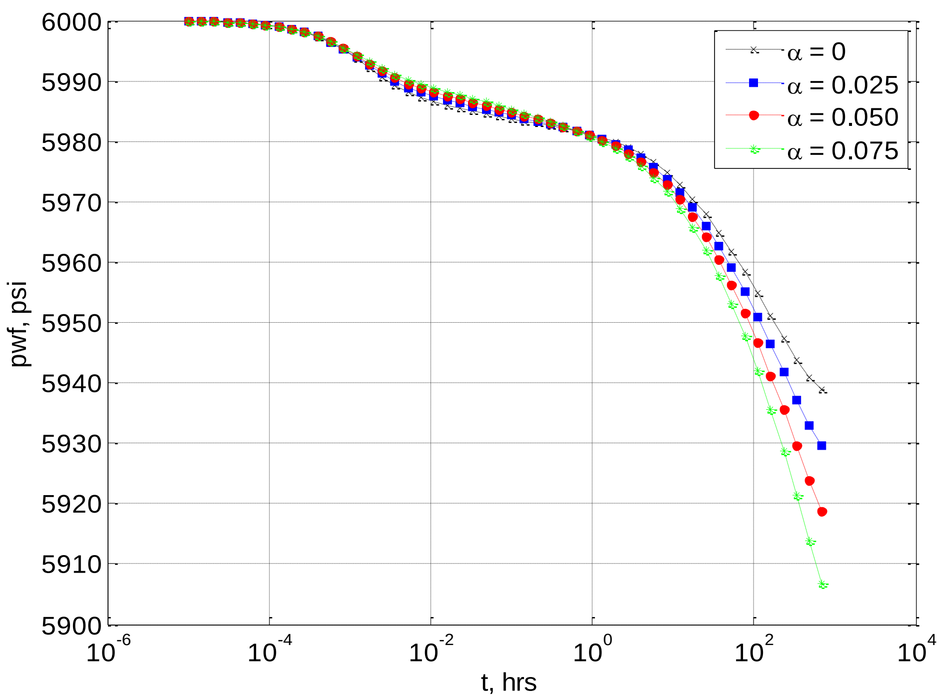

Figure 4.17: Semilog plot of bottom hole pressures versus time for M > 1 for constant pressure outer boundary

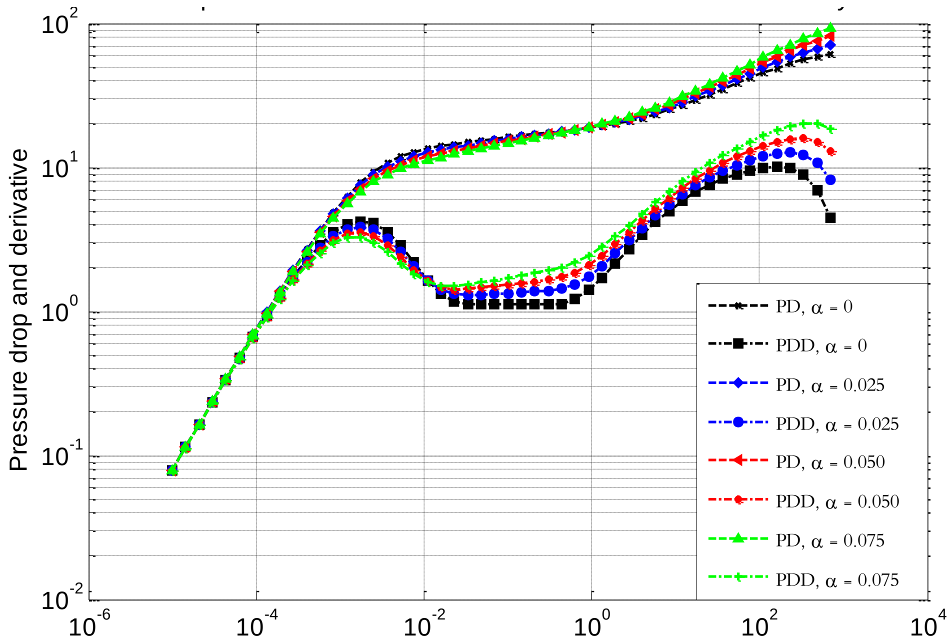

Figure 4.18: Pressure drop and pressure derivative for M > 1 for constant pressure outer boundary

Table 4.9: Reservoir and Wellbore parameters for two region radial composite reservoir with constant pressure OBC

Parameters	Value
$\dot{k}_1$	50
μ_1	0.8
B_1	1.2
c_{t1}	1e-5
φ_1	0.25
$\dot{k}_2$	200
μ_2	0.8
B_2	1.2
c_{t2}	1e-5
φ_2	0.25
r_f	400
r_e	1000
s	2
C	0.005
α	0, 0.025, 0.05, 0.075

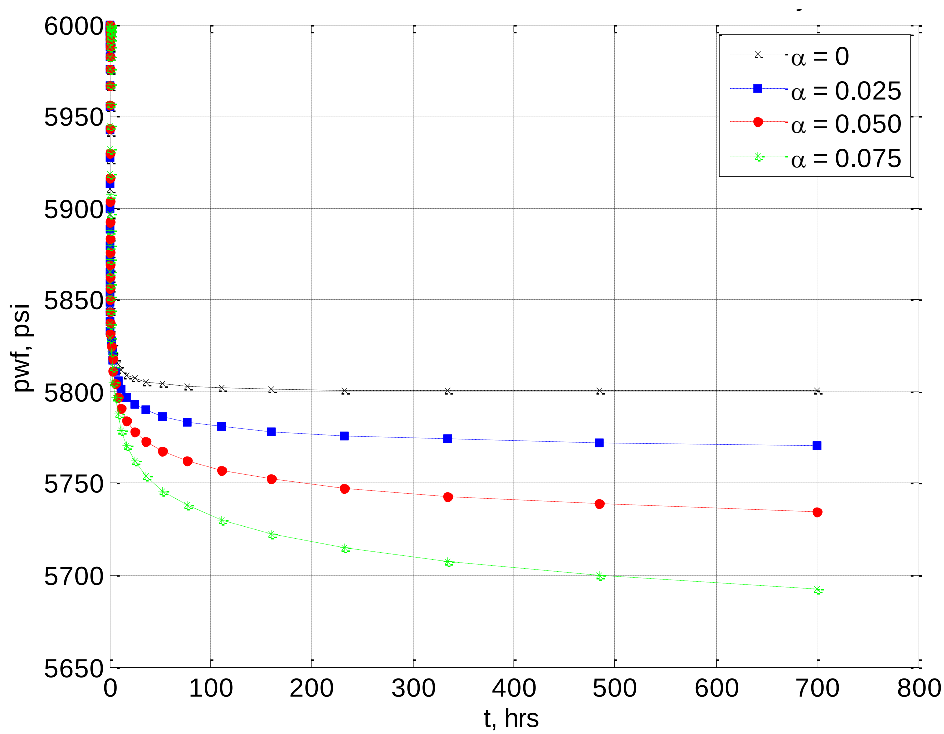

Figure 4.19: Bottom hole pressures versus for M < 1 for constant pressure outer boundary (Cartesian Plot)

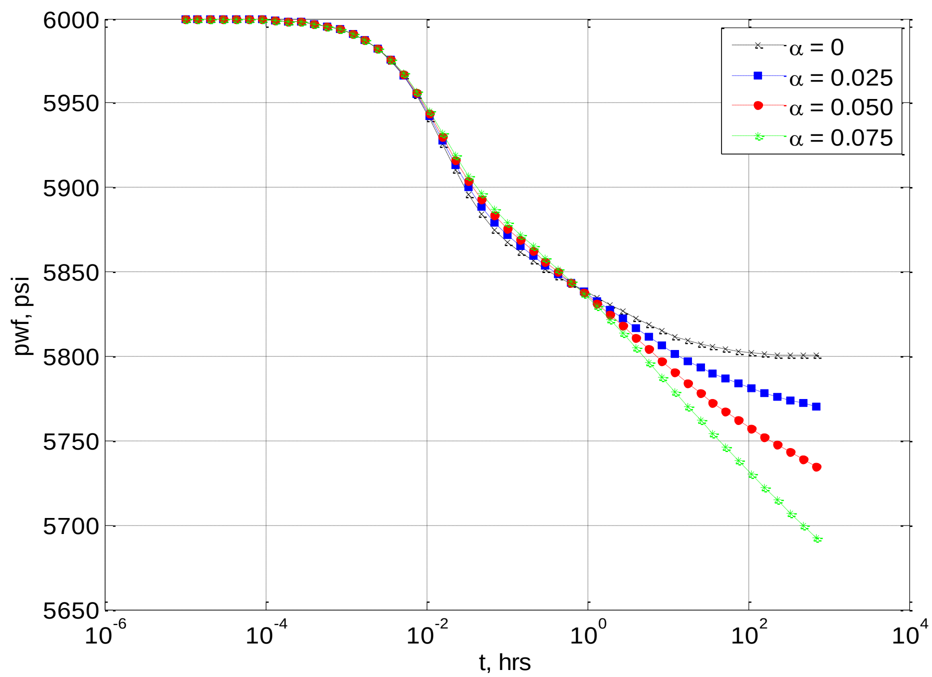

Figure 4.20: Semilog plot of bottom hole pressures versus time for M < 1 for constant pressure outer boundary

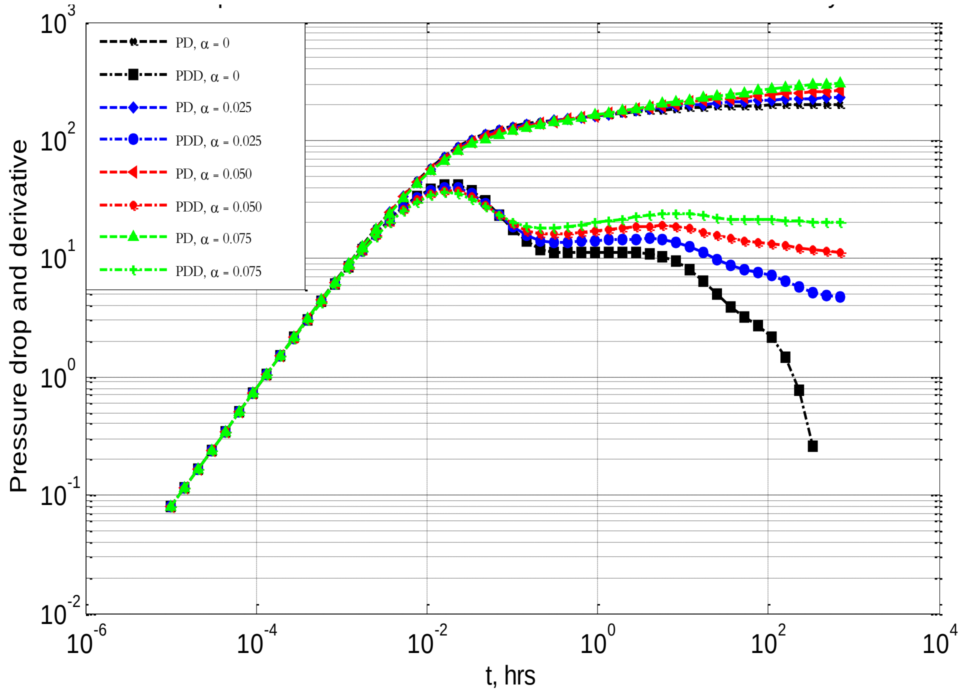

Figure 4.21: Pressure drop and pressure derivative for M< 1 for constant pressure outer boundary

CHAPTER 5

RESREVOIR PARAMETER ESTIMATION

Pressure transient analysis is mainly used to estimate the reservoir and well bore properties by identification of the reservoir system from the measured pressure response. For pressure transient analysis, a mathematical model is used that gives the same output pressure response of the actual reservoir system. This method is called as inverse analysis that has non uniqueness inherently.

Each reservoir system performs differently so a unique mathematical model is required for every reservoir system. However, due to limitations in modeling the diffusive nature of the pressure response in well test analysis, only a fixed number of mathematical models are available for studying the reservoir system. A number of theoretical explanations have been given on successes of well test analysis in real field experiences.

It is obvious that measured pressure data (the actual pressure response from the field) cannot be the same as the pressure response computed using a mathematical model because of the measurement errors and the simplified nature of model (Watson et *al.*, 1988). Nowadays measurement errors are greatly reduced by the use of advance electronic gauges that give accurate pressure measurements. On the other hand, modeling error is always present in the analysis due to simplicity and assumptions considered in development of a mathematical model.

The inconsistency between the measured pressure data (observed data) and the calculated pressure response from the model is intrinsic in well test analysis. These errors and non-uniqueness due to inverse nature of the problem is inherited. Hence, the final solution of the inverse problem is to find the most suitable model which gives the minimum error between measured pressure response and model pressure.

Nonlinear regression technique is being used in modern well testing for parameter estimation (Dastan, 2010). This technique became the standard industry practice in early 90's after the era of type curves. Nonlinear regression is also known as automated type curve matching. In this technique, the objective is to minimize the sum of squares of the difference between the observed pressure data and the model pressures. However this technique has disadvantage of getting trapped in local minima which is usually in the vicinity of initial guess.

5.1 Parameter Estimation using Levenberg-Marquardt Algorithm

In this section, we compute the reservoir and well bore parameters including the memory parameter α for three synthetic cases using Levenberg-Marquardt Algorithm. All the examples involved single-phase flow i.e. oil flow in the reservoir with a constant production well located at the center of a circular reservoir. The production of the well is 800 STB/D for all cases. The reservoir has a thickness of 100 ft and has two regions with different but homogeneous properties. These properties and reservoir external boundaries vary for each example. In all examples, there are 1500 data points to be matched. The objective function used in calculating various well and reservoir parameters is L2-norm. Evaluating the L2-Norm (also called as sum of error squares) for each possible solution requires calculating a model pressure. The results of model pressure are then used to compute the L2-Norm as follows:

$$Error = \sum_{\mathrm{i}=1}^{N_t} \frac{\left[p_{\mathrm{model}}\left(t_{\mathrm{i}}, k, s, C\right) - p_{\mathrm{measured}}\left(\mathrm{t_i}\right)\right]^2}{N_t}$$

Where p_{model} is calculated from the user's provided model p_{measured} is the measured pressure obtained from field data and. N_t is the total number of data points which is 1500 for all cases. The algorithm is run for two different sets of parameters, in one case parameters are estimated assuming some initial guess of memory parameter and in second case alpha is assumed to be zero.

5.1.1 Example 1: Two Region Radial Composite Reservoir with Infinite Acting Boundary Condition

In this example, we matched the pressure data generated using a two region radial composite reservoir with constant rate inner boundary condition and infinite acting external boundary condition also considering the fractional diffusion. Forward model (observed data) is shown in Figure 5.1. Wellbore storage and skin effects are also considered. Forward models are generated using solutions from previous chapter containing memory parameter α. Also Gaussian random noise is added into the final model calculations to mimic the field conditions. The reservoir is initially at a pressure of 6000 psia. A well under constant production of 800 STB/D is considered. Figure 5.2 shows the matched pressure data with the observed data. The Final match with consideration of α i.e. the case as shown in Figure 5.2 considering fractional diffusion gave very good match to the observed data and the values are identical to true values with very less value of error. However if we neglect α, it can be seen that the match is not good. The values are far from the true values and giving a high value of error compared to the case considering alpha in its calculation. So in case where anomalous diffusion is expected to occur, parameter estimation using normal diffusion may not give true results.

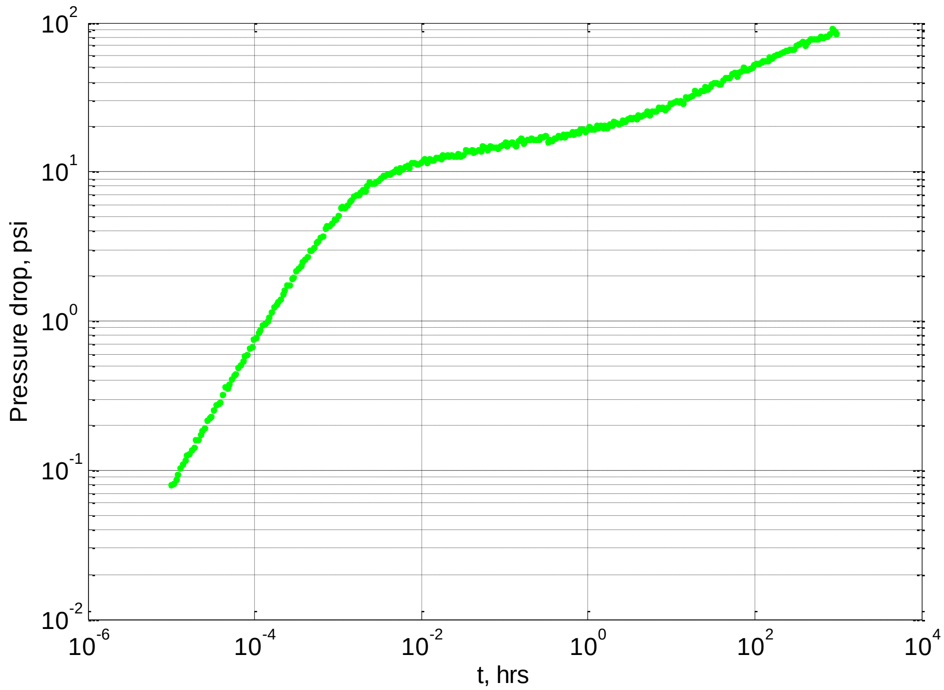

Figure 5.1: Forward Model for infinite acting case with noise

Table 5.1: Estimated reservoir and well bore parameters for infinite acting case

Parameters	**True Values**	**Initial guess**	**Final Match**	**Error**	**Initial guess**	**Final Match**	**Error**
		Fractional diffusion			**Normal diffusion**		
$\dot{k}_1$	500	560	500	0.5798	550	338.65	0.6654
$\dot{k}_2$	50	175	50		80	32.14	
r_f	400	170	400		300	407.95	
s	2	3	1.99		2.5	-0.426	
C	0.005	0.059	0.005		0.06	0.0039	
α	0.05	0.2	0.05		-	-	

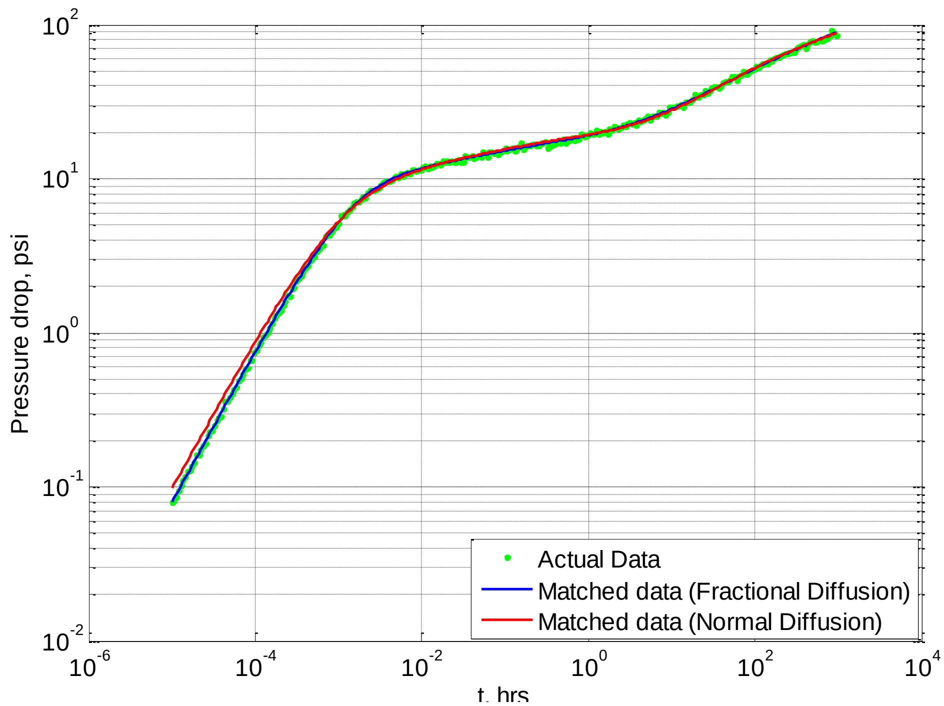

Figure 5.2: Final Matched data for infinite acting reservoir

5.1.2 Example 2: Two Region Radial Composite Reservoir with No-Flow Outer Boundary Condition

In this example, we matched the pressure data generated using a two region radial composite reservoir with constant rate inner boundary condition and no-flow external boundary condition also considering the fractional diffusion. Forward model (observed data) is shown in Figure 5.3. Wellbore storage and skin effects are also considered. Forward models are generated using solutions from previous chapter containing memory parameter α. Also random noise is added into the final model calculations to mimic the field conditions. The reservoir is initially at a pressure of 6000 psia. A well under constant production of 800 STB/D is considered. The Final match with consideration of α i.e. the case as shown in Figure 5.4 considering fractional diffusion gave very good match to the observed data and the values are identical to true values with very less value of error. However if we neglect α, it can be seen that the match is not good as compared to the parameters estimated considering α. The values are far from the true values and giving a high value of error compared to the case considering alpha in its calculation. So in case where anomalous diffusion is expected to occur, parameter estimation using normal diffusion may not give true results. It can also be seen that value of $\dot{k}_1$ comes out to be 643 $md/\sec^{\alpha}$ where as its true value is 600 $md/\sec^{\alpha}$, and skin value from inverse analysis came out to be 1.51 however its true value is 1.0. This shows the non-uniqueness that is inherent in parameter estimation from inverse analysis. Initial guesses are taken as random values for both the cases of fractional and normal diffusion. Figure 5.4 shows the matched pressure data with the observed data. The distortion in pressure drop at early time data is due to the presence of negative skin that cause numerical inversion problems.

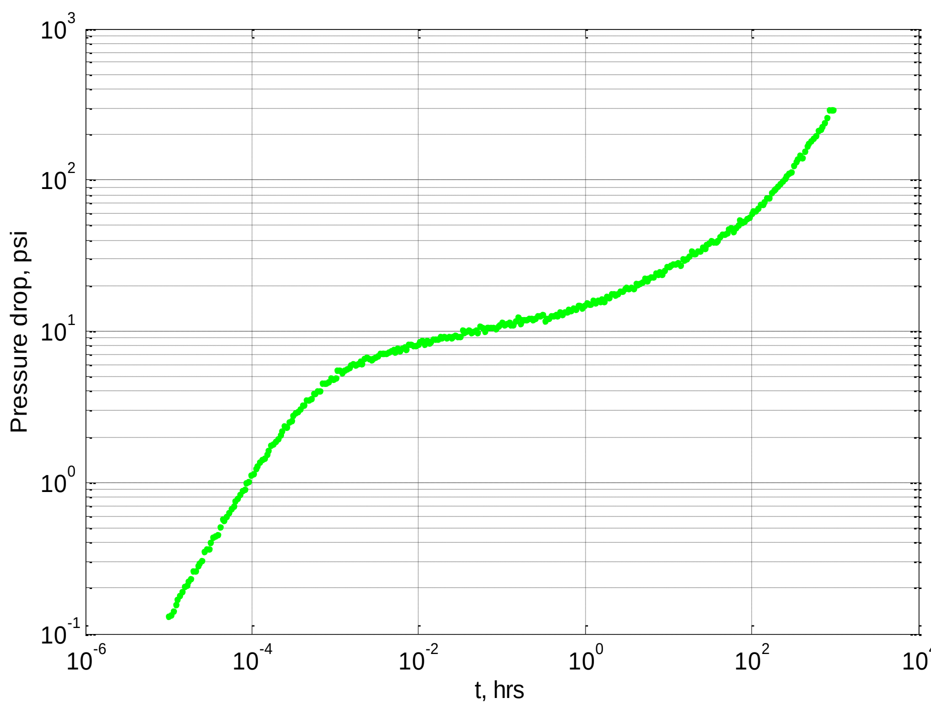

Figure 5.3: Forward Model for no-flow outer boundary case with noise

Table 5.2: Estimated reservoir and well bore parameters for no-flow outer boundary case

Parameters	True Values	Initial guess	Final Match	Error	Initial guess	Final Match	Error
		Fractional diffusion			Normal diffusion		
$\dot{k}_1$	600	560.67	643.20	2.439	550	469.98	2.565
$\dot{k}_2$	50	31.60	55.81		60	33.24	
r_f	300	83.41	301.53		250	294.94	
s	1	1.21	1.51		0.5	-0.568	
C	0.003	0.0052	0.0031		0.0055	0.0024	
r_e	1000	754.55	1003.27		800	990.34	
α	0.05	0.0223	0.0653		-	-	

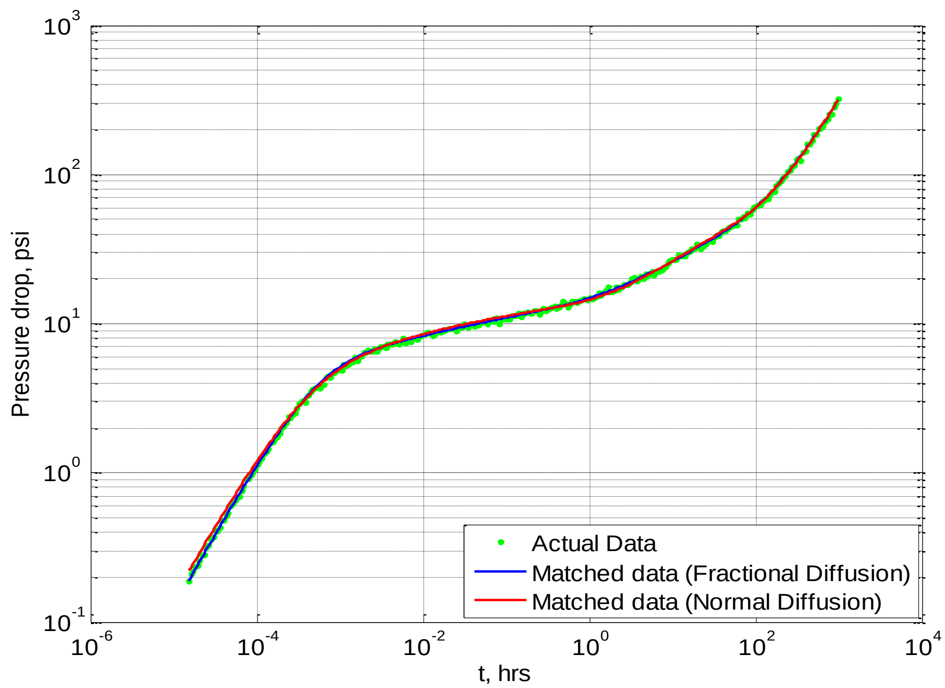

Figure 5.4: Matched data for no-flow Outer Boundary Condition reservoir

5.1.3 Example 3: Two Region Radial Composite Reservoir with Constant Pressure Outer Boundary Condition

In this example, we matched the pressure data generated using a two region radial composite reservoir with constant rate inner boundary condition and constant pressure external boundary condition also considering the fractional diffusion. Forward model (observed data) is shown in Figure 5.5. Wellbore storage and skin effects are also considered. Forward models are generated using solutions from previous chapter containing memory parameter α. Also random noise is added into the final model calculations to mimic the field conditions. The reservoir is initially at a pressure of 6000 psia. A well under constant production of 800 STB/D is considered. Figure 5.6 shows the matched pressure data with the observed data. Final match with consideration of alpha diffusion equations case as shown in Figure 5.6 gave very good match to the observed data and the values are identical to true values with very less value of error. However if we neglect alpha, it can be seen that the match is not good. The values are far from the true values and giving a high value of error compared to the case considering alpha in its calculation. Initial guesses are taken as random values for both the cases of fractional and normal diffusion.

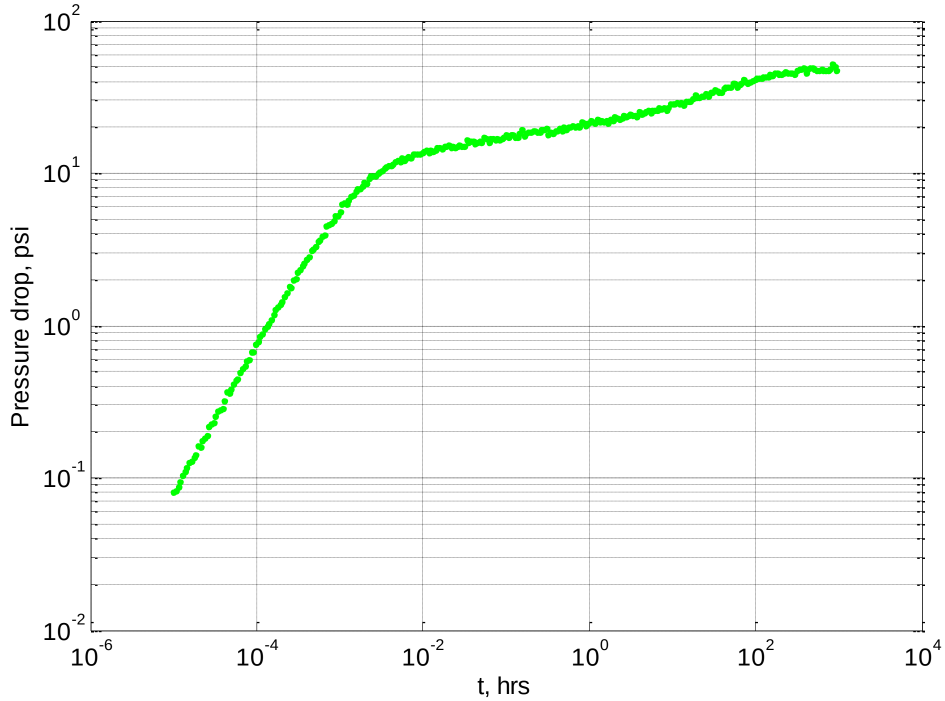

Figure 5.5: Forward Model for Constant Pressure outer boundary case with noise

Table 5.3: Estimated reservoir and well bore parameters for constant pressure outer boundary case

Parameters	**True Values**	**Initial guess**	**Final Match**	**Error**	**Initial guess**	**Final Match**	**Error**
		Fractional diffusion			**Normal diffusion**		
$\dot{k}_1$	500	422.85	489.94	0.351	550	366.79	0.386
$\dot{k}_2$	100	217.49	98.22		75	79.13	
r_f	500	25.11	503.20		350	374.07	
s	3	3.02	2.82		2.2	0.6139	
C	0.005	0.0153	0.0049		0.003	0.0044	
r_e	2000	1012.67	2025.74		1500	2825.41	
α	0.05	0.0193	0.0480		-	-	

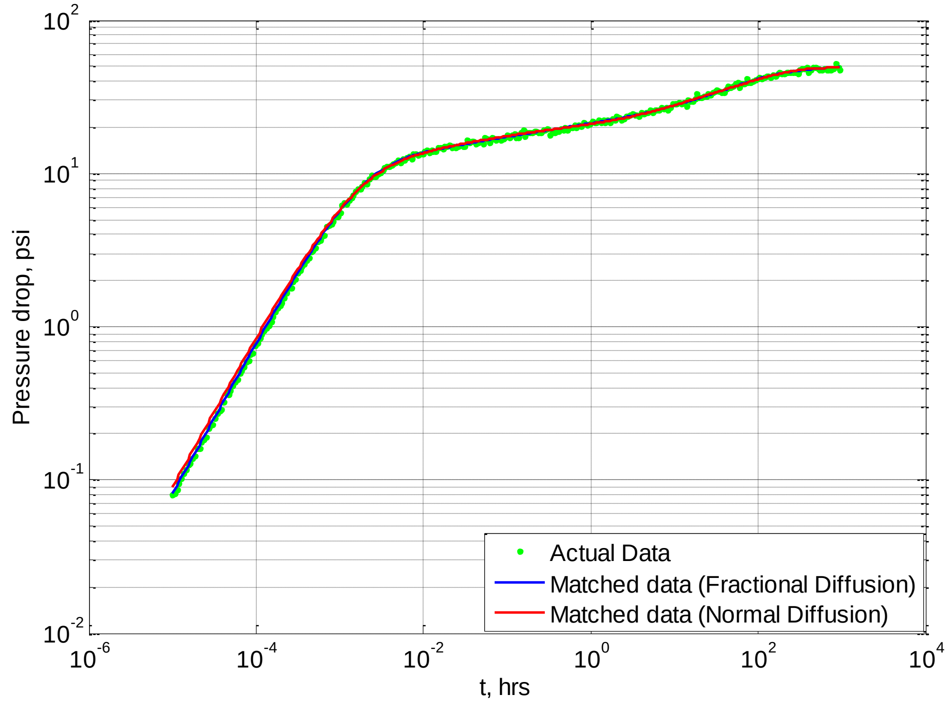

Figure 5.6: Matched data for Constant Pressure Outer Boundary Condition

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

This study has presented fractional diffusion models (analytical solutions) for two-region radial composite reservoirs. The most commonly encountered inner boundary i.e. constant rate production is considered. The external boundary condition can be infinite acting, no-flow or constant pressure. The solutions obtained include wellbore storage and skin effects. The ultimate goal of this research is to study transient pressure data and to estimate reservoir and well bore parameter in the presence of fractional diffusion. The effect of memory parameter α on pressure transient data has also been investigated.

Equations for fractional diffusion have been developed for two region composite reservoirs. Pressure transient analysis in conjunction with these models will constitute a significant addition to well test analysis methods for composite reservoirs in radial geometry. Reservoir parameters estimated from these models will have a greater degree of confidence interval. In summary, we may draw following conclusions from this study:

1. The bottom hole pressure is a function of various reservoir and well bore parameters. However, the memory parameter α also affects its value.
2. Small values of α will cause noticeable effect on the pressure drop. Based on observed pressure transient data from real fields, we do not expect the value of α

to be significantly greater than zero. Thus, the analysis in this work has been done based on a range of α from 0 to 0.075.

3. In this study, we have developed solutions to anomalous diffusion of slightly compressible fluids in porous media. The solutions developed account for both wellbore storage and skin effects.
4. Once the formalism proposed in this research is validated using experimental data, the results obtained in this research can be used for better reservoir description than those from normal diffusion.

6.2 Recommendations

Following recommendations can be made from this research:

1. The model developed in this research defines the analytical description of pressure transient behavior of two region radial composite reservoir. To further confirm the validity of the memory formalism proposed in this study, laboratory experiment is required.
2. Modified Darcy's law can be used in models other than radial composite reservoir such dual porosity, dual permeability, multilayered, hydraulically fractured reservoirs to see their pressure transient behavior in the presence of memory.

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