GLOBAL STABILIZATION OF DECENTRALIZED CONTROL SYSTEMS SUBJECT TO SATURATION

BY

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Dedicated to my wonderful parents,

M. Mahmood Hussain and Syeda Fatima Asra Mahmood

and my ever loving brothers,

M. Mahamid Hussain and M. Muzammil Hussain

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THESIS ABSTRACT

Name: Mohammed Masood Hussain.

Title: Global Stabilization of Decentralized Systems subject to Saturation.

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Practical systems are difficult to realize for advanced control techniques, both analysis and design, due to issues like saturation and interconnections. This thesis proposes techniques for designing controllers locally and then globally for linear decentralized control systems subject to input saturation using static and dynamic output feedback designs and H_∞ control design.

The analytical work presented is the development of overlapping decentralized control for both static and dynamic output feedback designs through LMI formulation. Further more using the idea of homotopy method a H_∞ control design for the same system was developed. Both the schemes are developed on the framework of Linear Matrix Inequalities(LMIs). A version of the inclusion principle was used for expanding and contracting the systems for overlapping decentralized control. An application of the proposed design scheme for static and dynamic output feedback was done on the Nuclear Power Plant model comprising of four subsystems. The proposed scheme showed promising results. For the H_∞ controller design on linear interconnected systems subject to input saturation the

idea of homotopy was used and the application was done on a Multi-zone Space Heating System comprised of three subsystems. The results for the H_∞ were compared to the developed decentralized LQR control design which were found to be favorable and better on comparison.

ملخص الرسالة

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النظم العملية من الصعوبة معرفتها بالنسبة الى تقنيات التحكم المتقدمة التحليل والتصميم معاً ، نتيجة الى مواضيع مثل التشبع و الترابط . ان غرض تقنيات هذه الأطروحة لتصميم متحكمات محلياً ومن ثم عالمياً بخصوص انظمة التحكم في اللاتركيزية الخطية مجال التشبع المغنطيسي باستخدام تصاميم تغذية استرجاعية و تصميم تحكم H∞

العمل التحليلي المقدم هو تطوير يتوافق مع التحكم اللاتركيزي للتصاميم الاستاتيكي والدينميكي من خلال صيغة ال ام آى . بالاضافة الى استخدام فكرة طريقة التماثل تصميم تحكم H∞ لنفس النظام تم تطويرها. المشاريع الاثنين تم تطويرهما في نطاق عدم تساوي المصفوفة (ال ام آى اس) تم استخدام نص متضمن القاعدة لنشر و تعاقد الانظمة في تحكم نطاق اللامركزية . تم تطبيق مشروع التصميم المقترح على ناتج التغذية الاسترجاعية الاستاتيكية والدينميكية على محطة طاقة نووية تتكون من ثلاثة انظمة فرعية. المشروع المقترح اظهر نتائج واعدة. بالنسبة الى تصميم تحكم H∞ على الانظمة المتداخلة الخاضعة الى ناتج الانظمة المشبعة استخدمت فكرة هومو توبي وتم التطبيق على نطاق مزدوج نظام تسخين يتألف من ثلاثة انظمة فرعية. النتائج بالنسبة الى H∞ بالمقارنة الى تصاميم اللامركزية المطورة التى وجدت حسنة ومفضلة بالمقارنة.

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Chapter 1

INTRODUCTION

1.1 Overview

Saturation is present in many physical systems and it tends to degrade the performance and may also destabilize the system. The research on decentralized control of interconnected systems has been a topic in research since late 1980's and still is done till the present day. Previously the research focused mainly on the stability of the interconnected system based on control design like state feedback, output feedback and later developed to linear quadratic regulator(LQR) problem. The research on large scale interconnected systems was initiated by

Wang and Davison [5] and since then has been the subject of intense study. Most recently the study on decentralized control has renewed interest because of its fundamental role in the problem coordinating motion of multiple autonomous agents which by itself has attracted significant attention.

Previously the study of stabilization using decentralized control was that of fixed modes. Fixed modes are the poles of the system which cannot be shifted just by using any decentralized feedback controllers. The same idea on fixed modes was done by Wang and Davison [5] who also showed that decentralized stabilization is possible if and only if the fixed modes are stable. Our perspective in this thesis is when the interconnected systems are subject to input saturation. It is known that every physically conceivable actuator has bounds on its output. Valves can only be operated between fully open and fully closed states, pumps and compressors have finite throughput capacity and tanks can only hold a certain volume. Ignoring such saturation elements in any control system can be detrimental to the stability and performance of the controlled system.

To overcome the problem of the interconnected systems when subject to input saturation and their effects on the stability and performance of the system we will, in this thesis, be designing a new decentralized control design for each local controller in the subsystem of the interconnected system and using the idea of inclusion principle and overlapping control globally stabilize the actual system.

1.2 Thesis Objective

The main objective of the thesis is to

- Design a decentralized controller for the interconnected system subject to input saturation which will locally stabilize each subsystem and with the idea of inclusion principle and overlapping design have global stability.
- Design of decentralized H_∞ controller using a method of homotopy to improve the stability of the system.

1.3 Problem Statement

In this thesis, we consider an LTI interconnected system composed of a finite number N of coupled subsystems and subject to input saturation is represented by:

$$\dot{x}(t) = Ax(t) + Bsat(u(t)) + h(t, x(t)) \tag{1.1}$$

$$y(t) = Cx(t) \tag{1.2}$$

where $x = [x_1^t, ..., x_N^t]^t \ \in \Re^n, \ n = \sum_{j=1}^N \ n_j$ is the overall system state, $sat(u) = [sat(u)_1^t, ..., sat(u)_N^t]^t \ \in \Re^m, \ m = \sum_{j=1}^N \ m_j$ is the saturated input of the overall system and $y = [y_1^t, ..., y_N^t]^t \ \in \Re^p, \ p = \sum_{j=1}^N \ p_j$ is the measured output of the overall system. The model matrices are $A = diag\{A_{11}, .., A_{NN}\}, \ A_{jj} \in \Re^{n_j \times n_j}$,

$B = diag\{B_1, .., B_N\},\ B_j \in \Re^{n_j \times m_j}$ and $C = diag\{C_1, .., C_N\},\ C_j \in \Re^{p_j \times n_j}$. The function

$$h(t, x(t)) = [h_1^t(t, x(t)), .., h_N^t(t, x(t))]^t$$

is a vector function piecewise-continuous in its arguments, it represents the coupling of the system. In the sequel, we assume that this function is uncertain and the available information is that, in the domains of continuity $\mathbf{G}$, it satisfies the global quadratic inequality

$$h^t(t, x(t))h(t, x(t)) \ \leq \ x^t(t)\widetilde{R}^t\widetilde{\Phi}^{-1}\widetilde{R}x(t) \tag{1.3}$$

where $\widetilde{R} = [\widetilde{R}_1^t, .., \widetilde{R}_N^t]^t,\ \widetilde{R}_j \in \Re^{r_j \times n}$ are constant matrices such that $h(t, 0) = 0$ and $x = 0$ is an equilibrium of system (1.1). With focus on the structural form of system (1.1), the jth subsystem model can be described by

$$\begin{aligned} \dot{x}_j(t) &= A_{jj}x_j(t) + B_j\mathbf{sat}(\mathbf{u_j})(t) + h_j(t, x) \\ y_j(t) &= C_jx_j(t) \end{aligned} \tag{1.4}$$

where $x_j(t) \in \Re^{n_j},\ u_j(t) \in \Re^{m_j},\ y_j(t) \in \Re^{p_j}$ are the subsystem state, input and measured output, respectively. The function $h_j \in \Re^{n_j}$ is a piecewise-continuous vector function in its arguments and in line of (1.3) it satisfies the quadratic inequality

$$h_j^t(t, x(t))h_j(t, x(t)) \ \leq \ \phi_j^2\, x^t(t)\widetilde{R}_j^t\widetilde{R}_jx(t) \tag{1.5}$$

where $\phi_j > 0$, $j \in \{1, ..., N\}$ are bounding parameters such that $\widetilde{\Phi} = diag\{\phi_1^{-2} I_{r_1}, .., \phi_N^{-2} I_{r_N}\}$ where $I_{m_j} \in \Re^{m_j \times m_j}$ represents identity matrix. From (1.3) and (1.5), it is always possible to find a matrix $\widetilde{\Phi}$ such that

$$h^t(t, x(t))h(t, x(t)) \leq x^t(t)R^t\widetilde{\Phi}^{-1}Rx(t) \tag{1.6}$$

where $R = diag\{R_1, .., R_N\}$. The saturation function $sat(u_j)$ is for $u \in \Re^m$ defined as,

$$\mathbf{sat}(\mathbf{u_j}) = \begin{cases} u_{jmax} & u_j \geq u_{jmax}, \\ u_j & u_{jmin} < u_j < u_{jmax}, \\ u_{jmin} & u_j \leq u_{jmin} \end{cases} \tag{1.7}$$

where u_{jmin} and u_{jmax} are chosen to correspond to actual input limits either by measurement or by estimation. Input saturation can also be applied as upper and lower limits of input constraints as u_{jmin} and u_{jmax}, respectively. It is also assumed that the pair (A_{jj}, B_j) is a controllable pair and (C_j, A_{jj}) is an observable pair for all $j \in I := 1, 2, \ldots, N$.

In this thesis we will design the controllers that achieves the objectives stated in section(1.2) and also solve the above described problem.

More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks, see [1] for a nice overview of application cases requiring a formal treatment of the saturation constraints.

This chapter addresses the latest research topics and theoretical advances on linear dynamical systems with saturation. The chapter is organized with the following sections showing the problem statement and theorems as well as assumptions used in all the literature addressed and Section III with concluding remarks.

2.2 Input Saturation

In this section we will be considering the important case of systems subject to Input/Actuator saturation. Other cases of systems subject to state saturation, systems subject to output saturation and systems presenting nested saturation will be dealt with in the sequel.

1.4 Thesis Organization

The thesis is organized as the chapter 2 showing the previous work done on systems subject to saturation and also the type of control design on them. In chapter 3 it shows the overlapping control design of interconnected systems subject to saturation with the idea of inclusion principle. In chapter 4 a H_∞ control design method will be used with an idea of homotopy method. In the following section conclusions will be drawn and directions for future research.

Notations: The Euclidean norm $|.|$ is used for vectors in the n-dimensional space $\Re^n$ and we denote by $||.||$ the corresponding induced matrix norm in $\Re^n$. We use W^t, W^{-1}, $\lambda_m(W)$ and $\lambda_M(W)$ to denote the transpose, the inverse, the minimum eigenvalue and the maximum eigenvalue of any square matrix W, respectively. We use $W < 0$ (≤ 0) to denote a symmetric negative definite (negative semidefinite) matrix W and I_j to denote the $n_j \times n_j$ identity matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry. Sometimes, the arguments of a function will be omitted when no confusion can arise.

Chapter 2

LITERATURE SURVEY

2.1 Literature Review

The behavior of linear, time-invariant (LTI) systems subject to actuator saturation has been extensively studied for several decades. It is known that saturation usually degrades the performance of a system and leads to instability. Over the last years systems subject to saturation has attracted a lot of researchers and a considerable amount of work has been done. Most of the study has been done on systems subject to actuator saturation, which involves problems as global, semi-global stabilization and local stabilization, anti-windup compensation, null controllable regions, to mention a few.

More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks, see [1] for a nice overview of application cases requiring a formal treatment of the saturation constraints. Most of the research efforts geared toward constructive linear or nonlinear control for saturated plants can be divided into two main strands. In the first one, called anti-windup design, a pre-designed controller is given, so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention, see [2, 3, 4] and the references therein. There are mainly two approaches to dealing with actuator saturation. One approach is to take control constraints into account at the outset of control design. A low-and-high gain method was presented in [4] to design linear semi globally stabilizing controllers.

It becomes increasingly apparent that saturation is an open topic of research in control systems and many researchers have done lot of work in this field of study. Available results can be broadly classified into

- Systems with actuator saturation, see [6, 8, 10, 11, 12, 17, 21, 39, 40, 43] and their references,
- Systems with input saturation, see [7, 16, 18, 26, 30, 32, 35, 36, 41, 42]

and their references,

- Systems with output saturation, see [13, 14, 15] and their references,

- Systems subject to state saturation, see [23, 27] and their references

Researchers also considered the study on large scale systems subject to multi layer nested saturations [18, 33, 37]. When saturation occurs global stability is an issue and it can never be ensured and usually semi-global stability [14, 15] was done, also the study on estimating the large domain of attraction was done in [2, 37, 39].

In some studies the set invariance and LMI based optimization was done [7, 10] and feedback designs for stabilizing the system using various feedback designs [6, 12, 16, 17, 18, 25, 28, 30, 32, 35, 36, 37, 41, 42] and various designs were developed too. The Anti-windup designs were done for the case of actuator saturation on systems [7, 8, 10] also resulting in a lot of interest in this part of the study.

A dynamic output feedback approach is developed in [6] for the controller design using the cone complementary linearization procedure. The paper dealt with the estimation of domain of attraction and then a method is described for the controller design of a LTI system in the presence of actuator saturation. The feasibility problem is solved using the cone complementary linearization method. The condition for set invariance with actuator saturation is also presented. They also considered the design of a controller for the system with saturation such that the estimated domain of attraction was maximized respect to prescribed bounded convex set. This presented a solution for the state feedback case, where all states are measurable, but when the states are not measurable then only the outputs are measurable for designing the output feedback controller.

In [10], an anti-windup technique was presented to enlarge the domain of attraction for systems subject to actuator saturation. They assumed a linear anti-windup compensator which stabilizes the system in the absence of actuator saturation and then used LMI techniques to enlarge the domain of attraction. A method for estimating the domain of attraction of the origin for the system under saturated linear feedback was discussed in [11]. A set invariance condition was derived and conditions for enlarging this invariant set was done. Using these conditions analysis and design was done for both closed loop stability and disturbance rejection. The condition they developed for the determination of invariant set was less conservative than that based on the circle criterion or the vertex analysis.

In [16], the goal of study was to design controllers for saturating decentralized systems that achieve not only stabilization but also achieve high performance. Their contribution to the work was to provide a broad and sufficient conditions for decentralized stabilization under saturation and they have shown that stabilization is possible whenever,

- the open loop eigen values are in closed unit disc,

- the eigen values on the closed unit circle disc are not in decentralized fixed modes, and
- these eigen values on unit circle have algebraic multiplicity of 1.

They tried to solve the problem of developing semi-global stabilization via decentralized control, which was not achieved, but semi-global stabilization was shown.

In [18], A study on the decentralized controllers for large-scale linear systems subjected to saturation control is done and also on the L_2 disturbance rejection. For a closed loop system under a saturating decentralized feedback law conditions were identified for which an ellipsoid is contractively invariant and also within the domain of attraction. A numerical algorithm was developed to solve the optimization BMI problems. The extension of the work was done for systems subjected to nested saturation case. Also, in this paper discussion on the various methods employed for weakly coupled and strongly coupled subsystems was discussed. The first phase of the paper dealt with the absence of actuator saturation and a design of a decentralized controller using the homotopy method [20]. In the second phase of the paper they considered the actuator saturation present. They used the decentralized feedback law as an initial controller a path following algorithm was developed which searches a new decentralized feedback law that could achieve a larger domain of attraction or stronger disturbance rejection capability.

In [35], they summarized observations about the control of decentralized systems with input saturation. They were able to show that time-invariant non-linear controllers cannot be used to move the fixed modes to zero. They used the singular perturbation method for model reduction and comment that this method is very promising for the first step in design of stabilizing controllers for decentralized systems with input saturation.

In [13], An LTI MIMO system was considered which is controllable and observable with each output component saturated. In this paper the output is first brought out of saturation, using a method which relies on the sign of the output. When the output comes out of the saturation, the state of the system is identified using the deadbeat control strategy. Since it was found to be difficult to bring all the outputs out of saturation at the same time in case of a MIMO system, it was found that it is better to use one output at a time out of the saturation regions, even if some others are in the saturation zone. After getting all the data for all the outputs at certain different times they merged the results and found the states of the systems. These states were brought to the origin using the deadbeat control.

It is described that basically global asymptotic stabilization is possible by output feedback and also it was illustrated that information from multiple output components at different points can be combined to identify the states of the system. But for the origin of a LTI MIMO system with saturated outputs to be globally asymptotic and stabilized it is necessary for the system to be controllable and observable.

In [15], they considered the problem of semi-globally stabilizing linear system using linear feedback of the saturated output measurement. They established that a SISO linear stabilizable and detectable system subject to output saturation can be semi-globally stabilized by linear output feedback if all the invariant zeros are in the closed left-half plane, no matter where open loop poles are. The linear feedback laws were designed in such a way that they used the saturated output to cause the system output to oscillate into the linear region of output saturation function and remain there in a finite time.

In [23], The study on the problem of stability analysis and controller design for continuous time linear systems with the consideration of full state saturation as well as partial state saturation was done. A new and tractable system was constructed showing that this system is with same domain of attraction as the original system. An LMI method is used for estimating the attraction domain of the origin for new constructed system with state saturation. An algorithm was developed for the designing of of output feedback controllers guaranteeing that the attraction domain of the origin for the closed-loop system is as large as possible.

In [27], they discussed the problem of stability analysis for linear systems under state constraints and some conditions were devised for global asymptotic stability of such systems. They achieved certain conditions under which linear systems defined on closed hypercube and linear systems with partial saturation are globally asymptotically stable at origin. Iterative LMI formulation was proposed for verifying the asymptotically stable system.

The problem of synthesizing fixed order anti-windup compensator which meet an L_2 performance bound was addressed in [7]. They used the linear anti-windup augmentation method to develop a controller. It was also shown that if and only if the plant is asymptotically stable this plant order anti-windup compensation is always feasible for large L_2 gain. They have demonstrated that the Lyapunov formulation of this problem can be taken as a non convex optimization problem.

A new saturation control technique is developed in [8] for anti-windup design for the case of exponentially unstable LTI system. The algorithm developed guaranteed regional stability in the presence of input saturation and improves performance too. It was also commented that systems with input non-linearities such as deadzone and hysteresis can also be treated using this approach.

In [12], a method for output feedback with saturation for stabilizing the system was presented. Also the enlargement of domain of attraction was done. They have used a non-linear output feedback controller expressed in quasi-LPV system form, establishing conditions for closed loop stability.

In [38], A piece-wise quadratic Lyapunov function is developed for the analysis of global and regional performances for systems with input saturation with an algebraic loop. The function incorporated the structure of the saturation non-linearity. Sector conditions were considered which are shown in the following section and also an introduction to three sector like conditions that were useful in this paper. A comparison with the non-quadratic Lyapunov function in [31] was done. They have addressed the problem of stability and performance analysis for linear systems which involve saturation/deadzone. They have compared the results from [6] with the ones produced by these new methods. In this paper they showed that the Lyapunov based approach on piece-wise quadratic function was developed to analyze the global, regional stability and performances for systems subject to saturation.

In [28], They presented the LMI based synthesis approach on output feedback design for input saturated linear systems using deadzone loops. The proposed approach will lead to regional stabilizing controllers if the plant is exponentially unstable, to semi-global stability if the plant is non-exponentially unstable, and to global stability if the plant is already exponentially stable, the requirement of the plant being detectable and stabilizable.

The design of decentralized controllers for interconnected linear systems subject to multi-layer nested saturations were considered in [37]. They formulated the decentralized state feedback laws that resulted in large domain of attractions. Their study was divided into two phases, the first phase they assumed no saturation and using the homotopy idea [20]. This decentralized control law, when subject to saturation still achieves local stabilization with guaranteed domain of attraction. The second phase, they consider the actuator saturation and designed the algorithms for larger domain of attraction.

In [33], they showed the problem of stability for systems presenting nested saturations. The generalized sector conditions were used for the stability analysis. This work proposed that it allows a more general nested saturation structure and addressed the global stability and global stabilization problems solved by LMI methods.

In [54], the authors developed a new LMI-based procedure for the design of decentralized dynamic output controllers for systems composed of linear subsystems coupled by uncertain nonlinear interconnections satisfying quadratic constraints. The scheme utilizes the general linear dynamic output feedback structure. The design procedure consists of two steps, the first providing the local Lyapunov matrices together with the corresponding robustness degrees, and the second the controller parameters providing a robustly stable overall system. A comprehensive review of decentralized control design techniques was provided in [55]. The synthesis of output feedback controllers with saturating inputs was studied in [61] where an observer based controller and a dynamic output feedback controller based on the circle criterion was developed via LMI formulation.

In [62], the authors presented a method for designing an output feedback law that stabilizes a linear system subject to actuator saturation with a large domain of attraction. This method applies to general linear systems including strictly unstable ones. A nonlinear output feedback controller is first expressed in the form of a quasi-LPV system. Conditions under which the closed-loop system is locally asymptotically stable are then established in terms of the coefficient matrices of the controller. The design of the controller (gain matrices) that maximizes an estimate of the domain of attraction is then formulated and solved as an optimization problem with LMI constraints.

2.2.1 Problem statement

Consider the LTI plant described by,

$$\begin{aligned} \dot{x_p}(t) &= A_p x_p(t) + B_p Sat(u) + Ew(t) \\ y(t) &= C_p x_p(t) + D_p Sat(u) \end{aligned} \tag{2.1}$$

where $x_p(t) \in \Re^{n_p}$ is the plant state, $u(t) \in \Re^{n_u}$ is the control input, $y(t) \in \Re^{n_y}$ is the plant output available for measurement and $w(t) \in \Re^{n_w}$ is the input disturbance.

In most of the works, the following assumptions were considered:

Assumption 2.2.1 *The following conditions hold*

- *The triple (A_p, B_p, C_p) is stabilizable and detectable,*
- *The matrices B_p^t and C_p have full row rank,*
- D_p*=0*

The common objective is to address the stability analysis and control design problems for system (2.1) under Assumption 2.2.1. In this paper, we survey available results pertaining to both problems using alternative approaches.

2.2.2 Main results

Initially, assume that the controller is given for the system with actuator saturation, the problem of interest is that finding the estimate of domain of attraction. The following theorem developed in [6] provides a basic result:

Theorem 2.2.1 *Given an ellipsoid* $\Xi(P,\rho)$*,* $P \in \Re^{n\times n}$ *, if there exists an* $H \in \Re^{n\times n}$*, such that,*

$$
\begin{aligned}
&(A_{cl} + B_{cl}(V_i C_{cl} + V_i^- H))^t P + \\
&P(A_{cl} + B_{cl}(V_i C_{cl} + V_i^- H)) < 0
\end{aligned} \tag{2.2}
$$

for all $V_i \in V$ *and* $\Xi(P,\rho) \subset \mathbf{P}(H)$*, that is,*

$$
|H_i x| \;\leq\; 1 \;\; \forall\, x \;\in\; \Xi(P,\rho)
$$

is contractively invariant set.

Remark 2.2.1 *System (2.1) was considered in [10] under Assumption 2.2.1. Introducing a typical anti-windup compensator involving a correction term of the form* $E_c(\sigma(u)-u)$ *leads the closed-loop system*

$$
\begin{aligned}
\dot{x}_c &= A_c x_c + B_c + E_c(\sigma(u)-u), \;\; x_c(0) = 0 \\
u &= C_c x_c + D_c y, \quad x_c \in \Re^{n_c}
\end{aligned} \tag{2.3}
$$

Based on Theorem (2.2.1), the problem estimating the domain of attraction is addressed. By using the matrix E as a free design parameter, it is shown that the domain of attraction can be enlarged via optimization procedure.

In the following theorem a condition of set invariance examined in [11] is presented:

Theorem 2.2.2 *Given an ellipsoid $\Xi(P,\rho)$, if there exists an $H \in \Re^{m\times n}$ such that,*

$$(A + BM(v,F,H))^t P + P(A + BM(v,F,H)) < 0 \tag{2.4}$$

where,

$$\begin{aligned} \{M(v,F,H) : v \in \nu\} &= \{H, \begin{bmatrix} h_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ h_2 \end{bmatrix}, F\} \\ \nu &= v \in \Re^m : v_i = 1 or 0 \end{aligned} \tag{2.5}$$

for all $v \in \nu$ and $\Xi(P,\rho) \subset \ell(H)$, that is,

$$|h_i x| \leq 1, \;\; \forall x \in \Xi(P,\rho), \; i \in [1,m]$$

then $\Xi(P,\rho)$ is a contractively invariant set.

The class of disturbances treated in the literature is characterized below

Assumption 2.2.2 *The input disturbance* $w(t) \in \Re^{n_w}$ *belongs to the bounded set* W *defined by*

$$\mathsf{W} := \{w(t) : w(t)^t w(t) \leq 1 \forall\, t \ \geq \ 0\}$$

An efficient method of determining disturbance rejection with guaranteed domain of attraction is summarized by the following theorem [11]:

Theorem 2.2.3 *Given two ellipsoids*

$$\Xi(P, \rho_1),\ \Xi(P, \rho_2),\ \rho_2 > \rho_1 > 0$$

If there exist matrices $H_1, H_2 \in \Re^{m \times n}$ *and a positive number* η *such that*

$$\begin{aligned}
&(A + BM(v, F, H_1))^t P + P(A + BM(v, F, H_1)) + \\
&\frac{1}{\eta} PEE^t P + \frac{\eta}{\rho_1} P < 0, \forall v \in \nu \qquad (2.6)\\
&(A + BM(v, F, H_2))^t P + P(A + BM(v, F, H_2)) + \\
&\frac{1}{\eta} PEE^t P + \frac{\eta}{\rho_2} P < 0, \forall v \in \nu \qquad (2.7)
\end{aligned}$$

and $\Xi(P.\rho_1) \subset L(H_1),\ \Xi(P.\rho_2) \subset L(H_2)$, *then for every* $\rho \in [\rho_1, \rho_2]$, *there exists a matrix* $H \in \Re^{m \times n}$ *such that*

$$\begin{aligned}
&(A + BM(v, F, H))^t P + P(A + BM(v, F, H)) + \\
&\frac{1}{\eta} PEE^t P + \frac{\eta}{\rho} P < 0, \forall v \in \nu \qquad (2.8)
\end{aligned}$$

and $\Xi(P,\rho) \in L(H)$. This implies that $\Xi(P,\rho)$ is also strictly invariant.

Next, in [16] the following system is considered

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{i=1}^{v} B_i Sat(u_i(k)) \\ y_i(k) &= C_i x(k), i = 1, \ldots v \end{aligned} \tag{2.9}$$

where $x \in \Re^n$ is the state, $u_i \in \Re^{m_i}, i = 1 \ldots v$ are control inputs, $y_i \in \Re^{p_i}$ are measured outputs. For system (2.9) an improved controller stabilization result is developed and summarized by the following theorem:

Theorem 2.2.4 *For system (2.9), there exists non-negative integers $s_1, s_2, \ldots, s_v$ such that for any given collection of compact sets $\mathcal{W} \in \Re^n$ and $\mathcal{S}_i \in \Re^{s_i}$ there exists v controllers of the form,*

$$\begin{aligned} z_i(k+1) &= K_i z_i(k) + L_i y_i(k) \\ u_i(k+1) &= M_i z_i(k) + N_i y_i(k) \end{aligned} \tag{2.10}$$

such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $\mathcal{W} \times \mathcal{S}_i \times \ldots \times S_v$ only if,

- *All fixed modes are in the open unit disc*
- *All eigen values of A are in the closed loop unit disc.*

An additional result on the conditions of semi-global stabilizability of system Theorem 2.2.4 using the set of controllers (2.10) is established by the theorem below:

Theorem 2.2.5 *Consider system (2.9), there exists non-negative integers* $s_1, s_2, \ldots, s_v$ *such that for any given collection of compact sets* $\mathcal{W} \in \Re^n$ *and* $\mathcal{S}_i \in \Re^{s_i}$ *there exists* v *controllers of the form (2.10), such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes* $\mathcal{W} \times \mathcal{S}_i \times \ldots \times S_v$ *if,*

- *All fixed modes are in the open unit disc*
- *All eigen values of* A *are in the closed loop unit disc with those eigen values on the unit disc having algebraic multiplicity equal to one.*

Remark 2.2.2 *Based on the condition for set invariance developed in [6], a result on the determination of disturbance tolerance capability of the closed loop system under state feedback law is reported in [18]. This result is stated in the theorem below.*

Theorem 2.2.6 *Consider system (2.1) under the state feedback law* $u = Fx$. *For a give positive definite matrix* P, *if*

$$\begin{aligned}&(A + B(D_sF + D_s^- H))^t P + P(A + B(D_sF + D_s^- H)) + \\ &\frac{1}{\eta} PEE^t P \leq 0, s \in I_{2^m}\end{aligned} \tag{2.11}$$

and $\Xi(P, 1+\alpha\eta) \subset L(H)$, then every trajectory of the closed loop system that starts from inside of $\Xi(P, 1)$ will remain inside of $\Xi(P, 1+\alpha\eta)$ for every $w \in W_\alpha^2$. Additionally, there exists a matrix $H \in \Re^{m\times n}$ and a positive number η such that (2.11) is satisfied and $\Xi(P, \alpha\eta)$ for every $w \in W_\alpha^2$.

The following theorem characterizes the conditions under which the linear system under actuator saturation (2.1) has L_2 gain less than or equal to γ.

Theorem 2.2.7 *Let α_{max} be the maximal tolerable disturbance level. Consider an $\alpha \in (0, \alpha_{max}]$. For a given constant $\gamma > 0$, if there exists a matrix $H \in \Re^{m\times n}$ such that,*

$$\begin{aligned}&(A + B(D_sF + D_s^-H))^tP + P(A + B(D_sF + D_s^-H)) + \\ &PEE^tP + \frac{1}{\gamma^2}C^tC \leq 0, \qquad s \in I_{2^m}\end{aligned} \tag{2.12}$$

and $\Xi(P, \alpha) \subset L(H)$, then the restricted L_2 gain from w to z over W_α^2 is less than or equal to γ.

In establishing Theorem 2.2.7, the procedure for stabilization of systems subject to nested saturations developed in [22] was employed.

2.2.3 Example 1

, Consider the system with,

$$\begin{aligned}
\mathbf{A_p} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B_p} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \\
\mathbf{C_p} &= \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{aligned}$$

Using a standard dynamic output feedback controller, it was shown in [6] that the gains are described by

$$A_k = -30,\ B_k = -22,\ C_k = -20,\ D_k = -30$$

Letting $R = I_{3\times 3}$, and solving an optimization problem, the feasible solution was attained at $\gamma^* = 118.0139$ and

$$\begin{aligned}
\mathbf{\Pi}^* &= \begin{bmatrix} 109.1588 & -0.4927 & 29.8610 \\ -0.4927 & 1.3420 & -2.6067 \\ 29.8610 - 2.6067 & 20.4395 & \end{bmatrix} \\
\mathbf{H}^* &= \begin{bmatrix} -7.7212 & -0.7368 & -0.4017 \end{bmatrix}
\end{aligned}$$

Considering the same system and setting, $\Xi = 2$, $N = 5$, $T = 5$, $\tau = 5$, algorithm 1 in [6], and the values obtained above, the efficiency of the ensuing design can be seen from the following results,

$$
\begin{aligned}
\mathbf{K} &= \begin{bmatrix} -33.4815 & -133.8522 \\ -2.1163 & -11.8956 \end{bmatrix}, \\
\mathbf{H}_1^* &= \begin{bmatrix} -7.0427 & -0.3701 & -0.3873 \end{bmatrix}, \\
\mathbf{\Pi}_1^* &= \begin{bmatrix} 56.5258 & -0.0227 & 5.4276 \\ -0.0227 & 2.1547 & -0.5192 \\ 5.4276 & -0.5192 & 1.5738 \end{bmatrix}
\end{aligned}
$$

2.3 Output Saturation

In what follows, we will examine the case of systems subject to output saturation.

2.3.1 Problem statement

Consider the LTI system with saturated outputs as,

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Sat(C(x)) \qquad (2.13)
\end{aligned}
$$

where, $x \in \Re^n$ is the state of the system $u \in \Re^m$ is the controller input and $y \in \Re^p$ is the output measurement.

2.3.2 Main results

Basic results are established in [13] and [15] and summarized by the following theorems:

Theorem 2.3.1 *The origin of system (2.13) is globally asymptotically stable, that is*

$$\begin{aligned} & x(0) = 0 \;\Rightarrow\; x(t) = 0 \;\forall\, t \;\geq 0 \; (equilibrium) \\ & such\; that\; ||x(0)|| \;\leq\; \sigma \; (stability) \\ & and \;\forall\; x(0),\; lim_{t\rightarrow \inf}\; x(t) = 0 \; (global\; attractivity). \end{aligned}$$

Theorem 2.3.2 *System (2.13) is semi-globally asymptotically stabilizable by linear feedback of the saturated output if*

- *The pair(A,B) is stabilizable,*
- *The pair (A,C) is detectable,*
- *All invariant zeros of the triple (A,B,C) are in the closed left-half plane.*

More specifically, for any a priori given bounded set $H_0 \subset \Re^{2n}$, there exists a linear dynamic output feedback law of the form,

$$\begin{aligned} \dot{z} &= Fz + Gy, \qquad z \in \Re^n \\ u &= Hz + H_0 y \end{aligned}$$

such that the equilibrium (x,z)=(0,0) of the closed loop system is asymptotically stable with H_0 contained in its domain of attraction.

2.3.3 Example 2

Consider the system,

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + x_3(t) \\
\dot{x}_2(t) &= u_1(t) \\
\dot{x}_3(t) &= u_2(t) \\
y_1(t) &= x_1(t) \\
y_2(t) &= x_1(t) + x_2(t)
\end{aligned}$$

The system has an eigen value with multiplicity one and an eigen value with multiplicity 2, both at the origin, thus the system is open-loop unstable. In [13], the controller algorithm is implemented using $T = 0.5,\ \rho = 1.1,\ h_1 = C_1^t$ and $h_2 = C_2^t$ with initial conditions were $x_1(0) = 2, x_2(0) = -4$ and $x_3(0) = 1$.

2.4 State Saturation

We now direct attention to the case of systems subject to state saturation.

2.4.1 Problem statement

Consider a class system with state saturation in the form

$$\begin{aligned} \dot{x_p} &= sat(A_p x_p) + B_p u \\ y &= C_p x_p \end{aligned} \tag{2.14}$$

where $x_p \in \Re^{n_p}$ is the plant state, $u \in \Re^{n_u}$ is the control input, $y \in \Re^{n_y}$ is the plant output available for measurement.

2.4.2 Main results

For system (2.14), sufficient conditions were derived in [27] to guarantee global asymptotic stability. The theoretical results are summarized by the following two theorems. The first theorem concerns the stability analysis:

Theorem 2.4.1 *If there exists a symmetric positive-definite matrix $P \in \Re^{n\times n}$ and a matrix $G \in \Re^{n\times n}$ such that,*

$$(D_i A + D_i^- G)^t P + P(D_i A + D_i^- G) < 0 \tag{2.15}$$

where G is (row) diagonally dominant and the diagonal is composed of negative *elements as specified below*

$$h(Ax + K) \ \in \ coD_i(Ax + K) + D_i^- Gx, \quad i \ \in \ [1, 2^n] \ \forall \ x \ \in D^n$$

for any matrix $K \in \Re^n$ *independent of x.*

The next theorem is for determining the globally stabilizing feedback gain F:

Theorem 2.4.2 *If there exists a symmetric positive-definite matrix* $P \in \Re^{n \times n}$ *and a matrix* $G \in \Re^{n \times n}$ *such that,*

$$\begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix}^t P + P \begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix} < 0 \ i \in [1, 2^n] \tag{2.16}$$

where $D_i \in D_m$ *and and* G *is (row) diagonally dominant and the diagonal is composed of negative elements, then the system is globally asymptotically stable at origin.*

2.4.3 Example 3

Consider the following system with state saturation

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -9.9 & 8 \\ 10 & 5 \end{bmatrix}, \\
\mathbf{B} &= \begin{bmatrix} 1 \\ -9 \end{bmatrix}, \\
\mathbf{C} &= \begin{bmatrix} 1 & 2 \end{bmatrix}
\end{aligned}
$$

Without considering the state saturation a controller is designed with gain parameters as,

$$
\begin{aligned}
\mathbf{A_k} &= \begin{bmatrix} -1 & 2.5 \\ 30 & -9 \end{bmatrix}, \\
\mathbf{B_k} &= \begin{bmatrix} -0.9 \\ -0.5 \end{bmatrix}, \\
\mathbf{C_k} &= \begin{bmatrix} 0.1 & -2 \end{bmatrix} \\
\mathbf{D_k} &= 1
\end{aligned}
$$

$$X_R = col[1100]^t, [1-100]^t, [-1100]^t, [-1-100]^t.$$

By Algorithm 1 in [23], the domain of attraction was estimated and the results were as follows,

$$\gamma^* = 1.6971$$

$$\begin{aligned}
\mathbf{Q}^* &= \begin{bmatrix} 3.1511 & 0.9275 & 0.3591 & -0.4926 \\ 0.9275 & 2.0033 & 0.4194 & -0.0770 \\ 0.3591 & 0.4194 & 0.1521 & -0.0429 \\ -0.4926 & -0.0770 & -0.0429 & 1.3383 \end{bmatrix}, \\
\mathbf{U}^* &= \begin{bmatrix} 0.1114 & 0 \\ 0 & 0.1111 \end{bmatrix}
\end{aligned}$$

To design the controller, Algorithm 2 was implemented to obtain the largest domain of attraction. The controller gains were

$$\begin{aligned}
\mathbf{A}_k^* &= \begin{bmatrix} -1.6255 & 4.4283 \\ 31.3529 & -22.3786 \end{bmatrix}, \\
\mathbf{B}_k^* &= \begin{bmatrix} -2.8123 \\ 4.5233 \end{bmatrix}, \\
\mathbf{C}_k^* &= \begin{bmatrix} 0.3996 & -3.1210 \end{bmatrix}, \\
\mathbf{D}_k^* &= 2.4213
\end{aligned}$$

and the result of the domain of attraction was found to be ,

$$\gamma^* = 0.4913$$

$$\begin{aligned}
\mathbf{Q}^* &= \begin{bmatrix} 12.2710 & -1.8511 & 0.4432 & 3.2522 \\ -1.8511 & 16.8187 & 2.9340 & -9.4093 \\ 0.4432 & 2.9340 & 0.7476 & -1.5786 \\ 3.2522 & -9.4093 & -1.5786 & 12.7334 \end{bmatrix}, \\
\mathbf{U}^* &= \begin{bmatrix} 0.7090 & 0 \\ 0 & 0.1629 \end{bmatrix}
\end{aligned}$$

It is demonstrated in [23] that by using Algorithm 2, the index of domain of attraction is improved than that which was obtained by Algorithm 1.

2.5 Systems Presenting Nested Saturation

Proceeding further, we deal with the case of systems presenting nested saturation.

2.5.1 Problem statement

Consider a linear system consisting of N interconnected subsystems, each subject to multi-layer nested saturation in its inputs,

$$\begin{aligned} \dot{x}_i &= A_i x_i + \sum_{j \neq 1} A_{ij} x_j + B_i Sat(F_{1i} x + K_{2i} Sat(F_{2i} x + \\ &+ \ldots + K_{pi} Sat(F_{pi} x))), \quad i \in I_N \end{aligned} \tag{2.17}$$

where $x_i \in \Re^{n_i}$ is the state of the ith subsystem. For a vector $u_i \in \Re^{m_i}$ where

$$Sat : \Re^{m_i} \rightarrow \Re^{m_i}$$

is the vector valued standard saturation function defined as

$$Sat(u_i) = [Sat(u_{i1}\ Sat(u_{i2}) \ldots Sat(u_{im_i}]^t$$

and

$$Sat(u_{il}) = sign(u_{il}) min|u_{il}|, 1,\ l \in I_{m_i}$$

2.5.2 Main results

Set invariance conditions were provided in [37] as presented by the following theorem:

Theorem 2.5.1 *Given an ellipsoid $\Xi(P,\rho)$. If there exists an $H \in \Re^{m\times n}$ such that,*

$$\begin{aligned} &(A + B(D_s F + D_s^- H))^t P + \\ &P(A + B(D_s F + D_s^- H)) \;\leq\; 0, \qquad s \in [1, 2^m] \end{aligned} \tag{2.18}$$

and $\Xi(P,\rho) \subset L(H)$. Then $\Xi(P,\rho)$ is an invariant set. If "'<"' holds for the aforementioned inequalities, then $\Xi(P,\rho)$ is a contractively invariant set towards the origin.

For the multi-layered nested saturations, the following theorem present a useful result [18]:

Theorem 2.5.2 *Consider the interconnected linear system 2.17. For a given ellipsoid $\Xi(P,1)$, if there exists some matrices $H_\ell,\ F_\ell\ \in \Re^{m\times n},\ \ \ell \in I_p$, such that, for all diagonal matrices $D_1,\ D_2,\ \ldots,\ D_{p+1} \in \Re^{m\times m}$ whose diagonal elements can only be 0 or 1, with $\sum_{i=1}^{p+1} D_i = 1$ such that*

$$\begin{aligned} &P(A + B(D_1 H_1 + D_2(F_1 + K_2 H_2) + \ldots + \\ &D_p(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) + \\ &D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p F_p))) + \\ &P(A + B(D_1 H_1 + D_2(F_1 + K_2 H_2) + \ldots + \\ &D_p(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p H_p) + \\ &D_{p+1}(F_1 + K_2 F_2 + \ldots + K_2 \ldots K_p F_p)))^t P < 0 \end{aligned} \tag{2.19}$$

and $\Xi(P,1) \subset \cap_{l=1}^{p} L(H_l)$*, then* $\Xi(P,1)$ *is contractively invariant set.*

Extending to the situation of disturbance rejection, the following theorem establishes a workable result

Theorem 2.5.3 *Consider the interconnected linear system 2.17 Let* P *be a positive definite matrix.*

- *If there exists a positive number* η *some matrices* $H_\ell,\ F_\ell\ \in \Re^{m\times n},\ \ \ell \in I_p$*, such that, for all diagonal matrices* $D_1,\ D_2,\ \ldots,\ D_{p+1} \in \Re^{m\times m}$ *whose diagonal elements can only be 0 or 1, with* $\sum_{i=1}^{p+1} D_i = 1$ *such that*

$$\begin{aligned}
&P(A + B(D_1H_1 + D_2(F_1 + K_2H_2) + \ldots + \\
&D_p(F_1 + K_2F_2 + \ldots + K_2 \ldots K_pH_p) + \\
&D_{p+1}(F_1 + K_2F_2 + \ldots + K_2 \ldots K_pF_p))) + \\
&P(A + B(D_1H_1 + D_2(F_1 + K_2H_2) + \ldots + \\
&D_p(F_1 + K_2F_2 + \ldots + K_2 \ldots K_pH_p) + \\
&D_{p+1}(F_1 + K_2F_2 + \ldots + K_2 \ldots K_pF_p)))^tP < 0
\end{aligned} \qquad (2.20)$$

  and $\Xi(P,1+\alpha\eta) \subset \cap_{l=1}^{p} L(H_l)$*, then every trajectory of the closed loop system that starts from inside of* $\Xi(P,1)$ *will remain inside of* $\Xi(P,1+\alpha\eta)$ *for every* $w\in W_\alpha^2$*.*

- *If there exists a positive number* η *some matrices* $H_\ell,\ F_\ell\ \in \Re^{m\times n},\ \ \ell \in I_p$*, such that, for all diagonal matrices* $D_1,\ D_2,\ \ldots,\ D_{p+1} \in \Re^{m\times m}$ *whose* *diagonal elements can only be 0 or 1, with* $\sum_{i=1}^{p+1} D_i = 1$, *inequality (2.20) is satisfied and* $\Xi(P, \alpha\eta) \subset \cap_{l=1}^{p} L(H_l)$, *then every trajectory of the closed loop system that starts from origin will remain inside of* $\Xi(P, \alpha\eta)$ *for every* $w \in W_\alpha^2$.

2.5.3 Example 4

For the case of multi layered nested saturation, let us consider a system with $w = 0$ and the following data

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 4 & 3 & 2 \\ 2 & 3 & 3 & 4 \\ 1 & 5 & 0 & 1 \\ 1 & 1 & 3 & 4 \end{bmatrix}, \\
\mathbf{B} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
p &= 2,\ K_2 = blkdiag[0.5,\ \ 0.5] \\
X_R &= [1\ 0.5\ 0.6\ 0.9]
\end{aligned}
$$

Consider the design algorithm for decentralized control for multi layered nested saturation in [37]. It is found that the feasible solution of the corresponding optimization problems was as follows:

$$\begin{aligned}
\mu^* &= 80.0127 \\
\mathbf{P}^* &= \begin{bmatrix} 8.7974 & 17.0317 & -1.3232 & 2.0154 \\ 17.0317 & 49.1759 & 3.6958 & 0.8127 \\ -1.3232 & 3.6958 & 29.3195 & 28.3716 \\ 2.0154 & 0.8127 & 28.3716 & 45.0384 \end{bmatrix}, \\
\mathbf{H}_1^* &= \begin{bmatrix} -2.1505 & -5.5006 & -3.3058 & -3.6132 \\ -1.8211 & -4.6616 & -3.8394 & -4.9424 \end{bmatrix}, \\
\mathbf{H}_2^* &= \begin{bmatrix} -0.4850 & -1.9498 & -4.0923 & -4.4794 \\ -1.9933 & -3.4248 & -0.0002 & -1.4339 \end{bmatrix}
\end{aligned}$$

2.6 Linear Systems with Deadzone

Finally, we deal with the case of systems containing deadzone.

2.6.1 Problem statement

Generally a system with saturation or deadzone is described as,

$$\begin{aligned}
\dot{x} &= Ax + B_q q + B_w w \\
y &= C_y x + D_{yq} q + D_{yw} w \\
z &= C_z x + D_{zq} q + D_{zw} w \\
q &= dz(y)
\end{aligned} \tag{2.21}$$

where $x \in \Re^n$, $q, y \in \Re^m$, $w \in \Re^r$, and $z \in \Re^p$. The deadzone function $dz(.) : \Re^m \rightarrow \Re^m$ is defined by

$$dz(y) := y - Sat(y), \quad \forall y \in \Re^m$$

where $Sat(.)$ is a vector saturation function with the saturation levels given by a vector

$$\bar{u} \in \Re^m, \quad \bar{u}_i > 0, \; i = 1, 2, \ldots, m$$

In [38], they considered an algebraic loop, when $D_{yq} \neq 0$ and a nonlinear algebraic loop imposed by

$$y = C_y x + D_{yq} dz(y) + D_{yw} w \tag{2.22}$$

Further analysis is based on the following facts:

Fact 1: For every diagonal matrix $\delta \in \Re^{s\times s}$, $\delta > 0$, the deadzone function $dz(.)$ satisfies

$$dz(v)^t\delta\{v - dz(v)\} \geq 0, \forall v \in \Re^s \tag{2.23}$$

Fact 2: Given $r \in \Re^m$ such that

$$-\bar{u}_i \leq r_i \leq \bar{u}_i,\ \forall i = 1, \ldots, m$$

the following inequality holds for any diagonal matrix $\delta \in \Re^{m\times m}$, $\delta > 0$:

$$dz(v)^t\delta\{v - dz(v) - r\} \geq 0, \forall v \in \Re^m \tag{2.24}$$

2.6.2 Main results

The following results were the sector-like conditions introduced to describe the properties of the algebraic loop with deadzone:

Result 1: In view of the non-decreasing properties of saturation and deadzone functions, the following inequality holds for every diagonal matrix $\delta \in \Re^{m\times m}$, $\delta > 0$

$$\begin{aligned}\{dz(v_1) - dz(v_2)\}^t\,\delta\,\{sat(v_1) - sat(v_2)\}\\ \geq 0,\ \ \forall\, v_1, v_2\ \in\ \Re^m.\end{aligned} \tag{2.25}$$

Result 2: For every diagonal matrix $\delta \in \Re^{m\times m}$, the following equalities hold almost everywhere:

$$\begin{aligned} \phi(x,w)^t\delta\left\{\dot{u}-\phi(x,w)\right\} &\equiv 0 \\ dz(u)^t\delta\left\{\dot{u}-\phi(x,w)\right\} &\equiv 0 \end{aligned} \tag{2.26}$$

where,

$$\dot{u} = C_yAx + C_yB_qdz(y) + C_yB_ww + D_{yq}\phi(x,w)$$

The conditions characterizing global analysis and regional analysis are mentioned in the following theorems:

Theorem 2.6.1 *Considering system 2.21, the following results hold true:*

1. *(Exponential Stability): If there exists a matrix* $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0$, *and diagonal matrices* $\Delta_i \in \Re^{m\times m}$, *i=1,* $\ldots, 5, \Delta_{i=1,2,3} > 0$, *satisfying the LMI,*

$$[I_{n+3m}\ 0_{(n+3m)\times r}]\bar{\Psi}\begin{bmatrix} I_{n+3m} \\ 0_{r\times(n+3m)} \end{bmatrix} < 0 \tag{2.27}$$

then for the Lyapunov function $V(x) = \xi(x)^tP\xi$, *there exists an* $\Xi > 0$ *such that* $\dot{V} < -\Xi||x||^2$ *for almost all* $x \in \Re^n$ *and* $w = 0$. *This guarantees the origin of the system is globally exponentially stable.*

2. *(Reachable Region): If there exists a matrix $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0$, and diagonal matrices $\Delta_i \in \Re^{m\times m}$, i=1, $\ldots, 5, \Delta_{i=1,2,3} > 0$, satisfying the LMI,*

$$\bar{\Psi} - \bar{\Psi}_6 \bar{\Psi}_6^t < 0 \tag{2.28}$$

then $\dot{V} < w^t w$ for almost all $x = \in \Re^n$ and all $w = \in \Re^r$. If $x(0) = 0$ and $||w||_2 \leq s$, then $\xi(x(t)) \in \Xi(P/s^2)$ for all t $\geq$0.

3. *(Global L_2 Gain): If there exists a matrix $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0$, and diagonal matrices $\Delta_i \in \Re^{m\times m}$, $i = 1, \ldots, 5, \Delta_{i=1,2,3} > 0$, satisfying the LMI,*

$$\begin{bmatrix} \bar{\Psi} - \bar{\Psi}_6 \gamma \bar{\Psi}_6^t & \bullet \\ \begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix} & -\gamma I \end{bmatrix} < 0 \tag{2.29}$$

then, $\dot{V} + \frac{1}{\gamma} z^t z < \gamma w^t w$ for almost all $x \in \Re^n$ and all w $w \in \Re^r$. If $x(0) = 0$, then $||z||_2 \leq \gamma ||w||_2$, that is, the global L_2 gain is bounded by γ.

The theorem for regional analysis [38] is as follows:

Theorem 2.6.2 *Considering system 2.21, the following results hold true:*

1. *(Exponential Stability): If there exists a matrix $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0, H_1, H_2 \in \Re^{m\times(n+m)}$, satisfying and diagonal matrices $\delta_i \in \Re^{m\times m}$,*

i=1, $\ldots, 5, \delta_{i=1,2,3} > 0$, *satisfying the LMI,*

$$[I_{n+3m}\ 0_{(n+3m)\times r}](\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^t) \bullet \begin{bmatrix} I_{n+3m} \\ 0_{r\times(n+3m)} \end{bmatrix} < 0 \tag{2.30}$$

then for the Lyapunov function V(x) $= \xi(x)^t P\xi$, *there exists an* $\Xi > 0$ *such that* $\dot{V} < -\Xi|x|^2$ *for almost all* $x \in \Re^n$ *and* $w = 0$. *Thus the origin of the system is globally exponentially stable. If* $\xi(x(0)) \in \Xi(P)$, *then* $\xi(x(t)) \in \Xi(P)$ *for all t >0. and lim*$_{t\to\inf}$*x(t)=0.*

2. *(Reachable Region): Let s >0. If there exists a matrix* $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0, H_1, H_2 \in \Re^{m\times(n+m)}$ *and diagonal matrices* $\delta_i \in \Re^{m\times m}$, *i=1,* $\ldots, 5, \delta_{i=1,2,3} > 0$, *satisfying the LMI,*

$$\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^t - \bar{\Psi}_6\bar{\Psi}_6^t < 0 \tag{2.31}$$

then $\dot{V} < w^t w$ *for almost all* $x \in \Re^n$ *and all* $w \in \Re^r$. *If* $\xi(x(0)) = 0$ *and* $||w||_2 \leq s$, *then* $\xi(x(t)) \in \Xi(P/s^2)$ *for all t ≥0.*

3. *(Regional* L_2 *Gain): Let s¿0. If there exists a matrix* $P \in \Re^{(n+m)\times(n+m)}$, $P = P^t > 0, H_1, H_2 \in \Re^{m\times(n+m)}$ *and diagonal matrices* $\delta_i \in \Re^{m\times m}$, *i=1,* $\ldots, 5, \delta_{i=1,2,3} > 0$, *satisfying the LMI,*

$$\left[\begin{array}{cc} \bar{\Psi} - \bar{\Omega} - \bar{\Omega}^t - \bar{\Psi}_6\bar{\Psi}_6^t & \bullet \\ \begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix} & -\gamma I \end{array} \right)] < 0 \tag{2.32}$$

then, $\dot{V} + \frac{1}{\gamma} z^t z < \gamma w^t w$ *for almost all* $\xi(x) \in \Xi(P/s^2)$ *and all* $w \in \Re^r$. *If* $\xi(x(0))$*=0, and*$||w||_2 \leq s$, *then* $||z||_2 \leq \gamma ||w||_2$

Conditions for LMI-feasibility testing are contained in the following theorem [28]:

Theorem 2.6.3 *Given s >0. Consider the linear plant subject to saturation/deadzone with* $||w||_2 \leq s$. *Let* $[N_1 \;\; N_2]^t$ *span the null space of* $[C_{py} \;\; D_{p,yw}]$. *If the following LMIs in the variables* $Q_{11}, P_{11} \in \Re^{n_p \times n_p}$, $Q_{11} = Q_{11}^t > 0$, $P_{11} = P_{11}^t > 0$, $Y_p \in \Re^{n_u \times n_p}, \gamma^2 > 0, \Xi \leq \frac{1}{s^2}$ *are feasible:*

$$\begin{bmatrix} A_p Q_{11} + B_p u Y_p & B_{pw} & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_{pz} Q_{11} + D_{p,zu} Y_p & D_{p,zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0$$

$$\begin{bmatrix} N_1 P_{11} A_p N_1^t + N_1 P_{11} B_{pw} N_1^t - \frac{1}{2} N_2 N_2^t & 0 \\ C_{pz} N_1^t + D_{p,zw} N_2^t & -\frac{\gamma^2 I}{2} \end{bmatrix}$$

$$\begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} > 0$$

$$\begin{bmatrix} \Xi \bar{u}^2 & Y_{pi} \\ Y_p i & Q_{11} \end{bmatrix} \geq 0 \; i = 1, \ldots, n_u$$

then there exists an output feedback controller in the form of

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y + E_1 dz(y_c) \\ y_c &= C_c x_c + D_c y + E_2 dz(y_c)\end{aligned}$$

of the order of n_p *, which guarantees the following three properties of the closed loop system:*

1. *the regional* L_2 *gain is bounded by* γ*,*
2. $\Xi(\xi Q_{11}^{-1}) \times 0$ *inside the domain of attraction,*
3. *the reachable set of the plant bounded by* $\Xi((s^2 Q_{11})^{-1})$*.*

where Y_{pi} *denotes the ith row of* Y_p

$Lyapunov Function$	$\begin{bmatrix} -3 & -1 \\ -2 & -4 \end{bmatrix}$	$\begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$	$\begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$
$Piecewise\ Quadratic$ [38]	15.13	17.19	25.86
$Convex\ Hull\ Quadratic$ [31]	17.06	19.33	31.67
$Max\ Quadratic$ [31]	17.37	20.78	42.34
$Quadratic\ via\ PDI$ [31]	38.96	170.15	∞
$Lure - Postinikov$ [38]	46.96	∞	∞
$Quadratic\ via\ NDI$ [7], [31]	46.96	∞	∞

2.6.3 Example 5

Consider system 2.21 with the following system parameters [38]:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

The results of using various Lyapunov type of functions are summarized in the following table:

2.7 Conclusions

In this chapter, we have presented a survey of the main results pertaining to linear dynamical systems subject to saturation including actuator, output and state types . The survey has outlined basic assumptions and has taken into considerations several technical views on the analysis and design procedures leading to global or semi-global stability results. A key feature has been the equal emphasis on the design of linear feedback laws, decentralized controllers. Results related stability with enlarging the domain of attraction and systems subject to multi-layered nested saturations have been provided. Some typical examples have been given to illustrate relevant issues.

Chapter 3

OVERLAPPING DECENTRALIZED CONTROL FOR INTERCONNECTED SYSTEMS SUBJECT TO SATURATION

3.1 Introduction

The development of techniques to design output feedback controllers for interconnected systems has been of great interest since the past few years [48]-[55]. Many techniques and design strategies were proposed and developed to form the control design of these systems. Various theorems were developed for calculating the unknowns of the dynamic output feedback controller gain matrix.Applications of multi-agent control designs subject to feedback have been generalized and looked upon. In the area of decentralized control designs it has been implied that the system with local feedback closed around the sub-systems is generally stable. Several methods were used for the development of static and dynamic output feedback designs as in [48, 49, 54, 55, 56, 59, 61, 62, 63]. LMI solution to the decentralized output feedback control problem for the interconnected non-linear systems was developed in [48], where the interacting non-linearity of each subsystem was considered o be bounded by a quadratic form of states of the overall system. Local output signals from each subsystems were used to generate the local control inputs. The robust stabilization problem of a class of nonlinear interconnected systems was considered in [49] and a decentralized dynamic output feedback controller was proposed. The authors formulated the controller design in the LMI framework, and used local sliding mode observers for the subsystems state estimation. However, the problem of designing local observers that are robust with respect to measurement noise is still unresolved. In [54], the authors developed a new LMI-based procedure for the design of decentralized dynamic output controllers for systems composed of linear subsystems coupled by uncertain nonlinear interconnections satisfying quadratic constraints. The scheme utilizes the general linear dynamic output feedback structure. The design procedure consists of two steps, the first providing the local Lyapunov matrices together with the corresponding robustness degrees, and the second the controller parameters providing a robustly stable overall system. A comprehensive review of decentralized control design techniques was provided in [55]. The synthesis of output feedback controllers with saturating inputs was studied in [61] where an observer based controller and a dynamic output feedback controller based on the circle criterion was developed via LMI formulation. In [62], the authors presented a method for designing an output feedback law that stabilizes a linear system subject to actuator saturation with a large domain of attraction. This method applies to general linear systems including strictly unstable ones. A nonlinear output feedback controller is first expressed in the form of a quasi-LPV system. Conditions under which the closed-loop system is locally asymptotically stable are then established in terms of the coefficient matrices of the controller. The design of the controller (gain matrices) that maximizes an estimate of the domain of attraction is then formulated and solved as an optimization problem with LMI constraints.

On another research front, the behavior of linear, time-invariant (LTI) systems subject to saturation has been extensively studied for several decades. It is known that saturation usually degrades the performance of a system and leads to instability. Over the last years systems subject to saturation has attracted a lot of researchers and a considerable amount of work has been done. Most of the study has been done on systems subject to actuator saturation, which involves problems as global, semi-global stabilization and local stabilization, anti-windup compensation, null controllable regions, to mention a few.

More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks, see [1] for a nice overview of application cases requiring a formal treatment of the saturation constraints. Most of the research efforts geared toward constructive linear or nonlinear control for saturated plants can be divided into two main strands. In the first one, called anti-windup design, a pre-designed controller is given,so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention, see [2, 3, 4] and the references therein. There are mainly two approaches to dealing with actuator saturation. One approach is to take control constraints into account at the outset of control design. A low-and-high gain method was presented in [4] to design linear semi-globally stabilizing controllers. The overlapping decomposition principle has been used extensively for interconnected systems for the design of the feedback controller [55].

In this chapter, we use the principle of overlapping decomposition to design output feedback controllers for each of the interconnected subsystems. This method helps us to differentiate each subsystem and design a local output feedback controller for the same and finally contract them to the original system comprising of all the subsystems. Controlling the interconnected systems with saturating inputs we have used the Overlapping design methodology and applied two types of controllers for the design, that is, static and dynamic output feedback schemes. For the design of controllers we firstly expand the system and carry on with the control design methods and after the controller has been designed we utilize the overlapping decomposition technique of contracting the expanded system to the original form.

3.2 Problem Statement

Consider an interconnected system composed of a finite number N of coupled subsystems and subject to input saturation is represented by:

$$\dot{x}(t) = Ax(t) + Bsat(u(t)) + h(t, x(t)) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

where $x = [x_1^t, ..., x_N^t]^t \in \Re^n,\ n = \sum_{j=1}^N n_j$ is the overall system state, $sat(u) = [sat(u)_1^t, ..., sat(u)_N^t]^t \in \Re^m,\ m = \sum_{j=1}^N m_j$ is the saturated input of the overall system and $y = [y_1^t, ..., y_N^t]^t \in \Re^p,\ p = \sum_{j=1}^N p_j$ is the measured output of the overall system. The model matrices are $A = diag\{A_{11}, .., A_{NN}\},\ A_{jj} \in \Re^{n_j \times n_j}$, $B = diag\{B_1, .., B_N\},\ B_j \in \Re^{n_j \times m_j}$ and $C = diag\{C_1, .., C_N\},\ C_j \in \Re^{p_j \times n_j}$. The function

$$h(t, x(t)) = [h_1^t(t, x(t)), .., h_N^t(t, x(t))]^t$$

is a vector function piecewise-continuous in its arguments. In the sequel, we assume that this function is uncertain and the available information is that, in the domains of continuity **G**, it satisfies the global quadratic inequality

$$h^t(t,x(t))h(t,x(t)) \;\; \leq \;\; x^t(t)\widetilde{R}^t\widetilde{\Phi}^{-1}\widetilde{R}x(t) \tag{3.3}$$

where $\widetilde{R} = [\widetilde{R}_1^t,..,\widetilde{R}_N^t]^t$, $\widetilde{R}_j \in \Re^{r_j \times n}$ are constant matrices such that $h(t,0) = 0$ and $x = 0$ is an equilibrium of system (3.1). With focus on the structural form of system (3.1), the jth subsystem model can be described by

$$\begin{aligned} \dot{x}_j(t) &= A_{jj}x_j(t) + B_j\mathbf{sat}(\mathbf{u_j})(t) + h_j(t,x) \\ y_j(t) &= C_jx_j(t) \end{aligned} \tag{3.4}$$

where $x_j(t) \in \Re^{n_j}$, $u_j(t) \in \Re^{m_j}$, $y_j(t) \in \Re^{p_j}$ are the subsystem state, input and measured output, respectively. The function $h_j \in \Re^{n_j}$ is a piecewise-continuous vector function in its arguments and in line of (3.3) it satisfies the quadratic inequality

$$h_j^t(t,x(t))h_j(t,x(t)) \;\; \leq \;\; \phi_j^2 \, x^t(t)\widetilde{R}_j^t\widetilde{R}_jx(t) \tag{3.5}$$

where $\phi_j > 0$, $j \in \{1,...,N\}$ are bounding parameters such that $\widetilde{\Phi} = diag\{\phi_1^{-2}I_{r_1},..,\phi_N^{-2}I_{r_N}\}$ where $I_{m_j} \in \Re^{m_j \times m_j}$ represents identity matrix. From (3.3) and (3.5), it is always possible to find a matrix $\widetilde{\Phi}$ such that

$$h^t(t,x(t))h(t,x(t)) \;\; \leq \;\; x^t(t)R^t\widetilde{\Phi}^{-1}Rx(t) \tag{3.6}$$

where $R = diag\{R_1, .., R_N\}$, $\widetilde{\Phi} = diag\{\delta_1 I_{r_1}, .., \delta_N I_{r_N}\}$ and $\delta_j = \phi_j^{-2}$. The saturation function $sat(u_j)$ is for $u \in \Re^m$ defined as,

$$\mathbf{sat(u_j)} = \begin{cases} u_{jmax} & u_j \geq u_{jmax}, \\ u_j & u_{jmin} < u_j < u_{jmax}, \\ u_{jmin} & u_j \leq u_{jmin} \end{cases} \tag{3.7}$$

where u_{jmin} and u_{jmax} are chosen to correspond to actual input limits either by measurement or by estimation. Input saturation can also be applied as upper and lower limits of input constraints as u_{jmin} and u_{jmax}, respectively. It is also assumed that the pair (A_{jj}, B_j) is a controllable pair and (C_j, A_{jj}) is an observable pair for all $j \in I := 1, 2, \ldots, N$.

Remark 3.2.1 *It is significant to observe that the local function $h_j(.,.)$ depends on the full state and delayed state vectors $x(t)$, $x(t-\tau)$ and therefore inequality (3.5) for $j = 1, ..., N$ represents a set of coupling relations that has to be manipulated simultaneously in order to achieve the desired objective.*

Our objective in this work is to design a static output feedback controller and a dynamic output feedback controller that stabilizes the interconnected continuous system 3.1,3.2 subject to input saturation using the overlapping decomposition technique.

3.3 Controller Design

In this section we will consider the two control designs for the interconnected system subject to input saturation, one being the Static Output Feedback Design and the other as Dynamic Output Feedback Design using the Overlapping Decomposition technique. In the following sub-section we will explain the overlapping decomposition principle also a version of the inclusion principle.

3.3.1 Overlapping Decomposition

In most of the systems, the subsystems share common parts. It is advantageous to use this fact for building and designing of a decentralized control using overlapping information from these subsystems. On using the overlapping technique it is clear that the overlapping subsystems appear as disjoint. Using each subsystem as different systems a decentralized control can be designed in the expanded space. These designed controls are later contracted for their implementation in the original system. For these kind of systems a mathematical framework known as *Inclusion Principle* is used. The Inclusion Principle was proposed in the early 1980s in the context of analysis and control of complex systems [55, 64, 65].

The main idea of the Inclusion Principle is to expand an initial system, with shared components, into higher dimensions in which overlapped subsystem appear as disjoint. Under certain conditions the expanded system contains the essential information about the initial system. The relation between the initial and the expanded system is constructed on the basis of appropriate linear transformations. These include a set of complementary matrices which have to satisfy well established necessary and sufficient conditions to ensure the Inclusion Principle. The conditions given in previous works [64, 65] on the complementary matrices to ensure the Inclusion Principle have a fundamental, implicit nature, in the sense that they have the form of matrix products from which it is not easy to select specific values for the matrices. The selection of these matrices helps in obtaining the expanded system and studying their properties. These matrices influence on properties like stability, controllability or even observability.

A version of the Inclusion Principle

A system $\tilde{S}$ includes the system S, denoted by $\tilde{S} \supset S$, if there exists a pair matrices (U,V) satisfying UV=I and such that for any initial state x_0 and any fixed point sat(u(t)) of S, the choice $\tilde{x}_0 = Vx_0$ of the system $\tilde{S}$ implies $x(t; x_0, sat(u)) = U\tilde{x}(t; \tilde{x}_0(t), sat(u)) \forall t \geq 0$. If $\tilde{S} \supset S$, then $\tilde{S}$ is said to be an expansion of S and S is a contraction of $\tilde{S}$.

Consider a system and cost function of the form:

$$\mathbf{S} : \dot{x} = Ax(t) + Bsat(u(t)) \tag{3.8}$$

$$J(x_0, sat(u)) = \int_{\infty}^{0} \left[x^t Q x + sat(u)^t R sat(u)\right] \tag{3.9}$$

Its expanded system with the cost functions are of the form

$$\tilde{\mathbf{S}} : \dot{\tilde{x}} = \tilde{A}\tilde{x}(t) + \tilde{B}sat(u(t)) \tag{3.10}$$

$$J(x_0, sat(u) = \int_{\infty}^{0} \left[\tilde{x}^T \tilde{Q}\tilde{x} + sat(\tilde{u})^T \tilde{R} sat(\tilde{u})\right] \tag{3.11}$$

where x(t) $\in \Re^n$ and sat(u(t)) $\in \Re^m$ are the states and the saturated inputs of S and $\tilde{x}(t) \in \Re^{\tilde{n}}$ and sat(u(t)) $\in \Re^m$ are corresponding to the expanded system $\tilde{S}$. The matrices A, B and $\tilde{A}, \tilde{B}$ are constant of dimensions $n \times n$, $n \times m$ and $\tilde{n} \times \tilde{n}, \tilde{n} \times m$, respectively.

The weighting matrices Q, $\tilde{Q}$ are symmetric positive definite and R, $\tilde{R}$ are symmetric positive definite. Suppose that the dimensions of the state vector x(t) are smaller than or equal to the vector $\tilde{x}(t)$ of system $\tilde{S}$. Let $x(t; x_0, sat(u))$ denote the unique solution of S for a fixed input sat(u(t)) and an initial state $x(0) = x_0$. Similar notation for $\tilde{x}(t; \tilde{x}(t), sat(u))$ is used for system $\tilde{S}$.

Let us consider the following transformations

$$V : \Re^n \rightarrow \Re^{\tilde{n}}, U : \Re^{\tilde{n}} \rightarrow \Re^n \tag{3.12}$$

Where rank(V)=n and such that $UV = I_n$, where I_n is identity matrix of indicated dimension. Given a matrix V, the pseudo inverse matrix U can be obtained by $U = (V^T V)^{-1} V^T$.

Definition 3.3.1 *The system $\tilde{S}$ is an extension of the system S, if there exists transformations as in 3.12 such that for any initial state $x_0 \in R^n$ and for any saturated input $sat(\tilde{u}(t)) \in R^{\tilde{m}}$, 0$\leq t < \inf$, the choice,*

$$\begin{aligned} \tilde{x}_0 &= Vx_0 \\ sat(u(t)) &= Usat(\tilde{u}(t)) \;\; \forall t \geq 0 \end{aligned} \tag{3.13}$$

implies that

$$\tilde{x}(t; \tilde{x}_0, sat(\tilde{u})) = Vxt; x_0, sat(u) \; \forall t \geq 0 \tag{3.14}$$

There are two particular cases within the inclusion principle called restriction and aggregation. They are defined as follows:

1. A system S is a restriction of $\tilde{S}$, if there exists a pair of matrices(U,V) satisfying UV=I such that for an initial state x_0 and any fixed input sat(u(t)) of S, the choice $\tilde{x}_0 = Vx_0$ implies

$$\tilde{x}(t; \tilde{x}_0, sat(u)) = Vx(t; x_0, sat(u))\forall t \geq 0$$

2. A system S is an aggregation of $\tilde{S}$ if there exists a pair of matrices(U,V) satisfying UV=I such that for an initial state $\tilde{x}_0$ and any fixed input u(t)

of $\tilde{S}$, the choice $x_0 = U\tilde{x}_0$ implies

$$\tilde{x}(t; \tilde{x}_0, sat(u)) = Vx(t; x_0, sat(u)) \forall t \geq 0$$

Complementary matrices

The expanded matrices $\tilde{A}, \tilde{B}, \tilde{Q}$ and $\tilde{R}$ of $\tilde{S}$ can be expressed as

$$\tilde{A} = VAU + M, \tilde{B} = VB + N \tag{3.15}$$

$$\tilde{Q} = U^t QU + M_Q, \tilde{R} = R + N_R \tag{3.16}$$

where M,N,M_Q and N_R are the complementary matrices. The designer have to choose the matrices M_Q and N_R is such a way that the corresponding expanded weighting matrices $\tilde{Q}$ and $\tilde{R}$ are symmetric positive semi-definite and symmetric positive definite matrices, respectively.

For $\tilde{S}$ to be an expansion of S, a proper choice of M and N is required [6, 7, 9, 10, 11]. In terms of complementary matrices, the previous definitions can also be written as follows,

1. A system S is a restriction of the system $\tilde{S}$ if and only if MV=0 and N=0.
2. A system S is an aggregation of the system $\tilde{S}$ if and only if UM=0 and UN=0.

By using complementary matrices different expanded systems $\tilde{S}$ can be obtained.

Theorem 3.3.1 $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{J}}) \supset (\boldsymbol{S}, \boldsymbol{J})$ *if either*

$$(i) \quad MV = 0, \quad N = 0, \quad V^t M_Q V = 0, \quad N_R = 0 \tag{3.17}$$

or

$$\begin{aligned} (ii) \quad UM^iV = 0, \quad M_Q M^{i-1} N = 0, \quad M_Q M^{i-1} V = 0 \\ UM^{i-1}N = 0, \quad N_R = 0, \quad \forall i \in \tilde{n} \end{aligned} \tag{3.18}$$

For proof see [54].

In the sequel, we choose $\tilde{Q}$, Q, $\tilde{R}$ and R as identity matrices although only one of them suffice. Both of the conditions of the aforementioned theorem are verified by choosing

$$M = 0, \quad M_Q = 0, \quad N = 0, \; N_R = 0 \tag{3.19}$$

with the state $\tilde{x}(t) \in \mathbf{R}^{\tilde{n}}$. In the sequel, we define the following relations

$$\begin{aligned} \tilde{A} = U\bar{A}V + M, \quad \tilde{B} = V\bar{B} + N \\ \tilde{Q} = U^t Q U + M_Q, \quad \tilde{R} = R + N_R \end{aligned} \tag{3.20}$$

with $UV = I$, and $\tilde{Q}$, Q, $\tilde{R}$ and R are the appropriate weighting matrices in conventional LQR designs. The matrices M, N, M_R and N_R are real matrices of appropriate dimensions.

3.3.2 Contractibility conditions

: A control law $u = -\tilde{K}\tilde{x}$ is contractible to control law $u = -K\bar{x}$ if and only if ([9])

$$FM^{i-1}V = 0, \quad FM^{i-1}N = 0 \ \ \forall i \in \tilde{n} \tag{3.21}$$

Where F is given by

$$\tilde{K} = KU + F \tag{3.22}$$

Since we have chosen $M = 0$ and $N = 0$, the aforementioned conditions are satisfied. by Corollary 8.14 in [9] if

$$MV = 0, \quad N = 0 \tag{3.23}$$

then, the contracted K can be obtained as:

$$K = \tilde{K}V \tag{3.24}$$

3.4 Application to A Nuclear Power Plant System

In what follows, we apply the foregoing decentralized control methodology to a nuclear power plant model [69, 70]:

3.4.1 System description

The system under consideration is described by the state-space model of the form (3.8): where the state vector $x \in \Re^{20}$ and input vector $u \in \Re^{4}$ are defined as follows:

$$\begin{aligned}
x &= [x_a \; x_b \;\; x_c]^t \\
x_a &= [\delta P \;\; \delta C_1 \;\; \delta C_2 \;\; \delta C_3 \;\; \delta C_4 \;\; \delta C_5 \;\; \delta C_6]^t \\
x_b &= [\delta T_f \;\; \delta T_{C1} \;\; \delta T_{C2} \;\; \delta P_P \;\; \delta T_m \;\; \delta T_p \;\; \delta P_s]^t \\
x_c &= [\delta T_{UP} \;\; \delta T_{HL} \;\; \delta T_{IP} \;\; \delta T_{OP} \;\; \delta T_{CL} \;\; \delta T_{LP}]^t
\end{aligned}$$

and

δP : deviation in reactor power from initial steady-state value

δC_i : deviation of normalized precursor concentrations, i=1:6

δT_f : deviation of fuel temperature in fuel node

δT_{C1} : deviation of temperature in the first coolant node

δT_{C2} : deviation of temperature in the second coolant node

δP_P : deviation of primary system pressure

δT_P : deviation of temperature of primary coolant node in the steam generator

δT_m : deviation of steam generator tube metal temperature

δP_s : deviation of steam pressure from its initial steady-state value

δT_{UP} : deviation of the reactor upper plenum temperature

δT_{LP} : deviation of the reactor lower plenum temperature

δT_{HL} : deviation of hot leg temperature

δT_{IP} : deviation of temperature of primary coolant in the steam generator or inlet plenum

δT_{OP} : deviation of temperature of primary coolant in the steam generator or outlet plenum

δT_{CL} : deviation of cold leg temperature

with

$$u = [\delta\rho_{rod} \;\; \delta W_{FW} \;\; \delta W_P \;\; \delta Q]^t$$

where

$\delta\rho_{rod}$: reactivity due to control rod movement

δW_{FW} : deviation of feed water flow rate in steam generator

δW_P : deviation of primary water flow rate to the steam generator

δQ : rate of heat addition to the pressurizer fluid with electric heater

Since the system order is high, the matrices A and B are arbitrary expressed in partitioned form as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} B \ = \ \begin{bmatrix} B_1 & B_2 \end{bmatrix}^t , \tag{3.25}$$

where the numerical values are given in Appendix A.

3.4.2 Dynamic behavior

Twenty eigen values of the system are $-400.1, -5.777, -2.86 \pm 0.07954i - 2.37 \pm 3.487i, -1.51 \pm 0.589i,$
$-1.04, -0.676 \pm 0.382i - 0.715, -0.37 \pm 0.103i, -0.286, -0.108, 0.0019, -0.012, -0.043,$ and -0.029. We see that there is one pole (0.0019) at right half plane, making the linearized system unstable.

3.4.3 Permutations

In order to successfully apply the decentralized control methodology, the input matrix B of a certain system should be in block-diagonal form. The system can then be decomposed into multiple subsystems with orders equal to the rows of the corresponding block of the input matrix B. It is obvious that in our case matrix B is not in the diagonal form. There are four inputs in the system and only four non-zero elements in the input matrix B all appearing in different columns. As such, the B matrix can be transformed to block-diagonal form by a set of permutations. After performing the permutations, the system described by (3.8) can be described as

$$\bar{\mathbf{S}}: \quad \dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u} \tag{3.26}$$

Since the columns of input matrix are not shuffled, the input vector remains unchanged, whereas the re-arranged system state vector $\bar{x}$ and resultant matrices $\bar{A}$ $\bar{B}$ after aforementioned permutations are mentioned in the followings.

$$\begin{aligned}
\bar{x} &= [\bar{x}_1 \;\; \bar{x}_2 \;\; \bar{x}_3 \;\; \bar{x}_4]^T \\
\bar{x}_1 &= [\delta P \;\; \delta C_1 \;\; \delta C_2 \;\; \delta C_3 \;\; \delta C_4\;]^T \\
\bar{x}_2 &= [\delta P_s \;\; \delta C_6 \;\; \delta T_f \;\; \delta T_{C1} \;\; \delta T_{C2}]^T \\
\bar{x}_3 &= [\delta T_{HL} \;\; \delta P_P \;\; \delta T_p \;\; \delta C_5 \;\; \delta T_f\;]^T \\
\bar{x}_4 &= [\delta C_4 \;\; \delta T_{IP} \;\; \delta T_{OP} \;\; \delta T_{CL} \;\; \delta T_{LP}]^T
\end{aligned}$$

and the numerical values of the permuted matrices are given in the Appendix A.

Since the system has four inputs, and therefore, to attain a maximum degree of decomposition, we decompose the system into four subsystems with certain degree of overlaps.To proceed, we first decompose the state vector $\bar{x}$ in to seven components as:

$$\bar{x} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \bar{x}_6 & \bar{x}_7 \end{bmatrix}^t \tag{3.27}$$

where $x_i \in \Re^{n_i}$, for $i = 1, 2, \ldots, 7$ and $n_1 = 5$, $n_2 = 2$, $n_3 = 3$, $n_4 = 2$, $n_5 = 3$, $n_6 = 2$ and $n_7 = 3$. This partition of the state $\bar{x}$ induces a partition of the the matrix $\bar{A}$ as

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} & \bar{A}_{15} & \bar{A}_{16} & \bar{A}_{17} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} & \bar{A}_{25} & \bar{A}_{26} & \bar{A}_{27} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & \bar{A}_{34} & \bar{A}_{35} & \bar{A}_{36} & \bar{A}_{37} \\ \bar{A}_{41} & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44} & \bar{A}_{45} & \bar{A}_{46} & \bar{A}_{47} \\ \bar{A}_{51} & \bar{A}_{52} & \bar{A}_{53} & \bar{A}_{55} & \bar{A}_{55} & \bar{A}_{56} & \bar{A}_{57} \\ \bar{A}_{61} & \bar{A}_{62} & \bar{A}_{63} & \bar{A}_{64} & \bar{A}_{65} & \bar{A}_{66} & \bar{A}_{67} \\ \bar{A}_{71} & \bar{A}_{72} & \bar{A}_{73} & \bar{A}_{74} & \bar{A}_{75} & \bar{A}_{76} & \bar{A}_{77} \end{bmatrix} \tag{3.28}$$

Where the sub matrices have appropriate dimensions. The seven components of the state vector $\bar{x}$ are arranged to four overlapping components as follows:

$$\begin{aligned} \tilde{x}_1 &= \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^t, \quad \tilde{x}_2 = \begin{bmatrix} \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \end{bmatrix}^t \\ \tilde{x}_3 &= \begin{bmatrix} \bar{x}_4 & \bar{x}_5 & \bar{x}_6 \end{bmatrix}^t, \quad \tilde{x}_4 = \begin{bmatrix} \bar{x}_6 & \bar{x}_7 \end{bmatrix}^t \end{aligned} \tag{3.29}$$

These four overlapping state vectors components constitute a new state vector

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 \end{bmatrix}^T \in R^{\tilde{n}}, \ \tilde{n} = 26 \tag{3.30}$$

The vector $\tilde{x}$ is related to $\bar{x}$ by a linear transformation

$$\tilde{x} = V\bar{x} \tag{3.31}$$

where V is the $\tilde{n} \times n$ matrix

$$V = \begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times2} & \mathbf{0}_{5\times3} & \mathbf{0}_{5\times2} & \mathbf{0}_{5\times3} & \mathbf{0}_{5\times2} & \mathbf{0}_{5\times3} \\ \mathbf{0}_{2\times5} & \mathbf{I}_2 & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times5} & \mathbf{I}_2 & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times5} & \mathbf{0}_{3\times2} & \mathbf{I}_3 & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{I}_2 & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{I}_2 & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times5} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} \\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{I}_2 & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} & \mathbf{I}_2 & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times5} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{I}_3 \end{bmatrix} \tag{3.32}$$

with $\tilde{n} = 26$, then V is a 26×20 matrix. I_i represents an identity matrix of order i, $\mathbf{0}_{i\times j}$, represents a $i \times j$ zero matrix and the matrix $U \in \mathbf{R}^{20\times26}$ is defined as $U = V^{\dagger}$, the pseudo-inverse of V.

Invoking the conditions in 3.15, the expanded system can be expressed as:

$$\tilde{\mathbf{S}} : \ \dot{\tilde{x}} \ = \ \tilde{A}\tilde{x} + \tilde{B}sat(u(t)) \tag{3.33}$$

$$\tilde{A} \ = \ V\bar{A}U, \quad \tilde{B} = V\bar{B} \tag{3.34}$$

Matrix $\tilde{A}$ is now expressed as

$$\tilde{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & 0 & 0 & 0 & \bar{A}_{13} & \bar{A}_{14} & \bar{A}_{15} & \bar{A}_{16} & \bar{A}_{17} \\ \bar{A}_{21} & \bar{A}_{22} & 0 & 0 & 0 & \bar{A}_{23} & \bar{A}_{24} & \bar{A}_{25} & \bar{A}_{26} & \bar{A}_{27} \\ \bar{A}_{21} & 0 & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} & 0 & 0 & \bar{A}_{25} & \bar{A}_{26} & \bar{A}_{27} \\ \bar{A}_{31} & 0 & \bar{A}_{32} & \bar{A}_{33} & \bar{A}_{34} & 0 & 0 & \bar{A}_{35} & \bar{A}_{36} & \bar{A}_{37} \\ \bar{A}_{41} & 0 & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44} & 0 & 0 & \bar{A}_{45} & \bar{A}_{46} & \bar{A}_{47} \\ \bar{A}_{41} & 0 & \bar{A}_{42} & \bar{A}_{43} & 0 & \bar{A}_{44} & \bar{A}_{45} & \bar{A}_{46} & 0 & \bar{A}_{47} \\ \bar{A}_{51} & 0 & \bar{A}_{52} & \bar{A}_{53} & 0 & \bar{A}_{54} & \bar{A}_{55} & \bar{A}_{56} & 0 & \bar{A}_{57} \\ \bar{A}_{61} & 0 & \bar{A}_{62} & \bar{A}_{63} & 0 & \bar{A}_{64} & \bar{A}_{65} & \bar{A}_{66} & 0 & \bar{A}_{67} \\ \bar{A}_{61} & 0 & \bar{A}_{62} & \bar{A}_{63} & 0 & \bar{A}_{64} & \bar{A}_{65} & 0 & \bar{A}_{66} & \bar{A}_{67} \\ \bar{A}_{71} & 0 & \bar{A}_{72} & \bar{A}_{73} & 0 & \bar{A}_{74} & \bar{A}_{75} & 0 & \bar{A}_{76} & \bar{A}_{77} \end{bmatrix} \tag{3.35}$$

The overlapping subsystems $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ and $\tilde{A}_4$ are now described in the followings:

$$\tilde{A}_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\ \bar{A}_{32} & \bar{A}_{33} & \bar{A}_{34} \\ \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{14} \end{bmatrix} \tag{3.36}$$

$$\tilde{A}_3 = \begin{bmatrix} \bar{A}_{44} & \bar{A}_{45} & \bar{A}_{46} \\ \bar{A}_{54} & \bar{A}_{55} & \bar{A}_{56} \\ \bar{A}_{64} & \bar{A}_{65} & \bar{A}_{66} \end{bmatrix}, \quad \tilde{A}_4 = \begin{bmatrix} \bar{A}_{66} & \bar{A}_{67} \\ \bar{A}_{76} & \bar{A}_{77} \end{bmatrix} \tag{3.37}$$

The interconnections among the overlapped subsystems can be easily obtained by simple inspection of $\tilde{A}$ in (3.35). Interconnection matrices $H_i, i = 1, 2, 3, 4$ associated with each of the subsystem is given by

$$
\begin{aligned}
\tilde{H}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{A}_{13} & \bar{A}_{14} & \bar{A}_{15} & \bar{A}_{16} & \bar{A}_{17} \\ 0 & 0 & 0 & 0 & 0 & \bar{A}_{23} & \bar{A}_{24} & \bar{A}_{25} & \bar{A}_{26} & \bar{A}_{27} \end{bmatrix} \\
\tilde{H}_2 &= \begin{bmatrix} \bar{A}_{21} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{25} & \bar{A}_{26} & \bar{A}_{27} \\ \bar{A}_{31} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{35} & \bar{A}_{36} & \bar{A}_{37} \\ \bar{A}_{41} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_{45} & \bar{A}_{46} & \bar{A}_{47} \end{bmatrix} \\
\tilde{H}_3 &= \begin{bmatrix} \bar{A}_{41} & 0 & \bar{A}_{42} & \bar{A}_{43} & 0 & 0 & 0 & 0 & 0 & \bar{A}_{47} \\ \bar{A}_{51} & 0 & \bar{A}_{52} & \bar{A}_{53} & 0 & 0 & 0 & 0 & 0 & \bar{A}_{57} \\ \bar{A}_{61} & 0 & \bar{A}_{62} & \bar{A}_{63} & 0 & 0 & 0 & 0 & 0 & \bar{A}_{67} \end{bmatrix} \\
\tilde{H}_4 &= \begin{bmatrix} \bar{A}_{61} & 0 & \bar{A}_{62} & \bar{A}_{63} & 0 & \bar{A}_{64} & \bar{A}_{65} & 0 & 0 & 0 \\ \bar{A}_{71} & 0 & \bar{A}_{72} & \bar{A}_{73} & 0 & \bar{A}_{74} & \bar{A}_{75} & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

The expanded input matrix $\tilde{B}$ is given by:

$$
\tilde{B} = \begin{bmatrix} \bar{B}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{B}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{B}_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{B}_4 \end{bmatrix} \tag{3.38}
$$

where $\bar{B}_i,\ i = 1, 2, 3, 4$ are given in the Appendix. Furthermore, the stabilizability check was performed for all the four subsystems which showed that all the subsystems are stabilizable although non of the subsystems is *controllable*. After the expansion of the original system we will now design the static and dynamic output feedback controls in order to stabilize each subsystem individually, which later on will be contracted to form the original stabilized system.

3.5 Static Output Feedback Design

We will consider the overall expanded system for the design of the static feedback controller where all the subsystems will be controlled by local static output feedback, which is given as

$$\begin{aligned} \tilde{\mathbf{S}} : \dot{\tilde{x}}(t) &= \tilde{A}_D \tilde{x}(t) + \tilde{B}_D sat(u(t)) + \tilde{H}(t, \tilde{x}) \\ y(t) &= \tilde{C}_D \tilde{x}(t) \end{aligned} \tag{3.39}$$

where, $\tilde{A}_D = diag(A_1, \ldots, A_N)$,$\tilde{B}_D = diag(B_1, \ldots, B_N)$,$\tilde{C}_D = diag(C_1, \ldots, C_N) and$ $\tilde{H}$ will be the diagonal interconnections as $\tilde{H} = diag(H_1, H_2, \ldots, H_N)$.

Also the interconnections bounds (3.5) will be as follows,

$$\tilde{h}_j^t(t, \tilde{x})\tilde{h}_j(t, \tilde{x}) \leq \tilde{x}^t(\sum_{i=1}^{N} \alpha_i^2 \tilde{H}_i^t \tilde{H}_i)\tilde{x} := \tilde{x}^t \Gamma^t \Gamma \tilde{x} \tag{3.40}$$

where $\Gamma^t\Gamma := \widetilde{R}^t\widetilde{\Phi}^{-1}\widetilde{R}$. The pair $(\tilde{A}_D, \tilde{B}_D)$ is controllable and the pair $(\tilde{C}_D, \tilde{A}_D)$ is observable, which is the direct result of each subsystem being controllable and observable. Using the above expanded system we will be designing a decentralized linear controller and a decentralized linear observer that will stabilize the system. We consider the following linear decentralized controller and observer,

$$\begin{aligned} \dot{\hat{x}}(t) &= \tilde{A}_D\hat{x}(t) + \tilde{B}_D sat(u(t)) + \tilde{L}_D(y - \tilde{C}_D)\hat{x}(t) \\ u(t) &= \tilde{K}_D\hat{x}(t) \end{aligned} \tag{3.41}$$

where $\tilde{K}_D = diag(\tilde{K}_1, \ldots, \tilde{K}_{\tilde{N}})$ and $\tilde{L}_D = diag(\tilde{L}_1, \ldots, \tilde{L}_{\tilde{N}})$ are the controller gain matrix and the observer gain matrix, respectively. The closed loop dynamics of the expanded system is,

$$\begin{aligned} \dot{\tilde{x}}(t) &= (\tilde{A}_D + \tilde{B}_D\tilde{K}_D)\tilde{x}(t) - \tilde{B}_D\tilde{K}_D\hat{x}(t) + \tilde{H}(t) \\ \dot{\hat{x}}(t) &= (\tilde{A}_D - \tilde{L}_D\tilde{C}_D)\hat{x}(t) + \tilde{H}(t) \end{aligned} \tag{3.42}$$

where, $\tilde{H}$(t) is the interconnection function for the expanded system. Let

$$\tilde{A}_c = \tilde{A}_D + \tilde{B}_D\tilde{K}_D \;, \tilde{A}_o = \tilde{A}_D - \tilde{L}_D\tilde{C}_D \tag{3.43}$$

The closed-loop dynamics will be in the form of,

$$\dot{\tilde{x}}(t) = \tilde{A}_c\tilde{x}(t) - \tilde{B}_D\tilde{K}_D\hat{x}(t) + \tilde{H}(t) \tag{3.44}$$

$$\dot{\hat{x}}(t) = \tilde{A}_o\hat{x}(t) + \tilde{H}(t) \tag{3.45}$$

For each subsystem, we determine the controller and observer gain matrices K_{D_i} and L_{D_i}. For simplicity in exposition, we focus on the **expanded system only** and after completing the design task for the expanded system($\tilde{\mathbf{S}}$), it will be contracted to the actual system($\mathbf{S}$) using the foregoing overlapping decomposition technique.

For some matrix $Y > 0$, define

$$
\begin{aligned}
\bar{M}_D &= K_D Y \\
\begin{bmatrix} S_1 & S_2 \end{bmatrix} &= \begin{bmatrix} -B_D K_D & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, F_c = \begin{bmatrix} W_c^t & Y H_1^t & \dots & Y H_N^t \\ \bullet & \gamma_1 I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \dots & -\gamma_N I \end{bmatrix} < 0, \\
F_o &= \begin{bmatrix} W_o & P_o \\ P_o & -I \end{bmatrix} < 0 \\
W_c^t &= Y A_D^t + A_D Y + (B_D K_D)^t + (B_D K_D) \\
W_o &= A_D^t P_o + P_o A_D - P_o L_D C_D - (P_o L_D C_D)^t
\end{aligned}
$$

the following theorem establishes the main design result

Theorem 3.5.1 *Consider the following optimization problem for finding the Controller K_{D_i} and Observer L_{D_i} of each subsystems. To determine Y, P_o, K_D,*

L_D and $\gamma_i, i \in I$, from the following optimization problem,

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} \gamma_i \\ \textit{subject to}\, Y > 0,\ & P_o > 0 \\ & \begin{bmatrix} F_c & S_1 & S_2 \\ \bullet & W_o & P_o \\ \bullet & \bullet & -I \end{bmatrix} < 0 \end{aligned} \tag{3.46}$$

The optimization problem (3.46) has to be solved by two steps [6]:

Step 1: Maximize the interconnection bounds $\alpha_i (= \frac{1}{\gamma_i})$ by solving the following optimization problem,

$$\begin{aligned} & \min \sum_{i=1}^{N} \gamma_i \\ & \textit{subject to}\, Y > 0,\ F_c < 0 \end{aligned} \tag{3.47}$$

This gives Y and $\bar{M}_D$. The control gain can then be calculated as,

$$K_D = \bar{M}_D Y^{-1} \tag{3.48}$$

Step 2: Using the K_D obtained from **Step 1**, find P_o and N_D by solving the following optimization problem:

$$\min \sum_{i=1}^{N} \beta_i \tag{3.49}$$
$$\textit{subject to } P_o > 0 \; \Lambda > 0$$
$$\begin{bmatrix} \Lambda F_c & S_1 & S_2 \\ \bullet & W_o & P_o \\ \bullet & \bullet & -I \end{bmatrix} < 0$$

where $\Lambda = diag(\beta_1 I_1, \ldots, \beta_N I_N)$, I_i denotes the $n_i \times n_i$ identity matrix, and $W_o = A_D^t P_o + P_o A_D - N_D C_D - (N_D C_D)^t$ and $N_D = P_o L_D$. The matrices F_c and S_1 in **Step 2** are obtained from **Step 1**. The observer gain L_D is obtained as:

$$L_D = P_o^{-1} N_D \tag{3.50}$$

Proof: We consider the following Lyapunov function candidate,

$$V(x, \bar{x}) = x^t P_c x + \bar{x}^t P_o x, \quad P_c > 0, \quad P_o > 0 \tag{3.51}$$

The time derivative of $V(x, \bar{x})$ along the trajectories of (3.45) is given by,

$$\dot{V}(x, \bar{x}) = \begin{bmatrix} x \\ \bar{x} \\ H \end{bmatrix}^t \begin{bmatrix} A_c^t \bar{P}_c + \bar{P}_c A_c & -\bar{P}_c B_D K_D & \bar{P}_c \\ \bullet & A_o^t \bar{P}_o + \bar{P}_o A_o & \bar{P}_o \\ \bullet & \bullet & 0 \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ H \end{bmatrix} \leq 0 \tag{3.52}$$

On considering the bounds on the interconnections (3.5), which can be written as,

$$\begin{bmatrix} x \\ \bar{x} \\ H \end{bmatrix} \begin{bmatrix} -\Gamma^t\Gamma & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & I \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ H \end{bmatrix} \leq 0. \tag{3.53}$$

The stabilization of system (3.45) requires that $\dot{V}(x, \bar{x}) < 0 \quad \forall x, \bar{x} \neq 0$. This entails from (3.52) and (3.53) with $\bar{P}_c > 0 \ \bar{P}_o > 0 \ \tau > 0$ via the **S**-procedure [10] that

$$\begin{bmatrix} A_c^t\bar{P}_c + \bar{P}_cA_c & -\bar{P}_cB_DK_D & \bar{P}_c \\ \bullet & A_o^t\bar{P}_o + \bar{P}_oA_o & \bar{P}_o \\ \bullet & \bullet & 0 \end{bmatrix} - \tau \begin{bmatrix} -\Gamma^t\Gamma & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & I \end{bmatrix} < 0 \tag{3.54}$$

With $P_c = \frac{\bar{P}_c}{\tau} \ \ P_o = \frac{P_o}{\tau}$, inequality (3.54) is equivalent to

$$\begin{bmatrix} A_c^tP_c + P_cA_c + \Gamma^t\Gamma & -P_cB_DK_D & P_c \\ \bullet & A_o^tP_o + P_oA_o & P_o \\ \bullet & \bullet & -I \end{bmatrix} < 0 \tag{3.55}$$

with $P_c > 0 \ \ P_o > 0$. Considering (3.40) and (3.43), and applying the Schur complement to inequality (3.55) we get,

$$\begin{bmatrix} W_c & -P_cB_DK_D & P_c & \alpha_1 H_1^t & \dots & \alpha_N H_N^t \\ \bullet & W_o & P_o & 0 & \dots & 0 \\ \bullet & \bullet & -I & 0 & \dots & 0 \\ \bullet & \bullet & \bullet & -I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \dots & -I \end{bmatrix} < 0 \tag{3.56}$$

Rearranging and scaling columns and rows related to H_i, i $\in I$ of (3.56) we obtain,

$$\begin{bmatrix} W_c & H_1^t & \dots & H_N^t & -P_cB_DK_D & P_c \\ \bullet & -\gamma I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \bullet & \bullet & \dots & -\gamma_N I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & W_o & P_o \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} \tag{3.57}$$

where $\gamma_i = \frac{1}{\alpha_i^2} > 0$. The optimization problem now becomes as follows,

$$\min \sum_{i=1}^{N} \gamma_i \quad \textit{subject to } (3.57) \tag{3.58}$$

This selection of the control gain matrix K_D and the observer gain matrix L_D does not only stabilize the overall system (3.45) but also simultaneously maximizes the interconnection bounds α_i.

Since there are coupled terms of matrix variables, P_c and K_D, and P_o and L_D in the inequality(3.57), the above inequality becomes a BMI. We will transform the inequality to a form which is affine in the unknown variables. To achieve this, we introduce variables,

$$M_D = P_c B_D K_D, \quad N_D = P_o L_D \tag{3.59}$$

Then the optimization problem (3.57) becomes,

$$\min \quad l\sum_{i=1}^{N} \gamma_i$$

$$\begin{bmatrix} W_c & H_1^t & \dots & H_N^t & -P_c B_D K_D & P_c \\ \bullet & -\gamma_1 I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \bullet & \bullet & \dots & -\gamma_N I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & W_o & P_o \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

The solution to the above optimization problem gives M_D and N_D. The controller and observer gain matrices were obtained as in [55] as,

$$L_D = P_0^{-1} N_D$$

The controller gain matrix K_D can be obtained if the matrix B_D is invertible,

$$K_D = B_D^{-1} P_C^{-1} M_D$$

If matrix B_D is not invertible, that is, very restrictive, then it becomes very difficult to obtain the control gain matrix K_D from (3.60). To overcome this we will pre and post multiply the (3.57) by diag(P_c^{-1}, I) and define $\mathrm{Y} = P_c^{-1}$ to obtain the following equivalent conditions:

$$\begin{bmatrix} W_c^t & YH_1^t & \dots & YH_N^t & -B_D K_D & I \\ \bullet & -\gamma_1 I & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \bullet & \bullet & \dots & -\gamma_N I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & W_o & P_o \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \qquad (3.60)$$

we retained the representation (3.46) which completes the proof.

3.6 Dynamic Output Feedback Design

Building on the foregoing design of static output feedback control, we now direct the attention to the design of the dynamic output feedback controller of the form

$$\begin{aligned} \mathbf{K} : \dot{z}(t) &= A_k z(t) + B_k y(t) \\ u(t) &= C_k z(t) + D_k y(t) \end{aligned} \tag{3.61}$$

where, z $\in \Re^{n_c}$ is the controller state, u(t) $\in \Re^m$ is the controller output. Similarly, for the expanded system (3.39), the dynamic output feedback controller of the form:

$$\begin{aligned} \tilde{\mathbf{K}}_{\mathbf{D}} : \dot{z}_i(t) &= A_{k_i} z_i(t) + B_{k_i} y_i(t) \\ u_i(t) &= C_{k_i} z_i(t) + D_{k_i} y_i(t) \end{aligned} \tag{3.62}$$

with $z_i(t) \in \Re^{\tilde{n_c}}$ with appropriate dimensions and

$$A_k = blockdiag\{A_{k1},\ ...,\ A_{kN}\},\ B_k = blockdiag\{B_{k1},\ ...,\ B_{kN}\}$$

$$C_k = blockdiag\{C_{k1},\ ...,\ C_{kN}\},\ D_k = blockdiag\{D_{k1}, ..., D_{kN}\}$$

Appending system (3.39) and controller (3.63), we obtain the closed-loop system:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_i(t) \\ \dot{z}_i(t) \end{bmatrix} &= \begin{bmatrix} \tilde{A}_{D_i} + \tilde{B}_{D_i} D_{k_i} \tilde{C}_{D_i} & \tilde{B}_{D_i} C_{k_i} \\ B_{k_i} \tilde{C}_{D_i} & A_{k_i} \end{bmatrix} \begin{bmatrix} x_i(t) \\ z_i(t) \end{bmatrix} \\
&+ \sum_{j \neq i} \begin{bmatrix} \tilde{A}_{D_{ij}} & 0_{n \times n_p} \\ 0_{n_p \times n} & 0_{n \times n_p} \end{bmatrix} \begin{bmatrix} x_i(t) \\ z_i(t) \end{bmatrix}
\end{aligned} \tag{3.63}
$$

The following theorem will be used for calculating the unknowns in the controller matrix $\tilde{K}_D$:

Theorem 3.6.1 *Given system (3.4), such that the pair* (A_j, B_j) *is stabilizable and pair* (C_j, A_j) *is detectable. If there exists a positive definite matrix* $K_{min} < I_m$, $K_{max} > I_m$ *and* $K = K_{max} - K_{min}$*, matrices* (A_k, B_k, C_k, D_k) *of suitable dimensions such that sat*$(D_k C x(t) + C_k z(t))$ *is sector bounded in* (K_{min}, K_{max})*, a symmetric positive definite matrix* $P \in \Re^{2n \times 2n}$ *and a positive scalar* ϵ *satisfying:*

$$
A_o^t P + P A_o + \epsilon P + \frac{1}{2}(F^t K - PB)(F^t K - PB)^t < 0 \tag{3.64}
$$

with:

$$
\begin{aligned}
A_o &= \begin{bmatrix} A_i - B_i K_{min} D_k C_i & -B_i K_{min} C_k \\ B_k C_i & A_k - B_k D K_{min} C_k \end{bmatrix} \\
B &= \begin{bmatrix} B_i \\ B_k D_i \end{bmatrix}, F^t = \begin{bmatrix} C_i^t D_k^t \\ C_k^t \end{bmatrix}
\end{aligned} \tag{3.65}
$$

then the saturated closed-loop system resulting from the interconnections of system (3.39)and the controller (3.63) is locally asymptotically stable.

Proof: The interconnection between systems (3.39)and the controller (3.63) is,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A_i & 0 \\ B_k C_i & A_k \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_i \\ B_k D \end{bmatrix} u \\ u &= \begin{bmatrix} D_k C_i & C_k \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned} \tag{3.66}$$

Then, if there exists a diagonal positive definite matrices $K_{min} < I_m$ and $K_{max} > I_m$, matrices (A_k, B_k, C_k, D_k) of suitable dimensions and such that $\text{sat}(D_k C_i x(t) + C_k z(t))$ is sector bounded in (K_{min}, K_{max}), a symmetric positive definite matrix P$\in \Re^{2n \times 2n}$ and a positive scalar ϵ satisfying inequality (3.64), then it follows the following proposition,

Proposition 3.6.1 .

Assume the existence of a triplet(F,K_{min}, K_{max}) with $K_{min} < I_m$ and $K_{max} \geq I_m$, and K= $K_{max} - K_{min}$, such that the matrix $A_i - B_i K_{min} F_i$ is Hurwitz, pair (F,A) is observable and sat(Fx) satisfies the following sector condition,

$$(\psi(t, y) - K_{min} y)^t (\psi(t, y) - K_{max} y) \leq 0 \; \forall t \geq 0, y \in S \subset \Re^m \tag{3.67}$$

then the system (3.39) is locally asymptotically stable in S(P,μ) defined by:

$$\begin{aligned} S(P,\mu) &= \{x \in \Re^n; x^t P x \le \mu\}\ , \mu > 0 \\ S(P,\mu) &\subset S(F, u_o^{K_{min}}) \\ &:= S(F, u_o^{K_{min}}) = \{x \in \Re^n; -\frac{u_0}{K_{min}} \le F_i \le \frac{u_0}{K_{max}}, i = 1, \dots, m\} \end{aligned}$$

Now in order to determine the unknowns of the dynamic output feedback matrix K, an LMI formulation using a linearizing change of variables is presented. Partition matrices P and P^{-1} are defined as,

$$P = \begin{bmatrix} \mathcal{Y} & N \\ N^t & \star \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \mathcal{X} & M^t \\ M & \star \end{bmatrix} \tag{3.68}$$

where $\mathcal{X}$ and $\mathcal{Y}$ belong to $\Re^{n\times n}$ and are symmetric positive definite. By $\star$ we denote terms which are not used in the linearizing change of variable, but which are, of course, depending on the matrices appearing in the partition of P and P_{-1}. This decomposition is general because no specific structure is assigned to the matrices in partition. Thus it does not reduce the choice for matrix P. Now, we define matrices,

$$\Pi_1 = \begin{bmatrix} \mathcal{X} & I_n \\ M^t & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} I_n & \mathcal{Y} \\ 0 & N^t \end{bmatrix} \tag{3.69}$$

It can be easily noticed that $P\Pi_1 = \Pi_2$ whatever the $\star$ terms are. Define the change of variables as:

$$\begin{aligned}
\mathcal{A} &= \mathcal{Y}(A - BK_{min}D_kC)\mathcal{X} + NA_kM^t + NB_kC\mathcal{X} - BK_{min}C_kCM^t - NB_kDK_{min}D_k \\
\mathcal{B} &= NB_k - \mathcal{Y}BK_{min}D_k \\
\mathcal{C} &= C_kM^t + D_kC\mathcal{X} \\
\mathcal{D} &= D_k
\end{aligned} \tag{3.70}$$

Now by pre-multiplying by Π_1^t and by post-multiplying by Π_1 in (3.64) and by using change of variables define in (3.70) we get the following inequality in the variables $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, which is an LMI for a fixed ϵ,

$$\begin{bmatrix} \mathcal{Q} & \mathcal{S} & -B + \mathcal{C}^tK \\ \mathcal{S}^t & \mathcal{R} & -\mathcal{Y}B - \mathcal{B}D + C^t\mathcal{D}\mathcal{K} \\ K\mathcal{C} - B^t & -B^t\mathcal{Y} - D^t\mathcal{B}^t + K\mathcal{D}C & -2I_m \end{bmatrix} < 0 \tag{3.71}$$

where,

$$\begin{aligned}
\mathcal{Q} &= A\mathcal{X} + \mathcal{X}A^t + -BK_{min}\mathcal{C} - \mathcal{C}^tK_{min}B^t + \epsilon\mathcal{X} \\
\mathcal{R} &= \mathcal{Y}A + A^t\mathcal{Y} - \mathcal{B}C - C^t\mathcal{B} + \epsilon\mathcal{Y} \\
\mathcal{S} &= \mathcal{A}^t + A - BK_{min}\mathcal{D}C + \epsilon I_n
\end{aligned} \tag{3.72}$$

In order to have P as positive definite, the following LMI must be added,

$$\Pi_1^tP\Pi_1 = \Pi_1^t\Pi_2 = \begin{bmatrix} \mathcal{X} & I_n \\ I_n\mathcal{Y} & \end{bmatrix} \tag{3.73}$$

Finally, by using the change of variables S(P,μ) included in S($[D_k C C_k], u_o^{K_{min}}$),

$$\begin{bmatrix} \mathcal{X} & I_N & \mathcal{C}^t \\ I_n & \mathcal{Y} & (\mathcal{D}C)^t \\ \mathcal{C} & \mathcal{D}C & \gamma\frac{u_0^2}{K_{min}^2} \end{bmatrix} \geq 0, \forall i = 1, \ldots, m \tag{3.74}$$

Now, to compute the unknowns in the dynamic output feedback controller (A_k, B_k, C_k, D_k) from $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ by the following steps,

Step1. Choose invertible matrices M and N such that $MN^t = I_n - \mathcal{X}\mathcal{Y}$ which is always possible by (3.73).

Step2. Compute Π_1 and Π_2 and finally P= $\Pi_2\Pi_1^{-1}$.

Step3. Compute matrices (A_k, B_k, C_k, D_k) as follows,

$$\begin{aligned} D_k &= \mathcal{D} \\ C_k &= (\mathcal{C} - D_k C\mathcal{X})M^{-T} \\ B_k &= N^{-1}(\mathcal{B} + \mathcal{Y}BK_{min}D_k) \\ A_k &= N^{-1}(\mathcal{A} + \mathcal{Y}(A - BK_{min}D_kC)\mathcal{X})M^{-T} - B_kC\mathcal{X}M^{-T} \\ &+ N^{-1}\mathcal{Y}BK_{min}C_k + B_kDK_{min}C_k \end{aligned} \tag{3.75}$$

The controller matrix ($\mathbf{K} = \{A_k, B_k, C_k, D_k\}$), calculated for the subsystems will then be contracted and formed back for the original system using the overlapping decomposition principle.

3.7 Simulation Results

Using the numerical data in the Appendix A and the developed control design algorithms, we proceed to perform MATLAB simulation for the original system after all the subsystems were taken and the expanded system was contracted by the overlapping decomposition technique. State trajectories of the of the nuclear power plant after the static and dynamic feedback control designs were plotted and compared in Figs. 3.1-3.10.

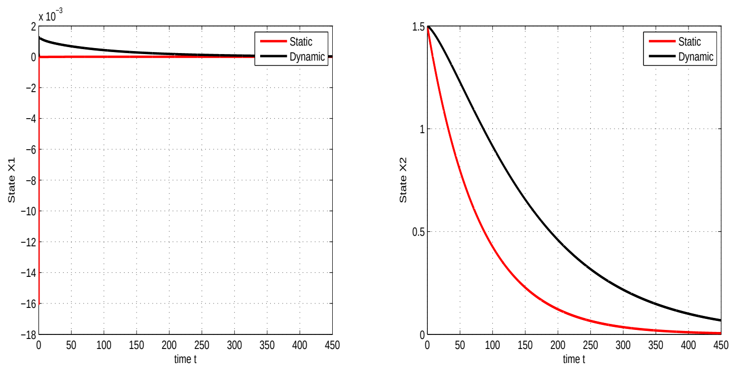

Figure 3.1: Trajectories of States x_1 (left) and x_2 (right)

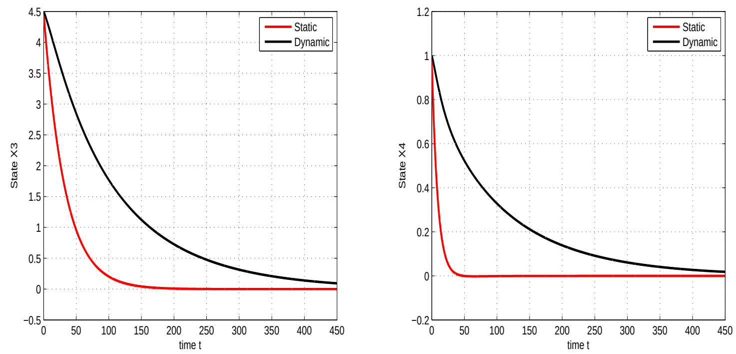

Figure 3.2: Trajectories of States x_3 (left) and x_4 (right)

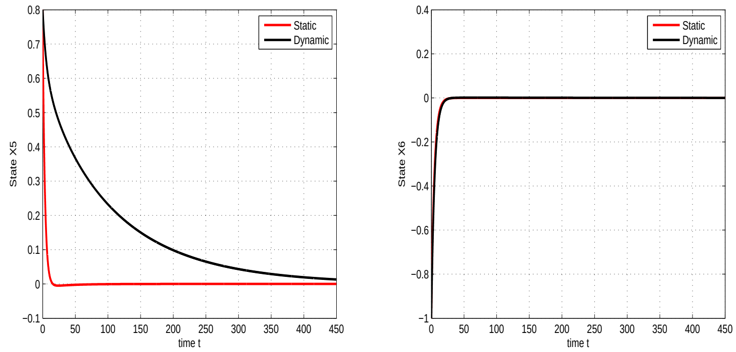

Figure 3.3: Trajectories of States x_5 (left) and x_6 (right)

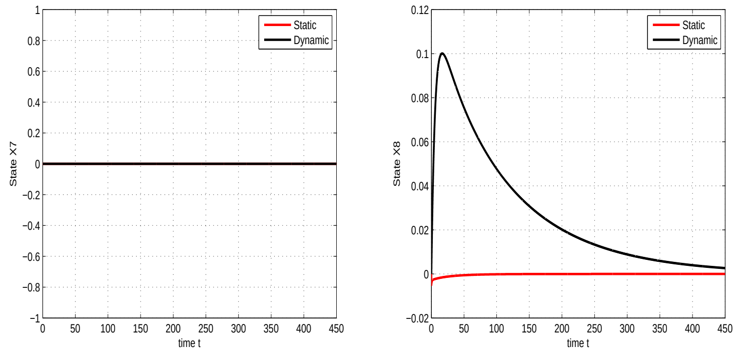

Figure 3.4: Trajectories of States x_7 (left) and x_8 (right)

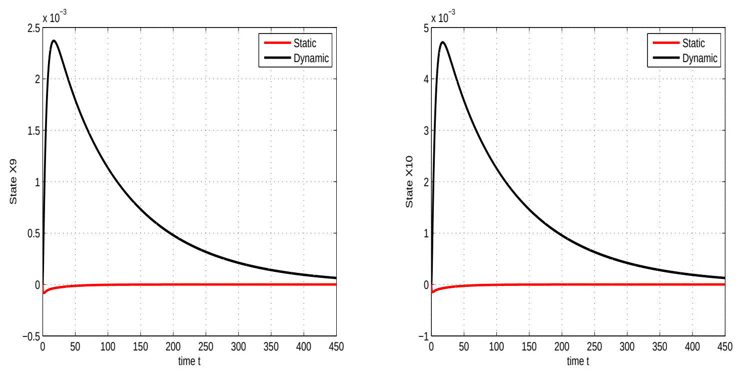

Figure 3.5: Trajectories of States x_9 (left) and x_{10} (right)

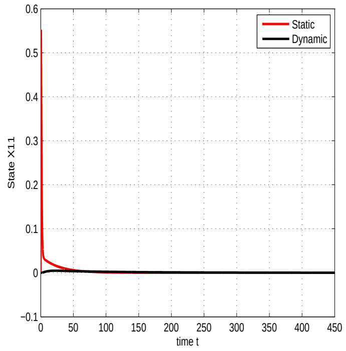

Figure 3.6: Trajectories of States x_{11} (left) and x_{12}

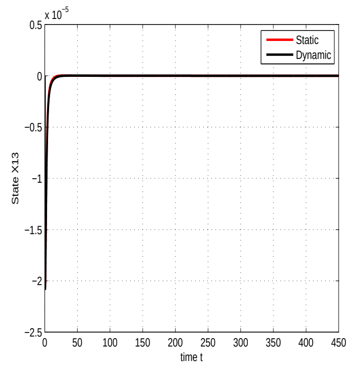

Figure 3.7: Trajectories of States x_{13} (left) and x_{14} (right)

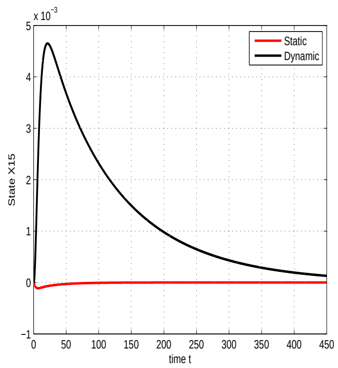

Figure 3.8: Trajectories of States x_{15} (left) and x_{16} (right)

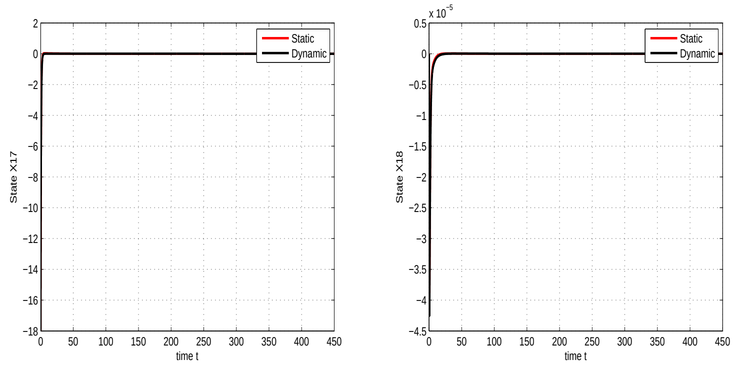

Figure 3.9: Trajectories of States x_{17} (left) and x_{18} (right)

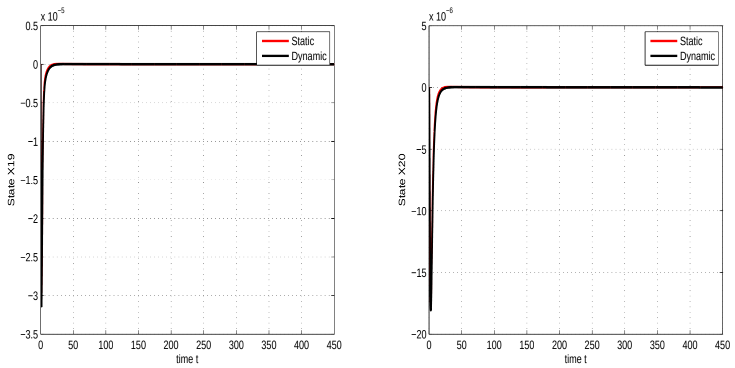

Figure 3.10: Trajectories of States x_{19} (left) and x_{20} (right)

The corresponding input trajectories of the nuclear power plant after the static and dynamic Control designs were compared in Figs. 3.11-3.12.

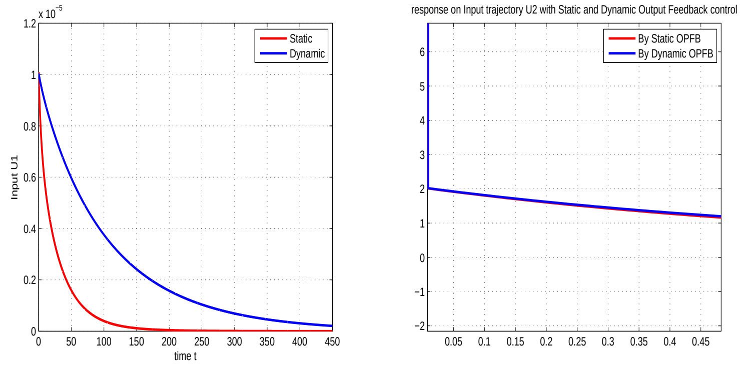

Figure 3.11: Trajectories of Inputs u_1 (left) and u_2 (right)

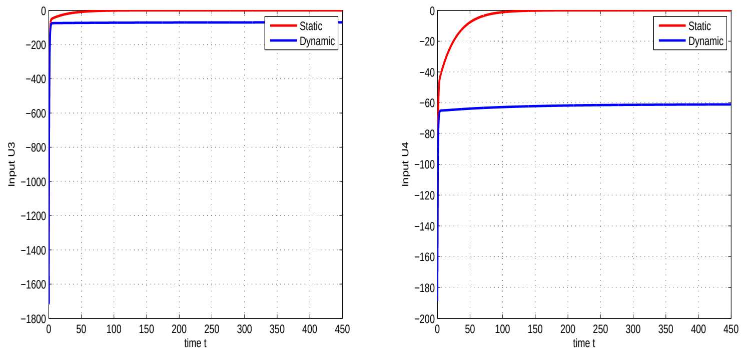

Figure 3.12: Trajectories of Inputs u_3 (left) and u_4 (right)

Finally, the trajectory of outputs of the nuclear power plant after the static and dynamic Control designs were compared in Figs. 3.13-3.14.

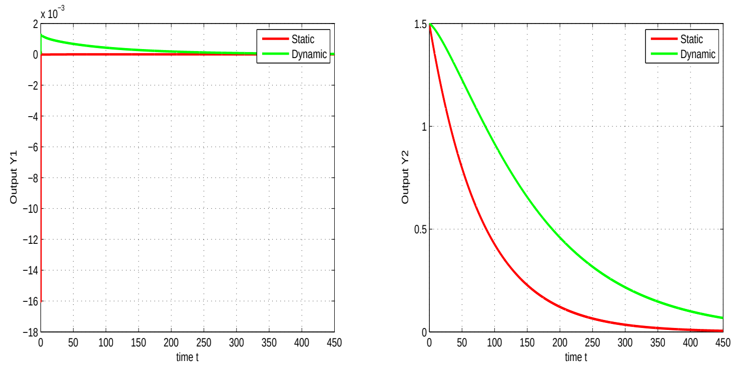

Figure 3.13: Trajectories of Outputs y_1 (left) and y_2 (right)

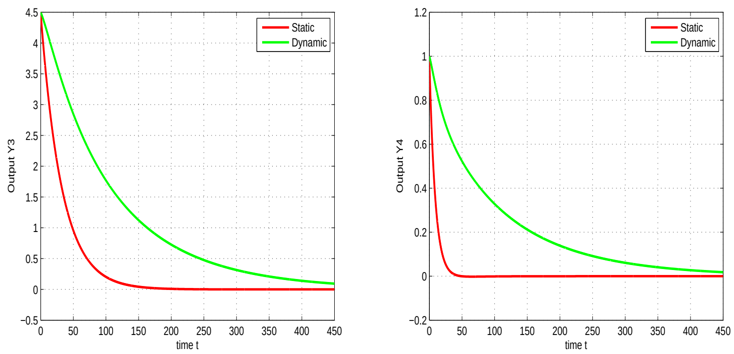

Figure 3.14: Trajectories of Outputs y_3 (left) and y_4 (right)

3.8 Conclusion

In this chapter, new results to the output feedback control design were provided for a class of linear interconnected continuous-time systems subject to input saturation. New schemes based on overlapping design methodology were developed for both static and dynamic output feedback control structures. In both cases, the expanded systems were taken for the control design and after completing the design procedure for the interconnected systems formed by the expanded system, they were contracted using the overlapping decomposition method. Finally the controllers were used for the original system. The theoretical developments were demonstrated by numerical simulations of a linearized nuclear power plant model. The simulation results clearly indicate that

- smooth behavior of the closed-loop system trajectories is guaranteed under overlapping design by either static or dynamic feedback control.
- in cases where static feedback control is superior to dynamic feedback control, the associated control input has larger magnitudes.

Chapter 4

DECENTRALIZED H_∞ CONTROLLER DESIGN FOR SYSTEMS SUBJECT TO SATURATION

4.1 Introduction

Water-loop heat pump system has been applied as a kind of energy saving for centralized air conditioning systems [75]. The main reason is that since it is considered energy efficient they can move heat from one location within a building to another location. It is composed of many water source heat pump units linked by a closed water-piping loop. The water in the loop can be thought as a heat source/sink for each of the heat pump units and it can store the heat inside a building and meet the different requirements of cooling and heating of each heat pump unit during a certain time of period. And the heat from the inner zone (interior zone or core area) in the building can be transferred to its outer zone (exterior zone or perimeter area) by the circulation water and heat recovery can be realized in this way [76].

A heat pump water heater (HPWH) operates on an electrically driven vapor-compression cycle and pumps energy from the air in its surroundings to water in a storage tank, thus raising the temperature of the water. HPWHs are a promising technology in both residential and commercial applications due to both improved efficiency and air conditioning benefits [77].

Residential HPWH units have been available for more than 20 years, but have experienced limited success in the marketplace. Commercial-scale HPWHs are also a very promising technology, while their present market share is extremely low. Typical disturbances acting on the zones can be classified into two groups:

(i) diurnal variations such as those that occur in outdoor air temperatures, solar radiation fluxes and wind velocities and

(ii) internal heat generating sources, viz. lights, occupants and equipments. The diurnal disturbances (we refer to them as external distances) are preview able and can be assumed to be known ahead of time with some degree of accuracy which increases as the preview time is decreased.

On the other hand, the internal heat sources (referred to as internal disturbances) are easy to predict since they are related to a building's operating schedule. Coupled with the fact that heating, ventilating and air conditioning (HVAC) systems (which are used to provide conditioned air to zones) have large time constants and therefore the effect of disturbances on the plant output are delayed, it seems appropriate to explore the application of preview control concepts to improve temperature regulation in buildings. Indoor environmental spaces or zones in large buildings are subjected to multiple disturbances during day-to-day operation. Therefore, good regulation of zone temperature in the presence of multiple disturbances is a problem of continued interest in the control of indoor environments of buildings.

Thus the temperature control of indoor environmental spaces in buildings is a practical problem of interest almost everywhere. Most of the indoor spaces are temperature controlled by the heating, ventilating and air-conditioning (HVAC) systems. The physical system of HVAC have a network distribution like modular structure which is repeated as many times as there are zones. the zones here are

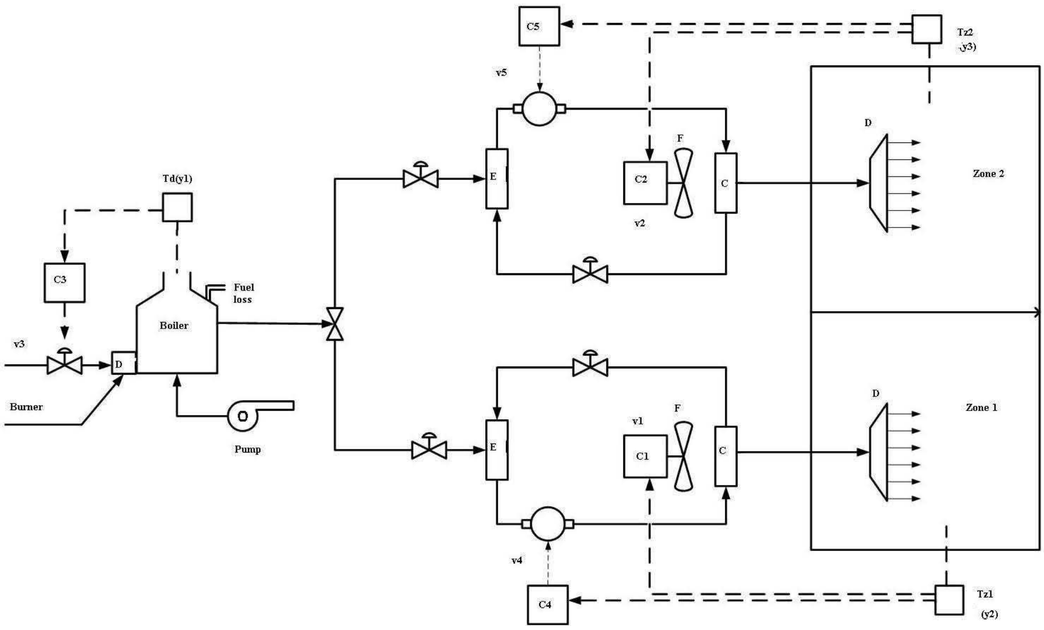

Figure 4.1: Schematic of multi-zone space heating system

the collection of spaces with identical heating/cooling requirements in a building. This is the reason for considering them as good candidates for decentralized control.

In this chapter, we consider a multi-zone space heating system(MZSH) shown in 4.1 for our chapter.The MZSH consists of a boiler which is used to supply the water at moderately warm temperatures (between 16-32°C) to the evaporative heat exchangers of the heat pumps. Each zone is installed with its own heat pump. The heat pump, working on the compression refrigeration cycle, receives the heat energy from the source water and elevates this energy to a higher temperature and delivers it to the condenser coil of the heat pump. A circulating fan and ductwork arrangement is used to extract the heat from the condenser coil and deliver it to the zone through the diffuser as shown in the figure. Thus the zone air is heated to offset the heating load acting on the zone due to cold ambient temperatures. Although two zones are shown in Fig. 4.1, it is obvious that the same arrangements hold for other zones in building with large number of zones. There has been some work done on modeling the field operating performance of space heating systems of the type shown in Fig. 4.1, which are also known as water-loop heating systems, done in [80, 81] however these studies do not address control issues. The approaches for control studies of HVAC systems have been usually centralized control as in [82]. As for the concept of decentralized control is concerned, it is used in practical HVAC systems [83] but only in a limited sense.

We present the design of decentralized H_∞ controller when the inputs of the MZSH are subjected to saturation using the idea of homotopy method presented in [79]. Our design method follows the work done using decentralized $H-\infty$ control of large scale systems [84, 85]. The MZSH model in this chapter is taken from [78], in which they had developed a bilinear model and linearized it about the operating points. The model is shown in section II and the control design in the following section with the simulation results in section IV.

4.2 Modeling the Multi-Zone Space Heating System

The MZSH can be characterized, as shown in Fig. 4.1, having three stations:

- Station 1: boiler, where the input is $u_1(t)(= v_3)$ to control output $y_1(t)$ via controller c_3,
- Station 2: zone-1, where the input $u_2(t)(= [v_1\ v_4]^T)$ to control output $y_2(t)$ via controller c_1, c_4,
- Station 3: zone-2, where the input $u_3(t)(= [v_2\ v_5]^T)$ to control output $y_3(t)$ via controller c_2, c_5

It is obvious that this constitutes a decentralized control problem of an interconnected system and it is the primary goal of this chapter to address the control design issues. In what follows, we provide the main relationships that govern the dynamics of the multi-zone space heating system. Using the energy conservation principle and identifying the energy that flows to and from each component of the MZSH, a bilinear model of the same is developed. If T_b is the boiler temperature and c_b its thermal capacity, the energy balance on the boiler is,

$$c_b T_b = v_3 v_{3max}(1 - \alpha T_b/T_{bmax}) - m_b cp_w(T_b - T_{l_1}) - m_b cp_w(T_b - T_{l_2}) - a_b(T_b - T_e) \quad (4.1)$$

where the rate of heat in the boiler is equated with the net energy input from the combustion chamber,
$v_3 v_{3max}(1-\alpha T_b/T_{bmax})$, minus the rate of the energy withdrawn from the boiler, $m_b c p_w(T_b - T_{l_1})$, and $m_b c p_w(T_b - T_{l_2})$, minus the condition losses of the boiler exterior surfaces to the surroundings, $a_b(T_b - T_e)$, see Table 5.2 for definitions of the symbols.

The heat pump can be modeled by two state equations describing the heat energy flows in the evaporator and the condenser. For heat pump-1 we have,

$$\begin{aligned} c_{l_1}\dot{T}_{l_1} &= -v_4 v_{4max}(P_1 - 1) + m_b c p_w(T_b - T_{l_1}) - a_{l_1}(T_{l_1} - T_e) \\ c_{h_1}\dot{T}_{h_1} &= -v_4 v_{4max} P_1 - v_1 v_{1max}\zeta(T_{h_1} - T_{z_1}) - a_{h_1}(T_{h_1} - T_e) \\ P_1 &= 1 + (P_{1max} - 1)(1 - (T_{h_1} - T_{z_1})/\Delta T_{1max}) \end{aligned} \tag{4.2}$$

where, P_1 is the COP of the heat pump-1.

It can be noticed from (4.2) that the rate of heat stored in the evaporator is equated to the energy withdrawn by the compression cycle, energy added by the source water and jacket losses to the surrounding space. Similarly in the same set of equations, the rate of heat stored in the condenser is equated to the energy input from the compression cycle, heat supplied to the zone air and conduction losses to the surroundings.

To describe the time rate of change in zone-1 temperature, T_{z_1}, we use,

$$c_{z_1}\dot{T}_{z_1} = -v_1 v_{1max}\zeta(T_{h_1} - T_{z_1}) - a_{z_1}(T_{z_1} - T_p) - a_{z_{12}}(T_{z_1} - T_{z_2}) \quad (4.3)$$

where the rate of heat stored in zone-1 air mass is equated to the heat supplied by the condenser, heat losses to outdoor air temperature from the enclosure surfaces, and the heat loss or gain to the adjacent zone (that is, zone-2). The state equations describing the heat pump-2 and the zone-2 are, For heat pump-2 we have,

$$\begin{aligned} c_{l_2}\dot{T}_{l_2} &= -v_5 v_{5max}(P_2 - 1) + m_b c p_w (T_b - T_{l_2}) - a_{l_2}(T_{l_2} - T_e) \\ c_{h_2}\dot{T}_{h_2} &= -v_5 v_{5max} P_2 - v_2 v_{2max}\zeta(T_{h_2} - T_{z_2}) - a_{h_2}(T_{h_2} - T_e) \\ P_2 &= 1 + (P_{2max} - 1)(1 - (T_{h_2} - T_{z_2})/\Delta T_{2max}) \\ c_{z_2}\dot{T}_{z_2} &= -v_2 v_{2max}\zeta(T_{h_2} - T_{z_2}) - a_{z_2}(T_{z_2} - T_p) - a_{z_{12}}(T_{z_1} - T_{z_2}) \quad (4.4) \end{aligned}$$

It can be observed that combining (4.2 and 4.4) together constitute a seventh order bilinear model of the MZSH. To attend for the decentralized control design, these equations were linearized at the operating points shown in Table 5.1 and the ensuing linear state model can be cast into the form:

$$\begin{aligned} \Delta\dot{x}(t) &= A\Delta x(t) + Bsat(\Delta u(t)) + E\Delta d(t) \\ \Delta y(t) &= C\Delta x(t) \quad (4.5) \end{aligned}$$

where

$$
\begin{aligned}
\Delta x(t) &= \begin{bmatrix} \Delta T_b & \Delta T_{l_1} & \Delta T_{h_1} & \Delta T_{z_1} & \Delta T_{l_2} & \Delta T_{h_2} & \Delta T_{z_2} \end{bmatrix}^T \\
\Delta u(t) &= \begin{bmatrix} \Delta v_3 & \Delta v_1 & \Delta v_4 & \Delta v_2 & \Delta v_5 \end{bmatrix}^T \\
\Delta d(t) &= \begin{bmatrix} \Delta T_e & \Delta T_p \end{bmatrix}^T
\end{aligned} \tag{4.6}
$$

where the $\Delta's$ represent the small variations about the operating point of the states (temperatures), control inputs (energy inputs and mass flow rates) and the external disturbances(step changes in ambient temperatures). The numerical values of the respective matrices are presented in the simulation section.

4.3 H_∞ Control Design

Since it is observed that all physical systems are subject to nonlinearities we consider the same in our problem for designing the H-infinity controller for MZSH. These nonlinearities affect the stability and performance of the system. The input of the system is bounded so that the saturating nonlinearity does not affect the design and performance of the decentralized control design of the system. Recall in MZSH that the inputs that are the flow rates have been given certain bounds where, if the inputs go beyond the limit of these bounds the system tends to saturate at the min/max level and the performance is not affected.

For simplicity in exposition while carrying out the design the H_∞ controller for the MZSH when the inputs are subject to saturation, we cast model (4.6) in what follows as an interconnected system of the form:

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_j sat(u_i) + E_i w_i \\ z_i &= C_{1i} x_j + D_{11i} w_i + D_{12i} sat(u_i) \\ y_i &= C_{2i} x_i + D_{21i} w_i\end{aligned} \tag{4.7}$$

where $x_i \in \Re^{n_i}$ is the state, $w_i \in \Re^{r_i}$ is the disturbance input, $z_i \in \Re^{p_i}$ is the controlled output, $sat(u) \in \Re^{m_i}$ is the saturated input and $y_i \in \Re^{q_i}$ is the measurement output. The matrices

$$A_i,\ B_i,\ E_i,\ C_{1i},\ C_{2i},\ D_{11i},\ D_{12i},\ D_{21i}$$

are constant and of appropriate dimensions for each subsystem. The saturation function $sat(u_i)$ is for $u_i \in \Re^{m_i}$ and is defined as,

$$sat(u_i) = \begin{cases} u_{imax} & u_i \geq u_{imax}, \\ u_i & u_{imin} < u_i < u_{imax}, \\ u_{imin} & u_i \leq u_{imin} \end{cases} \tag{4.8}$$

where u_{jmin} and u_{jmax} are chosen to correspond to actual input limits either by measurement or by estimation. Input saturation can also be applied as upper and lower limits of input constraints as u_{jmin} and u_{jmax}, respectively. It is also assumed that the pair (A_j, B_j) is a controllable pair and (C_j, A_j) is an observable pair for all $j \in I := 1, 2, \ldots, N$.

For system (4.7), we seek a decentralized output feedback controller of the form,

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{A}_i \hat{x}_i + \hat{B}_i y_i \\ \hat{u}_i &= \hat{C}_i \hat{x}_i + \hat{D}_i y_i \end{aligned} \tag{4.9}$$

where $\hat{x}_i \in \Re^{\hat{n}_i}$, is the state of the i^{th} local controller, and $\hat{A},\ \hat{B},\ \hat{C},\ \hat{D}$ are the constant matrices to be determined.

To proceed further, we need the following definition to express linear saturating feedback controllers on a convex hull:

$$L(H_1, H_2) = (\hat{x}, y) \in R^{n+p} : |H_{1i}\hat{x} + H_{2i}y| \leq 1, i \in [1, 2^m] \tag{4.10}$$

where H_{1i} and H_{2i} represent the i^{th} row of matrices H_1 and H_2respectively, and H_1, H_2 are the auxiallry feedback matrices. We can note that $L(H_1, H_2)$ is the region where the system does not saturate. Let V be a set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in V and elements in V are represented as V_i, i $\in [1, 2^m]$ and denote $V_i^- = I - V_i$. Clearly $V_i^- \in V$ if $V_i \in$ V.
The following lemma is used for providing the convex covering of saturating non-linearities.

Lemma 4.3.1 *Lemma For* $(\hat{x}_i, y_i) \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_\in)$

$sat(\hat{x}_i, y_i) \in co\ V_i(\hat{C}_i\hat{x}_i + \hat{D}_i y_i) + V_i^-(H_1\hat{x}_i + H_2 y_i), i \in [1, 2^m]$

Therefore by Lemma, the saturating input can be expressed as,

$$sat(u_i) = \sum_{i=1}^{2^m} \eta_i V_i(\hat{C}_i\hat{x}_i + \hat{D}_i y_i) + V_i^-(\hat{C}_i\hat{x}_i + \hat{D}_i y_i) \tag{4.11}$$

for some scalar $0 \leq \eta_i \leq 1, i \in [1, 2^m]$.

Appending the controller (4.9) to system (4.7) yields the closed-loop system:

$$\begin{aligned} \dot{x}_i &= (A_i + B_i\hat{D}_iC_{2i})x_i + B_i\hat{C}_i\hat{x}_i + (E + B_i\hat{D}_iD_{21i})w \qquad (4.12) \\ \dot{\hat{x}} &= \hat{B}C_2x + \hat{A}\hat{x} + \hat{B}D_{21}w \\ z &= (C_{1i} + D_{12i}\hat{D}_iC_{2i})x_i + D_{12i}\hat{C}_i\hat{x}_i + (D_{11i} + D_{12i}\hat{D}_iD_{21i})w \end{aligned}$$

we collect the state $\hat{x}_i$ and coefficient matrices $\hat{A}_i,\ \hat{B}_i,\ \hat{C}_i,\ \hat{D}_i$ as,

$$\begin{aligned} \hat{x} &= \begin{bmatrix} \hat{x}_2^T & \hat{x}_3^T & \dots & \hat{x}_n^T \end{bmatrix}^T, \ \hat{A}_D = diag\begin{bmatrix} \hat{A}_1 & \hat{A}_2 & \dots & \hat{A}_n \end{bmatrix} \\ \hat{B}_D &= diag\begin{bmatrix} \hat{B}_1 & \hat{B}_2 & \dots & \hat{B}_n \end{bmatrix}, \ \hat{C}_D = diag\begin{bmatrix} \hat{C}_1 & \hat{C}_2 & \dots & \hat{C}_n \end{bmatrix} \\ \hat{D}_D &= diag\begin{bmatrix} \hat{D}_1 & \hat{D}_2 & \dots & \hat{D}_n \end{bmatrix} \qquad (4.13) \end{aligned}$$

Define the matrices,

$$\begin{aligned}
B &= \begin{bmatrix} B_1 & B_2 & \dots & B_n \end{bmatrix} \\
C_2 &= \begin{bmatrix} C_{21}^T & C_{22}^T & \dots & C_{2n}^T \end{bmatrix}^T \\
D_{12} &= \begin{bmatrix} D_{121} & D_{122} & \dots & D_{12n} \end{bmatrix} \\
D_{21} &= \begin{bmatrix} D_{211}^T & D_{212}^T & \dots & D_{21n}^T \end{bmatrix}^T
\end{aligned} \tag{4.14}$$

then the closed loop system can be described as,

$$\begin{aligned}
\dot{x} &= (A + B_2\hat{D}_D C_2)x + B_2\hat{C}_D\hat{x} + (E + B_2\hat{D}_D D_{21})w \\
\dot{\hat{x}} &= \hat{B}_D C_2 x + \hat{A}_D\hat{x} + \hat{B}_D D_{21} w \\
z &= (C_1 + D_{12}\hat{D}_D C_2)x + D_{12}\hat{C}_D\hat{x} + (D_{11} + D_{12}\hat{D}_D D_{21})w
\end{aligned} \tag{4.15}$$

we write the matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ in one single matrix,

$$F_D = \begin{bmatrix} \hat{A}_D & \hat{B}_D \\ \hat{C}_D & \hat{D}_D \end{bmatrix} \tag{4.16}$$

and introduce the notations,

$$\begin{bmatrix} \bar{A} & \bar{E} & \bar{B} \\ \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \end{bmatrix} = \begin{bmatrix} A & 0 & \vdots & E & \vdots & 0 & B \\ 0 & 0 & \vdots & 0 & \vdots & I & 0 \\ \dots & \dots & \vdots & \dots & \vdots & \dots & \dots \\ C_1 & 0 & \vdots & D_{11} & \vdots & 0 & D_{12} \\ \dots & \dots & \vdots & \dots & \vdots & \dots & \dots \\ 0 & I_{\hat{n}} & \vdots & 0 & \vdots & & \\ C_2 & 0 & \vdots & D_{21} & \vdots & & \end{bmatrix} \tag{4.17}$$

Then system (4.15) can be written in compact form as,

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_{cl}\bar{x} + \bar{B}_{cl}w, \quad \bar{x} = \begin{bmatrix} x^T \ \hat{x}^T \end{bmatrix} \\ z &= \bar{C}_{cl}\bar{x} + \bar{D}_{cl}w \end{aligned} \tag{4.18}$$

$$\begin{aligned} \bar{A}_{cl} &= \bar{A} + \bar{B}F_D\bar{C}_2, \quad \bar{B}_{cl} = \bar{E} + \bar{B}F_D\bar{D}_{21} \\ \bar{C}_{cl} &= \bar{C}_1 + \bar{D}_{12}F_D\bar{C}_2 \ \ , \bar{D}_{cl} = \bar{D}_{11} + \bar{D}_{12}F_D\bar{D}_{21} \end{aligned} \tag{4.19}$$

The following preliminary result is recalled [87]:

Lemma 4.3.2 *The following statements are equivalent.*

- *A is a stable matrix and $||C(sI - A)^{-1}B + D||_\infty < \gamma$.*

- *There exists a positive-definite matrix P which satisfies the LMI:*

$$\begin{bmatrix} A^TP+PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \tag{4.20}$$

To apply this lemma to the closed loop system (4.19), we define,

$$\begin{aligned} \mathsf{F}(Q,\bar{P}) &= \begin{bmatrix} \bar{A}^T\bar{P}+\bar{P}\bar{A} & \bar{P}\bar{E} & \bar{C}_1{}^T \\ \bar{E}^T\bar{P} & -\gamma I_r & \bar{D}_{11}^T \\ \bar{C}_1 & \bar{D}_{11} & -\gamma I_p \end{bmatrix} + \begin{bmatrix} \bar{P}\bar{B} \\ 0 \\ \bar{D}_{12} \end{bmatrix} Q \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} & 0 \end{bmatrix}^T Q^T \begin{bmatrix} \bar{P}\bar{B} \\ 0 \\ \bar{D}_{12} \end{bmatrix}^T < 0 \end{aligned} \tag{4.21}$$

where Q=F$_D$. A simple consideration of (4.21) reveals that it bilinear matrix inequality (BLMI) [87]. In general, there has been no practical method for solving BMIs, especially for interconnected [88, 89, 90]. In this chapter, we adopt the idea of the homotopy method in the matrix inequality approach [79]. The main idea is that we initially consider a centralized $\mathcal{H}_\infty$ controller. An updating rule then attempts to reduce the off-diagonal term and converts the controller gradually from the centralized one to a decentralized one. At each step of the procedure, a linear matrix inequality (LMI) is obtained by suitably fixing one of the two matrix variables and subsequently solved. When the BMI problem is a feasible one, we can expect that there always exists a centralized

$\mathcal{H}_\infty$ controller for which the algorithm converges and presents a desired solution. To find such a suitable centralized $\mathcal{H}_\infty$ controller, this chapter suggests random search in the parametrized set of $\mathcal{H}_\infty$ controllers [91] with a proper dimension.

A preliminary step is to consider

$$F_o = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} \tag{4.22}$$

as a constant matrix of same size as F_D which we assume that it can be obtained by applying one of the $\mathcal{H}_\infty$ control design methods [92, 93]. The main control design result is summarized by the following theorem:

Theorem 4.3.1 *System (4.7) is stabilizable with the disturbance attenuation level γ via a decentralized controller (4.9) composed of $\hat{n}_i$-dimensional local controllers if and only if there exists a matrix F_D in (4.16) and a positive definite matrix $\bar{P}$ such that,*

$$H(F_D, \bar{P}) < 0 \tag{4.23}$$

$$H(F_D, \bar{P}) = \begin{cases} \mathsf{F}(F_o, \bar{P}) & \lambda = 0 \\ \mathsf{F}(F_D, \bar{P}) & \lambda = 1 \end{cases} \tag{4.24}$$

Applying the idea of homotopy method [84], we introduce a real number $\lambda \in$

$[0, 1]$ and consider the matrix function,

$$H(F_D, \bar{P}, \lambda) \quad = \quad \mathsf{F}((1-\lambda)F_o + \lambda F_D, \bar{P}) \tag{4.25}$$

Remark 4.3.1 *We note that the term in (4.23) defines a homotopy interpolating a centralized* $\mathcal{H}_\infty$ *controller and a desired decentralized* $\mathcal{H}_\infty$ *controller for the attenuation level* γ.

Equivalently stated, the solution to problem (4.23) amounts to that of finding a solution to the family of problems

$$H(F_D, \bar{P}, \lambda) < 0 \ \lambda \in [0, 1] \tag{4.26}$$

Remark 4.3.2 *We note that* λ *in (4.23) defines the initial solution* $\bar{P}_0$ *of the LMI* $H(F_o, \bar{P}) \equiv H(F_{Do}, \bar{P}) < 0$ *where* F_o *in (4.22) given initial estimator based on a centralized* $\mathcal{H}_\infty$ *control theory. This constitutes an important step of the homotopy method.*

Now our problem is to make the homotopy path to connect $(F_{D0}, \bar{P}_0)$ at λ=0 to $(F_D, \bar{P})$ at λ=1 in 4.26. Let **M** be a positive integer and consider (M+1) points $\lambda_k = k/M$, k=$0, 1, 2, \ldots, M$. in the interval [0,1] to generate the family of problems,

$$H(F_D, \bar{P}, \lambda_k) < 0 \ k = 0, 1, 2, \ldots, M. \tag{4.27}$$

If the problem at the kth point is feasible, we denote the solution by $(F_{Dk}, \bar{P}_k)$. Then we compute the solution $(F_{D(k+1)}, \bar{P}_{k+1})$ of $H(F_D, \bar{P}, \lambda_k) < 0$ by solving it as an LMI with one of the two variables being fixed as $F_D = F_{Dk}$ or $\bar{P} = \bar{P}_k$. If the family of the problems $H(F_D, \bar{P}, \lambda_k) < 0$ $k = 0, 1, 2, \ldots, M$, are all feasible, a solution to the BMI 4.23 is obtained at k=M. If it is not the case, that is, $H(F_D, \bar{P}, \lambda_{k+1}) < 0$ is not feasible for some k when we set $F_D = F_{Dk}$ and when $\bar{P} = \bar{P}_k$, we consider more points in the interval $[\lambda_k, 1]$, by increasing M and repeat the procedure from the solution of $(F_{Dk}, \bar{P}_k)$ at $\lambda = \lambda_k$.

The above idea is described in the following algorithm for computing the decentralized $\mathcal{H}_\infty$ controller:

1. Compute F_o using an existing method and them solve LMI $H(F_o, \bar{P}) < 0$ to obtain $\bar{P}_0$. Initialize M to a certain positive integer, and set a certain upper bound for the same.

2. Set k=0; $F_{Dk} := 0$ and $\bar{P}_k := \bar{P}_0$.

3. Set k=k+1 and $\lambda_k = k/M$. Compute a solution of F_D of $H(F_D, \bar{P}_{k-1}, \lambda_k) < 0$. If it is feasible, set $F_{Dk} = F_D$ and compute a solution of $\bar{P}$ of $H(F_{Dk}, \bar{P}, \lambda_k) < 0$. Then set $\bar{P}_k = \bar{P}$ and go to step 5. If $H(F_D, \bar{P}_{k-1}, \lambda_k) < 0$ is not feasible go to step 4.

4. Compute a solution of $\bar{P}$ of $H(F_{D(k-1)}, \bar{P}, \lambda_k) < 0$. If it is feasible, set $\bar{P}_k = \bar{P}$ and compute a solution of F_D of $H(F_D, \bar{P}_k, \lambda_k) < 0$. Then set $F_{Dk} = F_D$ and goto step 5.

5. If $k < M$, goto step 3,. If k=M, the obtained pair $(F_{DM}, \bar{P}_M)$ is a solution of the BMI 4.23.

When the BMI (4.23) is feasible, the convergence of the above algorithm to the solution F_D depends on the choice of F_o which is not unique and defines the starting point of the homotopy path described by (4.25).

Remark 4.3.3 *At each of steps 3 and 4, it is better to solve two LMIs obtained by fixing one of the two variables of the BMI.*

Remark 4.3.4 *The above algorithm can only be applied when a initial centralized $\mathcal{H}_\infty$ controller is of the dimension of $\hat{n} = \sum_{i=1}^{N} \hat{n}_i$ is obtained. However it is not so easy to compute such a centralized $\mathcal{H}_\infty$ controller if $\hat{n} < n$. To overcome this problem we consider to use a n-dimensional centralized $\mathcal{H}_\infty$ controller, which can be obtained easily, to compute a decentralized $\mathcal{H}_\infty$ controller of the dimensions $\hat{n} < n$. This can be done by augmenting the matrix F_D as,*

$$\hat{F}_D = \begin{bmatrix} \hat{A}_D & 0 & \vdots & \hat{B}_D \\ * & -I_l & \vdots & ** \\ \cdots & \cdots & \cdots & \cdots \\ \hat{C}_D & 0 & \vdots & \hat{D}_D \end{bmatrix} \tag{4.28}$$

where the notations $*, **$ are any sub matrices and $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ were defined in ([81]). Using this we can apply the same algorithm and the solution of decentralized H_∞ controller can be obtained.

4.4 Simulation Results

In what follows, we apply the foregoing algorithm to the decentralized control of multi-zone space heating system. The numerical values of the systems matrices are given below, and associated parameters and the operating points are given in the Appendix B:

$$
\begin{aligned}
A &= \begin{bmatrix}
-19.5250 & 7.9880 & 0 & 0 & 7.9880 & 0 & 0 \\
28.3610 & -29.7590 & 1.1340 & 0 & 0 & 0 & 0 \\
0 & 1.1340 & -9.5290 & -8.1300 & 0 & 0 & 0 \\
0 & 0 & 3.6350 & -5.0540 & 0 & 0 & 0.2364 \\
28.3610 & 0 & 0 & 0 & -29.7590 & 1.1340 & 0 \\
0 & 0 & 0 & 0 & 1.1340 & -10.5460 & 9.1470 \\
0 & 0 & 0 & 0.2659 & 0 & 4.0900 & -5.6850
\end{bmatrix} \\
B &= \begin{bmatrix}
80.6980 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -92.4420 & 0 & 0 & 0 & 0 \\
0 & -160.1030 & 133.2840 & & & & \\
0 & 71.5860 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -96.5860 \\
0 & 0 & 0 & 0 & 0 & 0 - 148.0330 & 137.4280 \\
0 & 0 & 0 & 0 & 0 & 066.1890 & 0
\end{bmatrix}, \\
C &= \begin{bmatrix} C_1^T & C_2^T & C_3^T \end{bmatrix}^T, \; E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}^T \\
C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
C_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; E_1 = \begin{bmatrix} 1.7740 & 0.2642 & 0.2642 & 0 & 0.2642 & 0.2642 & 0 \end{bmatrix}^T, \\
E_2 &= \begin{bmatrix} 0 & 0 & 0 & 1.1818 & 0 & 0 & 1.3296 \end{bmatrix}^T
\end{aligned}
$$

The simulation results of using the homotopy method are as follows: Step1 of the algorithm gives value of F_0 and P_0 for all subsystems

$$F_{01} = \begin{bmatrix} -19.8719 & 1.4407 \\ 0.0575 & 0 \end{bmatrix}$$

$$F_{02} = \begin{bmatrix} -30.9386 & -16.9757 & 32.2599 & 0 & 0 \\ -3.6802 & -22.2408 & 16.0523 & 0 & 0 \\ 2.9130 & 20.9935 & -36.6634 & 1.0e-006 * 0.1834 & 0 \\ 0.0243 & -0.2154 & -0.1855 & 0 & 0 \\ -0.1340 & -0.0608 & -0.2756 & 0 & 0 \end{bmatrix}$$

$$F_{03} = \begin{bmatrix} -25.5197 & -1.3077 & -10.2320 & 0 & 0 \\ -0.7631 & -9.3958 & 8.8478 & 0 & 0 \\ -1.8488 & 5.1291 & 0.9583 & 1.0e-006 * 0.1834 & 0 \\ 0.0669 & -0.1450 & 0.0549 & 0 & 0 \\ -0.0206 & -0.0220 & -0.1132 & 0 & 0 \end{bmatrix}$$

$$\bar{P}_{01} = 1.3752e-8 \,, \bar{P}_{02} = \begin{bmatrix} 2.8664 & 1.9192 & 4.6518 \\ 1.9192 & 2.7990 & 4.5402 \\ 4.6518 & 4.5402 & 11.9037 \end{bmatrix}$$

$$\bar{P}_{03} = \begin{bmatrix} 1.0300 & 0.1483 & 0.3654 \\ 0.1483 & 0.4933 & 0.0579 \\ 0.3654 & 0.0579 & 0.6297 \end{bmatrix}$$

In Step 2 we set k=0; and $\bar{P}_k = \bar{P}_0$ (respectively for each subsystem)with the integer value of M=8. In Step 3 it showed infeasibility so we go to step 4 and solve for $\bar{P}$ for all subsystems. Computing solutions for $\bar{P}$ and F_D in step 4 we get,

$$
\begin{aligned}
P_1 = 0.0174\ , P_2 &= \begin{bmatrix} 0.0546 & -0.0040 & 0 \\ -0.0040 & 0.1124 & 0 \\ 0 & 0 & 0.0263 \end{bmatrix} \\
P_3 &= \begin{bmatrix} 0.0200 & 0.0053 & 0.0067 \\ 0.0053 & 0.1330 & 0.1689 \\ 0.0067 & 0.1689 & 0.3733 \end{bmatrix}
\end{aligned}
$$

And F_D for each subsystem computed was,

$$
\begin{aligned}
F_{D1} &= -0.1666\ , F_{D2} = \begin{bmatrix} -0.0035 & -0.0033 & 0.0117 \\ -0.0054 & 0.0043 & 0.0176 \\ -0.0154 & -0.0055 & 0.0243 \end{bmatrix} \\
F_{D3} &= \begin{bmatrix} -0.0051 & -0.0007 & 0.0236 \\ -0.0074 & 0.0026 & 0.0139 \\ --0.1340 & -0.0608 & -0.2756 \end{bmatrix}
\end{aligned}
$$

The optimal H$_\infty$ performance of the three subsystems are given by γ_1^*=1.1818,

γ_2^*=2.1316 and γ_3^*=1.7936.

These results were compared with a LQR feedback controller designed for each of the 3 subsystems individually with the weighting matrices having the following values,

$$Q_1 = 2 \;\; R_1 = 5$$
$$Q_2 = 2 * eye(3) \;\; R_2 = 5 * eye(2)$$
$$Q_3 = Q_2 \;\; R_3 = R_2$$

The following are the simulation plots for the same

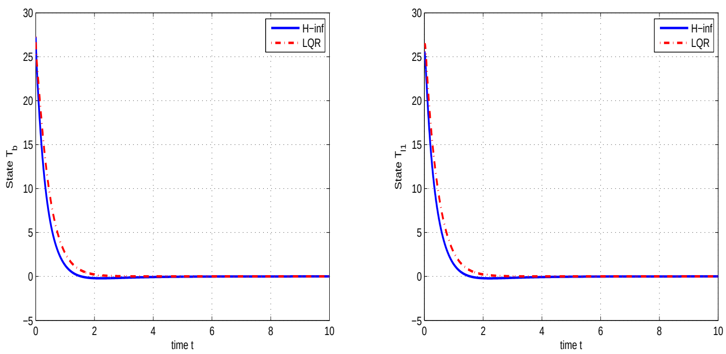

Figure 4.2: Trajectories of states x_1 (left) and x_2 (right)

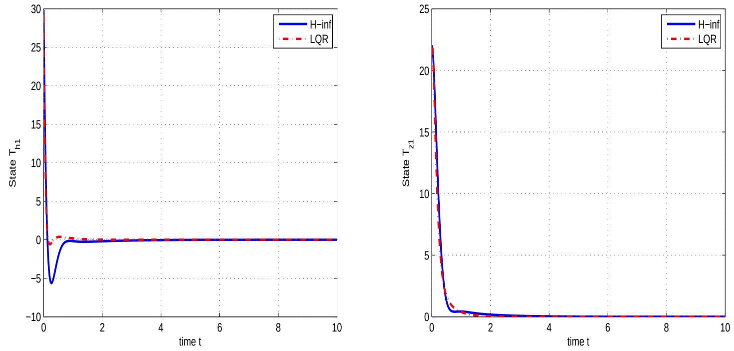

Figure 4.3: Trajectories of states x_3 (left) and x_4 (right)

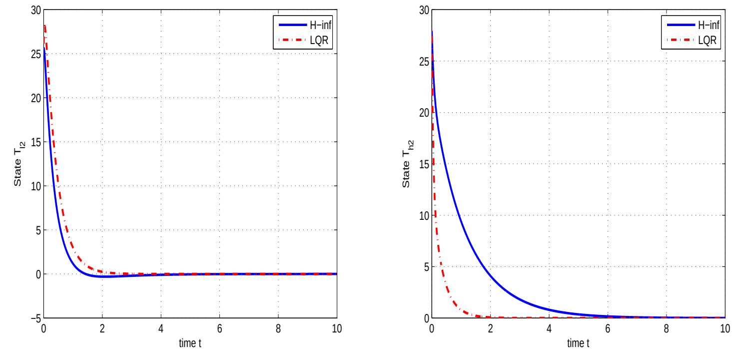

Figure 4.4: Trajectories of states x_5 (left) and x_6 (right)

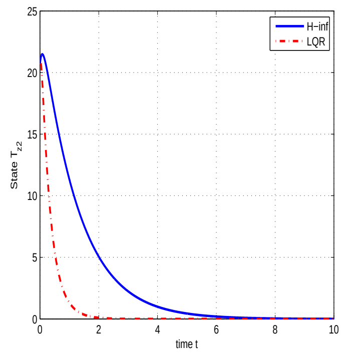

Figure 4.5: Trajectories of states x_7 (left)

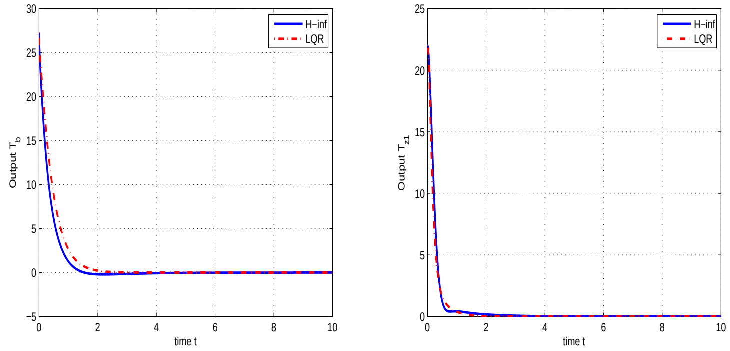

Figure 4.6: Trajectories of Outputs y_1 (left) and y_2 (right)

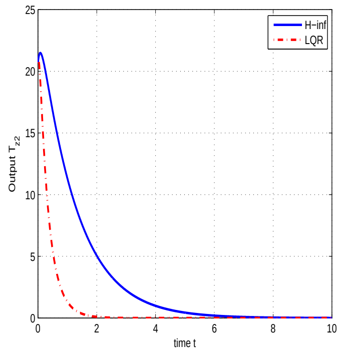

Figure 4.7: Trajectories of output y_3 (left)

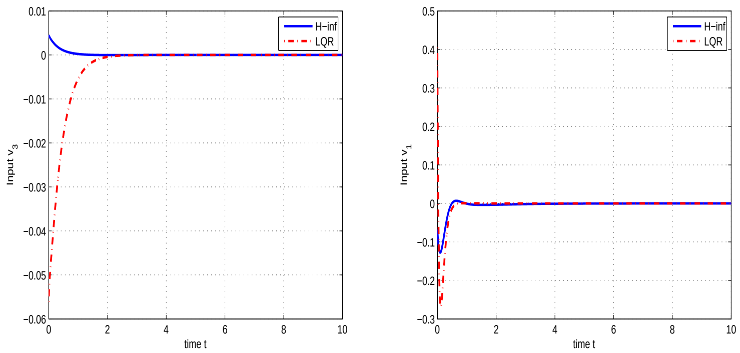

Figure 4.8: Trajectories of inputs u_1 (left) and u_2 (right)

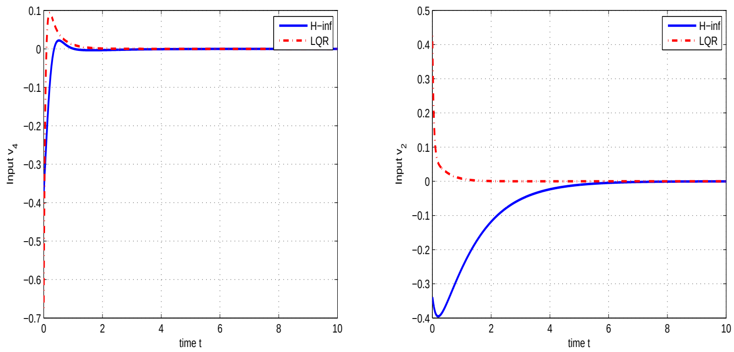

Figure 4.9: Trajectories of inputs u_3 (left) and u_4 (right)

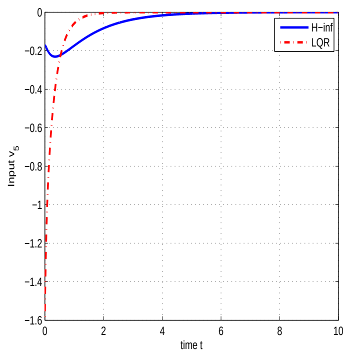

Figure 4.10: Trajectories of input u_5 (left)

4.5 Conclusion

In this chapter, we have addressed the problem of temperature control design in a multi zone space heating system (MZSH) as an interconnected dynamical system subject to external disturbances and input saturation. The design of decentralized $\mathcal{H}_\infty$ controllers have been accomplished using a homotopy method. Numerical simulations are presented and compared with an improved decentralized method based on linear quadratic theory. It is observed from the plots that

- The states trajectories corresponding to the temperatures of the boiler T_b, the evaporator and condenser temperature of heat pump-1 T_{l1} and T_{h1} have required relatively small time to settle (less than 2 sec) based on decentralized $\mathcal{H}_\infty$ control.
- The states trajectories corresponding to the temperature of zone-2 T_{z2} and the condenser of heatpump-2 T_{h2} took a bit longer duration for settling (around 4.5 sec) decentralized $\mathcal{H}_\infty$ control.
- From the output trajectories, it is quite clear that the decentralized $\mathcal{H}_\infty$ control design gives better and faster settling to the boiler and zone-1 temperatures as compared to the improved decentralized method based on linear quadratic theory.
- The situation is reversed in the case of zone-2 where improved decentralized method based on linear quadratic theory yields results.

- In general, the developed decentralized $\mathcal{H}_\infty$ control demands less cost than the improved decentralized method based on linear quadratic theory

Chapter 5

Conclusion and Future Work

In conclusion new results to the output feedback control design were provided for a class of linear interconnected continuous-time systems subject to input saturation. Schemes based on overlapping design methodology were developed for both static and dynamic output feedback control structures. In both cases, the expanded systems were taken for the control design and after completing the design procedure for the interconnected systems formed by the expanded system, they were contracted using the overlapping decomposition method. Finally the controllers were used for the original system. The theoretical developments were demonstrated by numerical simulations of a linearized nuclear power plant model.

Also we have addressed the problem of temperature control design in a multi zone space heating system (MZSH) as an interconnected dynamical system subject to external disturbances and input saturation. The design of decentralized $\mathcal{H}_\infty$ controllers have been accomplished using a homotopy method. Numerical simulation are presented and compared with an improved decentralized method based on linear quadratic theory. Suggestions for future work would be

- Designing controllers for the non-linear systems subject to saturation.
- Desig and analysis for systems presenting saturation and delays.
- Stability analysis and control design for systems subject to saturation using H_2 control.
- LTI interconnected systems when subject to input and output saturations, together.

5.1 Appendix A

5.1.1 Original data of the nuclear power plant

$$A_{11} = \begin{bmatrix} -400 & 0.0125 & 0.0305 & 0.111 & 0.301 & 1.14 & 3.01 \\ 13.125 & -0.0125 & 0 & 0 & 0 & 0 & 0 \\ 87.5 & 0 & -0.0305 & 0 & 0 & 0 & 0 \\ 78.125 & 0 & 0 & -0.111 & 0 & 0 & 0 \\ 158.125 & 0 & 0 & 0 & -0.301 & 0 & 0 \\ 46.25 & 0 & 0 & 0 & 0 & -1.14 & 0 \\ 16.875 & 0 & 0 & 0 & 0 & 0 & -3.01 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} -1781 & -13700 & -13700 & 0.111 & 411 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{13} = [0]_{6\times 7}$$

$$A_{21} = \begin{bmatrix} 0.0756 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -0.16466 & 0.16466 & 0 & 0 & 0 & 0 & 0 \\ 0.05707 & -24403 & 0 & 0 & 0 & 0 & 0 \\ 0.05707 & 23262 & -23832 & 0 & 0 & 0 & 0 \\ 0.0207 & -0.0207 & 0.0103 & 0 & 0.634 & -0.509 & 0 \\ 0 & 0 & 0 & 0 & -53657 & 307017 & 0.3372 \\ 0 & 0 & 0 & 0 & 0.53819 & -0.76642 & 0 \\ 0 & 0 & 0 & 0 & 1.349 & 0 & -0.2034 \end{bmatrix}$$

$$A_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.3832 \\ 0.240 & -0.279 & -0.130 & -0.116 & 0.0235 & 0.121 \\ 0 & 0 & 0.2238 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{31} = [0]_{6\times 7}$$

$$A_{32} = \begin{bmatrix} 0 & 0 & 0.33645 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.45 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{33} = \begin{bmatrix} -0.33645 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & -2.5 & 0 & 0 & 0 & 0 \\ 0 & 1.45 & -1.45 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.45 & 0 & 0 \\ 0 & 0 & 0 & 1.48 & -1.48 & 0 \\ 0 & 0 & 0 & 0 & 0.516 & -1.516 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.03843 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0016 & 0 & 0 & 0 \\ -0.0062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

5.1.2 Permuted data

$$\bar{A}_{11} = \begin{bmatrix} -400.0000 & 0.0125 & 0.0305 & 0.1110 & 0.3010 \\ 13.1250 & -0.0125 & 0 & 0 & 0 \\ 87.5000 & 0 & -0.0305 & 0 & 0 \\ 78.1250 & 0 & 0 & -0.1110 & 0 \\ 158.1250 & 0 & 0 & 0 & -0.3010 \end{bmatrix}$$

$$\bar{A}_{12} = \begin{bmatrix} 0 & 30 & -1781 & -13700 & -13700 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_{13} = \begin{bmatrix} 0 & 0 & 0 & 1.140 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{A}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 16.8750 & 0 & 0 & 0 & 0 \\ 0.0756 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_{22} = \begin{bmatrix} -0.2034 & 0 & 0 & 0 & 0 \\ 0 & -3.0100 & 0 & 0 & 0 \\ 0 & 0 & -0.1647 & 0.1647 & 0 \\ 0 & 0 & 0.0571 & -2.4403 & 0 \\ 0 & 0 & 0.0571 & 2.3262 & -2.3832 \end{bmatrix}$$

$$\bar{A}_{23} = \begin{bmatrix} 0 & 1.3490 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{A}_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.3832 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_{33} = \begin{bmatrix} -2.5000 & 0 & 0 & 0 & 2.5000 \\ 0 & -5.3657 & 3.0702 & 0 & 0 \\ 0 & 0.5382 & -0.7664 & 0 & 0 \\ 0 & 0 & 0 & -1.1400 & 0 \\ 0 & 0 & 0 & 0 & -0.3365 \end{bmatrix}$$

$$\bar{A}_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2238 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_{41} = [0]_{5\times 5}$$

$$\bar{A}_{42} = \begin{bmatrix} 0 & 0.027 & 0 & -0.0207 & 00103 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_{43} = \begin{bmatrix} -0.2790 & 0.6340 & -0.5090 & 0 & 0.2400 \\ 1.4500 & 0 & 0 & 0 & 0 \\ 0 & 1.4500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_{44} = \begin{bmatrix} 0 & -0.1300 & -0.1160 & 0.0235 & 0.1210 \\ 0 & -1.4500 & 0 & 0 & 0 \\ 0 & 0 & -1.4500 & 0 & 0 \\ 0 & 0 & 1.4800 & -1.4800 & 0 \\ 0 & 0 & 0 & 0.5160 & -0.5160 \end{bmatrix}$$

$$
\begin{aligned}
\bar{B} &= diag(\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4) \\
\bar{B}_1 &= \begin{bmatrix} 10^6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} -0.03843 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\bar{B}_3 &= \begin{bmatrix} 0.0016 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_4 = \begin{bmatrix} -0.00062 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

5.2 Appendix B

5.2.1 Table showing System operating points.

Table 5.1: System Operating Points.

Temperature ° C	operating point
T_{b_0}	27.266
$T_{l_{10}}$	25.584
$T_{h_{10}}$	29.843
$T_{z_{10}}$	21.966
$T_{l_{20}}$	25.512
$T_{h_{20}}$	27.944
$T_{z_{20}}$	20.662
T_{e_0}	20.0
T_{p_0}	-2.0

5.2.2 Table showing design parameters of the MZSH system.

Table 5.2: Design parameters of the MZSH system.

Variable	Symbol	Magnitude,Units
Zone-1 heat loss coeff.	a_{z_1}	122.935 W/°C
Zone-2 heat loss coeff.	a_{z_2}	138.32 W/°C
Evaporator heat loss coeff.	$a_{l_1} = a_{l_2}$	12.29 W/°C
Condensor heat loss coeff.	$a_{h_1} = a_{h_2}$	12.29 W/°C
Boiler heat loss coeff.	a_b	12.29 W/°C
Interzone heat loss coeff.	$a_{z_{12}}$	12.29 W/°C
Thermal capacity of the zones	$c_{z_1} = c_{z_2}$	374.48 kJ/°C
Thermal capacity of the evaporator	$c_{l_1} = c_{l_2}$	167.44 kJ/°C
Thermal capacity of the condensors	$c_{h_1} = c_{h_2}$	167.44 kJ/°C
Thermal capacity of the boiler	c_b	594.55 kJ/°C
Max. air flowrate of water	$v_{1max} = v_{2max}$	1.575 kJ/s
Burner capacity	v_{3max}	5.86 kJ/s
Heat pump capacity	$v_{4max} = v_{5max}$	3.8 kJ/s
Mass flow rate of water	m_b	0.3151 kJ/s
Specific heat of water	cp_w	4.186 kJ/s
Heat exchanger coeff.	$\zeta = \zeta_1 = \zeta_2$	0.6 kJ/kg °C
Max. coeff. of performance	$P_{1max} = P_{2max}$	3.5
Max. temperature differential	$\Delta T_{1max} = \Delta T_{2max}$	45° C
Max. temperature of the boiler	T_{bmax}	60 °C
Boiler flu loss coeff.	α	0.1

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