APPROXIMATION BY WAVELETS

by

MOHAMMAD RAFIQUL ISLAM

A Thesis Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCES

In

MATHEMATICS

May 2005

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This Thesis, written by **Mohammad Rafiqul Islam** under the direction of his thesis supervisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfilment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICAL SCIENCES**.

Thesis Committee

Prof. Abul Hasan Siddiqi (Chairman)

Dr. Khaled M. Furati (Co-chairman)

23/5/05

Prof. Fiazuddin Zaman (Member)

23/5/05

Prof. Y.A. Fiagbedzi (Member)

23-5-2005

Prof. M. El-Gebeily (Member)

Dr. Khaled M. Furati

Department Chairman

Dr. Mohammad A. Al-Ohali

Dean of Graduate Studies

Date

1 JUNE 2005

Dedicated to my wife

and

to my sons Redwan and Nayeem.

ACKNOWLEDGEMENT

All praise be to almighty Allah, without whose mercy and clemency nothing would have been possible. I wish to express my appreciation to Professor Abul Hasan Siddiqi , my advisor, for introducing me to the subject and also for giving me all the necessary support I needed to complete this work, without him this work would not have been accomplished. I am deeply obliged to Dr. Khaled M. Furati, my thesis co-advisor, for giving his valuable suggestions and constructive criticism. Without their help this work would not been accomplished.

Acknowledgement is due to King Fahd University of Petroleum & Minerals for supporting this research work and in particular, to the Department of Mathematical Sciences for giving me access to all its available facilities which makes the completion of this work much easier.

I wish to express my thanks and gratitude to Prof. M. El-Gebeily for his constructive suggestions, cooperation and criticism. I also express my thanks and gratitude to Prof. F. Zaman, Prof. Y.A. Fiagbedzi for their advice and constructive criticism. My thanks also extend to Dr. Anwar Joarder, Mr. Khaled Saifullah, Dr. G.K. Beg, Zeid Al-Muhiameed, M. Abdulwahab, A. Al-Kafi, A.T.M. Jahiruddin and the Bangladeshi community in general for making my living in Saudi Arabia a memorable one.

I would like to acknowledge Dr. Khaled M. Furati again as a Chairman of the Department, he has given his valuable time for me to successfully complete my thesis.

Table of Contents

Dedication **iii**

Acknowledgement **iv**

List of Figures **viii**

Abstract (English) **ix**

Abstract (Arabic) **x**

Preface **xi**

1 Basic Review **1**

1.1 The Approximation Problem 1

1.2 The Best Approximation Problem 2

1.3 Uniqueness of best approximation 2

1.4 Function Spaces 3

1.5 Relation Between Smoothness and Approximation order 7

1.6 Wavelet Theory 9

1.6.1 Wavelet Transforms 9

1.6.2 Continuous wavelet transform 10
1.6.3 Discrete wavelet transform 11
1.6.4 Multiresolution Analysis (MRA) 13

2 The Haar System 16

2.1 Haar Wavelet 16
2.2 The Dyadic Intervals 17
2.3 The Haar System 18
2.3.1 The Haar Scaling Function 18
2.3.2 The Haar Wavelet Function 19
2.4 Orthogonality of the Haar System 20
2.5 Comparison of Haar series with Fourier series 24
2.5.1 Behavior of Haar Coefficients Near Jump Discontinuities 25
2.5.2 Haar Coefficients and Global Smoothness 29

3 Approximation by Wavelets in Different Spaces 30

3.1 Approximation space 30
3.2 Approximation by Haar wavelets in Different Spaces 31
3.2.1 $L_2(\mathbb{R})$ Spaces 31
3.2.2 $L_p(\mathbb{R})$ Spaces 32
3.2.3 $Lip(\alpha, L_p)$ Spaces 33
3.2.4 Sobolev Spaces $H^m(\mathbb{R})$ 34
3.2.5 Besov spaces $B_q^{\alpha,r}(L_p(\mathbb{R}))$ 35
3.3 Approximation by Daubechies wavelets 36
3.4 Approximation by Coifman wavelets 39

3.5 Jackson and Bernstein Theorems for Wavelets 40

4 Walsh System, Wavelet Packets, Walsh-type Wavelet Packets and their Approximation 43

4.1 Walsh system 43

4.2 Approximation by Walsh Polynomial 46

4.3 Wavelet Packets 48

4.4 Examples of Wavelet Packets 51

4.4.1 Shannon Wavelet Packets 51

4.4.2 Daubechies Wavelet Packets 53

4.4.3 Walsh Type Wavelet Packets 54

5 Variants of Haar wavelet 55

5.1 Rationalized Haar wavelet 55

5.2 Non-uniform Haar wavelet 58

5.2.1 Haar scaling function and non-uniform Haar wavelet 58

5.2.2 Non-uniform Haar wavelet from MRA 59

5.2.3 Approximation by Non-uniform Haar wavelets 61

6 Open Problems and Further Research 66

Bibliography 67

Vita 72

List of Figures

2.1 The Haar function 17

2.2 (a) Haar scaling function for j (b) Haar scaling function for $j = j'$, where $j' > j$. 18

2.3 (a) Haar wavelet function for j (b) Haar wavelet function for $j = j'$, where $j' > j$. 19

2.4 A wavelet series expansion of $f(t) = t^2$ using Haar wavelets. 23

4.1 Walsh Functions 45

4.2 Example of a wavelet packet 50

5.1 A set of the RH functions (r=0 to 7) 57

5.2 (a) Haar scaling function, (b) Non-uniform Haar wavelet. 59

5.3 The function f and its uniform Haar approximation f_j for $j = 2$. 64

5.4 The function f and its non-uniform Haar approximation f_j for $j = 2$. . 64

ABSTRACT

Name: Mohammad Rafiqul Islam

Title: Approximation by wavelets.

Major Field: Mathematics

Date of Degree: May 2005

In this dissertation, we studied existence and uniqueness of the best approximation and its order, where the underlying space is the space of Lipschitz continuous functions or Besov spaces and the subspaces are the set of wavelets and Walsh type wavelet packet polynomials.

MASTER OF SCIENCE DEGREE

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

Dhahran, Saudi Arabia

May 2005

الخلاصة

الإسم: محمدرفيق الإسلام.

العنوان:التقريب بالمويجات

التخصص الرئيسي: رياضيات.

تاريخ نيل الدرجة: مايو 2005 .

في هذه الرسالة درسنا وجود و أحادية أفضل تقريب في فضاءات ليبيج وبيسوف من خلال تحت فضاءات مويجية أو مجموعة مويجات من نوع كثيرات حدود والش.

درجة ماجستير العلوم

جامعة الملك فهد للبترول والمعادن

الظهران- المملكة العربية السعودية

مايو- 2005

Preface

Approximation theory is the branch of mathematics which studies the process of approximating general functions by simple functions such as polynomials and different kinds of series [3, 4, 5, 26, 29, 39, 41]. It plays a central role in the numerical analysis and its applications to partial differential equations representing real world systems.

Approximation theory can be treated as a link between modern mathematics and classical mathematics, particularly analysis. Origin of approximation theory can be traced back to Weierstrass approximation theorem proved by Weierstrass in 1885. Since then several celebrated mathematicians notably Picard, Volterra, Lebesgue, Mittag-Leffler, Fejér, Lerch, Landau, de LaVallee Poussin and Bernstein provided alternative proofs of this theorem. The Weierstrass theorem was extended to functions of n variables by M. H. Stone in 1937. In fact, he proved the Weierstrass theorem in the setting of algebra. Theorems in which the rate of decrease of the best approximation is established for a family of functions and different class as of polynomials are generally called theorems of Jackson type in recognition of the pioneering research work of Jackson between 1911 and 1930. The theorems in which the smoothness of f is deduced from the best approximation are called theorems of Bernstein type. The memoir of Bernstein dealing with these results fetched a very prestigious prize in 1913.

It may be remarked that a systematic study of approximation theory was initiated by Natanson et al [29] in the fifties. Results concerning approximation by trigonometric polynomials of different classes of functions can be found in the famous book of Zygmund [48]. For a detailed historical note one can refer to Cheney [4]. By the seventies, the subject became very popular in view of its wide applications. The celebrated finite element method developed by engineers in early fifties found close connection with the approximation theory. French mathematician Céa observed in early sixties that an error estimation of finite element is an approximation problem in Sobolev spaces. Approximation by Spline functions [23] attracted the attention of several eminent mathematicians during the seventies and eighties. These functions are not only convenient and suitable for computer calculations, but also provide optimal theoretical solution to the estimation of functions from limited data.

Wavelet theory has been developed in the last two decades. This is a refinement of Fourier analysis and is applied to different fields [5, 8, 9, 10, 11, 12, 13, 15, 18, 22, 24, 26, 27, 28, 38, 41, 42, 45, 47, 44]. From the viewpoint of approximation theory and harmonic analysis, the wavelet theory is important on several counts. It provides simple and elegant unconditional wavelet bases for function spaces (Lebesgue, Sobolev, Besov, etc).

A recent development of approximation theory is approximation of an arbitrary function by wavelet polynomials. There is a number of examples of wavelet such as Haar wavelet, Mexican-hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer wavelet etc.

In this thesis, we focus on approximation by Haar wavelet and its variants. Haar function is the simplest example of wavelet and reflects significant features of the general wavelet. Well known Daubechies wavelet is a generalization of Haar wavelet.

In chapter 1 we introduce the basic concepts of approximation theory and wavelet analysis. Chapter 2 is devoted to the Haar system and its properties. In chapter 3 we discuss approximation of general functions by the Haar wavelet in different smoothness space such as Lebesgue, Lipschitz, Sobolev and Besov spaces. This chapter also contains some results concerning Daubechies and Coiffman's wavelet. In chapter 4, we discuss Walsh function and wavelet packet which are the generalization of Haar wavelet. Chapter 5 deals with the variants of the Haar wavelet. Finally, Chapter 6 is devoted to open problems and further research scope in this area.

Chapter 1

Basic Review

1.1 The Approximation Problem

Approximation of a given arbitrary function by a polynomial amounts to finding a polynomial which is close to the function. The Approximation problem is concerned with how one can choose a polynomial that approximates a given function more accurately [3, 14]. Weierstrass proved the following theorem in 1885.

Theorem 1.1 *[14] (Weierstrass theorem) Let f be a continuous function defined on a closed and bounded interval I of $\mathbb{R}$. Given $\varepsilon > 0$, we can find a polynomial $P_n(x)$ (of sufficiently high degree) for which $|f(x) - P_n(x)| \leq \varepsilon, \ \forall x \in I$.*

The approximating polynomial $P_n(x)$ depends on three factors:

i) How well we want f approximated, i.e., how small ε is;

ii) The behavior of f ; strong oscillations in f usually force $P_n(x)$ to be of a higher degree and

iii) The length of the interval I; enlarging the interval in general forces us to choose polynomials of higher degree if a certain accuracy has to be obtained.

1.2 The Best Approximation Problem

The minimal distance requirement is the criterion for the best approximation in a normed space.

Definition 1.2 *Let X be a normed space and A be a subspace of X. We say that $a_1 \in A$ is a better approximation than $a_2 \in A$ of $f \in X$, if $\|a_1 - f\|_X \leqq \|a_2 - f\|_X$. We define $a^* \in A$ to be a best approximation to $f(x)$ if the condition $\|a^* - f\|_X \leqq \|a - f\|_X$ holds for all $a \in A$.*

i.e. $dist(f, A)_X := \inf_{a\epsilon A} \|f - a\|_X$.

We call $dist(f, A)_X$ or $d(f, A)_X$ or $E(f)_X$ the distance from f to A. The best approximation indicates that the error norm of the given function will be minimum. Best approximations does not always exist. Even if it exists, it may not be unique.

Theorem 1.3 *[14] Let X be a normed space. For given f and n linearly independent elements $x_1, x_2, x_3, ..., x_n$ in X, the problem of finding $\min_{a_i} \left\| f - \sum_{i=1}^{n} a_i x_i \right\|_X$ has a solution, where $a_1, a_2, ..., a_n$ are constants.*

1.3 Uniqueness of best approximation

In order to approximate a point or a function f by an element of a set A, it is usual to choose conditions that define a particular approximation. Best approximation with respect to an appropriate distance function is often suitable, but sometimes there are several best approximations. In fact, the best approximations form a convex set.

Definition 1.4 *(Convex set) Let X be a linear space. A subset $C \in X$ is called convex set if $x_1, x_2 \in C$ and $tx_1 + (1-t)x_2 \in C$, for all $0 \leqslant t \leqslant 1$.*

Theorem 1.5 *[34] Let A be a convex set in a normed linear space X and $f \in X$ such that there exists a best approximation of f in A. Then the set of best approximation is convex.*

A fairly extensive sufficient condition can be given which assures the uniqueness of the best approximation. If a normed linear space is strictly convex, then there is a unique best approximation.

Definition 1.6 *(Strictly convex set) A normed linear space X is called strictly convex if for every $x,\ y \in X$ and $r > 0$, $\|x\| \leqslant r$ and $\|y\| \leqslant r$, imply $\|x+y\| < 2r$ unless $x = y$.*

Theorem 1.7 *[34] Let A be a compact and strictly convex set in a normed linear space X. Then for all $f \in X$, there is a unique best approximation of f in A.*

Theorem 1.8 *[34] Let A be a strictly convex set in a normed linear space X. Then for all $f \in X$, there is at most one best approximation of f in A.*

1.4 Function Spaces

There are many different ways of measuring the smoothness of a function f. With particular measure of smoothness, one associates a class of function spaces: for $\Omega \subset \mathbb{R}^n$,

$C^m(\Omega)$ to be the spaces of continuous functions which have bounded and continuous partial derivatives $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_1^{\alpha_n}}$, $|\alpha| \leq m$. This space equipped with the norm

$$\|f\|_{C^m(\Omega)} = \sup_{x \in \Omega} |f(x)| + \sum_{|\alpha|=m} \sup_{x \in \Omega} |D^\alpha f(x)| \tag{1.1}$$

is a Banach space.

Definition 1.9 *The space of all functions* $f : [a, b] \longrightarrow \mathbb{R}$ *such that* $\int_a^b |f(x)|^p \, dx < \infty$, $1 \leq p < \infty$, *is denoted by* $L_p[a, b]$. *If* $p = 1$, *the class is denoted by* $L_1[a, b]$ *or* $L[a, b]$.

Definition 1.10 *The space of all functions* $f : \Omega \longrightarrow \mathbb{R}$ *such that* $\int_\Omega |f(x)|^p \, dx < \infty$, $1 \leq p < \infty$, *is denoted by* $L_p(\Omega)$.

Definition 1.11 *For each* $0 \leq \alpha \leq 1$ *and* $M > 0$, *the set of Lipschitz continuous functions of order* α *denoted by* $Lip_M(\alpha)$, *is the set of all functions* f *on* $\mathbb{R}$ *such that* $|f(x_1) - f(x_2)| \leqslant M |x_1 - x_2|^\alpha$.

Definition 1.12 *The space* $Lip_M(\alpha, L_p)$, $0 < \alpha \leqslant 1$, $1 \leq p < \infty$, *is the set of all functions* $f \in L_p(\Omega)$ *for which* $\|f(\cdot + h) - f(\cdot)\|_{L_p(\Omega)} \leqslant M h^\alpha$, $0 < h < 1$, $M > 0$.

Definition 1.13 *(p-modulus of continuity) Let* f *be a real-valued function defined on* $\mathbb{R}$. *For* $1 \leq p \leq \infty$ *and* $\delta > 0$, *let*

$$\omega_p(f; \delta) = \sup_{0 < |h| \leq \delta} \|f(x) - f(x - h)\|_p . \tag{1.2}$$

The function $\omega_p(f; \delta)$ *or* $\omega_p(\delta)$ *is called the p-modulus of continuity of* f.

Definition 1.14 *Sobolev space $H^m(\Omega)$*

The set $H^m(\Omega) = \{f \in L_2(\Omega)/\ D^\alpha f \in L_2(\Omega), |\alpha| \leqslant m\}$, m being any positive integer is called the Sobolev space of order m.

$H^m(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f,\ g\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leqslant m} \langle D^\alpha f,\ D^\alpha g\rangle_{L_2(\Omega)}\,. \tag{1.3}$$

For $m = 1$ and $\Omega = (a, b)$, $D^\alpha f = \frac{df}{dx}$, and

$$\langle f,\ g\rangle_{H^1(a,b)} = \langle f,\ g\rangle_{L_2(a,b)} + \left\langle \frac{df}{dx},\ \frac{dg}{dx}\right\rangle_{L_2(a,b)}$$

For $m = 2$,

$$\langle f,\ g\rangle_{H^2(a,b)} = \sum_{|\alpha|\leqslant 2} \langle D^\alpha f,\ D^\alpha g\rangle_{L_2(\Omega)}$$

for $\alpha = (0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0)$.

The equivalent norm is

$$\|f\|_{H^m(\Omega)} = \sqrt{\langle f,\ g\rangle_{H^m(\Omega)}} = \left(\sum \|D^\alpha f\|_{L_2(\Omega)}\right)^{\frac{1}{2}}.$$

Besov space $B_q^{\alpha,r}(\Omega)$:

We introduce here the notion of Besov space and equivalence of the Besov norm with a norm defined by wavelet coefficients. For any $h \in \mathbb{R}^2$, we define

$$
\begin{aligned}
\triangle_h^0 f(x) &= f(x) \\
\triangle_h^1 f(x) &= \triangle(\triangle_h^0) f(x) = f(x+h) - f(x) \\
\triangle_h^2 f(x) &= \triangle(\triangle_h^1) f(x) = f(x+2h) - 2f(x+h) + f(x) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\triangle_h^{r+1} f(x) &= \triangle_h^r f(x+h) - \triangle_h^r f(x),\ r = 0, 1, 2, \ldots
\end{aligned}
$$

Now, we define the $r-$th modulus of continuity in L_p as

$$
\omega_r(f,t)_p = \sup_{|h|\leq t} \left(\int\limits_{I_{rh}} |\triangle_h^r f(x)|^p \, dx \right)^{\frac{1}{p}},
$$

where $I_{rh} = \left\{ x \in \frac{I}{x} + rh \in I,\ I = [0,1] \times [0,1] \right\},\ 0 < p \leq \infty$; with usual change to an essential supremum when $p = \infty$.

Given $\alpha > 0,\ 0 < p \leqslant \infty,\ 0 < q \leqslant \infty$, choose $q > \alpha$. The space $B_q^{\alpha,r}(L_p(\Omega))$ is called Besov space, which consists of those functions f for which

$$
\|f\|_{B_q^{\alpha,r}(L_p(\Omega))} = \|f\|_{L_p(\Omega)} + \left(\int\limits_0^\infty [t^{-\alpha}\omega_r(f,t)_p]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,\ when\ q < \infty \tag{1.4}
$$

and

$$
\|f\|_{B_\infty^{\alpha,r}(L_p(\Omega))} = \|f\|_{L_p(\Omega)} + \sup_{t>0} [t^{-\alpha}\omega_r(f,t)_p] < \infty,\ when\ q = \infty. \tag{1.5}
$$

where $\omega_r(f,t)_p$ is the $r-$modulus of continuity in L_p.

Properties of Besov spaces:

i) If $0 < p < 1,\ or\ \ 0 < q < 1$, then $\|f\|_{B_q^{\alpha,r}(L_p(\Omega))}$ does not satisfy the triangular inequality. However, there exist a constant C such that for all $f, g \in B_q^{\alpha,r}(L_p(\Omega))$,

$$\|f+g\|_{B_q^{\alpha,r}(L_p(\Omega))} \leq C(\|f\|_{B_q^{\alpha,r}(L_p(\Omega))} + \|g\|_{B_q^{\alpha,r}(L_p(\Omega))}) \tag{1.6}$$

ii) Since, for any $r > \alpha,\ r_1 > r,\ \|f\|_{B_q^{\alpha,r}(L_p(\Omega))}\ and\ \ \|f\|_{B_q^{\alpha,r_1}(L_p(\Omega))}$ are equivalent norms, we defined in the Besov space $B_q^{\alpha}(L_p(\Omega))$ to be $B_q^{\alpha,r}(L_p(\Omega))$ for $r > \alpha$.

iii) For $p = q = 2,\ B_2^{\alpha}(L_2(\Omega))$, is the Sobolev space $H^{\alpha}(\Omega)$.

iv) For $\alpha < 1,\ \ 1 \leqslant p < \infty$ and $q = \infty$, then $B_p^{\alpha}(L_p(\Omega))$ is
$Lip(\alpha,\ L_p(\Omega)) = \left\{ f \in L_p(\Omega) / \left\| f(x+h) - f(x) \right\|_{L_p} \leqslant kh^{\alpha},\ k > 0 \text{ cons tan t} \right\}.$

v) $\|f\|_{B_q^{\alpha}(L_2(\Omega))}$ is equivalent to the norm $\left(\sum\limits_k \sum\limits_j 2^{\alpha k} \left| d_{j,k} \right|^q \right)^{\frac{1}{q}}$, where $d_{j,k}$ are wavelet coefficients of f.

vi) $\|f\|_{B_q^{\alpha}(L_q(\Omega))}$ is equivalent to the norm $\left(\sum\limits_k \sum\limits_j 2^{\alpha k} \left| d_{j,k} \right|^q \right)^{\frac{1}{q}}$, where $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$.

1.5 Relation Between Smoothness and Approximation order

The following theorems provide the relationship between smoothness and order of approximation of the given function f.

Let $E_n(f)$ denote the error in the best uniform approximation of $f \in C[a, b]$ by a polynomial of degree $\leq n$, either algebraic or trigonometric, depending on the context.

Theorem 1.15 *([14] Jackson theorem I) Let* $f \in C^p[-\pi,\pi]$ *and suppose that* $f^{(p)}(x)$ *satisfies Lipschitz condition of order* $\alpha,\ 0<\alpha\leq 1$. *If*

$$E_n(f)=\min_{c_k,d_k}\max_{-\pi\leq x\leq\pi}\left|f(x)-\sum_{k=0}^{n}(c_k\cos kx+d_k\sin kx)\right| \tag{1.7}$$

then $E_n(f)\leq \frac{const}{n^{p-1+\alpha}}$.

The estimate in (1.7) is obtained by using the partial sum of the Fourier series of f. There is no reason to suppose that these are the most efficient trigonometric polynomials of order n to use. In this context D. Jackson has found that the following polynomials lead to a better estimate.

Theorem 1.16 *([14] Jackson theorem II) Let* $f\in C[a,b]$ *and* $x_0,x_1,x_2,...,x_n$ *are linearly independent elements in* $C[a,b]$ *and if*

$$E_n(f)=\min_{a_i}\max_{-1\leq x\leq 1}\left|f(x)-\sum_{i=0}^{n}a_ix_i\right|,$$

then $E_n(f)\leq\left(1+\frac{\pi^2}{2}\right)\omega\left(\frac{1}{n}\right)$.

Theorem 1.17 *([4] Bernstein's theorem I) If* $f\in C[-\pi,\pi]$ *and* $E_n(f)\leq An^{-\alpha}$ *with* $\alpha\in(0,1)$ *and* n *is the degree of approximating polynomial of* f, *then* $f\in Lip_M(\alpha)$.

Theorem 1.18 *([4] Bernstein's theorem II) If* $f\in C[-\pi,\pi]$ *and* $E_n(f)\leq\frac{A}{n}$, *then the modulus of continuity of* f *satisfies* $\omega(\delta)\leq\lambda\delta\left|\log\delta\right|$, *for small* δ.

Theorem 1.19 *([4] Bernstein's theorem III) If* $f\in C[-\pi,\pi]$ *and* $E_n(f)<An^{-p-\alpha}$, *where* p *is a natural number and* $\alpha\in(0,1)$, *then* f *possesses continuous derivatives of order* $1,2,3,...p$, *and the last one belongs to* $Lip_M(\alpha)$.

1.6 Wavelet Theory

Definition 1.20 *(Wavelet) A wavelet means a small wave (the sinusoids used in Fourier analysis are big waves). In brief, a wavelet is an oscillation that decays quickly. Sufficient mathematical conditions are :*

$$\int_{\mathbb{R}} |\psi(t)|^2 \, dt < \infty, \tag{1.8}$$

$$\int_{\mathbb{R}} \psi(t) dt = 0, \tag{1.9}$$

$$\int_{\mathbb{R}} \frac{\left|\widehat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty, \tag{1.10}$$

where $\widehat{\psi}(\omega)$ *is the Fourier Transform of* $\psi(t)$. (1.10) *is called the admissibility condition.*

1.6.1 Wavelet Transforms

Jean Morlet in 1982, introduced the idea of the wavelet transform and provided a new mathematical tool for seismic wave analysis. Morlet first considered wavelets as a family of functions constructed from translations and dilations of a single function called the "mother wavelet", $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \ a \neq 0 \tag{1.11}$$

The parameter a is the scaling parameter or scale, and it measures the degree of compression. The parameter b is the translation parameter which determines the time location of the wavelet. If $|a| < 1$, then the wavelet in (1.11) is the compressed version (smaller support in time- domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when $|a| > 1$, then $\psi_{a,b}(t)$ has a larger time-width than $\psi(t)$ and corresponds to lower frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of the Morlet wavelets in signal processing and time-frequency analysis.

1.6.2 Continuous wavelet transform

The wavelet transform of $f \in L_2(\mathbb{R})$ can be defined as

$$\begin{aligned} T_\psi f(a,b) &= |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\overline{\psi}\left(\frac{t-b}{a}\right) dt \\ &= \langle f,\ \psi_{a,b} \rangle \end{aligned} \tag{1.12}$$

where $\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\, \psi(\frac{t-b}{a})$. Let $\overline{\psi}$ denotes the complex conjugate of ψ. $T_\psi f(a,b)$ is called the wavelet transform of f in $L_2(\mathbb{R})$.

$\psi_{a,b}(t)$ plays the same role as the kernel $e^{i\omega t}$ in the Fourier transform. Like the Fourier transform, the continuous wavelet transform T_ψ is linear. However, unlike the Fourier transform, the continuous wavelet transform is not a single transform, but any transform obtained in this way.

The inverse wavelet transform can be defined [15] by the formula

$$f(t) = C_\psi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\psi f(a,b) \psi_{a,b}(t)(a^{-2}da)db, \tag{1.13}$$

provided that C_ψ satisfies the admissibility condition, where $C_\psi = 2\pi \int\limits_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ and $\widehat{\psi}(\omega)$ is the Fourier Transform of $\psi(t)$[41].

1.6.3 Discrete wavelet transform

The continuous wavelet transform can be computed at discrete grid points (a_m, b_n), $m, n \in \mathbb{Z}$. To do this, a general wavelet system can be defined as

$$\psi_{m,n}(t) = a_0^{\frac{m}{2}} \psi(a_0^m t - b_0 n),\ m, n \in \mathbb{Z}, \tag{1.14}$$

where $a_0 > 1$ *and* $b_0 > 1$ are fixed parameters. For such a family, two vital questions arise:

i) Does the sequence $\{\langle f,\ \psi_{m,n}\rangle\}_{m,n\in\mathbb{Z}}$ completely characterize the function f?

ii) Is it possible to recover f from this sequence in a stable manner?

These are closely related to the concept of frames.

Definition 1.21 *(Frames) A sequence $\{\varphi_n\}$ in a Hilbert space H is called a frame if there exist positive constants A and B such that*

$$A\|f\|^2 \leqq \sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2 \leqq B\|f\|^2, \quad for \ \ all\ f \in H. \tag{1.15}$$

The constants A and B are called frame bounds. The sequence $\{\varphi_n\}_{n=1}^{\infty}$ is called a tight frame if (1.15) *holds with $A = B$.*

Definition 1.22 *(Orthogonal basis and Orthonormal basis) A basis $\{\psi_n\}_{n=1}^{\infty}$ of a*

Hilbert space H is called orthogonal if

$$\langle \psi_n, \psi_m \rangle = 0 \ \ for \ n \neq m. \tag{1.16}$$

An orthogonal basis is orthonormal if $\langle \psi_n, \psi_m \rangle = 1 \ \ for \ n = m.$

$$\tag{1.17}$$

Theorem 1.23 *An orthogonal basis $\{\psi_n\}_{n=1}^{\infty}$ is complete if and only if $\langle \psi_n, \psi \rangle = 0$, for all $n \in \mathbb{N}$ implies $\psi = 0$.*

Definition 1.24 *(Approximation operator): For each $j \in \mathbb{Z}$, the approximation operator P_j on $f(t) \in L_2(\mathbb{R})$ is*

$$P_j f(t) = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{j,k} \rangle \, \varphi_{j,k}(t) \tag{1.18}$$

Definition 1.25 *(Detail operator): For each $j \in \mathbb{Z}$, the detail operator Q_j is defined by $P_{j+1} - P_j$.*

Definition 1.26 *Let f and g be two functions defined on $\mathbb{R}$. Then the convolution of f and g denoted by $h = f * g$ is defined by*

$$f * g(x) = \int_{\mathbb{R}} f(t) g(x - t) dt, \tag{1.19}$$

whenever the integral makes sense.

Definition 1.27 *Dilation operator $D_a f(t) = a^{\frac{1}{2}} f(at)$.*

Definition 1.28 *Translation of operator $T_k f(t) = f(t - k)$.*

For each $j, k \in \mathbb{Z}$, define $f_{j,k}(t) = D_{2^j} T_k f(t) = 2^{\frac{j}{2}} f(2^j t - k)$.

The following theorem provides a method for constructing a new wavelet from a given one.

Theorem 1.29 *[41] If ψ is a wavelet and φ is a bounded integrable function, then the convolution function $\psi * \varphi$ is also a wavelet.*

Definition 1.30 *(Wavelet series and wavelet coefficients) If a function $f \in L_2(\mathbb{R})$, the series*

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(t)$$

is called the wavelet series of f, and $\langle f, \psi_{j,k} \rangle = d_{j,k} = \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt$ are called the wavelet coefficients of f.

1.6.4 Multiresolution Analysis (MRA)

A multiresolution analysis (as defined by S. Mallat in 1989) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed linear subspaces of $L_2(\mathbb{R})$, such that the following properties are satisfied:

i) The sequence is nested. i.e. $\forall\, j, \quad V_j \subset V_{j+1}$

ii) The spaces are related to each other by dyadic scaling.

i.e. $f \in V_j \Leftrightarrow f(2x) \in V_{j+1} \Leftrightarrow f(2^{-j}x) \in V_0$

iii) The union of the spaces is dense in $L_2(\mathbb{R})$, i.e. $\bigcup_{j\in\mathbb{Z}} V_j = L_2(\mathbb{R})$.

i.e. $\forall\, f \in L_2(\mathbb{R}), \quad \lim_{j\to\,+\infty} \|f - P_j f\|_{L_2} = 0$

where P_j is the orthonormal projection onto V_j.

iv) The intersection of spaces is reduced to the null function. i.e. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

i.e. $\lim_{j\to\,-\infty} \|P_j f\|_{L_2} = 0$

v) There exist a function $\varphi \in V_0$ such that the family $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$, is a Riesz basis of V_0.

The function φ is called the scaling function for $\{V_j\}$. If $\{V_j\}$ is an MRA of $L_2(\mathbb{R})$ and if V_0 is the closed subspace generated by the integer translations of a single function φ, then we say that φ generates the MRA.

From condition (v) for every $f \in V_0$, there exists a unique sequence $\{c_k\}_{k=-\infty}^{\infty} \in l_2(\mathbb{Z})$ $\left(l_2(\mathbb{Z}) := \{x_i\}_{i=1}^{\infty} / \sum_{i=1}^{\infty} |x_i|^2 < \infty \right)$ such that $f(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k)$ with convergence in $L_2(\mathbb{R})$ and there exist two positive constants A and B independent of $f \in V_0$ such that $A \sum_{k\in\mathbb{Z}} |c_k|^2 \leqq \|f\|^2 \leqq B \sum_{k\in\mathbb{Z}} |c_k|^2$, where $0 < A < B < \infty$. In this case, we have an MRA with a Riesz basis.

Theorem 1.31 *Suppose $\{V_j\}$ is an MRA with scaling function φ. Then for any $j \in \mathbb{Z}$, the set of functions $\left\{\varphi_{j,k}(x)\right\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j.*

Lemma 1.32 *For all continuous functions f on $\mathbb{R}$ with compact support*

$$\begin{aligned} (a) \quad \lim_{j\longrightarrow\infty} \|P_j f - f\|_{L_2} &= 0, \text{ and} \\ (b) \quad \lim_{j\longrightarrow-\infty} \|P_j f\|_{L_2} &= 0. \end{aligned}$$

Lemma 1.33 *Let $\{V_j\}$ be an MRA with scaling function φ. There exists a sequence $\{h_k\}$ in $l_2(\mathbb{Z})$ such that*

$$\varphi(x) = \sum_k h_k 2^{\frac{1}{2}} \varphi(2x-k) \tag{1.20}$$

is in $L_2(\mathbb{R})$. Furthermore,

$$\widehat{\varphi}(\alpha) = m_0(\frac{\alpha}{2})\widehat{\varphi}(\frac{\alpha}{2}), \tag{1.21}$$

where $m_0(\alpha) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-2\pi i k\alpha}$ and $\widehat{\varphi}(\alpha)$ is the Fourier Transform of $\varphi(x)$.

Definition 1.34 *Let φ be a scaling function of an MRA $\{V_j\}$. The sequence $\{h_k\}$ in $l_2(\mathbb{Z})$ is called the scaling filter associated with φ. The function $m_0(\alpha)$ defined in the above lemma, is called the auxiliary function associated with φ.*

Theorem 1.35 *Let $\{V_j\}$ be an MRA with scaling function φ and scaling filter $\{h_k\}$ which satisfies $\sum_{k\in\mathbb{Z}} |h_k| < \infty$. Let $g_k = (-1)^k h_{1-k}$ be the wavelet filter, then the wavelet $\psi(x)$ can be written as*

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k 2^{\frac{1}{2}} \varphi(2x - k). \tag{1.22}$$

Chapter 2

The Haar System

2.1 Haar Wavelet

The Hungarian mathematician Alfred Haar first introduced the Haar function in 1909 in his Ph.D. thesis.

Definition 2.1 *(Haar function) A function defined on the real line* $\mathbb{R}$ *as*

$$\psi(t) = \begin{cases} 1 \; for \; t \in \left[0, \frac{1}{2}\right) \\ -1 \; for \; t \in \left[\frac{1}{2}, 1\right) \\ 0 \qquad otherwise \end{cases} \tag{2.1}$$

is known as the Haar function.

The Haar function $\psi(t)$ is the simplest example of a Haar wavelet. The Haar function $\psi(t)$ is a wavelet because it satisfies all the conditions of wavelet. This fundamental example has all the major features of the general wavelet theory. Haar wavelet is discontinuous at $t = 0, \frac{1}{2}, 1$ and it is very well-localized in the time domain.

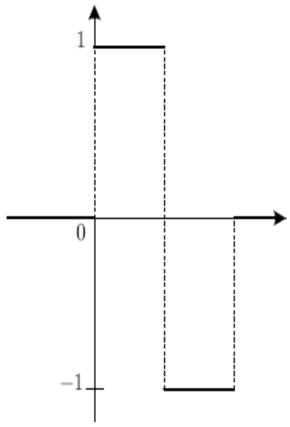

Figure 2.1: The Haar function

2.2 The Dyadic Intervals

Definition 2.2 *(Dyadic interval) For each* $j, k \in \mathbb{Z}$*, the intervals* $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ *is known as a dyadic interval. The collection of all such intervals is called dyadic subintervals of* $\mathbb{R}$.

Note that when $(j, k) \neq (j_1, k_1)$, then either

$$i)\ I_{j,k} \cap I_{j_1,k_1} = \phi, \ or \tag{2.2}$$

$$ii)\ I_{j,k} \subseteq I_{j_1,k_1} \ or \ I_{j_1,k_1} \subseteq I_{j,k} \tag{2.3}$$

Definition 2.3 *Given a dyadic interval at scale* j*,* $I_{j,k} = I^l_{j,k} \cup I^r_{j,k}$*, where* $I^l_{j,k}$ *and* $I^r_{j,k}$ *are dyadic intervals at scale* $j+1$*, to denote the left half and right half of the interval* $I_{j,k}$*. In fact,* $I^l_{j,k} = I_{j+1,2k}$ *and* $I^r_{j,k} = I_{j+1,2k+1}$.

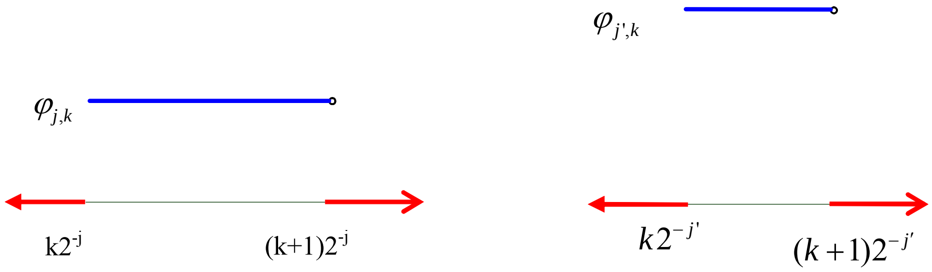

Figure 2.2: (a) Haar scaling function for j (b) Haar scaling function for $j = j'$, where $j' > j$.

2.3 The Haar System

2.3.1 The Haar Scaling Function

The Haar scaling function can be defined as

$$\varphi(t) = \chi_{[0,1)}(t) = \begin{cases} 1 & if\ 0 \leqq t < 1 \\ 0, & otherwise. \end{cases}$$

Definition 2.4 *The collection* $\{\varphi_{j,k}(t)\}_{j,k\in\mathbb{Z}}$ *is referred to as the system of Haar scaling functions. The collection* $\{\varphi_{j,k}(t)\}_{k\in\mathbb{Z}}$ *is referred to as the system of scale* j *Haar scaling functions.*

The system of Haar scaling functions satisfies the following properties:

i) $\varphi_{j,k}(t)$ is supported on the interval $I_{j,k}$.

ii) For each $j, k \in \mathbb{Z}$, $\int\limits_{\mathbb{R}} \varphi_{j,k}(t)dt = 2^{-\frac{j}{2}}$.

iii) For each $j, k \in \mathbb{Z}$, $\int\limits_{\mathbb{R}} \left|\varphi_{j,k}(t)\right|^2 dt = 1$.

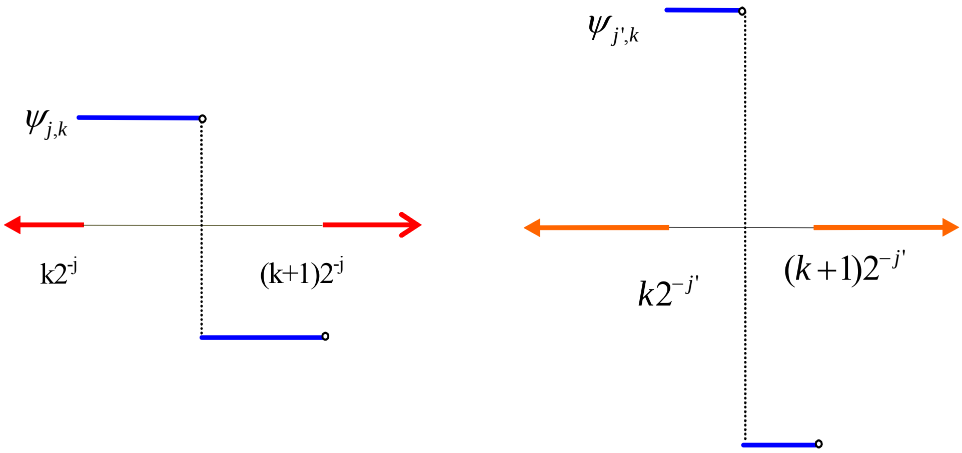

Figure 2.3: (a) Haar wavelet function for j (b) Haar wavelet function for $j = j'$, where $j' > j$.

2.3.2 The Haar Wavelet Function

Haar wavelet function (Haar wavelet) ψ can be written as $\psi(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},\,1)}(t)$.

Definition 2.5 *The collection $\{\psi_{j,k}(t)\}_{j,k\in\mathbb{Z}}$ is referred to as the Haar wavelet system on $\mathbb{R}$. For each $j \in \mathbb{Z}$, the collection $\{\psi_{j,k}(t)\}_{k\in\mathbb{Z}}$ is referred to as the system of Haar wavelet functions at scale j.*

The Haar wavelet system satisfies the following properties:

i) $\psi_{j,k}(t)$ is supported on $I_{j,k}$.

ii) $\psi_{j,k}(t) \ = 2^{\frac{j}{2}}\left(\chi_{I_{j+1,2k}}(t) - \chi_{I_{j+1,2k+1}}(t)\right)$

iii) $\int\limits_{\mathbb{R}} \psi_{j,k}(t)dt = \int\limits_{I_{j,k}} \psi_{j,k}(t)dt = 0$

iii) $\int\limits_{\mathbb{R}} \left|\psi_{j,k}(t)\right|^2 dt = \int\limits_{I_{j,k}} \left|\psi_{j,k}(t)\right|^2 dt = 1$

2.4 Orthogonality of the Haar System

Theorem 2.6 *The Haar system* $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *is an orthonormal system on* $\mathbb{R}$.

Proof. We have the inner product

$$\left\langle \psi_{j,k}\ ,\psi_{j_1,k_1}\right\rangle = \int_{-\infty}^{\infty} \psi_{j,k}(t)\psi_{j_1,k_1}(t)dt. \tag{2.4}$$

We first show orthonormality within a given scale j. By the properties of dyadic intervals, if $k \neq k_1$ then $\psi_{j,k}\ \psi_{j,k_1} \equiv 0$, and thus

$$\left\langle \psi_{j,k}\ ,\psi_{j,k_1}\right\rangle = 0 \tag{2.5}$$

If $k = k_1$, *then*

$$\left\langle \psi_{j,k}\ ,\psi_{j,\ k_1}\right\rangle = \int_{-\infty}^{\infty} \psi_{j,k}(t)\psi_{j,k}(t)dt = \int_{I_{j,k}} \left|\psi_{j,k}(t)\right|^2 dt = 1 \tag{2.6}$$

Next we show orthonormality between different scales. Let $j \neq j_1$ and $j < j_1$,then

$$\left\langle \psi_{j,k}\ ,\psi_{j_1,k_1}\right\rangle = \int_{-\infty}^{\infty} 2^{\frac{j}{2}}\psi(2^j t - k)2^{\frac{j_1}{2}}\psi(2^{j_1}t - k_1)dt \tag{2.7}$$

Let $2^j t - k = u$, so that $t = (k+u)2^{-j}$, then $\left\langle \psi_{j,k}\ ,\psi_{j_1,k_1}\right\rangle = \int_{-\infty}^{\infty} 2^{\frac{j_1-j}{2}}\psi(u)\psi(2^{j_1-j}u + 2^{j_1-j}k - k_1)du$.

Let $j_1 - j = s$ *and* $k_1 - k2^s = r$, then

$$\left\langle \psi_{j,k}\ , \psi_{j_1,k_1} \right\rangle = \int_{-\infty}^{\infty} 2^{\frac{s}{2}} \psi(u) \psi(2^s u - r) du = \int_{-\infty}^{\infty} 2^{\frac{s}{2}} \psi(t) \psi(2^s t - r) dt \tag{2.8}$$

According to properties (2.2) and (2.3), there are three possibilities:

(i) $I_{j,k}\ \cap\ I_{j_1,k_1} = \phi,$ *so that* $\left\langle \psi_{j,k}\ , \psi_{j_1,k_1} \right\rangle = \int_{-\infty}^{\infty} \psi_{j,k}(t) \psi_{j_1,k_1}(t) dt = 0.$

(ii) $I_{j,k}\ \subseteq\ I^l_{j_1,k_1}$.In this case ψ_{j_1,k_1} is the constant 1 on $I^l_{j_1,k_1}$. Since $I_{j,k}\ \subset\ I_{j_1,k_1}$ it is also identically 1 on $I_{j,k}$. Since $\psi_{j,k}(t)$ is supported on $I_{j,k}$, hence

$$\left\langle \psi_{j,k}\ , \psi_{j_1,k_1} \right\rangle = \int_{-\infty}^{\infty} \psi_{j,k}(t) \psi_{j_1,k_1}(t) dt = \int_{I_{j,k}} \psi_{j,k}(t) dt = 0. \tag{2.9}$$

(iii) $I_{j,k}\ \subseteq\ I^r_{j_1,k_1}$,in this case $\psi_{j_1,k_1}(t)$ is the constant -1 on $I^r_{j_1,k_1}$ and on $I_{j,k}$. Thus

$$\left\langle \psi_{j,k}\ , \psi_{j_1,k_1} \right\rangle = \int_{-\infty}^{\infty} \psi_{j,k}(t) \psi_{j_1,k_1}(t) dt = \int_{I_{j,k}} \psi_{j,k}(t) dt = 0 \tag{2.10}$$

If $j > j_1$,the above three possibilities will come again. So the Haar system is an orthonormal system on $\mathbb{R}$. ■

Theorem 2.7 *[44] The Haar system on $\mathbb{R}$ is a complete orthonormal system on $\mathbb{R}$.*

Lemma 2.8 *[44] Given $j \in \mathbb{Z}$, and a function f continuous with compact support on $\mathbb{R}$,*

$$Q_j f(t) = \sum_{k \in \mathbb{Z}} \left\langle f, \psi_{j,k} \right\rangle \psi_{j,k}(t), \tag{2.11}$$

where the sum is finite and Q_j is the detail operator defined in sec. (1.5.3).

Definition 2.9 *(Haar Wavelet series and wavelet coefficient) If f is defined on* $[0,1]$, *then it has an expansion in terms of Haar functions as follows. Given any integer* $J \geq 0$,

$$\begin{aligned} f(t) &= \sum_{k=0}^{2^J-1} \langle f, \varphi_{J,k} \rangle \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \\ &= \sum_{k=0}^{2^J-1} c_{J,k} \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t), \end{aligned} \tag{2.12}$$

the series (2.12) *is known as the Haar wavelet series for* f. $d_{j,k}$ *and* $c_{J,k}$ *are known as the Haar wavelet coefficients and the Haar scaling coefficients respectively.*

Example 2.10 *(Haar wavelet series)*

Consider a simple function

$$f(t) = \begin{cases} t^2, \ 0 \leq t < 1 \\ 0, \ otherwise \end{cases}.$$

Let the starting scale be $J = 0$. So the scaling coefficient $c_{0,0} = \int_0^1 t^2 \varphi_{0,0}(t)dt = \int_0^1 t^2 dt = \frac{1}{3}$ and the wavelet coefficients

$$d_{0,0} = \int_0^1 t^2 \psi_{0,0}(t)dt = \int_0^{0.5} t^2 dt - \int_{0.5}^1 t^2 dt = -\frac{1}{4}$$

$$d_{1,0} = \int_0^1 t^2 \psi_{1,0}(t)dt = \int_0^{0.25} t^2 \sqrt{2} dt - \int_{0.25}^{0.5} t^2 \sqrt{2} dt = -\frac{\sqrt{2}}{32}$$

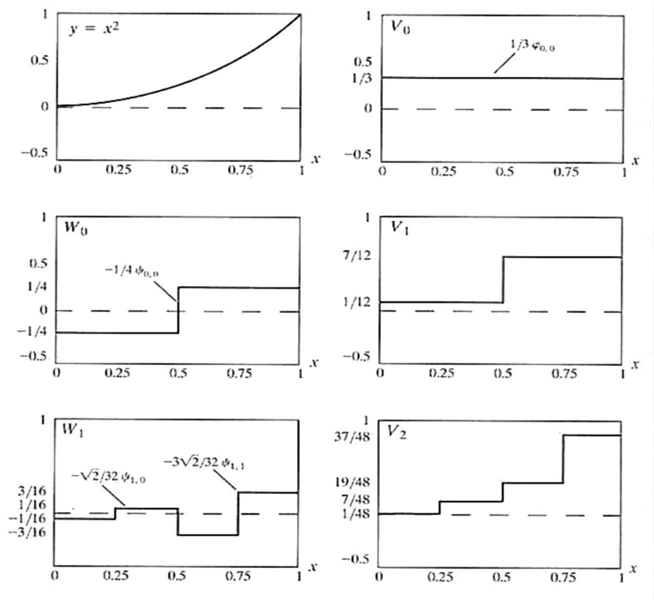

Figure 2.4: A wavelet series expansion of $f(t) = t^2$ using Haar wavelets.

$$d_{1,1} = \int_0^1 t^2 \psi_{1,1}(t)dt = \int_{0.5}^{0.75} t^2 \sqrt{2}dt - \int_{0.75}^{1} t^2 \sqrt{2}dt = -\frac{3\sqrt{2}}{32}$$

and so on.

Therefore

$$f(t) = t^2 = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(t)}_{V_0} + \underbrace{[-\frac{1}{4}\psi_{0,0}(t)]}_{W_0}}_{V_1=V_0\oplus W_0} + \underbrace{[-\frac{\sqrt{2}}{32}\psi_{1,0}(t) - \frac{3\sqrt{2}}{32}\psi_{1,1}(t)]}_{W_1}}_{V_2=V_1\oplus W_1=V_0\oplus W_0\oplus W_1} + ...$$

where V_j and W_j, $j \geqslant 0$ are the orthogonal subspaces of $L_2[0,1]$ and $\oplus$ is the direct sum.

2.5 Comparison of Haar series with Fourier series

Let $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ be the dyadic interval. The function $\psi_{j,k}$ vanishes outside the interval $I_{j,k}$. The length of the interval $I_{j,k}$ is 2^{-j}. For large j , the length of $I_{j,k}$ is small. This implies that the function $\psi_{j,k}$ is well localized in time (or, depending on the context, well localized in space). This property is to be contrasted with the trigonometric basis $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$. Each element of the trigonometric basis has absolute value 1 for every $t \in [0, 1)$ and so it never vanishes for any t.

For good time localization of the Haar basis, the function f vanishes outside a small subinterval (a, b) of $[0, 1)$. So most of its Haar coefficients $\langle f, \psi_{j,k}\rangle$ are identically 0 out side the subinterval (a, b). But in the case of Fourier series, even if a function f is supported inside a small subinterval (a, b) of $[0, 1)$, most of its Fourier coefficients would be nonzero. Of course, in both cases, we are dealing with infinite series with infinitely many coefficients.

In the case of the Haar series, let us fix an integer $j \geq 0$ and note that there are 2^j functions $\psi_{j,k}$ in the Haar system on $[0, 1)$. For a given function f supported in an interval (a, b), then for this j, the Haar coefficients $\langle f, \psi_{j,k}\rangle = 0 \;\; if \;\; t \notin (a, b)$,

that is, either $(k + 1)2^{-j} \leq a$ or $k2^{-j} \geq b$,

that is, either $k \leq 2^j a - 1$ or $k \geq 2^j b$

and $\langle f, \psi_{j,k}\rangle \neq 0 \; if \; t \in (a, b)$ i.e. $2^j a - 1 < k < 2^j b$.

The number of integers k satisfying the above inequality, which we will denote by N_j, is

$$2^j(b - a) < N_j < 2^j(b - a) + 1.$$

Hence,

$$b - a < \frac{N_j}{2^j} < (b-a) + 2^{-j}.$$

therefore $\lim_{j\to\infty} \frac{N_j}{2^j} = b - a$.

Thus, we conclude that the fraction of possibly nonzero Haar coefficients for a function vanishing outside an interval is approximately proportional to the length of that interval.

2.5.1 Behavior of Haar Coefficients Near Jump Discontinuities

Suppose that $f(t)$ is a function defined on $[0,1]$, with a jump discontinuity at $t_0 \in (0,1)$ and continuous at all other points in $[0,1]$. Here we analyze the behavior of Haar coefficients when t_0 is inside or outside the dyadic interval $I_{j,k}$. In particular, we can find the location of a jump discontinuity just by examining the absolute value of the Haar coefficients.

For simplicity, let us assume that $f(t)$ is C^2 on $[0,t_0]$ and $[t_0,1]$. This means that both f' and f'' exist, are continuous functions, and hence are bounded on each of these intervals. For fixed $j \geq 0$ and $0 \leq k \leq 2^j - 1$, and let $t_{j,k}$ be the mid point of the interval $I_{j,k}$; that is, $t_{j,k} = (k + \frac{1}{2})2^{-j}$. There are now two possibilities, either $t_0 \in I_{j,k}$ *or* $t_0 \notin I_{j,k}$.

Case 1: If $t_0 \notin I_{j,k}$, then expanding $f(t)$ about $t_{j,k}$ by Taylor's formula

$$f(t) = f(t_{j,k}) + f'(t_{j,k})(t - t_{j,k}) + \frac{1}{2}f''(\xi_{j,k})(t - t_{j,k})^2,$$

where $\xi_{j,k} \in I_{j,k}$. Now using the fact that $\int \psi_{j,k}(t)dt = 0$,

$$\begin{aligned}
\langle f, \psi_{j,k} \rangle &= \int\limits_{I_{j,k}} f(t)\psi_{j,k}(t)dt \\
&= f(t_{j,k}) \int\limits_{I_{j,k}} \psi_{j,k}(t)dt + f'(t_{j,k}) \int\limits_{I_{j,k}} \psi_{j,k}(t)(t - t_{j,k})dt \\
&\quad + \frac{1}{2} \int\limits_{I_{j,k}} f''(\xi_{j,k})(t - t_{j,k})^2 \psi_{j,k}(t)dt \\
&= f'(t_{j,k}) \int\limits_{I_{j,k}} t\psi_{j,k}(t)dt + r_{j,k}(t), \qquad (2.13)
\end{aligned}$$

where $|r_{j,k}(t)| = \frac{1}{2} \left| \int\limits_{I_{j,k}} f''(\xi_{j,k})(t - t_{j,k})^2 \psi_{j,k}(t)dt \right|$.

Now

$$\begin{aligned}
\int\limits_{I_{j,k}} t\psi_{j,k}(t)dt &= \int\limits_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} 2^{\frac{j}{2}} t dt - \int\limits_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} 2^{\frac{j}{2}} t dt \\
&= 2^{\frac{j}{2}} \left\{ \left[\frac{t^2}{2}\right]_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} - \left[\frac{t^2}{2}\right]_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} \right\} \\
&= 2^{\frac{j}{2}} . 2^{-2j} . \frac{1}{2} \left\{ (k + \frac{1}{2})^2 - k^2 - (k+1)^2 + (k + \frac{1}{2})^2 \right\} \\
&= -\frac{1}{4} 2^{-\frac{3j}{2}}. \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14)

$$\langle f, \psi_{j,k} \rangle = -\frac{1}{4} 2^{-\frac{3j}{2}} f'(t_{j,k}) + r_{j,k}(t).$$

Now

$$
\begin{aligned}
|r_{j,k}(t)| &\leq \frac{1}{2}\max_{t\in I_{j,k}}|f''(t)|\int\limits_{I_{j,k}}(t-t_{j,k})^2\left|\psi_{j,k}(t)\right|dt, \\
&\leq \frac{2^{\frac{j}{2}}}{2}\max_{t\in I_{j,k}}|f''(t)|\int\limits_{k2^{-j}}^{(k+1)2^{-j}}(t-t_{j,k})^2dt, \\
&= \frac{2^{\frac{j}{2}}}{2}.\frac{2^{-3j}}{3.4}.\max_{t\in I_{j,k}}|f''(t)| \\
&= \frac{1}{24}2^{-\frac{5j}{2}}.\max_{t\in I_{j,k}}|f''(t)|.
\end{aligned}
$$

For large j, $2^{-\frac{5j}{2}}$ is very small compared with $2^{-\frac{3j}{2}}$. So

$$
\left|\langle f,\psi_{j,k}\rangle\right| \approx \frac{1}{4}2^{-\frac{3j}{2}}\left|f'(t_{j,k})\right| = O(2^{-\frac{3j}{2}}). \tag{2.15}
$$

Case 2: If $t_0 \in I_{j,\ k}$, then either it is in $I^l_{j,\ k}$ or in $I^r_{j,\ k}$. We assume that $t_0 \in I^l_{j,\ k}$, and the other case is similar. Now expanding $f(t)$ about t_0 by Taylor's formula, we have

$$
\begin{aligned}
f(t) &= f(t_0^-) + f'(\xi^-)(t-t_0),\ t\in[0,t_0),\ \xi^-\in[t,t_0] \\
f(t) &= f(t_0^+) + f'(\xi^+)(t-t_0),\ t\in[t_0,1),\ \xi^+\in[t_0,t].
\end{aligned}
$$

Therefore

$$\begin{aligned}
\langle f, \psi_{j,k} \rangle &= \int\limits_{I_{j,k}} f(t)\psi_{j,k}(t)dt \\
&= 2^{\frac{j}{2}} \int\limits_{k2^{-j}}^{t_0} f(t_0^-)dt + 2^{\frac{j}{2}} \int\limits_{t_0}^{(k+\frac{1}{2})2^{-j}} f(t_0^+)dt - 2^{\frac{j}{2}} \int\limits_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} f(t_0^+)dt + \varepsilon_{j,k} \\
&= 2^{\frac{j}{2}}(t_0 - k2^{-j})\{f(t_0^-) - f(t_0^+)\} + \varepsilon_{j,k}, \qquad (2.16)
\end{aligned}$$

where

$$\varepsilon_{j,k} = \int\limits_{k2^{-j}}^{t_0} f'(\xi^-)(t - t_0)\psi_{j,k}dt + \int\limits_{t_0}^{(k+1)2^{-j}} f'(\xi^+)(t - t_0)\psi_{j,k}dt.$$

Thus

$$\begin{aligned}
|\varepsilon_{j,k}| &\leq \max_{t \in I_{j,k} \setminus \{t_0\}} |f'(t)| \int\limits_{I_{j,k}} |t - t_0| \left|\psi_{j,k}(t)\right| dt \\
&\leq 2^{\frac{j}{2}} \max_{t \in I_{j,k} \setminus \{t_0\}} |f'(t)| \int\limits_{I_{j,k}} |t - t_0| \, dt \\
&\leq 2^{\frac{j}{2}} \max_{t \in I_{j,k} \setminus \{t_0\}} |f'(t)| \frac{1}{4} 2^{-2j} \\
&= \frac{1}{4} \max_{t \in I_{j,k} \setminus \{t_0\}} |f'(t)| \,.2^{-\frac{3j}{2}}.
\end{aligned}$$

For large j, $2^{-\frac{3j}{2}}$ is very small compared with $2^{-\frac{j}{2}}$. So

$$\langle f, \psi_{j,k} \rangle \approx 2^{\frac{j}{2}} \left|t_0 - k2^{-j}\right| \left|f(t_0^-) - f(t_0^+)\right|.$$

The quantity $|t_0 - k2^{-j}|$ is very small if t_0 is close to the left end point of $I_{j,k}^l$ and can even be zero. However, we can expect that in most cases, t_0 will be in the middle of

$I_{j,k}^l$ so that $|t_0 - k2^{-j}| \approx \frac{1}{4} \cdot 2^{-j}$. Thus, for large j,

$$\left|\left\langle f, \psi_{j,k}\right\rangle\right| \approx \frac{1}{4} 2^{-\frac{j}{2}} \left|f(t_0^-) - f(t_0^+)\right| = O(2^{-\frac{j}{2}}). \tag{2.17}$$

Comparing (2.15) with (2.17), we see that the decay of $\left|\left\langle f, \psi_{j,k}\right\rangle\right|$ for large j is considerably slower if $t_0 \in I_{j,k}$ than if $t_0 \notin I_{j,k}$. That is, large coefficients in the Haar expansion of a function f that persist for all scales suggest the presence of a jump discontinuity in the interval $I_{j,k}$ corresponding to the large coefficients.

2.5.2 Haar Coefficients and Global Smoothness

We know that the global smoothness of a function f defined on [0,1] is reflected in the decay of its Fourier coefficients. Specifically, if f is periodic and C^k on $\mathbb{R}$, then there exists a constant M depending on f such that for all $n \in \mathbb{Z}$, $|c_n| \leq M\,|n|^{-k}$, where c_n are the Fourier coefficients of f. This can be regarded as a statement about the frequency content of smooth functions, namely that smoother functions tend to have smaller high frequency components than do functions that are not smooth.

However, no such estimate holds for the Haar series. To see this, simply note that the function $f(t) = e^{it}$ has period 1 and is C^∞ on $\mathbb{R}$ with all of its derivatives bounded by 1. But have

$$\left|\left\langle f, \psi_{j,k}\right\rangle\right| = 2^{-\frac{j}{2}} \frac{\sin^2((\frac{1}{4})2^{-j})}{((\frac{1}{4})2^{-j})},$$

and since $\sin^2((\frac{1}{4})2^{-j}) \approx (\frac{1}{4})2^{-j}$ for large j, this means that $\left|\left\langle f, \psi_{j,k}\right\rangle\right| \approx (\frac{1}{4})2^{-\frac{3j}{2}}$ for large j. But this is the same rate of decay observed for functions continuous but with a discontinuous first derivative. Hence, global smoothness of a function does not affect the rate of decay of its Haar coefficients.

Chapter 3

Approximation by Wavelets in Different Spaces

3.1 Approximation space

If $A(u)$ and $B(u)$ are functions of a set u of parameters, we shall often use the notation $A(u) \lesssim B(u)$, to express that there exists a constant $c > 0$ such that $A(u) \leq cB(u)$ independent of the parameters. We use $A(u) \sim B(u)$ to express that $A(u) \lesssim B(u)$ and $B(u) \lesssim A(u)$. Important problems of approximation theory have in common the following general setting: Let $(S_n)_{n \geqslant 0}$ be a family of sub-spaces of a normed space X, and for $f \in X$, we consider the best approximation error:

$$E_n(f)_X = dist(f, S_n)_X = \inf_{g \in S_n} \|f - g\|_X . \tag{3.1}$$

Typically, n represents the number of parameters which are needed to describe an element in S_n and in most cases of interest, $E_n(f)$ goes to zero as this number tends to infinity. If in addition $E_n(f) \lesssim n^{-\alpha}$ for some $\alpha > 0$, we say that f is approximated at rate α.

Given such a setting, the central problem of approximation theory is to characterize by some analytic (typically smoothness) condition those functions f which are approximated at some prescribed rate $\alpha > 0$.

3.2 Approximation by Haar wavelets in Different Spaces

3.2.1 $L_2(\mathbb{R})$ Spaces

Theorem 3.1 *[3] Assume that $f : [0,1] \longrightarrow \mathbb{R}$ is continuous. Then the Haar wavelet series of f is*

$$f(t) = \langle f, \chi_{[0,1)} \rangle \chi_{[0,1)}(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \tag{3.2}$$

where $d_{j,k} = \langle f, \psi_{j,k} \rangle$ are the wavelet coefficients.

For computation we need a finite sum. Let $N = 2^J$, $J \in \mathbb{N}$. This means that we consider $j = 0, 1, 2, 3, ..., J-1$. In the case of Haar wavelet we have seen in sec. (2.5.1) for each j only one of the coefficients in (3.2) is non zero and its size is $d_{j,k} = \langle f, \psi_{j,k} \rangle \sim 2^{-\frac{j}{2}}$.

Theorem 3.2 *Let f be continuous in $L_2(\mathbb{R})$ and the partial sum of the Haar wavelet series is*

$$g = \langle f, \chi_{[0,1)} \rangle \chi_{[0,1)}(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t),$$

for fixed $j = J \in \mathbb{N}$. Then the error of the approximation in $L_2(\mathbb{R})$ is $O(2^{-\frac{J}{2}})$.

Proof. *The error of the approximation in $L_2(\mathbb{R})$ is*

$$\begin{aligned}
\|f-g\|_{L_2(\mathbb{R})} &= \left\| f - \left\langle f, \chi_{[0,1)} \right\rangle \chi_{[0,1)}(t) - \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \right\|_{L_2} \\
&= \left\| \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \right\|_{L2} \\
&= \left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} |d_{j,k}|^2 \right)^{\frac{1}{2}} \\
&\sim \left(\sum_{j=J}^{\infty} 2^{-j} \right)^{\frac{1}{2}} \sim 2^{-\frac{J}{2}} \sim \frac{1}{\sqrt{N}} = O(2^{-\frac{J}{2}}) \qquad (3.3)
\end{aligned}$$

■

3.2.2 $L_p(\mathbb{R})$ Spaces

Theorem 3.3 *If $f \in L_p(\mathbb{R})$ and the partial sum of the Haar wavelet series is*

$$g = \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} \left\langle f, \psi_{j,k} \right\rangle \psi_{j,k}(t) = \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t),$$

for fixed $j = J \in \mathbb{N}$, then the error of the approximation in $L_p(\mathbb{R})$ is $O(2^{-\frac{J}{2}})$.

Proof. *The error of the approximation in* $L_p(\mathbb{R})$ *is*

$$\begin{aligned}
\|f-g\|_{L_p(\mathbb{R})} &= \left\| f - \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \right\|_{L_p} \\
&= \left\| \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \right\|_{Lp} \\
&= \left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} |d_{j,k}|^p \right)^{\frac{1}{p}} \\
&\sim \left(\sum_{j=J}^{\infty} 2^{-\frac{jp}{2}} \right)^{\frac{1}{p}} \sim \left(2^{-\frac{J}{2}}\right) = O(2^{-\frac{J}{2}}) \qquad (3.4)
\end{aligned}$$

■

3.2.3 $Lip(\alpha, L_p)$ Spaces

Theorem 3.4 *If* $f \in Lip(\alpha, L_p)$, $0 < \alpha \leqslant 1,\ 1 < p \leqslant \infty,$ *and*

$$g(t) = \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t)$$

is the Haar wavelet series of f *for some* $J \in \mathbb{N}$, *then the error of the approximation in* $Lip(\alpha, L_p)$ *is* $O(2^{-J\alpha})$.

Proof. We have from [17] if $f \in Lip(\alpha, L_p),\ 0 < \alpha \leqslant 1$ and $1 < p \leqslant \infty$ then

$$\begin{aligned}
dist(f, S_n)_p &\leq \inf_{g \in S_n} \|f-g\|_p \\
&\leq C_p\,|f|_{Lip(\alpha,\ L_p)}\,\delta^{\alpha},\ \text{where } \delta = \max_{0\leq k<N}|t_{k+1}-t_k| \qquad (3.5)
\end{aligned}$$

According to (3.5)here the error of the approximation in $Lip(\alpha, L_p)$ is

$$\begin{aligned}
\|f-g\|_{L_p} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(t) \right\|_{L_p} \\
&\leq C_p \, |f|_{Lip(\alpha,\ L_p)} \, (2^{-J})^{\alpha} \\
&\leqslant M(2^{-J})^{\alpha} = O(2^{-J\alpha}), \qquad (3.6)
\end{aligned}$$

where constant $M \geqslant 0$ and C_p depending at most on p. ■

3.2.4 Sobolev Spaces $H^m(\mathbb{R})$

Theorem 3.5 *If $f \in H^m(\mathbb{R})$ and $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{N-1} d_{j,k}\psi_{j,k}(t)$ is the finite Haar wavelet series of f for some $J \in \mathbb{N}$, then the error of the approximation is $O(2^{-\frac{mN}{2}})$, where $N = 2^J$.*

Proof. *The error of the approximation is*

$$\begin{aligned}
\|f - \ g\|_{L_2(\mathbb{R})} &= \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} |d_{j,k}|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \frac{2^{mk}}{2^{mN}} |d_{j,k}|^2 \right)^{\frac{1}{2}} \\
&\leqslant \left(2^{-mN} \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |d_{j,k}|^2 \right)^{\frac{1}{2}}. \qquad (3.7)
\end{aligned}$$

From the properties of Besov space (in sec.1.4) we have,

$$\|f\|_{H^m(L_2(\mathbb{R}))} \simeq \left(\sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |d_{j,k}|^2 \right)^{\frac{1}{2}}.$$

Therefore (3.7) *implies that*

$$\|f - g\|_{L_2(\mathbb{R})} \leqslant 2^{-\frac{mN}{2}} \|f\|_{H^m(L_2(\mathbb{R}))} = O(2^{-\frac{mN}{2}}).$$

Hence the proof. ■

3.2.5 Besov spaces $B_q^{\alpha,r}(L_p(\mathbb{R}))$

Theorem 3.6 *If* $f \in B_q^{\alpha}(L_q(\mathbb{R}))$, $\alpha > 0$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ *and* $g(t) = \sum_{j=0}^{J-1}\sum_{k=0}^{N-1} d_{j,k}\psi_{j,k}(t)$ *is the finite Haar wavelet series of* f, *then the error of the approximation is* $O(2^{-\frac{\alpha N}{q}})$, *where* $N = 2^J$.

Proof. *The error of the approximation is*

$$\begin{aligned}\|f - g\|_{L_q(\mathbb{R})} &= \left(\sum_{j=J}^{\infty}\sum_{k=0}^{N-1} |d_{j,k}|^q\right)^{\frac{1}{q}} \leqslant \left(\sum_{j=J}^{\infty}\sum_{k=0}^{N-1} \frac{2^{\alpha k}}{2^{\alpha N}} |d_{j,k}|^q\right)^{\frac{1}{q}} \\ &\leqslant 2^{-\frac{\alpha N}{q}} \left(\sum_{j=J}^{\infty}\sum_{k=0}^{N-1} 2^{\alpha k} |d_{j,k}|^q\right)^{\frac{1}{q}}. \qquad (3.8)\end{aligned}$$

From the properties of Besov space (in sec.1.4) we have,

$$\|f\|_{B_q^{\alpha}(L_q(\mathbb{R}))} \simeq \left(\sum_{j=J}^{\infty}\sum_{k=0}^{N-1} 2^{\alpha k} |d_{j,k}|^q\right)^{\frac{1}{q}}.$$

Therefore (3.8) *implies that*

$$\|f - g\|_{L_q(\mathbb{R})} \leqslant 2^{-\frac{\alpha N}{q}} \|f\|_{B_q^{\alpha}(L_q(\mathbb{R}))} = O(2^{-\frac{\alpha N}{q}}),$$

where $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$. *Hence the proof.* ■

Conclusion 3.7 *The above theorems show that the approximation order will improve if the smoothness of the approximation spaces is improved.*

3.3 Approximation by Daubechies wavelets

I. Daubechies ([9, 12]) has constructed the family of orthonormal wavelets with compact support. This family has many interesting properties and it can be constructed to have a given number of derivatives and to have a given number of vanishing moments. For an arbitrary integer N, an orthonormal basis for $L_2(\mathbb{R})$ of the form $2^{\frac{j}{2}}\psi(2^j x - k),\ \ j,k \in \mathbb{Z}$ has the following properties:

(i) The support of ψ_N is contained in $[-N,\ N]$.

(ii) $\int\limits_{-\infty}^{\infty} \psi_N(x)dx = \int\limits_{-\infty}^{\infty} x\psi_N(x)dx = ... = \int\limits_{-\infty}^{\infty} x^N\psi_N(x)dx = 0.$

In fact, we have the following theorem:

Theorem 3.8 *[12] There exists a constant K such that for each $N = 2, 3, 4,...$, there exists an MRA $\{V_j\}$ with the scaling function φ and an associated wavelet ψ such that*

(i) φ and ψ belong to $C^N[-KN,\ KN]$.

(ii) φ and ψ are compactly supported and both supp φ and supp ψ are contained in $[-KN,\ KN]$.

(iii) $\int\limits_{-\infty}^{\infty} \psi_N(x)dx = \int\limits_{-\infty}^{\infty} x\psi_N(x)dx = ... = \int\limits_{-\infty}^{\infty} x^N\psi_N(x)dx = 0.$

Let P_j denote the orthogonal projection of $L_2(\mathbb{R})$ onto V_j and Q_j denote the orthogonal projection of $L_2(\mathbb{R})$ onto W_j, where $W_j,\ j \in \mathbb{Z}$ are the orthogonal subspace of $L_2(\mathbb{R})$. We know that $L_2(\mathbb{R}) = \bigoplus\limits_{j\in\mathbb{Z}} W_j$.

For every integer $N \geq 1$, Daubechies constructed a pair of functions φ and ψ that are, respectively, $\chi_{[0,1)}$ and the Haar function for $N = 1$. She also generalize these functions for $N > 1$. The construction takes the following steps.

Step 1: Construct a finite sequence $h_0, h_1, h_2, ..., h_{2N-1}$ satisfying the conditions

$$\sum_{k=0}^{2N-1} h_k h_{k+2m} = \delta_m \quad for\ every \text{ integer } m, \tag{3.9}$$

$$\sum_{k=0}^{2N-1} h_k = \sqrt{2} \tag{3.10}$$

$$\sum_{k=0}^{2N-1} g_k k^m = 0, \tag{3.11}$$

whenever $0 \leq m \leq N-1,\ where\ g_k = (-1)^k h_{1-k}$.

It can be observed that (3.9) and (3.10) imply (3.11) for $m = 0$.

Step2: Construct the trigonometric polynomial $m_0(y) = \sqrt{2} \sum\limits_{k=0}^{2N-1} h_k e^{iky}$

Step3: Construct the scaling function φ so that its Fourier transform $\widehat{\varphi}$ satisfies

$$\widehat{\varphi}(y) = \left(\frac{1}{\sqrt{2\pi}}\right) \prod_{k\geq 1}^{2N-1} m_0(2^{-k}y).$$

Step4: Construct the wavelet ψ by $\psi(x) = \sum\limits_{k=0}^{2N-1} g_k \varphi(2x-k)$

For $N > 1$, we have the following:

1. $\varphi_{n,k} = \sum\limits_{j\in\mathbb{Z}} h_{j-2k}\varphi_{n+1,j}\ and\ \psi_{n,k} = \sum\limits_{j\in\mathbb{Z}} g_{j-2k}\varphi_{n+1,j}$,
2. $supp\ \varphi_{n,k} = [k2^{-n},\ (k+2N-1)2^{-n}]$,

   $supp\ \psi_{n,k} = [(k+1-N)2^{-n},\ (k+N)2^{-n}]$,
3. $\int \psi_{j,k}(x) x^m dx = 0$ for all integer j and k and any integer $0 \leq m \leq N-1$ and
4. $\varphi_{n,k}$ and $\psi_{n,k} \in C^{\lambda(n)} = Lip\lambda(n)$ with exponent $\lambda(n)$, where $\lambda(2) = 2 - \log_2(1+\sqrt{3}) \simeq 0.5500,\ \lambda(3) \simeq 1.087833,\ \lambda(4) \simeq 1.617926\ and\ \lambda(N) \simeq 0.3485N$ for large N.

The following theorem is the Jackson type theorem for Daubechies wavelets:

Theorem 3.9 *[41] If $f \in C_0^\infty(\mathbb{R})$, $\psi \in H^m(\mathbb{R})$, then there exists a constant $K > 0$ such that*

$$\|f - P_j(f)\|_{H^m} \leqslant K2^{-j(N-m)} = O\left(2^{-j(N-m)}\right)$$

where $P_j(f)$ denotes the orthogonal projection of $L_2(\mathbb{R})$ on V_j and N is the order of Daubechies wavelet ψ.

Proof. Let $f \in C^N(\mathbb{R})$ and φ and $\psi \in H^m(\mathbb{R})$, $0 < m < N$. Then the approximation error is $\|f - P_j(f)\|_{H^m} \leqslant \sum_{l \geqslant j} \|P_{l+1}f - P_l f\|_{H^m}$

$$\text{Now } \|P_{l+1}f - P_l f\|_{H^m} \leqslant \|f - P_l f\|_{H^m} + \|f - P_{l+1}f\|_{H^m} \tag{3.12}$$

By using inverse estimate [5], $\|f - P_j f\|_{H^m} \lesssim 2^{jm} \|f\|_{L_p}$. So (3.12) becomes

$$\|P_{l+1}f - P_l f\|_{H^m} \leqslant 2^{jm} \|f\|_{L_p} + 2^{jm} \|f\|_{L_p} \tag{3.13}$$

Again by using the direct estimate [5], $\|f - P_j f\|_{L_p} \lesssim 2^{-jN} |f|_{H^m}$. So (3.13) becomes

$$\begin{aligned} \|P_{l+1}f - P_l f\|_{H^m} &\leqslant 2^{jm} \|f\|_{L_p} + 2^{jm} \|f\|_{L_p} \\ &\lesssim 2^{-jN} 2^{jm} |f|_{H^m} + 2^{-jN} 2^{jm} |f|_{H^m} \\ &= 2^{-j(N-m)} |f|_{H^m} \lesssim K2^{-j(N-m)} = O(2^{-j(N-m)}). \end{aligned}$$

Therefore $\|f - P_j(f)\|_{H^m} \leqslant K2^{-j(N-m)} = O\left(2^{-j(N-m)}\right)$. Hence the proof. ■

3.4 Approximation by Coifman wavelets

Coifman wavelets are similar to Daubechies wavelets in that they have maximal number of vanishing moments; however in Coifman wavelets, the vanishing moments are equally distributed between the scaling function and the wavelet. These are very useful for numerical solutions of partial differential equations as they have very good order of approximation (see [38]).

Definition 3.10 *(Coifman wavelets or Coiflets). An orthonormal wavelet system with compact support is called a Coifman wavelet system of degree N if the moments of the associated scaling function φ and wavelet ψ satisfy the conditions*

$$\begin{aligned} Mom_0(\varphi) &= \int \varphi(x)dx = 1 \quad if \quad l = 0 \\ Mom_l(\varphi) &= \int x^l \varphi(x)dx = 0 \quad if \quad l = 1, 2, 3, ..., N \\ Mom_l(\psi) &= \int x^l \psi(x)dx = 0 \quad if \quad l = 0, 1, 2, ..., N. \end{aligned}$$

It may be observed that those conditions are equivalent to the following conditions:

$$\sum_{k\in\mathbb{Z}} (2k)^l h_{2k} = \sum_{k\in\mathbb{Z}} (2k+1)^l h_{2k+1} = 0, \quad for\ l = 1, 2, 3, ..., N$$

$$\sum_{k\in\mathbb{Z}} h_{2k} = \sum_{k\in\mathbb{Z}} h_{2k+1} = 1,$$

where h_k is the scaling filter of φ.

Theorem 3.11 *[41] (Tain and Wells Jr.,1997). For an orthogonal Coifman wavelet system of degree N with scaling function φ, let $\{h_k\}$ be a finite scaling filter of φ in*

$l_2(\mathbb{Z})$. *For* $f \in C^N(\mathbb{R})$ *having compact support, define*

$$\begin{aligned} S_j f(x) &= 2^{-\frac{j}{2}} \sum_{k\in\mathbb{Z}} f\left(k2^{-j}\right) \varphi_{J,k}(x), \ \forall\ j \in \mathbb{Z} \\ \textit{then} \quad \|f - S_j f\|_{L_2} &\leq C2^{-jN} = O(2^{-jN}), \end{aligned}$$

where C *depends only on* f *and* φ.

3.5 Jackson and Bernstein Theorems for Wavelets

Jackson in 1911 has given some theorems relating $E_n(f)$ to the smoothness properties of given function f. From the Jackson theorem, if $f \in Lip_M(\alpha)$ with $0 < \alpha \leq 1$ then $E_n(f)$ must converge to zero with at least the rapidity of $n^{-\alpha}$. The converse of Jackson theorem is Bernstein theorem. If $0 < \alpha < 1$ and $\{n^{-\alpha}E_n(f)\}$ is bounded, then $f \in Lip_M(\alpha)$.

Suppose we have an MRA with a C^1 scaling function φ such that

$$|\varphi(x)| \leq c(1+|x|)^{-A} \tag{3.14}$$

and

$$|\varphi'(x)| \leq c(1+|x|)^{-A} \ \text{ for some } A > 3. \tag{3.15}$$

According to assumption (3.14) we see that the series $\sum_{k\in\mathbb{Z}} a_k\varphi(x-k)$ is absolutely and almost uniformly convergent for all sequence $\{a_k\}_{k\in\mathbb{Z}}$ satisfying $|a_k| \leq c + |k|^{\beta}$ with $\beta < A - 1$.

Theorem 3.12 *[47] (Jackson's inequality) If* f *has a p-modulus of continuity and* P_j

(1.18) *is the projection onto* V_j,

$$P_j f(x) = \int_{-\infty}^{\infty} f(t) 2^j \varphi(2^j t, 2^j x) dt,$$

then

$$\|f - P_j f\|_{L_p} \leq c\, \omega_p(f; 2^{-j}), \tag{3.16}$$

for all $j \in \mathbb{Z}$, *where* c *is a constant.*

Remark 3.13 *In this theorem we do not assume that* $f \in L_p(\mathbb{R})$.

Theorem 3.14 *[47] (Bernstein's inequality) For each* p, $1 \leq p \leq \infty$, *there exists a constant* c *such that for* $f \in V_j$ *we have*

$$\omega_p(f; t) \leq c \min(2^j t, 1) \|f\|_p . \tag{3.17}$$

Theorem 3.15 *[47] Suppose we have an MRA with a scaling function* φ *satisfying* (3.14) *and* (3.15) *and an associated wavelet* ψ *also satisfying*

$$|\psi(x)| \leq c(1 + |x|)^{-A}. \tag{3.18}$$

Let Q_j *be the projection defined by* (2.11) *. For* $0 < \alpha < 1$ *and* $1 \leq p,\ s \leq \infty$ *and any* *function* $f \in V_j$ *such that* $\|f\|_{B_p^{\alpha,s}} < \infty$ *and all the following conditions hold:*

$$\left(\sum_{j\in\mathbb{Z}}[2^{j\alpha}s_j^p(f)]^s\right)^{\frac{1}{s}} < \infty, \tag{3.19}$$

$$\left(\sum_{j\in\mathbb{Z}}[2^{j\alpha}\left\|f - P_j f\right\|_p]^s\right)^{\frac{1}{s}} < \infty; \tag{3.20}$$

$$\left(\sum_{j\in\mathbb{Z}}[2^{j\alpha}\left\|Q_j f\right\|_p]^s\right)^{\frac{1}{s}} < \infty; \tag{3.21}$$

$$\left(\sum_{j\in\mathbb{Z}}\left[2^{j\alpha}\left(\sum_{k\in\mathbb{Z}}2^{j(\frac{1}{2}-\frac{1}{p})p}\left|\langle f,\psi_{j,k}\rangle\right|^p\right)^{\frac{1}{p}}\right]^s\right)^{\frac{1}{s}} < \infty; \tag{3.22}$$

where $s_j^p(f) = \inf\left\{\|f-g\|_p : g \in V_j\right\}$.

Conversely, if f is a function such that

$$\omega_p(P_j f;1) \longrightarrow 0 \ as \ j \longrightarrow -\infty, \tag{3.23}$$

and any one of (3.19) – (3.22) holds, then $\|f\|_{B_p^{\alpha,s}} < \infty$.

Chapter 4

Walsh System, Wavelet Packets, Walsh-type Wavelet Packets and their Approximation

4.1 Walsh system

We present two equivalent definitions of the Walsh system [30, 40].

Definition 4.1 *(Rademacher function) Let R_0 be the 1-periodic function (i.e. $R_0(x+1) = R_0(x)$) whose value on $[0,1)$ is*

$$R_0(x) = \chi_{[0,\ \frac{1}{2})}(x) - \chi_{[\frac{1}{2},\ 1)}(x).$$

Define a sequence of functions $R_1,\ R_2,\ \ldots\ R_k, \ldots$ by

$$R_k(x) = R_0(2^k x), where\ k = 1, 2, 3, \ldots \tag{4.1}$$

The functions $R_k,\ k = 1, 2, 3, \ldots$ *are called the Rademacher's functions. Rademacher functions can be used to construct Walsh function.*

Definition 4.2 *(Walsh system) Let Walsh function* W_n *be the 1-periodic function. The Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is defined in terms of the Rademacher functions as follows: For* $n = 0,\ W_0(x) = \chi_{[0,\ 1)}(x) = 1,$ *and other values of n,*

$$n = \sum_{k=1}^{j_n} a_k 2^{k-1} = a_1 + a_2 2^1 + a_3 2^2 + \ldots. + a_{j_n} 2^{j_n - 1}, \tag{4.2}$$

where a_k *is* 0 *or* $1,\ k = 1, 2, \ldots, j_n$ *, we define*

$$\begin{aligned} W_n(x) &= \prod_{k=1}^{j_n} (R_k(x))^{a_k} \\ &= (R_1(x))^{a_1} (R_2(x))^{a_2} \ldots (R_{j_n}(x))^{a_{j_n}},\ n = 1, 2, \ldots \end{aligned} \tag{4.3}$$

The set $\{W_n\}_{n=0}^{\infty}$ *is known as Walsh system.*

Alternative definition of Walsh system:

The *Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is defined recursively on* $\mathbb{R}$ *by letting*

$$W_0(x) = \chi_{[0,\ 1)}(x) = 1 \text{ and} \tag{4.4}$$

$$W_{2n}(x) = W_n(2x) + W_n(2x+1),\ n = 1, 2, 3, \ldots \tag{4.5}$$

$$W_{2n+1}(x) = W_n(2x) - W_n(2x+1),\ n = 0, 1, 2, \ldots \tag{4.6}$$

The family $\{W_n(x)\}_{n=0}^{\infty}$ is an orthogonal system of $L_2(\mathbb{R})$ and is called the Walsh system. The Walsh system is the basic wavelet packet associated with the Haar MRA.

Examples of some walsh functions:

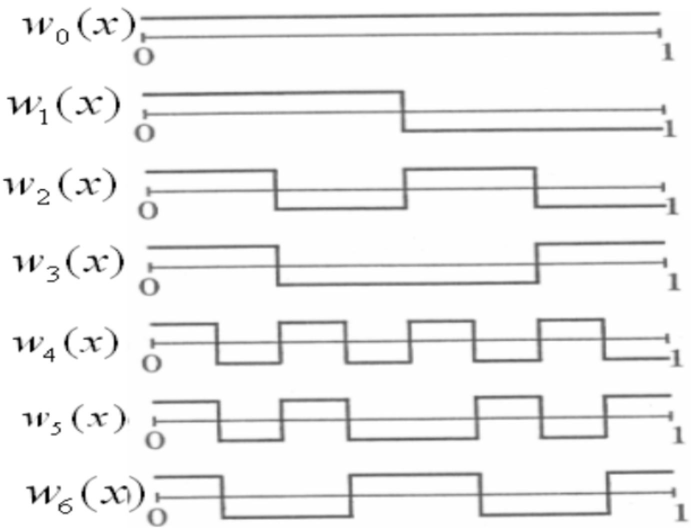

Figure 4.1: Walsh Functions

$$
\begin{aligned}
W_1(x) &= W_0(2x) - W_0(2x+1) = \chi_{[0,\ 1)}(2x) - \chi_{[1,\ 2)}(2x) \\
&= \chi_{[0,\ \frac{1}{2})}(x) - \chi_{[\frac{1}{2},\ 1)}(x) \\
W_2(x) &= W_1(2x) + W_1(2x+1) \\
&= \chi_{[0,\ \frac{1}{4})}(x) - \chi_{[\frac{1}{4},\ \frac{1}{2})}(x) + \chi_{[\frac{1}{2},\ \frac{3}{4})}(x) - \chi_{[\frac{3}{4},\ 1)}(x)
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= W_1(2x) - W_1(2x+1) \\
&= \chi_{[0,\ \frac{1}{2})}(2x) - \chi_{[\frac{1}{2},\ 1)}(2x) - \chi_{[1,\ \frac{3}{2})}(2x+1) - \chi_{[\frac{3}{2},\ 2)}(2x+1) \\
&= \chi_{[0,\ \frac{1}{4})}(x) - \chi_{[\frac{1}{4},\ \frac{1}{2})}(x) - \chi_{[\frac{1}{2},\ \frac{3}{4})}(x) + \chi_{[\frac{3}{4},\ 1)}(x) \\
W_4(x) &= W_2(2x) + W_2(2x+1) \\
&= \chi_{[0,\ \frac{1}{8})}(x) - \chi_{[\frac{1}{8},\ \frac{1}{4})}(x) + \chi_{[\frac{1}{4},\ \frac{3}{8})}(x) - \chi_{[\frac{3}{8},\ \frac{1}{2})}(x) + \chi_{[\frac{1}{2},\frac{5}{8})}(x) - \\
&\quad \chi_{[\frac{5}{8},\ \frac{3}{4})}(x) + \chi_{[\frac{3}{4},\frac{7}{8})}(x) - \chi_{[\frac{7}{8},\ 1)}(x)
\end{aligned}
$$

and so on.

Theorem 4.3 *[30] The Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is a complete orthonormal system in* $L_2[\mathbb{R})$.

Let $f \in L_2[0,1]$, *then* $\sum\limits_{n=0}^{\infty} \langle f, W_n \rangle W_n(x) = \sum\limits_{n=0}^{\infty} c_n W_n(x)$ is called the Walsh series for f. Where $c_n = \langle f, W_n \rangle$, $n = 1, 2, 3, ...$ are called the Walsh coefficients.

Theorem 4.4 *[40] If f is continuous on $[0,1]$, then the Walsh series converges to f a.e. in $[0,1]$.*

4.2 Approximation by Walsh Polynomial

Definition 4.5 *(Walsh Polynomials) A linear combination of Walsh functions is known as Walsh polynomial.*

Definition 4.6 *[40] (Dyadic Group G) Let the dyadic group G be the set of all sequences $\{x_n\}$, $x_n = 0, 1$ for $n = 1, 2, 3, ...$*

For each $n \in \mathbb{N}$, let

$$P_n := \{f \in C(G) : \ supp(f) \subseteq [0, n)\},$$

where $C(G)$ is the continuous function defined in G. Thus P_n is the collection of Walsh polynomials of order less than n.

Let $(X, \|\cdot\|_X)$ be a Banach space over the dyadic group G. Given an operator T on X we shall denote its operator norm by

$$\|T\|_X := \sup_{f \in X, \|f\|_{X \leq 1}} \|Tf\|_X$$

For each P_n is an n-dimensional subspace of X and that the partial sum operator S_n is a projection of X onto P_n, let $S_n g = P$ for all $g \in P_n$, and $S_n f \in P_n$ for all $f \in X$.

Theorem 4.7 *[39] Let $n \in \mathbb{N}$ and $T_n : X \longrightarrow P_n$ be a projection. Then the operator norms of S_n and T_n are related by $\|S_n\|_X \leq \|T_n\|_X$.*

To measure of the rate of approximation of an $f \in X$ by polynomials in P_n, define by (3.1).

Since each P_n is a finite dimensional subspace of X, it is clear that for every $f \in X$ there is at least one polynomial $g_n \in P_n$ such that

$$E_n(f, X) := \|f - g_n\|_X .$$

i.e., the infimum above is attained.

Such a polynomial g_n will be called a best approximation of f in P_n. It need not be unique.

If $f \in L_2(G)$, then $E_n(f, L_2(G)) := \inf_{g \in P_n} \|f - g\|_{L_2(G)}$

For each $n \in \mathbb{N}$ and $f \in X$ it is clear that

$$\|f - S_n f\|_X \leq \|f - g_n\|_X + \|S_n(f - g_n)\|_X ,$$

for any polynomial $g_n \in P_n$. Therefore,

$$\|f - S_n f\|_X \leq (1 + \|S_n\|_X) E_n(f, X). \tag{4.7}$$

A sharp estimate can be obtained for $X := L_p(G)$, $1 < p < \infty$. Indeed for each such p there exists a constant C_p (depending only on p) such that

$$\|S_n\|_p \leq C_p, \quad (n \in \mathbb{N}).$$

Consequently, (4.7) implies

$$\|f - S_n f\|_p \leq (1 + C_p) E_n(f, L_p(G)), \tag{4.8}$$

for $f \in L_p(G),\ n \in \mathbb{N},$ and $1 < p < \infty$. Thus for a given $f \in L_p(G), 1 < p < \infty$, the rate of approximation by Walsh polynomial of order n is no better than that of the Walsh -Fourier partial sum $S_n f$.

Theorem 4.8 *[39] Let $f \in X,\ \alpha > 0$ and for given $j \in \mathbb{Z},$ let $N = 2^j$. Then the following conditions are equivalent :*

i) $f \in Lip_M(\alpha,\ X),$

ii) $\|f - S_{2^j}\|_p = O(2^{-j\alpha})$ *as* $j \longrightarrow \infty,$

iii) $E_{2^j}(f, X) = O(2^{-\alpha j})$ *as* $j \longrightarrow \infty,$

iii) $E_N(f, X) = O(N^{-\alpha})$ *as* $N \longrightarrow \infty,$

iv) $\omega(f,\ 2^{-j})_X = O(2^{-j\alpha})$ *as* $j \longrightarrow \infty.$

Theorem 4.9 *[39] Suppose $f \in Lip(\alpha,\ X)$ and $\alpha > 0$. Then*

$$\|E_n f - f\|_X = \begin{cases} O(n^{-\alpha}),\ 0 < \alpha < 1 \\ O\left(\frac{\log n}{n}\right), \quad \alpha = 1 \\ O\left(\frac{1}{n}\right), \quad \alpha > 1. \end{cases} \qquad \text{as } n \longrightarrow \infty.$$

4.3 Wavelet Packets

From the previous chapters we know that orthonormal wavelet bases have a frequency localization which is proportional to 2^j at the resolution level j. The wavelet bases have poor frequency localization when j is large. For some applications, specially for speech signal processing, it is more convenient to have orthonormal bases with better frequency localization. This will be provided by the wavelet packets, which are obtained from wavelets associated with MRAs. Wavelet packets are the generalization of wavelets. A family of Walsh functions is an example of wavelet packets.

Definition 4.10 *(Wavelet packets) Let* $h = \{h_k : k \in \mathbb{Z}\}$ *and* $g = \{g_k : k \in \mathbb{Z}\}$ *be two sequences in* $l_2(\mathbb{Z})$*. Fix the initial functions* $w_0,\ w_1 \in L_2(\mathbb{R})$ *and for each integer* $n > 0$*, define*

$$\begin{aligned} w_{2n}(x) &= \sum_{k \epsilon \mathbb{Z}} h_k w_n(2x-k) = H w_n(x) && (4.9) \\ w_{2n+1}(x) &= \sum_{k \epsilon \mathbb{Z}} g_k w_n(2x-k) = G w_n(x). && (4.10) \end{aligned}$$

The collection of functions $w_n(x-k)$ *form an orthogonal basis of* $L_2(\mathbb{R})$ *if* $\varphi = w_0$ *and* $\psi = w_1$ *are the scaling function and mother wavelet, respectively, of an orthogonal multiresolution analysis of* $L_2(\mathbb{R})$*. The operators* H, G *are defined by sequences* $h,\ g$ *satisfying the following conditions for all integers* n *and* m*:*

$$\begin{aligned} \sum_{k \epsilon \mathbb{Z}} h_k h_{(k+2n)} &= 2\delta(n), && (4.11) \\ \sum_{k \epsilon \mathbb{Z}} g_k g_{(k+2n)} &= 2\delta(n), && (4.12) \\ \sum_{k \epsilon \mathbb{Z}} g_k h_{(k+2n)} &= 0, && (4.13) \\ \sum_{k \epsilon \mathbb{Z}} [h_{(n+2k)} h_{(m+2k)} + g_{(n+2k)} g_{(m+2k)}] &= 2\delta(n-m), && (4.14) \end{aligned}$$

where $\delta(n) = \begin{cases} 1, & for\ n = 0, \\ 0, & for\ n \neq 0. \end{cases}$

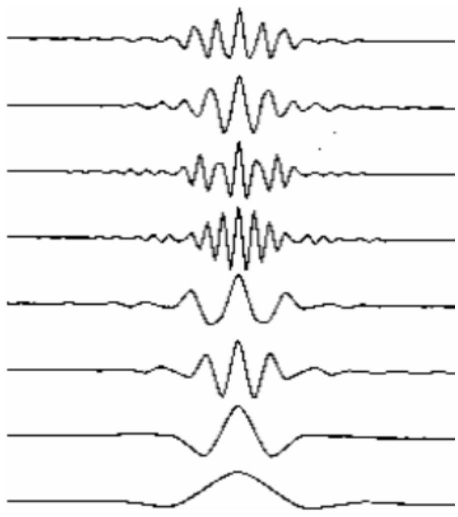

Figure 4.2: Example of a wavelet packet

The collection of functions $\{w_n(x-k)\}_{n\geq 0, k\in\mathbb{Z}}$ defined by (4.9) and (4.10) is called a *wavelet packet*. Sequences $h,\ g$ satisfying (4.11)-(4.13) are called orthogonal conjugate quadrature filters (orthogonal CQF).

Theorem 4.11 [30] *The family of wavelet packets* $\{w_n(x-k)\}_{n\geq 0, k\in\mathbb{Z}}$ *is an orthonormal basis of* $L_2(\mathbb{R})$, $\forall\ n \geqslant 0$.

For a scaling function φ with associated wavelet ψ we have constructed the corresponding wavelet packets given by (4.9) and (4.10). The set

$$\left\{2^{\frac{j}{2}} w_n(2^j x - k) : \ j \in \mathbb{Z},\ n = 0, 1, 2, ...\right\} \tag{4.15}$$

is overcomplete in $L_2(\mathbb{R})$. In fact, this system (4.15) contains the wavelet basis

$$\left\{2^{\frac{j}{2}} \psi(2^j x - k) : \ j, k \in \mathbb{Z}\right\} \text{(choose } n = 1\text{)},$$

and the wavelet packets $\{w_n(x-k): \ k \in \mathbb{Z}, \ n = 0, 1, 2, ...\}$(choose $j = 0$).

Advantage of wavelet packets:

1. Wavelet interpreted two parameters scale and frequency, but wavelet packets interpreted three parameters position, scale and frequency.

2. Wavelet packet bases offers a particular way of coding of a given signals.

3. Wavelet packet method gives the efficient decomposition selection of a given signal.

4. Reconstruction by wavelet packet method gives more exact features than general wavelet method.

5. Wavelet packet method improve the poor frequency localization of wavelet bases.

4.4 Examples of Wavelet Packets

Walsh functions are special cases of wavelet packets, where $h_0 = h_{-1} = g_0 = -g_{-1} = 1$, with $h_k = g_k = 0 \ for \ k \ \notin \{0, -1\}$ to define H and G, and functions $\varphi = 1$ and $\psi = G\omega_0 = G.1 = G$.

4.4.1 Shannon Wavelet Packets

For each $n \geq 0,$ the Shannon function is defined by

$$S_n(x) = \frac{\sin\left[\pi(n+1)(x-\frac{1}{2})\right] - \sin\left[\pi n(x-\frac{1}{2})\right]}{\pi(x-\frac{1}{2})}. \tag{4.16}$$

The Shannon functions are the elements of the doubly-indexed set $\{S_{nk}\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ defined by $S_{nk} = S_n(x-k)$. It is an orthonormal basis for $L_2(\mathbb{R})$[45, 46].

Shannon functions can also be obtained by the above recursion (4.9) and (4.10), if the conditions (4.11)-(4.13) are removed. Take

$$h_k = \frac{\sin\left[\frac{\pi}{2}(k-\frac{1}{2})\right]}{\frac{\pi}{2}(k-\frac{1}{2})}; \quad g_k = (-1)^k \frac{\sin\left[\frac{\pi}{2}(k-\frac{1}{2})\right]}{\frac{\pi}{2}(k-\frac{1}{2})}, \tag{4.17}$$

to define H and G, and

$$\varphi(x) = \frac{\sin\left[\pi(x-\frac{1}{2})\right]}{\pi(x-\frac{1}{2})}; \quad \psi(x) = \frac{\sin\left[2\pi(x-\frac{1}{2})\right] - \sin\left[\pi(x-\frac{1}{2})\right]}{\pi(x-\frac{1}{2})}, \tag{4.18}$$

for the initial functions.

The operators H and G act as Fourier multipliers:

$$\widehat{\omega}_{2n}(\xi) = \frac{1}{2} m_0\left(\frac{\xi}{2}\right) \widehat{\omega}_n\left(\frac{\xi}{2}\right); \quad \widehat{\omega}_{2n+1}(\xi) = \frac{1}{2} m_1\left(\frac{\xi}{2}\right) \widehat{\omega}_n\left(\frac{\xi}{2}\right), \tag{4.19}$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i k\xi}$ and $m_1(\xi) = \sum_{k\in\mathbb{Z}} g_k e^{-2\pi i k\xi}$. Functions m_0 and m_1 are 1-periodic, and trigonometric polynomials whenever h and g are finitely supported.

In the Walsh case,

$$m_0 = 1 + e^{2\pi i\xi} = 2e^{\pi i\xi}\cos\pi\xi,$$

and

$$m_1 = 1 - e^{2\pi i\xi} = -2e^{\pi i\xi}\sin\pi\xi.$$

In the Shannon case, one can take

$$m_0(\xi) = \begin{cases} 2, \ if\ k-\frac{1}{4} \le \xi < k+\frac{1}{4}\ for\ some\ \text{integer}\ k \\ 0, \ \ otherwise; \end{cases}$$

$$m_1(\xi) = \begin{cases} 2, \; if \; k+\frac{1}{4} \le \xi < k+\frac{3}{4} \; for \; some \text{ integer } k \\ 0, \; otherwise; \end{cases} = 2 - m_0(\xi).$$

4.4.2 Daubechies Wavelet Packets

The filters h and g that define the compactly-supported orthonormal wavelets of Daubechies [10] can be used here. For example, the Daubechies filters of length 4, which produce a scaling function supported in $[0,3]$ that satisfies $\varphi = H\varphi$, and a mother wavelet also supported in $[0,3]$ that satisfies $\psi = G\varphi$, are

$$h_k = \begin{cases} \frac{1+\sqrt{3}}{4}, \; if \; k=0; \\ \frac{3+\sqrt{3}}{4}, \; if \; k=-1; \\ \frac{3-\sqrt{3}}{4}, \; if \; k=-2; \\ \frac{1-\sqrt{3}}{4}, \; if \; k=-3; \\ 0, \; otherwise; \end{cases}, \quad g_k = \begin{cases} \frac{1-\sqrt{3}}{4}, \; if \; k=0; \\ -\frac{3-\sqrt{3}}{4}, \; if \; k=-1; \\ \frac{3+\sqrt{3}}{4}, \; if \; k=-2; \\ -\frac{1+\sqrt{3}}{4}, \; if \; k=-3; \\ 0, \; otherwise; \end{cases}.$$

Note that $g_k = (-1)^k h_{-3-k}$.

For every positive integer $N > 1$, there is a Daubechies wavelet supported in $[0, 2N]$ which belongs to the smoothness class C^d for $d \approx \frac{N}{5}$[10]. Since Daubechies' wavelets form an orthonormal MRA, the associated wavelet packets $\{w_{nk}\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ form an orthonormal basis for $L_2(\mathbb{R})$, and they are just as smooth as the mother wavelet and the scaling function, because the filters are finitely supported. Unfortunately, though they are smooth, these wavelet packets are not uniformly bounded.

4.4.3 Walsh Type Wavelet Packets

We now define a class of wavelet packets that can be seen as a natural generalization of Walsh functions. In particular, each wavelet packet system in the class turns out to be equivalent to the Walsh functions in $L_p(\mathbb{R}),\ 1 < p < \infty$.

Definition 4.12 *(Haar Filter): The pair of conjugate quadrature filters (CQF) given by* $h_0 = h_1 = \frac{1}{2},\ h_k = 0$ *otherwise, and* $g_k = (-1)^k h_{1-k}$ *are called the Haar filters.*

Definition 4.13 *(Walsh Type Wavelet Packets): If the pair of CQFs* (h^J, g^J) *satisfy the Haar filters for sufficiently large* $J \geq J_0$*. The resulting wavelet packets are called Walsh-type wavelet packets.*

Definition 4.14 *(Shannon Type Wavelet Packets): If the pair of CQFs* (h^J, g^J) *satisfy the Shannon filters for sufficiently large* $J \geq J_0$*. The resulting wavelet packets are called Shannon-type wavelet packets.*

Theorem 4.15 *[31, 32] The Walsh and Shannon Type Wavelet Packet series converge pointwise almost everywhere.*

For $f \in L_2[0,1]$, the Walsh-type wavelet packet series is

$$f = \sum_{n \geq 0, |k| \leq N} \langle f, W_{n,k} \rangle \, W_{n,k}(x), \tag{4.20}$$

where $\langle f, W_{n,k} \rangle$ are the Walsh-type wavelet packet coefficients.

Chapter 5

Variants of Haar wavelet

In chapter 2 we have discussed the Haar wavelet and its properties. In this chapter we will discuss two new variants of the Haar wavelet, called Rationalized Haar wavelet and Non-uniform Haar wavelet [19, 33, 35].

5.1 Rationalized Haar wavelet

Lynch et al. [25, 37] have rationalized the Haar transform by deleting the irrational numbers and introducing the integer powers of two. This modification results in what is called the rationalized Haar (RH) wavelet. The rationalized Haar wavelet preserves all the properties of the original Haar wavelet.

Definition 5.1 *The rationalized Haar functions* $RH(r,t), r = 1, 2, 3, ...$ *are composed* *of only three values* $+1, -1$ *and* 0 *and can be defined on the interval* $[0, T)$ *as [33].*

$$RH(r,t) = \begin{cases} 1, \ 0 \leq t < T \quad and\ r = 0 \\ 1, \quad J_1 \leq t < J_{1/2} \\ -1, \quad J_{1/2} \leq t < J_0 \\ 0, \quad otherwise, \end{cases} \tag{5.1}$$

where $J_u = \frac{j-u}{2^i}T,\ u = 0, \frac{1}{2}, 1.$ *The value of* r *is defined by two parameters* i *and* j *as* $r = 2^i + j - 1, \quad i = 0, 1, 2, 3, ...\ , j = 1, 2, 3, ..., 2^i.$

For example,

$$\begin{aligned} i &= 0,\ j = 1,\ then\ J_1 = 0,\ J_{\frac{1}{2}} = \frac{1}{2}T,\ J_0 = T, \\ i &= 1,\ j = \begin{cases} 1,\ then\ J_1 = 0,\ J_{\frac{1}{2}} = \frac{1}{4}T,\ J_0 = \frac{1}{2}T, \\ 2,\ then\ J_1 = \frac{1}{2}T,\ J_{\frac{1}{2}} = \frac{3}{4}T,\ J_0 = T, \end{cases} \\ i &= 2,\ j = \begin{cases} 1,\ then\ J_1 = 0,\ J_{\frac{1}{2}} = \frac{1}{8}T,\ J_0 = \frac{1}{4}T, \\ 2,\ then\ J_1 = \frac{1}{4}T,\ J_{\frac{1}{2}} = \frac{3}{8}T,\ J_0 = \frac{1}{2}T, \\ 3,\ then\ J_1 = \frac{1}{2}T,\ J_{\frac{1}{2}} = \frac{5}{8}T,\ J_0 = \frac{3}{4}T, \\ 4,\ then\ J_1 = \frac{3}{4}T,\ J_{\frac{1}{2}} = \frac{7}{8}T,\ J_0 = T. \end{cases} \end{aligned}$$

The first eight RH functions are shown in Fig.5.1 where $r = 0, 1, 2, ..., 7$.

The RHF is a wavelet because it satisfies all the conditions of wavelet.

The RH functions satisfy the orthogonal relation

$$\int_0^T RH(r,t)RH(v,t)dt = \begin{cases} 2^{-i}T \quad for\ r = v \\ 0 \quad for \quad r \neq v, \end{cases} \tag{5.2}$$

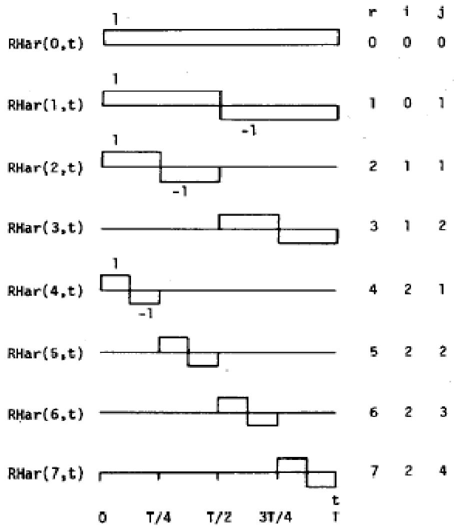

Figure 5.1: A set of the RH functions (r=0 to 7)

where $v = 2^n + m - 1,\ n = 0, 1, 2, 3, ...,\ \ m = 1, 2, 3, ..., 2^n.$

Now we will approximate any general function by Rationalized Haar wavelet. Let $f(t) \in L_2[0, T]$ and the RH wavelet series of $f(t)$ can be expanded as

$$f(t) = \sum_{r=0}^{\infty} c_r RH(r, t), \tag{5.3}$$

where the RH wavelet coefficients are

$$c_r = \frac{2^i}{T} \int_0^T f(t) RH(r, t) dt,\ r = 2^i + j - 1,\quad i = 0, 1, 2, 3, ..., j = 1, 2, 3, ...2^i.$$

For computation we need a finite sum. Consider the finite sum of the series (5.3), for fixed integer α, $\ i = 0, 1, 2, 3, ...\alpha,$ then $\ j = 2^\alpha$ and $\ r = 2^\alpha + 2^\alpha - 1 = 2^{\alpha+1} - 1 =$

$N-1$, where $N = 2^{\alpha+1}$,

$$g(t) = \sum_{r=0}^{N-1} c_r RH(r,t).$$

Then the error of the approximation is

$$\begin{aligned}
\|f-g\|_{L_2}^2 &= \left\| f - \sum_{r=0}^{N-1} c_r RH(r,t) \right\|_{L_2}^2 \\
&= \left\| \sum_{r=N}^{\infty} c_r RH(r,t) \right\|_{L_2}^2 \\
&\leq \sum_{r=N}^{\infty} |c_r|^2 \, \|RH(r,t)\|_{L_2}^2 \\
&= \sum_{r=N}^{\infty} |c_r|^2 \\
&\sim \sum_{r=N}^{\infty} 2^{-r} \sim 2^{-N} = O(2^{-N}). \qquad (5.4)
\end{aligned}$$

The RH coefficients in each r level is $c_r \sim 2^{-r}$ (for details one can see [33]). So it is obvious that the approximation order is improved for the RH wavelet in comparison to uniform Haar wavelet [35].

5.2 Non-uniform Haar wavelet

5.2.1 Haar scaling function and non-uniform Haar wavelet

Recall that the Haar scaling function is $\varphi(x) = \chi_{[0,1)}(x)$. For any $\alpha_1, \alpha_2 \in [0,1]$, such that $\alpha_1 < \alpha_2$, we have $\varphi(\frac{x-\alpha_1}{\alpha_2-\alpha_1}) = \chi_{[\alpha_1,\ \alpha_1)}(x)$. Hence, for any $\alpha \in (0,\ 1)$, we obtain $\varphi(x) = \chi_{[0,\alpha)}(x) + \chi_{[\alpha,1)}(x) = \varphi(\frac{x}{\alpha}) + \varphi(\frac{x-\alpha}{1-\alpha})$.

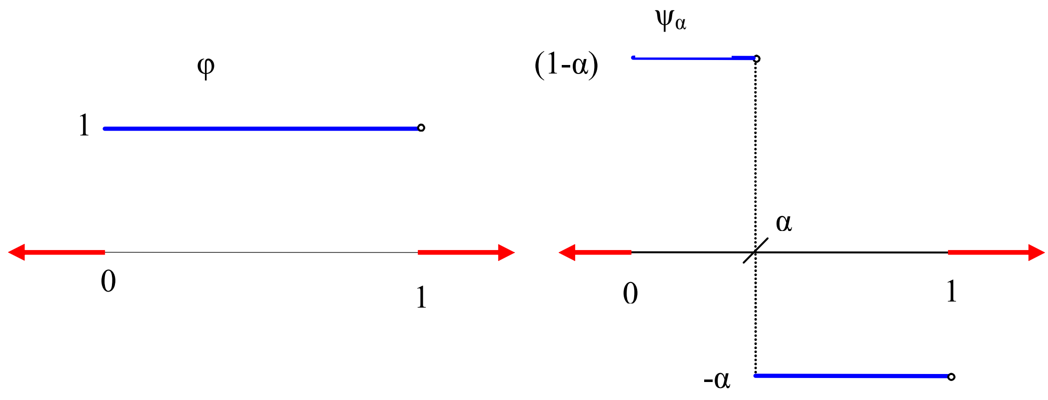

Figure 5.2: (a) Haar scaling function, (b) Non-uniform Haar wavelet.

We define the non-uniform Haar wavelet ψ_α by

$$\begin{aligned}\psi_\alpha(x) &= (1-\alpha)\chi_{[0,\ \alpha)}(x) - \alpha\chi_{[\alpha,1)}(x) \\ &= (1-\alpha)\varphi(\frac{x}{\alpha}) - \alpha\varphi(\frac{x-\alpha}{1-\alpha}). \end{aligned} \tag{5.5}$$

The graphs of φ and the wavelet ψ_α are in Fig. 5.2.

Note that for any $n \in \mathbb{N}$
$\int\limits_{-\infty}^{\infty} |\varphi(x)|^n \, dx = 1$ and $\int\limits_{-\infty}^{\infty} |\psi_\alpha(x)|^n \, dx = \alpha(1-\alpha)\left\{(1-\alpha)^{n-1} + \alpha^{n-1}\right\}.$

5.2.2 Non-uniform Haar wavelet from MRA

Let $\{\Delta_m\}_{m\in\mathbb{Z}}$ be a family of partitions of $\mathbb{R}$ such that the partition Δ_{m+1} is finer than Δ_m for any $m \in \mathbb{Z}$. Let $\Delta_m = \{x_k^m\}_{m,k\in\mathbb{Z}}$ such that

$$x_k^m = x_{2k}^{m+1} < x_{2k+1}^{m+1} < x_{2k+2}^{m+1} = x_{k+1}^m, \tag{5.6}$$

and $\lim\limits_{k\longrightarrow-\infty} x_k^m=-\infty, \quad \lim\limits_{k\longrightarrow\infty} x_k^m=\infty$

For each $m,\ k\in\mathbb{Z}$, we get

$$\varphi_k^m(x)=\varphi\left(\frac{x-x_k^m}{x_{k+1}^m-x_k^m}\right)=\chi_{[x_k^m,\ x_{k+1}^m\ )}(x)=\begin{cases}1,\ x\in[x_k^m,\ x_{k+1}^m\ ),\\ 0,\quad otherwise,\end{cases} \tag{5.7}$$

and for $\alpha_k^m=\alpha\in(0,\ 1)$ define

$$\alpha_k^m=\frac{x_{2k+1}^{m+1}-x_{2k}^{m+1}}{x_{2k+2}^m-x_{2k}^m}=\frac{x_{2k+1}^{m+1}-x_k^m}{x_{k+1}^m-x_k^m}, \tag{5.8}$$

we get

$$\psi_k^m(x)=\psi_{\alpha_k^m}^m\left(\frac{x-x_k^m}{x_{k+1}^m-x_k^m}\right)=\begin{cases}(1-\alpha_k^m),\ x\in[x_{2k}^{m+1},\ x_{2k+1}^{m+1}\ ),\\ -\alpha_k^m,\ x\in[x_{2k+1}^{m+1},\ x_{2k+2}^{m+1}\ )\ ,\\ 0,\qquad otherwise.\end{cases} \tag{5.9}$$

Properties for the functions φ_k^m and ψ_k^m:

i) $\int\limits_{-\infty}^{+\infty}\varphi_k^m(x)dx=x_{k+1}^m-x_k^m$,

ii) $\int\limits_{-\infty}^{+\infty}\psi_k^m(x)dx=0.$

and for $n\in\mathbb{Z}$

iii) $\int\limits_{-\infty}^{+\infty}|\varphi_k^m(x)|^n\,dx=x_{k+1}^m-x_k^m$,

iv) $\int\limits_{-\infty}^{+\infty}|\psi_k^m(x)|^n\,dx=\alpha_k^m(1-\alpha_k^m)\left\{(1-\alpha_k^m)^{n-1}+(\alpha_k^m)^{n-1}\right\}\left(x_{k+1}^m-x_k^m\right).$

For any $m,\ n\in\mathbb{Z}$ and $k,\ l\in\mathbb{Z}$ we have

v) $\int\limits_{-\infty}^{+\infty}\varphi_k^m(x)\psi_l^n(x)dx=0$,

vi) $\int\limits_{-\infty}^{+\infty}\varphi_k^m(x)\varphi_l^n(x)dx=\delta_{kl}^{mn}\left(x_{k+1}^m-x_k^m\right)$,

vii) $\int\limits_{-\infty}^{+\infty} \psi_k^m(x)\psi_l^n(x)dx = \delta_{kl}^{mn}\alpha_k^m(1-\alpha_k^m)\left(x_{k+1}^m - x_k^m\right),$

$$where\ \delta_{kl}^{mn} = \begin{cases} 1, & if\ m = n\ and\ k = l, \\ 0, & elswhere. \end{cases}$$

Theorem 5.2 *[19] The functions* $\varphi_k^{m-1},\ \psi_k^{m-1}, \varphi_{2k}^m\ ,\ \varphi_{2k+1}^m$ *are related as follows*

$$\varphi_k^{m-1}(x) = \varphi_{2k}^m(x) + \varphi_{2k+1}^m(x), \tag{5.10}$$

$$\psi_k^{m-1}(x) = (1-\alpha_k^{m-1})\varphi_{2k}^m(x) - \alpha_k^{m-1}\varphi_{2k+1}^m(x), \tag{5.11}$$

$$and\ \varphi_{2k}^m(x) = \alpha_k^{m-1}\varphi_k^{m-1} + \psi_k^{m-1}, \tag{5.12}$$

$$\varphi_{2k+1}^m(x) = (1-\alpha_k^{m-1})\varphi_k^{m-1}(x) - \psi_k^{m-1}(x). \tag{5.13}$$

Let us define the vector spaces $V_m = \overline{span}\left\{\varphi_k^m(x)\ /k \in \mathbb{Z}\right\}$ and $W_m = \overline{span}\left\{\psi_k^m(x)\ /k \in \mathbb{Z}\right\}$ for any $m \in \mathbb{Z}$. From the previous properties we obtain, $V_m = V_{m-1} \oplus W_{m-1}$.

5.2.3 Approximation by Non-uniform Haar wavelets

To obtain the Haar wavelet transform, let $f^m(x) \in V_m$ such that $f^m(x) = \sum\limits_{k\in\mathbb{Z}} a_k^m\varphi_k^m(x)$ by using theorem (5.2) we obtain,

$$\begin{aligned} f^m(x) &= \sum_{k\in\mathbb{Z}} \left[a_{2k}^m\varphi_{2k}^{m+1}(x) + a_{2k+1}^m\varphi_{2k+1}^{m+1}(x)\right] \\ &= \sum_{k\in\mathbb{Z}}[a_{2k}^m\left\{\alpha_k^m\varphi_k^m(x) + \psi_k^m(x)\right\} \\ &\quad +a_{2k+1}^m\left\{(1-\alpha_k^m)\varphi_k^m(x) - \psi_k^m(x)\right\}] \\ &= \sum_{k\in\mathbb{Z}} a_k^m\varphi_k^m(x) + \sum_{k\in\mathbb{Z}} c_k^m\psi_k^m(x) \end{aligned} \tag{5.14}$$

where $a_k^m = \alpha_k^m a_{2k}^m + (1-\alpha_k^m) a_{2k+1}^m$,

$c_k^m = a_{2k}^m - a_{2k+1}^m$,

$\varphi_k^m \in V_m$ *and* $\psi_k^m \in W_m$.

We know that compression means only reducing the number of coefficients needed to represent a function. Let our given function be

$$f^M(x) = \sum_{m=0}^{M-1} \sum_{k=0}^{2^M-1} a_k^M \varphi_k^M(x).$$

We obtain by decomposition, the representation with respect to an orthogonal basis

$\{\varphi_0^0\} \cup \{\psi_k^m / k = 0, 1, 2, ...2^m - 1;\ m = 0, 1, 2, ...M-1\ \}$ of V_M

$$f^M(x) = a_0^0 \varphi_0^0(x) + \sum_{m=0}^{M-1} \sum_{k=0}^{2^M-1} c_k^m \psi_k^m(x). \tag{5.15}$$

Then the error of the approximation in $L_2(\mathbb{R})$ is

$$\begin{aligned} \left\| f - f^M(x) \right\|_{L_2}^2 &= \left\| \sum_{m=2^M}^{\infty} \sum_{k=0}^{2^M-1} c_k^m \psi_k^m(x) \right\|_{L_2}^2 \\ &\le c \sum_{m=2^M}^{\infty} \sum_{k=0}^{2^M-1} |c_k^m|^2 \left\| \psi_k^m(x) \right\|_{L_2}^2 \\ &\le c \sum_{m=2^M}^{\infty} \sum_{k=0}^{2^M-1} 2^{-k} \left\| \psi_k^m(x) \right\|_{L_2}^2, \end{aligned}$$

where $c_k^m \sim 2^{-\frac{k}{2}}$(for details see sec. 2.5.1). Now by using properties (vii) for the functions ψ_k^m,we get

$$\begin{aligned}\|\psi_k^m(x)\|_{L_2}^2 &= \left\|\alpha_k^m(1-\alpha_k^m)\left(x_{k+1}^m - x_k^m\right)\right\|_{L_2}^2 \\ &= |\alpha(1-\alpha)|^2 \cdot 1, \text{ by using } (5.8) \\ &= O((\alpha(1-\alpha))^2), \end{aligned} \tag{5.16}$$

where $\alpha_k^m = \alpha \in (0,\ 1)$.

Therefore

$$\left\|f - f^M(x)\right\|_{L_2} = O(2^{-\frac{M}{2}}\alpha(1-\alpha)). \tag{5.17}$$

If we consider $\alpha = \frac{1}{2}$, we get the uniform Haar wavelet and in this case the approximation order is $O(2^{-\frac{M}{2}})$.

If $\alpha \neq \frac{1}{2}$, we get the non-uniform Haar wavelet, For example say $\alpha = \frac{1}{3}, \frac{1}{4}$, then the approximation order are $O(3^{-2}.2^{-\frac{M}{2}})$ and $O(4^{-2}.2^{-\frac{M}{2}})$ respectively. So it is clear that in the case of non-uniform Haar wavelet, the approximation order is improved in comparison to uniform Haar wavelet.

Example 5.3 *Let $f(x) = x^2$ and $x \in [0, 1]$.*

In the case of uniform Haar wavelet, we approximate f on $[0, 1]$ by $f_j = \sum\limits_{n=0}^{2^j-1} c_n\varphi_n$, where $c_n = 2^j \int\limits_{n2^{-j}}^{(n+1)2^{-j}} f(x)dx$ and φ_n are the scaling functions of the Haar wavelet. By using MATLAB with $j = 2$, we obtain $\|f - f_2\|_{L_2} = 0.8545$ and the graph shown in Fig.5.3.

In the case of non-uniform Haar wavelet with $j = 2$, we obtain $\|f - f_2\|_{L_2} = 0.7552$ and the graph shown in Fig.5.4.

Conclusion 5.4 *In this section we have seen that by using rationalized Haar wavelet*

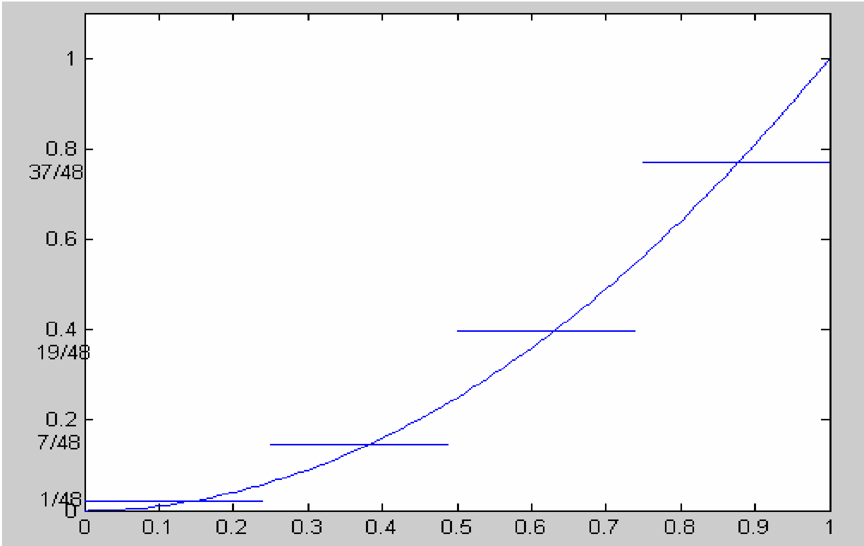

Figure 5.3: The function f and its uniform Haar approximation f_j for $j = 2$.

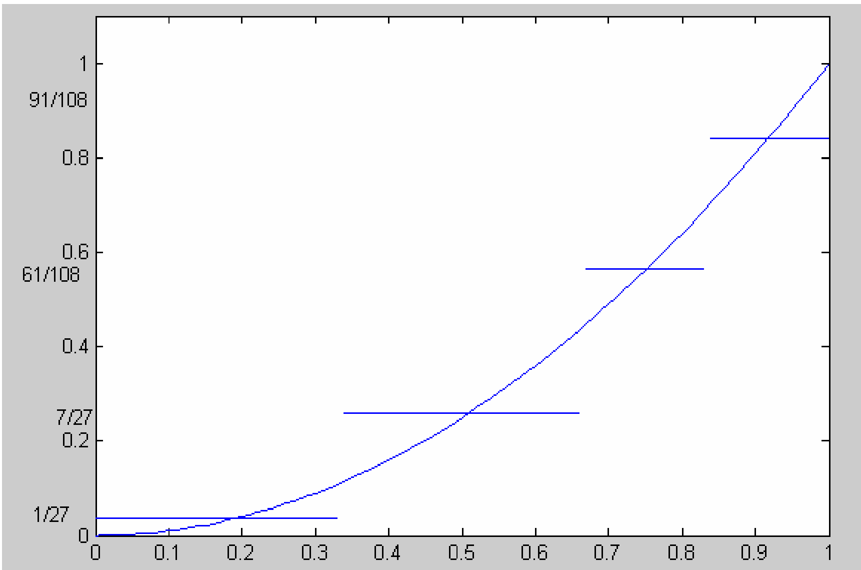

Figure 5.4: The function f and its non-uniform Haar approximation f_j for $j = 2$.

and non-uniform Haar wavelet, the approximation order is improved compare with simple uniform Haar wavelet.

Chapter 6

Open Problems and Further Research

Approximation by Walsh polynomials has been studied but their extensions to Walsh type wavelet packets have not been studied till now. One may be tempted to extend the Theorems 4.8 and 4.9 to Walsh type wavelet packets. M.V. Wickerhauser [45, 46] observed that Daubechies wavelet packet series convergence almost everywhere. This property also holds for Walsh and Shannon type wavelet packets.

One can also study the approximation by rationalized Haar and non-uniform Haar wavelet in different function spaces.

Bibliography

[1] P. Burt and E. Adelson, *A multiplication spline with application to image mosaics*, ACM Trans. Graphics, **2**(1983), pp. 217-236.

[2] P. Burt and E. Adelson, *The Laplacian Pyramid as a compact image code*, IEEE Trans. commun. **31**(1983), pp. 482-540.

[3] O. Christensen and K.L. Christensen, *Approximation Theory, From Taylor polynomials to wavelets*, Birkhäuser, Boston, 2004.

[4] E.W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, Inc., 1966.

[5] A. Cohen, *Numerical Analysis of wavelet methods*, Elsevier, 2003.

[6] A. Cohen, I. Daubechies and J. Feauveau, *Biorthogonal basis of compactly supported wavelets*, Commun. Pure and Appl. Math.,**45**(1992), pp. 485-560.

[7] R.R. Coifman, Y. Meyer and M.V. Wickerhauser, *Size properties of wavelet packets*, (1992) , pp. 453-470, Wavelets and Their Applications, Jones and Bartlett.

[8] I. Daubechies, A. Grossmann, and J. Morlet, *Painless non-orthogonal expansion*, J. Math. Phys.,**27**(1986), pp. 1271-1283.

[9] I. Daubechies, *Time-frequency localization operators: A geometric phase space approach*, IEEE Trans. Inform. Theory. **34**(1988), pp. 605-612.

[10] I. Daubechies, *Orthogonal bases of compactly supported wavelets*, Commun. Pure & Appl. Math. **41**(1988), pp. 909-996.

[11] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory. **36**(1990), pp. 961-1005.

[12] I. Daubechies, *Ten lecturers on wavelets*, SIAM, Philadelphia, 1992.

[13] I. Daubechies, *Orthonormal basis of compactly supported wavelets*, Commun. Pure Appl. Math.**4**(1998), pp. 909-996.

[14] P. J. Davis, *Interpolation and Approximation*, Dover publications, Inc., 1975.

[15] L. Debnath, *Wavelet transformation and their applications*, Birkhäuser Boston, 2002.

[16] L.T. Dechevsky, N. Grip and L. Persson, *Sharp error estimates for approximation by wavelet frames in Lebesgue space*, J. Analysis and Application, **1**(2003), pp. 11-31.

[17] R. A. deVore, *Non-linear approximation*, Acta Numerica, Cambridge University Press, **7**(1998), pp. 51-150.

[18] D. Donoho, *Interpolating wavelet transforms*, J. Appl. Comput. Harmonic Analy, **2**(1994), pp. 1-12.

[19] F. Dubeau, S. Elmejdani and R. Ksantini, *Non-uniform Haar wavelets*, Appl. Math. comput. **159**(2004), pp. 675-693.

[20] K. M. Furati, M.Z. Nashed, A.H. Siddiqi (eds), *Mathematical Models and Methods for real World system*, Taylor and Francis/Marcel-Dekker/CRC/Chapmann Hall, New York, 2005.

[21] W. Glabisz, *Direct Walsh-wavelet packet method for variational problems*, Appl. Math. comput. **159**(2004), pp. 769-7681.

[22] A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape*, SIAM J. Anal.**15**(1984), pp. 723-736.

[23] K. Hollig, *Finite element method in B-Spline*, SIAM, Philadelphia, 2003.

[24] J. Lippus, *Wavelet coefficients of functions of generalized Lipschitz classes*, In Sampling theory & Appl., F.A. Marvasti, ed. Ins. of Electronic & computer science, Riga, Latvia, (1995), pp. 167-172.

[25] R.T. Lynch and J.J. Reis, *Haar transform image coding*, in Pro. National Telecommun. Conf., Dallas, TX, (1976), pp. 44.3-1-44.3-5.

[26] S. Mallat, *A wavelet tour of signal processing*, Academic Press, 1999.

[27] Y. Meyer, *Ondelettes et Operateurs*, Vol. **1** and **2**(1990) Hermann, Paris.Tour of Signal Processing, Academic Press, New York.

[28] Y. Meyer, *Wavelets: their past and their future*, Progress in wavelet analysis and its applications ,Gif-sur-Yvette,(1993), pp 9-18.

[29] I.P. Natanson, *Constructive function theory,* Mosco, 1949.

[30] M. Nielsen, *Size properties of wavelet packets*, Ph.D. thesis, Washington University, U.S.A., 1999.

[31] M. Nielson, *Walsh type wavelet Packet Expansions*, Applied and Computational Harmonic Analysis, **9(**2000), pp. 265-285.

[32] M. Nielson, *On Convergence of wavelet packet expansions*, Approximation theory Appl., **18(1)**(2002), pp. 34-50.

[33] M. Ohkita and Y. Kobayashi, *An application of Rationalized Haar functions to solution of linear differential equations*, IEEE Trans. on Circuits and Systems, **9**(1986), pp. 853-862.

[34] M.J.D. Powell, *Approximation Theory and methods*, Cambridge University Press,Cambridge,New York, 1991.

[35] M. Razzaghi and Y. Ordokhani, *An application of rationalized Haar functions for variational problems*, Appl. Math. comput. **122**(2001), pp. 353-364.

[36] M. Razzaghi and J. Nazarzadeh, *Walsh functions, Wiley Encyclopedia of Electrical and Electronics Engineering*, Wiley, New York. **23**(1999), pp. 429-440.

[37] J.J. Reis, R.T. Lynch and J. Butman, *Adaptive Haar transform video bandwidth reduction system for RPV's*, in Proc. Ann. Meeting Soc. Photo Optic Ins. Eng.(SPIE), (1976), San Diego, Canada.

[38] HL. Resnikoff and RO. Walls, *In wavelet analysis, the Scalable Structure of information*, New York, Berlin, Heidelberg: Springer, 1998.

[39] F. Schipp, W. R. Wade, P. Simon, *An Introduction to Dyadic Harmonic Analysis*, Academi Budapest and Adam Hilger, Bristol, New York, 1990.

[40] A. H. Siddiqi, *Walsh Functions*, AMU Publication, Aligarh, 1977.

[41] A. H. Siddiqi, *Applied Functional Analysis*, Marcel- Dekker Publ., New York, 2004.

[42] A. H. Siddiqi, G. Korvin, U. Mosco, S. Dahlke, W. Frieden (editors), *Theme issues Wavelet and Fractal Methods in Science and Engineering: Part I*, AJSE, **28(1c)**(2003), pp. 1-5.

[43] W. R. Wade, *Dyadic Harmonic Analysis*, Contemporary Mathematics, AMS Publ. **208**(1997), pp. 313-350.

[44] D.F. Walnut, *An Introduction to Wavelet Analysis*, Birkhäuser, Boston, 2001.

[45] M.V. Wickerhauser, *Basic and convergence properties of wavelet packets*, International Conference on Wavelet Analysis and its Applications, AMS/IP Studies in Advanced Mathematics. **25**(2002), pp. 279-287.

[46] M. V. Wickerhauser, *Some problems related to wavelet packet bases and convergence*, Theme issues Wavelet and Fractal Methods in Science and Engineering, AJSE, **28(1c)**(2003), pp. 47-58.

[47] P. Wojtaszczyk, *A Mathematical introduction to wavelets*, Cambridge University press, 1997.

[48] A. Zygmund, *Trigonometric Series*, 2d. ed., Cambridge, London, 1959.

VITA

- Md. Rafiqul Islam
- Born in Bangladesh on July 11, 1966.
- Received B.Sc. (Honours) in Mathematics from University of Rajshahi, Bangladesh in 1990.
- Received M.Sc. in Applied Mathematics from University of Rajshahi, Bangladesh in 1992.
- Joined as a Lecturer in Mathematics in Computer Science and Engineering Discipline, Khulna University, Bangladesh in 1994.
- Joined as an Assistant professor in Mathematics in Computer Science and Engineering Discipline, Khulna University, Bangladesh in 1997.
- Joined as an Associate professor in Mathematics Discipline, Khulna University, Bangladesh in 2002.
- Received M.S. Degree in Mathematics in May, 2005.
- E-mail: mrislam@kfupm.edu.sa, mrislam_66@yahoo.com