B-SPLINE COLLOCATION METHODS FOR COUPLED NONLINEAR SCHRÖDINGER EQUATION

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by

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LIST OF ABBREVIATIONS

BCuBSM	Besse Cubic B-Spline Collocation Method
BCuTBSM	Besse Cubic Trigonometric B-Spline Collocation Method
BRM	Besse Relaxation Method
BVP	Boundary Value Problem
CAGD	Computer-Aided Geometric Design
CBVM	Compact Boundary Value Method
CN	Crank-Nicolson
CNLSE	Coupled Nonlinear Schrödinger Equation
CuBS	Cubic B-Spline
CuBSM	Cubic B-Spline Collocation Method
CuTBS	Cubic Trigonometric B-Spline
CuTBSM	Cubic Trigonometric B-Spline Collocation Method
DE	Differential Equation
FBCuBSM	Fourth-Order Besse Cubic B-Spline Collocation Method
FBCuTBSM	Fourth-Order Besse Cubic Trigonometric B-Spline Collocation Method
FCuBSM	Fourth-Order Cubic B-Spline Collocation Method
FCuTBSM	Fourth-Order Cubic Trigonometric B-Spline Collocation Method
FDM	Finite Difference Method
FEM	Finite Element Method

FTCS	Forward Time Centred Space
FVM	Finite Volume Method
KdV	Korteweg-de Vries
KG	Klein-Gordon
NLSE	Nonlinear Schrödinger Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

LIST OF SYMBOLS

$B_{w,j}$	j-th B-Spline basis function of degree w
C	continuity
C_j	time dependent unknown at x_j
C_x	vector contains x-coordinates
C_y	vector contains y-coordinates
D	differential operator
D_j	time dependent unknown at x_j
D	vector of D_j
E	shift operator
E_j	time dependent unknown at x_j
E	vector of E_j
F_j	time dependent unknown at x_j
F	vector of F_j
I_1, I_2	mass conservations
I_3	momentum conservation
I_4	energy conservation
L_2	Euclidean norm
L_∞	maximum absolute error
M	total number of t interval

N total number of x interval

R_j approximated solution of r at x_j

R vector of R_j

S_j approximated solution of s at x_j

S vector of S_j

$T_{w,j}$ j-th Trigonometric B-Spline basis function of degree w

U_j approximated solution of u at x_j

U vector contains approximated solution U_j

U_B approximated solution of u using CuBSM / BCuBSM

$(\tilde{U}_B)_{xx}$ new approximated solution of u_{xx} using FCuBSM / FBCuBSM

U_T approximated solution of u using CuTBSM / BCuTBSM

$(\tilde{U}_T)_{xx}$ new approximated solution of u_{xx} using FCuTBSM / FBCuTBSM

V_j approximated solution of v at x_j

V vector contains approximated solution V_j

V_B approximated solution of v using CuBSM / BCuBSM

$(\tilde{V}_B)_{xx}$ new approximated solution of v_{xx} using FCuBSM / FBCuBSM

V_T approximated solution of v using CuTBSM / BCuTBSM

$(\tilde{V}_T)_{xx}$ new approximated solution of v_{xx} using FCuTBSM / FBCuTBSM

e exponential

h space interval / x interval

i imaginary unit

j	space level
k	time level
p	given sine function
q	given sine function
r	exact solution (additional dependent variable)
s	exact solution (additional dependent variable)
t	time (independent variable)
u	wave amplitude (dependent variable)
v	wave amplitude (dependent variable)
w	degree of B-Spline basis
x	space (independent variable)
Δt	time interval
δ	amplification factor / eigenvalue
η	mode number
ε	cross-phase modulation coefficient
Θ	numerical order of convergence
θ	free parameter
ϑ	function of partial differential equation
$\mathcal{B}$	matrix of CuBS basis
$\mathcal{K}$	matrix of CuTBS basis

KAEDAH-KAEDAH KOLOKASI SPLIN-B UNTUK PERSAMAAN TAK LINEAR SCHRÖDINGER GANDINGAN

ABSTRAK

Dalam kajian ini, persamaan tak linear Schrödinger gandingan (PTLSG) yang memodelkan penyebaran gelombang cahaya pada gentian optik diselesaikan dengan menggunakan kaedah berangka iaitu Kaedah Perbezaan Terhingga dan kaedah kolokasi Splin-B. Persamaan itu didiskret mengikut ruang dan masa. Sebutan tak linear pada PTLSG didiskretkan menggunakan pendekatan Taylor dan pendekatan yang baru dibangunkan disebut sebagai Besse. Kaedah teta berwajaran digunakan bagi penyeluruhan skema di mana skema Crank-Nicolson (iaitu $\theta = 0.5$) dipilih. Terbitan masa didiskretkan dengan penghampiran beza ke depan. Untuk setiap pendekatan, dimensi ruang kemudiannya didiskretasi oleh lima kaedah kolokasi yang berbeza secara berasingan. Kaedah pertama bagi pendekatan Taylor adalah berdasarkan Kaedah Perbezaan Terhingga yang mana terbitan ruang digantikan dengan penghampiran beza pusat. Dua kaedah lain berasal dari Splin-B peringkat kedua yang dikenali sebagai Kaedah Kolokasi Splin-B Kubik dan Kaedah Kolokasi Splin-B Trigonometrik Kubik. Kemudian, dua kaedah baru yang ditambah baik dengan ketepatan peringkat keempat dibangunkan dan diperkenalkan sebagai Kaedah Kolokasi Splin-B Kubik Peringkat Keempat dan Kaedah Kolokasi Splin-B Trigonometrik Kubik Peringkat Keempat (KSTKuE). Sementara itu, bagi pendekatan Besse, kaedah peringkat kedua yang dicadangkan adalah Kaedah Relaksasi Besse, Kaedah Kolokasi Splin-B Kubik Besse dan Kaedah Kolokasi Splin-B Trigonometrik Kubik Besse. Di samping itu, pendekatan ini menggunakan Kaedah Kolokasi Splin-B Kubik Besse Peringkat Keempat dan Kaedah Kolokasi Splin-B Trigonometrik Kubik Besse Peringkat Keempat (KSTKuBE) pada PTLSG.

Mengikuti kesemua perkara di atas, kestabilan tanpa syarat bagi semua kaedah yang dicadangkan dibuktikan dengan menggunakan kaedah von Neumann. Analisis ralat, analisis konsistensi dan analisis penumpuan bagi kesemua kaedah kecuali KSTKuE dan KSTKuBE dilakukan. Kecekapan kesemua kaedah juga dinilai melalui aplikasi kepada dua masalah. Ralat mutlak bagi masalah tersebut dikira. Semua kaedah didapati menghasilkan anggaran yang baik. Peringkat penumpuan berangka juga dikira dan telah membuktikan penyataan teori yang berkaitan. Kesemua kuantiti konservasi dipelihara dengan sangat baik. Kesemua kaedah didapati menghasilkan penyelesaian anggaran yang tepat dan boleh digunakan untuk menyelesaikan PTLSG.

B-SPLINE COLLOCATION METHODS FOR COUPLED NONLINEAR SCHRÖDINGER EQUATION

ABSTRACT

In this study, the Coupled Nonlinear Schrödinger Equation (CNLSE) which models the propagation of light waves in optical fiber is solved using numerical methods namely Finite Difference Method (FDM) and B-Spline collocation methods. The equation was discretized in space and time. We propose the discretization of the nonlinear terms in the CNLSE following the Taylor approach and a newly developed approach called Besse. The theta-weighted method is used to generalize the scheme whereby the Crank-Nicolson scheme (i.e $\theta = 0.5$) is chosen. The time derivatives are discretized by forward difference approximation. For each approach, the space dimension is then discretized by five different collocation methods independently. The first method for Taylor approach is based on FDM whereby the space derivatives are replaced by central difference approximation. Another two methods come from second-order B-Spline known as Cubic B-Spline and Cubic Trigonometric B-Spline Collocation Methods. Then, two newly improved methods with fourth-order accuracy are developed and introduced as Fourth-Order Cubic B-Spline Method and Fourth-Order Cubic Trigonometric B-Spline Collocation Method (FCuTBSM). Meanwhile, for Besse approach, the proposed second-order methods are Besse Relaxation Method, Besse Cubic B-Spline and Besse Cubic Trigonometric B-Spline Collocation Methods. In addition, the approach implemented Fourth-Order Besse Cubic B-Spline Method and Fourth-Order Besse Cubic Trigonometric B-Spline Collocation Method (FBCuTBSM) on the CNLSE. Following all of the above, the unconditional stability of all the proposed methods is proven using the von Neumann method. Error analysis, consistency and convergence analysis for all methods excluding FCuTBSM and FBCuTBSM are conducted. Their efficiency is also evaluated through the application on two test problems. The absolute error for the problems is calculated. They are found to produce good approximations. The numerical order of convergence is also computed and proved the corresponding theoretical statement. All the conservation quantities are conserved well. All methods are found to produce reliable approximate solutions and are feasible for solving the CNLSE.

CHAPTER 1

INTRODUCTION

1.1 Introduction to Waves

A wave is a disturbance that moves through space or matter. For instance, water waves, sound and light. Wave transfers energy from one place to another place provided that it has velocity. The disturbance can be affected in the form of pressure, electrical intensity and others. There are several types of waves that can be mentioned according to how they move. They are given by longitudinal waves and transverse waves (Anyaegbunam, 2013). For longitudinal waves, the disturbance has the same direction as the wave. In other words, the waves oscillate parallel to the direction of energy transfer. For examples, sound waves and spring. Meanwhile, for transverse waves such as light, the disturbance is at the right angles to the wave's direction. These waves are also said to oscillate perpendicular to the direction in which the waves transmit the energy. In addition, the properties of classical light waves are reflection, refraction and diffraction. Waves always take the "sine waves" shape as in Figure 1.1. Otherwise, they can also have other interesting representation like in Figure 1.2 due to the interaction between waves. They can be added to each other in effect.

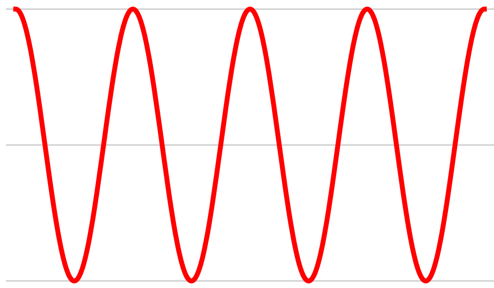


Figure 1.1: Sine waves shape

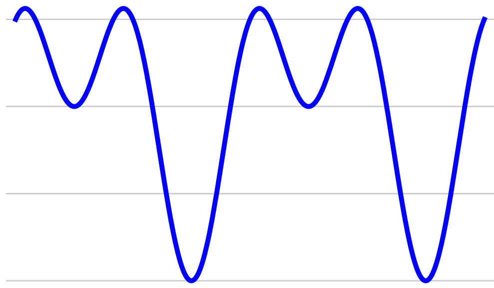


Figure 1.2: Interaction of waves

In general, a wave has three basic characteristics which are amplitude, wavelength and frequency. Amplitude is the height from the center line to the peak. Whereas wavelength is the length from one peak to the next peak. These characteristics are described in Figure 1.3. Then, frequency can be defined as a number of waves that pass a fixed point per unit of time. They are depicted in Figures 1.4(a) and 1.4(b) for low and high frequencies, respectively.

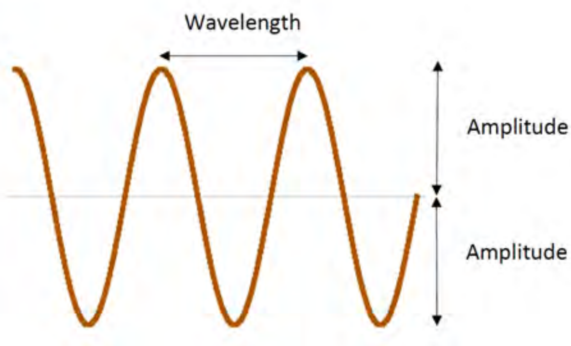
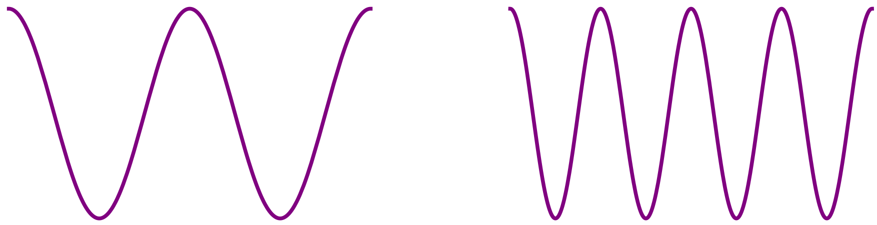


Figure 1.3: Characteristics of waves



(a) (b)

Figure 1.4: Waves with low frequency in (a) and high frequency in (b)

In addition, waves can be classified into mechanical waves and electrical waves. The waves that require a medium to travel through are called mechanical waves such as sound and earthquake waves. These waves can't travel through a vacuum. On the other hand, electrical waves such as radio, microwaves and light, are able to travel through a vacuum. The simplest wave model that is represented in the form of a Differential Equation (DE) is Korteweg-de Vries (KdV) equation (Knobel, 2000) given by

$$u_t + u u_x + u_{xxx} = 0,$$

where $u(x,t)$ is the wave amplitude of fluid while x and t are the spatial and temporal independent variables, respectively. This equation is the combination of dispersion and nonlinearity. It was first derived in 1872 by a French mathematician named Joseph Boussinesq to model the surface waves on shallow water. After two decades, two Dutchmen, Diedrik Korteweg (PhD advisor) and Custav de Vries (student) rediscovered the equation where it is named after them (Olver, 2016). KdV equation is classified as the nonlinear evolution equation. Following this, further exploration of wave phenomena which appeared in coupled vector form was derived. It started with modelling the two-layered fluid waves near ocean shores. In addition, it described the interaction process of two long waves. It is called coupled KdV equation and the general formula is (Ahmed & Biswas, 2013),

$$\begin{aligned} u_t + \alpha_1 \, u u_x + \beta_1 \, u_{xxx} + \mathscr{K}_1 \, v v_x &= 0, \\ v_t + \alpha_2 \, u v_x + \beta_2 \, v_{xxx} &= 0, \end{aligned}$$

where $u(x,t)$ and $v(x,t)$ are the wave amplitudes for two layers of fluid, α_i, β_i, $i = 1, 2$ and $\mathscr{K}_1$ are the constant coefficients. u_t and v_t are the evolution terms, $u u_x$ and $v v_x$ are the nonlinear terms, u_{xxx} and v_{xxx} are the dispersion terms, and $u v_x$ is the cross-nonlinear term. Several methods like eigen spectrum of contant potentials, differential transformation method, Hirota's direct method and numerical adaptive moving mesh PDEs method have been developed by researchers to produce the wave solution of coupled KdV equation (Esfandyari & Jafarizadeh, 2001; Gökdogan et al., 2012; Jaradat et al., 2017; Abdelrahman et al., 2020). Many nonlinear phenomena from various research fields can be modeled by this equation. In this study, we will explore the modelling of waves in optical fiber and further details will be discussed in the next section.

1.2 Optical Fiber and Its Modelling

There are numerous physical phenomena involving waves. One of them is the propagation of light wave in optical fiber. Optical fiber can be defined as a flexible and transparent fiber made by a silica or plastic which has a diameter slightly thicker than a human hair. It can be bent or twisted. The structures of optical fiber are jacket, cladding and core (Govind, 2002). Light ray will travel inside the core. They are pictured in Figure 1.5.

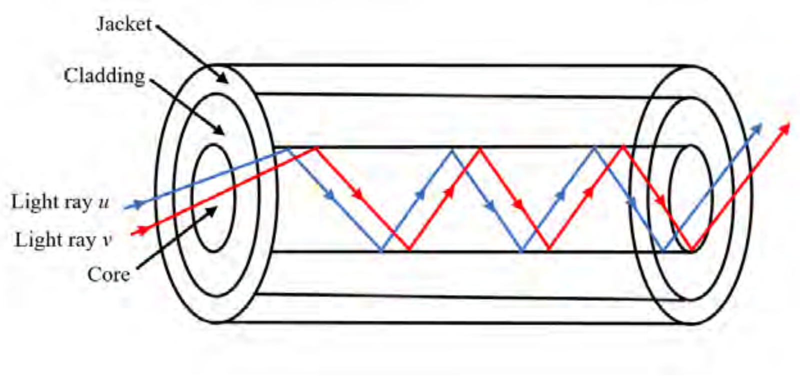


Figure 1.5: Structures of optical fiber

There are two types of optical fiber which are single-mode and multimode (Saleh & Teich, 1991). One signal can only be transmitted for the case of single-mode whereas multimode allows several signals to be transmitted. Not only that, single-mode and multimode have very small and large core diameters, respectively. Light propagation commonly exists in fiber optic communication in which the information is transferred from one place to another place. The information can be in term of voice, data and video (Akpan, 2014).

This type of communication is applied to various fields like public network, military applications, industrial, computers and many more (Kumari, 2017). This is due to its advantages in which optical fiber can carry information with very low power consumption. However, high-skilled installers are needed to implement it. This physical phenomenon becomes important and can be modelled using a DE. For instance, the propagation of pulses in two-mode nonlinear optic fibers is governed by Coupled Nonlinear Schrödinger Equation (CNLSE) (Ismail, 2008; Wang, 2014)

$$\mathbf{i}\, u_t + \frac{1}{2} u_{xx} + (|u|^2 + \varepsilon |v|^2) u = 0, \tag{1.1}$$

$$\mathbf{i}\, v_t + \frac{1}{2} v_{xx} + (\varepsilon |u|^2 + |v|^2) v = 0, \tag{1.2}$$

subject to the initial conditions

$$u(x,0) = f_1(x), \quad a \le x \le b, \tag{1.3}$$

$$v(x,0) = f_2(x), \quad a \le x \le b, \tag{1.4}$$

with the boundary conditions

$$u_x(a,t) = u_x(b,t) = 0, \quad t > 0, \tag{1.5}$$

$$v_x(a,t) = v_x(b,t) = 0, \quad t > 0, \tag{1.6}$$

where $\mathbf{i} = \sqrt{-1}$, ε is the cross-phase modulation coefficient, u and v represent the wave amplitudes in two polarizations.

1.3 Introduction to Differential Equations

Any real-life phenomenon involving physical, sociological or economic can be explained through mathematical terms. This is known as a mathematical model which is commonly presented through the DE. DE can be defined as an equation that includes derivatives of one or more unknown functions (i.e dependent variables) in relation to one or more independent variables. There are two types of DEs which are Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). An ODE is a DE with ordinary derivatives of one or more unknown functions in relation to one independent variable. Whereas, a PDE is an equation containing partial derivatives of one or more unknown functions in relation to two or more independent variables. PDE can take the following general form (Strauss, 2007):

$$\vartheta\left(x,t,u(x,t),\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t^2},\cdots\right)=0, \tag{1.7}$$

where $u(x,t)$ is the solution for equation (1.7). To identify the order of certain DE, one can look into the order of the highest derivative in the equation. DE is said to be linear if the function ϑ is a function of u only, no function of derivatives of u except coefficient multiply with derivatives and no multiplication of the derivative of u or no multiplication of u and its derivatives. Otherwise, it is nonlinear. Consider the general form of linear second-order PDE to be

$$\hat{a}\,\frac{\partial^2 u}{\partial x^2}+\hat{b}\,\frac{\partial^2 u}{\partial x\,\partial t}+\hat{c}\,\frac{\partial^2 u}{\partial t^2}+\hat{d}\,\frac{\partial u}{\partial x}+\hat{e}\,\frac{\partial u}{\partial t}+\hat{f}\,u=\hat{g}, \tag{1.8}$$

where coefficients $\hat{a},\hat{b},\hat{c},\hat{d},\hat{e},\hat{f}$ and $\hat{g}$ are the functions of x and t. When $\hat{g}=0$, equation (1.8) is considered to be homogeneous. Otherwise, it is nonhomogeneous. Solutions of PDE can be obtained and approximated through the application of analytical or numerical methods. However, sufficient conditions are required. There are two types of conditions which are initial and boundary conditions. Initial conditions describe the state of the solution u at the initial time t. Meanwhile, for boundary conditions, they describe how the solutions behave on their boundaries for all time. There are three basic types of boundary conditions which are Dirichlet boundary condition, Neumann boundary condition and Robin boundary condition. In this study, Neumann boundary condition will be used since the CNLSE is a PDE and it will give value of the derivative of solution on the boundary of the domain (Ismail, 2008; Wang, 2014). This condition is going to be discussed in Chapter 4.

1.4 B-Spline Approach for Approximating Solution of Differential Equation

Most of the real-life phenomena that experiencing changes are represented by DEs. There are many well-developed techniques used to obtain the solutions of DEs. Frequently, the systems portrayed by DEs are complex and have a long description. Analytical solutions are usually hard to get. In addition, for the case of solutions of PDEs, they can be very challenging, depending on the type of equation, the number of independent variables, the boundary and initial conditions and other factors. Thus, numerical methods become useful. The three classical choices for the numerical solutions of PDEs are Finite Difference Method (FDM), Finite Element Method (FEM) and Finite Volume Method (FVM).

FDM is the oldest and simplest numerical method. It uses local Taylor expansion to approximate the equations. Each derivative term in the equation will be replaced by certain difference-quotient approximations. They are given by forward difference scheme, backward difference scheme and central difference scheme. Consequently, a system of algebraic equations is generated and need to be solved. This method has second-order convergence. Unfortunately, FDM can only approximate the solutions at discrete points. Thus, approximation using piecewise-polynomial becomes a flexible alternative method.

One of the well-known piecewise-polynomial approximations is B-Spline. It is constructed by several curve segments of polynomial functions that join each consecutive knots or control points smoothly. B-Spline is initially introduced by Schoenberg

(1946) for the case of uniform knots. After that, Curry (1947) reviewed an article on the B-Spline over nonuniform knots. A few years later, de Boor (2001) began to use B-Spline as geometric representation. Then, a recursive evaluation of the B-Spline curve is named after him which is the de Boor algorithm.

Many researchers implemented this B-Spline approach. It has local support property in a specific range. It can be applied to different kinds of problems. For instance, a study on singular two-point boundary problems was done by Kadalbajoo and Aggarwal (2005). After that, Caglar and Caglar (2006) proved that the B-Spline approach can be extended either for solving homogeneous or nonhomogeneous equations. A third-degree of B-Spline function was tested and compared with FDM, FEM and FVM through a study by Caglar et al. (2006). Among the four of them, B-Spline was found to produce the best approximate solutions.

In addition to the above, B-Spline can also be applied to the nonlinear equations to get the approximate solutions. Mat Zin et al. (2014a) studied the efficiency of new trigonometric spline approach on the Nonlinear Klein-Gordon equation. More complicated study of implementing Cubic B-Spline Collocation Method (CuBSM) on Nonlinear Klein-Gordon equation and Klein-Gordon-Schrödinger equation was conducted by Mittal and Bhatia (2015). Chanthrasuwan et al. (2017) and Ahmad et al. (2017) applied the same B-Spline approach on Nonlinear Buckmaster equation and NLSE, respectively. Other than single equation, coupled equations can also be tested for the B-Spline approach. A few studies from several researchers have been shown to produce good results. CuBSM was implemented by Mittal and Arora (2011) on Coupled Viscous Burgers' equation while Ismail and Ashi (2014) worked on Coupled

KdV equation.

Besides that, in certain cases, Bézier curves and surfaces become popular in the designing world. However, when it comes to the collection of control points, it needs to implement a single order polynomial of high degree to approximate the control polygon or polyhedron. It makes the computation more complicated as it produced long and unstable computations. Contrary to B-Spline curves and surfaces, they only need low-degree approximations. This advantage will speed up the computation and save the cost involved. Additionally, even though splines can be developed by performing piecewise Bézier curves and surfaces, one needs to determine the location of control points carefully so that the Bézier segments have smooth continuous joins. Conversely with the B-Spline approach, curves and surfaces meet smoothly at joins for any arbitrary collection of control points. It is due to its continuity property.

1.5 Motivation and Issues

There are various types of numerical methods used to approximate DEs. The simplest one is finite difference approximation as it is easy for users to understand and implement. However, this method can only approximate solutions at selected points. Because of this, B-Spline approximation has become much used in recent studies. It is constructed by piecewise equations. Due to its structure, B-Spline is able to approximate solutions in the domain especially for the case of solving PDEs. In addition, it can model the solution curve up to certain continuity. It can design curves and surfaces through Computer Aided Geometric Design. In comparison to the FDM, FEM and FVM, B-Spline has a higher accuracy in approximation. Besides, both FDM and

B-Spline approximations are commonly tested and applied on single equation but few researchers have worked on coupled equations especially for the case of CNLSE. It might be due to the complexity of the problem. Hence, this leads to the idea of combining the finite difference approach with B-Spline collocation methods for solving CNLSE.

1.6 Objectives of The Study

The objectives of the study are:

1. To develop and implement newly improved methods of Taylor approach (i.e Fourth-Order Cubic B-Spline Collocation Method (FCuBSM) and Fourth-Order Cubic Trigonometric B-Spline Collocation Method (FCuTBSM)) on CNLSE.

2. To develop and implement Besse B-Spline methods (i.e Besse Relaxation Method (BRM), Besse Cubic B-Spline Collocation Method (BCuBSM) and Besse Cubic Trigonometric B-Spline Collocation Method (BCuTBSM)) on CNLSE.

3. To develop and implement newly improved methods of Besse approach (i.e Fourth-Order Besse Cubic B-Spline Collocation Method (FBCuBSM) and Fourth-Order Besse Cubic Trigonometric B-Spline Collocation Method (FBCuTBSM)) on CNLSE.

4. To investigate the performances on two test problems, von Neumann stability analysis and error analysis of all proposed methods.

1.7 Methodology

This study will begin with the literature review on the numerical methods used for solving the differential equations. It will highlight the methods of Cubic B-Spline (CuBS) and Cubic Trigonometric B-Spline (CuTBS). Their advantages and disadvantages are identified. Thus, this leads to the main study of this thesis which is applying the collocation methods for solving the CNLSE.

The equations will be introduced together with the boundary and initial conditions. Then, they will undergo time and space discretizations. Two approaches will be applied on the time dimension namely Taylor and Besse. For the Taylor approach, the time derivatives of CNLSE will be discretized by finite difference approximation. The theta-weighted method is also applied and the nonlinear terms of CNLSE will be linearized by Taylor expansion. After that, five different collocation methods are proposed and implemented independently for the space dimension.

These methods consist of FDM, two second-order B-Spline methods and two newly improved fourth-order B-Spline methods. All of them are categorized as Taylor B-Spline methods. For FDM, this method is chosen as a basic comparative method tested. Central difference approximation will be applied to the space dimension. Systems of linear equations are produced and need to be solved for the approximate solutions of CNLSE. For the case of second-order B-Spline methods, CuBSM and CuTBSM will be used to approximate the solutions at the collocation points. Both of them will generate systems of linear equations. Unknown coefficients are obtained by solving them in which later they are substituted back to the second-order B-Spline functions for the approximate solutions of CNLSE.

After that, newly improved fourth-order methods will be derived from the classical second-order B-Spline functions. The order of accuracy for the second derivative of B-Spline approximations is improved. This leads to the development of methods called FCuBSM and FCuTBSM. Both of them will also produce systems of linear equations and by solving them, the values for the unknown coefficients are generated. As a consequence, the B-Spline functions from this improvement are able to approximate the solutions of CNLSE.

Meanwhile, for the case of Besse approach, the same theta-weighted method and finite difference approximation are used to discretize the time dimension. However, the nonlinear terms of CNLSE will be replaced by additional variables instead of the Taylor expansion. Subsequently, five collocation methods are then implemented for the space discretization. They are listed as BRM, BCuBSM, BCuTBSM, FBCuBSM and FBCuTBSM. BRM will be derived from the application of finite difference approximation. Whereas for BCuBSM and BCuTBSM, they will use second-order B-Spline functions to collocate the solutions of CNLSE at points. Similarly, for the case of FBCuBSM and FBCuTBSM, the same fourth-order methods from the previous study are taken into place. All of them will also produce systems of linear equations but with extra unknown coefficients need to be solved. Their values are then substituted back into the B-Spline functions to reveal the approximate solutions.

On top of all the above, the stability analysis for all of the proposed methods will be analyzed using the von Neumann analysis. The error analysis is also determined through the investigation on truncation errors, consistency and convergence analyses. Lastly, the efficiency of all the proposed methods will be evaluated through two test problems. The approximate solutions are compared with the exact solutions. They will be tabulated in tables and illustrated through graph plotting. The numerical order of convergence and conservation of the methods will also be computed.

1.8 Outline of Thesis

This thesis consists of six chapters. Initially, brief introductions of waves, optical fiber and its modelling are done in Chapter 1. This has lead to the explanation on PDEs. Some well-known numerical methods used in dealing with the differential equations are mentioned and the B-Spline approach is highlighted. The advantages of applying it are discussed. This brings us to the motivation of this study. Not only that, the objectives, motivation and outline of this thesis will be described briefly.

Then, in Chapter 2, a review of the literature is presented. A survey on the numerical methods used by many researchers will be covered to identify the limitations of using them. They are the studies on the methods used for solving NLSE and CNLSE, CuBSM and CuTBSM for solving various equations. The results produced will be revealed and summarized in this chapter.

Meanwhile, Chapter 3 will explain three main basic concepts of the proposed methods for this thesis starting with the derivation of the FDM. Furthermore, the formulation and properties of CuBS and CuTBS basis functions will be discussed briefly. An example is chosen to illustrate the interpolation technique.

Numerical methods using the Taylor linearization approach will be presented in Chapter 4 for solving the CNLSE. FDM is selected as a comparative method. Second-order B-Spline methods which are CuBSM and CuTBSM will be implemented to obtain the approximate solutions. Then, two newly improved fourth-order B-Spline methods will be derived. They are known as FCuBSM and FCuTBSM. In addition, stability and error analyses will be investigated. The accuracy of the methods in this chapter is then evaluated through two test problems.

An approach called Besse will be introduced in Chapter 5 to replace the previous Taylor linearization approach. Five numerical methods will be proposed to obtain the approximate solutions of CNLSE. Three of them are second-order methods which are BRM, BCuBSM and BCuTBSM. While the other two methods are derived and implemented as fourth-order methods called FBCuBSM and FBCuTBSM. All of the proposed methods will undergo stability and error analyses. The same test problems in Chapter 4 are used to check the efficiency of the methods from this approach. A comparative study is done between all of the proposed methods in this thesis. Lastly, Chapter 6 will conclude all the findings from this research. Future works will also be suggested at the end of this chapter for further exploration in the future.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The NLSE and CNLSE are the nonlinear PDEs that are to be studied. In some cases, the analytical solutions are hard to compute, therefore numerical methods are becoming more favourable in solving the nonlinear equations. Many researchers developed the numerical methods for NLSE but few studies were done on CNLSE. Hence, studies utilising the B-Spline basis on CNLSE is executed to fill in the gap after going through the literature. This chapter will summarize the earlier recent studies on CNLSE, particularly the approximate methods used for solving it; CuBS and CuTBS.

2.2 Methods for Solving NLSE and CNLSE

NLSE and CNLSE belong to the wave equations as introduced in Chapter 1. Various methods can be used to solve them such as analytical, semi-analytical and numerical methods. Each method has its advantages and disadvantages in solving a certain equation. In this section, the literature on NLSE and CNLSE are presented as early as 2004.

A relaxation scheme was introduced by Besse (2004) for solving NLSE. This approach preserved the density and energy quantities. It also avoids the complications of dealing with nonlinear systems by replacing the NLSE with a system of two equations. Hence, a linear Schrödinger equation was obtained. The convergence analysis is shown to be second-order. Besides that, the method was conducted on several numerical tests and found to have good results.

The same approach as Besse (2004) applied on NLSE occurring in plasma physics by Oelz and Trabelsi (2014). This method satisfied the mass and energy conservations. The convergence analysis is proven theoretically. It required a low cost of computations and no particular spatial discretization was carried out.

Bhatt and Khaliq (2014) developed a new version of the Cox and Matthews third-order exponential time differencing Runge-Kutta scheme based on the $(1,2)$-Padé approximation to the exponential function. To improve the accuracy of the temporal direction, they used local extrapolation from the developed scheme that resulted in the method to be fourth-order in time. The space direction also has fourth-order accuracy after applying the Numorov/Douglas approximation in discretizing the space derivative. Besides, they tested on several examples such as systems of two NLSE including single soliton, the interaction of two solitons and system of four NLSE involving the interaction of four solitons. L_∞ and L_2 error norms were computed for each example. The mass conservation is proven to ensure the simulation of the scheme will not blow-up, as confirmed by the numerical tests. The scheme can be used to solve higher-dimensional problems.

In the same year, Wang (2014) proposed a compact finite difference scheme to solve the CNLSE. The scheme has high order accuracy. Nevertheless, there's a limitation in dealing with the nonlinear terms using the classical way. Hence, the research generated a vector form from the difference scheme. To prove the mass conservation theoretically, the energy method is used together with some lemmas. Besides, the convergence analysis was found to be $O(h^4 + \Delta t^2)$ where h and Δt are the mesh size and time step, respectively. By applying the proposed scheme on several test problems, the results conserved the discrete masses and energy.

Wang et al. (2015) applied fractional centred difference on space fractional CNLSE. The method is implicit and conserved discrete mass and energy. The fractional Sobolev space $H^{\alpha/2}$ and a fractional norm equivalence in $H^{\alpha/2}$ were created to obtain L_∞ error norm. Then, the convergence is shown to be of order two for both space and time directions by fractional Sobolev inequality. Furthermore, they tested the method on systems of coupled and decoupled equations. Both systems are shown to be of second-order in the maximum norm, as expected.

About a few years later, Wang and Wang (2017) presented a conservative scheme generated from Fourier pseudospectral, Crank-Nicolson and leap-frog schemes. The Fourier pseudospectral method was used to discretize the spatial direction. While for the time direction discretization, Crank-Nicolson and leap-frog scheme were used for the linear and nonlinear terms, respectively. The advantages of the presented scheme are: it can be decoupled, linearized and applied for the parallel computation as well as it preserved the mass and energy. Moreover, the energy method together with the classical interpolation theory were used to prove the error estimation of the presented scheme theoretically. As a result, there is no restriction on the grid ratio. To verify the finding numerically, they applied the scheme on two test problems which are single soliton and collisions of two solitons. The numerical results confirmed the theoretical convergence of the presented scheme.

Iqbal et al. (2020) made use of Galerkin finite element method to solve CNLSE. The method is derived from cubic B-spline function and weight functions. It is tested on three numerical experiments involving single solitary wave, collision of two solitary waves and collision of three solitary waves. Comparison between approximated and analytical solutions is done. The results are also compared with other published numerical results. In order to check the accuracy of the method, order of convergence is calculated. Not only that, the same approach is conducted on CNLSE by Iqbal et al. (2021) but by using different B-spline which is quintic B-spline function. The capability of the method is tested by computing the maximum errors, norms and conserved quantities of three numerical problems.

Most of the literature in this section used methods based on finite difference scheme. It is because the method is considered as simple and easy to implement for the DE cases. The idea behind it is the derivative of the DE is replaced by the differential quotients. Space and time are divided into a certain interval and approximated solutions are obtained at space and time points. However, this approach has a limitation when the domain is changed. The system needed to be modified all over again that will cause costly computations and time-consuming. Hence, this leads to the study on the method that has flexibility in obtaining solutions of differential equations which will be discussed in the next section.

2.3 Cubic B-Spline Method for Solving Various Equations

In general, B-Spline is a piecewise function that is joined by certain points called knots. It has minimal support to a given degree, smoothness and domain partition. The literature on this area started from the year 2006 onward and keep being explored actively until today.

CuBSM was applied on the two-point Boundary Value Problem (BVP) by Caglar et al. (2006). The method was compared with FDM, FEM and FVM. From the numerical results obtained, CuBSM showed that it has better accuracy in interpolating the smooth functions compared to the others due to the big difference of errors between them.

Following this, Caglar and Caglar (2006) found the numerical solutions of homogenous and non-homogenous singular BVP. The B-Spline approximation was used to solve BVP after a modification of the equation at a singular point. Four problems from previous studies involving one homogenous singular BVP and three non-homogenous BVP were tested and compared with the exact solutions. As a result, the numerical method showed that the approximated solutions have good agreement with the exact solutions.

Goh et al. (2011) used CuBSM to solve one-dimensional heat and wave equations. The temporal dimension was discretized using finite difference approximation while the CuBS was used for the spatial dimension. The efficiency of the method was tested by computing the numerical errors in which the approximated solutions were compared with the exact solutions. The heat and wave equations are proven to have convergence of $O(h^2+\Delta t^2)$ and $O(h^2+\Delta t)$, respectively. To illustrate the accuracy of the method, one-dimensional heat and wave equations of different mesh sizes were tested and compared with Forward Time Centred Space (FTCS) approach. As a result, the CuBSM showed less error than the FTCS.

Shortly after, Mittal and Arora (2011) made use of CuBSM to solve coupled system known as coupled viscous Burger's equation. Crank-Nicolson (CN) scheme was applied on the equation for the time discretization and the nonlinear terms undergo linearization. For the space discretization, CuBS functions were used. In this work, the von Neumann stability analysis was applied and the method is proved to be unconditionally stable. Apart from that, the order of convergence was computed and showed to have second-order in space. Several problems were tested to evaluate the accuracy of the proposed method through error norms computation. In conclusion, the approach is considered as a simple and straight-forward method.

Goh et al. (2012) studied CuBSM applied on one-dimensional heat and advection-diffusion equations to get the approximated solutions. Finite difference scheme and CuBS were selected to discretize the time and space dimensions, respectively. The stability of the method was analyzed by von Neumann stability analysis and it is shown to be unconditionally stable for the case of $\theta = 0.5$. Numerical results obtained from the test problems were compared with CN scheme and Compact Boundary Value Method (CBVM). CBVM is found to have better approximation than CuBSM. However, for CuBSM and CN scheme, they can be considered as comparable methods.

An attempt to use CuBSM for obtaining the numerical solutions of one-dimensional telegraph equation was made by Rashidinia et al. (2014). Two-level explicit difference scheme was generated based on the application of the finite difference scheme and CuBS on temporal and spatial derivatives, respectively. The approach leads to a linear tridiagonal system. Three examples of linear telegraph equations were considered. L_∞ norm, L_2 norm and Root Mean Square of errors were measured. The results indicated that the numerical method is more accurate compared to Quartic B-spline collocation method.

On the other hand, Ahmad et al. (2017) discussed the approximated numerical solutions of NLSE using two methods which are FDM and CuBSM. For FDM, the time derivative of NLSE was approximated using forward difference while for space, second-order central difference approximation was used. Whereas, CuBSM considered finite difference and CuBS approaches to discretize the time and space dimension, respectively. CuBSM is proved to be stable by von Neumann stability analysis. Both methods were analyzed by calculating the L_2 and L_∞ error norms of single soliton problem. The numerical results validated the stability analysis of the method. On top of that, CuBSM is found to have more accurate results than FDM.

The most recent literature is solving nonlinear singular BVP using new CuBSM by Iqbal et al. (2018a). The method was generated from CuBSM and new approximation of second-order derivative. As results, they obtained fifth-order accurate solutions. In contrast to FDM, new CuBSM can approximate solutions at any point in the domain and not limited for selected knots through the existence of piecewise function and singularity. However, this approach is novel for second-order singular BVP. Examples in the area of physiological sciences were tested and they found that new CuBSM produced better approximation compared to other methods in the literature such as

FDM, CuBSM and Exponential CuBSM. The new CuBSM is considered as simple, straight-forward and low-cost.

This method is also further studied by Iqbal et al. (2018b) for solving third-order Emden-Flower type equations. The space discretization was done by CuBSM together with new approximations for second-order and third-order derivatives. Approximated solutions from several third-order Emden-Flower type equations were computed to evaluate the accuracy of the method. Also, they converged to the exact solutions when the mesh sizes are reduced.

In conclusion, this section indicates that CuBSM became a popular method in solving different types of DEs including PDE, homogenous and non-homogenous DEs. The properties of the method enable researchers to get the solutions needed at any point in the domain. Next section will discuss an extended method that can improve the accuracy of approximated solutions for some cases.

2.4 Cubic Trigonometric B-Spline Method for Solving Various Equations

Trigonometric B-Spline is constructed based on trigonometric functions instead of polynomial functions in B-Spline. Geometric properties of trigonometric B-Spline are local support, smoothness and the ability to model the local phenomena. CuTBSM is believed to be a suitable method to handle problems involving trigonometric nature instead of CuBSM. It can also solve some linear and nonlinear PDEs. Hence, it has attracted the attention of many researchers in the literature.

Abd Hamid et al. (2010) used CuTBSM to solve linear two-point BVP of order two. Four initial examples were tested to check the feasibility of the method. Both L_∞ and L_2 error norms were computed for each example by comparing the approximated solutions with the exact solutions. One of the tested examples showed better results compared to the CuBSM because of the existence of trigonometric in the DE. Hence, to confirm the analysis, further problems with trigonometric nature were tested. As results, all of them produced slightly better approximation solutions than solutions obtained by CuBSM. It can be concluded that CuTBSM is the right method to solve trigonometric problems compared to CuBSM.

The approximated solutions of the one-dimensional hyperbolic equation which is a wave equation were produced using CuTBSM by Abbas et al. (2014). The trigonometric B-Spline approach is believed to have more accurate approximations of linear and nonlinear BVP problems compared to the classical B-Spline functions. The temporal discretization was done by central difference approximation. While CuTBSM was utilized for the spatial discretization. The von Neumann stability analysis showed that the method is unconditionally stable. Several test problems were chosen to analyze the efficiency of the method. The approximated solutions were compared with the exact solutions to obtain the error norms. As results, CuTBSM showed less errors compared to other previous methods such as CuBSM, 3 point explicit method, the optimal explicit method and others. Besides, it required less CPU time and smaller storage in computing the approximated solutions. The CuTBSM is declared as a simple and straightforward method. It can be used to get the solutions at any intermediate point in the space dimension.